Chapter 12

Equations with Three Space Variables in Spherical Coordinates

I have answered three questions, and that is enough.

Lewis Carroll
"Alice's Adventures in Wonderland"

12.1 Domain $0 \leq r < \infty$, $-1 \leq \mu \leq 1$, $0 \leq \varphi \leq 2\pi$

Consider the time-fractional diffusion-wave equation with a source term in spherical coordinates r, θ, and φ:

$$\frac{\partial^\alpha T}{\partial t^\alpha} = a \left[\frac{\partial^2 T}{\partial r^2} + \frac{2}{r} \frac{\partial T}{\partial r} + \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial \theta} \left(\sin\theta \frac{\partial T}{\partial \theta} \right) \right.$$
$$\left. + \frac{1}{r^2 \sin^2\theta} \frac{\partial^2 T}{\partial \varphi^2} \right] + \Phi(r, \theta, \varphi, t), \tag{12.1}$$
$$0 \leq r < \infty, \quad 0 \leq \theta \leq \pi, \quad 0 \leq \varphi \leq 2\pi.$$

Change of variable $\mu = \cos\theta$ leads to the equation

$$\frac{\partial^\alpha T}{\partial t^\alpha} = a \left\{ \frac{\partial^2 T}{\partial r^2} + \frac{2}{r} \frac{\partial T}{\partial r} + \frac{1}{r^2} \frac{\partial}{\partial \mu} \left[\left(1 - \mu^2\right) \frac{\partial T}{\partial \mu} \right] \right.$$
$$\left. + \frac{1}{r^2 \left(1 - \mu^2\right)} \frac{\partial^2 T}{\partial \varphi^2} \right\} + \Phi(r, \mu, \varphi, t),$$
$$0 \leq r < \infty, \quad -1 \leq \mu \leq 1, \quad 0 \leq \varphi \leq 2\pi. \tag{12.2}$$

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In the subsequent text we will consider immediately Eq. (12.2). For this equation the initial conditions are prescribed:

$$t = 0: \quad T = f(r, \mu, \varphi), \qquad 0 < \alpha \le 2, \tag{12.3}$$

$$t = 0: \quad \frac{\partial T}{\partial t} = F(r, \mu, \varphi), \quad 1 < \alpha \le 2. \tag{12.4}$$

As usually, the zero condition at infinity is assumed:

$$\lim_{r \to \infty} T(r, \mu, \varphi, t) = 0. \tag{12.5}$$

The solution to the initial-value problem (12.2)–(12.5) is written as:

$$\begin{aligned} T(r, \mu, \varphi, t) &= \int_0^{2\pi} \int_{-1}^{1} \int_0^{\infty} f(\rho, \zeta, \phi)\, \mathcal{G}_f(r, \mu, \varphi, \rho, \zeta, \phi, t)\, \rho^2 \,\mathrm{d}\rho \,\mathrm{d}\zeta \,\mathrm{d}\phi \\ &+ \int_0^{2\pi} \int_{-1}^{1} \int_0^{\infty} F(\rho, \zeta, \phi)\, \mathcal{G}_F(r, \mu, \varphi, \rho, \zeta, \phi, t)\, \rho^2 \,\mathrm{d}\rho \,\mathrm{d}\zeta \,\mathrm{d}\phi. \\ &+ \int_0^{t} \int_0^{2\pi} \int_{-1}^{1} \int_0^{\infty} \Phi(\rho, \zeta, \phi, \tau)\, \mathcal{G}_\Phi(r, \mu, \varphi, \rho, \zeta, \phi, t-\tau)\, \rho^2 \,\mathrm{d}\rho \,\mathrm{d}\zeta \,\mathrm{d}\phi \,\mathrm{d}\tau. \end{aligned} \tag{12.6}$$

The fundamental solution $\mathcal{G}_f(r, \mu, \varphi, \rho, \zeta, \phi, t)$ is the solution to the following problem:

$$\frac{\partial^\alpha \mathcal{G}_f}{\partial t^\alpha} = a \left\{ \frac{\partial^2 \mathcal{G}_f}{\partial r^2} + \frac{2}{r} \frac{\partial \mathcal{G}_f}{\partial r} + \frac{1}{r^2} \frac{\partial}{\partial \mu} \left[\left(1 - \mu^2\right) \frac{\partial \mathcal{G}_f}{\partial \mu} \right] + \frac{1}{r^2 \left(1 - \mu^2\right)} \frac{\partial^2 \mathcal{G}_f}{\partial \varphi^2} \right\}, \tag{12.7}$$

$$t = 0: \quad \mathcal{G}_f = p_0 \frac{\delta(r - \rho)}{r^2} \,\delta(\mu - \zeta)\, \delta(\varphi - \phi), \quad 0 < \alpha \le 2, \tag{12.8}$$

$$t = 0: \quad \frac{\partial \mathcal{G}_f}{\partial t} = 0, \qquad 1 < \alpha \le 2. \tag{12.9}$$

Now we introduce the new looked-for function $v = \sqrt{r}\, \mathcal{G}_f$. In terms of this function, the initial value problem (12.7)–(12.9) is rewritten as

$$\frac{\partial^\alpha v}{\partial t^\alpha} = a \left\{ \frac{\partial^2 v}{\partial r^2} + \frac{1}{r} \frac{\partial v}{\partial r} - \frac{1}{4r^2} v + \frac{1}{r^2} \frac{\partial}{\partial \mu} \left[\left(1 - \mu^2\right) \frac{\partial v}{\partial \mu} \right] + \frac{1}{r^2 \left(1 - \mu^2\right)} \frac{\partial^2 v}{\partial \varphi^2} \right\}, \tag{12.10}$$

$$t = 0: \quad v = \frac{p_0}{r^{3/2}} \,\delta(r - \rho)\, \delta(\mu - \zeta)\, \delta(\varphi - \phi), \quad 0 < \alpha \le 2, \tag{12.11}$$

$$t = 0: \quad \frac{\partial v}{\partial t} = 0, \qquad 1 < \alpha \le 2. \tag{12.12}$$

Next, we use the Laplace transform (2.1) with respect to time t, the finite Fourier transform for 2π-periodic functions (2.72) with respect to the angular coordinate φ, the Legendre transform (2.75) with respect to the coordinate μ, and the Hankel transform (2.78) of the order $n+1/2$ with respect to the radial coordinate r. It should be emphasized that the order of integral transforms is important. In the transforms domain we get

$$\widehat{\widetilde{v}}^{**}(\xi, m, n, \rho, \phi, s) = \frac{p_0}{\sqrt{\rho}}\, J_{n+1/2}(\rho\xi) P_n^m(\zeta)\cos[m(\varphi-\phi)]\,\frac{s^{\alpha-1}}{s^\alpha + a\xi^2}. \tag{12.13}$$

Inversion of all the integral transforms and bringing back to the fundamental solution $\mathcal{G}_f = v/\sqrt{r}$ gives [169]:

$$\mathcal{G}_f(r,\mu,\varphi,\rho,\zeta,\phi,t) = \frac{p_0}{\pi\sqrt{r\rho}} \sum_{n=0}^{\infty} {\sum_{m=0}^{n}}' \frac{2n+1}{2}\,\frac{(n-m)!}{(n+m)!}\, P_n^m(\mu)\, P_n^m(\zeta)$$
$$\times \cos[m(\varphi-\phi)] \int_0^\infty E_\alpha\left(-a\xi^2 t^\alpha\right)\, J_{n+1/2}(r\xi)\, J_{n+1/2}(\rho\xi)\,\xi\, d\xi, \tag{12.14}$$

where the prime near the summation symbol denotes that the term corresponding to $m=0$ in the sum should be multiplied by the factor $1/2$.

Dependence of the fundamental solution (12.14) on the radial coordinate r is presented in Fig. 12.1. The following nondimensional quantities have been

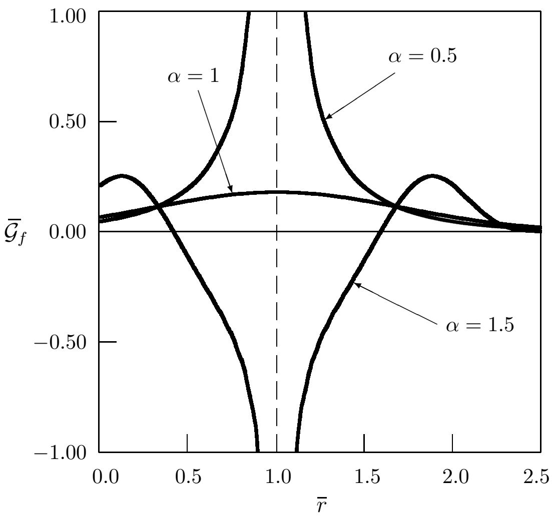

Figure 12.1: Dependence of the fundamental solution to the first Cauchy problem in an infinite medium in spherical coordinates on the radial coordinate r ($\kappa = 0.5$, $\varphi = 0$, $\phi = 0$, $\zeta = 0$, $\mu = 0$)

introduced:

$$\overline{\mathcal{G}}_f = \frac{\rho^3}{p_0}\,\mathcal{G}_f, \qquad \bar{r} = \frac{r}{\rho}, \qquad \kappa = \frac{\sqrt{a}\,t^{\alpha/2}}{\rho}. \tag{12.15}$$

The fundamental solutions to the second Cauchy problem and to the source problem are obtained in a similar way and are expressed as [169]

$$\begin{pmatrix} \mathcal{G}_F(r,\mu,\varphi,\rho,\zeta,\phi,t) \\ \mathcal{G}_\Phi(r,\mu,\varphi,\rho,\zeta,\phi,t) \end{pmatrix} = \frac{1}{\pi\sqrt{r\rho}} \sum_{n=0}^{\infty} {\sum_{m=0}^{n}}' \frac{2n+1}{2}\,\frac{(n-m)!}{(n+m)!}$$
$$\times P_n^m(\mu)\,P_n^m(\zeta)\,\cos[m(\varphi-\phi)]$$
$$\times \int_0^\infty \begin{pmatrix} w_0 t\, E_{\alpha,2}(-a\xi^2 t^\alpha) \\ q_0 t^{\alpha-1} E_{\alpha,\alpha}(-a\xi^2 t^\alpha) \end{pmatrix} J_{n+1/2}(r\xi)\, J_{n+1/2}(\rho\xi)\,\xi\,\mathrm{d}\xi. \tag{12.16}$$

The nondimensional fundamental solutions

$$\overline{\mathcal{G}}_F = \frac{\rho^3}{w_0 t}\,\mathcal{G}_F \quad \text{and} \quad \overline{\mathcal{G}}_\Phi = \frac{\rho^3}{q_0 t^{\alpha-1}}\,\mathcal{G}_\Phi \tag{12.17}$$

are shown in Figs. 12.2–12.4 and Figs. 12.5–12.7, respectively.

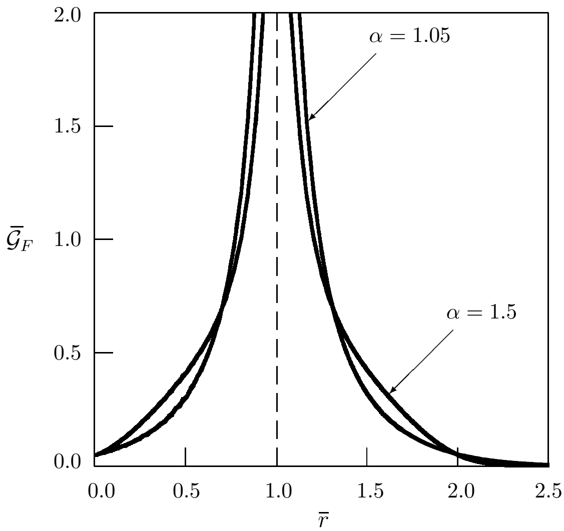

Figure 12.2: Dependence of the fundamental solution to the second Cauchy problem in an infinite medium in spherical coordinates on the radial coordinate r ($\kappa = 0.5$, $\varphi = 0$, $\phi = 0$, $\zeta = 0$, $\mu = 0$)

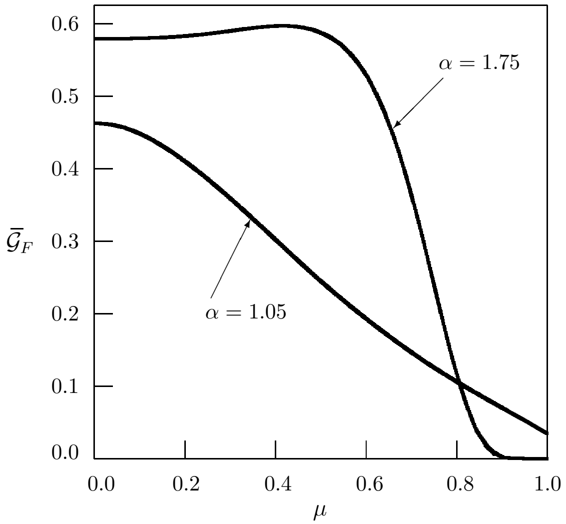

Figure 12.3: Dependence of the fundamental solution to the second Cauchy problem in an infinite medium in spherical coordinates on the coordinate μ ($\kappa = 0.5$, $\varphi = 0$, $\phi = 0$, $\zeta = 0$, $\bar{r} = 0.6$)

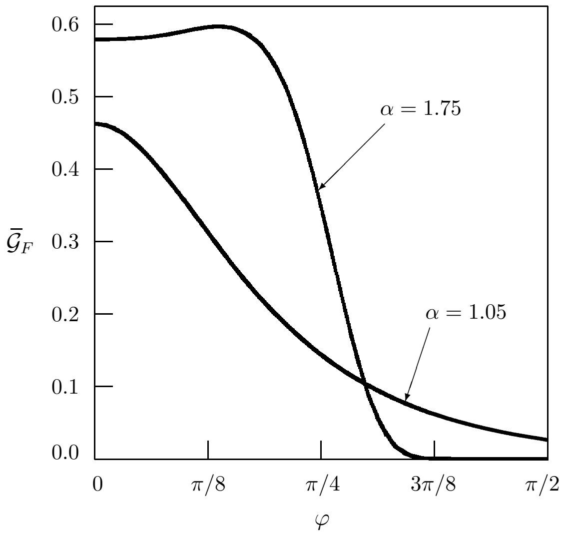

Figure 12.4: Dependence of the fundamental solution to the second Cauchy problem in an infinite medium in spherical coordinates on the angular coordinate φ ($\kappa = 0.5$, $\mu = 0$, $\phi = 0$, $\zeta = 0$, $\bar{r} = 0.6$)

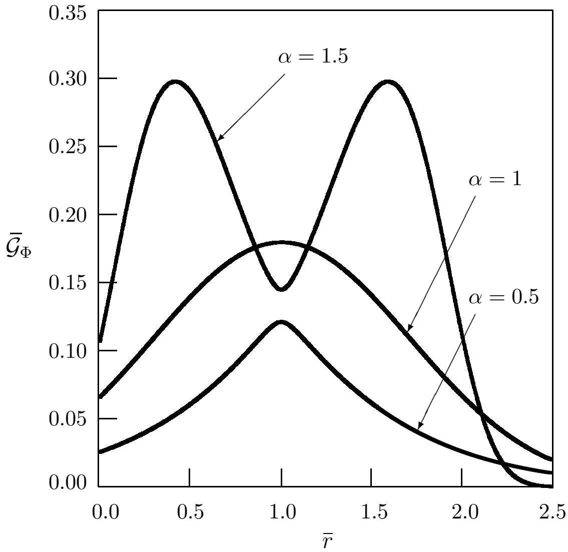

Figure 12.5: Dependence of the fundamental solution to the source problem in an infinite medium in spherical coordinates on the radial coordinate r ($\kappa = 0.5$, $\mu = 0$, $\phi = 0$, $\zeta = 0$, $\bar{r} = 0.6$)

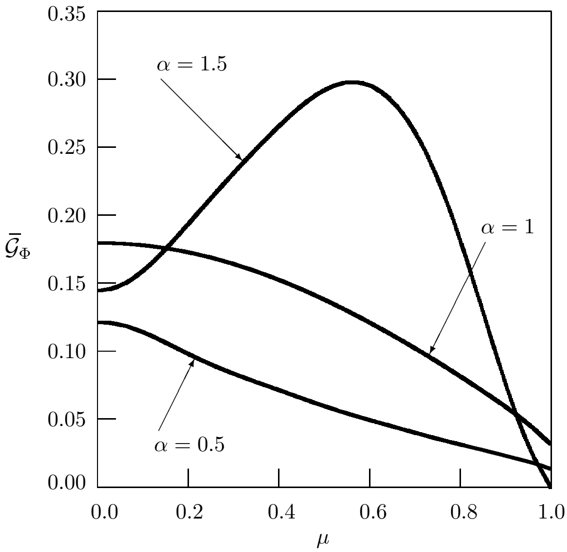

Figure 12.6: Dependence of the fundamental solution to the source problem in an infinite medium in spherical coordinates on the coordinate μ ($\kappa = 0.5$, $\mu = 0$, $\phi = 0$, $\zeta = 0$, $\bar{r} = 0.6$)

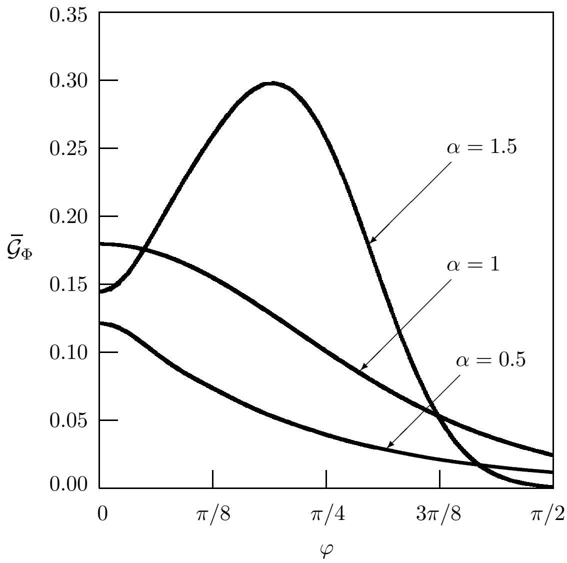

Figure 12.7: Dependence of the fundamental solution to the source problem in an infinite medium in spherical coordinates on the angular coordinate φ ($\kappa = 0.5$, $\mu = 0$, $\phi = 0$, $\zeta = 0$, $\bar{r} = 0.6$)

12.2 Domain $0 \le r < R,\ -1 \le \mu \le 1,\ 0 \le \varphi \le 2\pi$

12.2.1 Dirichlet boundary condition

$$\frac{\partial^\alpha T}{\partial t^\alpha} = a \left\{ \frac{\partial^2 T}{\partial r^2} + \frac{2}{r}\,\frac{\partial T}{\partial r} + \frac{1}{r^2}\,\frac{\partial}{\partial \mu}\left[\left(1-\mu^2\right)\,\frac{\partial T}{\partial \mu}\right] \right.$$
$$\left. + \ \frac{1}{r^2\,(1-\mu^2)}\,\frac{\partial^2 T}{\partial \varphi^2} \right\} + \Phi(r,\mu,\varphi,t), \tag{12.18}$$

$$t = 0: \quad T = f(r,\mu,\varphi), \qquad 0 < \alpha \le 2, \tag{12.19}$$

$$t = 0: \quad \frac{\partial T}{\partial t} = F(r,\mu,\varphi), \quad 1 < \alpha \le 2, \tag{12.20}$$

$$r = R: \quad T = g(\mu,\varphi,t). \tag{12.21}$$

The solution:

$$T(r,\mu,\varphi,t) = \int_0^{2\pi}\int_{-1}^{1}\int_0^{R} f(\rho,\zeta,\phi)\,\mathcal{G}_f(r,\mu,\varphi,\rho,\zeta,\phi,t)\,\rho^2\,\mathrm{d}\rho\,\mathrm{d}\zeta\,\mathrm{d}\phi$$

$$+ \int_0^{2\pi} \int_{-1}^{1} \int_0^{R} F(\rho, \zeta, \phi)\, \mathcal{G}_F(r, \mu, \varphi, \rho, \zeta, \phi, t)\, \rho^2 \, d\rho\, d\zeta\, d\phi$$

$$+ \int_0^{t} \int_0^{2\pi} \int_{-1}^{1} \int_0^{R} \Phi(\rho, \zeta, \phi, \tau)\, \mathcal{G}_\Phi(r, \mu, \varphi, \rho, \zeta, \phi, t-\tau)\, \rho^2 \, d\rho\, d\zeta\, d\phi\, d\tau$$

$$+ \int_0^{t} \int_0^{2\pi} \int_{-1}^{1} g(\zeta, \phi, \tau)\, \mathcal{G}_g(r, \mu, \varphi, \zeta, \phi, t-\tau)\, d\zeta\, d\phi\, d\tau, \tag{12.22}$$

with the fundamental solutions obtained by the Laplace transform to with respect time, the finite Fourier transform for 2π-periodic functions (2.72) with respect to the angular coordinate φ, the Legendre transform (2.75) with respect to the coordinate μ and the finite Hankel transform (2.84) with respect to the radial coordinate r:

$$\begin{pmatrix} \mathcal{G}_f(r, \mu, \varphi, \rho, \zeta, \phi, t) \\ \mathcal{G}_F(r, \mu, \varphi, \rho, \zeta, \phi, t) \\ \mathcal{G}_\Phi(r, \mu, \varphi, \rho, \zeta, \phi, t) \end{pmatrix} = \frac{1}{\pi\sqrt{r\rho}R^2} \sum_{k=1}^{\infty} \sum_{n=0}^{\infty} {\sum_{m=0}^{n}}' (2n+1)\, \frac{(n-m)!}{(n+m)!}$$

$$\times P_n^m(\mu)\, P_n^m(\zeta)\, \cos[m(\varphi-\phi)]$$

$$\times \begin{pmatrix} p_0\, E_\alpha(-a\xi_{nk}^2 t^\alpha) \\ w_0 t\, E_{\alpha,2}(-a\xi_{nk}^2 t^\alpha) \\ q_0 t^{\alpha-1} E_{\alpha,\alpha}(-a\xi_{nk}^2 t^\alpha) \end{pmatrix} \frac{J_{n+1/2}(r\xi_{nk})\, J_{n+1/2}(\rho\xi_{nk})}{\left[J'_{n+1/2}(R\xi_{nk})\right]^2}, \tag{12.23}$$

where ξ_{nk} are the positive roots of the equation $J_{n+1/2}(R\xi_{nk}) = 0$.

The fundamental solution to the Dirichlet problem has the following form [176]:

$$\mathcal{G}_g(r, \mu, \varphi, \zeta, \phi, t) = -\frac{a g_0 t^{\alpha-1}}{\pi\sqrt{rR}} \sum_{n=0}^{\infty} \sum_{k=1}^{\infty} {\sum_{m=0}^{n}}' (2n+1)\, \frac{(n-m)!}{(n+m)!}\, P_n^m(\mu)\, P_n^m(\zeta)$$

$$\times \cos[m(\varphi-\phi)]\, \frac{J_{n+1/2}(r\xi_{nk})}{J'_{n+1/2}(R\xi_{nk})}\, \xi_{nk}\, E_{\alpha,\alpha}\left(-a\xi_{nk}^2 t^\alpha\right). \tag{12.24}$$

Two known particular cases can be obtained from Eq. (12.24).

Classical diffusion equation ($\alpha = 1$)

$$\mathcal{G}_g(r,\mu,\varphi,\zeta,\phi,t) = -\frac{a g_0}{\pi\sqrt{rR}} \sum_{n=0}^{\infty}\sum_{k=1}^{\infty}{\sum_{m=0}^{n}}{}' (2n+1)\,\frac{(n-m)!}{(n+m)!}\,P_n^m(\mu)\,P_n^m(\zeta)$$

$$\times \cos[m(\varphi-\phi)]\,\frac{J_{n+1/2}(r\xi_{nk})}{J'_{n+1/2}(R\xi_{nk})}\,\xi_{nk}\,\exp\left(-a\xi_{nk}^2 t\right). \tag{12.25}$$

This solution is presented in [20, 26].

Wave equation ($\alpha = 2$)

$$\mathcal{G}_g(r,\mu,\varphi,\zeta,\phi,t) = -\frac{\sqrt{a} g_0}{\pi\sqrt{rR}} \sum_{n=0}^{\infty}\sum_{k=1}^{\infty}{\sum_{m=0}^{n}}{}' (2n+1)\,\frac{(n-m)!}{(n+m)!}\,P_n^m(\mu)\,P_n^m(\zeta)$$

$$\times \cos[m(\varphi-\phi)]\,\frac{J_{n+1/2}(r\xi_{nk})}{J'_{n+1/2}(R\xi_{nk})}\,\sin\left(\sqrt{a}\xi_{nk} t\right). \tag{12.26}$$

Dependence of nondimensional fundamental solution $\overline{\mathcal{G}}_g$ on the coordinates $\bar{r}$, μ and φ is displayed in Figs. 12.8–12.10. Here

$$\overline{\mathcal{G}}_g = \frac{R^2}{a g_0 t^{\alpha-1}}\,\mathcal{G}_g, \qquad \bar{r} = \frac{r}{R}, \qquad \kappa = \frac{\sqrt{a} t^{\alpha/2}}{R}. \tag{12.27}$$

12.2.2 Neumann boundary condition

$$\frac{\partial^\alpha T}{\partial t^\alpha} = a\left\{\frac{\partial^2 T}{\partial r^2} + \frac{2}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial}{\partial \mu}\left[\left(1-\mu^2\right)\frac{\partial T}{\partial \mu}\right] + \frac{1}{r^2\left(1-\mu^2\right)}\frac{\partial^2 T}{\partial \varphi^2}\right\} + \Phi(r,\mu,\varphi,t), \tag{12.28}$$

$$t = 0: \quad T = f(r,\mu,\varphi), \qquad 0 < \alpha \le 2, \tag{12.29}$$

$$t = 0: \quad \frac{\partial T}{\partial t} = F(r,\mu,\varphi), \quad 1 < \alpha \le 2, \tag{12.30}$$

$$r = R: \quad \frac{\partial T}{\partial r} = g(\mu,\varphi,t). \tag{12.31}$$

The solution:

$$T(r,\mu,\varphi,t) = \int_0^{2\pi}\int_{-1}^{1}\int_0^{R} f(\rho,\zeta,\phi)\,\mathcal{G}_f(r,\mu,\varphi,\rho,\zeta,\phi,t)\,\rho^2\,\mathrm{d}\rho\,\mathrm{d}\zeta\,\mathrm{d}\phi$$

$$+ \int_0^{2\pi}\int_{-1}^{1}\int_0^{R} F(\rho,\zeta,\phi)\,\mathcal{G}_F(r,\mu,\varphi,\rho,\zeta,\phi,t)\,\rho^2\,\mathrm{d}\rho\,\mathrm{d}\zeta\,\mathrm{d}\phi$$

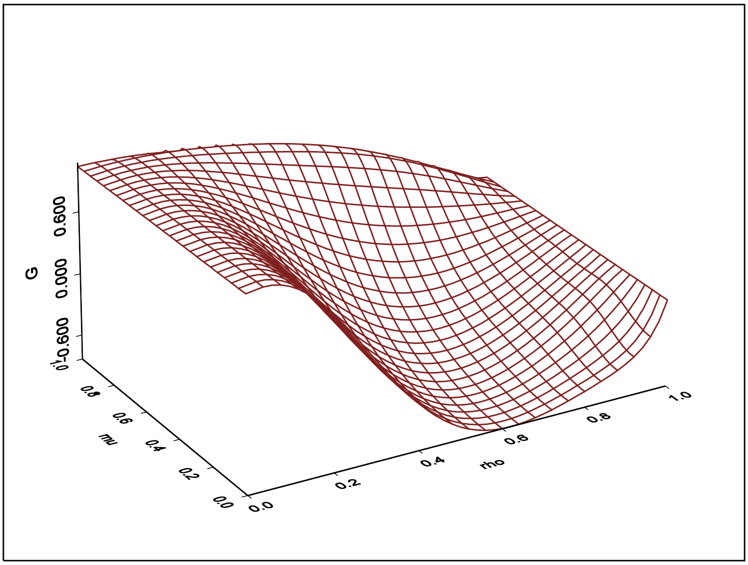

Figure 12.8: Dependence of the fundamental solution to the Dirichlet problem for a sphere $\mathcal{G}_g(r, \mu, \varphi, \zeta, \phi, t)$ on coordinates r and μ for $\zeta = 0$, $\phi = 0$, $\varphi = 0$, $\kappa = 0.5$ [176]

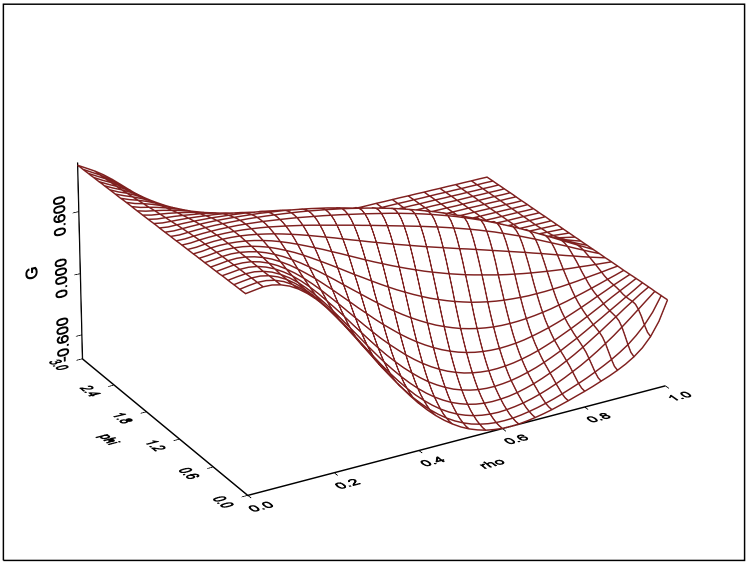

Figure 12.9: Dependence of the fundamental solution to the Dirichlet problem for a sphere $\mathcal{G}_g(r, \mu, \varphi, \zeta, \phi, t)$ on coordinates r and φ for $\zeta = 0$, $\phi = 0$, $\mu = 0$, $\kappa = 0.5$ [176]

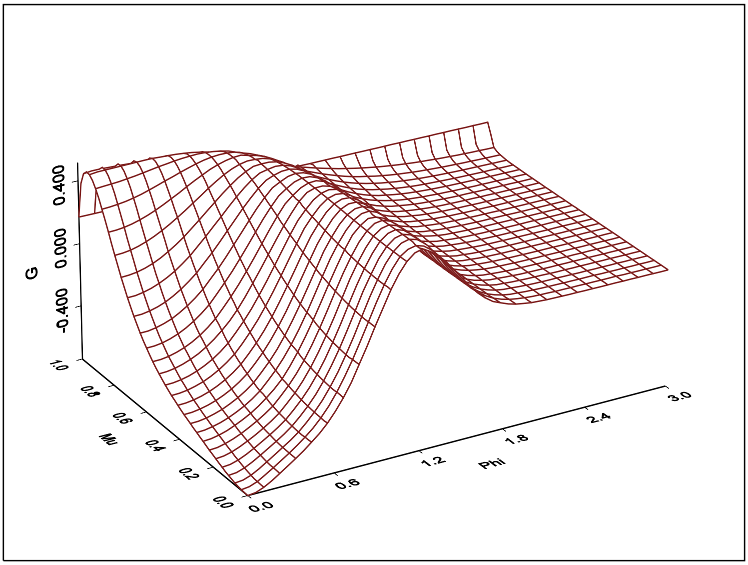

Figure 12.10: Dependence of the fundamental solution to the Dirichlet problem for a sphere $\mathcal{G}_g(r, \mu, \varphi, \zeta, \phi, t)$ on coordinates μ and φ for $\zeta = 0$, $\phi = 0$, $\bar{r} = 0.75$, $\kappa = 0.5$ [176]

$$+ \int_0^t \int_0^{2\pi} \int_{-1}^{1} \int_0^R \Phi(\rho, \zeta, \phi, \tau)\, \mathcal{G}_\Phi(r, \mu, \varphi, \rho, \zeta, \phi, t-\tau)\, \rho^2 \,\mathrm{d}\rho \,\mathrm{d}\zeta \,\mathrm{d}\phi \,\mathrm{d}\tau$$

$$+ \int_0^t \int_0^{2\pi} \int_{-1}^{1} g(\zeta, \phi, \tau)\, \mathcal{G}_g(r, \mu, \varphi, \zeta, \phi, t-\tau) \,\mathrm{d}\zeta \,\mathrm{d}\phi \,\mathrm{d}\tau. \qquad (12.32)$$

The fundamental solutions are obtained using the Laplace transform with respect to time t, the finite Fourier transform (2.72) for 2π-periodic functions with respect to the angular coordinate φ, the Legendre transform (2.75) with respect to the coordinate μ and the finite Hankel transform (2.88) of the order $n + 1/2$ with respect to the radial coordinate r:

$$\begin{pmatrix} \mathcal{G}_f(r, \mu, \varphi, \rho, \zeta, \phi, t) \\ \mathcal{G}_F(r, \mu, \varphi, \rho, \zeta, \phi, t) \\ \mathcal{G}_\Phi(r, \mu, \varphi, \rho, \zeta, \phi, t) \end{pmatrix} = \frac{1}{\pi\sqrt{r\rho}} \sum_{k=1}^{\infty} \sum_{n=0}^{\infty} {\sum_{m=0}^{n}}' (2n+1) \frac{(n-m)!}{(n+m)!}$$

$$\times P_n^m(\mu)\, P_n^m(\zeta)\, \cos[m(\varphi - \phi)]$$

$$\times \begin{pmatrix} p_0 \, E_\alpha(-a\xi_{nk}^2 t^\alpha) \\ w_0 t \, E_{\alpha,2}(-a\xi_{nk}^2 t^\alpha) \\ q_0 t^{\alpha-1} \, E_{\alpha,\alpha}(-a\xi_{nk}^2 t^\alpha) \end{pmatrix} \frac{\xi_{nk} \, J_{n+1/2}(r\xi_{nk}) \, J_{n+1/2}(\rho\xi_{nk})}{[R^2\xi_{nk}^2 - (n+1/2)^2] \, J_{n+1/2}^2(R\xi_{nk})}, \tag{12.33}$$

where ξ_{nk} are the positive roots of the equation $J'_{n+1/2}(R\xi_k) = 0$.

The fundamental solution to the mathematical and physical Neumann problems can be calculated as

$$\mathcal{G}_m(r,\mu,\varphi,\zeta,\phi,t) = \frac{ag_0R}{q_0} \, \mathcal{G}_\Phi(r,\mu,\varphi,\rho,\zeta,\phi,t)\Big|_{\rho=R}, \tag{12.34}$$

$$\mathcal{G}_p(r,\mu,\varphi,\zeta,\phi,t) = \frac{ag_0R}{p_0} \, \mathcal{G}_f(r,\mu,\varphi,\rho,\zeta,\phi,t)\Big|_{\rho=R}. \tag{12.35}$$

12.3 Domain $R \le r < \infty$, $-1 \le \mu \le 1$, $0 \le \varphi \le 2\pi$

12.3.1 Dirichlet boundary condition

$$\frac{\partial^\alpha T}{\partial t^\alpha} = a \left\{ \frac{\partial^2 T}{\partial r^2} + \frac{2}{r} \frac{\partial T}{\partial r} + \frac{1}{r^2} \frac{\partial}{\partial \mu} \left[\left(1-\mu^2\right) \frac{\partial T}{\partial \mu} \right] \right.$$

$$\left. + \frac{1}{r^2\left(1-\mu^2\right)} \frac{\partial^2 T}{\partial \varphi^2} \right\} + \Phi(r,\mu,\varphi,t), \tag{12.36}$$

$$t = 0: \quad T = f(r,\mu,\varphi), \qquad 0 < \alpha \le 2, \tag{12.37}$$

$$t = 0: \quad \frac{\partial T}{\partial t} = F(r,\mu,\varphi), \quad 1 < \alpha \le 2, \tag{12.38}$$

$$r = R: \quad T = g(\mu,\varphi,t). \tag{12.39}$$

The solution:

$$T(r,\mu,\varphi,t) = \int_0^{2\pi} \int_{-1}^{1} \int_R^{\infty} f(\rho,\zeta,\phi) \, \mathcal{G}_f(r,\mu,\varphi,\rho,\zeta,\phi,t) \, \rho^2 \, \mathrm{d}\rho \, \mathrm{d}\zeta \, \mathrm{d}\phi$$

$$+ \int_0^{2\pi} \int_{-1}^{1} \int_R^{\infty} F(\rho,\zeta,\phi) \, \mathcal{G}_F(r,\mu,\varphi,\rho,\zeta,\phi,t) \, \rho^2 \, \mathrm{d}\rho \, \mathrm{d}\zeta \, \mathrm{d}\phi$$

$$+ \int_0^{t} \int_0^{2\pi} \int_{-1}^{1} \int_R^{\infty} \Phi(\rho,\zeta,\phi,\tau) \, \mathcal{G}_\Phi(r,\mu,\varphi,\rho,\zeta,\phi,t-\tau) \, \rho^2 \, \mathrm{d}\rho \, \mathrm{d}\zeta \, \mathrm{d}\phi \, \mathrm{d}\tau$$

$$+ \int_0^t \int_0^{2\pi} \int_{-1}^1 g(\zeta, \phi, \tau)\, \mathcal{G}_g(r, \mu, \varphi, \zeta, \phi, t-\tau)\, \mathrm{d}\zeta\, \mathrm{d}\phi\, \mathrm{d}\tau \tag{12.40}$$

with the fundamental solutions [170]

$$\begin{pmatrix} \mathcal{G}_f(r, \mu, \varphi, \rho, \zeta, \phi, t) \\ \mathcal{G}_F(r, \mu, \varphi, \rho, \zeta, \phi, t) \\ \mathcal{G}_\Phi(r, \mu, \varphi, \rho, \zeta, \phi, t) \end{pmatrix} = \frac{1}{\pi\sqrt{r\rho}} \sum_{n=0}^{\infty} {\sum_{m=0}^{n}}' \frac{2n+1}{2} \frac{(n-m)!}{(n+m)!}$$

$$\times P_n^m(\mu)\, P_n^m(\zeta)\, \cos[m(\varphi-\phi)] \int_0^\infty \begin{pmatrix} p_0\, E_\alpha(-a\xi^2 t^\alpha) \\ w_0 t\, E_{\alpha,2}(-a\xi^2 t^\alpha) \\ q_0 t^{\alpha-1} E_{\alpha,\alpha}(-a\xi t^\alpha) \end{pmatrix}$$

$$\times \frac{J_{n+1/2}(r\xi)\, Y_{n+1/2}(R\xi) - Y_{n+1/2}(r\xi) J_{n+1/2}(R\xi)}{J^2_{n+1/2}(R\xi) + Y^2_{n+1/2}(R\xi)}$$

$$\times \Big[J_{n+1/2}(\rho\xi)\, Y_{n+1/2}(R\xi) - Y_{n+1/2}(\rho\xi) J_{n+1/2}(R\xi) \Big]\, \xi\, \mathrm{d}\xi \tag{12.41}$$

obtained using the Laplace transform with respect to time t, the finite Fourier transform (2.72) for 2π-periodic functions with respect to the angular coordinate φ, the Legendre transform (2.75) with respect to the coordinate μ and the Weber transform (2.108), (2.111) of the order $n + 1/2$ with respect to the radial coordinate r.

The fundamental solution to the Dirichlet problem has the form

$$\mathcal{G}_g(r, \mu, \varphi, \zeta, \phi, t) = -\frac{a g_0 t^{\alpha-1}\sqrt{R}}{\pi^2\sqrt{r}} \sum_{n=0}^{\infty} {\sum_{m=0}^{n}}' (2n+1) \frac{(n-m)!}{(n+m)!}$$

$$\times P_n^m(\mu)\, P_n^m(\zeta)\, \cos[m(\varphi-\phi)] \int_0^\infty E_{\alpha,\alpha}(-a\xi^2 t^\alpha)$$

$$\times \frac{J_{n+1/2}(r\xi)\, Y_{n+1/2}(R\xi) - Y_{n+1/2}(r\xi) J_{n+1/2}(R\xi)}{J^2_{n+1/2}(R\xi) + Y^2_{n+1/2}(R\xi)}\, \xi\, \mathrm{d}\xi. \tag{12.42}$$

Figure 12.11 presents the dependence of the fundamental solution to the second Cauchy problem on the radial coordinate with $\bar{\mathcal{G}}_F = R^3 \mathcal{G}_F/(t w_0)$. The fundamental solution to the source problem $\bar{\mathcal{G}}_\Phi = R^3 \mathcal{G}_\Phi/(q_0 t^{\alpha-1})$ is depicted in Figs. 12.12–12.13. The fundamental solution to the Dirichlet problem is presented in Fig. 12.14 with $\bar{\mathcal{G}}_g = t\mathcal{G}_g/g_0$.

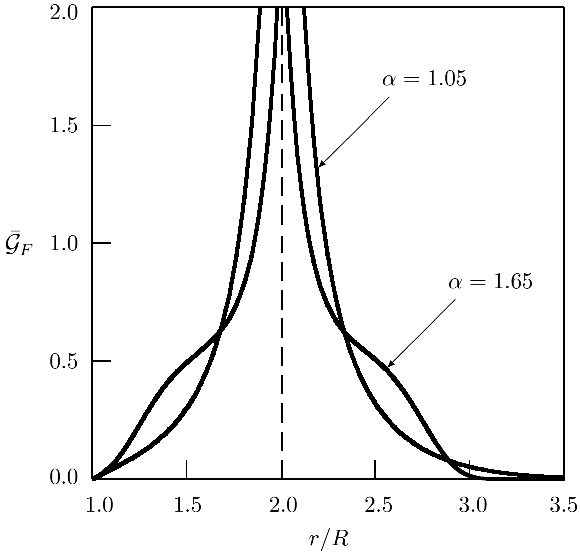

Figure 12.11: Dependence of the fundamental solution to the second Cauchy problem for a solid with a spherical hole $\mathcal{G}_F(r, \mu, \varphi, \rho, \zeta, \phi, t)$ on the radial coordinate r for $\mu = 0$, $\varphi = 0$, $\rho/R = 2$, $\zeta = 0$, $\phi = 0$, and $\kappa = 0.5$ [170]

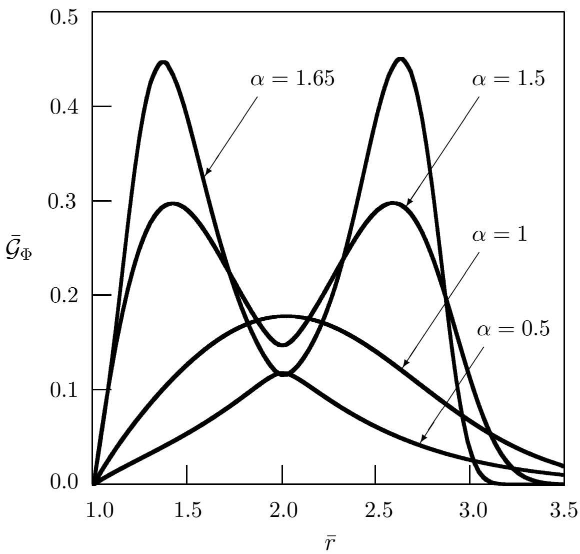

Figure 12.12: Dependence of the fundamental solution to the source problem for a solid with a spherical hole $\mathcal{G}_\Phi(r, \mu, \varphi, \rho, \zeta, \phi, t)$ on the radial coordinate r for $\mu = 0$, $\varphi = 0$, $\rho/R = 2$, $\zeta = 0$, $\phi = 0$, and $\kappa = 0.5$ [170]

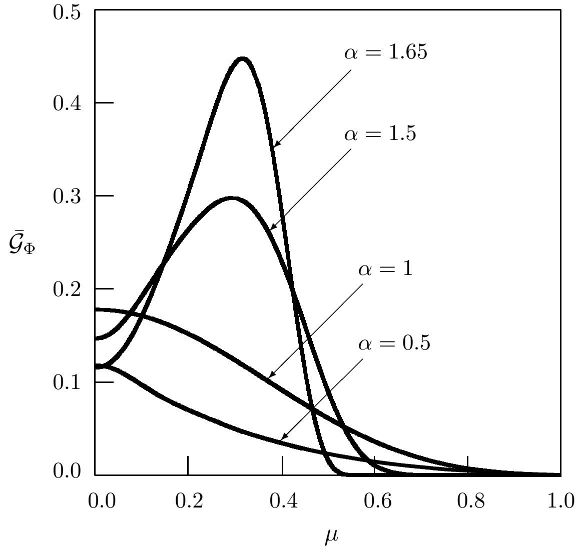

Figure 12.13: Dependence of the fundamental solution to the source problem for a solid with a spherical hole $\mathcal{G}_\Phi(r, \mu, \varphi, \rho, \zeta, \phi, t)$ on the coordinate μ for $r/R = 2$, $\varphi = 0$, $\rho/R = 2$, $\zeta = 0$, $\phi = 0$, and $\kappa = 0.5$ [170]

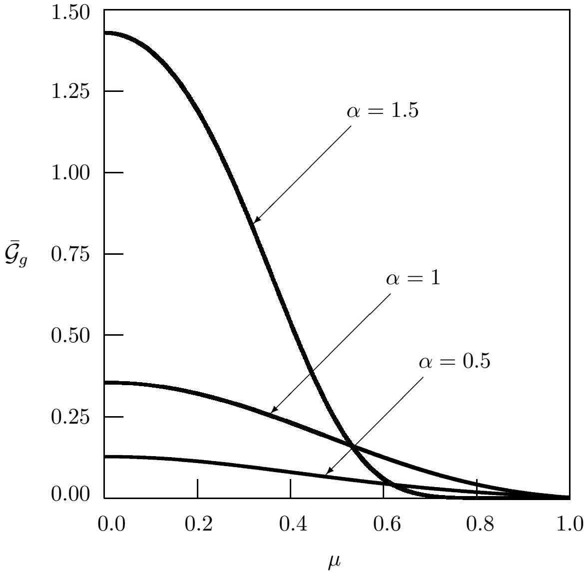

Figure 12.14: Dependence of the fundamental solution to the Dirichlet problem for a solid with a spherical hole $\mathcal{G}_g(r, \mu, \varphi, \zeta, \phi, t)$ on the coordinate μ for $r/R = 2$, $\varphi = 0$, $\zeta = 0$, $\phi = 0$, and $\kappa = 0.5$ [170]

12.3.2 Neumann boundary condition

$$\frac{\partial^\alpha T}{\partial t^\alpha} = a\left\{\frac{\partial^2 T}{\partial r^2} + \frac{2}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial}{\partial \mu}\left[\left(1-\mu^2\right)\frac{\partial T}{\partial \mu}\right] + \frac{1}{r^2\left(1-\mu^2\right)}\frac{\partial^2 T}{\partial \varphi^2}\right\} + \Phi(r,\mu,\varphi,t), \tag{12.43}$$

$$t=0: \quad T = f(r,\mu,\varphi), \qquad 0<\alpha\le 2, \tag{12.44}$$

$$t=0: \quad \frac{\partial T}{\partial t} = F(r,\mu,\varphi), \quad 1<\alpha\le 2, \tag{12.45}$$

$$r=R: \quad -\frac{\partial T}{\partial r} = g(\mu,\varphi,t). \tag{12.46}$$

The solution is obtained using the Laplace transform with respect to time t, the finite Fourier transform (2.72) for 2π-periodic functions with respect to the angular coordinate φ, the Legendre transform (2.75) with respect to the coordinate μ and the Weber transform (2.108), (2.113) of the order $n+1/2$ with respect to the radial coordinate r:

$$\begin{aligned} T(r,\mu,\varphi,t) &= \int_0^{2\pi}\int_{-1}^{1}\int_R^{\infty} f(\rho,\zeta,\phi)\,\mathcal{G}_f(r,\mu,\varphi,\rho,\zeta,\phi,t)\,\rho^2\,\mathrm{d}\rho\,\mathrm{d}\zeta\,\mathrm{d}\phi \\ &+ \int_0^{2\pi}\int_{-1}^{1}\int_R^{\infty} F(\rho,\zeta,\phi)\,\mathcal{G}_F(r,\mu,\varphi,\rho,\zeta,\phi,t)\,\rho^2\,\mathrm{d}\rho\,\mathrm{d}\zeta\,\mathrm{d}\phi \\ &+ \int_0^{t}\int_0^{2\pi}\int_{-1}^{1}\int_R^{\infty} \Phi(\rho,\zeta,\phi,\tau)\,\mathcal{G}_\Phi(r,\mu,\varphi,\rho,\zeta,\phi,t-\tau)\,\rho^2\,\mathrm{d}\rho\,\mathrm{d}\zeta\,\mathrm{d}\phi\,\mathrm{d}\tau \\ &+ \int_0^{t}\int_0^{2\pi}\int_{-1}^{1} g(\zeta,\phi,\tau)\,\mathcal{G}_g(r,\mu,\varphi,\zeta,\phi,t-\tau)\,\mathrm{d}\zeta\,\mathrm{d}\phi\,\mathrm{d}\tau, \end{aligned} \tag{12.47}$$

with the fundamental solutions

$$\begin{pmatrix} \mathcal{G}_f(r,\mu,\varphi,\rho,\zeta,\phi,t) \\ \mathcal{G}_F(r,\mu,\varphi,\rho,\zeta,\phi,t) \\ \mathcal{G}_\Phi(r,\mu,\varphi,\rho,\zeta,\phi,t) \end{pmatrix} = \frac{1}{\pi\sqrt{r\rho}} \sum_{n=0}^{\infty} {\sum_{m=0}^{n}}' \frac{2n+1}{2}\frac{(n-m)!}{(n+m)!}$$

$$\times\, P_n^m(\mu)\,P_n^m(\zeta)\,\cos[m(\varphi-\phi)] \int_0^{\infty} \begin{pmatrix} p_0\,E_\alpha(-a\xi^2 t^\alpha) \\ w_0 t\,E_{\alpha,2}(-a\xi^2 t^\alpha) \\ q_0 t^{\alpha-1}\,E_{\alpha,\alpha}(-a\xi t^\alpha) \end{pmatrix}$$

$$\times \frac{J_{n+1/2}(r\xi)\, Y'_{n+1/2}(R\xi) - Y_{n+1/2}(r\xi) J'_{n+1/2}(R\xi)}{\left[J'_{n+1/2}(R\xi)\right]^2 + \left[Y'_{n+1/2}(R\xi)\right]^2}$$

$$\times \left[J_{n+1/2}(\rho\xi)\, Y'_{n+1/2}(R\xi) - Y_{n+1/2}(\rho\xi) J'_{n+1/2}(R\xi)\right] \xi \, \mathrm{d}\xi. \tag{12.48}$$

The fundamental solutions to the mathematical and physical Neumann problems have the form

$$\begin{pmatrix} \mathcal{G}_m(r,\mu,\varphi,\zeta,\phi,t) \\ \mathcal{G}_p(r,\mu,\varphi,\zeta,\phi,t) \end{pmatrix} = \frac{a g_0 \sqrt{R}}{\pi^2 \sqrt{r}} \sum_{n=0}^{\infty} {\sum_{m=0}^{n}}' (2n+1) \frac{(n-m)!}{(n+m)!}$$

$$\times\ P_n^m(\mu)\, P_n^m(\zeta)\, \cos[m(\varphi-\phi)] \int\limits_0^\infty \begin{pmatrix} t^{\alpha-1} E_{\alpha,\alpha}(-a\xi^2 t^\alpha) \\ E_\alpha(-a\xi t^\alpha) \end{pmatrix}$$

$$\times \frac{J_{n+1/2}(r\xi)\, Y'_{n+1/2}(R\xi) - Y_{n+1/2}(r\xi) J'_{n+1/2}(R\xi)}{\left[J'_{n+1/2}(R\xi)\right]^2 + \left[Y'_{n+1/2}(R\xi)\right]^2} \, \mathrm{d}\xi. \tag{12.49}$$