Chapter 8

Equations in Polar Coordinates

I'm very good at integral and differential calculus,
I know the scientific names of beings animalculous.

W.S. Gilbert

8.1 Domain $0 \le r < \infty$, $0 \le \varphi \le 2\pi$

$$\frac{\partial^\alpha T}{\partial t^\alpha} = a\left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \varphi^2}\right) + \Phi(r,\varphi,t), \tag{8.1}$$

$$t = 0: \quad T = f(r,\varphi), \qquad 0 < \alpha \le 2, \tag{8.2}$$

$$t = 0: \quad \frac{\partial T}{\partial t} = F(r,\varphi), \qquad 1 < \alpha \le 2, \tag{8.3}$$

$$\lim_{r\to\infty} T(r,\varphi,t) = 0. \tag{8.4}$$

The solution:

$$\begin{aligned} T(r,\varphi,t) = & \int_0^{2\pi}\int_0^{\infty} f(\rho,\phi)\,\mathcal{G}_f(r,\varphi,\rho,\phi,t)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\phi \\ & + \int_0^{2\pi}\int_0^{\infty} F(\rho,\phi)\,\mathcal{G}_F(r,\varphi,\rho,\phi,t)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\phi \\ & + \int_0^{t}\int_0^{2\pi}\int_0^{\infty} \Phi(\rho,\phi,\tau)\,\mathcal{G}_\Phi(r,\varphi,\rho,\phi,t-\tau)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}\tau, \end{aligned} \tag{8.5}$$

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where $\mathcal{G}_f(r,\rho,\varphi,t)$ is the fundamental solution to the first Cauchy problem, $\mathcal{G}_F(r,\rho,\varphi,t)$ is the fundamental solution to the second Cauchy problem, and $\mathcal{G}_\Phi(r,\rho,\varphi,t)$ is the fundamental solution to the source problem.

Consider the fundamental solution to the first Cauchy problem.

$$\frac{\partial^\alpha \mathcal{G}_f}{\partial t^\alpha} = a\left(\frac{\partial^2 \mathcal{G}_f}{\partial r^2} + \frac{1}{r}\frac{\partial \mathcal{G}_f}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \mathcal{G}_f}{\partial \varphi^2}\right), \tag{8.6}$$

$$t=0: \quad \mathcal{G}_f = p_0 \frac{\delta(r-\rho)}{r}\,\delta(\varphi-\phi), \qquad 0<\alpha\le 2, \tag{8.7}$$

$$t=0: \quad \frac{\partial \mathcal{G}_f}{\partial t} = 0, \qquad 1<\alpha\le 2. \tag{8.8}$$

The Laplace transform with respect to time t gives

$$s^\alpha \mathcal{G}_f^* - s^{\alpha-1} p_0 \frac{\delta(r-\rho)}{r}\,\delta(\varphi-\phi) = a\left(\frac{\partial^2 \mathcal{G}_f^*}{\partial r^2} + \frac{1}{r}\frac{\partial \mathcal{G}_f^*}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \mathcal{G}_f^*}{\partial \varphi^2}\right). \tag{8.9}$$

Next we use the finite Fourier transform (2.72) with respect to the angular coordinate φ for 2π-periodic functions, thus obtaining

$$s^\alpha \widetilde{\mathcal{G}}_f^* - s^{\alpha-1} p_0 \frac{\delta(r-\rho)}{r}\,\cos\left[n(\varphi-\phi)\right] = a\left(\frac{\partial^2 \widetilde{\mathcal{G}}_f^*}{\partial r^2} + \frac{1}{r}\frac{\partial \widetilde{\mathcal{G}}_f^*}{\partial r} - \frac{n^2}{r^2}\widetilde{\mathcal{G}}_f^*\right). \tag{8.10}$$

The Hankel transform (2.78) with respect to the radial variable r leads to the solution in the transform domain

$$\widehat{\widetilde{\mathcal{G}}}_f^* = p_0\, J_n(\rho\xi)\cos\left[n(\varphi-\phi)\right]\frac{s^{\alpha-1}}{s^\alpha + a\xi^2}. \tag{8.11}$$

The inverse integral transforms result in

$$\begin{aligned}\mathcal{G}_f(r,\varphi,\rho,\phi,t) &= \frac{p_0}{\pi}\sum_{n=0}^{\infty}{}' \cos\left[n(\varphi-\phi)\right] \\ &\times \int_0^\infty E_\alpha\left(-a\xi^2 t^\alpha\right) J_n(r\xi)\, J_n(\rho\xi)\,\xi\, d\xi. \end{aligned}\tag{8.12}$$

Helmholtz equation ($\alpha \to 0$)

Evaluating integral (A.35) from Appendix, we get

$$\mathcal{G}_f = \frac{p_0}{a\pi}\sum_{n=0}^{\infty}{}'\cos\left[n(\varphi-\phi)\right]\begin{cases} I_n(r/\sqrt{a})\,K_n(\rho/\sqrt{a}), & 0\le r<\rho, \\ I_n(\rho/\sqrt{a})\,K_n(r/\sqrt{a}), & \rho<r<\infty, \end{cases} \tag{8.13}$$

where $I_n(r)$ and $K_n(r)$ are the modified Bessel functions. The sum in the right-hand side of (8.13) can be evaluated analytically taking into account that [196]

$$\sum_{n=0}^{\infty}{}' \, I_n(r)\, K_n(\rho) \cos(n\varphi) = \frac{1}{2} K_0 \left(\sqrt{r^2 + \rho^2 - 2r\rho \cos\varphi} \right).$$

Hence

$$\mathcal{G}_f = \frac{p_0}{2a\pi} K_0 \left(\sqrt{[r^2 + \rho^2 - 2r\rho \cos(\varphi - \phi)]/a} \right). \tag{8.14}$$

Classical diffusion equation ($\alpha = 1$)

It follows from (A.34) that

$$\mathcal{G}_f = \frac{1}{2\pi a t} \exp\left(-\frac{r^2 + \rho^2}{4at} \right) \sum_{n=0}^{\infty}{}' \cos\left[n(\varphi - \phi) \right] I_n\left(\frac{r\rho}{2at} \right). \tag{8.15}$$

Evaluating the sum in the right-hand side of (8.15) taking into account that [196]

$$\sum_{n=0}^{\infty}{}' \, I_n(r) \cos(n\varphi) = \frac{1}{2} \mathrm{e}^{r \cos\varphi},$$

we finally obtain [144]

$$\mathcal{G}_f = \frac{1}{4\pi a t} \exp\left[-\frac{r^2 + \rho^2 - 2r\rho\cos(\varphi - \phi)}{4at} \right]. \tag{8.16}$$

In a similar way we get the fundamental solutions to the second Cauchy problem and to the source problem:

$$\begin{pmatrix} \mathcal{G}_F(r, \varphi, \rho, \phi, t) \\ \mathcal{G}_\Phi(r, \varphi, \rho, \phi, t) \end{pmatrix} = \frac{1}{\pi} \sum_{n=0}^{\infty}{}' \cos\left[n(\varphi - \phi) \right]$$

$$\times \int_0^\infty \begin{pmatrix} w_0 t\, E_{\alpha,2}(-a\xi^2 t^\alpha) \\ q_0 t^{\alpha-1} E_{\alpha,\alpha}(-a\xi^2 t^\alpha) \end{pmatrix} J_n(r\xi)\, J_n(\rho\xi)\, \xi \, \mathrm{d}\xi. \tag{8.17}$$

8.2 Domain $0 \le r < R,\, 0 \le \varphi \le 2\pi$

8.2.1 Dirichlet boundary condition

$$\frac{\partial^\alpha T}{\partial t^\alpha} = a \left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r} \frac{\partial T}{\partial r} + \frac{1}{r^2} \frac{\partial^2 T}{\partial \varphi^2} \right) + \Phi(r, \varphi, t), \tag{8.18}$$

$$t = 0: \quad T = f(r, \varphi), \qquad 0 < \alpha \le 2, \tag{8.19}$$

$$t = 0: \quad \frac{\partial T}{\partial t} = F(r, \varphi), \qquad 1 < \alpha \le 2, \tag{8.20}$$

$$r = R: \quad T = g(\varphi, t). \tag{8.21}$$

The solution:

$$\begin{aligned} T(r, \varphi, t) = & \int_0^{2\pi}\int_0^R f(\rho, \phi)\, \mathcal{G}_f(r, \varphi, \rho, \phi, t)\, \rho \,\mathrm{d}\rho\, \mathrm{d}\phi \\ & + \int_0^{2\pi}\int_0^R F(\rho, \phi)\, \mathcal{G}_F(r, \varphi, \rho, \phi, t)\, \rho \,\mathrm{d}\rho\, \mathrm{d}\phi \\ & + \int_0^{t}\int_0^{2\pi}\int_0^R \Phi(\rho, \phi, \tau)\, \mathcal{G}_\Phi(r, \varphi, \rho, \phi, t-\tau)\, \rho \,\mathrm{d}\rho\, \mathrm{d}\phi\, \mathrm{d}\tau \\ & + \int_0^{t}\int_0^{2\pi} g(\phi, \tau)\, \mathcal{G}_g(r, \varphi, \phi, t-\tau)\, \mathrm{d}\phi\, \mathrm{d}\tau. \end{aligned} \tag{8.22}$$

The fundamental solutions are obtained using the Laplace transform with respect to time t, the finite Fourier transform with respect to the angular coordinate φ (2.72) for 2π-periodic functions and the finite Hankel transform (2.84) with respect to the radial coordinate r:

$$\begin{aligned} \begin{pmatrix} \mathcal{G}_f(r, \varphi, \rho, \phi, t) \\ \mathcal{G}_F(r, \varphi, \rho, \phi, t) \\ \mathcal{G}_\Phi(r, \varphi, \rho, \phi, t) \end{pmatrix} = & \frac{2}{\pi R^2} {\sum_{n=0}^{\infty}}' \sum_{k=1}^{\infty} \begin{pmatrix} p_0\, E_\alpha(-a\xi_{nk}^2 t^\alpha) \\ w_0 t\, E_{\alpha,2}(-a\xi_{nk}^2 t^\alpha) \\ q_0 t^{\alpha-1}\, E_{\alpha,\alpha}(-a\xi_{nk}^2 t^\alpha) \end{pmatrix} \\ & \times \frac{J_n(r\xi_{nk})\, J_n(\rho\xi_{nk})}{[J_n'(R\xi_{nk})]^2} \cos[n(\varphi - \phi)], \end{aligned} \tag{8.23}$$

where ξ_{nk} are the positive roots of the transcendental equation

$$J_n(R\xi_{nk}) = 0. \tag{8.24}$$

The fundamental solution to the Dirichlet problem has the following form:

$$\begin{aligned} \mathcal{G}_g(r, \varphi, \phi, t) = & -\frac{2ag_0 t^{\alpha-1}}{\pi R} {\sum_{n=0}^{\infty}}' \sum_{k=1}^{\infty} E_{\alpha,\alpha}\left(-a\xi_{nk}^2 t^\alpha\right) \\ & \times \frac{\xi_{nk}\, J_n(r\xi_{nk})}{J_n'(R\xi_{nk})} \cos\left[n(\varphi - \phi)\right]. \end{aligned} \tag{8.25}$$

8.2.2 Neumann boundary condition

$$\frac{\partial^\alpha T}{\partial t^\alpha} = a\left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \varphi^2}\right) + \Phi(r,\varphi,t), \tag{8.26}$$

$$t = 0: \quad T = f(r,\varphi), \qquad 0 < \alpha \le 2, \tag{8.27}$$

$$t = 0: \quad \frac{\partial T}{\partial t} = F(r,\varphi), \qquad 1 < \alpha \le 2, \tag{8.28}$$

$$r = R: \quad \frac{\partial T}{\partial r} = g(\varphi, t). \tag{8.29}$$

The solution:

$$\begin{aligned} T(r,\varphi,t) &= \int_0^{2\pi}\int_0^R f(\rho,\phi)\,\mathcal{G}_f(r,\varphi,\rho,\phi,t)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\phi \\ &+ \int_0^{2\pi}\int_0^R F(\rho,\phi)\,\mathcal{G}_F(r,\varphi,\rho,\phi,t)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\phi \\ &+ \int_0^t\int_0^{2\pi}\int_0^R \Phi(\rho,\phi,\tau)\,\mathcal{G}_\Phi(r,\varphi,\rho,\phi,t-\tau)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}\tau \\ &+ \int_0^t\int_0^{2\pi} g(\phi,\tau)\,\mathcal{G}_g(r,\varphi,\phi,t-\tau)\,\mathrm{d}\phi\,\mathrm{d}\tau. \end{aligned} \tag{8.30}$$

The fundamental solutions are obtained using the Laplace transform with respect to time t, the finite Fourier transform with respect to the angular coordinate φ (2.72) for 2π-periodic functions and the finite Hankel transform (2.88) with respect to the radial coordinate r:

$$\begin{aligned} \begin{pmatrix} \mathcal{G}_f(r,\varphi,\rho,\phi,t) \\ \mathcal{G}_F(r,\varphi,\rho,\phi,t) \\ \mathcal{G}_\Phi(r,\varphi,\rho,\phi,t) \end{pmatrix} &= \frac{1}{\pi R^2}\begin{pmatrix} p_0 \\ w_0 t \\ q_0 t^{\alpha-1}/\Gamma(\alpha) \end{pmatrix} \\ &+ \frac{2}{\pi}\sum_{n=0}^{\infty}{}'\sum_{k=1}^{\infty}\begin{pmatrix} p_0\,E_\alpha(-a\xi_{nk}^2 t^\alpha) \\ w_0 t\,E_{\alpha,2}(-a\xi_{nk}^2 t^\alpha) \\ q_0 t^{\alpha-1}\,E_{\alpha,\alpha}(-a\xi_{nk}^2 t^\alpha) \end{pmatrix} \\ &\times \frac{\xi_{nk}^2 J_n(r\xi_{nk})\,J_n(\rho\xi_{nk})}{(R^2\xi_{nk}^2 - n^2)\,[J_n(R\xi_{nk})]^2}\cos[n(\varphi-\phi)], \end{aligned} \tag{8.31}$$

where ξ_{nk} are the positive roots of the transcendental equation

$$J_n'(R\xi_{nk}) = 0. \tag{8.32}$$

The fundamental solutions to the mathematical and physical Neumann problems under zero initial conditions are expressed as

$$\mathcal{G}_m(r,\varphi,\phi,t) = \frac{aRg_0}{q_0}\mathcal{G}_\Phi(r,\rho,\varphi,t)\Big|_{\rho=R}, \tag{8.33}$$

$$\mathcal{G}_p(r,\varphi,\phi,t) = \frac{aRg_0}{p_0}\mathcal{G}_f(r,\rho,\varphi,t)\Big|_{\rho=R}. \tag{8.34}$$

8.2.3 Robin boundary condition

$$\frac{\partial^\alpha T}{\partial t^\alpha} = a\left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \varphi^2}\right) + \Phi(r,\varphi,t), \tag{8.35}$$

$$t = 0: \quad T = f(r,\varphi), \qquad 0 < \alpha \le 2, \tag{8.36}$$

$$t = 0: \quad \frac{\partial T}{\partial t} = F(r,\varphi), \qquad 1 < \alpha \le 2, \tag{8.37}$$

$$r = R: \quad \frac{\partial T}{\partial r} + HT = g(\varphi,t). \tag{8.38}$$

The solution:

$$\begin{aligned} T(r,\varphi,t) &= \int_0^{2\pi}\int_0^R f(\rho,\phi)\,\mathcal{G}_f(r,\varphi,\rho,\phi,t)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\phi \\ &+ \int_0^{2\pi}\int_0^R F(\rho,\phi)\,\mathcal{G}_F(r,\varphi,\rho,\phi,t)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\phi \\ &+ \int_0^t\int_0^{2\pi}\int_0^R \Phi(\rho,\phi,\tau)\,\mathcal{G}_\Phi(r,\varphi,\rho,\phi,t-\tau)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}\tau \\ &+ \int_0^t\int_0^{2\pi} g(\phi,\tau)\,\mathcal{G}_g(r,\varphi,\phi,t-\tau)\,\mathrm{d}\phi\,\mathrm{d}\tau. \end{aligned} \tag{8.39}$$

The fundamental solutions are obtained using the Laplace transform with respect to time t, the finite Fourier transform with respect to the angular coordinate φ

(2.72) for 2π-periodic functions and the finite Hankel transform (2.92) with respect to the radial coordinate r:

$$\begin{pmatrix} \mathcal{G}_f(r,\varphi,\rho,\phi,t) \\ \mathcal{G}_F(r,\varphi,\rho,\phi,t) \\ \mathcal{G}_\Phi(r,\varphi,\rho,\phi,t) \end{pmatrix} = \frac{2}{\pi} \sum_{n=0}^{\infty}{}' \sum_{k=1}^{\infty} \begin{pmatrix} p_0\, E_\alpha(-a\xi_{nk}^2 t^\alpha) \\ w_0 t\, E_{\alpha,2}(-a\xi_{nk}^2 t^\alpha) \\ q_0 t^{\alpha-1}\, E_{\alpha,\alpha}(-a\xi_{nk}^2 t^\alpha) \end{pmatrix}$$

$$\times \frac{\xi_{nk}^2 J_n(r\xi_{nk})\, J_n(\rho\xi_{nk})}{(R^2H^2 + R^2\xi_{nk}^2 - n^2)\,[J_n(R\xi_{nk})]^2} \cos[n(\varphi-\phi)], \tag{8.40}$$

where ξ_{nk} are the positive roots of the transcendental equation

$$\xi_{nk} J_n'(R\xi_{nk}) + H J_n(R\xi_{nk}) = 0. \tag{8.41}$$

The fundamental solution to the mathematical Robin problem under zero initial conditions is calculated as

$$\mathcal{G}_g(r,\varphi,\phi,t) = \frac{aRg_0}{q_0} \mathcal{G}_\Phi(r,\rho,\varphi,t)\Big|_{\rho=R}. \tag{8.42}$$

8.3 Domain $R < r < \infty,\ 0 \le \varphi \le 2\pi$

8.3.1 Dirichlet boundary condition

$$\frac{\partial^\alpha T}{\partial t^\alpha} = a\left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial T}{\partial \varphi^2}\right) + \Phi(r,\varphi,t), \tag{8.43}$$

$$t = 0: \quad T = f(r,\varphi), \qquad 0 < \alpha \le 2, \tag{8.44}$$

$$t = 0: \quad \frac{\partial T}{\partial t} = F(r,\varphi), \quad 1 < \alpha \le 2, \tag{8.45}$$

$$r = R: \quad T = g(\varphi,t), \tag{8.46}$$

$$\lim_{r\to\infty} T(r,\varphi,t) = 0. \tag{8.47}$$

The solution:

$$
\begin{aligned}
T(r,\varphi,t) &= \int_0^{2\pi}\int_R^{\infty} f(\rho,\phi)\,\mathcal{G}_f(r,\varphi,\rho,\phi,t)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\phi \\
&+ \int_0^{2\pi}\int_R^{\infty} F(\rho,\phi)\,\mathcal{G}_F(r,\varphi,\rho,\phi,t)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\phi \\
&+ \int_0^{t}\int_0^{2\pi}\int_R^{\infty} \Phi(\rho,\phi,\tau)\,\mathcal{G}_\Phi(r,\varphi,\rho,\phi,t-\tau)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}\tau \\
&+ \int_0^{t}\int_0^{2\pi} g(\phi,\tau)\,\mathcal{G}_g(r,\varphi,\phi,t-\tau)\,\mathrm{d}\phi\,\mathrm{d}\tau. \qquad (8.48)
\end{aligned}
$$

The fundamental solutions are obtained using the Laplace transform with respect to time t, the finite Fourier transform for 2π-periodic functions with respect to the angular coordinate φ (2.72) and the Weber transform (2.108), (2.111) with respect to the radial variable r:

$$
\begin{aligned}
&\begin{pmatrix} \mathcal{G}_f(r,\varphi,\rho,\phi,t) \\ \mathcal{G}_F(r,\varphi,\rho,\phi,t) \\ \mathcal{G}_\Phi(r,\varphi,\rho,\phi,t) \end{pmatrix} = \frac{1}{\pi}\sum_{n=0}^{\infty}{}' \int_0^{\infty} \begin{pmatrix} p_0\,E_\alpha(-a\xi^2t^\alpha) \\ w_0 t\,E_{\alpha,2}(-a\xi^2t^\alpha) \\ q_0 t^{\alpha-1}\,E_{\alpha,\alpha}(-a\xi^2t^\alpha) \end{pmatrix} \\
&\times \frac{J_n(r\xi)\,Y_n(R\xi) - Y_n(r\xi)\,J_n(R\xi)}{J_n^2(R\xi)+Y_n^2(R\xi)} \cos\left[n(\varphi-\phi)\right] \\
&\times \Big[J_n(\rho\xi)\,Y_n(R\xi) - Y_n(\rho\xi)\,J_n(R\xi)\Big]\,\xi\,\mathrm{d}\xi. \qquad (8.49)
\end{aligned}
$$

The fundamental solution to the Dirichlet problem under zero initial conditions reads:

$$
\begin{aligned}
\mathcal{G}_g(r,\varphi,\phi,t) &= -\frac{2ag_0t^{\alpha-1}}{\pi^2}\sum_{n=0}^{\infty}{}' \int_0^{\infty} E_{\alpha,\alpha}\left(-a\xi^2t^\alpha\right)\cos\left[n(\varphi-\phi)\right] \\
&\times \frac{J_n(r\xi)\,Y_n(R\xi) - Y_n(r\xi)\,J_n(R\xi)}{J_n^2(R\xi)+Y_n^2(R\xi)}\,\xi\,\mathrm{d}\xi. \qquad (8.50)
\end{aligned}
$$

8.3.2 Neumann boundary condition

$$\frac{\partial^\alpha T}{\partial t^\alpha} = a\left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial T}{\partial \varphi^2}\right) + \Phi(r,\varphi,t), \tag{8.51}$$

$$t = 0: \quad T = f(r,\varphi), \qquad 0 < \alpha \le 2, \tag{8.52}$$

$$t = 0: \quad \frac{\partial T}{\partial t} = F(r,\varphi), \quad 1 < \alpha \le 2, \tag{8.53}$$

$$r = R: \quad -\frac{\partial T}{\partial r} = g(\varphi,t), \tag{8.54}$$

$$\lim_{r\to\infty} T(r,\varphi,t) = 0. \tag{8.55}$$

The solution:

$$\begin{aligned}
T(r,\varphi,t) &= \int_0^{2\pi}\int_R^{\infty} f(\rho,\phi)\,\mathcal{G}_f(r,\varphi,\rho,\phi,t)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\phi \\
&+ \int_0^{2\pi}\int_R^{\infty} F(\rho,\phi)\,\mathcal{G}_F(r,\varphi,\rho,\phi,t)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\phi \\
&+ \int_0^{t}\int_0^{2\pi}\int_R^{\infty} \Phi(\rho,\phi,\tau)\,\mathcal{G}_\Phi(r,\varphi,\rho,\phi,t-\tau)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}\tau \\
&+ \int_0^{t}\int_0^{2\pi} g(\phi,\tau)\,\mathcal{G}_g(r,\varphi,\phi,t-\tau)\,\mathrm{d}\phi\,\mathrm{d}\tau.
\end{aligned} \tag{8.56}$$

The fundamental solutions are obtained using the Laplace transform with respect to time t, the finite Fourier transform for 2π-periodic functions with respect to the angular coordinate φ (2.72) and the Weber transform (2.108), (2.113) with respect to the radial variable r:

$$\begin{aligned}
\begin{pmatrix} \mathcal{G}_f(r,\varphi,\rho,\phi,t) \\ \mathcal{G}_F(r,\varphi,\rho,\phi,t) \\ \mathcal{G}_\Phi(r,\varphi,\rho,\phi,t) \end{pmatrix} &= \frac{1}{\pi}\sum_{n=0}^{\infty}{}' \int_0^{\infty} \begin{pmatrix} p_0\,E_\alpha(-a\xi^2 t^\alpha) \\ w_0 t\,E_{\alpha,2}(-a\xi^2 t^\alpha) \\ q_0 t^{\alpha-1}\,E_{\alpha,\alpha}(-a\xi^2 t^\alpha) \end{pmatrix} \\
&\times \frac{J_n(r\xi)\,Y_n'(R\xi) - Y_n(r\xi)\,J_n'(R\xi)}{[J_n'(R\xi)]^2 + [Y_n'(R\xi)]^2}\cos[n(\varphi-\phi)] \\
&\times \Big[J_n(\rho\xi)\,Y_n'(R\xi) - Y_n(\rho\xi)\,J_n'(R\xi)\Big]\,\xi\,\mathrm{d}\xi.
\end{aligned} \tag{8.57}$$

The fundamental solutions to the mathematical and physical Neumann problems under zero initial conditions have the following form:

$$\mathcal{G}_m(r,\varphi,\phi,t) = \frac{2ag_0 t^{\alpha-1}}{\pi^2}\sum_{n=0}^{\infty}{}' \int_0^{\infty} E_{\alpha,\alpha}\left(-a\xi^2 t^{\alpha}\right)\cos\left[n(\varphi-\phi)\right]$$
$$\times \frac{J_n(r\xi)\,Y_n'(R\xi)-Y_n(r\xi)\,J_n'(R\xi)}{[J_n'(R\xi)]^2+[Y_n'(R\xi)]^2}\,\mathrm{d}\xi; \tag{8.58}$$

$$\mathcal{G}_p(r,\varphi,\phi,t) = \frac{2ag_0}{\pi^2}\sum_{n=0}^{\infty}{}' \int_0^{\infty} E_{\alpha}\left(-a\xi^2 t^{\alpha}\right)\cos\left[n(\varphi-\phi)\right]$$
$$\times \frac{J_n(r\xi)\,Y_n'(R\xi)-Y_n(r\xi)\,J_n'(R\xi)}{[J_n'(R\xi)]^2+[Y_n'(R\xi)]^2}\,\mathrm{d}\xi. \tag{8.59}$$

8.3.3 Robin boundary condition

$$\frac{\partial^{\alpha}T}{\partial t^{\alpha}} = a\left(\frac{\partial^2 T}{\partial r^2}+\frac{1}{r}\frac{\partial T}{\partial r}+\frac{1}{r^2}\frac{\partial T}{\partial\varphi^2}\right)+\Phi(r,\varphi,t), \tag{8.60}$$

$$t=0: \quad T=f(r,\varphi), \qquad 0<\alpha\le 2, \tag{8.61}$$

$$t=0: \quad \frac{\partial T}{\partial t}=F(r,\varphi), \quad 1<\alpha\le 2, \tag{8.62}$$

$$r=R: \quad -\frac{\partial T}{\partial r}+HT=g(\varphi,t), \tag{8.63}$$

$$\lim_{r\to\infty} T(r,\varphi,t)=0. \tag{8.64}$$

The solution:

$$T(r,\varphi,t) = \int_0^{2\pi}\int_R^{\infty} f(\rho,\phi)\,\mathcal{G}_f(r,\varphi,\rho,\phi,t)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\phi$$
$$+ \int_0^{2\pi}\int_R^{\infty} F(\rho,\phi)\,\mathcal{G}_F(r,\varphi,\rho,\phi,t)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\phi$$
$$+ \int_0^{t}\int_0^{2\pi}\int_R^{\infty} \Phi(\rho,\phi,\tau)\,\mathcal{G}_{\Phi}(r,\varphi,\rho,\phi,t-\tau)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}\tau$$

$$+ \int_0^t \int_0^{2\pi} g(\phi, \tau)\, \mathcal{G}_g(r, \varphi, \phi, t-\tau)\, \mathrm{d}\phi\, \mathrm{d}\tau. \tag{8.65}$$

The fundamental solutions are obtained using the Laplace transform with respect to time t, the finite Fourier transform for 2π-periodic functions with respect to the angular coordinate φ (2.72) and the Weber transform (2.108), (2.115) with respect to the radial variable r:

$$\begin{pmatrix} \mathcal{G}_f(r,\varphi,\rho,\phi,t) \\ \mathcal{G}_F(r,\varphi,\rho,\phi,t) \\ \mathcal{G}_\Phi(r,\varphi,\rho,\phi,t) \end{pmatrix} = \frac{1}{\pi} {\sum_{n=0}^{\infty}}' \int_0^\infty \begin{pmatrix} p_0\, E_\alpha(-a\xi^2 t^\alpha) \\ w_0 t\, E_{\alpha,2}(-a\xi^2 t^\alpha) \\ q_0 t^{\alpha-1}\, E_{\alpha,\alpha}(-a\xi^2 t^\alpha) \end{pmatrix} \cos\left[n(\varphi-\phi)\right]$$

$$\times \frac{J_n(r\xi)[\xi Y_n'(R\xi) - HY_n(R\xi)] - Y_n(r\xi)[\xi J_n'(R\xi) - HJ_n(R\xi)]}{[\xi Y_n'(R\xi) - HY_n(R\xi)]^2 + [\xi J_n'(R\xi) - HJ_n(R\xi)]^2}$$

$$\times \Big\{ J_n(\rho\xi)[\xi Y_n'(R\xi) - HY_n(R\xi)] - Y_n(\rho\xi)[\xi J_n'(R\xi) - HJ_n(R\xi)] \Big\} \xi\, \mathrm{d}\xi. \tag{8.66}$$

The fundamental solution to the mathematical Robin problem under zero initial conditions is written as:

$$\mathcal{G}_g(r,\varphi,\phi,t) = \frac{2ag_0 t^{\alpha-1}}{\pi^2} {\sum_{n=0}^{\infty}}' \int_0^\infty E_{\alpha,\alpha}\left(-a\xi^2 t^\alpha\right) \cos\left[n(\varphi-\phi)\right]$$

$$\times \frac{J_n(r\xi)[\xi Y_n'(R\xi) - HY_n(R\xi)] - Y_n(r\xi)[\xi J_n'(R\xi) - HJ_n(R\xi)]}{[\xi Y_n'(R\xi) - HY_n(R\xi)]^2 + [\xi J_n'(R\xi) - HJ_n(R\xi)]^2}\, \xi\, \mathrm{d}\xi. \tag{8.67}$$

8.4 Domain $0 \leq r < \infty$, $0 < \varphi < \varphi_0$

8.4.1 Dirichlet boundary condition

$$\frac{\partial^\alpha T}{\partial t^\alpha} = a\left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \varphi^2}\right) + \Phi(r,\varphi,t), \tag{8.68}$$

$$t = 0: \quad T = f(r,\varphi), \qquad 0 < \alpha \leq 2, \tag{8.69}$$

$$t = 0: \quad \frac{\partial T}{\partial t} = F(r,\varphi), \qquad 1 < \alpha \leq 2, \tag{8.70}$$

$$\varphi = 0: \quad T = g_1(r,t), \tag{8.71}$$

$$\varphi = \varphi_0: \quad T = g_2(r,t), \tag{8.72}$$

$$\lim_{r\to\infty} T(r,\varphi,t) = 0. \tag{8.73}$$

The solution:

$$
\begin{aligned}
T(r,\varphi,t) &= \int_0^{\varphi_0}\int_0^{\infty} f(\rho,\phi)\,\mathcal{G}_f(r,\varphi,\rho,\phi,t)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\phi \\
&+ \int_0^{\varphi_0}\int_0^{\infty} F(\rho,\phi)\,\mathcal{G}_F(r,\varphi,\rho,\phi,t)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\phi \\
&+ \int_0^{t}\int_0^{\varphi_0}\int_0^{\infty} \Phi(\rho,\phi,\tau)\,\mathcal{G}_\Phi(r,\varphi,\rho,\phi,t-\tau)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}\tau \\
&+ \int_0^{t}\int_0^{\infty} g_1(\rho,\tau)\,\mathcal{G}_{g_1}(r,\varphi,\rho,t-\tau)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\tau \\
&+ \int_0^{t}\int_0^{\infty} g_2(\rho,\tau)\,\mathcal{G}_{g_2}(r,\varphi,\rho,t-\tau)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\tau. \qquad (8.74)
\end{aligned}
$$

To obtain the fundamental solutions we use the Laplace transform with respect to time t, the finite sin-Fourier transform (2.44) with respect to the angular coordinate φ and the Hankel transform (2.78) with respect to the radial variable r with $\nu = n\pi/\varphi_0$. Thus, we get [190]

$$
\begin{pmatrix} \mathcal{G}_f(r,\varphi,\rho,\phi,t) \\ \mathcal{G}_F(r,\varphi,\rho,\phi,t) \\ \mathcal{G}_\Phi(r,\varphi,\rho,\phi,t) \end{pmatrix} = \frac{2}{\varphi_0}\sum_{n=1}^{\infty}\int_0^{\infty} \begin{pmatrix} p_0\,E_\alpha(-a\xi^2 t^\alpha) \\ w_0 t\,E_{\alpha,2}(-a\xi^2 t^\alpha) \\ q_0 t^{\alpha-1}\,E_{\alpha,\alpha}(-a\xi^2 t^\alpha) \end{pmatrix}
$$

$$
\times \sin\left(\frac{n\pi\varphi}{\varphi_0}\right)\sin\left(\frac{n\pi\phi}{\varphi_0}\right) J_{n\pi/\varphi_0}(r\xi)\,J_{n\pi/\varphi_0}(\rho\xi)\,\xi\,\mathrm{d}\xi. \qquad (8.75)
$$

The fundamental solution to the first Dirichlet problem under zero initial conditions is expressed as

$$
\mathcal{G}_{g1}(r,\varphi,\rho,t) = \frac{2ag_0 t^{\alpha-1}}{\varphi_0\rho^2}\sum_{n=1}^{\infty}\left(\frac{n\pi}{\varphi_0}\right)\sin\left(\frac{n\pi\varphi}{\varphi_0}\right)\int_0^{\infty} E_{\alpha,\alpha}\left(-a\xi^2 t^\alpha\right)
$$

$$
\times J_{n\pi/\varphi_0}(r\xi)\,J_{n\pi/\varphi_0}(\rho\xi)\,\xi\,\mathrm{d}\xi. \qquad (8.76)
$$

The fundamental solution to the second Dirichlet problem under zero initial conditions is obtained from (8.76) by multiplying each term by $(-1)^{n+1}$.

8.4.2 Neumann boundary condition

$$\frac{\partial^\alpha T}{\partial t^\alpha} = a\left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \varphi^2}\right) + \Phi(r,\varphi,t), \tag{8.77}$$

$$t = 0: \quad T = f(r,\varphi), \qquad 0 < \alpha \le 2, \tag{8.78}$$

$$t = 0: \quad \frac{\partial T}{\partial t} = F(r,\varphi), \qquad 1 < \alpha \le 2, \tag{8.79}$$

$$\varphi = 0: \quad -\frac{1}{r}\frac{\partial T}{\partial \varphi} = g_1(r,t), \tag{8.80}$$

$$\varphi = \varphi_0: \quad \frac{1}{r}\frac{\partial T}{\partial \varphi} = g_2(r,t), \tag{8.81}$$

$$\lim_{r\to\infty} T(r,\varphi,t) = 0. \tag{8.82}$$

The solution:

$$\begin{aligned}
T(r,\varphi,t) = &\int_0^{\varphi_0}\int_0^\infty f(\rho,\phi)\,\mathcal{G}_f(r,\varphi,\rho,\phi,t)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\phi \\
&+ \int_0^{\varphi_0}\int_0^\infty F(\rho,\phi)\,\mathcal{G}_F(r,\varphi,\rho,\phi,t)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\phi \\
&+ \int_0^t\int_0^{\varphi_0}\int_0^\infty \Phi(\rho,\phi,\tau)\,\mathcal{G}_\Phi(r,\varphi,\rho,\phi,t-\tau)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}\tau \\
&+ \int_0^t\int_0^\infty g_1(\rho,\tau)\,\mathcal{G}_{g_1}(r,\varphi,\rho,t-\tau)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\tau \\
&+ \int_0^t\int_0^\infty g_2(\rho,\tau)\,\mathcal{G}_{g_2}(r,\varphi,\rho,t-\tau)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\tau.
\end{aligned} \tag{8.83}$$

To obtain the fundamental solutions we use the Laplace transform with respect to time t, the finite cos-Fourier transform (2.48) with respect to the angular coordinate φ and the Hankel transform (2.78) with respect to the radial variable r with $\nu = n\pi/\varphi_0$. As a result, we get [190]

$$\begin{pmatrix} \mathcal{G}_f(r,\varphi,\rho,\phi,t) \\ \mathcal{G}_F(r,\varphi,\rho,\phi,t) \\ \mathcal{G}_\Phi(r,\varphi,\rho,\phi,t) \end{pmatrix} = \frac{2}{\varphi_0}\sum_{n=0}^{\infty}{}' \int_0^\infty \begin{pmatrix} p_0\, E_\alpha(-a\xi^2 t^\alpha) \\ w_0 t\, E_{\alpha,2}(-a\xi^2 t^\alpha) \\ q_0 t^{\alpha-1}\, E_{\alpha,\alpha}(-a\xi^2 t^\alpha) \end{pmatrix}$$

$$\times \cos\left(\frac{n\pi\varphi}{\varphi_0}\right) \cos\left(\frac{n\pi\phi}{\varphi_0}\right) J_{n\pi/\varphi_0}(r\xi)\, J_{n\pi/\varphi_0}(\rho\xi)\, \xi\, \mathrm{d}\xi. \tag{8.84}$$

The fundamental solution to the first mathematical Neumann problem under zero initial conditions has the form [190]:

$$\mathcal{G}_{m1}(r,\varphi,\rho,t) = \frac{2ag_0 t^{\alpha-1}}{\varphi_0 \rho} {\sum_{n=0}^{\infty}}' \cos\left(\frac{n\pi\varphi}{\varphi_0}\right) \int_0^{\infty} E_{\alpha,\alpha}\left(-a\xi^2 t^{\alpha}\right)$$
$$\times\, J_{n\pi/\varphi_0}(r\xi)\, J_{n\pi/\varphi_0}(\rho\xi)\, \xi\, \mathrm{d}\xi. \tag{8.85}$$

Classical diffusion equation ($\alpha = 1$)

Using (A.34) from the Appendix, we obtain [20, 144]

$$\mathcal{G}_{g_1}(r,\varphi,\rho,t) = \frac{g_0}{\rho\varphi_0 t} \exp\left(-\frac{r^2+\rho^2}{4at}\right)$$
$$\times \left[\frac{1}{2} I_0\left(\frac{r\rho}{2at}\right) + \sum_{n=1}^{\infty} I_{n\pi/\varphi_0}\left(\frac{r\rho}{2at}\right) \cos\left(\frac{n\pi\varphi}{\varphi_0}\right)\right]. \tag{8.86}$$

Wave equation ($\alpha = 2$)

$$\mathcal{G}_{g_1}(r,\varphi,\rho,t) = \frac{2\sqrt{a}g_0}{\varphi_0\rho} {\sum_{n=0}^{\infty}}' \cos\left(\frac{n\pi\varphi}{\varphi_0}\right) \Psi(r,\rho), \tag{8.87}$$

where

a) $\sqrt{a}t < \rho$

$$\Psi(r,\rho) = \begin{cases} 0, & 0 \le r < \rho - \sqrt{a}t, \\ \dfrac{1}{2\sqrt{r\rho}} P_{n\pi/\varphi_0 - 1/2}\left(\dfrac{r^2+\rho^2-at^2}{2r\rho}\right), & \rho - \sqrt{a}t < r < \rho + \sqrt{a}t, \\ 0, & \rho + \sqrt{a}t < r < \infty; \end{cases}$$

b) $\sqrt{a}t = \rho$

$$\Psi(r,\rho) = \begin{cases} \dfrac{1}{2\sqrt{r\rho}} P_{n\pi/\varphi_0 - 1/2}\left(\dfrac{r}{2\rho}\right), & 0 < r < 2\rho, \\ 0, & 2\rho < r < \infty; \end{cases}$$

c) $\sqrt{a}t > \rho$

$$\Psi(r,\rho) = \begin{cases} -\dfrac{1}{\pi\sqrt{r\rho}} \cos\left(\dfrac{n\pi^2}{\varphi_0}\right) Q_{n\pi/\varphi_0 - 1/2}\left(\dfrac{at^2 - r^2 - \rho^2}{2r\rho}\right), \\ \qquad\qquad\qquad\qquad 0 \le r < \sqrt{a}t - \rho, \\ \dfrac{1}{2\sqrt{r\rho}} P_{n\pi/\varphi_0 - 1/2}\left(\dfrac{r^2 + \rho^2 - at^2}{2r\rho}\right), \\ \qquad\qquad\qquad\qquad \sqrt{a}t - \rho < r < \rho + \sqrt{a}t, \\ 0, \qquad\qquad\qquad\quad \rho + \sqrt{a}t < r < \infty, \end{cases}$$

where $P_\nu(r)$ and $Q_\nu(r)$ are the Legendre functions of the first and second kind, respectively.

For the physical Neumann problem, the boundary condition at $\varphi = 0$ is formulated in terms of the normal component of the heat flux:

$$\varphi = 0: \quad -\frac{1}{r} D_{RL}^{1-\alpha} \frac{\partial G_{p_1}}{\partial \varphi} = g_0 \frac{\delta(r-\rho)}{r} \delta(t), \quad 0 < \alpha \le 1, \tag{8.88}$$

$$\varphi = 0: \quad -\frac{1}{r} I^{\alpha-1} \frac{\partial G_{p_1}}{\partial \varphi} = g_0 \frac{\delta(r-\rho)}{r} \delta(t), \quad 1 < \alpha \le 2. \tag{8.89}$$

The solution is expressed as

$$\mathcal{G}_{p1}(r,\varphi,\rho,t) = \frac{2ag_0}{\varphi_0\rho} \sum_{n=0}^{\infty}{}' \cos\left(\frac{n\pi\varphi}{\varphi_0}\right) \int_0^\infty E_\alpha\left(-a\xi^2 t^\alpha\right) \times J_{n\pi/\varphi_0}(r\xi) J_{n\pi/\varphi_0}(\rho\xi)\, \xi \,\mathrm{d}\xi. \tag{8.90}$$

The fundamental solutions (8.85) and (8.90) are shown in Fig. 8.1 and Fig. 8.2, respectively, for $\varphi = 0$.

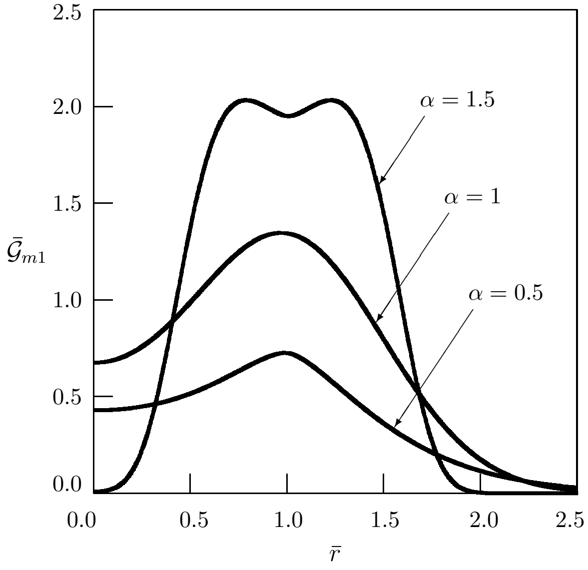

Figure 8.1: Dependence of the fundamental solution to the mathematical Neumann problem in a wedge on the radial coordinate [190]

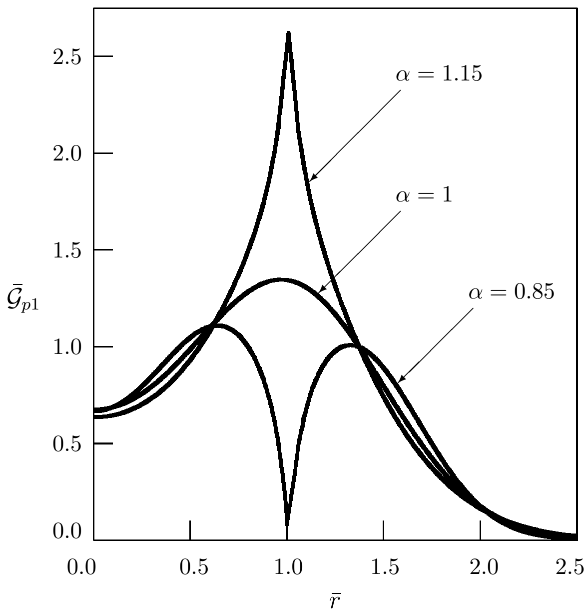

Figure 8.2: Dependence of the fundamental solution to the physical Neumann problem in a wedge on the radial coordinate [190]

8.5 Domain $0 \le r < R,\ 0 < \varphi < \varphi_0$

8.5.1 Dirichlet boundary condition

$$\frac{\partial^\alpha T}{\partial t^\alpha} = a\left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \varphi^2}\right) + \Phi(r,\varphi,t), \tag{8.91}$$

$$t = 0: \quad T = f(r,\varphi), \qquad 0 < \alpha \le 2, \tag{8.92}$$

$$t = 0: \quad \frac{\partial T}{\partial t} = F(r,\varphi), \qquad 1 < \alpha \le 2, \tag{8.93}$$

$$r = R: \quad T = g_1(\varphi,t), \tag{8.94}$$

$$\varphi = 0: \quad T = g_2(r,t), \tag{8.95}$$

$$\varphi = \varphi_0: \quad T = g_3(r,t). \tag{8.96}$$

The solution:

$$\begin{aligned}
T(r,\varphi,t) = & \int_0^{\varphi_0}\int_0^R f(\rho,\phi)\,\mathcal{G}_f(r,\varphi,\rho,\phi,t)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\phi \\
& + \int_0^{\varphi_0}\int_0^R F(\rho,\phi)\,\mathcal{G}_F(r,\varphi,\rho,\phi,t)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\phi \\
& + \int_0^t\int_0^{\varphi_0}\int_0^R \Phi(\rho,\phi,\tau)\,\mathcal{G}_\Phi(r,\varphi,\rho,\phi,t-\tau)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}\tau \\
& + \int_0^t\int_0^{\varphi_0} g_1(\phi,\tau)\,\mathcal{G}_{g_1}(r,\varphi,\phi,t-\tau)\,\mathrm{d}\phi\,\mathrm{d}\tau \\
& + \int_0^t\int_0^R g_2(\rho,\tau)\,\mathcal{G}_{g_2}(r,\varphi,\rho,t-\tau)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\tau \\
& + \int_0^t\int_0^R g_3(\rho,\tau)\,\mathcal{G}_{g_3}(r,\varphi,\rho,t-\tau)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\tau.
\end{aligned} \tag{8.97}$$

The problem is solved using the Laplace transform with respect to time t, the finite sin-Fourier transform (2.44) with respect to the angular coordinate φ and the finite

Hankel transform (2.84) with respect to the radial variable r with $\nu = n\pi/\varphi_0$. For the fundamental solutions we get [188]

$$\begin{pmatrix} \mathcal{G}_f(r,\varphi,\rho,\phi,t) \\ \mathcal{G}_F(r,\varphi,\rho,\phi,t) \\ \mathcal{G}_\Phi(r,\varphi,\rho,\phi,t) \end{pmatrix} = \frac{4}{\varphi_0 R^2} \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} \begin{pmatrix} p_0\, E_\alpha(-a\xi_{nk}^2 t^\alpha) \\ w_0 t\, E_{\alpha,2}(-a\xi_{nk}^2 t^\alpha) \\ q_0 t^{\alpha-1}\, E_{\alpha,\alpha}(-a\xi_{nk}^2 t^\alpha) \end{pmatrix}$$

$$\times \sin\left(\frac{n\pi\varphi}{\varphi_0}\right) \sin\left(\frac{n\pi\phi}{\varphi_0}\right) \frac{J_{n\pi/\varphi_0}(r\xi_{nk})\, J_{n\pi/\varphi_0}(\rho\xi_{nk})}{\left[J'_{n\pi/\varphi_0}(R\xi_{nk})\right]^2}, \tag{8.98}$$

where ξ_{nk} are the positive roots of the transcendental equation

$$J_{n\pi/\varphi_0}(R\xi_{nk}) = 0. \tag{8.99}$$

For the sake of simplicity, we have used the notation ξ_{nk} for the roots (not $\xi_{n\pi/\varphi_0,k}$).

The fundamental solutions to the first and second Dirichlet problem under zero initial conditions have the following form [188]

$$\mathcal{G}_{g1}(r,\varphi,\phi,t) = -\frac{4at^{\alpha-1}}{\varphi_0 R} \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} E_{\alpha,\alpha}\left(-a\xi_{nk}^2 t^\alpha\right) \sin\left(\frac{n\pi\varphi}{\varphi_0}\right) \sin\left(\frac{n\pi\phi}{\varphi_0}\right)$$

$$\times \frac{\xi_{nk}\, J_{n\pi/\varphi_0}(r\xi_{nk})}{J'_{n\pi/\varphi_0}(R\xi_{nk})}; \tag{8.100}$$

$$\mathcal{G}_{g2}(r,\varphi,\rho,t) = \frac{4at^{\alpha-1}}{\varphi_0 R^2 \rho^2} \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} \frac{n\pi}{\varphi_0}\, E_{\alpha,\alpha}\left(-a\xi_{nk}^2 t^\alpha\right) \sin\left(\frac{n\pi\varphi}{\varphi_0}\right)$$

$$\times \frac{J_{n\pi/\varphi_0}(r\xi_{nk})\, J_{n\pi/\varphi_0}(\rho\xi_{nk})}{\left[J'_{n\pi/\varphi_0}(R\xi_{nk})\right]^2}. \tag{8.101}$$

The fundamental solution to the third Dirichlet problem is obtained by multiplying each term in (8.101) by $(-1)^{n+1}$.

8.5.2 Neumann boundary condition

$$\frac{\partial^\alpha T}{\partial t^\alpha} = a\left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \varphi^2}\right) + \Phi(r,\varphi,t), \tag{8.102}$$

$$t = 0: \quad T = f(r,\varphi), \qquad 0 < \alpha \le 2, \tag{8.103}$$

$$t = 0: \quad \frac{\partial T}{\partial t} = F(r,\varphi), \qquad 1 < \alpha \le 2, \tag{8.104}$$

$$r = R: \quad \frac{\partial T}{\partial r} = g_1(\varphi,t), \tag{8.105}$$

$$\varphi = 0: \quad -\frac{1}{r}\frac{\partial T}{\partial \varphi} = g_2(r,t), \tag{8.106}$$

$$\varphi = \varphi_0: \quad \frac{1}{r}\frac{\partial T}{\partial \varphi} = g_3(r,t). \tag{8.107}$$

The solution:

$$\begin{aligned}
T(r,\varphi,t) = & \int_0^{\varphi_0}\int_0^R f(\rho,\phi)\,\mathcal{G}_f(r,\varphi,\rho,\phi,t)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\phi \\
& + \int_0^{\varphi_0}\int_0^R F(\rho,\phi)\,\mathcal{G}_F(r,\varphi,\rho,\phi,t)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\phi \\
& + \int_0^t\int_0^{\varphi_0}\int_0^R \Phi(\rho,\phi,\tau)\,\mathcal{G}_\Phi(r,\varphi,\rho,\phi,t-\tau)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}\tau \\
& + \int_0^t\int_0^{\varphi_0} g_1(\phi,\tau)\,\mathcal{G}_{g_1}(r,\varphi,\phi,t-\tau)\,\mathrm{d}\phi\,\mathrm{d}\tau \\
& + \int_0^t\int_0^R g_2(\rho,\tau)\,\mathcal{G}_{g_2}(r,\varphi,\rho,t-\tau)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\tau \\
& + \int_0^t\int_0^R g_3(\rho,\tau)\,\mathcal{G}_{g_3}(r,\varphi,\rho,t-\tau)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\tau.
\end{aligned} \tag{8.108}$$

The problem is solved using the Laplace transform with respect to time t, the finite cos-Fourier transform (2.48) with respect to the angular coordinate φ and the finite

Hankel transform (2.88) with respect to the radial variable r with $\nu = n\pi/\varphi_0$. For the fundamental solutions we get [188]

$$\begin{pmatrix} \mathcal{G}_f(r,\varphi,\rho,\phi,t) \\ \mathcal{G}_F(r,\varphi,\rho,\phi,t) \\ \mathcal{G}_\Phi(r,\varphi,\rho,\phi,t) \end{pmatrix} = \frac{2}{\varphi_0 R^2} \begin{pmatrix} p_0 \\ w_0 t \\ q_0 t^{\alpha-1}/\Gamma(\alpha) \end{pmatrix}$$

$$+ \frac{4}{\varphi_0} \sum_{n=0}^{\infty}{}' \sum_{k=1}^{\infty} \begin{pmatrix} p_0\, E_\alpha(-a\xi_{nk}^2 t^\alpha) \\ w_0 t\, E_{\alpha,2}(-a\xi_{nk}^2 t^\alpha) \\ q_0 t^{\alpha-1}\, E_{\alpha,\alpha}(-a\xi_{nk}^2 t^\alpha) \end{pmatrix} \cos\left(\frac{n\pi\varphi}{\varphi_0}\right) \cos\left(\frac{n\pi\phi}{\varphi_0}\right)$$

$$\times \frac{\xi_{nk}^2\, J_{n\pi/\varphi_0}(r\xi_{nk})\, J_{n\pi/\varphi_0}(\rho\xi_{nk})}{\left[R^2\xi_{nk}^2 - (n\pi/\varphi_0)^2\right] \left[J_{n\pi/\varphi_0}(R\xi_{nk})\right]^2}, \tag{8.109}$$

where ξ_{nk} are the positive roots of the transcendental equation

$$J'_{n\pi/\varphi_0}(R\xi_{nk}) = 0. \tag{8.110}$$

The fundamental solutions to the mathematical and physical Neumann problems under zero initial conditions have the following form

$$\mathcal{G}_{m1}(r,\varphi,\phi,t) = \frac{aRg_{01}}{q_0}\, \mathcal{G}_\Phi(r,\varphi,\rho,\phi,t)\Big|_{\rho=R}, \tag{8.111}$$

$$\mathcal{G}_{m2}(r,\varphi,\rho,t) = \frac{ag_{02}}{\rho\, q_0}\, \mathcal{G}_\Phi(r,\varphi,\rho,\phi,t)\Big|_{\phi=0}, \tag{8.112}$$

$$\mathcal{G}_{m3}(r,\varphi,\rho,t) = \frac{ag_{03}}{\rho\, q_0}\, \mathcal{G}_\Phi(r,\varphi,\rho,\phi,t)\Big|_{\phi=\varphi_0}, \tag{8.113}$$

$$\mathcal{G}_{p1}(r,\varphi,\phi,t) = \frac{aRg_{01}}{p_0}\, \mathcal{G}_f(r,\varphi,\rho,\phi,t)\Big|_{\rho=R}, \tag{8.114}$$

$$\mathcal{G}_{p2}(r,\varphi,\rho,t) = \frac{ag_{02}}{\rho\, p_0}\, \mathcal{G}_f(r,\varphi,\rho,\phi,t)\Big|_{\phi=0}, \tag{8.115}$$

$$\mathcal{G}_{p3}(r,\varphi,\rho,t) = \frac{ag_{03}}{\rho\, p_0}\, \mathcal{G}_f(r,\varphi,\rho,\phi,t)\Big|_{\phi=\varphi_0}. \tag{8.116}$$

8.6 Domain $R \le r < \infty,\ 0 < \varphi < \varphi_0$

8.6.1 Dirichlet boundary condition

$$\frac{\partial^\alpha T}{\partial t^\alpha} = a\left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \varphi^2}\right) + \Phi(r,\varphi,t), \tag{8.117}$$

$$t = 0: \quad T = f(r,\varphi), \qquad 0 < \alpha \le 2, \tag{8.118}$$

$$t = 0: \quad \frac{\partial T}{\partial t} = F(r,\varphi), \qquad 1 < \alpha \le 2, \tag{8.119}$$

$$r = R: \quad T = g_1(\varphi,t), \tag{8.120}$$

$$\varphi = 0: \quad T = g_2(r,t), \tag{8.121}$$

$$\varphi = \varphi_0: \quad T = g_3(r,t). \tag{8.122}$$

The solution:

$$\begin{aligned}
T(r,\varphi,t) &= \int_0^{\varphi_0}\int_R^{\infty} f(\rho,\phi)\,\mathcal{G}_f(r,\varphi,\rho,\phi,t)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\phi \\
&+ \int_0^{\varphi_0}\int_R^{\infty} F(\rho,\phi)\,\mathcal{G}_F(r,\varphi,\rho,\phi,t)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\phi \\
&+ \int_0^{t}\int_0^{\varphi_0}\int_R^{\infty} \Phi(\rho,\phi,\tau)\,\mathcal{G}_\Phi(r,\varphi,\rho,\phi,t-\tau)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}\tau \\
&+ \int_0^{t}\int_0^{\varphi_0} g_1(\phi,\tau)\,\mathcal{G}_{g_1}(r,\varphi,\phi,t-\tau)\,\mathrm{d}\phi\,\mathrm{d}\tau \\
&+ \int_0^{t}\int_R^{\infty} g_2(\rho,\tau)\,\mathcal{G}_{g_2}(r,\varphi,\rho,t-\tau)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\tau \\
&+ \int_0^{t}\int_R^{\infty} g_3(\rho,\tau)\,\mathcal{G}_{g_3}(r,\varphi,\rho,t-\tau)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\tau.
\end{aligned} \tag{8.123}$$

The problem is solved using the Laplace transform with respect to time t, the finite sin-Fourier transform (2.44) with respect to the angular coordinate φ and the Weber transform (2.108), (2.111) with respect to the radial variable r with

$\nu = n\pi/\varphi_0$. For the fundamental solutions to the first and second Cauchy problems and to the source problem under zero Dirichlet boundary condition we get

$$\begin{pmatrix} \mathcal{G}_f(r,\varphi,\rho,\phi,t) \\ \mathcal{G}_F(r,\varphi,\rho,\phi,t) \\ \mathcal{G}_\Phi(r,\varphi,\rho,\phi,t) \end{pmatrix} = \frac{2}{\varphi_0} \sum_{n=1}^{\infty} \int_0^{\infty} \begin{pmatrix} p_0\, E_\alpha(-a\xi^2 t^\alpha) \\ w_0 t\, E_{\alpha,2}(-a\xi^2 t^\alpha) \\ q_0 t^{\alpha-1}\, E_{\alpha,\alpha}(-a\xi^2 t^\alpha) \end{pmatrix} \sin\left(\frac{n\pi\varphi}{\varphi_0}\right)$$
$$\times \sin\left(\frac{n\pi\phi}{\varphi_0}\right) \frac{J_{n\pi/\varphi_0}(r\xi)\, Y_{n\pi/\varphi_0}(R\xi) - Y_{n\pi/\varphi_0}(r\xi)\, J_{n\pi/\varphi_0}(R\xi)}{J^2_{n\pi/\varphi_0}(R\xi) + Y^2_{n\pi/\varphi_0}(R\xi)}$$
$$\times \left[J_{n\pi/\varphi_0}(\rho\xi)\, Y_{n\pi/\varphi_0}(R\xi) - Y_{n\pi/\varphi_0}(\rho\xi)\, J_{n\pi/\varphi_0}(R\xi) \right] \xi\, \mathrm{d}\xi. \tag{8.124}$$

The fundamental solutions to the Dirichlet problems under zero initial conditions are calculated as

$$\mathcal{G}_{g1}(r,\varphi,\phi,t) = -\frac{4ag_{01}t^{\alpha-1}}{\pi\varphi_0} \sum_{n=1}^{\infty} \int_0^{\infty} E_{\alpha,\alpha}(-a\xi^2 t^\alpha) \sin\left(\frac{n\pi\varphi}{\varphi_0}\right) \sin\left(\frac{n\pi\phi}{\varphi_0}\right)$$
$$\times \frac{J_{n\pi/\varphi_0}(r\xi)\, Y_{n\pi/\varphi_0}(R\xi) - Y_{n\pi/\varphi_0}(r\xi)\, J_{n\pi/\varphi_0}(R\xi)}{J^2_{n\pi/\varphi_0}(R\xi) + Y^2_{n\pi/\varphi_0}(R\xi)} \xi\, \mathrm{d}\xi, \tag{8.125}$$

$$\mathcal{G}_{g2}(r,\varphi,\rho,t) = \frac{ag_{02}}{\rho^2 q_0} \left.\frac{\partial \mathcal{G}_\Phi(r,\varphi,\rho,\phi,t)}{\partial \phi}\right|_{\phi=0}, \tag{8.126}$$

$$\mathcal{G}_{g3}(r,\varphi,\rho,t) = -\frac{ag_{03}}{\rho^2 q_0} \left.\frac{\partial \mathcal{G}_\Phi(r,\varphi,\rho,\phi,t)}{\partial \phi}\right|_{\phi=\varphi_0}. \tag{8.127}$$

8.6.2 Neumann boundary condition

$$\frac{\partial^\alpha T}{\partial t^\alpha} = a\left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \varphi^2}\right) + \Phi(r,\varphi,t), \tag{8.128}$$

$$t = 0: \quad T = f(r,\varphi), \qquad 0 < \alpha \le 2, \tag{8.129}$$

$$t = 0: \quad \frac{\partial T}{\partial t} = F(r,\varphi), \qquad 1 < \alpha \le 2, \tag{8.130}$$

$$r = R: \quad -\frac{\partial T}{\partial r} = g_1(\varphi,t), \tag{8.131}$$

$$\varphi = 0: \quad -\frac{1}{r}\frac{\partial T}{\partial \varphi} = g_2(r,t), \tag{8.132}$$

$$\varphi = \varphi_0: \quad \frac{1}{r}\frac{\partial T}{\partial \varphi} = g_3(r,t). \tag{8.133}$$

The solution:

$$\begin{aligned} T(r,\varphi,t) &= \int_0^{\varphi_0}\int_R^{\infty} f(\rho,\phi)\,\mathcal{G}_f(r,\varphi,\rho,\phi,t)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\phi \\ &+ \int_0^{\varphi_0}\int_R^{\infty} F(\rho,\phi)\,\mathcal{G}_F(r,\varphi,\rho,\phi,t)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\phi \\ &+ \int_0^{t}\int_0^{\varphi_0}\int_R^{\infty} \Phi(\rho,\phi,\tau)\,\mathcal{G}_\Phi(r,\varphi,\rho,\phi,t-\tau)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}\tau \\ &+ \int_0^{t}\int_0^{\varphi_0} g_1(\phi,\tau)\,\mathcal{G}_{g_1}(r,\varphi,\phi,t-\tau)\,\mathrm{d}\phi\,\mathrm{d}\tau \\ &+ \int_0^{t}\int_R^{\infty} g_2(\rho,\tau)\,\mathcal{G}_{g_2}(r,\varphi,\rho,t-\tau)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\tau \\ &+ \int_0^{t}\int_R^{\infty} g_3(\rho,\tau)\,\mathcal{G}_{g_3}(r,\varphi,\rho,t-\tau)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\tau. \end{aligned} \tag{8.134}$$

The problem is solved using the Laplace transform with respect to time t, the finite cos-Fourier transform (2.48) with respect to the angular coordinate φ and the Weber transform (2.108), (2.113) with respect to the radial variable r with $\nu = n\pi/\varphi_0$. For the fundamental solutions to the first and second Cauchy problems and to the source problem under zero Neumann boundary condition we get

$$\begin{aligned} \begin{pmatrix} \mathcal{G}_f(r,\varphi,\rho,\phi,t) \\ \mathcal{G}_F(r,\varphi,\rho,\phi,t) \\ \mathcal{G}_\Phi(r,\varphi,\rho,\phi,t) \end{pmatrix} &= \frac{2}{\varphi_0}\sum_{n=0}^{\infty}{}'\int_0^{\infty} \begin{pmatrix} p_0\,E_\alpha(-a\xi^2t^\alpha) \\ w_0 t\,E_{\alpha,2}(-a\xi^2t^\alpha) \\ q_0 t^{\alpha-1}\,E_{\alpha,\alpha}(-a\xi^2t^\alpha) \end{pmatrix} \cos\left(\frac{n\pi\varphi}{\varphi_0}\right) \\ &\times \cos\left(\frac{n\pi\phi}{\varphi_0}\right) \frac{J_{n\pi/\varphi_0}(r\xi)\,Y'_{n\pi/\varphi_0}(R\xi) - Y_{n\pi/\varphi_0}(r\xi)\,J'_{n\pi/\varphi_0}(R\xi)}{[J'_{n\pi/\varphi_0}(R\xi)]^2 + [Y'_{n\pi/\varphi_0}(R\xi)]^2} \\ &\times \left[J_{n\pi/\varphi_0}(\rho\xi)\,Y'_{n\pi/\varphi_0}(R\xi) - Y_{n\pi/\varphi_0}(\rho\xi)\,J'_{n\pi/\varphi_0}(R\xi)\right]\xi\,\mathrm{d}\xi. \end{aligned} \tag{8.135}$$

The fundamental solutions to the mathematical and physical Neumann problems under zero initial conditions have the following form:

$$\mathcal{G}_{m1}(r,\varphi,\phi,t) = \frac{4ag_{01}t^{\alpha-1}}{\pi\varphi_0} \sum_{n=0}^{\infty}{}' \int_0^{\infty} E_{\alpha,\alpha}(-a\xi^2 t^{\alpha}) \cos\left(\frac{n\pi\varphi}{\varphi_0}\right) \cos\left(\frac{n\pi\phi}{\varphi_0}\right)$$
$$\times \frac{J_{n\pi/\varphi_0}(r\xi)\, Y'_{n\pi/\varphi_0}(R\xi) - Y_{n\pi/\varphi_0}(r\xi)\, J'_{n\pi/\varphi_0}(R\xi)}{[J'_{n\pi/\varphi_0}(R\xi)]^2 + [Y'_{n\pi/\varphi_0}(R\xi)]^2}\, \mathrm{d}\xi, \tag{8.136}$$

$$\mathcal{G}_{m2}(r,\varphi,\rho,t) = \frac{ag_{02}}{\rho\, q_0}\, \mathcal{G}_{\Phi}(r,\varphi,\rho,\phi,t)\Big|_{\phi=0}, \tag{8.137}$$

$$\mathcal{G}_{m3}(r,\varphi,\rho,t) = \frac{ag_{03}}{\rho\, q_0}\, \mathcal{G}_{\Phi}(r,\varphi,\rho,\phi,t)\Big|_{\phi=\varphi_0}, \tag{8.138}$$

$$\mathcal{G}_{p1}(r,\varphi,\phi,t) = \frac{4ag_{01}}{\pi\varphi_0} \sum_{n=0}^{\infty}{}' \int_0^{\infty} E_{\alpha}(-a\xi^2 t^{\alpha}) \cos\left(\frac{n\pi\varphi}{\varphi_0}\right) \cos\left(\frac{n\pi\phi}{\varphi_0}\right)$$
$$\times \frac{J_{n\pi/\varphi_0}(r\xi)\, Y'_{n\pi/\varphi_0}(R\xi) - Y_{n\pi/\varphi_0}(r\xi)\, J'_{n\pi/\varphi_0}(R\xi)}{[J'_{n\pi/\varphi_0}(R\xi)]^2 + [Y'_{n\pi/\varphi_0}(R\xi)]^2}\, \mathrm{d}\xi, \tag{8.139}$$

$$\mathcal{G}_{p2}(r,\varphi,\rho,t) = \frac{ag_{02}}{\rho\, p_0}\, \mathcal{G}_{f}(r,\varphi,\rho,\phi,t)\Big|_{\phi=0}, \tag{8.140}$$

$$\mathcal{G}_{p3}(r,\varphi,\rho,t) = \frac{ag_{03}}{\rho\, p_0}\, \mathcal{G}_{f}(r,\varphi,\rho,\phi,t)\Big|_{\phi=\varphi_0}. \tag{8.141}$$