Chapter 5
Design of Reactionless Mechanisms with Counter-Rotary Counter-Masses

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Abstract In this chapter a new method to find the force and moment balancing conditions based on Natural Coordinates is introduced. The method is simple and can be highly automated, it is very prone to be used in combination with a system for the manipulation of symbolic expressions. These conditions can be interpreted and used for the creation of dynamic balanced linkages by design. The application of the method is demonstrated through the dynamic balancing of a simple pendulum (open-loop linkage) and a general four-bar mechanism (closed-loop linkage), particularly by the design of counter-rotary counter-masses applying optimization. The resulting designs are presented and their virtual prototypes simulated using a general multibody dynamics simulation software (ADAMS), specifying the resulting geometry (dimensions), shaking force, shaking moment, and driving torque.

Keywords Dynamic balancing • Counter rotary counter-masses • Optimization • Planar mechanisms • Natural coordinates

5.1 Introduction

Force and moment balancing (dynamic balancing) of rigid body linkages with constant mass links is a traditional but still very active area of research in mechanical engineering. Its benefits are well known as machine vibrations often occur due to dynamic unbalance inducing noise, wear, fatigue problems [1], limiting the full potential of many machines. Mechanisms that are dynamically balanced do not transmit vibrations to the base, a useful property in hand tools, in objects and vehicles moving in free space, and in robotics.

But dynamic balancing of linkages has some difficulties and drawbacks. First finding the balancing conditions may be complicated [2] and second a substantial

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amount of mass and inertia must generally be added [3, 4]. A complete overview of dynamic balancing techniques and methods can be found in [5–7].

An increment in mass or inertia implies more power to drive the mechanism, so research has therefore been focused on reducing these disadvantages. A way to reduce the necessary mass and inertia to balance the mechanisms can be to use the counter-masses, necessary for the force balance of the linkage, also for balancing the moment. This principle, compared with other balancing principles, has shown effective in the reduction of the additional mass and inertia, see [4, 8], and used effectively to synthesize different dynamic balanced mechanisms using the double pendulum as building element [9].

In this work a completely new general method to find the dynamic balancing conditions, based on the use of Natural Coordinates [10], is introduced. The method is direct and very easy to automate, and can be used to obtain the shaking force and the shaking moment balancing conditions for the linkages in the plane and in space, although at this time is presented only for planar mechanisms. Once the balancing conditions are found, these are used to the effective design of reactionless mechanisms with counter-rotary counter-masses.

The chapter is organized as follows: in Sect. 5.2 the new method is introduced, showing how Natural Coordinates are very useful to directly obtain the balancing conditions. In Sects. 5.3 and 5.4, the application of the method to the dynamic balancing of a single pendulum and of a four-bar mechanism are presented. In Sect. 5.5 some numerical examples are solved to obtain specific mechanisms designs, and the results obtained from dynamic simulations made with ADAMS are presented. Finally some concluding remarks are made in Sect. 5.6.

5.2 Balancing Conditions Using Natural Coordinates

For the effective design of reactionless mechanisms with counter-rotary counter-masses it is necessary first to obtain the dynamic balancing conditions. In this section a new method based on Natural Coordinates is presented. The method is straightforward and can be easily automated, mainly it has the advantage of being suitable for the application of a computer algebra system.

A dynamic balanced mechanism must be completely force and moment balanced. In fact the mechanism that is not balanced by force first, cannot be balanced by moments.

A mechanism is force balanced if its linear momentum, $\mathbf{l}_m$, is conserved. This condition in general can be expressed as:

$$\mathbf{l}_m = \sum_{i=1}^{n} \mathbf{l}_i = cnt. \tag{5.1}$$

where n is the number of total moving elements in the linkage.

When working in reference-point coordinates (Cartesian Coordinates) the linear momentum of body i can be calculated as:

$$\mathbf{l}_i = \sum_{i=1}^{n} m_i \mathbf{v}_i \tag{5.2}$$

where $\mathbf{v}_i$ and m_i are the velocity of the center of mass and the mass of the ith moving body, respectively.

On the other hand, a mechanism is moment balanced if its angular momentum, $\mathbf{h}_m$, is conserved. This condition is expressed by:

$$\mathbf{h}_m = \sum_{i=1}^{n} \mathbf{r}_i \times (\mathbf{l}_i) + \mathbf{h}_i = cnt. \tag{5.3}$$

where $\mathbf{r}_i$ is the position vector of the center of mass. Again when working in reference-point coordinates the angular momentum of body i can be calculated as:

$$\mathbf{h}_i = \mathbf{I}_i \boldsymbol{\omega}_i \tag{5.4}$$

where $\mathbf{I}_i$ and $\boldsymbol{\omega}_i$ are the inertia tensor with respect to the center of mass and the angular velocity of the ith body, respectively.

So it is necessary to find equivalent expressions to Eqs. (5.2) and (5.4) in natural coordinates to calculate the linear and angular momentum of the mechanism. In the next subsections these expressions are developed.

5.2.1 Linear Momentum of a Body Using Natural Coordinates

Equations (5.1) and (5.3) are in general well recognized when using reference-point coordinates (Cartesian Coordinates), but in this work we are interested on its form in Natural Coordinates. So the first step to find this form is to develop each part of the equations according to our goal.

When dealing with mechanical systems in the plain, Natural coordinates introduce a set of points to define a body, the basic points, see [10] for a detailed explanation. So a body can be modeled in Natural Coordinates with a pair of points, i and j, as seen in Fig. 5.1. In this figure it can be noted an inertial fixed reference frame **XY**, a local reference frame **xy** attached to the moving body at the basic point i, $(0, 0)$. It is also noted that the second basic point j has its position in local coordinates at $(l, 0)$, and that the center of mass of the body, point g, at (x, y).

Using this pair of points and considering that the body has a total mass concentrated at g equal to m, and a moment of inertia I_i with respect to the origin of the local reference frame (point i), the constant mass matrix of a body can be expressed as (see [10]):

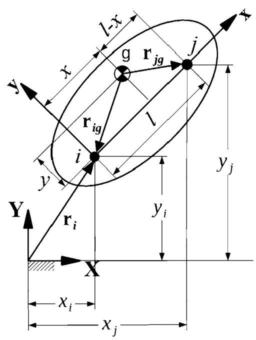

Fig. 5.1 A general model of a body using Natural Coordinates. Two basic points, *i* and *j*, with a local moving reference frame attached to the body at point *i*, the origin

$$\mathbf{M} = \begin{bmatrix} m - \frac{2mx}{l} + \frac{I_i}{l^2} & 0 & \frac{mx}{l} - \frac{I_i}{l^2} & -\frac{my}{l} \\ 0 & m - \frac{2mx}{l} + \frac{I_i}{l^2} & \frac{my}{l} & \frac{mx}{l} - \frac{I_i}{l^2} \\ \frac{mx}{l} - \frac{I_i}{l^2} & \frac{my}{l} & \frac{I_i}{l^2} & 0 \\ -\frac{my}{l} & \frac{mx}{l} - \frac{I_i}{l^2} & 0 & \frac{I_i}{l^2} \end{bmatrix} \tag{5.5}$$

The pair of basic points introduce a vector of four coordinates represented by $\mathbf{q}_k$, the positions, and its time derivative $\dot{\mathbf{q}}$, the velocities:

$$\mathbf{q} = \left[x_i \; y_i \; x_j \; y_j \right]^T \tag{5.6}$$

$$\dot{\mathbf{q}} = \left[\dot{x}_i \; \dot{y}_i \; \dot{x}_j \; \dot{y}_j \right]^T \tag{5.7}$$

So it is possible to calculate a set of the linear momentum vectors associated with the basic points in the body as:

$$\mathbf{M}\dot{\mathbf{q}} = \begin{bmatrix} m - \frac{2mx}{l} + \frac{I_i}{l^2} & 0 & \frac{mx}{l} - \frac{I_i}{l^2} & -\frac{my}{l} \\ 0 & m - \frac{2mx}{l} + \frac{I_i}{l^2} & \frac{my}{l} & \frac{mx}{l} - \frac{I_i}{l^2} \\ \frac{mx}{l} - \frac{I_i}{l^2} & \frac{my}{l} & \frac{I_i}{l^2} & 0 \\ -\frac{my}{l} & \frac{mx}{l} - \frac{I_i}{l^2} & 0 & \frac{I_i}{l^2} \end{bmatrix} \begin{bmatrix} \dot{x}_i \\ \dot{y}_i \\ \dot{x}_j \\ \dot{y}_j \end{bmatrix} = \begin{bmatrix} \mathbf{l}_i \\ \mathbf{l}_j \end{bmatrix} \tag{5.8}$$

where $\mathbf{l}_i$ and $\mathbf{l}_j$ are the linear momentum associated with the points *i* and *j*, respectively. They can be expressed as:

$$\mathbf{l}_i = \begin{bmatrix} \left(\frac{I_i}{l^2} - \frac{2mx}{l} + m\right)\dot{x}_i + \left(\frac{mx}{l} - \frac{I_i}{l^2}\right)\dot{x}_j - \frac{my}{l}\dot{y}_j \\ \left(\frac{I_i}{l^2} - \frac{2mx}{l} + m\right)\dot{y}_i + \frac{my}{l}\dot{x}_j + \left(\frac{mx}{l} - \frac{I_i}{l^2}\right)\dot{y}_j \end{bmatrix} \tag{5.9}$$

$$\mathbf{l}_j = \begin{bmatrix} \left(\frac{mx}{l} - \frac{I_i}{l^2}\right)\dot{x}_i + \frac{my}{l}\dot{y}_i + \left(\frac{I_i}{l^2}\right)\dot{x}_j \\ -\frac{my}{l}\dot{x}_i + \left(\frac{mx}{l} - \frac{I_i}{l^2}\right)\dot{y}_i + \left(\frac{I_i}{l^2}\right)\dot{y}_j \end{bmatrix} \tag{5.10}$$

Then the total linear momentum of the body is:

$$\mathbf{l} = \mathbf{l}_i + \mathbf{l}_j = \begin{bmatrix} \left(m - \frac{mx}{l}\right)\dot{x}_i + \left(\frac{my}{l}\right)\dot{y}_i + \left(\frac{mx}{l}\right)\dot{x}_j - \left(\frac{my}{l}\right)\dot{y}_j \\ \left(\frac{my}{l}\right)\dot{x}_i + \left(m - \frac{mx}{l}\right)\dot{y}_i + \left(\frac{my}{l}\right)\dot{x}_j - \left(\frac{mx}{l}\right)\dot{y}_j \end{bmatrix} \tag{5.11}$$

5.2.2 Angular Momentum of a Body Using Natural Coordinates

The angular momentum of the body, represented by a pair of masses on points i and j, with respect to its center of mass g can be calculated as:

$$\mathbf{h}_g = \mathbf{r}_{ig} \times \mathbf{l}_i + \mathbf{r}_{jg} \times \mathbf{l}_j \tag{5.12}$$

where $\mathbf{r}_{ig} = [-x \;\; -y]^T$ and $\mathbf{r}_{jg} = [(x-l) \;\; -y]^T$ are the position vector of points i and j with respect to the center of mass of the body expressed in the global fixed reference frame, and can be calculated by:

$$\mathbf{r}_{ig} = \mathbf{A}\bar{\mathbf{r}}_{ig}; \;\; \mathbf{r}_{jg} = \mathbf{A}\bar{\mathbf{r}}_{jg} \tag{5.13}$$

where $\bar{\mathbf{r}}_{ig}$ and $\bar{\mathbf{r}}_{jg}$ are the position vector of i and j expressed in local coordinates, while $\mathbf{A}$ is the rotation matrix:

$$\mathbf{A} = \frac{1}{l}\begin{bmatrix} (x_j - x_i) & (y_i - y_j) \\ (y_j - y_i) & (x_j - x_i) \end{bmatrix} \tag{5.14}$$

So the general form of the angular momentum of the body with respect to the global fixed reference frame can be calculated as:

$$\mathbf{h} = \mathbf{r}_g \times \mathbf{l} + \mathbf{h}_g \tag{5.15}$$

where $\mathbf{r}_g = \mathbf{r}_i - \mathbf{r}_{ig}$.

5.3 Dynamic Balancing of a Single Pendulum

5.3.1 Linear and Angular Momentum

As an example of the application of the previous developed equations consider a general single pendulum, Fig. 5.2, rotating at constant angular velocity ω.

In this case points A and B can be identified as basic points, so the linear momentum of the system can be obtained substituting the corresponding values in Eq. (5.11):

$$\mathbf{l} = \mathbf{l}_A + \mathbf{l}_B = \begin{bmatrix} \left(m - \frac{mx}{l}\right)\dot{x}_A + \left(\frac{my}{l}\right)\dot{y}_A + \left(\frac{mx}{l}\right)\dot{x}_B - \left(\frac{my}{l}\right)\dot{y}_B \\ \left(\frac{my}{l}\right)\dot{x}_A + \left(m - \frac{mx}{l}\right)\dot{y}_A + \left(\frac{my}{l}\right)\dot{x}_B - \left(\frac{mx}{l}\right)\dot{y}_B \end{bmatrix} \tag{5.16}$$

But the point A is fixed, so its velocity is zero. Substituting this result in the previous equation the linear momentum of the single pendulum expressed in natural coordinates can be obtained:

$$\mathbf{l} = \begin{bmatrix} \left(\frac{mx}{l}\right)\dot{x}_B - \left(\frac{my}{l}\right)\dot{y}_B \\ \left(\frac{my}{l}\right)\dot{x}_B - \left(\frac{mx}{l}\right)\dot{y}_B \end{bmatrix} \tag{5.17}$$

This result clearly indicates that to have a constant (invariant) linear momentum for the pendulum, the location of the center of mass must be the origin of the local reference system, point A, that coincides with the origin of the global fixed coordinate system. This means $(x, y) = (0, 0)$. In practice it is necessary to add a counterweight.

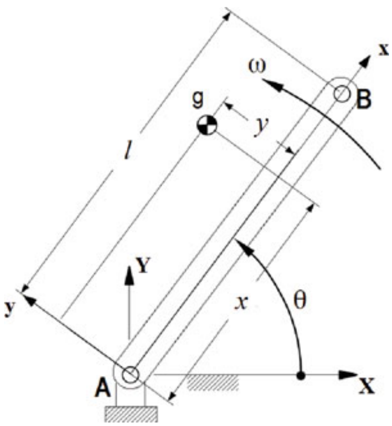

Fig. 5.2 A general model of a single pendulum rotating at constant angular velocity

On the other hand, the angular momentum of the pendulum can be obtained by substituting the corresponding values in Eq. (5.15), considering that the point A is fixed, so finally obtaining:

$$\mathbf{h} = \frac{I_A}{l^2}\left(\dot{y}_B\, x_B - \dot{x}_B\, y_B\right) \tag{5.18}$$

By substituting the corresponding values of the coordinates of point B and its derivatives:

$$\begin{aligned} x_B &= l\,\cos(\theta) \\ y_B &= l\,\sin(\theta) \\ \dot{x}_B &= -\omega\, l\,\sin(\theta) \\ \dot{y}_B &= \omega\, l\,\cos(\theta) \end{aligned}$$

in Eq. (5.18) it is possible to obtain the angular momentum of the single pendulum expressed in reference-point coordinates. For example, in the case of a uniform bar with its center of mass at the middle of its length:

$$\mathbf{h} = \omega\, I_A \tag{5.19}$$

but $I_A = I_g + m\left(\frac{l}{2}\right)^2$ so:

$$\mathbf{h} = \omega\left[I_g + m\left(\frac{l}{2}\right)^2\right] \tag{5.20}$$

This last results can indicate that to obtain a moment balanced pendulum, it is necessary to add a counter-inertia moving with an opposite angular velocity.

It can also be seen that the equations of linear momentum and angular momentum in Natural Coordinates have very simple forms and, as should be expected, they are correct only if the positions and velocities used are consistent with the kinematic constraints of the system.

5.3.2 Dynamic Balancing

For the dynamic balancing of a single pendulum it is necessary the addition of a counterweight, Eq. (5.17), and the addition of a counter-inertia, Eq. (5.18). Both the counterweight and the counter-inertia can be added as single counter-rotary counter-mass that rotates with an opposite angular velocity with respect to the angular velocity of the pendulum, as can be seen in Fig. 5.3. In Fig. 5.3, body one is the extended bar that works as the pendulum, defined by two basic points, A and B. The counter-rotary counter-mass used to dynamic balance the pendulum is the second

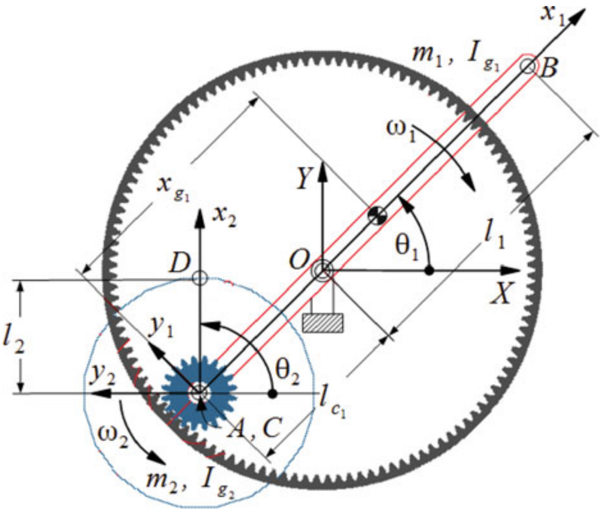

Fig. 5.3 A single pendulum, dynamic balanced by a counter-rotary counter-mass

body, defined by two basic points, C and D. It is important to note that points A and C have the same position and velocity, so strictly they are the same point.

Using Eq. (5.1) and applying Eq. (5.11) for each body, the linear momentum of the system in Fig. 5.3 can be calculated. The following result can be obtained, taking into account that in a general solution, the center of mass of the system must be in the line defined by points A and B ($y_{g_1} = 0$):

$$\mathbf{l}_m = \begin{bmatrix} \left(-\frac{m_1\, x_{g1}}{lc_1 + l_1} + m_2 + m_1\right) \dot{x}_A + \frac{m_1\, x_{g1}}{lc_1 + l_1}\, \dot{x}_B \\ \left(-\frac{m_1\, x_{g1}}{lc_1 + l_1} + m_2 + m_1\right) \dot{y}_A + \frac{m_1\, x_{g1}}{lc_1 + l_1}\, \dot{y}_B \end{bmatrix} \tag{5.21}$$

where x_{g_1} is the location of the center of mass of the bar with respect to the local reference frame of body one, with the origin at A.

As mentioned before, Eq. (5.21) will be correct as long as the correct values of the velocities are substituted. This velocities should come from the solution of velocities problem, this means to solve the time derivative of the constraint equations of the system (generally the closed-loop equations). Then in this case the corresponding values of the velocities are:

$$\begin{aligned} \dot{x}_A &= lc_1\, \omega_1\, \sin(\theta_1) \\ \dot{y}_A &= -lc_1\, \omega_1\, \cos(\theta_1) \\ \dot{x}_B &= -l_1\, \omega_1\, \sin(\theta_1) \\ \dot{y}_B &= l_1\, \omega_1\, \cos(\theta_1) \end{aligned} \tag{5.22}$$

so the linear momentum of this system is:

$$\mathbf{l}_m = \begin{bmatrix} \{\omega_1\ \sin(\theta_1)\}\left\{lc_1\ \left(-\frac{m_1\, x_{g1}}{lc_1+l_1} + m_2 + m_1\right) - \frac{l_1\, m_1\, x_{g1}}{lc_1+l_1}\right\} \\ -\{\omega_1\ \cos(\theta_1)\}\left\{lc_1\ \left(-\frac{m_1\, x_{g1}}{lc_1+l_1} + m_2 + m_1\right) - \frac{l_1\, m_1\, x_{g1}}{lc_1+l_1}\right\} \end{bmatrix} \tag{5.23}$$

which is invariant if:

$$lc_1\ \left(-\frac{m_1\, x_{g_1}}{lc_1 + l_1} + m_2 + m_1\right) - \frac{l_1\, m_1\, x_{g_1}}{lc_1 + l_1} = 0$$

meaning that the force balancing condition of the system is:

$$x_{g_1} = lc_1\left(\frac{m_2}{m_1} + 1\right) \tag{5.24}$$

On the other hand, using Eq. (5.3) and applying Eq. (5.18) for each body, the angular momentum of the system can be formulated. And substituting the force balancing condition, Eq. (5.24), the velocities at Eq. (5.22), and the corresponding positions:

$$\begin{aligned} x_A &= -lc_1\ \cos(\theta_1) \\ y_A &= -lc_1\ \sin(\theta_1) \\ x_B &= l_1\ \cos(\theta_1) \\ y_B &= l_1\ \sin(\theta_1) \end{aligned} \tag{5.25}$$

the final form of the angular momentum can be obtained as:

$$\mathbf{h}_m = \omega_1\left[I_1 - lc_1^2\,(m_1 + m_2)\right] + \omega_2\, I_2 \tag{5.26}$$

where I_1 is the inertia moment with respect to the local coordinate system of body one at point A, and I_2 is the inertia moment with respect to the local coordinate system of body two at point C. In this case:

$$\begin{aligned} I_1 &= I_{g_1} + lc_1^2\left(\frac{m_2^2}{m_1} + 2\,m_2 + m_1\right) \\ I_2 &= I_{g_2} \end{aligned}$$

where I_{g_1} and I_{g_2} are the moments of inertia with respect to the corresponding center on mass of each body. Additionally it is known that:

$$\omega_2 = -\left(\frac{lc_1}{R_2}\right)\omega_1 \tag{5.27}$$

so the angular momentum of the system is:

$$\mathbf{h}_m = \omega_1\left[I_{g_1} + lc_1^2\left(\frac{m_2^2}{m_1} + m_2\right) - \left(\frac{lc_1}{R_2}\right)I_{g2}\right] \tag{5.28}$$

Then to obtain an invariant angular momentum the following moment balancing condition must be maintained:

$$\left[I_{g_1} + lc_1^2\left(\frac{m_2^2}{m_1} + m_2\right) - \left(\frac{lc_1}{R_2}\right)I_{g2}\right] = 0 \tag{5.29}$$

In this way Eqs. (5.24) and (5.29) are the design conditions to obtain a dynamic balanced single pendulum.

5.4 Dynamic Balancing of a Four-Bar Mechanisms

Consider now a general four-bar mechanism as the one shown in Fig. 5.4, that is modeled in Natural Coordinates. In this case body one is defined with points A and B, body two is defined with points B and C, and body three is defined with points C and D. The origin of the local reference frames is also indicated, being at A, B, and C, respectively.

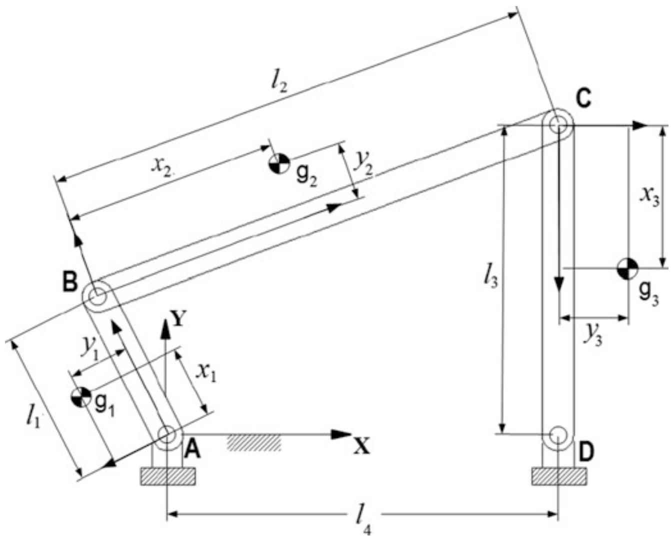

Fig. 5.4 A four-bar mechanism modeled in Natural Coordinates

The total linear momentum of this mechanism can be obtained by calculating the linear momentum of each body and then applying the Eq. (5.1) to get:

$$\mathbf{l}_m = \begin{bmatrix} a\,\dot{x}_B + b\,\dot{y}_B + c\,\dot{x}_C + d\,\dot{y}_C \\ \\ a\,\dot{y}_B + b\,\dot{x}_B + c\,\dot{y}_C + d\,\dot{x}_C \end{bmatrix} \tag{5.30}$$

where:

$$\begin{aligned} a &= \left(-\frac{m_2\,x_2}{l_2} + \frac{m_1\,x_1}{l_1} + m_2\right) \\ b &= \left(\frac{m_2\,y_2}{l_2} - \frac{m_1\,y_1}{l_1}\right) \\ c &= \left(-\frac{m_3\,x_3}{l_3} + \frac{m_2\,x_2}{l_2} + m_3\right) \\ d &= \left(\frac{m_3\,y_3}{l_3} - \frac{m_2\,y_2}{l_2}\right) \end{aligned} \tag{5.31}$$

The Eq. (5.31), equated to zero, are the force balancing conditions of the four-bar mechanism:

$$\begin{aligned} \left(-\frac{m_2\,x_2}{l_2} + \frac{m_1\,x_1}{l_1} + m_2\right) &= 0 \\ \left(\frac{m_2\,y_2}{l_2} - \frac{m_1\,y_1}{l_1}\right) &= 0 \\ \left(-\frac{m_3\,x_3}{l_3} + \frac{m_2\,x_2}{l_2} + m_3\right) &= 0 \\ \left(\frac{m_3\,y_3}{l_3} - \frac{m_2\,y_2}{l_2}\right) &= 0 \end{aligned} \tag{5.32}$$

This result is exactly the same as the one obtained in [11].

On the other hand, the angular momentum of the mechanisms can be obtained using Eq. (5.3), calculating previously the angular momentum of each body applying Eq. (5.15). In this case the angular momentum of the system is:

$$\begin{aligned} \mathbf{h}_m = {} & \dot{x}_B\,(-e\,y_B - h\,x_C + f\,y_C) \\ & + \dot{y}_B\,(e\,x_B + f\,x_C - h\,y_C) \\ & + \dot{x}_C\left(h\,x_B + f\,y_B - g\,y_C - \frac{l_4}{l_3} m_3\,y_3\right) \\ & + \dot{y}_C\left(e\,x_B + h\,y_B + g\,x_C - \frac{l_4}{l_3^2} I_3 + \frac{l_4}{l_3} m_3\,x_3\right) \end{aligned} \tag{5.33}$$

where

$$e = \left(\frac{I_2}{l_2^2} + \frac{I_1}{l_1^2} - \frac{2\,m_2\,x_2}{l_2} + m_2 \right) \tag{5.34}$$

$$f = \left(\frac{I_2}{l_2^2} - \frac{m_2\,x_2}{l_2} \right) \tag{5.35}$$

$$g = \left(\frac{I_3}{l_3^2} + \frac{I_2}{l_2^2} - \frac{2\,m_3\,x_3}{l_3} + m_3 \right) \tag{5.36}$$

$$h = \left(\frac{m_2\,y_2}{l_2} \right) \tag{5.37}$$

5.4.1 Dynamic Balancing of the Parallel Mechanism

A special case of a four-bar mechanisms is when the crank and the rocker have the same length and move parallel to each other, a parallel four-bar mechanisms, as the one in Fig. 5.5.

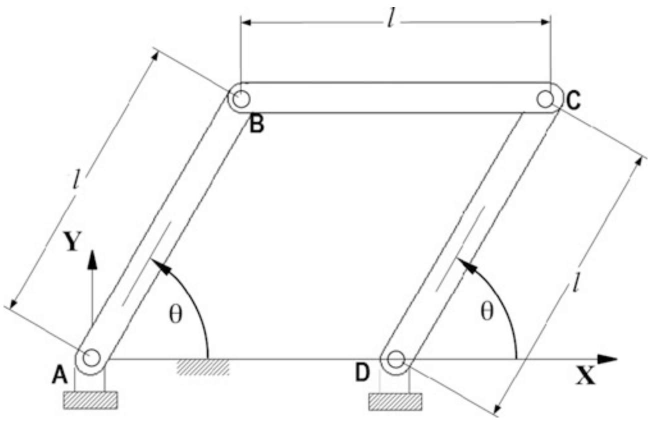

Fig. 5.5 A parallel four-bar mechanism

This is an inline[1] four-bar mechanism, and some of its geometric characteristics can be used to simplify force and moment balancing conditions to get an efficient design for the dynamic balanced system.

Considering that in a parallel inline four-bar mechanism $y_1 = y_2 = y_3 = 0$ and $l_1 = l_2 = l_3 = l$, the linear momentum equation, Eq. (5.30), can be expressed as:

$$\mathbf{l}_m = \begin{bmatrix} \dot{x}_B \left(-\frac{m_2 x_2}{l} + \frac{m_1 x_1}{l} + m_2\right) + \dot{x}_C \left(-\frac{m_3 x_3}{l} + \frac{m_2 x_2}{l} + m_3\right) \\ \dot{y}_B \left(-\frac{m_2 x_2}{l} + \frac{m_1 x_1}{l} + m_2\right) + \dot{y}_C \left(-\frac{m_3 x_3}{l} + \frac{m_2 x_2}{l} + m_3\right) \end{bmatrix} \tag{5.38}$$

but $\dot{x}_B = \dot{x}_C$ and $\dot{y}_B = \dot{y}_C$, so Eq. (5.38) can be expressed in terms of the velocity of point C as:

$$\mathbf{l}_m = \begin{bmatrix} \dot{x}_C \left(\frac{m_1 x_1}{l} - \frac{m_3 x_3}{l} + m_2 + m_3\right) \\ \dot{y}_C \left(\frac{m_1 x_1}{l} - \frac{m_3 x_3}{l} + m_2 + m_3\right) \end{bmatrix} \tag{5.39}$$

This equation indicates that this system can be force balanced by a single counterweight at body three (body one could be chosen in the same way), and the balancing condition is:

$$\frac{m_1 x_1}{l} - \frac{m_3 x_3}{l} + m_2 + m_3 = 0 \tag{5.40}$$

On the other hand, taking into account the angular momentum of a four-bar mechanism, Eq. (5.33), and considering that $y_1 = y_2 = y_3 = 0$ and $l_1 = l_2 = l_3 = l$, as in the case of the linear momentum, and that $x_C = x_B + l$, $y_C = y_B$, $\dot{x}_C = \dot{x}_B$, and $\dot{y}_C = \dot{y}_B$, the angular momentum of a parallel inline four-bar mechanism can be expressed as:

$$\begin{aligned} \mathbf{h}_m = & -\dot{x}_B\, y_B \left(\frac{I_3}{l^2} + \frac{I_1}{l^2} - \frac{2\, m_3\, x_3}{l} + m_3 + m_2\right) \\ & + \dot{y}_B\, x_B \left(\frac{I_3}{l^2} + \frac{I_1}{l^2} - \frac{2\, m_3\, x_3}{l} + m_3 + m_2\right) \\ & - m_3\, x_3 + m_2\, x_2 + l\, m_3 \end{aligned} \tag{5.41}$$

Equations (5.40) and (5.41) can help us to the design of different dynamic balanced parallel four-bar mechanisms. This can be done by assigning different values to x_1, x_2, and x_3, the location of the center of mass of each moving link.

[1]The term "inline" means that the centers of mass of the links must lie on the line connecting the pivots (which can be extended beyond the pivots). The links need not be symmetrical in any way [12].

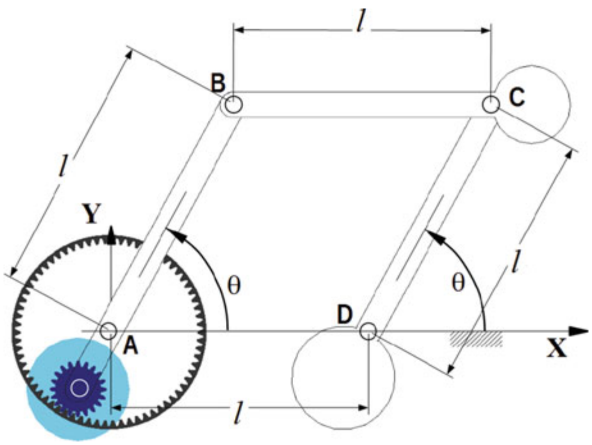

Fig. 5.6 A proposed design of an inline parallel four-bar mechanism with two counter-masses and one counter-rotary counter-mass

Consider first a mechanism with a counter-rotary counter-mass at the crank, a counterweight at the coupler, and a counterweight at the rocker. Figure 5.6 is a representation of this design, previously reported also in [13].

In this case $x_1 = 0$ and $x_2 = l$ were chosen, meaning that the center of mass of link one is at joint A and the center of mass of link two is at joint C, see Fig. 5.5. Substituting these values in Eq. (5.40), the corresponding value of x_3 can be found:

$$x_3 = l\left(\frac{m_2}{m_3} + 1\right)$$

Substituting the previous values for x_1, x_2, and x_3, and considering that:

$$\begin{aligned} x_B &= l\cos(\theta) \\ y_B &= l\sin(\theta) \\ \dot{x}_B &= -l\sin(\theta)\omega \\ \dot{y}_B &= l\cos(\theta)\omega \end{aligned} \tag{5.42}$$

the angular momentum, Eq. (5.41), finally results in:

$$\mathbf{h}_m = \left(\frac{I_3}{l^2} + \frac{I_1}{l^2} - m_3\, l^2 - m_2\, l^2\right)\omega$$

where ω is the angular velocity of the crank and the rocker. Then for the shaking moment balancing of the system it is necessary to add a counter-rotary counter-mass with the same magnitude but in opposite direction to the value of $\mathbf{h}_m$.

An alternative more efficient design could be a mechanism with a counter-rotary counter-mass at the crank and a counterweight at the rocker. Figure 5.7 is a representation of this design, also reported in [14].

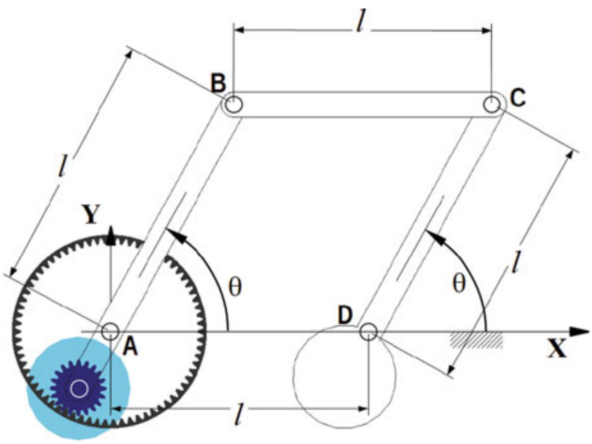

Fig. 5.7 A proposed design of an inline parallel four-bar mechanism with one counter-mass and one counter-rotary counter-mass

In this case $x_1 = -\frac{l m_2}{2 m_1}$ and $x_2 = l/2$ were chosen. Substituting these values in Eq. (5.40), the corresponding value of x_3 can be found as:

$$x_3 = l + \frac{l m_2}{2 m_3}$$

Substituting the previous values for x_1, x_2, and x_3, and considering the positions and velocities of point B, Eq. (5.42), the angular momentum, Eq. (5.41), finally results in:

$$\mathbf{h}_m = \left(\frac{I_3}{l^2} + \frac{I_1}{l^2} - m_3 l^2 \right) \omega$$

where ω is the angular velocity of the crank and the rocker. Again for the shaking moment balancing of the system it is necessary to add a counter-rotary counter-mass with the same magnitude but in opposite direction to the found value of $\mathbf{h}_m$.

5.4.2 *Dynamic Balancing of the Inline Four-Bar Mechanism*

Another special case is the inline four-bar mechanisms balanced by two counter-rotary counter-masses. This case has been previously studied in detail in [12] and the proposed design is similar to the one presented in Fig. 5.8.

In an inline four-bar mechanism $y_1 = y_2 = y_3 = 0$, so the linear momentum equation, Eq. (5.30), can be expressed as:

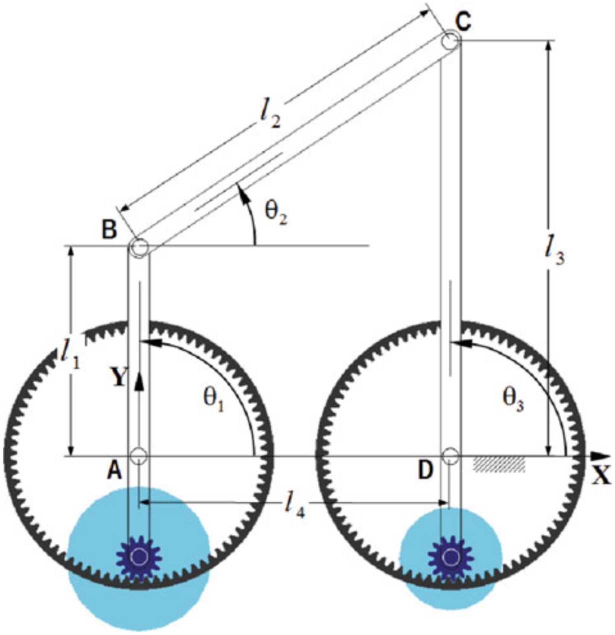

Fig. 5.8 Design proposal for the dynamic balancing of an inline four-bar mechanism

$$\mathbf{l}_m = \begin{bmatrix} \dot{x}_B\left(-\frac{m_2\,x_2}{l_2} + \frac{m_1\,x_1}{l_1} + m_2\right) + \dot{x}_C\left(-\frac{m_3\,x_3}{l_3} + \frac{m_2\,x_2}{l_2} + m_3\right) \\ \dot{y}_B\left(-\frac{m_2\,x_2}{l_2} + \frac{m_1\,x_1}{l_1} + m_2\right) + \dot{y}_C\left(-\frac{m_3\,x_3}{l_3} + \frac{m_2\,x_2}{l_2} + m_3\right) \end{bmatrix} \tag{5.43}$$

If the coupler is left without change, this equation indicates that this system can be force balanced by a two counterweights, one at body one and one at body three. And their location can be determined by the following balancing conditions:

$$\left(-\frac{m_2\,x_2}{l_2} + \frac{m_1\,x_1}{l_1} + m_2\right) = 0$$
$$\left(-\frac{m_3\,x_3}{l_3} + \frac{m_2\,x_2}{l_2} + m_3\right) = 0 \tag{5.44}$$

On the other hand taking into account the angular momentum of a four-bar mechanism, Eq. (5.33), and considering that $y_1 = y_2 = y_3 = 0$, the angular momentum of an inline four-bar mechanism can be expressed as:

$$
\begin{aligned}
\mathbf{h}_m = &- \dot{x}_B \left\{ y_B \left(\frac{I_2}{l_2^2} + \frac{I_1}{l_1^2} - \frac{2\, m_2\, x_2}{l_2} + m_2 \right) + y_C \left(\frac{I_2}{l_2^2} - \frac{2\, m_2\, x_2}{l_2} \right) \right\} \\
&+ \dot{y}_B \left\{ x_B \left(\frac{I_2}{l_2^2} + \frac{I_1}{l_1^2} - \frac{2\, m_2\, x_2}{l_2} + m_2 \right) + x_C \left(\frac{I_2}{l_2^2} - \frac{2\, m_2\, x_2}{l_2} \right) \right\} \\
&- \dot{x}_C \left\{ y_C \left(\frac{I_3}{l_3^2} + \frac{I_2}{l_2^2} - \frac{2\, m_3\, x_3}{l_3} + m_3 \right) + y_B \left(\frac{I_2}{l_2^2} - \frac{2\, m_2\, x_2}{l_2} \right) \right\} \\
&+ \dot{y}_C \left\{ x_B \left(\frac{I_3}{l_3^2} + \frac{I_2}{l_2^2} - \frac{2\, m_3\, x_3}{l_3} + m_3 \right) + x_C \left(\frac{I_2}{l_2^2} - \frac{2\, m_2\, x_2}{l_2} \right) \right\} \\
&+ \dot{y}_C \left(-\frac{l_4\, I_3}{l_3^2} + \frac{l_4\, m_3\, x_3}{l_3} \right)
\end{aligned} \tag{5.45}
$$

but if the coupler is considered a physical pendulum (as done in [12]) then $x_2 = l_2/2$ and $I_2 = l_2\, m_2\, x_2$, and substituting the force balancing conditions, Eq. (5.44), the angular momentum of this system is:

$$
\begin{aligned}
\mathbf{h}_m = &-\dot{x}_B y_B \left(\frac{I_1}{l_1^2} - \frac{m_2}{2} \right) + \dot{y}_B x_B \left(\frac{I_1}{l_1^2} - \frac{m_2}{2} \right) \\
&-\dot{x}_C y_C \left(\frac{I_3}{l_3^2} + \frac{m_2}{2} + m_3 \right) + \dot{y}_C x_C \left(\frac{I_3}{l_3^2} + \frac{m_2}{2} + m_3 \right) \\
&+\dot{y}_C \left(-\frac{l_4\, I_3}{l_3^2} + \frac{l_4\, m_2}{2} + l_4\, m_3 \right)
\end{aligned} \tag{5.46}
$$

Finally considering the positions and velocities of the points B and C:

$$
\begin{aligned}
x_B &= l_1 \cos(\theta_1); & y_B &= l_1 \sin(\theta_1) \\
\dot{x}_B &= - l_1 \sin(\theta_1)\omega_1; & \dot{y}_B &= l_1 \sin(\theta_1)\omega_1 \\
x_C &= l_3 \cos(\theta_3) + l_4; & y_C &= l_3 \sin(\theta_3) \\
\dot{x}_C &= - l_3 \sin(\theta_1)\omega_3; & \dot{y}_C &= l_3 \sin(\theta_3)\omega_3
\end{aligned}
$$

and substituting in Eq. (5.46), after some reductions the resulting expression for the angular momentum is:

$$
\mathbf{h}_m = \left(I_1 + \frac{l_1^2\, m2}{2} \right) \omega_1 + \left(I_3 - \frac{l_3^2\, m2}{2} - l_3^2\, m_3 \right) \omega_3 \tag{5.47}
$$

This equations clearly show that an inline four-bar mechanism, with a physical pendulum as coupler, can be dynamic balanced just with two counter-rotary counter-masses, as reported in [12].

5.5 Design Examples and Simulation Results

5.5.1 Dynamic Balancing of a Single Pendulum

Let us suppose that it is desired the dynamic balancing of a single pendulum as the one shown in Fig. 5.9, using the results given in Eqs. (5.24) and (5.29).

The pendulum is made of aluminum with density, mass, and moment of inertia as indicated in Fig. 5.9. The force balancing of this system implies to comply with the balancing condition, the Eq. (5.24).

Applying a solution similar to the one proposed in Fig. 5.3, the bar OB is modified to get the bar AB, made also of aluminum. Setting $l_{c_1} = 15\,\text{cm}$, $x_{g_1} = 32.5\,\text{cm}$, both with respect to the local reference frame of the pendulum at point O. Then this new bar has a total mass $m_1 = 0.36467\,\text{kg}$ and a moment of inertia $I_{g_1} = 0.013471\,\text{kg}\,\text{m}^2$. Substituting this values at Eq. (5.24) the corresponding value of m_2 can be found:

$$\begin{aligned} m_2 &= m_1 \left(\frac{x_{g_1}}{l_{c_1}} - 1 \right) \\ m_2 &= 0.36467 \left(\frac{0.325}{0.15} - 1 \right) \\ m_2 &= 0.425445\,\text{kg} \end{aligned} \tag{5.48}$$

Note that this change implies an increment of near 1.8 times the original mass.

The resulting mass, m_2, has to be distributed between the gear and the disk that form the counter-rotary counter-mass, maintaining the moment balancing condition in Eq. (5.29).

The gear material (the density), diameter, and thickness depend on the mechanical design rules, while the physical characteristics of the disk depend on the convenience of the designer. On the other hand, a high value for the angular velocity

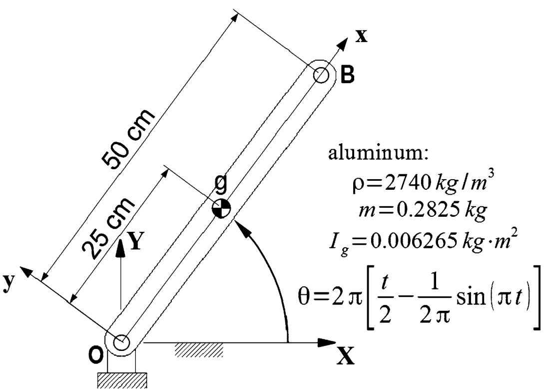

Fig. 5.9 A general single pendulum. In the equation of θ, t stands for time

ω_2, see Eq. (5.27), helps to reduce the inertia required in the counter-rotary counter-mass. Taking these factors into account and using a modulus $m = 2$, a gear with 20 teeth and $R_2 = 2\,\text{cm}$ is chosen. A 170 teeth crown gear results.

All previous values are substituted in Eq. (5.29) to obtain an appropriate value for I_{g_2} :

$$\left[I_{g_1} + lc_1^2\left(\frac{m_2^2}{m_1} + m_2\right) - \left(\frac{lc_1}{R_2}\right) I_{g_2}\right] = 0$$

$$\left[0.013471 + 0.15\left(\frac{(0.425445)^2}{0.36467} + 0.425445\right) - \left(\frac{0.15}{0.02}\right) I_{g_2}\right] = 0$$

then,

$$I_{g_2} = 0.004562\,\text{kg}\,\text{m}^2$$

This inertia moment corresponds to both the gear and the disk, so:

$$I_{g_2} = I_g + I_d \tag{5.49}$$

where I_g is the moment of inertia of the gear and I_d is the moment of inertia of the disk. In the same way the mass of the counter-rotary counter-mass should be:

$$m_2 = m_g + m_d \tag{5.50}$$

where m_g and m_d are the mass of the gear and the disk, respectively.

The gear is chosen made of steel, and its moment of inertia determined by its design, in this case $m_g = 0.09803\,\text{kg}$ and $I_g = 0.00001961\,\text{kg}\,\text{m}^2$. On the other hand, the mass and the moment of inertia of the disk can be calculated by:

$$m_d = \pi\, R_d^2\, t_d\, \rho_d$$

$$I_d = \frac{1}{2} m_d R_d^2$$

where R_d, t_d, and ρ_d are the radius, the thickness, and the density of the disk, respectively.

Substituting the previous values in Eqs. (5.50) and (5.50), considering that the disk is made of brass (8,545 kg m^2), the following two equations are obtained:

$$m_2 = 0.09803 + 8545\,\pi\, t_d\, R_d^2$$

$$0.425445 = 0.09803 + 8545\,\pi\, t_d\, R_d^2 \tag{5.51}$$

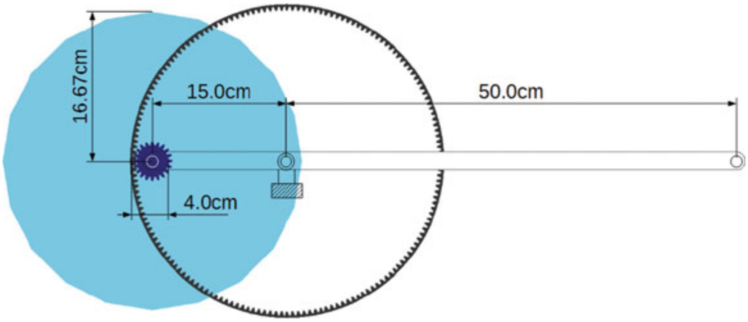

Fig. 5.10 Resulting design in the dynamic balancing of a pendulum

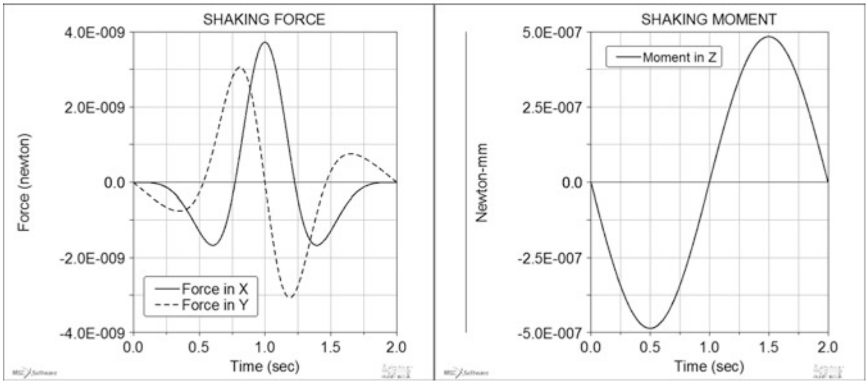

Fig. 5.11 Resulting shaking force and shaking moment in the dynamic balancing of a pendulum

and

$$
\begin{aligned}
I_{g_2} &= \frac{1}{2} 8545\, \pi\, t_d\, R_d^4 \\
0.004562 &= \frac{1}{2} 8545\, \pi\, t_d\, R_d^4
\end{aligned}
\tag{5.52}
$$

that can be solved simultaneously to get $R_d = 166.566$ mm and $t_d = 0.43961$ mm.

The final design is shown in Fig. 5.10. The resulting shaking force and shaking moment are shown in Fig. 5.11. In Fig. 5.12 the shaking force of the not balanced pendulum is shown, additionally the driving torque required to move the original pendulum and the driving torque necessary to move the balanced pendulum are compared.

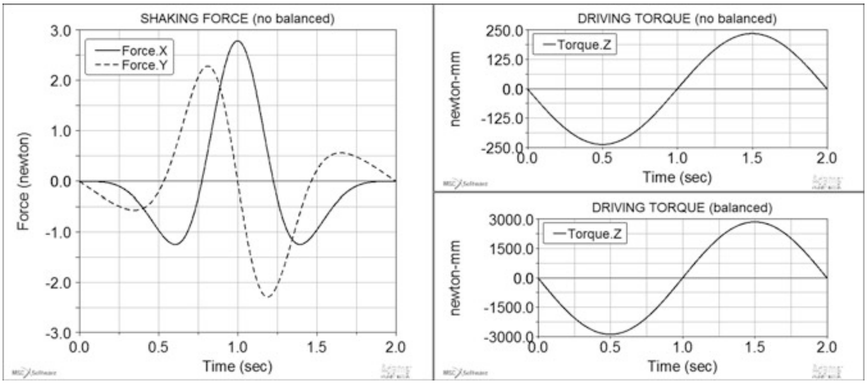

Fig. 5.12 Resulting reaction force and driving torque in the non-balanced pendulum. The driving torque of the balanced system is also included for comparison

5.5.2 Dynamic Balancing of a Four-Bar Mechanism

Frequently, when designing a dynamic balanced mechanism, it is important to maintain the counterweights near to the fixed joints (fixed pivots) attached to ground. This practice reduces the total additional inertia introduced in the balancing process and helps to reduce the increment in the driving torque.

In this example the dynamic balancing of a four-bar mechanisms is solved, by applying the design proposed in Fig. 5.8, an inline four-bar mechanism. This solution complies with the conditions exposed in the previous paragraph and works fine with the application of two counter-rotary counter-masses near the base, one at the crank and one at the rocker.

Consider the mechanism in Fig. 5.13. All elements are made of aluminum and their cross section is equal for all (2 × 1 cm), the values for the corresponding mass and moment of inertia are indicated in the figure. Its motion is defined by the function specified for angle θ_1.

To get the full dynamic balancing of this mechanism it is necessary first to balance it by forces, imposing the force balancing conditions expressed by Eq. (5.44). These conditions assure that the center of mass of the system will be stationary at the origin of the fixed reference frame, point A. Note that in this case the coupler of the mechanism (element 2) will be changed to be a physical pendulum, so x_2 and m_2 are completely determined.

On the other hand, at the same time it is necessary to impose the moment balancing condition. This can be obtained by taking into account the total angular momentum of the system, Eq. (5.47). Meaning that in order to moment balancing this system it is necessary to make the total angular momentum equal to zero. This clearly can be done by adding two counter-rotary counter-masses, one at the crank and one at the rocker.

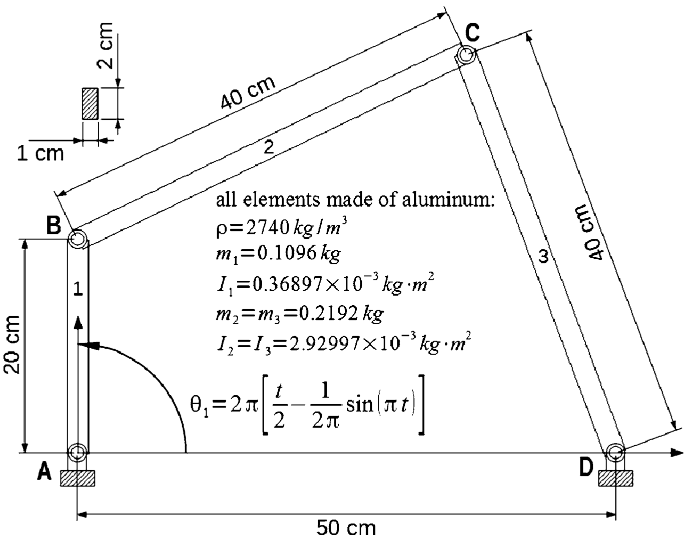

Fig. 5.13 Original four-bar mechanism to be dynamic balanced

In this way the four dynamic balancing conditions are:

$$\left(-\frac{m_2\,x_2}{l_2}+\frac{m_1\,x_1}{l_1}+m_2\right)=0 \tag{5.53}$$

$$\left(-\frac{m_3\,x_3}{l_3}+\frac{m_2\,x_2}{l_2}+m_3\right)=0 \tag{5.54}$$

$$\left(I_1+\frac{l_1^2\,m2}{2}\right)\omega_1-I_{PC_1}(k_1\omega_1)=0 \tag{5.55}$$

$$\left(I_3-\frac{l_3^2\,m2}{2}-l_3^2\,m_3\right)\omega_3-I_{PC_3}(k_3\omega_3)=0 \tag{5.56}$$

where I_{c_1} and I_{c_3} are the moment of inertia of the counter-rotary counter-masses attached to the crank and the rocker, respectively. As can be noted these counter-rotary counter-masses must rotate with an angular velocity in the opposite direction with respect to their associated elements. The counter-rotation can be achieved by introducing gears, belts, etc., in this case a set of gears are chosen, giving a design similar to the one presented in Fig. 5.13, so

$$k_1=\frac{d_1}{R_{P_1}};\quad k_3=\frac{d_3}{R_{P_3}}$$

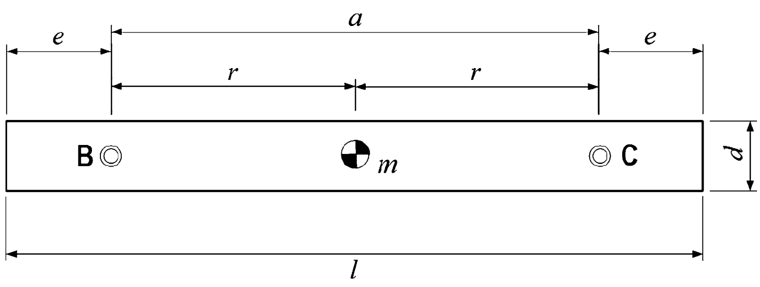

Fig. 5.14 Rectangular bar predesigned to be a physical pendulum

It can also be seen from the figure that the coupler has been changed to be a physical pendulum. So the original coupler has been modified to have its moment of inertia with respect to its center of mass equal to the moment of inertia generated by two equal punctual masses located at points B and C, respectively:

$$I_{CM_2} = m_2 \frac{l_2^2}{2}$$

In this case the physical pendulum is made by extending the original rectangular bar satisfying the following equation, [12], (see Fig. 5.14):

$$\frac{e}{h} = \frac{1}{2}\sqrt{3\left(\frac{a}{h}\right)^2 - 1} - \frac{a}{2h}$$

In this case the original coupler has $a = 400$ mm and $h = 20$ mm, then $e = 146.266$ mm, and the new coupler will have a total length $l = 692.532$ mm, and a mass $m_2 = 0.3795$ kg.

The new coupler is dynamically equivalent to a pair of masses located at B and at C, then the crank and the rocker can be balanced independently.

5.5.2.1 Balancing of the Crank

For the dynamic balancing of the crank, Eqs. (5.54) and (5.56) must be solved simultaneously to find the appropriate counter-rotary counter-mass that balance for forces and moments.

The counter-rotary counter-mass, I_{C_1}, is made by a gear (the pinion) with radius R_{P_1} and thickness t_{P_1}, and a disk with radius R_{C_1} and thickness t_{C_1}. On the other hand, the length of the crank should be increased by a distance d_1 to connect to these new elements, as seen in Fig. 5.15. All these variables form a set of five unknowns, so three of them must be set by election.

From the mechanical design point of view the pinion diameter and its thickness are determined by the general design rules for the gears. So choosing a modulus for

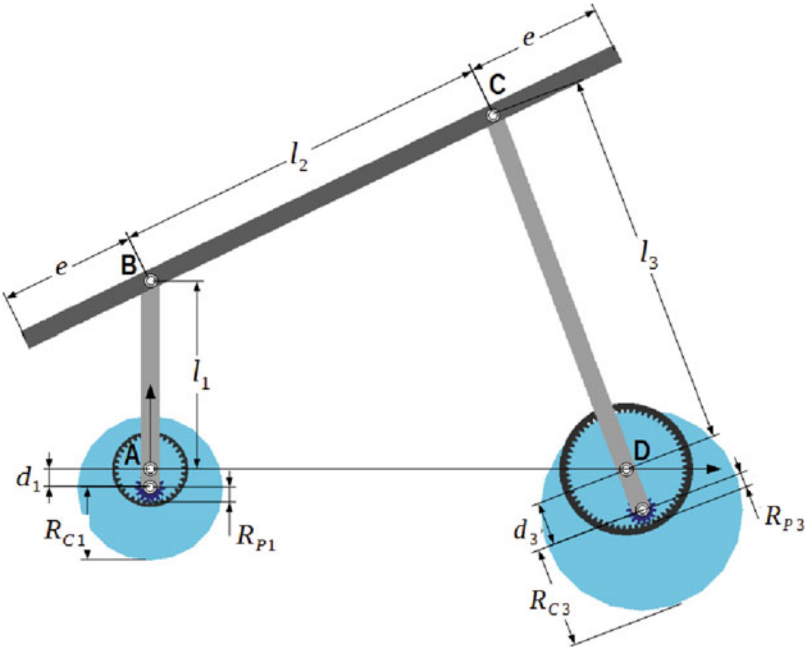

Fig. 5.15 Proposed design for an inline dynamic balanced four-bar mechanism

the crown-pinion set automatically sets the radius, thickness, and number of teeth. In this case a modulus $m = 2$ has been chosen, $R_{P_1} = 14\,\text{mm}$, and $t_{P_1} = 10\,\text{mm}$.

To determine the remaining variables: d_1, R_{C_1}, and t_{C_1}, an optimization problem is solved. It has been chosen to minimize Eq. (5.56) subject to the following constraints: Eq. (5.54), $R_{C_1} > 0$, $t_{C_1} > 0$ and $d_1 > 0$. But to do this the corresponding values for lengths, masses, and moments of inertia were substituted:

$$m_2 = 0.3795\,\text{kg};\ l_2 = 400\,\text{mm}$$

$$m_1 = m_c + m_{d_1} + m_{P_1} + m_{C_1}$$

$$I_1 = I_c + \frac{1}{4} m_c\, l_1^2 + I_{d_1} + \frac{1}{4} m_{d_1}\, d_1^2 + I_{P_1} + m_{P_1}\, d_1^2 + I_{C_1} + m_{C_1}\, d_1^2$$

$$I_{PC_1} = I_{P_1} + I_{C_1}$$

where m_c, m_{d_1}, m_{P_1}, and m_{C_1} are the corresponding mass of the crank, the added piece of bar, the pinion, and the disk at the crank, respectively. In similar way I_c, I_{d_1}, I_{P_1}, and I_{C_1} are the moments of inertia of the crank, the added piece of bar, the pinion, and the disk, respectively. I_{PC_1} is precisely the counter-inertia at the crank.

All these masses and moments of inertia are determined by the geometry and density of the bodies. In this case the bar is considered made of aluminum ($\rho_a = 2{,}740\,{}^{\text{kg}}/_{\text{m}^3}$), the pinion of steel ($\rho_s = 7{,}801\,{}^{\text{kg}}/_{\text{m}^3}$), and the disk of brass ($\rho_b = 8{,}545\,{}^{\text{kg}}/_{\text{m}^3}$), so:

$$m_c = htl_1\rho_a; \quad I_c = \frac{1}{12}ht\rho_a l_1^3$$

$$m_{d_1} = htd_1\rho_a; \quad I_{d_1} = \frac{1}{12}ht\rho_a d_1^3$$

$$m_{P_1} = \pi R_{P_1}^2 t_{P_1}\rho_s; \quad I_{P_1} = \frac{1}{2}\pi R_{P_1}^4 t_{P_1}\rho_s$$

$$m_{C_1} = \pi R_{C_1}^2 t_{C_1}\rho_b; \quad I_{C_1} = \frac{1}{2}\pi R_{C_1}^4 t_{C_1}\rho_b$$

where $h = 20$ mm and $t = 10$ mm as specified in Fig. 5.13.

Substituting all values in Eqs. (5.54) and (5.56), the force and moment balancing conditions finally are:

$$-8545\,\pi\, d_1\, {R_{C_1}}^2\, t_{C_1} - 0.274\, d_1^2 - 0.01529\,\pi\, d_1 + 0.048911 = 0 \tag{5.57}$$

$$\begin{aligned} &- 305178.57\,\pi\, d\, Rc^4\, tc + 8545\,\pi\, d^2\, Rc^2\, tc + 0.1827\, d^3 \\ &+ 0.01529\,\pi\, d^2 - 1.0702972 \times 10^{-4}\,\pi\, d + 0.0090515 = 0 \end{aligned} \tag{5.58}$$

Finally solving the optimization problem, using an open source code implementation of the method introduced in [15], the following results are obtained:

$$R_{C_1} = 76.59\,\text{mm}, \quad t_{C_1} = 15.191\,\text{mm}, \quad d_1 = 20.0\,\text{mm}$$

5.5.2.2 Balancing of the Rocker

For the dynamic balancing of the rocker, Eqs. (5.55) and (5.56) must be solved simultaneously to find the appropriate counter-rotary counter-mass that balance for forces and moments.

The counter-rotary counter-mass, I_{C_3}, is made by a gear (the pinion) with radius R_{P_3} and thickness t_{P_3}, and a disk with radius R_{C_3} and thickness t_{C_3}. On the other hand, the length of the crank should be increased by a distance d_3 to connect to these new elements, as seen in Fig. 5.15. All these variables form a set of five unknowns, so three of them must be set by election.

From the mechanical design point of view the pinion diameter and its thickness are determined by the general design rules for the gears. So choosing a modulus for the crown-pinion set automatically sets the radius, thickness, and number of teeth. In this case a modulus $m = 2$ has been chosen, $R_{P_3} = 14$ mm, and $t_{P_3} = 10$ mm.

To determine the remaining variables: d_3, R_{C_3}, and t_{C_3}, an optimization problem is solved. It has been chosen to minimize Eq. (5.56) subject to the following constraints: Eq. (5.55), $R_{C_3} > 0$, $t_{C_3} > 0$ and $d_3 > 0$. But to do this the corresponding values for lengths, masses, and moments of inertia were substituted:

$$m_2 = 0.3795\,\text{kg};\; l_2 = 400\,\text{mm}$$
$$m_3 = m_r + m_{d_3} + m_{P_3} + m_{C_3}$$
$$I_1 = I_r + \frac{1}{4}m_r\,l_1^2 + I_{d_3} + \frac{1}{4}m_{d_3}\,d_3^2 + I_{P_3} + m_{P_3}\,d_3^2 + I_{C_3} + m_{C_3}\,d_3^2$$
$$I_{PC_3} = I_{P_3} + I_{C_3}$$

where m_r, m_{d_3}, m_{P_3}, and m_{C_3} are the corresponding mass of the rocker, the added piece of bar, the pinion, and the disk at the rocker, respectively. In similar way I_r, I_{d_3}, I_{P_3}, and I_{C_3} are the moments of inertia of the rocker, the added piece of bar, the pinion, and the disk, respectively. I_{PC_3} is precisely the counter-inertia at the rocker.

All these masses and moments of inertia are determined by the geometry and density of the bodies. The bar is considered made of aluminum($\rho_a = 2,740\,{}^{\text{kg}}/_{\text{m}^3}$), the pinion of steel($\rho_s = 7,801\,{}^{\text{kg}}/_{\text{m}^3}$), and the disk of brass ($\rho_b = 8,545\,{}^{\text{kg}}/_{\text{m}^3}$), so:

$$m_r = htl_3\rho_a; \quad I_r = \frac{1}{12}ht\rho_a l_3^3$$
$$m_{d_3} = htd_3\rho_a; \quad I_{d_3} = \frac{1}{12}ht\rho_a d_3^3$$
$$m_{P_3} = \pi R_{P_3}^2 t_{P_3}\rho_s; \quad I_{P_3} = \frac{1}{2}\pi R_{P_3}^4 t_{P_3}\rho_s$$
$$m_{C_3} = \pi R_{C_3}^2 t_{C_3}\rho_b; \quad I_{C_3} = \frac{1}{2}\pi R_{C_3}^4 t_{C_3}\rho_b$$

where $h = 20\,\text{mm}$ and $t = 10\,\text{mm}$ as specified in Fig. 5.13.

Substituting all values in Eqs. (5.55) and (5.56), noting that the origin of the local reference frame of the rocker is at point C, so all distances have to be taken with respect to this point, the force and moment balancing conditions finally are:

$$-21362.5\,\pi\,d_3\,R_{C_3}^2\,t_{C_3} - 0.685\,d_3^2 - 0.038225\,\pi\,d_3 + 0.2994 = 0 \tag{5.59}$$

$$\begin{aligned} &-305178.57\,\pi\,d_3\,Rc^4\,tc + 8545.0\,\pi\,d_3^2\,Rc^2\,tc \\ &\quad + 6836.0\,\pi\,d\,Rc^2\,tc + 0.1827\,d_3^3 + 0.01529\,\pi\,d_3^2 \\ &\quad + 0.2192\,d_3^2 + 0.012125\,\pi\,d_3 - 0.05374192 = 0 \end{aligned} \tag{5.60}$$

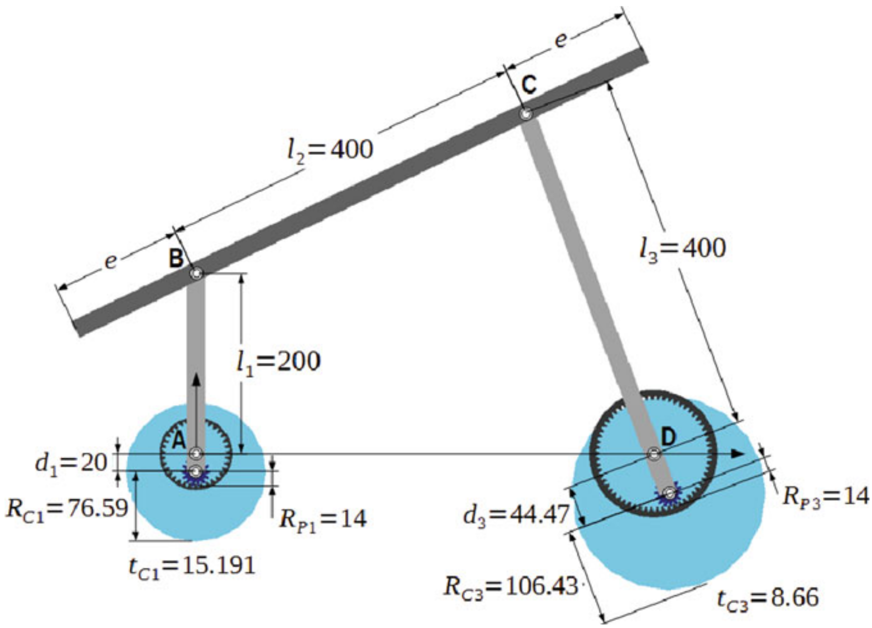

Fig. 5.16 Resulting design for an inline dynamic balanced four-bar mechanism. All measures in mm

Finally solving the optimization problem, using an open source code implementation of the method introduced in [15], the following results are obtained:

$$R_{C_3} = 106.43\,\text{mm}, \quad t_{C_3} = 8.66\,\text{mm}, \quad d_3 = 44.47\,\text{mm}$$

5.5.2.3 Resulting Inline Four-Bar Mechanism

The final design of the proposed inline four-bar mechanisms can be seen at Fig. 5.16. This results in an increment of 9.7 times the original mass of the system.

The comparison of the shaking force, the shaking moment and the driving torque, can be seen in Figs. 5.17, 5.18, and 5.19, respectively.

5.6 Concluding Remarks

In this chapter a completely new general method to find the dynamic balancing conditions based on the use of Natural Coordinates has been introduced. The method can be used for linkages in the plane and in space, although it is presented for planar mechanisms.

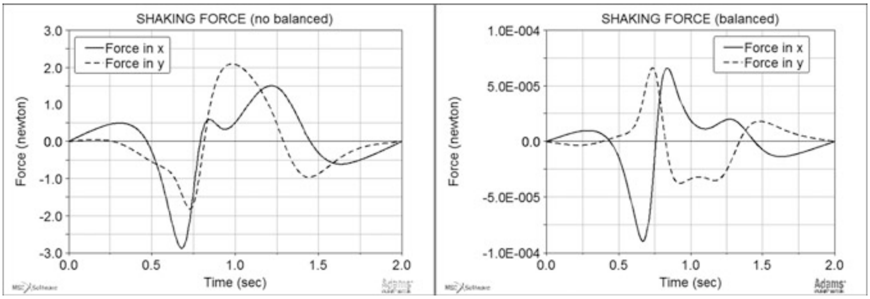

Fig. 5.17 Resulting shaking force in the inline dynamic balanced four-bar mechanism

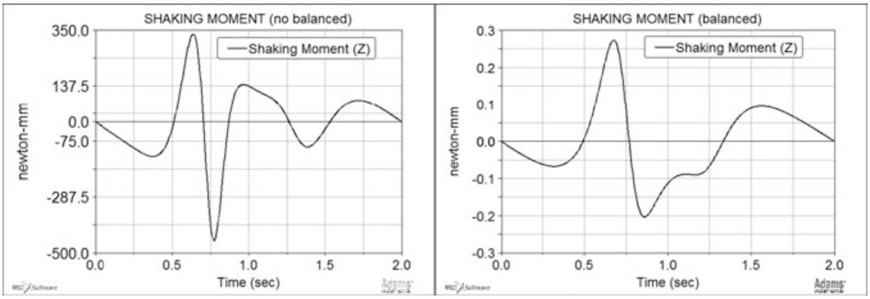

Fig. 5.18 Resulting shaking moment in the inline dynamic balanced four-bar mechanism

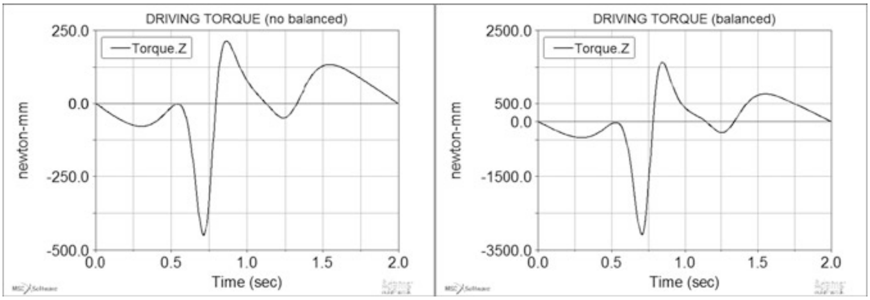

Fig. 5.19 Resulting driving torque in the inline dynamic balanced four-bar mechanism

The method is simple and can be highly automated, and it has been shown in its application to the design of dynamic balanced planar mechanisms using counter-rotary counter-masses. In particular the resulting equations of a general simple pendulum and of a general four-bar mechanism are presented. It is shown that these equations must be solved simultaneously in order to obtain a feasible design, and that in a more general case the use of optimization could be better from the mechanical design point of view. Detailed results obtained from the dynamic simulations made using virtual prototypes defined in ADAMS are included, showing the validity and applicability of the method, and helping the reader to deeply understand the concepts, maybe repeating the examples and proposing variations of the presented designs.

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