Chapter 6
Shaking Force and Shaking Moment Balancing of Six- and Eight-Bar Planar Mechanisms

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Abstract This chapter presents the dynamic balancing technique for shaking force and shaking moment balancing of six- and eight-bar planar mechanisms. Shaking force is balance by the method of redistribution of mass and shaking moment by geared inertia elements. The planetary gears used to balance shaking moment of links not directly connected to the frame in earlier methods are mounted on the base of the mechanism which is constructively more efficient. The proposed method is illustrated by numerical examples and it is observed that better results are obtained than those of the previous method.

Keywords Shaking force • Shaking moment • Dynamic balancing • Watt mechanisms • Self-balanced Slider-crank mechanism

6.1 Introduction

Mechanisms, particularly those which run at high speeds, generate variable forces on their foundations. These forces may cause noise, vibration, and unnecessary wear and fatigue. If these devices were balanced they would run more smoothly due to a reduction in these undesirable qualities. The balancing of a linkage would eliminate the vibration and noise and maintains a peaceful and productive environment, and it also minimizes the alternating components of the dynamic forces acting on the frame of the mechanism and machine. Therefore, the problems of shaking force and shaking moment balancing have attracted the attention of the machine and mechanism designers for a long time. Balancing of shaking force and shaking moment in high-speed mechanisms/machines reduces the forces transmitted to the frame. In effect, this reduces the noise and wear, improves the dynamic performance, and extends the fatigue life of the mechanisms. A considerable amount of research on balancing of shaking force and shaking moment in planar mechanisms has been carried out in the past.

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6.1.1 Shaking Force in a Mechanism

Of special interest to the designer are the forces transmitted to the frame or foundation of the machine owing to the inertia of moving links and other machine members. When these forces vary in magnitude or direction, they tend to shake or vibrate the machine, and consequently they are called shaking forces. Thus the shaking forces are the forces, which act upon the frame of a machine owing only to the inertia forces of the moving parts. Shaking forces and shaking moments are the unbalanced forces and moments generated when the planar and spatial mechanisms are in motion. The study of these forces and moments is important when they run at high speeds. These undesirable qualities of the mechanism reduce the performance of the mechanism. The shaking force generated by the mechanism can be determined as follows:

If a four-bar linkage is considered, as an example, with links 2, 3, and 4 as the moving members and link 1 as the frame, then the inertia forces associated with the moving members are $-m_2A_{G_2}, -m_3A_{G_3}, -m_4A_{G_4}$. Therefore, taking the moving members as a free body, it can be immediately written as

$$\sum F = F_{12} + F_{14} + (-m_2A_{G_2}) + (-m_3A_{G_3}) + (-m_4A_{G_4}) = 0$$

Using "F_S" as a symbol for the resulting shaking force, it is defined as equal to the resultant of all the reaction forces on the ground link 1,

$$F_S = F_{21} + F_{41}$$

Therefore, from the previous equation, it can be written as

$$F_S = -\left(m_2A_{G_2} + m_3A_{G_3} + m_4A_{G_4}\right)$$

Thus a general equation for the shaking forces in any machine is

$$F_S = -\sum_{2}^{n} m_nA_{G_n}$$

where it is understood that link 1 is always the frame and where "n" is the number of members making up the machine.

6.1.2 Shaking Moment of the Mechanism

The shaking moment of a linkage can be described as the time rate of change of the total angular momentum with respect to the reference origin "O." It is

$M = I\alpha$, where M is the shaking moment w.r.t. point "O"; α is the angular acceleration; and I is the mass moment of inertia.

6.1.3 Methods of Balancing

Balancing of linkages is an important step in the design of machinery. When shaking forces and shaking moments of the whole mechanisms are to be balanced then balancing of sub-linkages is considered. The linkages consist of different sub-linkages; this study considers two sub-linkages as most of the mechanisms are formed by them. Many methods [1–75] have been developed for the balancing of shaking force and shaking moment of planar linkages:

1. Method of redistribution of mass [1–6]
2. Method of double crank with symmetrical properties [7]
3. Method of active balancing [8–16]
4. Methods of balancing by planetary systems attached to the coupler [17–30]
5. Method of balancing by minimizing vibration [31–41]
6. Computational methods of optimization for balancing [42–68]
7. Methods for the minimization of shaking moments [69–74]
8. Balancing by opposite movements [75]

This chapter deals with the shaking force and shaking moment balancing of single degree of freedom planar mechanisms. Specifically, the author employs the traditional technique of addition of counterweights and counter-rotating inertias in order to balance six- and eight-bar linkages through the development of analytical expressions. This chapter is the extension of the work carried out by the authors [18–23].

6.2 Articulation Dyad

6.2.1 Complete Shaking Force and Shaking Moment Balancing of an Articulation Dyad

An open kinematic chain of two binary links and one joint is called a dyad. When two links are articulated by a joint so that movement is possible that arrangement of links is known as articulation dyad. The well-known scheme of complete shaking force and shaking moment balancing of an articulation dyad [18–23] is shown in Fig. 6.1.

For shaking force balancing link 2 is dynamically replaced by two point masses. A counterweight $m_{\mathrm{CW}_2} = (m_2 l_{\mathrm{A}S_2})/r_{\mathrm{CW}_2}$ is added to link 2 which permits the displacement of the center of mass of link 2 to joint A. Then, by means of a counterweight with mass $m_{\mathrm{CW}_1} = [(m_2 + m_{\mathrm{CW}_2})\, l_{\mathrm{OA}} + m_1 l_{\mathrm{O}S_1}]/r_{\mathrm{CW}_1}$ a complete balancing of shaking force is achieved. A complete shaking moment balance is realized through four gear inertia counterweights 3–6, one of them being of the planetary type and mounted on link 2.

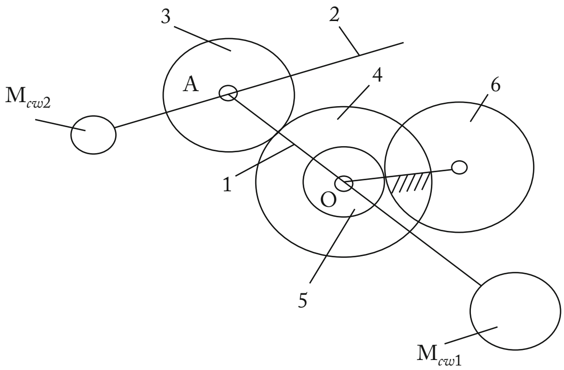

Fig. 6.1 Complete shaking force and shaking moment balancing of an articulation dyad

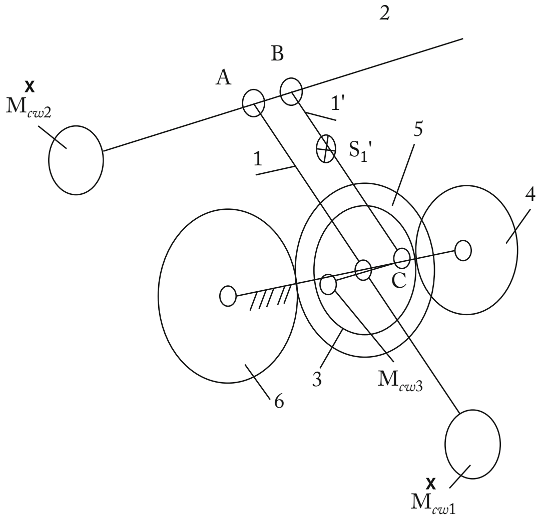

Fig. 6.2 Complete shaking force and shaking moment balancing of an articulation dyad by gear inertia counterweights mounted on the base

6.2.2 Complete Shaking Force and Shaking Moment Balancing of an Articulation Dyad by Gear Inertia Counterweights Mounted on the Base

The scheme used in this work (Fig. 6.2) is distinguished from the earlier scheme by the fact that gear 3 is mounted on the base and is linked kinematically with link 2 through link $1'$.

To prove the merits of such a balancing, the application of the new system with the mass of link $1'$ not taken into account is considered. In this case (compared to the usual method in Fig. 6.1), the mass of the counterweight of link 1 will be reduced by an amount

$$\delta m_{\mathrm{cw}_1} = {m_3 l_{\mathrm{OA}}} \Big/ {r_{\mathrm{cw}_1}} \tag{6.1}$$

where m_3 is the mass of gear 3.

l_{OA} is the distance between the centers of hinges O and A.

r_{cw_1} is the rotation radius of the center of mass of the counterweight.

It is obvious that the moment of inertia of the links is correspondingly reduced. If the gear inertias are made in the form of heavy rims in order to obtain a large moment of inertia, the moments of inertia of the gear inertia counterweights may be presented as

$$I = \frac{m_i D_i^2}{4}\,(i = 3 \ldots 6)\,.$$

Consequently, the mass of gear 6 will be reduced by an amount

$$\delta m_6 = 4\left(m_3 l_{\mathrm{OA}}^2 + \delta m_{\mathrm{cw}_1} r_{\mathrm{cw}_1}^2\right)\frac{T_6}{D_6^2 T_5} \tag{6.2}$$

where

T_5 and T_6 are the numbers of teeth of the corresponding gears. Thus, the total mass of the system will be reduced by an amount

$$\delta m = \delta m_{\mathrm{cw}_1} + \delta m_6 \tag{6.3}$$

Here the complete shaking force and shaking moment balancing of the articulation dyad with the mass and inertia of link $1'$ taken into account are considered. For this purpose initially, statically replace mass $m_1^{'}$ of link $1'$ by two point masses m_{B} and m_{c} at the centers of the hinges B and C:

$$\begin{aligned} m_{\mathrm{B}} &= m_{1'} l_{\mathrm{C}S_{1'}} / l_{\mathrm{BC}} \\ m_C &= m_{1'} l_{\mathrm{B}S_{1'}} / l_{\mathrm{BC}} \end{aligned} \tag{6.4}$$

where

l_{BC} is the length of link 1.

$l_{\mathrm{C}S_{1'}}$ and $l_{\mathrm{B}S_1'}$ are the distances between the centers of joints C and B and the center of mass $S_1^{'}$ of link $1'$, respectively.

After such an arrangement of masses the moment of inertia of link $1'$ will be equal to

$$I_{S_1'}^{*} = I_{S_1'} - m_{1'} l_{\mathrm{B}S_1'} l_{\mathrm{C}S_1'} \tag{6.5}$$

where

$I_{S_1'}$ is the moment of inertia of link $1'$ about the center of mass $S_1^{'}$ of the link.

Thus a new dynamic model of the system is obtained, where the link $1'$ is represented by two point masses $m_{\mathrm{B}}, m_{\mathrm{C}}$ and has a moment of inertia $I_{S_1'}^{*}$.

This fact allows for an easy determination of the parameters of the balancing elements as follows:

$$m^{*}_{\mathrm{CW}_2} = (m_2 l_{\mathrm{AS}_2} + m_{\mathrm{B}} l_{\mathrm{AB}}) / r_{\mathrm{CW}_2} \tag{6.6}$$

where

m_2 is the mass of link 2.

l_{AB} is the distance between the centers of the hinges A and B

l_{AS_2} is the distance of the center of hinge A from the center mass of S_2 of link 2.

r_{CW_2} is the rotation radius of the center of mass of the counterweight with respect to A and

$$m^{*}_{\mathrm{CW}_1} = [(m_2 + m_{\mathrm{CW}_2} + m_{\mathrm{B}})\, l_{\mathrm{OA}} + m_1 l_{\mathrm{OS}_1}] / r_{\mathrm{CW}_1} \tag{6.7}$$

where m_1 is the mass of link 1.

l_{OS_1} is the distance of the joint center O from the center of mass S_1 of link 1.

Also,

$$m_{\mathrm{CW}_3} = m_{\mathrm{C}} l_{\mathrm{OC}} / r_{\mathrm{CW}_3} \tag{6.8}$$

where

$$l_{\mathrm{OC}} = l_{\mathrm{AB}}$$

r_{CW_3} is the rotation radius of the center of mass of the counterweight.

Taking into account the mass of link $1'$ brings about the correction in Eq. (6.3) in this case,

$$\delta m = \delta m_{\mathrm{CW}_1} + \delta m_6 - \delta m_1' \tag{6.9}$$

where $\delta m_1'$ is the value characterizing the change in the distribution of the masses of the system links resulting from the addition of link $1'$.

6.3 Asymmetric Link with Three Rotational Pairs

A link with three nodes is called ternary link, where nodes are points for attachment to other links. In previous work by Gao Feng [18] relating to balancing of linkages with a dynamic substitution of the masses of the link by three rotational pairs shown in Fig. 6.3 two replacement points A and B are considered. This results in the need to increase the mass of the counterweight. However, such a solution may be avoided by considering the problem of dynamic substitution of link masses by three point masses. Usually the center of mass of such an asymmetric link is located inside a triangle formed by these points.

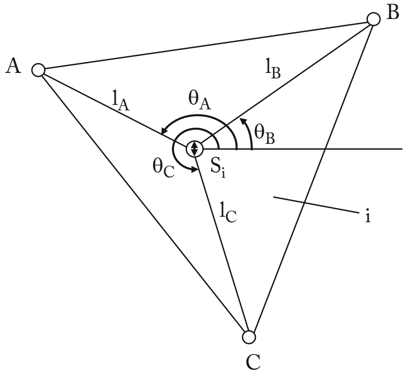

Fig. 6.3 Dynamic substitution of the masses of the link by three rotational pairs

The conditions for dynamic substitution of masses are the following:

$$\begin{bmatrix} 1 & 1 & 1 \\ l_A e^{i\theta_A} & l_B e^{i\theta_B} & l_C e^{i\theta_C} \\ l_A^2 & l_B^2 & l_C^2 \end{bmatrix} \begin{bmatrix} m_A \\ m_B \\ m_C \end{bmatrix} = \begin{bmatrix} m_i \\ 0 \\ I_{S_i} \end{bmatrix}$$

where m_A, m_B, and m_C are point masses.

l_A, l_B and l_C are the moduli of radius vectors of corresponding points.

θ_A, θ_B and θ_C are angular positions of radius vectors; m_i is the mass of link.

I_{S_i} is the moment of inertia of the link about an axis through S_i (axial moment of inertia of link).

From this system of equations the masses are obtained:

$$m_A = D_A/D_i; m_B = D_B/D_i; m_C = D_C/D_i \tag{6.10}$$

where D_A, D_B, D_C and D_i are determinants of the third order obtained from the above system of equations.

6.4 Summary

The complete shaking force is balanced by the method of redistribution of mass and making the total mass center of the mechanism stationary. The complete shaking moment is balanced by geared inertia counterweights. The planetary gears which are mounted on the links not directly connected to the frame in earlier method are mounted on the frame of the mechanism by connecting the planetary gear and the corresponding link by a link of known mass, center of mass, and mass moment of inertia. This arrangement makes the balanced mechanism constructively more efficient and compact and yields better results over the Gao Feng method.

6.5 Watt Mechanism with Three Fixed Points Linkage

Watt mechanism consists of six links; out of them two are ternary and the remaining four are binary links. In Watt mechanism two ternary links are directly connected to one another. This mechanism is obtained when one of the ternary links in the basic Watt chain is fixed. This is a simple mechanism as the radii of path curvature of all motion transfer points are known. This mechanism is used in steam engines and is also used to oscillate the agitator in some washing machines. In the Watt mechanism with three fixed points shown in Fig. 6.4, link 1 and 3 are ternary links and all other links are binary links. The balanced Watt mechanism with three fixed points is shown in Fig. 6.5.

6.5.1 *Shaking Force Balancing of the Mechanism*

For shaking force balancing link 3 is dynamically replaced by three point masses m_{B3}, m_{C3} and m_{D3} and then the problems of sub-linkages OAB and DEF are considered.

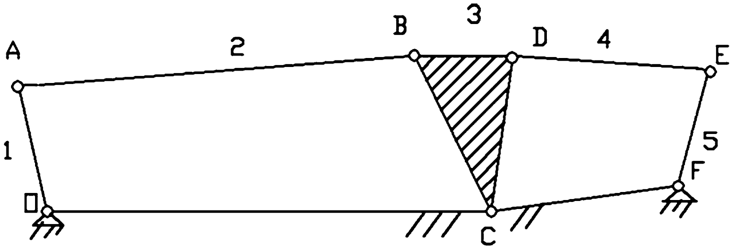

Fig. 6.4 Watt mechanism with three fixed points

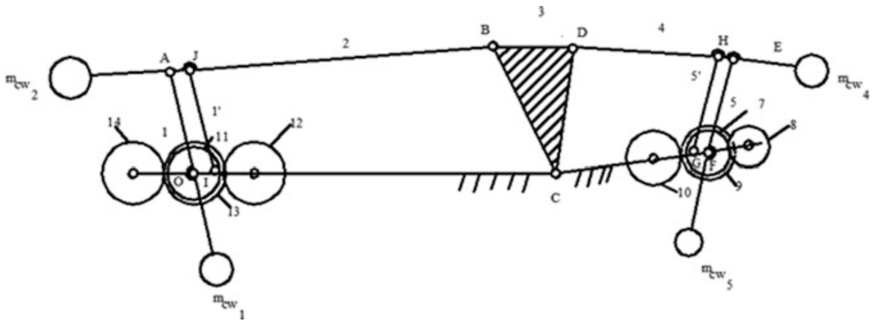

Fig. 6.5 Balanced Watt mechanism with three fixed points

The dynamic conditions for link 3 to be replaced by three point masses are

$$\begin{bmatrix} 1 & 1 & 1 \\ l_{\mathrm{B}}\mathrm{e}^{i\theta_{\mathrm{B}}} & l_{\mathrm{C}}\mathrm{e}^{i\theta_{\mathrm{C}}} & l_{\mathrm{D}}\mathrm{e}^{i\theta_{\mathrm{D}}} \\ l_{\mathrm{B}}^2 & l_{\mathrm{C}}^2 & l_{\mathrm{D}}^2 \end{bmatrix} \begin{bmatrix} m_{\mathrm{B3}} \\ m_{\mathrm{C3}} \\ m_{\mathrm{D3}} \end{bmatrix} = \begin{bmatrix} m_3 \\ 0 \\ I_{\mathrm{S}_3} \end{bmatrix}$$

$$m_{\mathrm{B3}} = {D_{\mathrm{B}}}\Big/{D_3}\,;\, m_{\mathrm{C3}} = {D_{\mathrm{C}}}\Big/{D_3}\,;\, m_{\mathrm{D3}} = {D_{\mathrm{D}}}\Big/{D_3} \tag{6.11}$$

where

$l_{\mathrm{B}}, l_{\mathrm{C}}, l_{\mathrm{D}}$ are the moduli of radius vectors of corresponding points.

$\theta_{\mathrm{B}}, \theta_{\mathrm{C}}, \theta_{\mathrm{D}}$ are the angular positions of radius vectors.

m_3 is the mass of link 3.

I_{S_3} is the mass moment of inertia link 3 about its center of mass.

$D_{\mathrm{B}}, D_{\mathrm{C}}, D_{\mathrm{D}}$ and D_3 are the third-order determinants obtained from the system of equations.

For sub-linkage DEF link 4 is dynamically replaced by two point masses m_{D4} and m_{P4} and then kinematically linked link 4 and its corresponding gear inertia counterweight 7 by link 5′ and link 5′ is statically replaced by two point masses m_{G} and m_{H} and attached a counterweight m_{CW_4} against link 4. Then link 5 has been dynamically replaced by two point masses $m_{\mathrm{E5}}, m_{\mathrm{P5}}$ and attached a counterweight m_{CW_5} against it.

For link 4 to be dynamically replaced by two point masses the condition to be satisfied is $k_4^2 = l_{\mathrm{DS}_4} l_{\mathrm{P}_4\mathrm{S}_4}$

where

k_4 is the radius of gyration of link 4 about its center of mass.

l_{DS_4} is arbitrarily fixed.

$l_{\mathrm{P}_4\mathrm{S}_4}$ is obtained from the above condition:

$$m_{\mathrm{D4}} = {m_4 l_{\mathrm{P}_4\mathrm{S}_4}}\Big/{(l_{\mathrm{DS}_4} + l_{\mathrm{P}_4\mathrm{S}_4})}$$

$$m_{\mathrm{P4}} = {m_4 l_{\mathrm{DS}_4}}\Big/{(l_{\mathrm{DS}_4} + l_{\mathrm{P}_4\mathrm{S}_4})}$$

For link 5 to be dynamically replaced by two point masses the condition to be satisfied is

$$k_5^2 = l_{\mathrm{ES}_5} l_{\mathrm{P}_5\mathrm{S}_5}$$

where

k_5 is the radius of gyration of link 5 about its center of mass.

l_{ES_5} is arbitrarily fixed.

$l_{\mathrm{P}_5 S_5}$ is obtained from the above condition:

$$m_{\mathrm{E5}} = {m_5 l_{\mathrm{P}_5\mathrm{S}_5}}\Big/{(l_{\mathrm{ES}_5} + l_{\mathrm{P}_5\mathrm{S}_5})}$$

$$m_{\mathrm{P5}} = {m_5 l_{\mathrm{ES}_5}}\Big/{(l_{\mathrm{ES}_5} + l_{\mathrm{P}_5\mathrm{S}_5})}$$

and counterweight mass against "G" is equal to

$$m_{\mathrm{CW}_7} = {m_{\mathrm{G}} l_{\mathrm{FG}}} \Big/ {r_{\mathrm{CW}_7}} \tag{6.12}$$

$$\begin{aligned}
m_{\mathrm{G}} &= {m'_5 l_{\mathrm{HS}_5}} \Big/ {l_{\mathrm{GH}}} \\
m_{\mathrm{H}} &= {m'_5 l_{\mathrm{GS}_5}} \Big/ {l_{\mathrm{GH}}} \\
I'^{*}_{\mathrm{S}_5} &= I'_{\mathrm{S}_5} - m'_5 l_{\mathrm{GS}_5} l_{\mathrm{HS}_5} \qquad (6.13) \\
m_{\mathrm{CW}_4} &= {(m_4 l_{\mathrm{ES}_4} + m_{\mathrm{D}3} l_{\mathrm{DE}} + m_{\mathrm{H}} l_{\mathrm{EH}})} \Big/ {r_{\mathrm{CW}_4}} \\
m_{\mathrm{CW}_5} &= {((m_4 + m_{\mathrm{D}3} + m_{\mathrm{H}} + m_{\mathrm{CW}_4})\, l_{\mathrm{EF}} + m_5 l_{\mathrm{FS}_5})} \Big/ {r_{\mathrm{CW}_5}}
\end{aligned}$$

where $r_{\mathrm{CW}_4} = (l_{\mathrm{P}_4\mathrm{S}_4} - l_{\mathrm{ES}_4})$ is the radius of rotation of counterweight m_{CW_4} and $r_{\mathrm{CW}_5} = (l_{\mathrm{P}_5\mathrm{S}_5} - l_{\mathrm{FS}_5})$ is the radius of rotation of counterweight m_{CW_5}.

For sub-linkage OAB link 2 is dynamically replaced by two point masses $m_{\mathrm{B}2}, m_{\mathrm{P}2}$ and then kinematically linked link 2 and its corresponding gear inertia counterweight 11 by link $1'$ and link $1'$ is statically replaced by two point masses $m_{\mathrm{I}}, m_{\mathrm{J}}$ and attached a counterweight m_{CW_2} against link 2. Then link 1 is dynamically replaced by two point masses $m_{\mathrm{A}1}, m_{\mathrm{P}1}$ and attached a counterweight m_{CW_1} against it.

For link 2 to be dynamically replaced by two point masses the condition to be satisfied is $k_2^2 = l_{\mathrm{BS}_2} l_{\mathrm{P}_2\mathrm{S}_2}$

where k_2 is the radius of gyration of link 2 about its center of mass.

l_{BS_2} is arbitrarily fixed and $l_{\mathrm{P}_2\mathrm{S}_2}$ is obtained from the above condition:

$$\begin{aligned}
m_{\mathrm{B}2} &= {m_2 l_{\mathrm{P}_2\mathrm{S}_2}} \Big/ {(l_{\mathrm{BS}_2} + l_{\mathrm{P}_2\mathrm{S}_2})} \\
m_{\mathrm{P}2} &= {m_2 l_{\mathrm{BS}_2}} \Big/ {(l_{\mathrm{BS}_2} + l_{\mathrm{P}_2\mathrm{S}_2})}
\end{aligned}$$

For link 1 to be dynamically replaced by two point masses the condition to be satisfied is

$$k_1^2 = l_{\mathrm{AS}_1} l_{\mathrm{P}_1\mathrm{S}_1}$$

where

k_1 is the radius of gyration of link 1 about its center of mass.

l_{AS_1} is arbitrarily fixed.

$l_{\mathrm{P}_1\mathrm{S}_1}$ is obtained from the above condition:

$$m_{\mathrm{A1}} = {m_1 r_{\mathrm{P_1S_1}}}\Big/{(l_{\mathrm{AS_1}} + l_{\mathrm{P_1S_1}})}$$

$$m_{\mathrm{P1}} = {m_1 l_{\mathrm{AS_1}}}\Big/{(l_{\mathrm{AS_1}} + l_{\mathrm{P_1S_1}})}$$

$$m_{\mathrm{I}} = {m_1' l_{\mathrm{IS_5}}}\Big/{l_{\mathrm{IJ}}}$$

$$m_{\mathrm{J}} = {m_1' l_{\mathrm{JS_5}}}\Big/{l_{\mathrm{IJ}}}$$

$$m_{\mathrm{CW_{11}}} = {m_{\mathrm{I}} l_{\mathrm{OI}}}\Big/{r_{\mathrm{CW_{11}}}}$$

$$I'^{*}_{\mathrm{S_1}} = I'_{\mathrm{S_1}} - m_1' l'_{\mathrm{JS_1}} l_{\mathrm{IS_1}}$$

$$m_{\mathrm{CW_2}} = {(m_2 l_{\mathrm{AS_2}} + m_{\mathrm{B3}} l_{\mathrm{AB}} + m_{\mathrm{J}} l_{\mathrm{AJ}})}\Big/{r_{\mathrm{CW_2}}}$$

$$m_{\mathrm{CW_1}} = {((m_2 + m_{\mathrm{B3}} + m_{\mathrm{J}} + m_{\mathrm{CW_2}})\, l_{\mathrm{OA}} + m_1 l_{\mathrm{OS_1}})}\Big/{r_{\mathrm{CW_1}}}$$

where $m_{\mathrm{CW_{11}}}$ is the counterweight attached against point mass m_{I}.

$r_{\mathrm{CW_2}} = (l_{\mathrm{P_2S_2}} - l_{\mathrm{AS_2}})$ is the radius of rotation of counterweight $m_{\mathrm{CW_2}}$, and $r_{\mathrm{CW_1}} = (l_{\mathrm{P_1S_1}} - l_{\mathrm{OS_1}})$ is the radius of rotation of counterweight $m_{\mathrm{CW_1}}$.

6.5.2 Shaking Moment Balancing of the Mechanism

The shaking moments generated by links 1, 2, 4, and 5 are given in Eq. (6.14). The links 2 and 4 are not directly connected to the frame, and the geared inertia counterweights required to balance the shaking moments of these two links are mounted on the base of the mechanism, by kinematically linking them to the corresponding links by links of known mass and center of mass.

The shaking moment generated by the linkage is determined by the sum

$$\begin{aligned}
&M^{\mathrm{int}} = M_1^{\mathrm{int}} + M_5^{\mathrm{int}} + M_2^{\mathrm{int}} + M_4^{\mathrm{int}} \\
&M_1^{\mathrm{int}} = \left(I_{\mathrm{S_1}} + m_1 l^2_{\mathrm{OS_1}} + (m_{\mathrm{CW_2}} + m_{\mathrm{J}} + m_2 + m_{\mathrm{B3}})\, l^2_{\mathrm{OA}} + m_{\mathrm{CW_1}} r^2_{\mathrm{CW_1}} + I'^{*}_{\mathrm{S_1}} + m_1' l^2_{\mathrm{OS_1}}\right) \alpha_1 \\
&M_5^{\mathrm{int}} = \left(I_{\mathrm{S_5}} + m_5 l^2_{\mathrm{FS_5}} + (m_{\mathrm{CW_4}} + m_{\mathrm{H}} + m_4 + m_{\mathrm{D3}})\, l^2_{\mathrm{EF}} + m_{\mathrm{CW_5}} r^2_{\mathrm{CW_5}} + I'^{*}_{\mathrm{S_5}} + m_5' l'^2_{\mathrm{FS_5}}\right) \alpha_5 \\
&M_2^{\mathrm{int}} = \left(2 m_{\mathrm{I}} l^2_{\mathrm{OI}}\right) \alpha_2 \\
&M_4^{\mathrm{int}} = \left(2 m_{\mathrm{G}} l^2_{\mathrm{FG}}\right) \alpha_4
\end{aligned} \tag{6.14}$$

where

$M_1^{\text{int}}, M_5^{\text{int}}$ are the shaking moments of rotating links 1 and 5, respectively.

I_{S_1}, I_{S_5} are mass moments of inertia of links 1 and 5 about their centers of masses, respectively.

$I'^*_{S_1}, I'^*_{S_5}$ are the changed moments of inertia of links $1', 5'$, respectively.

$\alpha_1, \alpha_2, \alpha_{4,} \alpha_5$ are the angular accelerations of links 1, 2, 4, and 5, respectively.

For shaking moment balancing eight gear inertia counterweights are used, four at F and four at O.

6.6 Watt Mechanism with Two Fixed Points

The Watt mechanism with two fixed points is obtained when one of the binary links in the basic Watt chain is fixed. This mechanism is generally used in steam engines. In the Watt mechanism with two fixed points shown in Fig. 6.6, links 2 and 3 are ternary links and all other links are binary links. The balanced Watt mechanism with two fixed points is shown in Fig. 6.7.

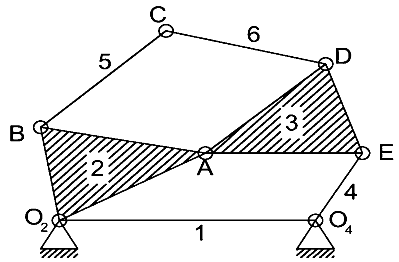

Fig. 6.6 Watt mechanism with two fixed points

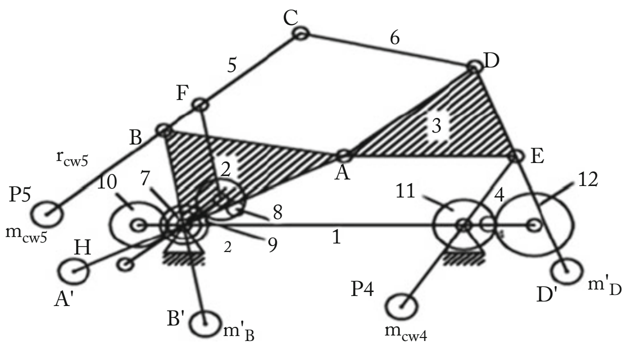

Fig. 6.7 Balanced Watt mechanism with two fixed points

6.6.1 Shaking Force Balancing of the Mechanism

Here the link 2 is dynamically replaced by three point masses $m_{A_2}, m_{B_2}, m_{O_2}$ by using the following conditions:

$$\begin{bmatrix} 1 & 1 & 1 \\ l_{O_2}e^{i\theta_{O_2}} & l_A e^{i\theta_A} & l_B e^{i\theta_B} \\ {l_{O_2}}^2 & {l_A}^2 & {l_B}^2 \end{bmatrix} \begin{bmatrix} m_{O_2} \\ m_{A_2} \\ m_{B_2} \end{bmatrix} = \begin{bmatrix} m_2 \\ 0 \\ I_{S_2} \end{bmatrix}$$

$$m_{O_2} = \frac{D_{O_2}}{D_2},\ m_{A_2} = \frac{D_{A_2}}{D_2},\ m_{B_2} = \frac{D_{B_2}}{D_2}$$

where $l_{O_2},\ l_A,\ l_B$ are the moduli of radius vectors of corresponding points.
$\theta_{O_2},\ \theta_A,\ \theta_B$ are the angular positions of radius vectors.
m_2 is the mass of link 2.
I_{S_2} is the mass moment of inertia of link 2 about its center of mass.
$D_{O_2},\ D_{A_2},\ D_{B_2}$ and D_2 are the third-order determinants obtained from the system of equations.

For link 6 to be statically replaced by the point masses m_{C_6} and m_{D_6}

$$m_{C_6} = \frac{m_6 l_{DS_6}}{l_{CD}}$$

$$m_{D_6} = \frac{m_6 l_{CS_6}}{l_{CD}}$$

Changed mass moment of inertia $I'^{*}_{S_6} = I'_{S_6} - m_6 l_{DS_6} l_{CS_6}$

For link 5 to be dynamically replaced by two point masses m_{C_5} and m_{P_5} the condition to be satisfied is

$$k_5{}^2 = l_{CS_5} l_{P_5S_5}$$

where l_{CS_5} is arbitrarily taken and $l_{P_5S_5}$ is obtained from the above condition:

$$m_{C_5} = \frac{m_5 l_{P_5S_5}}{(l_{P_5S_5} + l_{CS_5})}$$

$$m_{P_5} = \frac{m_5 l_{CS_5}}{(l_{P_5S_5} + l_{DS_5})}$$

After link 5 is dynamically replaced by two point masses it is kinematically connected to its corresponding gear inertia counterweight 8 by link 2′; moreover link 2′ is statically replaced by two point masses m_G and m_F:

$$m_G = \frac{m'_2 l'_{FS_2}}{l_{FG}}$$

$$m_{\mathrm{F}} = \frac{m_2' l'_{\mathrm{GS}_2}}{l_{\mathrm{FG}}}$$

Counterweight m_{CW_5} can be obtained as

$$m_{\mathrm{CW}_5} = \frac{(m_{\mathrm{C6}} l_{\mathrm{BC}} + m_{\mathrm{F}} l_{\mathrm{BF}} + m_5\, l_{\mathrm{BS}_5})}{r_{\mathrm{CW}_5}} \tag{6.15}$$

where $r_{\mathrm{CW}_5} = l_{\mathrm{P}_5\mathrm{S}_5} - l_{\mathrm{CS}_5}$ is radius of rotation of counterweight m_{CW_5}.

Link 3 is dynamically replaced by three point masses $m_{\mathrm{A3}}, m_{\mathrm{D3}}, m_{\mathrm{E3}}$ by using the following conditions:

$$\begin{bmatrix} 1 & 1 & 1 \\ l_{\mathrm{A}} e^{i\theta_{\mathrm{A3}}} & l_{\mathrm{A}} e^{i\theta_{\mathrm{D}}} & l_{\mathrm{B}} e^{i\theta_{\mathrm{E}}} \\ l_{\mathrm{A}}{}^2 & l_{\mathrm{D}}{}^2 & l_{\mathrm{E}}{}^2 \end{bmatrix} \begin{bmatrix} m_{\mathrm{A}_3} \\ m_{\mathrm{D}_3} \\ m_{\mathrm{E}_3} \end{bmatrix} = \begin{bmatrix} m_3 \\ 0 \\ I_{\mathrm{S}_3} \end{bmatrix}$$

$$m_{\mathrm{A3}} = \frac{D_{\mathrm{A3}}}{D_3},\ m_{\mathrm{D3}} = \frac{D_{\mathrm{D3}}}{D_3},\ m_{\mathrm{E3}} = \frac{D_{\mathrm{E3}}}{D_3} \tag{6.16}$$

where $l_{\mathrm{A}}, l_{\mathrm{D}}, l_{\mathrm{E}}$ are the moduli of radius vectors of corresponding points.

$\theta_{\mathrm{A3}}, \theta_{\mathrm{D}}, \theta_{\mathrm{E}}$ are the angular positions of radius vectors.

m_3 is the mass of link 3.

I_{S3} is the mass moment of inertia of link 2 about its center of mass.

$D_{\mathrm{A3}}, D_{\mathrm{D3}}, D_{\mathrm{E3}}$ and D_2 are the third-order determinants obtained from the system of equations.

Counterweight against point B of link 2 can be obtained as

$$m'_{\mathrm{B}} = (m_{\mathrm{CW}_5} + m_{\mathrm{F}} + m_5 + m_{\mathrm{C6}})\, l_{\mathrm{O_2B}} \Big/ l'_{\mathrm{O_2B}}$$

where $l'_{\mathrm{O_2B}}$ is arbitrarily fixed.

Counterweight against point A of link 3 can be obtained as

$$m'_{\mathrm{A}} = (m_{\mathrm{A2}} + m_{\mathrm{A3}})\, l_{\mathrm{O_2A}} \Big/ l'_{\mathrm{O_2A}}$$

where $l'_{\mathrm{O_2A}}$ is arbitrarily chosen.

Counterweight against point D of link 3 can be obtained as t

$$m'_{\mathrm{D}} = (m_{\mathrm{D3}} + m_{\mathrm{D6}})\, l_{\mathrm{DE}} \Big/ l'_{\mathrm{DE}}$$

where l'_{DE} is arbitrarily chosen.

For link 4 to be dynamically replaced by two point masses m_{E4}, m_{P4} the condition to be satisfied is $k_4^2 = l_{ES_4} l_{P_4S_4}$, where l_{ES_4} is arbitrarily is chosen and $l_{P_4S_4}$ is obtained from the above condition:

$$m_{E4} = {m_4 l_{P_4S_4}} \Big/ {(l_{P_4S_4} + l_{ES_4})}\,; \quad m_{P4} = {m_4 l_{ES_4}} \Big/ {(l_{P_4S_4} + l_{ES_4})}$$

Counterweight against link 4 can be obtained as

$$m_{CW_4} = \frac{\left(m_{E3} + m_{D6} + m'_D\right) l_{O_4E}}{r_{CW_4}}$$

where $r_{CW_4} = l_{P_4S_4} - l_{O_4S_4}$ is the radius of rotation of counterweight m_{CW_4}.

6.6.2 Shaking Moment Balancing of the Mechanism

The shaking moments generated by links 2, 4, and 5 are given in Eq. (6.17).

The shaking moment generated by the mechanism can be determined by the sum

$$M^{int} = M_2^{int} + M_5^{int} + M_4^{int} \tag{6.17}$$

where

$$\begin{aligned} M_2^{int} =& \Big(I_{S_2} + I'^{*}_{S_2} + m'_2 l'_{GS_2} l'_{FS_2} \\ &+ (m_{A2} + m_{A3})\, l^2_{O_2A} + (m_{CW_5} + m_F + m_5 + m_{C6})\, l^2_{O_2B} + m'_B l'^2_{O_2B}\Big)\, \alpha_2 \\ M_4^{int} =& \left(I_{S_4} + m_4 l^2_{O_4S_4} + m_{CW_4} r^2_{CW_4} \left(m'_D + m_{D3} + m_{D6} + m_{E3}\right) l^2_{O_4E}\right) \alpha_4 \\ M_5^{int} =& \left(2 m_G l^2_{O_2G}\right) \alpha_5 \end{aligned}$$

$M_2^{int}, M_4^{int}, M_5^{int}$ are the shaking moments of rotating links 2, 4, and 5, respectively.

I_{S_2}, I_{S_4} are the mass moment of inertia of links 2 and 4, respectively.

$\alpha_{2,}, \alpha_4, \alpha_5$ are the angular accelerations of links 2, 4, and 5, respectively.

For shaking moment balancing six gear inertia counterweights are used, four at O_2 and two at O_4.

Shaking force of the mechanism by the proposed method:

$$F_{Proposed} = -\left(m_2 A_{G2} + m_3 A_{G3} + m_4 A_{G4} + m_5 A_{G5} + m_6 A_{G6} + m'_2 A'_{G2}\right)$$

Shaking moment of the mechanism by the proposed method:

$$M^{int}_{proposed} = M_2^{int} + M_4^{int} + M_5^{int}$$

Shaking force of the mechanism by Gao Feng's method:

$$F_{\text{Gaofeng}} = -\left(m_2 A_{G2} + m_3 A_{G3} + m_4 A_{G4} + m_5 A_{G5} + m_6 A_{G6} + m_{G8} A_{G8}\right)$$

Shaking moment of the mechanism by Gao Feng's method:

$$M_{\text{Gaofeng}}^{\text{int}} = M_2^{\text{int}} + M_4^{\text{int}} + M_5^{\text{int}} + \left(I_{S8} + 2m_{G8} l_{FG}^2\right)\alpha_2$$

Numerical example: The Watt mechanism with two fixed points shown in Fig. 6.6 has the following parameters:

$m_2 = 2$ kg, $k_2 = 0.1198$ m, $m_3 = 1.8$ kg, $k_3 = 0.1178$ m, $m_4 = 7$ kg, $k_4 = 0.237$m, $m_5 = 2.8$ kg, $k_5 = 0.934$ m, $m_6 = 3$ kg, $k_6 = 0.369$ m, $l_{A3} = 3.7$ m, $l_E = 5.8$ m, $l_D = 5.6$ m, $\theta_A = 0°$, $\theta_B = 117°$, $\theta_{O_2} = 262°$, $l_{O_2} = 5$ m, $l_A = 2.6$ m, $l_B = 3.7$, $l_{O_2B} = 5$, $l_{O_2A} = 2.3$ m, $l_{AB} = 9$ m, $l_{BC} = 8$ m, $l_{CD} = 6$ m, $l_{BF} = 2.1$ m, $l_{O_4E} = 9$ m, $l_{DE} = 7$ m, $l_{AE} = 5$ m, $l_{AD} = 2.5$ m, $\theta_E = 0°$, $\theta_{A\#} = 208°$, $\theta_D = 147°$, $m_2' = 0.5$ kg, $\omega_2 = 10$ rad$/s$, $\alpha_2 = 10$ rad$/s^2$

6.6.3 Comparison Between the Results of Proposed and Gao Feng Methods

The results of shaking force and shaking moment by proposed method and Gao Feng method for Watt mechanism with two fixed points are shown in Tables 6.1 and 6.2.

The shaking forces in Watt mechanism with two fixed points are determined at intervals of 90°. At all positions better results are produced by proposed method. Shaking force of the mechanism is maximum 2,726.43 N, at 0°, and minimum 791.96 N, at 180° in the proposed method. The shaking force gradually decreases from maximum at 0° to minimum at 180° and again gradually increases to

Table 6.1 Shaking force comparison of Watt mechanism with two fixed points

Crank angle(deg)	Shaking force generated in proposed method (N)	Shaking force generated in Gao Feng's method (N)
0	2,726.43	15,285.24
90	1,840.32	14,399.16
180	791.96	13,350.82
270	923.45	13,482.31
360	2,726.43	15,285.22

Table 6.2 Shaking moment comparison of Watt mechanism with two fixed points

Crank angle(deg)	Shaking moment generated in proposed method $\times 10^5$ N m	Shaking moment generated in Gao Feng's method $\times 10^5$ N m
0	−468.22	−468.19
90	−5.23	−5.18
180	33.17	33.23
270	39.41	39.46
360	−468.25	−468.19

maximum at 360°. The shaking moment of Watt mechanism with two fixed points is maximum 468.2×10^5 N m, at 0°, and minimum -5.23×10^5 N m, at 90°. The shaking moment gradually decreases from 0° to 90° and again increases to maximum at 360°. It can be observed that shaking forces by proposed method are very much less at all intervals of crank angle than those by Gao Feng's method. As there is only one planetary gear 8 to be mounted on the base of the mechanism, there is a little improvement in the shaking moment balancing, but the shaking forces have been substantially reduced. Though the results of a numerical example are not available in the literature to make a comparison in Tables 6.1 and 6.2, the balanced mechanisms of both the proposed and Gao Feng methods can be compared construction-wise. It can be observed that the balanced mechanism of proposed method is constructively more efficient and compact and occupies less space.

6.7 Self-Balanced Slider-Crank Mechanism

In the two identical slider-crank mechanism shaking forces are automatically balanced as the movements of the two slider-crank mechanisms are opposite to each other, so it is called as self-balanced slider-crank system. These mechanical systems find successful applications in engines, agricultural machines, mills, and various automatic machines (Fig. 6.8).

6.7.1 Self-Balanced Slider-Crank System with an Imagined Articulation Dyad

Figure 6.9 shows a self-balanced slider-crank system with an imagined articulation dyad B′D′E, which forms a pantograph with the initial system. The similarity factor of the formed pantograph is $k = {l_{\mathrm{AD}}}/{l_{\mathrm{AB}}} = 1$ and $l_{\mathrm{BB'}} = l_{\mathrm{DD'}}, l_{\mathrm{B'D'}} = l_{\mathrm{AD}} + l_{\mathrm{AB}}$.

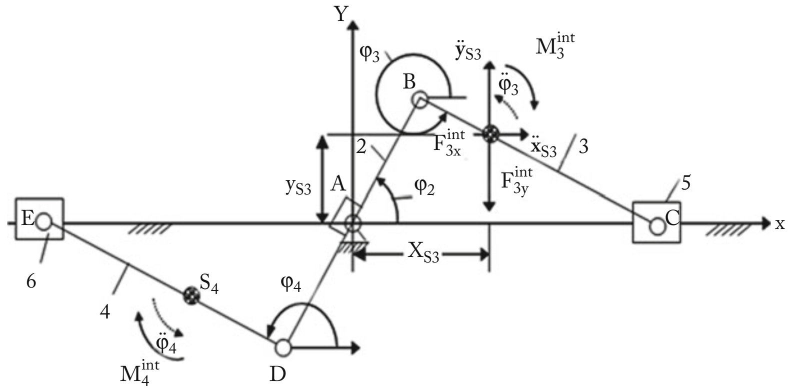

Fig. 6.8 Self-balanced slider-crank system

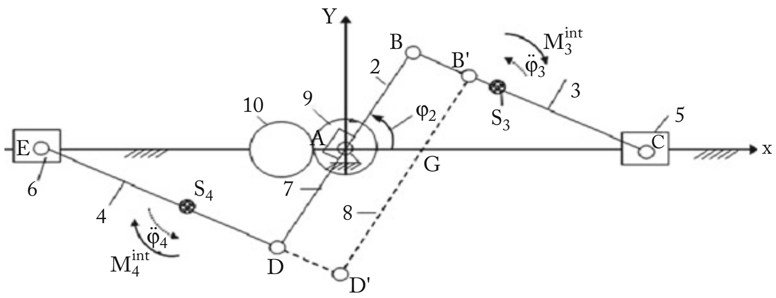

Fig. 6.9 Self-balanced slider-crank system with an imagined articulation dyad B′D′E

By substituting dynamically the mass m_3 of the connecting coupler 3 by point masses at the centers B, B′ and C and using the following condition

$$\begin{bmatrix} 1 & 1 & 1 \\ l_{BS_3} & -l_{CS_3} & l_{B'S_3} \\ {l_{BS_3}}^2 & {l_{CS_3}}^2 & {l_{B''S_3}}^2 \end{bmatrix} \begin{bmatrix} m_B \\ m_C \\ m_{B'} \end{bmatrix} = \begin{bmatrix} m_3 \\ 0 \\ I_{S_3} \end{bmatrix}$$

where $l_{BS_3}, l_{CS_3}, l_{B'S_3}$ are the distances of joint centers B, C, and B' from the centers of masses S_3 of the link 3.

I_{S_3} is the axial moment of inertia of link 3; we determine the value of the point masses

$$m_B = D_B/D_3; m_C = {}^{D_C}\big/_{D_3}\,; m_{B'} = D_{B'}/D_3 \tag{6.18}$$

where $D_B, D_C, D_{B'}, D_3$ are determinants of the third order obtained from the system of equations.

We now require imagined link B′D′ to be balanced about point G of the pantograph, i.e.,

$$m_{\mathrm{D}'} = m_{\mathrm{B}'} l_{\mathrm{B}'\mathrm{G}} / l_{\mathrm{D}'\mathrm{G}}$$

The concentrated point masses $m_{\mathrm{G},} m_{\mathrm{C}}, m_{\mathrm{E}}$ to be balanced about center A, i.e.,

$$m_{\mathrm{E}} = \left(m_{\mathrm{G}} l_{\mathrm{BB}'} + m_{\mathrm{C}} l_{\mathrm{BC}}\right) / l_{\mathrm{DE}}$$

where $l_{\mathrm{BB}',} l_{\mathrm{BC}}$ are the distances of joint centers B′, C from the joint center B, and l_{DE} is the distance of joint center D from the joint center E:

$$m_{\mathrm{G}} = m_{\mathrm{B}'} + m_{\mathrm{D}'} \tag{6.19}$$

Finally the concentrated point masses $m_{\mathrm{B}}, m_{\mathrm{D}}$ are also to be balanced about center A, i.e.,

$m_{\mathrm{D}} = m_{\mathrm{B}} l_{\mathrm{AB}} / l_{\mathrm{AD}}$. Thus we obtain the values of three concentrated point masses $m_{\mathrm{D}'}, m_{\mathrm{D}}, m_{\mathrm{E}}$ which allow the determination of the mass and inertia parameters of the connecting coupler 4

where $l^*_{\mathrm{DS}_4} = l_{\mathrm{DE}} - l_{\mathrm{ES}_4}; l_{\mathrm{D}'\mathrm{S}_4} = l_{\mathrm{D}'\mathrm{E}} - l_{\mathrm{ES}_4}$.

6.7.2 Shaking Moment Balancing of the Mechanism

The shaking moment transmitted to the frame by links 2 and 7 is calculated using the angular acceleration of link 2. The shaking moment transmitted to the frame by connecting rods 3 and 4 is calculated using angular acceleration of link 8, as their point masses are brought to link 8:

$$M_2^{\mathrm{int}} + M_7^{\mathrm{int}} = \left(I_{\mathrm{S}_2} + m_2 l^2_{\mathrm{AS}_2} + m_{\mathrm{B}} l^2_{\mathrm{AB}} + m_{\mathrm{D}} l^2_{\mathrm{AD}} + m_7 l^2_{\mathrm{AS}_7} + I_{\mathrm{S}_7}\right) \alpha_2$$

$$M_8^{\mathrm{int}} = \left(I_{\mathrm{S}_8} + m_8 l^2_{\mathrm{GS}_8} + m_{\mathrm{D}'} l^2_{\mathrm{D}'\mathrm{G}} + m_{\mathrm{B}'} l^2_{\mathrm{B}'\mathrm{G}}\right) \alpha_8 \tag{6.20}$$

Total shaking moment generated by the mechanism:

$$M^{\mathrm{int}} = M_2^{\mathrm{int}} + M_7^{\mathrm{int}} + M_8^{\mathrm{int}} \tag{6.21}$$

The shaking moment generated by the mechanism is balanced by addition of gear inertia counterweights 9 and 10.

For any mechanism with the given numerical values of link mass, length, mass moment of inertia, and radius of gyration, the shaking force and shaking moment can be calculated using the above equations. To balance the shaking moment generated by the mechanism geared inertia counterweights with the equal amount of inertia moment can be mounted on the frame of the mechanism.

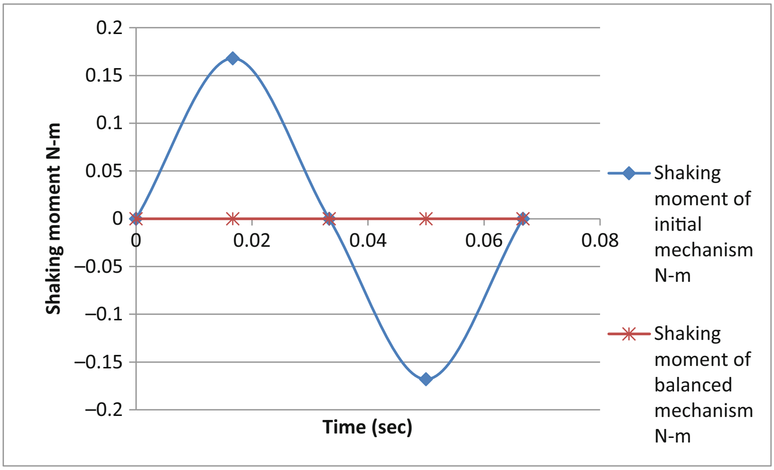

Fig. 6.10 Time vs. shaking moment

Numerical example:
The parameters of the self-balanced slider-crank system are the following:

$$l_{AB} = l_{AD} = 0.05m; l_{BC} = l_{DE} = 0.2m; l_{CS_3} = l_{ES_4} = 0.1m; m_3 = m_4 = 0.35 \text{ kg};$$

$$m_5 = m_6 = 2 \text{ kg}; I_{S_3} = I_{S_4} = 0.005 \text{ kg} - m^2; \omega_{AB} = 30\pi/s; \alpha_{AB} = 15 \text{ rad}/s^2;$$

$$m_2 = m_7 = 0.3 \text{ kg}; I_{S_2} = I_{S_7} = 0.003 \text{ kg}; l_{AS_2} = l_{AS_7} = 0.025m;$$

$$I_{S_8} = 0.006 \text{ kg} - m^2; m_8 = 0.6 \text{ kg}$$

Figure 6.10 shows the variations of the shaking moment of the initial mechanical system. For cancellation of the shaking moment it is necessary to redistribute the masses of the second connecting coupler. By dynamically substituting the mass m_3 of the connecting coupler 3 by point masses at centers B, B′, C and taking into account conditions $m_D'; m_E; m_D$, we calculate the mass and inertia parameters of the connecting coupler 4. Figure 6.10 illustrates the obtained results. So by mounting geared inertia counterweights the shaking moment is cancelled. The shaking moment of initial mechanism was +0.168 N m at 90° and −0.168 N m at 270°. It has been observed that the shaking moment of self-balanced slider-crank mechanism has been zero at all angular positions of the crank.

If the driving torque and time are plotted for both the unbalanced and balanced linkages then it can be observed that the driving torque is slightly higher for the balanced mechanism, as the inertia elements are mounted on the base of the mechanism for shaking moment balancing. On the other hand the performance of the balanced mechanism improves considerably and it also increases the fatigue life of the mechanism.

6.8 Eight-Bar Mechanism with Three Fixed Points and Three Ternary Links (Mechanism with Low Degree of Complexity)

The eight-bar mechanism with three fixed points and three ternary links shown in Fig. 6.11 has eight links, four ternary links and four binary links; one of the ternary links is fixed. It has ten binary joints. The degree of freedom of this mechanism is one. It is a mechanism with low degree of complexity as the path curvature of motion transfer point “E” is not known. Links 2, 4, 6, and 7 are binary links and 1, 3, 5, and 8 are four ternary links; among them link 8 is fixed link. The links 2, 4, 6, and 7 are not directly connected to the frame. The geared inertia counterweights required to balance the shaking moments generated by links 2, 4, 6, and 7 are mounted on the frame of the mechanism by kinematically linking geared inertia counterweights and the corresponding links by links of known mass and center of mass. The balanced eight-bar mechanism with three fixed points and three ternary links is shown in Fig. 6.12.

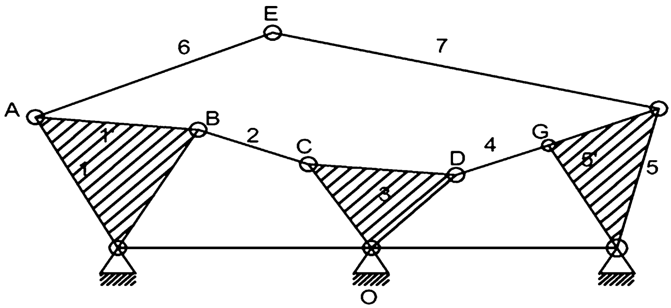

Fig. 6.11 Eight-bar mechanism with three fixed points and three ternary links

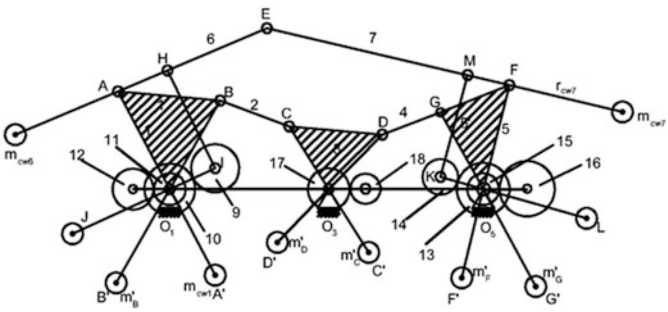

Fig. 6.12 Balanced eight-bar mechanism with three fixed points and three ternary links

6.8.1 Shaking Force Balancing of the Mechanism

Here link 1 is dynamically replaced by three point masses m_{O_1}, m_{A_1}, m_{B_1} by using the following conditions:

$$\begin{bmatrix} 1 & 1 & 1 \\ l_{O_1} e^{i\theta_{O_1}} & l_A e^{i\theta_A} & l_B e^{i\theta_B} \\ {l_{O_1}}^2 & {l_A}^2 & {l_B}^2 \end{bmatrix} \begin{bmatrix} m_{O_1} \\ m_{A_1} \\ m_{B_1} \end{bmatrix} = \begin{bmatrix} m_1 \\ 0 \\ I_{S_1} \end{bmatrix}$$

$$m_{O_1} = \frac{D_{O_1}}{D_1},\ m_{A_1} = \frac{D_{A_1}}{D_1},\ m_{B_1} = \frac{D_{B_1}}{D_1} \tag{6.22}$$

where l_{O_1}, l_A, l_B are the moduli of radius vectors of corresponding points.

θ_{O_1}, θ_A, θ_B, are the angular positions of radius vectors.

m_1 is the mass of link 1.

I_{S_1} is the mass moment of inertia of link 1 about its center of mass.

D_{O_1}, D_{A_1}, D_{B_1} and D_1 are the third-order determinants obtained from the system of equations.

Link 2 is statically replaced by two point masses m_{B_2} and m_{C_2}:

$$m_{B_2} = \frac{m_2\, l_{CS_2}}{l_{BC}};\ m_{C_2} = \frac{m_2\, l_{BS_2}}{l_{BC}}$$

Changed mass moment of inertia $I'^{*}_{S_2} = I_{S_2} - m_2 l_{CS_2} l_{BS_2}$.

For link 6 to be dynamically replaced by two point masses m_{E_6} and m_{P_2} the condition to be satisfied is

$$k_6{}^2 = l_{ES_6} l_{P_6S_6}$$

where k_6 is the radius of gyration of link 6 about its center of mass.

l_{ES_6} is arbitrarily fixed and $l_{P_6S_6}$ is obtained from the above condition:

$$m_{E_6} = \frac{m_6\, l_{P_6S_6}}{(l_{ES_6} + l_{P_6S_6})};\ m_{P_6} = \frac{m_6\, l_{ES_6}}{(l_{ES_6} + l_{P_6S_6})}$$

After dynamically replacing link 6 by two point masses, it is kinematically connected to its corresponding gear 9 by link 1′ and it is statically replaced by two point masses m_H and m_I:

$$m_H = \frac{m'_1 l_{IS'_1}}{l_{HI}};\ m_I = \frac{m'_1 l_{HS'_1}}{l_{HI}}$$

Changed mass moment of inertia $I'^{*}_{S_1} = I'_{S_1} - m'_1 l'_{\mathrm{IS}_1} l'_{\mathrm{H}S_1}$

where I'_{S_1} is the original mass moment of inertia link 1′.

Counterweight m_{CW_6} against link 6 is calculated by using the formula

$$m_{\mathrm{CW}_6} = \frac{(m_6 l_{\mathrm{AS}_6} + m_{\mathrm{H}} l_{\mathrm{AH}})}{r_{\mathrm{CW}_6}} \tag{6.23}$$

where $r_{\mathrm{CW}_6} = l_{\mathrm{P}_6\mathrm{S}_6} - l_{\mathrm{AS}_6}$ is the radius of rotation of counterweight m_{CW_6}.

Counterweight against point mass B can be obtained as

$$m_{\mathrm{B}'} = \frac{(m_{\mathrm{B1}} + m_{\mathrm{B2}})\, l_{\mathrm{BO}_1}}{l_{\mathrm{B}'\mathrm{O}_1}}$$

Counterweight against link 1 is calculated by using the formula

$$m_{\mathrm{CW}_1} = \frac{(m_{\mathrm{CW}_6} + m_{\mathrm{H}} + m_6)}{r_{\mathrm{CW}_1}} \tag{6.24}$$

where r_{CW_1} is radius of rotation of counterweight against link 1 which is arbitrarily taken.

For link 3 to be dynamically replaced by three point masses $m_{\mathrm{C}_3},\ m_{\mathrm{D}_3}, m_{\mathrm{O}_3}$ the conditions to be satisfied are

$$\begin{bmatrix} 1 & 1 & 1 \\ l_{\mathrm{O}_3} \mathrm{e}^{i\theta_{\mathrm{O}_3}} & l_{\mathrm{C}} \mathrm{e}^{i\theta_{\mathrm{C}}} & l_{\mathrm{D}} \mathrm{e}^{i\theta_{\mathrm{D}}} \\ l_{\mathrm{O}_3}{}^2 & l_{\mathrm{C}}{}^2 & l_{\mathrm{D}}{}^2 \end{bmatrix} \begin{bmatrix} m_{\mathrm{O}_3} \\ m_{\mathrm{C}_3} \\ m_{\mathrm{D}_3} \end{bmatrix} = \begin{bmatrix} m_3 \\ 0 \\ I_{\mathrm{S}_3} \end{bmatrix}$$

$$m_{\mathrm{O}_3} = \frac{D_{\mathrm{O}_3}}{D_3},\ m_{\mathrm{C}_3} = \frac{D_{\mathrm{C}_3}}{D_3},\ m_{\mathrm{D}_3} = \frac{D_{\mathrm{D}_3}}{D_3} \tag{6.25}$$

where $l_{\mathrm{O}_3},\ l_{\mathrm{C}},\ l_{\mathrm{D}}$ are the moduli of radius vectors of corresponding points.

$\theta_{\mathrm{O}_3},\ \theta_{\mathrm{C}},\ \theta_{\mathrm{D}}$ are the angular positions of radius vectors.

m_3 is the mass of link 3.

I_{S_3} is the mass moment of inertia of link 3 about its center of mass.

$D_{\mathrm{O}_3},\ D_{\mathrm{C}_3},\ D_{\mathrm{D}_3},\ D_3$ are the third-order determinants obtained from the system of equations.

Counterweight against point C can be obtained as

$$m_{\mathrm{C}'} = \frac{(m_{\mathrm{C}_2} + m_{\mathrm{C}_3})\, l_{\mathrm{O}_3 c}}{l_{\mathrm{C}'\mathrm{O}_3}}$$

For link 4 to be statically replaced by two point masses m_{D_4} and m_{G_4} the conditions to be satisfied are

$$m_{\mathrm{D}_4} = \frac{m_4 l_{\mathrm{GS}_4}}{l_{\mathrm{DG}}}; \; m_{\mathrm{G}_4} = \frac{m_4 l_{\mathrm{DS}_4}}{l_{\mathrm{DG}}}$$

Changed mass moment of inertia of link 4 can be obtained as $I'^{*}_{\mathrm{S}_4} = I_{\mathrm{S}_4} - m_4 l_{\mathrm{DS}_4} l_{\mathrm{GS}_4}$

Counterweight against point D can be obtained as

$$m_{\mathrm{D}'} = \frac{(m_{\mathrm{D}_3} + m_{\mathrm{D}_4})\, l_{\mathrm{DO}_3}}{l_{\mathrm{D}'\mathrm{O}_3}}$$

For link 5 to be dynamically replaced by three point masses $m_{\mathrm{O}_5},\ m_{\mathrm{G}_5}, m_{\mathrm{F}_5}$ the conditions to be satisfied are

$$\begin{bmatrix} 1 & 1 & 1 \\ l_{\mathrm{O}_5} \mathrm{e}^{i\theta_{\mathrm{O}_5}} & l_{\mathrm{G}} \mathrm{e}^{i\theta_{\mathrm{G}}} & l_{\mathrm{F}} \mathrm{e}^{i\theta_{\mathrm{F}}} \\ {l_{\mathrm{O}_5}}^2 & {l_{\mathrm{G}}}^2 & {l_{\mathrm{F}}}^2 \end{bmatrix} \begin{bmatrix} m_{\mathrm{O}_5} \\ m_{\mathrm{G}_5} \\ m_{\mathrm{F}_5} \end{bmatrix} = \begin{bmatrix} m_5 \\ 0 \\ I_{\mathrm{S}_5} \end{bmatrix}$$

$$m_{\mathrm{O}_5} = \frac{D_{\mathrm{O}_5}}{D_5},\ m_{\mathrm{G}_5} = \frac{D_{\mathrm{G}_5}}{D_5},\ m_{\mathrm{F}_5} = \frac{D_{\mathrm{F}_5}}{D_5} \tag{6.26}$$

where $l_{\mathrm{O}_5},\ l_{\mathrm{G}},\ l_{\mathrm{F}}$ are the moduli of radius vectors of corresponding points.

$\theta_{\mathrm{O}_5},\ \theta_{\mathrm{G}},\ \theta_{\mathrm{F}}$ are the angular positions of radius vectors.

m_5 is the mass of link 5.

I_{S_5} is the mass moment of inertia of link 5 about its center of mass.

$D_{\mathrm{O}_5},\ D_{\mathrm{G}_5},\ D_{\mathrm{F}_5},\ D_5$ are the third-order determinants obtained from the system of equations.

For link 7 to be dynamically replaced by two point masses m_{E_7} and m_{P_7} the condition to be satisfied is

$${k_7}^2 = l_{\mathrm{FS}_7} l_{\mathrm{P}_7\mathrm{S}_7}$$

where k_7 is the radius of gyration of link 7 about its center of mass.

l_{FS_7} is the arbitrarily fixed and $l_{\mathrm{P}_7\mathrm{S}_7}$ is obtained from the above condition:

$$m_{\mathrm{E}_7} = \frac{m_7 l_{\mathrm{P}_7\mathrm{S}_7}}{l_{\mathrm{FS}_7} + l_{\mathrm{P}_7\mathrm{S}_7}}; \; m_{\mathrm{P}_7} = \frac{m_7 l_{\mathrm{FS}_7}}{l_{\mathrm{FS}_7} + l_{\mathrm{P}_7\mathrm{S}_7}}$$

After link 7 is dynamically replaced by two point masses, it is kinematically linked to its corresponding gear 13 by link 5′ and moreover link 5′ is statically replaced by two point masses m_{M} and m_{K}:

$$m_{\mathrm{M}} = \frac{m'_5 l_{\mathrm{KS}'_5}}{l_{\mathrm{KM}}}; \; m_{\mathrm{K}} = \frac{m'_5 l_{\mathrm{MS}'_5}}{l_{\mathrm{KM}}}$$

Counterweight m_{CW_7} against link 7 is calculated using the formula

$$m_{\mathrm{CW}_7} = \frac{(m_{\mathrm{M}} l_{\mathrm{FM}} + m_7 l_{\mathrm{FS}_7})}{r_{\mathrm{CW}_7}}$$

where $r_{\mathrm{CW}_7} = l_{\mathrm{P}_7\mathrm{S}_7} - l_{\mathrm{FS}_7}$ is the radius of rotation counterweight.

Counterweight against point G can be obtained as

$$m'_{\mathrm{G}} = \frac{(m_{\mathrm{G}_4} + m_{\mathrm{G}_5})\, l_{\mathrm{O}_5\mathrm{G}}}{l_{\mathrm{O}_5\mathrm{G}'}}$$

where $l_{\mathrm{O}_5\mathrm{G}'}$ is arbitrarily taken.

Counterweight against point F can be obtained as

$$m'_{\mathrm{F}} = \frac{(m_7 + m_{\mathrm{CW}_7} +)\, l_{\mathrm{O}_5\mathrm{F}}}{l_{\mathrm{O}_5\mathrm{F}'}}$$

where $l_{\mathrm{O}_5\mathrm{F}'}$ is arbitrarily taken

6.8.2 Shaking Moment Balancing of the Mechanism

The shaking moments of links 1, 3, 5, 6, and 7 are given as follows:

The shaking moment generated by the linkage is determined by the sum

$$M^{\mathrm{int}} = M_1^{\mathrm{int}} + M_6^{\mathrm{int}} + M_3^{\mathrm{int}} + M_5^{\mathrm{int}} + M_7^{\mathrm{int}} \tag{6.27}$$

$$M_1^{\mathrm{int}} = \left(I_{\mathrm{S}_1} + m_1 l_{\mathrm{O}_1\mathrm{S}_1}^2 + I'^{*}_{\mathrm{S}_1} + m'_1 l_{\mathrm{IS}'_1}^2 + m_{\mathrm{B}2} l_{\mathrm{O}_1\mathrm{B}}^2 + m'_{\mathrm{B}} l_{\mathrm{O}_1\mathrm{B}'}^2 + m_{\mathrm{CW}_1} r_{\mathrm{CW}_1}^2\right) \alpha_1$$

$$M_6^{\mathrm{int}} = \left(I_{\mathrm{S}_6} + m_{\mathrm{CW}_6} r_{\mathrm{CW}_6}^2 + m_6 l_{\mathrm{AS}_6}^2 + m_{\mathrm{H}} l_{\mathrm{AH}}^2 + 2 m_{\mathrm{J}} l_{\mathrm{O}_1\mathrm{J}}^2\right) \alpha_6$$

$$M_3^{\mathrm{int}} = \left(I_{\mathrm{S}_3} + m_3 l_{\mathrm{O}_3\mathrm{S}_3}^2 + m_{\mathrm{D}_4} l_{\mathrm{O}_3\mathrm{D}}^2 + m'_{\mathrm{D}} l_{\mathrm{O}_3\mathrm{D}'}^2 + m'_{\mathrm{C}} l_{\mathrm{O}_3\mathrm{C}'}^2\right) \alpha_3$$

$$M_5^{\mathrm{int}} = \left(I_{\mathrm{S}_5} + m_5 l_{\mathrm{O}_5\mathrm{S}_5}^2 + m_{\mathrm{G}4} l_{o5\mathrm{G}}^2 + m'_{\mathrm{G}} l_{\mathrm{O}_5\mathrm{G}'}^2 + m'_{\mathrm{F}} l_{\mathrm{O}_5\mathrm{F}'}^2 + + m'_5 l_{\mathrm{KS}'_5}^2\right) \alpha_5$$

$$M_7^{\mathrm{int}} = \left(I_{\mathrm{S}_7} + m_7 l_{\mathrm{FS}_7}^2 + m_{\mathrm{CW}_7} r_{\mathrm{CW}_7}^2 + m_{\mathrm{M}} l_{\mathrm{FM}}^2 + 2 m_{\mathrm{K}} l_{\mathrm{O}_5\mathrm{K}}^2\right) \alpha_7$$

where M_1^{int}, M_3^{int}, M_5^{int}, M_6^{int}, M_7^{int} are the shaking moments generated by links 1, 3, 5, 6, and 7, respectively.

I_{S_1}, I_{S_3}, I_{S_5}, I_{S_6}, and I_{S_7} are the mass moment of inertias of links 1, 3, 5, 6, and 7, respectively.

α_1, α_3, α_5, α_6 and α_7 are the angular accelerations of links 1, 3, 5, 6, and 7, respectively.

For shaking moment balancing ten geared inertia counterweights are used, four at O_5, two at O_3, and four at O_1.

6.9 Summary

Self-balanced slider-crank mechanism has been studied with numerical example and it is observed that shaking moment is completely balanced. Shaking force and shaking moment balancing expressions are developed for eight-bar mechanism with three fixed points and three ternary links.

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