Chapter 1

Introduction

The world-known specialist on parabolic equations Professor Marian Krzyżański should present a talk at "Professor Seminar" at the Jagello University of Cracaw.

The well-known specialist on elliptic equations Professor Frantiszek Leja came to a lecture hall and asked "What will be your talk about?"

"About parabolic equations," the speaker answered.

"It is not interesting," claimed the Dean Leja and went out of the auditorium.

I laughed loudly and, of course, was asked why? "In fact, only hyperbolic equations are of great interest."

Tadeusz Trajdos

Partial differential equations arise in various fields of science. Today, the literature on these equations is unbounded. Usually partial differential equations are divided into three basic types – elliptic, parabolic and hyperbolic. The simplest example of an elliptic equation is the Laplace equation, but we begin our consideration from another representative of this type equations – the Helmholtz equation

$$T = a\Delta T. \tag{1.1}$$

The simplest example of a parabolic equation is the diffusion equation (heat conduction equation)

$$\frac{\partial T}{\partial t} = a\Delta T. \tag{1.2}$$

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The well-known example of a hyperbolic equation is the wave equation

$$\frac{\partial^2 T}{\partial t^2} = a\Delta T. \tag{1.3}$$

It should be emphasized that the solutions of the equations belonging to each of the above-mentioned types have their own characteristic features. For example, dissipation is common to parabolic equations, wave fronts and finite speed of propagation are specific for hyperbolic equations. There are many excellent books and textbooks devoted to classical partial differential equations, some of them ([2, 39, 63, 74, 75, 216, 219]) are quoted in the References.

Elliptic partial differential equations are studied in [50, 64, 65, 88], among others. Solutions to the parabolic heat conduction equation in various spatial domains are presented in the books [26, 31, 98, 140]. The books [5, 16, 86, 199] are devoted to investigation of hyperbolic partial differential equations. The fundamental solutions to standard partial differential equations were considered in [21, 38, 85]. Many solutions have also been given in collections of problems [20, 87, 211].

In the last few decades, considerable research efforts have been expended to study fractional differential and integral equations – equations with operators of differentiation and integration of fractional (not integer) order [36, 56, 77, 82, 99, 118, 132, 133, 143, 202, 206].

The notion of the Riemann–Liouville fractional integral is introduced as a natural generalization of the repeated integral written in a convolution type form:

$$I^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau)\,\mathrm{d}\tau, \qquad \alpha > 0, \tag{1.4}$$

where $\Gamma(\alpha)$ is the gamma function.

The Riemann–Liouville derivative of the fractional order α is defined as left-inverse to the fractional integral I^{α}:

$$D^{\alpha}_{RL} f(t) = D^m I^{m-\alpha} f(t) = \frac{\mathrm{d}^m}{\mathrm{d}t^m}\left[\frac{1}{\Gamma(m-\alpha)} \int_0^t (t-\tau)^{m-\alpha-1} f(\tau)\,\mathrm{d}\tau\right], \quad m-1 < \alpha < m. \tag{1.5}$$

The Caputo fractional derivative is defined as

$$\begin{aligned} D^{\alpha}_{C} f(t) &\equiv \frac{\mathrm{d}^{\alpha} f(t)}{\mathrm{d}t^{\alpha}} = I^{m-\alpha} D^m f(t) \\ &= \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-\tau)^{m-\alpha-1} \frac{\mathrm{d}^m f(\tau)}{\mathrm{d}\tau^m}\,\mathrm{d}\tau, \quad m-1 < \alpha < m. \end{aligned} \tag{1.6}$$

The Caputo fractional derivative is a regularization in the time origin for the Riemann–Liouville fractional derivative by incorporating the relevant initial conditions [57]. The major utility of the Caputo fractional derivative is caused by the treatment of differential equations of fractional order for physical applications, where the initial conditions are usually expressed in terms of a given function and its derivatives of integer (not fractional) order, even if the governing equation is of fractional order [105, 143]. If care is taken, the results obtained using the Caputo formulation can be recast to the Riemann–Liouville version and vice versa according to the following relation [56]:

$$D^{\alpha}_{RL} f(t) = D^{\alpha}_{C} f(t) + \sum_{k=0}^{m-1} \frac{t^{k-\alpha}}{\Gamma(k-\alpha+1)} f^{(k)}(0^+), \quad m-1 < \alpha < m. \tag{1.7}$$

It should be also emphasized that [56]

$$\frac{\mathrm{d}^{\alpha} 1}{\mathrm{d}t^{\alpha}} = 0, \qquad \alpha > 0, \tag{1.8}$$

whereas

$$D^{\alpha}_{RL}\, 1 = \frac{t^{-\alpha}}{\Gamma(1-\alpha)}, \qquad \alpha \geq 0. \tag{1.9}$$

Fractional differential equations have many applications in physics, geophysics, geology, chemistry, rheology, engineering, bioengineering, robotics, medicine and finance (see, for example, the books [23, 99, 104, 132, 143, 198, 221, 230, 235]; the monographs [11, 70, 121, 142, 204]; the extensive surveys [102, 103, 114, 115, 200, 201, 217, 234)]; and several papers [9, 10, 22, 24, 32, 46, 51, 81]). The interested reader is also referred to a historical survey [218] and a survey of useful formulas [223].

The time-fractional diffusion-wave equation

$$\frac{\partial^{\alpha} T}{\partial t^{\alpha}} = a \Delta T, \qquad 0 < \alpha \leq 2, \tag{1.10}$$

describes many important physical phenomena in different media. The limiting case $\alpha = 0$ in (1.10) corresponding to the Helmholtz equation (1.1) and is associated with localized diffusion (localized heat conduction). The subdiffusion regime is characterized by the value $0 < \alpha < 1$. Of course, the standard diffusion equation (standard heat conduction equation) (1.2) corresponds to $\alpha = 1$. The superdiffusion regime is associated with $1 < \alpha < 2$. The limiting case $\alpha = 2$ corresponding to the wave equation (1.3) is also known as ballistic diffusion (ballistic heat conduction).

Various mathematical aspects relating to existence, uniqueness, correctness, well-posedness of solutions to fractional diffusion-wave equations were considered by many authors. Here we refer to [6, 17, 40–42, 49, 66–68, 76, 78–80, 83, 91, 92, 124, 143, 205, 222, 224–229, 233], among others.

Starting from the pioneering papers [45, 100, 101, 209, 231], considerable interest has been shown in finding solutions to time-fractional diffusion-wave equations (1.10). Fujita [45] treated integrodifferential equations which interpolate the diffusion equation and the wave equation. The fundamental solution for the fractional diffusion-wave equation in one space dimension was obtained by Mainardi [100, 101], who also considered the signaling problem and the evolution of the initial box-signal. Schneider and Wyss [209] converted the diffusion-wave equation with appropriate initial conditions into the integrodifferential equation and found the corresponding Green functions in terms of Fox functions. Wyss [231] obtained solutions to the Cauchy problem in terms of H-functions using the Mellin transform. The studies mentioned above do not consider solutions to the two-dimensional and three-dimensional diffusion-wave equation in finite domains. Presently, in the literature there exists no book devoted to the diffusion-wave equation. This book, which in large part is based on the author's investigations [97, 145–193], bridges the gaps in this field. Presenting the solutions to the time-fractional diffusion-wave equation, we follow the encyclopedical book of Polyanin [144], where the corresponding results for standard partial differential equations are given.