Chapter 2

Mathematical Preliminaries

Да это ж математика богов
Владимир Высоцкий[1]

2.1 Integral transforms

The integral transform technique allows us to remove partial derivatives from the considered equations and to obtain the algebraic equation in a transform domain. Here we briefly recall the integral transforms which are used in this book to reduce the differential operators to an algebraic form. The Laplace transform with respect to time is marked by an asterisk, the Fourier transforms are denoted by a tilde, the Hankel transforms are indicated by a hat and the Legendre transform is designated by a star. Additional information concerning integral transforms can be found in [34, 37, 48, 140, 212], among others.

2.1.1 Laplace transform

The Laplace transform is defined as

$$\mathcal{L}\left\{f(t)\right\} = f^*(s) = \int_0^\infty f(t)\,\mathrm{e}^{-st}\,\mathrm{d}t, \tag{2.1}$$

where s is the transform variable.

[1] This is mathematics of gods.

Vladimir Vysotsky

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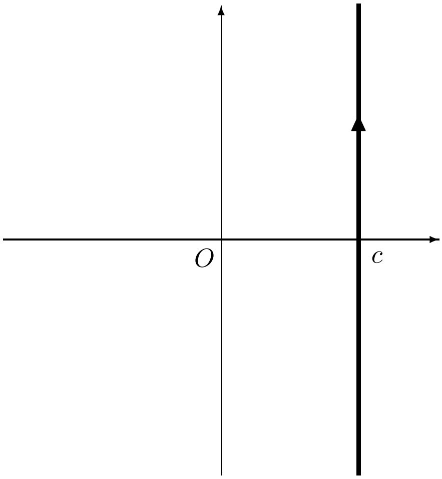

Figure 2.1: The Bromwich path of integration in the complex s-plane

The inverse Laplace transform is carried out according to the Fourier–Mellin formula

$$\mathcal{L}^{-1}\left\{f^*(s)\right\} = f(t) = \frac{1}{2\pi i}\int\limits_{c-i\infty}^{c+i\infty} f^*(s)\,\mathrm{e}^{st}\,\mathrm{d}s, \quad t > 0, \tag{2.2}$$

where c is a positive fixed number. The transform $f^*(s)$ is assumed analytical for $\mathfrak{Re}\, s > c$, all the singularities of $f^*(s)$ must lie to the left of the vertical line known as the Bromwich path of integration (see Fig. 2.1).

For the primitive of a function $f(t)$

$$If(t) = \int\limits_0^t f(\tau)\,\mathrm{d}\tau \tag{2.3}$$

we have

$$\mathcal{L}\left\{If(t)\right\} = \frac{1}{s}\,f^*(s), \tag{2.4}$$

whereas in the case of the m-fold primitive of a function $f(t)$,

$$I^m f(t) = \int\limits_0^t \mathrm{d}t_1 \int\limits_0^{t_1} \mathrm{d}t_2 \cdots \int\limits_0^{t_{m-1}} f(t_m)\,\mathrm{d}t_m, \tag{2.5}$$

the Laplace transform rule reads

$$\mathcal{L}\left\{I^m f(t)\right\} = \frac{1}{s^m}\,f^*(s). \tag{2.6}$$

The Laplace transform of the first derivative $f'(t)$ is easily obtained integrating the appropriate formula by parts which leads to

$$\mathcal{L}\{f'(t)\} = sf^*(s) - f(0^+), \tag{2.7}$$

and for the mth derivative $f^{(m)}(t)$

$$\mathcal{L}\left\{f^{(m)}(t)\right\} = s^m f^*(s) - \sum_{k=0}^{m-1} f^{(k)}(0^+)s^{m-1-k}. \tag{2.8}$$

The Laplace transform rule for the fractional integral (1.4) is similar to the rule (2.6):

$$\mathcal{L}\{I^\alpha f(t)\} = \frac{1}{s^\alpha} f^*(s). \tag{2.9}$$

The Riemann–Liouville derivative of the fractional order α (1.5) for its Laplace transform rule requires knowledge of the initial values of the fractional integral $I^{m-\alpha} f(t)$ and its derivatives of the order $k = 1, 2, \ldots, m-1$

$$\mathcal{L}\{D^\alpha_{RL} f(t)\} = s^\alpha f^*(s) \; - \sum_{k=0}^{m-1} D^k I^{m-\alpha} f(0^+)s^{m-1-k},$$
$$m - 1 < \alpha < m. \tag{2.10}$$

The Laplace transform rule for the Caputo derivative (1.6) has the following form

$$\mathcal{L}\left\{\frac{\mathrm{d}^\alpha f(t)}{\mathrm{d}t^\alpha}\right\} = s^\alpha f^*(s) - \sum_{k=0}^{m-1} f^{(k)}(0^+)s^{\alpha-1-k}, \quad m-1 < \alpha < m. \tag{2.11}$$

The convolution theorem, often used for inversion of the Laplace transform, reads as

$$\mathcal{L}^{-1}\{f^*(s)\, g^*(s)\} = \int_0^t f(t-\tau)\, g(\tau)\, \mathrm{d}\tau = \int_0^t f(\tau)\, g(t-\tau)\, \mathrm{d}\tau. \tag{2.12}$$

If the transform $f^*(s)$ can be expanded into the absolutely convergent series

$$f^*(s) = \sum_{k=0}^{\infty} \frac{c_k}{s^{\lambda_k}} \tag{2.13}$$

with arbitrary powers $0 < \lambda_0 < \lambda_1 < \lambda_2 < \cdots$ (need not be integers), then the inverse transform $f(t)$ has the expansion

$$f(t) = \sum_{k=0}^{\infty} \frac{c_k}{\Gamma(\lambda_k)}\, t^{\lambda_k - 1}. \tag{2.14}$$

If the transform $f^*(s)$ can be expanded into the absolutely convergent series

$$f^*(s) = \sum_{k=0}^{\infty} c_k s^{\lambda_k} \tag{2.15}$$

with arbitrary powers $0 < \lambda_0 < \lambda_1 < \lambda_2 < \cdots$ (need not be integers), then the inverse transform $f(t)$ for $t \to \infty$ has the asymptotic expansion

$$f(t) \sim \sum_{k=0}^{\infty} \frac{c_k}{\Gamma(-\lambda_k)}\, t^{-\lambda_k - 1}. \tag{2.16}$$

To calculate the inverse Laplace transform the Cauchy residue theorem is of fundamental importance.

Cauchy residue theorem. If $f(z)$ is analytic within and on a simple, closed contour $\mathfrak{C}$ except at finitely many points $z_1, z_2, \ldots, z_m$ lying in the interior of $\mathfrak{C}$, then

$$\frac{1}{2\pi i} \int_{\mathfrak{C}} f(z)\, \mathrm{d}z = \sum_{k=1}^{m} \operatorname{Res}_{z_k} f(z), \tag{2.17}$$

where integration is carried out in the positive direction.

Now choose the integration contour $\mathfrak{C}$ shown in Fig. 2.2 containing the portion of the vertical line $\mathfrak{Re}\, s = c$, two parts of the circle of radius R (designating as $\mathfrak{C}_R$), and a loop which starts from $-\infty$ along the upper side of the negative real axis, encircles a small circle of the radius ε in the negative direction and ends at $-\infty$ along the lower side of the negative real axis.

For a sufficiently "good" function $f^*(s)$

$$\lim_{R\to\infty} \int_{\mathfrak{C}_R} f^*(s)\, \mathrm{e}^{st}\, \mathrm{d}s = 0. \tag{2.18}$$

Hence

$$f(t) = \lim_{\varepsilon\to 0} \frac{1}{2\pi i} \int_{Ha_\varepsilon} f^*(s)\, \mathrm{e}^{st}\, \mathrm{d}s + \sum_{k=1}^{m} \operatorname{Res}_{s_k} f^*(s)\, \mathrm{e}^{st}, \tag{2.19}$$

where the Hankel path of integration Ha_ϵ is a loop which starts from $-\infty$ along the lower side of the negative real axis, encircles a small circle in the positive direction and ends at $-\infty$ along the upper side of the negative real axis (see Fig. 2.3). It should be noted that multiplying $f^*(s)$ by e^{st} does not affect the poles of $f^*(s)$.

2.1.2 Exponential Fourier transform

The exponential Fourier transform

$$\mathcal{F}\{f(x)\} = \widetilde{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \mathrm{e}^{ix\xi} \mathrm{d}x \tag{2.20}$$

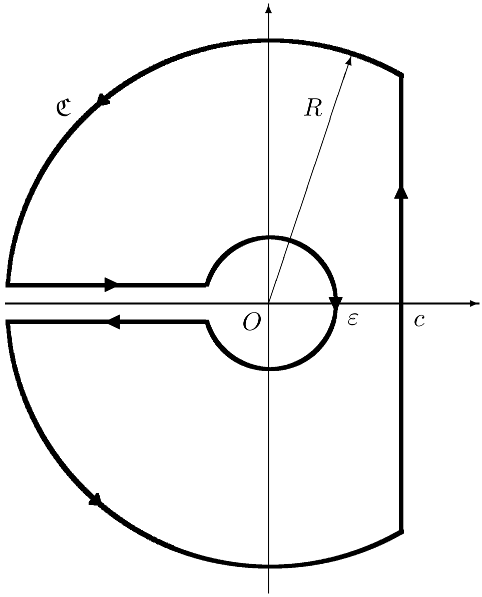

Figure 2.2: The closed path of integration in the complex s-plane

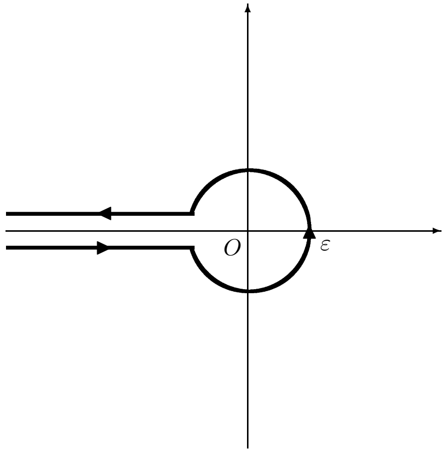

Figure 2.3: The Hankel path of integration in the complex s-plane

is used in the domain $-\infty < x < \infty$ and has the inverse

$$\mathcal{F}^{-1}\{\widetilde{f}(\xi)\} = f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \widetilde{f}(\xi) \mathrm{e}^{-ix\xi} \mathrm{d}\xi. \tag{2.21}$$

The Fourier transform of the mth derivative of a function has the form

$$\mathcal{F}\left\{\frac{\mathrm{d}^m f(x)}{\mathrm{d}x^m}\right\} = (-i\xi)^m \widetilde{f}(\xi), \tag{2.22}$$

in particular,

$$\mathcal{F}\left\{\frac{\mathrm{d}^2 f(x)}{\mathrm{d}x^2}\right\} = -\xi^2 \widetilde{f}(\xi). \tag{2.23}$$

The convolution theorem for the exponential Fourier transform reads:

$$\begin{aligned} \mathcal{F}^{-1}\left\{\widetilde{f}(\xi)\,\widetilde{g}(\xi)\right\} &= \frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{\infty} f(x-u)\,g(u)\,\mathrm{d}u \\ &= \frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{\infty} f(u)\,g(x-u)\,\mathrm{d}u. \end{aligned} \tag{2.24}$$

2.1.3 Sin-Fourier transform

The sin-Fourier transform is defined as

$$\mathcal{F}\{f(x)\} = \widetilde{f}(\xi) = \int\limits_0^{\infty} f(x)\sin(x\xi)\,\mathrm{d}x \tag{2.25}$$

with the inverse

$$\mathcal{F}^{-1}\{\widetilde{f}(\xi)\} = f(x) = \frac{2}{\pi}\int\limits_0^{\infty} \widetilde{f}(\xi)\sin(x\xi)\,\mathrm{d}\xi. \tag{2.26}$$

The sin-Fourier transform is used in the domain $0 \le x < \infty$ for Dirichlet boundary condition with the prescribed boundary value of a function, since for the second derivative of a function we get

$$\mathcal{F}\left\{\frac{\mathrm{d}^2 f(x)}{\mathrm{d}x^2}\right\} = -\xi^2 \widetilde{f}(\xi) + \xi\, f(x)\bigg|_{x=0}. \tag{2.27}$$

2.1.4 Three-fold Fourier transform in the case of spherical symmetry

If the considered function $f(x,y,z)$ depends only on the radial coordinate $r = \left(x^2+y^2+z^2\right)^{1/2}$, then the three-fold Fourier transform (2.20) can be simplified. Introducing the spherical coordinates

$$\begin{aligned} &x = r\sin\varphi\cos\vartheta, \quad y = r\sin\varphi\sin\vartheta, \quad z = r\cos\varphi, \\ &\xi = \varrho\sin\phi\cos\theta, \quad \eta = \varrho\sin\phi\sin\theta, \quad \zeta = \varrho\cos\phi, \end{aligned} \tag{2.28}$$

we have

$$\widetilde{f}(\xi,\eta,\zeta) = \frac{1}{(2\pi)^{3/2}} \int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} f(x,y,z)\,\mathrm{e}^{i(x\xi+y\eta+z\zeta)}\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$

$$= \frac{1}{(2\pi)^{3/2}} \int\limits_{0}^{\infty} r^2 f(r)\,\mathrm{d}r \int\limits_{0}^{\pi} \mathrm{e}^{ir\varrho\cos\varphi\cos\phi}\,\sin\varphi\,\mathrm{d}\varphi$$

$$\times \int\limits_{0}^{2\pi} \mathrm{e}^{ir\varrho\sin\varphi\sin\phi\cos(\vartheta-\theta)}\,\mathrm{d}\vartheta. \tag{2.29}$$

Due to the periodic character of the third integrand

$$\int\limits_{0}^{2\pi} \mathrm{e}^{ir\varrho\sin\varphi\sin\phi\cos(\vartheta-\theta)}\,\mathrm{d}\vartheta = \int\limits_{0}^{2\pi} \mathrm{e}^{ir\varrho\sin\varphi\sin\phi\cos\vartheta}\,\mathrm{d}\vartheta.$$

Using the integral representation of the Bessel function of the first kind of the zeroth order [1]

$$\int\limits_{0}^{2\pi} \mathrm{e}^{iz\cos\vartheta}\mathrm{d}\vartheta = 2\pi J_0(z), \tag{2.30}$$

we get

$$\widetilde{f}(\xi,\eta,\zeta) = \frac{1}{\sqrt{2\pi}} \int\limits_{0}^{\infty} r^2 f(r)\,\mathrm{d}r$$

$$\times \int\limits_{0}^{\pi} \sin\varphi\cos\left(r\varrho\cos\varphi\cos\phi\right)\, J_0\left(r\varrho\sin\varphi\sin\phi\right)\,\mathrm{d}\varphi$$

$$= \sqrt{\frac{2}{\pi}} \int\limits_{0}^{\infty} r^2 f(r)\,\mathrm{d}r \int\limits_{0}^{1} \cos\left(r\varrho v\cos\phi\right) J_0\left(r\varrho\sqrt{1-v^2}\sin\phi\right)\,\mathrm{d}v.$$

Next, we use the integral [196]

$$\int\limits_{0}^{1} \cos\left(av\right) J_0\left(b\sqrt{1-v^2}\right)\,\mathrm{d}v = \frac{1}{\sqrt{a^2+b^2}}\,\sin\sqrt{a^2+b^2},$$

and for the three-fold Fourier transform in the central symmetric case we obtain the following pair of equations:

$$\mathcal{F}\left\{f(x,y,z)\right\} = \mathcal{F}\left\{f(r)\right\} = \widetilde{f}(\varrho) = \sqrt{\frac{2}{\pi}} \int\limits_{0}^{\infty} r\,f(r)\,\frac{\sin(r\varrho)}{\varrho}\,\mathrm{d}r, \tag{2.31}$$

$$\mathcal{F}^{-1}\left\{\widetilde{f}(\varrho)\right\} = f(r) = \sqrt{\frac{2}{\pi}}\int\limits_0^\infty \varrho\,\widetilde{f}(\varrho)\,\frac{\sin(r\varrho)}{r}\,\mathrm{d}\varrho. \tag{2.32}$$

This result coincides with the particular case of the m-fold Fourier transform in the central symmetric case obtained by another method in [212]:

$$\varrho^{\frac{1}{2}m-1}\widetilde{f}(\varrho) = \int\limits_0^\infty r^{\frac{1}{2}m-1}\,f(r)\,J_{\frac{1}{2}m-1}(r\varrho)\,r\,\mathrm{d}r, \tag{2.33}$$

$$r^{\frac{1}{2}m-1}f(r) = \int\limits_0^\infty \varrho^{\frac{1}{2}m-1}\,\widetilde{f}(\varrho)\,J_{\frac{1}{2}m-1}(r\varrho)\,\varrho\,\mathrm{d}\varrho, \tag{2.34}$$

where $J_\nu(r)$ is the Bessel function.

For $m = 3$, taking into account that the Bessel function of the order one-half is represented as [1]

$$J_{1/2}(z) = \sqrt{\frac{2z}{\pi}}\,\frac{\sin z}{z}, \tag{2.35}$$

from (2.33) and (2.34) we get (2.31) and (2.32).

In this case

$$\mathcal{F}\left\{\frac{\mathrm{d}^2 f(r)}{\mathrm{d}r^2} + \frac{2}{r}\,\frac{\mathrm{d}f(r)}{\mathrm{d}r}\right\} = -\varrho^2\widetilde{f}(\varrho). \tag{2.36}$$

The pair of transform equations (2.31) and (2.32) seems like the pair of sin-Fourier transform equations (2.25) and (2.26) for the function $rf(r)$ (accurate to constant multipliers), but Eq. (2.31) does not need the value of a function at $r = 0$ as in Eq. (2.27). This allows us to consider also functions with singularities at $r = 0$ on condition that the integral in (2.31) is convergent.

2.1.5 Cos-Fourier transform

For the cos-Fourier transform we have

$$\mathcal{F}\{f(x)\} = \widetilde{f}(\xi) = \int\limits_0^\infty f(x)\cos(x\xi)\mathrm{d}x, \tag{2.37}$$

$$\mathcal{F}^{-1}\{\widetilde{f}(\xi)\} = f(x) = \frac{2}{\pi}\int\limits_0^\infty \widetilde{f}(\xi)\cos(x\xi)\mathrm{d}\xi. \tag{2.38}$$

The cos-Fourier transform is used in the domain $0 \le x < \infty$ in the case of Neumann boundary condition with the prescribed boundary value of the normal derivative of a function, since for the second derivative of a function it leads to the following formula:

$$\mathcal{F}\left\{\frac{\mathrm{d}^2 f(x)}{\mathrm{d}x^2}\right\} = -\xi^2\widetilde{f}(\xi) - \left.\frac{\mathrm{d}f(x)}{\mathrm{d}x}\right|_{x=0}. \tag{2.39}$$

2.1.6 Sin-cos-Fourier transform

In the case of the Robin boundary conditions with the prescribed boundary value of linear combination of a function and its normal derivative, the sin-cos-Fourier transform is employed:

$$\mathcal{F}\left\{f(x)\right\} = \widetilde{f}(\xi) = \int\limits_0^\infty K(x,\xi)\, f(x)\, \mathrm{d}x, \tag{2.40}$$

$$\mathcal{F}^{-1}\left\{\widetilde{f}(\xi)\right\} = f(x) = \frac{2}{\pi}\int\limits_0^\infty K(x,\xi)\, \widetilde{f}(\xi)\, \mathrm{d}\xi \tag{2.41}$$

with the kernel

$$K(x,\xi) = \frac{\xi\cos(x\xi) + H\sin(x\xi)}{\sqrt{\xi^2 + H^2}}. \tag{2.42}$$

In classical heat conduction the quantity H is usually connected with the heat transfer coefficient, in the case of spherical coordinates the quantity $1/R$ often stands in place of H.

Application of sin-cos-Fourier transform to the second derivative of a function gives

$$\mathcal{F}\left\{\frac{\mathrm{d}^2 f(x)}{\mathrm{d}x^2}\right\} = -\xi^2\widetilde{f}(\xi) + \frac{\xi}{\sqrt{\xi^2 + H^2}}\left[-\frac{\mathrm{d}f(x)}{\mathrm{d}x} + H f(x)\right]\Bigg|_{x=0}. \tag{2.43}$$

It is obvious that for $H \to \infty$ the sin-cos-Fourier transform turns into the standard sin-Fourier transform, while for $H \to 0$ it turns into the standard cos-Fourier transform.

2.1.7 Finite sin-Fourier transform

The finite sin-Fourier transform is the convenient reformulation of the sin-Fourier series in the domain $0 \le x \le L$:

$$\mathcal{F}\{f(x)\} = \widetilde{f}(\xi_k) = \int\limits_0^L f(x)\sin(x\xi_k)\, \mathrm{d}x, \tag{2.44}$$

$$\mathcal{F}^{-1}\{\widetilde{f}(\xi_k)\} = f(x) = \frac{2}{L}\sum_{k=1}^{\infty} \widetilde{f}(\xi_k)\sin(x\xi_k), \tag{2.45}$$

where

$$\xi_k = \frac{k\pi}{L}. \tag{2.46}$$

The finite sin-Fourier transform is used in the case of the Dirichlet boundary condition as for the second derivative of a function we have

$$\mathcal{F}\left\{\frac{\mathrm{d}^2 f(x)}{\mathrm{d}x^2}\right\} = -\xi_k^2 \widetilde{f}(\xi_k) + \xi_k \left[f(0) - (-1)^k f(L)\right]. \tag{2.47}$$

2.1.8 Finite cos-Fourier transform

The finite cos-Fourier transform is the convenient reformulation of the cos-Fourier series in the domain $0 \leq x \leq L$:

$$\mathcal{F}\{f(x)\} = \widetilde{f}(\xi_k) = \int_0^L f(x)\cos(x\xi_k)\,\mathrm{d}x, \tag{2.48}$$

$$\begin{aligned}\mathcal{F}^{-1}\{\widetilde{f}(\xi_k)\} = f(x) &= \frac{1}{L}\widetilde{f}(0) + \frac{2}{L}\sum_{k=1}^{\infty}\widetilde{f}(\xi_k)\cos(x\xi_k) \\ &= \frac{2}{L}{\sum_{k=0}^{\infty}}{}'\,\widetilde{f}(\xi_k)\cos(x\xi_k),\end{aligned} \tag{2.49}$$

where the prime near the sum denotes that the term corresponding to $k = 0$ should be multiplied by 1/2 and as in (2.46)

$$\xi_k = \frac{k\pi}{L}. \tag{2.50}$$

The finite cos-Fourier transform is used in the case of Neumann boundary condition as

$$\mathcal{F}\left\{\frac{\mathrm{d}^2 f(x)}{\mathrm{d}x^2}\right\} = -\xi_k^2\,\widetilde{f}(\xi_k) - \left.\frac{\mathrm{d}f(x)}{\mathrm{d}x}\right|_{x=0} + (-1)^k \left.\frac{\mathrm{d}f(x)}{\mathrm{d}x}\right|_{x=L}. \tag{2.51}$$

2.1.9 Finite sin-cos-Fourier transform

The finite sin-cos-Fourier transform is used in the case of the Robin boundary condition:

$$\mathcal{F}\{f(x)\} = \widetilde{f}(\xi_k) = \int_0^L f(x)\,\frac{\xi_k\cos(x\xi_k) + H\sin(x\xi_k)}{\sqrt{\xi_k^2 + H^2 + \frac{2H}{L}}}\,\mathrm{d}x, \tag{2.52}$$

$$\mathcal{F}^{-1}\{\widetilde{f}(\xi_k)\} = f(x) = \frac{2}{L}\sum_{k=1}^{\infty}\widetilde{f}(\xi_k)\,\frac{\xi_k\cos(x\xi_k) + H\sin(x\xi_k)}{\sqrt{\xi_k^2 + H^2 + \frac{2H}{L}}}, \tag{2.53}$$

where ξ_k are the positive roots of the transcendental equation

$$\tan(L\xi_k) = \frac{2H\xi_k}{\xi_k^2 - H^2} \tag{2.54}$$

and

$$\begin{aligned} \mathcal{F}\left\{\frac{\mathrm{d}^2 f(x)}{\mathrm{d}x^2}\right\} &= -\xi_k^2 \widetilde{f}(\xi_k) \\ &+ \frac{\xi_k}{\sqrt{\xi_k^2 + H^2 + \frac{2H}{L}}} \left[-\frac{\mathrm{d}f(x)}{\mathrm{d}x} + Hf(x)\right]\Bigg|_{x=0} \\ &+ \frac{\xi_k}{\sqrt{\xi_k^2 + H^2 + \frac{2H}{L}}} \frac{\xi_k^2 + H^2}{\xi_k^2 - H^2} \cos(L\xi_k) \left[\frac{\mathrm{d}f(x)}{\mathrm{d}x} + Hf(x)\right]\Bigg|_{x=L}. \end{aligned} \tag{2.55}$$

We have restricted ourselves to the case of the same H in the Robin boundary conditions at $x = 0$ and $x = L$. The general case of different coefficients H_1 and H_2 is considered in [48].

2.1.10 Finite sin-Fourier transform for a sphere

This type of finite sin-Fourier transform is convenient for central symmetric problems for a sphere $0 \le r \le R$. In the case of the Dirichlet boundary condition:

$$\mathcal{F}\{f(r)\} = \widetilde{f}(\xi_k) = \int_0^R r\, f(r)\, \frac{\sin(r\xi_k)}{\xi_k}\, \mathrm{d}r, \tag{2.56}$$

$$\mathcal{F}^{-1}\{\widetilde{f}(\xi_k)\} = f(r) = \frac{2}{R} \sum_{k=1}^{\infty} \xi_k\, \widetilde{f}(\xi_k)\, \frac{\sin(r\xi_k)}{r}, \tag{2.57}$$

where

$$\xi_k = \frac{k\pi}{R} \tag{2.58}$$

and

$$\mathcal{F}\left\{\frac{\mathrm{d}^2 f(r)}{\mathrm{d}r^2} + \frac{2}{r}\frac{\mathrm{d}f(r)}{\mathrm{d}r}\right\} = -\xi_k^2\, \widetilde{f}(\xi_k) + (-1)^{k+1} R f(R). \tag{2.59}$$

For the Neumann boundary condition

$$\mathcal{F}\{f(r)\} = \widetilde{f}(\xi_k) = \int_0^R r\, f(r)\, \frac{\sin(r\xi_k)}{\xi_k}\, \mathrm{d}r, \tag{2.60}$$

$$\mathcal{F}^{-1}\{\widetilde{f}(\xi_k)\} = f(r) = \frac{3}{R^3}\widetilde{f}(0) + \frac{2}{R} \sum_{k=1}^{\infty} \widetilde{f}(\xi_k) \frac{\xi_k}{\sin^2(R\xi_k)} \frac{\sin(r\xi_k)}{r}, \tag{2.61}$$

where ξ_k are the positive roots of the transcendental equation

$$\tan(R\xi_k) = R\xi_k \tag{2.62}$$

and

$$\mathcal{F}\left\{\frac{\mathrm{d}^2 f(r)}{\mathrm{d}r^2} + \frac{2}{r}\,\frac{\mathrm{d}f(r)}{\mathrm{d}r}\right\} = -\xi_k^2\,\widetilde{f}(\xi_k) + \frac{R\sin(R\xi_k)}{\xi_k}\,\frac{\mathrm{d}f(r)}{\mathrm{d}r}\bigg|_{r=R}. \tag{2.63}$$

For the Robin boundary condition

$$\mathcal{F}\{f(r)\} = \widetilde{f}(\xi_k) = \int_0^R f(r)\,\frac{\sin(r\xi_k)}{\xi_k}\,r\,\mathrm{d}r, \tag{2.64}$$

$$\mathcal{F}^{-1}\{\widetilde{f}(\xi_k)\} = f(r) = 2\sum_{k=1}^{\infty}\widetilde{f}(\xi_k)\frac{\xi_k^2}{R\xi_k - \sin(R\xi_k)\cos(R\xi_k)}\,\frac{\sin(r\xi_k)}{r}, \tag{2.65}$$

where ξ_k are the positive roots of the transcendental equation

$$\tan(R\xi_k) = \frac{R\xi_k}{1 - RH}, \tag{2.66}$$

and for the Laplace operator in the case of central symmetric problem we obtain

$$\mathcal{F}\left\{\frac{\mathrm{d}^2 f(r)}{\mathrm{d}r^2} + \frac{2}{r}\,\frac{\mathrm{d}f(r)}{\mathrm{d}r}\right\}$$
$$= -\xi_k^2\,\widetilde{f}(\xi_k) + \frac{R\sin(R\xi_k)}{\xi_k}\left[\frac{\mathrm{d}f(r)}{\mathrm{d}r} + Hf(r)\right]\bigg|_{r=R}. \tag{2.67}$$

2.1.11 Finite Fourier transform for 2π-periodic functions

Consider series development of the 2π-periodic function in the interval $[0, 2\pi]$

$$f(\varphi) = \frac{1}{2}\,a_0 + \sum_{m=1}^{\infty}\left[a_m\cos(m\varphi) + b_m\sin(m\varphi)\right], \tag{2.68}$$

where

$$a_m = \frac{1}{\pi}\int_0^{2\pi} f(\eta)\cos(m\eta)\,\mathrm{d}\eta, \qquad m = 0, 1, 2, \ldots$$

$$b_m = \frac{1}{\pi}\int_0^{2\pi} f(\eta)\sin(m\eta)\,\mathrm{d}\eta, \qquad m = 1, 2, \ldots \tag{2.69}$$

Now we insert the coefficients (2.69) into the equality (2.68), thus obtaining

$$f(\varphi) = \frac{1}{2\pi} \int_0^{2\pi} f(\eta)\,\mathrm{d}\eta + \frac{1}{\pi} \sum_{m=1}^{\infty} \int_0^{2\pi} f(\eta)\, \cos[m(\varphi - \eta)]\,\mathrm{d}\eta \tag{2.70}$$

or

$$f(\varphi) = \frac{1}{\pi} \sum_{m=0}^{\infty}{}' \int_0^{2\pi} f(\eta)\, \cos[m(\varphi - \eta)]\,\mathrm{d}\eta, \tag{2.71}$$

where the prime near the sum denotes that the term corresponding to $m = 0$ should be multiplied by $1/2$.

Formula (2.71) can be considered as the integral transform

$$\mathcal{F}\{f(\varphi)\} = \widetilde{f}(\varphi, m) = \int_0^{2\pi} f(\eta)\, \cos[m(\varphi - \eta)]\,\mathrm{d}\eta \tag{2.72}$$

with the inverse

$$\mathcal{F}^{-1}\{\widetilde{f}(\varphi, m)\} = f(\varphi) = \frac{1}{\pi} \sum_{m=0}^{\infty}{}' \widetilde{f}(\varphi, m). \tag{2.73}$$

This transform is used for solving equations in polar, cylindrical and spherical coordinates as the following equation is fulfilled:

$$\mathcal{F}\left\{ \frac{\mathrm{d}^2 f(\varphi)}{\mathrm{d}\varphi^2} \right\} = -m^2 \widetilde{f}(\varphi, m). \tag{2.74}$$

2.1.12 Legendre transform

The Legendre transform is applied to solve equations in spherical coordinates and reads:

$$\mathcal{P}\left\{f(\mu, m)\right\} = f^{\star}(n, m) = \int_{-1}^{1} f(\mu, m)\, P_n^m(\mu)\,\mathrm{d}\mu, \tag{2.75}$$

where $P_n^m(\mu)$ is the associated Legendre function of the first kind of degree n and order m. The inverse Legendre transform has the form

$$\begin{aligned} \mathcal{P}^{-1}\{f^{\star}(n, m)\} &= f(\mu, m) \\ &= \sum_{n=0}^{\infty} \frac{2n+1}{2} \frac{(n-m)!}{(n+m)!} P_n^m(\mu)\, f^{\star}(n, m), \quad n \geq m. \end{aligned} \tag{2.76}$$

The importance of this integral transform results from the following formula:

$$\mathcal{P}\left\{ \frac{\partial}{\partial \mu} \left[\left(1 - \mu^2\right) \frac{\partial f}{\partial \mu} \right] - \frac{m^2}{1 - \mu^2}\, f \right\} = -n(n+1) f^{\star}(n, m). \tag{2.77}$$

2.1.13 Hankel transform

The Hankel transform is used to solve problems in cylindrical coordinates in the domain $0 \leq r < \infty$ and is defined as

$$\mathcal{H}\{f(r)\} = \widehat{f}(\xi) = \int_0^\infty f(r)\, J_\nu(r\xi)\, r\, \mathrm{d}r, \tag{2.78}$$

$$\mathcal{H}^{-1}\{\widehat{f}(\xi)\} = f(r) = \int_0^\infty \widehat{f}(\xi)\, J_\nu(r\xi)\, \xi\, \mathrm{d}\xi, \tag{2.79}$$

$$\mathcal{H}\left\{\frac{\mathrm{d}^2 f(r)}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}f(r)}{\mathrm{d}r} - \frac{\nu^2}{r^2} f(r)\right\} = -\xi^2 \widehat{f}(\xi), \tag{2.80}$$

where $J_\nu(r)$ is the Bessel function of the order ν.

2.1.14 Two-fold Fourier transform in the case of axial symmetry

If the considered function $f(x, y)$ depends only on the radial coordinate

$$r = \left(x^2 + y^2\right)^{1/2},$$

then the two-fold Fourier transform (2.20) can be simplified. Introducing the polar coordinates

$$\begin{aligned} x &= r \sin\varphi, & y &= r\cos\varphi, \\ \xi &= \varrho \sin\phi, & \eta &= \varrho\cos\phi, \end{aligned} \tag{2.81}$$

we have

$$\widetilde{f}(\xi, \eta) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x, y)\, \mathrm{e}^{i(x\xi + y\eta)} \mathrm{d}x\mathrm{d}y = \frac{1}{2\pi}\int_0^\infty r f(r) \mathrm{d}r \int_0^{2\pi} \mathrm{e}^{ir\varrho\cos(\varphi - \phi)} \mathrm{d}\varphi.$$

Due to the periodic character of the second integrand

$$\int_0^{2\pi} \mathrm{e}^{ir\varrho\cos(\varphi-\phi)}\, \mathrm{d}\varphi = \int_0^{2\pi} \mathrm{e}^{ir\varrho\cos\varphi}\, \mathrm{d}\varphi.$$

Using the integral representation of the Bessel function of the first kind of the zeroth order (2.30) we get

$$\mathcal{F}\left\{f(x,y)\right\} = \widetilde{f}(\xi,\eta) = \mathcal{H}\left\{f(r)\right\} = \widehat{f}(\varrho) = \int\limits_0^\infty r f(r)\, J_0(r\varrho)\,\mathrm{d}r, \tag{2.82}$$

$$\mathcal{F}^{-1}\left\{\widetilde{f}(\xi,\eta)\right\} = f(x,y) = \mathcal{H}^{-1}\left\{\widehat{f}(\varrho)\right\} = f(r) = \int\limits_0^\infty \varrho \widehat{f}(\varrho)\, J_0(r\varrho)\,\mathrm{d}\varrho. \tag{2.83}$$

Hence, in the case of axial symmetry the two-fold Fourier transform with respect to the Cartesian coordinates is reduced to the Hankel transform with respect to the radial coordinate. Formulae (2.82) and (2.83) can also be obtained from the general formulae (2.33) and (2.34) for $m = 2$ [212].

2.1.15 Finite Hankel transform

The Fourier–Bessel and Dini series can be interpreted in terms of finite Hankel transform used in cylindrical coordinates in the domain $0 \le r \le R$. The specific form of the finite Hankel transform depends on the type of boundary conditions at $r = R$. For the Dirichlet boundary condition with the given boundary value of a function at $r = R$ we have

$$\mathcal{H}\{f(r)\} = \widehat{f}(\xi_{\nu k}) = \int\limits_0^R f(r)\, J_\nu(r\xi_{\nu k})\, r\,\mathrm{d}r, \tag{2.84}$$

$$\mathcal{H}^{-1}\{\widehat{f}(\xi_{\nu k})\} = f(r) = \frac{2}{R^2}\sum_{k=1}^{\infty} \widehat{f}(\xi_{\nu k}) \frac{J_\nu(r\xi_{\nu k})}{[J'_\nu(R\xi_{\nu k})]^2}, \tag{2.85}$$

where $\xi_{\nu k}$ are the positive roots of the transcendental equation

$$J_\nu(R\xi_{\nu k}) = 0 \tag{2.86}$$

and

$$\mathcal{H}\left\{\frac{\mathrm{d}^2 f(r)}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}f(r)}{\mathrm{d}r} - \frac{\nu^2}{r^2}\, f(r)\right\} = -\,\xi_{\nu k}^2 \widehat{f}(\xi_{\nu k}) - R\xi_{\nu k} J'_\nu(R\xi_{\nu k}) f(R). \tag{2.87}$$

In the case of the Neumann boundary conditions with the given boundary value of a normal derivative of a function we have

$$\mathcal{H}\{f(r)\} = \widehat{f}(\xi_{\nu k}) = \int\limits_0^R f(r)\, J_\nu(r\xi_{\nu k})\, r\,\mathrm{d}r, \tag{2.88}$$

$$\mathcal{H}^{-1}\{\widehat{f}(\xi_{\nu k})\} = f(r) = \frac{2}{R^2}\sum_{k=0}^{\infty} \widehat{f}(\xi_{\nu k})\, \frac{R^2\xi_{\nu k}^2}{R^2\xi_{\nu k}^2 - \nu^2}\, \frac{J_\nu(r\xi_{\nu k})}{[J_\nu(R\xi_{\nu k})]^2}, \tag{2.89}$$

where $\xi_{\nu k}$ are positive roots of the transcendental equation

$$J'_\nu(R\xi_{\nu k}) = 0. \tag{2.90}$$

It should be noted that for $\nu = 0$ there also appears the zero root $\xi_{00} = 0$, which must be taken into account in (2.89).

The basic equation for this integral transform reads:

$$\mathcal{H}\left\{\frac{\mathrm{d}^2 f(r)}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}f(r)}{\mathrm{d}r} - \frac{\nu^2}{r^2} f(r)\right\} = -\xi_{\nu k}^2 \widehat{f}(\xi_{\nu k}) + RJ_\nu(R\xi_{\nu k})\left(\frac{\mathrm{d}f}{\mathrm{d}r}\right)\bigg|_{r=R}. \tag{2.91}$$

For the Robin boundary condition with the given linear combination of values of function and its normal derivative at the boundary the corresponding finite Hankel transform has the following form:

$$\mathcal{H}\{f(r)\} = \widehat{f}(\xi_{\nu k}) = \int_0^R f(r)\, J_\nu(r\xi_{\nu k})\, r\, \mathrm{d}r, \tag{2.92}$$

$$\mathcal{H}^{-1}\{\widehat{f}(\xi_{\nu k})\} = f(r) = \frac{2}{R^2}\sum_{k=1}^{\infty} \widehat{f}(\xi_{\nu k})\, \frac{R^2\xi_{\nu k}^2}{R^2H^2 + (R^2\xi_{\nu k}^2 - \nu^2)}\, \frac{J_\nu(r\xi_{\nu k})}{[J_\nu(R\xi_{\nu k})]^2}, \tag{2.93}$$

where $\xi_{\nu k}$ are positive roots of the transcendental equation

$$\xi_{\nu k} J'_\nu(R\xi_{\nu k}) + HJ_\nu(R\xi_{\nu k}) = 0 \tag{2.94}$$

and

$$\mathcal{H}\left\{\frac{\mathrm{d}^2 f(r)}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}f(r)}{\mathrm{d}r} - \frac{\nu^2}{r^2} f(r)\right\}$$
$$= -\,\xi_{\nu k}^2 \widehat{f}(\xi_{\nu k}) + RJ_\nu(R\xi_{\nu k})\left[\frac{\mathrm{d}f(r)}{\mathrm{d}r} + Hf(r)\right]\bigg|_{r=R}. \tag{2.95}$$

Now we consider the particular cases of the finite Hankel transform of the zeroth order. For the Dirichlet boundary condition we get

$$\mathcal{H}\{f(r)\} = \widehat{f}(\xi_k) = \int_0^R f(r)\, J_0(r\xi_k)\, r\, \mathrm{d}r, \tag{2.96}$$

$$\mathcal{H}^{-1}\{\widehat{f}(\xi_k)\} = f(r) = \frac{2}{R^2}\sum_{k=1}^{\infty} \widehat{f}(\xi_k)\frac{J_0(r\xi_k)}{[J_1(R\xi_k)]^2} \tag{2.97}$$

with the sum over all positive roots of the zeroth-order Bessel function

$$J_0(R\xi_k) = 0, \tag{2.98}$$

and

$$\mathcal{H}\left\{\frac{\mathrm{d}^2 f(r)}{\mathrm{d}r^2}+\frac{1}{r}\frac{\mathrm{d}f(r)}{\mathrm{d}r}\right\}=-\xi_k^2\widehat{f}(\xi_k)+R\xi_k J_1(R\xi_k)f(R). \tag{2.99}$$

In the case of the Neumann boundary condition

$$\mathcal{H}\{f(r)\}=\widehat{f}(\xi_k)=\int_0^R r f(r) J_0(r\xi_k)\,\mathrm{d}r, \tag{2.100}$$

$$\begin{aligned}\mathcal{H}^{-1}\{\widehat{f}(\xi_k)\}=f(r)&=\frac{2}{R^2}\sum_{k=0}^{\infty}\widehat{f}(\xi_k)\frac{J_0(r\xi_k)}{[J_0(R\xi_k)]^2}\\ &=\frac{2}{R^2}\,\widehat{f}(0)+\frac{2}{R^2}\sum_{k=1}^{\infty}\widehat{f}(\xi_k)\frac{J_0(r\xi_k)}{[J_0(R\xi_k)]^2},\end{aligned} \tag{2.101}$$

where ξ_k are nonnegative roots of the equation

$$J_1(R\xi_k)=0. \tag{2.102}$$

To obtain the correct results it should be emphasized that this equation also has the root $\xi_0=0$. This root should be taken into consideration, and sometimes it is convenient to treat it separately (see Eq. (2.101)).

The fundamental equation for this transform has the form

$$\mathcal{H}\left\{\frac{\mathrm{d}^2 f(r)}{\mathrm{d}r^2}+\frac{1}{r}\frac{\mathrm{d}f(r)}{\mathrm{d}r}\right\}=-\xi_k^2\widehat{f}(\xi_k)+RJ_0(R\xi_k)\left.\left(\frac{\mathrm{d}f}{\mathrm{d}r}\right)\right|_{r=R}. \tag{2.103}$$

For the Robin boundary condition we have

$$\mathcal{H}\{f(r)\}=\widehat{f}(\xi_k)=\int_0^R r f(r) J_0(r\xi_k)\,\mathrm{d}r \tag{2.104}$$

with the inverse

$$\mathcal{H}^{-1}\{\widehat{f}(\xi_k)\}=f(r)=\frac{2}{R^2}\sum_{k=1}^{\infty}\widehat{f}(\xi_k)\,\frac{\xi_k^2}{H^2+\xi_k^2}\,\frac{J_0(r\xi_k)}{[J_0(R\xi_k)]^2}, \tag{2.105}$$

where ξ_k are the positive roots of the transcendental equation

$$\xi_k J_1(R\xi_k)=HJ_0(R\xi_k). \tag{2.106}$$

In this instance

$$\mathcal{H}\left\{\frac{\mathrm{d}^2 f(r)}{\mathrm{d}r^2}+\frac{1}{r}\frac{\mathrm{d}f(r)}{\mathrm{d}r}\right\}=-\xi_k^2\widehat{f}(\xi_k)+\;RJ_0(R\xi_k)\left.\left[\frac{\mathrm{d}f(r)}{\mathrm{d}r}+Hf(r)\right]\right|_{r=R}. \tag{2.107}$$

2.1.16 Weber transform

The Weber integral transform of order ν is defined as

$$\mathcal{W}\{f(r)\} = \widehat{f}(\xi) = \int_R^\infty K_\nu(r, R, \xi) f(r)\, r\, \mathrm{d}r \tag{2.108}$$

having the inverse

$$\mathcal{W}^{-1}\{\widehat{f}(\xi)\} = f(r) = \int_0^\infty K_\nu(r, R, \xi) \widehat{f}(\xi)\, \xi\, \mathrm{d}\xi. \tag{2.109}$$

The significance of the Weber transform for problems in the domain $R \le r < \infty$ is due to the formula

$$\begin{aligned} &\mathcal{W}\left\{\frac{\mathrm{d}^2 f(r)}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}f(r)}{\mathrm{d}r} - \frac{\nu^2}{r^2} f(r)\right\} \\ &\quad = -\xi^2 \widehat{f}(\xi) + R f(R) \left.\frac{\partial K_\nu(r, R, \xi)}{\partial r}\right|_{r=R} - R K_\nu(R, R, \xi) \left.\frac{\mathrm{d}f(r)}{\mathrm{d}r}\right|_{r=R}. \end{aligned} \tag{2.110}$$

The specific expression of the kernel $K_\nu(r, R, \xi)$ depends on the boundary conditions at $r = R$.

For the Dirichlet boundary condition the kernel is chosen as

$$K_\nu^{(\mathrm{D})}(r, R, \xi) = \frac{J_\nu(r\xi) Y_\nu(R\xi) - Y_\nu(r\xi) J_\nu(R\xi)}{\sqrt{J_\nu^2(R\xi) + Y_\nu^2(R\xi)}}, \tag{2.111}$$

where $J_\nu(r)$ and $Y_\nu(r)$ are the Bessel functions of the first and second kind, respectively.

Since

$$K_\nu^{(\mathrm{D})}(R, R, \xi) = 0,$$

$$\frac{\partial K_\nu^{(\mathrm{D})}(r, R, \xi)}{\partial r} = \frac{J'_\nu(r\xi) Y_\nu(R\xi) - Y'_\nu(r\xi) J_\nu(R\xi)}{\sqrt{J_\nu^2(R\xi) + Y_\nu^2(R\xi)}}\, \xi,$$

and [1]

$$J_\nu(z) Y'_\nu(z) - Y_\nu(z) J'_\nu(z) = \frac{2}{\pi z},$$

then

$$\mathcal{W}\left\{\frac{\mathrm{d}^2 f(r)}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}f(r)}{\mathrm{d}r} - \frac{\nu^2}{r^2} f(r)\right\} = -\xi^2 \widehat{f}(\xi) - \frac{2}{\pi} \frac{1}{\sqrt{J_\nu^2(R\xi) + Y_\nu^2(R\xi)}} f(R). \tag{2.112}$$

Similarly, in the case of the Neumann boundary condition

$$K_\nu^{(\mathrm{N})}(r,R,\xi) = \frac{J_\nu(r\xi)Y_\nu'(R\xi) - Y_\nu(r\xi)J_\nu'(R\xi)}{\sqrt{[J_\nu'(R\xi)]^2 + [Y_\nu'(R\xi)]^2}}, \tag{2.113}$$

and

$$\mathcal{W}\left\{\frac{\mathrm{d}^2 f(r)}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}f(r)}{\mathrm{d}r} - \frac{\nu^2}{r^2}\,f(r)\right\}$$
$$= -\,\xi^2 \widehat{f}(\xi) - \frac{2}{\pi\xi}\,\frac{1}{\sqrt{[J_\nu'(R\xi)]^2 + [Y_\nu'(R\xi)]^2}}\,\left[\frac{\mathrm{d}f(r)}{\mathrm{d}r}\right]\Bigg|_{r=R}. \tag{2.114}$$

For the Robin boundary condition

$$K_\nu^{(\mathrm{R})}(r,R,\xi)$$
$$= \frac{J_\nu(r\xi)[\xi Y_\nu'(R\xi) - HY_\nu(R\xi)] - Y_\nu(r\xi)[\xi J_\nu'(R\xi) - HJ_\nu(R\xi)]}{\sqrt{[\xi J_\nu'(R\xi) - HJ_\nu(R\xi)]^2 + [\xi Y_\nu'(R\xi) - HY_\nu(R\xi)]^2}}, \tag{2.115}$$

and

$$\mathcal{W}\left\{\frac{\mathrm{d}^2 f(r)}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}f(r)}{\mathrm{d}r} - \frac{\nu^2}{r^2}\,f(r)\right\} = -\,\xi^2 \widehat{f}(\xi)$$
$$+\,\frac{2}{\pi}\,\frac{1}{\sqrt{[\xi J_\nu'(R\xi) - HJ_\nu(R\xi)]^2 + [\xi Y_\nu'(R\xi) - HJ_\nu(R\xi)]^2}}$$
$$\times\left[-\frac{\mathrm{d}f(r)}{\mathrm{d}r} + Hf(r)\right]\Bigg|_{r=R}. \tag{2.116}$$

The formulae above simplify considerably in the case $\nu = 0$. For the Dirichlet boundary condition

$$K_0^{(\mathrm{D})}(r,R,\xi) = \frac{J_0(r\xi)Y_0(R\xi) - Y_0(r\xi)J_0(R\xi)}{\sqrt{J_0^2(R\xi) + Y_0^2(R\xi)}} \tag{2.117}$$

and

$$\mathcal{W}\left\{\frac{\mathrm{d}^2 f(r)}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}f(r)}{\mathrm{d}r}\right\} = -\,\xi^2 \widehat{f}(\xi) - \frac{2}{\pi}\,\frac{1}{\sqrt{J_0^2(R\xi) + Y_0^2(R\xi)}}\,f(R). \tag{2.118}$$

For the Neumann boundary condition

$$K_0^{(\mathrm{N})}(r,R,\xi) = -\frac{J_0(r\xi)Y_1(R\xi) - Y_0(r\xi)J_1(R\xi)}{\sqrt{J_1^2(R\xi) + Y_1^2(R\xi)}} \tag{2.119}$$

and

$$\mathcal{W}\left\{\frac{\mathrm{d}^2 f(r)}{\mathrm{d}r^2}+\frac{1}{r}\frac{\mathrm{d}f(r)}{\mathrm{d}r}\right\}$$
$$=-\xi^2\widehat{f}(\xi)-\frac{2}{\pi\xi}\frac{1}{\sqrt{J_1^2(R\xi)+Y_1^2(R\xi)}}\left[\frac{\mathrm{d}f(r)}{\mathrm{d}r}\right]\bigg|_{r=R}. \tag{2.120}$$

In the case of the Robin boundary condition

$$K_0^{(\mathrm{R})}(r,R,\xi)$$
$$=\frac{Y_0(r\xi)[\xi J_1(R\xi)+HJ_0(R\xi)]-J_0(r\xi)[\xi Y_1(R\xi)+HY_0(R\xi)]}{\sqrt{[\xi J_1(R\xi)+HJ_0(R\xi)]^2+[\xi Y_1(R\xi)+HY_0(R\xi)]^2}} \tag{2.121}$$

and

$$\mathcal{W}\left\{\frac{\mathrm{d}^2 f(r)}{\mathrm{d}r^2}+\frac{1}{r}\frac{\mathrm{d}f(r)}{\mathrm{d}r}\right\}=-\,\xi^2\widehat{f}(\xi)$$
$$+\frac{2}{\pi}\frac{1}{\sqrt{[\xi J_1(R\xi)+HJ_0(R\xi)]^2+[\xi Y_1(R\xi)+HJ_0(R\xi)]^2}}$$
$$\times\left[-\frac{\mathrm{d}f(r)}{\mathrm{d}r}+Hf(r)\right]\bigg|_{r=R}. \tag{2.122}$$

2.2 Mittag-Leffler function

The Mittag-Leffler function in one parameter α [119, 120] (see also [43, 56, 59, 77, 143])

$$E_\alpha(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+1)},\qquad \alpha>0,\ \ z\in C, \tag{2.123}$$

provides a generalization of the exponential function

$$\mathrm{e}^z=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(k+1)},\qquad z\in C. \tag{2.124}$$

The generalized Mittag-Leffler function in two parameters α and β [43, 56, 59, 71, 72, 77, 143] is described by the series representation

$$E_{\alpha,\beta}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+\beta)},\qquad \alpha>0,\ \beta>0,\ z\in C. \tag{2.125}$$

Here we recall several particular cases of the Mittag-Leffler functions for negative real values of argument:

$$E_0(-x)\;=\;\frac{1}{1+x}, \tag{2.126}$$

$$E_{1/2}(-x) = \mathrm{e}^{x^2}\mathrm{erfc}\,(x), \tag{2.127}$$

$$E_1(-x) = \mathrm{e}^{-x}, \tag{2.128}$$

$$E_2(-x) = \cos\sqrt{x}, \tag{2.129}$$

$$E_{1/2,\,1/2}(-x) = \frac{1}{\sqrt{\pi}} - x\mathrm{e}^{x^2}\mathrm{erfc}\,(x), \tag{2.130}$$

$$E_{0,2}(-x) = \frac{1}{1+x}, \tag{2.131}$$

$$E_{1/2,\,3/2}(-x) = \frac{1}{x}\left[1 - \mathrm{e}^{x^2}\mathrm{erfc}\,(x)\right], \tag{2.132}$$

$$E_{1/2\,,\,2}(-x) = \frac{1}{x^2}\left[\frac{2x}{\sqrt{\pi}} + \mathrm{e}^{x^2}\mathrm{erfc}\,(x) - 1\right], \tag{2.133}$$

$$E_{1,2}(-x) = \frac{1-\mathrm{e}^{-x}}{x}, \tag{2.134}$$

$$E_{2,2}(-x) = \frac{\sin\sqrt{x}}{\sqrt{x}}. \tag{2.135}$$

The Mittag-Leffler functions with the index 1/2 often appear in applications. It is convenient to obtain the helpful integral representations of these functions. For example, we have

$$E_{1/2}(-x) = \mathrm{e}^{x^2}\mathrm{erfc}\,(x) = \mathrm{e}^{x^2}\frac{2}{\sqrt{\pi}}\int_x^\infty \mathrm{e}^{-t^2}\mathrm{d}t.$$

Substitution $t = u + x$ leads to

$$E_{1/2}(-x) = \frac{2}{\sqrt{\pi}}\int_0^\infty \mathrm{e}^{-u^2-2ux}\mathrm{d}u. \tag{2.136}$$

Similarly,

$$E_{1/2\,,\,1/2}(-x) = \frac{2}{\sqrt{\pi}}\int_0^\infty \mathrm{e}^{-u^2-2ux}u\,\mathrm{d}u. \tag{2.137}$$

Several functional relations between different Mittag-Leffler functions can be found in [43]. We present the relations which will be used in the following:

$$E_{\alpha,\beta}(z) = \frac{1}{\Gamma(\beta)} + zE_{\alpha,\alpha+\beta}(z), \tag{2.138}$$

$$\frac{\mathrm{d}\left[z^{\beta-1}E_{\alpha,\beta}\left(z^\alpha\right)\right]}{\mathrm{d}z} = z^{\beta-2}E_{\alpha,\beta-1}\left(z^\alpha\right), \tag{2.139}$$

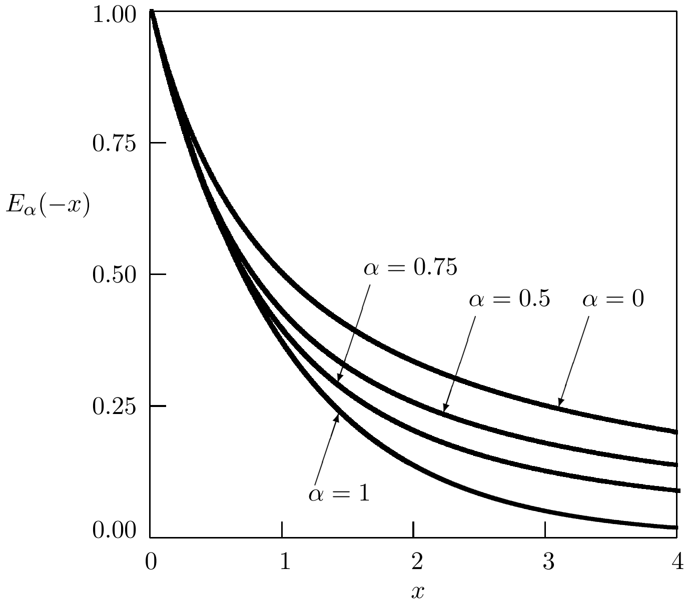

Figure 2.4: The Mittag-Leffler functions $E_\alpha(-x)$ for $0 \leq \alpha \leq 1$

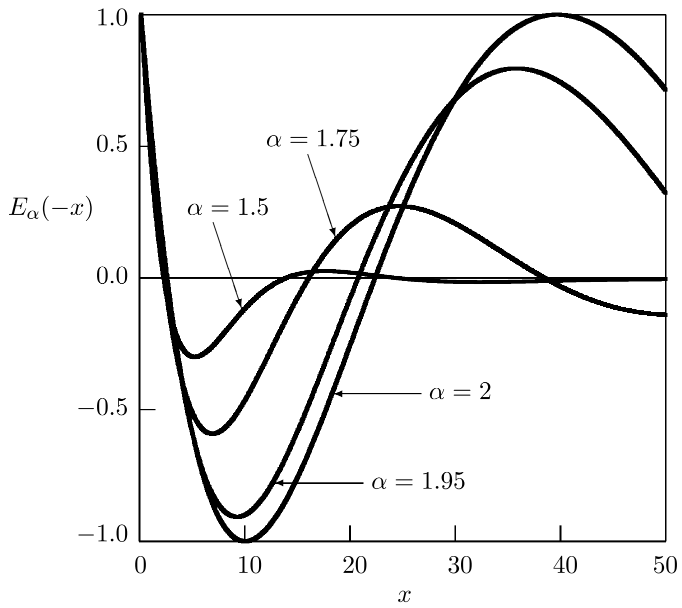

Figure 2.5: The Mittag-Leffler functions $E_\alpha(-x)$ for $1 < \alpha \leq 2$

$$\frac{\mathrm{d}\left[z^{\beta-1}E_{1,\beta}\left(z\right)\right]}{\mathrm{d}z} = z^{\beta-2}E_{1,\beta-1}\left(z\right). \tag{2.140}$$

The essential role of the Mittag-Leffler functions in fractional calculus results from the formula for the inverse Laplace transform [56, 77, 143]:

$$\mathcal{L}^{-1}\left\{\frac{s^{\alpha-\beta}}{s^{\alpha}+b}\right\} = t^{\beta-1}\,E_{\alpha,\beta}(-bt^{\alpha}). \tag{2.141}$$

For three important particular cases $\beta = 1$, $\beta = 2$ and $\beta = \alpha$, respectively, we get

$$\mathcal{L}^{-1}\left\{\frac{s^{\alpha-1}}{s^{\alpha}+b}\right\} = E_{\alpha}(-bt^{\alpha}), \tag{2.142}$$

$$\mathcal{L}^{-1}\left\{\frac{s^{\alpha-2}}{s^{\alpha}+b}\right\} = t\,E_{\alpha,2}(-bt^{\alpha}), \tag{2.143}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^{\alpha}+b}\right\} = t^{\alpha-1}\,E_{\alpha,\alpha}(-bt^{\alpha}). \tag{2.144}$$

The series representation of the Mittag-Leffler functions is inconvenient for numerical calculation. The integral representations of these functions suitable for such calculation were obtained in [52, 56]. In the subsequent discussion we restrict ourselves to the case of negative real values of argument. We have

$$\begin{aligned} E_{\alpha}(-bt^{\alpha}) &= \mathcal{L}^{-1}\left\{\frac{s^{\alpha-1}}{s^{\alpha}+b}\right\} = \frac{1}{2\pi i}\int_{Br}\mathrm{e}^{st}\frac{s^{\alpha-1}}{s^{\alpha}+b}\,\mathrm{d}s \\ &= \frac{1}{2\pi i}\int_{Ha}\mathrm{e}^{st}\frac{s^{\alpha-1}}{s^{\alpha}+b}\,\mathrm{d}s + \sum_{k}\mathrm{Res}_{\,s_k}\left(\mathrm{e}^{st}\frac{s^{\alpha-1}}{s^{\alpha}+b}\right) \\ &= f_{\alpha}(b,t) + g_{\alpha}(b,t). \end{aligned} \tag{2.145}$$

On the upper and lower sides of the Hankel path

$$s = r\mathrm{e}^{\pm i\pi} \pm i\varepsilon. \tag{2.146}$$

If $\varepsilon \to 0$, then

$$\mathrm{d}r = -\mathrm{d}s, \qquad s^{\alpha} = r^{\alpha}\left[\cos(\alpha\pi) \pm i\sin(\alpha\pi)\right]$$

and

$$\begin{aligned} f_{\alpha}(b,t) &= \frac{1}{\pi}\,\mathfrak{Im}\int_{0}^{\infty}\mathrm{e}^{-tr}\frac{r^{\alpha-1}[\cos(\alpha\pi)+i\sin(\alpha\pi)]}{r^{\alpha}[\cos(\alpha\pi)+i\sin(\alpha\pi)]+b}\,\mathrm{d}r \\ &= \frac{\sin(\alpha\pi)}{\pi}\int_{0}^{\infty}\mathrm{e}^{-tr}\frac{br^{\alpha-1}}{r^{2\alpha}+2r^{\alpha}b\cos(\alpha\pi)+b^{2}}\,\mathrm{d}r. \end{aligned} \tag{2.147}$$

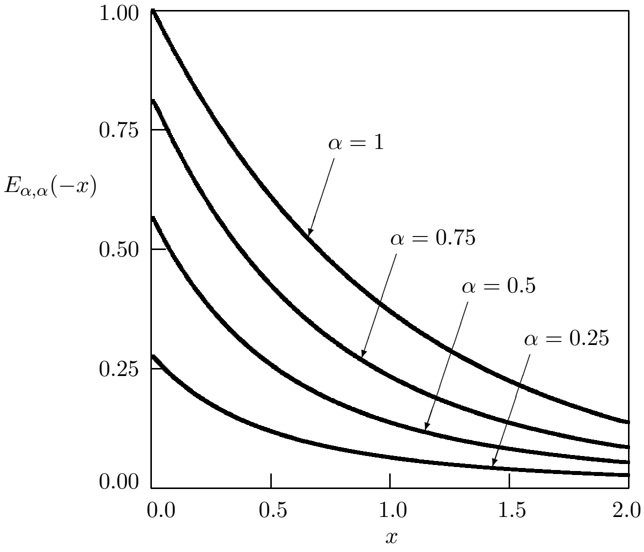

Figure 2.6: The Mittag-Leffler functions $E_{\alpha,\alpha}(-x)$ for $0 \leq \alpha \leq 1$

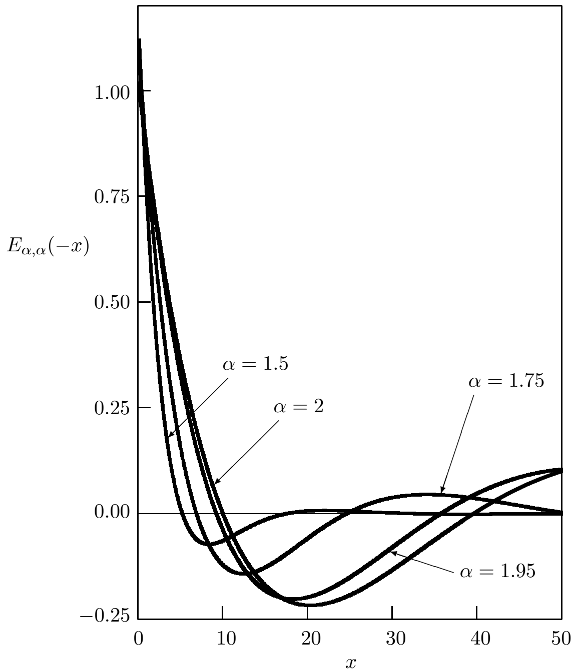

Figure 2.7: The Mittag-Leffler functions $E_{\alpha,\alpha}(-x)$ for $1 < \alpha \leq 2$

It is worthwhile introducing the substitution $r = b^{1/\alpha}u$ which leads to

$$f_\alpha(b,t) = \frac{\sin(\alpha\pi)}{\pi}\int_0^\infty \mathrm{e}^{-tb^{1/\alpha}u}\,\frac{u^{\alpha-1}}{u^{2\alpha}+2u^\alpha\cos(\alpha\pi)+1}\,\mathrm{d}u. \tag{2.148}$$

To investigate the poles of $\dfrac{s^{\alpha-1}}{s^\alpha+b}$ it should be mentioned that

$$s_k = b^{1/\alpha}\left[\cos\frac{(2k+1)\pi}{\alpha} + i\sin\frac{(2k+1)\pi}{\alpha}\right], \tag{2.149}$$

but only the poles situated in the main Riemann sheet are relevant, i.e., those s_k for which

$$-\pi < \frac{(2k+1)\pi}{\alpha} < \pi.$$

For $0<\alpha<1$ there are no such poles and

$$g_\alpha(b,t) = 0. \tag{2.150}$$

For $1<\alpha<2$ there are two such poles:

$$b^{1/\alpha}\left[\cos\left(\frac{\pi}{\alpha}\right) \pm i\sin\left(\frac{\pi}{\alpha}\right)\right],$$

and

$$g_\alpha(b,t) = \frac{2}{\alpha}\exp\left[tb^{1/\alpha}\cos\left(\frac{\pi}{\alpha}\right)\right]\,\cos\left[tb^{1/\alpha}\sin\left(\frac{\pi}{\alpha}\right)\right]. \tag{2.151}$$

Finally we arrive at the following result [52, 56]:

$$E_\alpha(-x) = \begin{cases} \dfrac{\sin(\alpha\pi)}{\pi}\displaystyle\int_0^\infty \mathrm{e}^{-x^{1/\alpha}u}\frac{u^{\alpha-1}}{u^{2\alpha}+2u^\alpha\cos(\alpha\pi)+1}\,\mathrm{d}u, & \\ \hfill 0<\alpha<1; & \\ \dfrac{\sin(\alpha\pi)}{\pi}\displaystyle\int_0^\infty \mathrm{e}^{-x^{1/\alpha}u}\frac{u^{\alpha-1}}{u^{2\alpha}+2u^\alpha\cos(\alpha\pi)+1}\,\mathrm{d}u & \\ +\dfrac{2}{\alpha}\exp\left[x^{1/\alpha}\cos\left(\dfrac{\pi}{\alpha}\right)\right]\,\cos\left[x^{1/\alpha}\sin\left(\dfrac{\pi}{\alpha}\right)\right], & \\ \hfill 1<\alpha<2. & \end{cases} \tag{2.152}$$

Similarly, for $1<\alpha<2$ we obtain

$$E_{\alpha,2}(-x) = -\frac{\sin(\alpha\pi)}{\pi\,x^{1/\alpha}}\int_0^\infty \mathrm{e}^{-x^{1/\alpha}u}\frac{u^{\alpha-2}}{u^{2\alpha}+2u^\alpha\cos(\alpha\pi)+1}\,\mathrm{d}u$$

$$+\frac{2}{\alpha x^{1/\alpha}}\exp\left[x^{1/\alpha}\cos\left(\frac{\pi}{\alpha}\right)\right]\,\cos\left[x^{1/\alpha}\sin\left(\frac{\pi}{\alpha}\right)-\frac{\pi}{\alpha}\right] \tag{2.153}$$

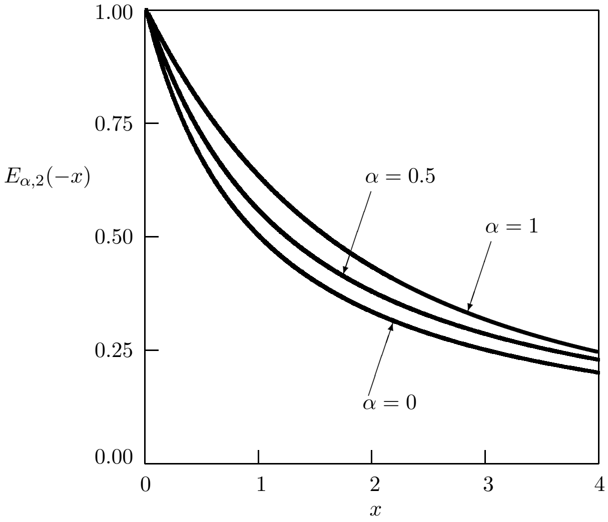

Figure 2.8: The Mittag-Leffler functions $E_{\alpha,2}(-x)$ for $0 \leq \alpha \leq 1$

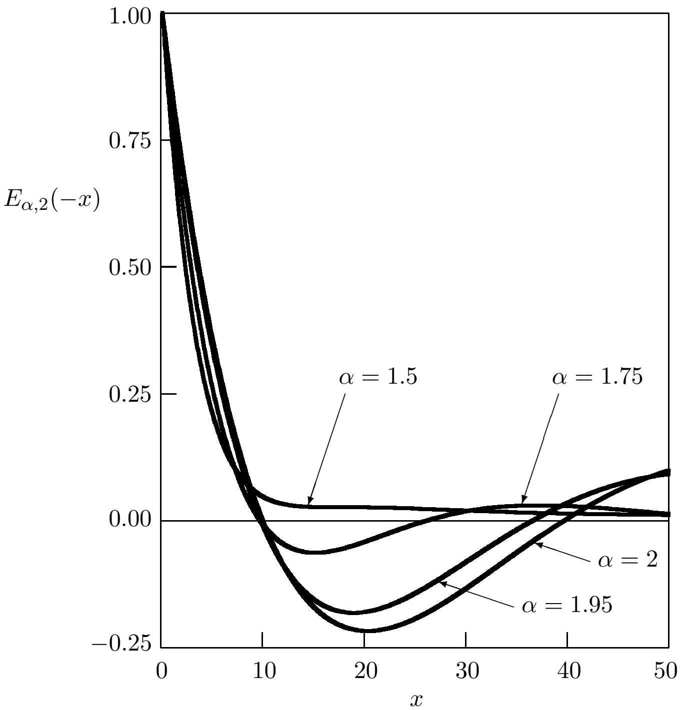

Figure 2.9: The Mittag-Leffler functions $E_{\alpha,2}(-x)$ for $1 < \alpha \leq 2$

and

$$E_{\alpha,\alpha}(-x) = \begin{cases} \dfrac{\sin(\alpha\pi)}{\pi x^{(\alpha-1)/\alpha}} \displaystyle\int_0^\infty \mathrm{e}^{-x^{1/\alpha}u} \dfrac{u^\alpha}{u^{2\alpha} + 2u^\alpha \cos(\alpha\pi) + 1}\, \mathrm{d}u, \\ \qquad\qquad 0 < \alpha < 1; \\ \dfrac{\sin(\alpha\pi)}{\pi x^{(\alpha-1)/\alpha}} \displaystyle\int_0^\infty \mathrm{e}^{-x^{1/\alpha}u} \dfrac{u^\alpha}{u^{2\alpha} + 2u^\alpha \cos(\alpha\pi) + 1}\, \mathrm{d}u \\ -\dfrac{2}{\alpha x^{(\alpha-1)/\alpha}} \exp\left[x^{1/\alpha}\cos\left(\dfrac{\pi}{\alpha}\right)\right] \cos\left[x^{1/\alpha}\sin\left(\dfrac{\pi}{\alpha}\right) + \dfrac{\pi}{\alpha}\right], \\ \qquad\qquad 1 < \alpha < 2. \end{cases} \tag{2.154}$$

Typical curves for $E_\alpha(-x)$ are presented in Figs. 2.4 and 2.5; for $E_{\alpha,\alpha}(-x)$ are shown in Figs. 2.6 and 2.7; $E_{\alpha,2}(-x)$ are depicted in Figs. 2.8 and 2.9 for various values of α.

In the general case, the integral representation of the generalized Mittag-Leffler function $E_{\alpha,\beta}$ can be obtained for $\alpha > 0$, $\beta > 0$, $\beta < \alpha + 1$:

$$E_{\alpha,\beta}(-x) = \begin{cases} \dfrac{1}{\pi x^{(\beta-1)/\alpha}} \displaystyle\int_0^\infty \mathrm{e}^{-x^{1/\alpha}u}\, u^{\alpha-\beta}\, \dfrac{u^\alpha \sin(\beta\pi) + \sin[(\beta-\alpha)\pi]}{u^{2\alpha} + 2u^\alpha \cos(\alpha\pi) + 1}\, \mathrm{d}u, \\ \qquad\qquad 0 < \alpha < 1; \\ \dfrac{1}{\pi x^{(\beta-1)/\alpha}} \displaystyle\int_0^\infty \mathrm{e}^{-x^{1/\alpha}u}\, u^{\alpha-\beta}\, \dfrac{u^\alpha \sin(\beta\pi) + \sin[(\beta-\alpha)\pi]}{u^{2\alpha} + 2u^\alpha \cos(\alpha\pi) + 1}\, \mathrm{d}u \\ +\dfrac{2}{\alpha x^{(\beta-1)/\alpha}} \exp\left[x^{1/\alpha}\cos\left(\dfrac{\pi}{\alpha}\right)\right] \cos\left[x^{1/\alpha}\sin\left(\dfrac{\pi}{\alpha}\right) + (1-\beta)\dfrac{\pi}{\alpha}\right] \\ \qquad\qquad 1 < \alpha < 2. \end{cases} \tag{2.155}$$

To investigate convergence of integral containing the Mittag-Leffler function it may be useful to have their asymptotic representations for large negative values of argument. Such a representation can be obtained expanding $(s^{\alpha-\beta})/(s^\alpha + b)$ in series for small s taking into account that

$$\frac{1}{s^\alpha + b} = \frac{1}{b}\left[1 - \frac{s^\alpha}{b} + \frac{s^{2\alpha}}{b^2} - \frac{s^{3\alpha}}{b^3} + \cdots\right]. \tag{2.156}$$

For $t \to \infty$ (see (2.15) and (2.16)) we have

$$\mathcal{L}^{-1}\left\{\frac{s^{\alpha-1}}{s^\alpha + b}\right\} \sim \frac{1}{\Gamma(1-\alpha)bt^\alpha} - \frac{1}{\Gamma(1-2\alpha)b^2t^{2\alpha}}, \tag{2.157}$$

$$\mathcal{L}^{-1}\left\{\frac{s^{\alpha-2}}{s^\alpha+b}\right\} \sim \frac{1}{\Gamma(2-\alpha)bt^{\alpha-1}} - \frac{1}{\Gamma(2-2\alpha)b^2t^{2\alpha-1}}, \tag{2.158}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^\alpha+b}\right\} \sim -\frac{1}{\Gamma(-\alpha)b^2t^{\alpha+1}}, \tag{2.159}$$

$$\mathcal{L}^{-1}\left\{\frac{s^{\alpha-\beta}}{s^\alpha+b}\right\} \sim \frac{1}{\Gamma(\beta-\alpha)bt^{\alpha-\beta+1}} - \frac{1}{\Gamma(\beta-2\alpha)b^2t^{2\alpha-\beta+1}}. \tag{2.160}$$

Hence, for $x\to\infty$ the desired results read as follows:

$$E_\alpha(-x) \sim \frac{1}{\Gamma(1-\alpha)x} - \frac{1}{\Gamma(1-2\alpha)x^2}, \tag{2.161}$$

$$E_{\alpha,2}(-x) \sim \frac{1}{\Gamma(2-\alpha)x} - \frac{1}{\Gamma(2-2\alpha)x^2}, \tag{2.162}$$

$$E_{\alpha,\alpha}(-x) \sim -\frac{1}{\Gamma(-\alpha)x^2}, \tag{2.163}$$

$$E_{\alpha,\beta}(-x) \sim \frac{1}{\Gamma(\beta-\alpha)x} - \frac{1}{\Gamma(\beta-2\alpha)x^2}. \tag{2.164}$$

2.3 Wright function and Mainardi function

The Wright function is defined as [43, 53, 54, 77, 90, 100, 101, 107, 143]

$$W(\alpha,\beta;z) = \sum_{k=0}^{\infty}\frac{z^k}{k!\,\Gamma(\alpha k+\beta)}, \qquad \alpha>-1, \quad z\in C. \tag{2.165}$$

Its integral representation has the following form [100, 143]:

$$W(\alpha,\beta;z) = \frac{1}{2\pi i}\int_{Ha} s^{-\beta}e^{s+zs^{-\alpha}}\,\mathrm{d}s, \qquad \alpha>-1, \quad z\in C, \tag{2.166}$$

where Ha denotes the Hankel path of integration in the complex s-plane.

The Wright function is a generalization of the exponential function and the Bessel functions (see [100, 143]):

$$W(0,1;z) = e^z, \tag{2.167}$$

$$W\left(-\frac{1}{2},\frac{1}{2};-z\right) = \frac{1}{\sqrt{\pi}}\exp\left(-\frac{z^2}{4}\right), \tag{2.168}$$

$$W\left(1,\nu+1;-\frac{z^2}{4}\right) = \left(\frac{2}{z}\right)^\nu J_\nu(z), \tag{2.169}$$

$$W\left(1,\nu+1;\frac{z^2}{4}\right) = \left(\frac{2}{z}\right)^{\nu} I_\nu(z). \tag{2.170}$$

Comparison of the definition of the Wright function (2.165) and the series expansion of the complementary error function [1]

$$\operatorname{erfc}(z) = \sum_{k=0}^{\infty} \frac{(-1)^k (2z)^k}{k!\,\Gamma\left(-\frac{1}{2}k+1\right)} \tag{2.171}$$

allows us to obtain the additional relation

$$W\left(-\frac{1}{2},1;-z\right) = \operatorname{erfc}\left(\frac{z}{2}\right). \tag{2.172}$$

The Wright function satisfies the equations

$$\alpha z W(\alpha,\alpha+\beta;z) = W(\alpha,\beta-1;z) + (1-\beta)W(\alpha,\beta;z), \tag{2.173}$$

$$\frac{\mathrm{d}W(\alpha,\beta;z)}{\mathrm{d}z} = W(\alpha,\alpha+\beta;z). \tag{2.174}$$

The Mainardi function $M(\alpha;z)$ [100, 101, 143] is the particular case of the Wright function

$$M(\alpha;z) = W(-\alpha,1-\alpha;-z) = \sum_{k=0}^{\infty} \frac{(-1)^k z^k}{k!\,\Gamma[-\alpha k+(1-\alpha)]},$$

$$0<\alpha<1, \qquad z\in C, \tag{2.175}$$

and also

$$M(\alpha;z) = \frac{1}{\alpha z} W(-\alpha,0;-z), \qquad 0<\alpha<1. \tag{2.176}$$

For $\alpha = 1/q$, where $q \geq 2$ is a positive integer, the Mainardi function can be expressed in terms of simpler functions, for example [100, 101]:

$$M\left(\frac{1}{2};z\right) = \frac{1}{\sqrt{\pi}} \exp\left(-\frac{z^2}{4}\right), \tag{2.177}$$

$$M\left(\frac{1}{3};z\right) = 3^{2/3}\operatorname{Ai}\left(\frac{z}{3^{1/3}}\right). \tag{2.178}$$

Similarly (see [67]):

$$M\left(\frac{2}{3};z\right) = \exp\left(-\frac{2z^3}{27}\right)\left[3^{-1/3} z \operatorname{Ai}\left(\frac{z^2}{3^{4/3}}\right) - 3^{1/3}\operatorname{Ai}'\left(\frac{z^2}{3^{4/3}}\right)\right], \tag{2.179}$$

where $\operatorname{Ai}(z)$ is the Airy function, the prime denotes the derivative.

The Mainardi and Wright functions appear in the formulae for the inverse Laplace transform [100, 101]

$$\mathcal{L}^{-1}\left\{\exp\left(-\lambda s^{\alpha}\right)\right\} = \frac{\alpha\lambda}{t^{\alpha+1}} M\left(\alpha; \lambda t^{-\alpha}\right), \quad 0 < \alpha < 1, \quad \lambda > 0, \tag{2.180}$$

$$\mathcal{L}^{-1}\left\{s^{\alpha-1}\exp\left(-\lambda s^{\alpha}\right)\right\} = \frac{1}{t^{\alpha}} M\left(\alpha; \lambda t^{-\alpha}\right), \quad 0 < \alpha < 1, \quad \lambda > 0, \tag{2.181}$$

and [214] (see also [47, 100, 101, 117])

$$\mathcal{L}^{-1}\left\{s^{-\beta}\exp\left(-\lambda s^{\alpha}\right)\right\} = t^{\beta-1} W\left(-\alpha, \beta; -\lambda t^{-\alpha}\right), \quad 0 < \alpha < 1, \quad \lambda > 0. \tag{2.182}$$

The Laplace transform of the Wright function is expressed in terms of the Mittag-Leffler function [43, 77, 143]

$$\mathcal{L}\left\{W(\alpha, \beta; t)\right\} = \frac{1}{s}\, E_{\alpha,\beta}\left(\frac{1}{s}\right). \tag{2.183}$$

Integration of (2.174) gives

$$\int\limits_0^{\infty} W(\alpha, \beta; -x)\, \mathrm{d}x = \frac{1}{\Gamma(\beta - \alpha)}, \tag{2.184}$$

in particular

$$\int\limits_0^{\infty} M(\alpha; x)\, \mathrm{d}x = 1. \tag{2.185}$$

The Mittag-Leffler function and the Mainardi function are related by the cos-Fourier transform:

$$M\left(\frac{\alpha}{2}; x\right) = \frac{2}{\pi} \int\limits_0^{\infty} E_{\alpha}(-\xi^2)\, \cos(x\xi)\, \mathrm{d}\xi, \qquad 0 < \alpha < 2. \tag{2.186}$$

Similar relations are valid for the following Wright functions:

$$W\left(-\frac{\alpha}{2}, 2 - \frac{\alpha}{2}; -x\right) = \frac{2}{\pi} \int\limits_0^{\infty} E_{\alpha,2}(-\xi^2)\, \cos(x\xi)\, \mathrm{d}\xi, \quad 0 < \alpha < 2, \tag{2.187}$$

$$W\left(-\frac{\alpha}{2}, \frac{\alpha}{2}; -x\right) = \frac{2}{\pi} \int\limits_0^{\infty} E_{\alpha,\alpha}(-\xi^2)\, \cos(x\xi)\, \mathrm{d}\xi, \quad 0 < \alpha < 2. \tag{2.188}$$

The relation above are proved in Chapter 4 (see (4.11) and (4.14), (4.27) and (4.28), (4.34) and (4.35)).