Chapter 2
Design of Reactionless Mechanisms Without Counter-rotations

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Abstract This chapter presents methods and principles used for balancing of planar mechanisms without counter-rotations. The fundamentals of balancing are described at first. Balancing only by counterweights provides only the force balance of mechanisms. Several basic methods which balance linkages by internal mass redistribution or adding of counterweights are introduced. These methods are the principal vector method, linearly independent vector method, complex mass method, and linear momentum method. The principles of these methods are explained in the example of the four-bar linkage and some extensions and important outcomes of these methods are added.

Keywords Balancing • Principal vector method • Linearly independent vector method • Complex mass method • Linear momentum method • Four-bar linkage

2.1 Principles of Balancing

During the working process of mechanisms, inertia forces and inertia torques are generated which are exerted to the base as reaction forces and moments. These reactions cause vibrations, inducing noise, wear, fatigue, poor product quality, and other undesired effects. Vibration suppression is usually achieved by applications of damping or other means. However, these solutions do not prevent the origin of vibrations. The balancing compared with, for example, damping eliminates or reduces the inertia forces and moments that cause the vibrations.

The sum of inertia forces that exert on the base of the mechanism is named shaking force and the sum of inertia torques is named shaking moment. The elimination of the shaking force acting on the base for any motion of the mechanism is called force balancing. The elimination of shaking moment is called moment balancing. The combination of force and moment balancing is called dynamic balancing which is synonymous to the terms complete balancing or reactionless.

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The research on balancing is very extensive and many principles and methods were described in literature. One of the basic ideas and the most general approach to the dynamic balancing resulted from the conservation of linear momentum law and the conservation of angular momentum law. The first law states that linear momentum p is conserved if the resultant force F is zero, and the second law states that angular momentum h is conserved if the resultant moment M is zero:

$$\frac{dp}{dt} = F = 0 \tag{2.1}$$

$$\frac{dh}{dt} = M = 0 \tag{2.2}$$

The opposites of the terms on the left-hand side of Eqs. (2.1) and (2.2) are the shaking force and shaking moment of the system. The shaking force and the shaking moment of the mechanism will vanish if the linear momentum p and the angular momentum h of a mechanism remain constant for any motion at all times. The linear and angular momentum of a completely balanced mechanism can be written as follows:

$$p = \sum_{i=1}^{n} m_i \dot{r}_i = \text{const} \tag{2.3}$$

$$h = \sum_{i=1}^{n} (I_i \dot{\varphi}_i + r_i \times m_i \dot{r}_i) = \text{const} \tag{2.4}$$

with i being the number of the link of the mechanism, m_i the mass of the link, r_i the position vector of its center of mass, I_i the moment of inertia, and φ_i the angle of rotation. These two constraints are necessary and sufficient conditions for a reactionless mechanism.

From Eq. (2.3), it implies that the center of mass of the force balanced mechanism performs a constant velocity motion or it is stationary. In practical situations, the second possibility is more convenient and the right-hand side of Eq. (2.3) is then set to zero. Similarly, the angular momentum in Eq. (2.4) is usually set to zero.

Constant linear momentum and angular momentum of the system also mean that there are no forces and moments between the system of moving links and the base. Internal forces and moments act within the mechanisms. They include the reaction forces and moments between the links, internal collisions, internal springs, friction, and other. External forces and moments act from outside of the mechanism, for example, gravity force, magnetic force, external springs, and collisions. The sum of all internal forces and moments is zero; therefore, they do not affect the linear and angular momentum and also the dynamic balancing.

The term static balancing is not the same as force balancing. The static balancing implies that the mechanism is in static equilibrium for any motion at all times, which means that the potential energy of a statically balanced mechanism remains constant. A forced balanced mechanism with a stationary center of mass has constant potential energy, so it is also statically balanced. The force balancing, therefore, implies static balancing but the opposite is not true. Constant potential energy can be achieved by using springs; however, there still exists a reaction force and moment on the base.

The positive effects of dynamic balancing are elimination of shaking force and moment and thus reducing vibrations and noise. Balancing has also some undesirable effects that cannot be neglected. The drawbacks of balancing are mainly addition of mass and inertia, influence on input torque, modification of machine design, and costs. Because of these disadvantages, the balancing of mechanisms in practical situations cannot be very often complete and is only partial. The shaking force and shaking moment are then reduced to an acceptable level and the solution is usually obtained by using an optimization procedure.

There are several methods for deriving the conditions for dynamic balancing. The first method is based on calculation of the linear and angular momentum and the conditions for which they are zero (constant). The next method calculates the position of the center of mass of a mechanism and the conditions for which it is stationary. Both methods are convenient for analytic solution of the problem, however, the second only for the force balancing. The conditions for balancing can also be derived using calculation of the shaking force and shaking moment for which they are zero. This method is especially suitable for numeric computations and partial balancing conditions.

There are many principles, methods, and practical solutions for designing reactionless mechanisms. The choice of methods described in this chapter is limited to methods which do not use counter-rotations and methods described in other chapters, for example, duplicate mechanisms, counter-rotations, and input torque balancing mechanisms. The stationary center of mass of a mechanism and thus the force balancing are usually accomplished by the addition of counterweights or redistribution of internal mass. Methods based on these principles are principal vector method [1], method of linearly independent vectors [2], and complex mass method [3]. Balancing of shaking moment without counter-rotations is still challenging. One solution is deriving balancing conditions from the angular momentum equation of the center of mass. Direct balancing of shaking moment is achieved by using noncircular gears or cam mechanisms.

2.2 Principal Vector Method

The principal vector method was published by O. Fisher in 1902 [1] and afterwards it was extended many times. The motion of the center of mass of a mechanism is described analytically and the parameters for which the center of mass is stationary are determined. The position of the center of mass is given by a series of vectors directed along the links of a mechanism. The magnitude of these vectors depends on the mass of each link and its center of mass position. The principal vectors create an augmented mechanism of parallelogram structure which contains the total center of mass.

The position of the total center of mass of the mechanism $\boldsymbol{r_t}$ is given by

$$r_{\mathrm{t}} = \frac{1}{M}\sum_{i=1}^{n} m_i r_{\mathrm{t}i} \tag{2.5}$$

where m_i is the mass of a moving link i, M is the total mass of n moving links, and $\boldsymbol{r_{ti}}$ is the position vector of link center of mass. The position vectors of the first and k center of mass r_{t1} and r_{tk} can be expressed as

$$r_{\mathrm{t1}} = b_1 e_1, \quad r_{\mathrm{tk}} = b_{\mathrm{k}} e_{\mathrm{k}} + \sum_{i=1}^{k-1} a_i e_i \tag{2.6}$$

with e_i being the unit vector directed along the link i, a_i the length of the link, and b_i the distance between the center of mass and the link joint.

This method is, for example, applied on the four-bar linkage as it is shown in Fig. 2.1. The position of the center of mass according to Eqs. (2.5) and (2.6) of this four-bar mechanism is

$$r_t = \frac{1}{M}\left(m_1 b_1 + m_2 a_1 + m_3 a_1\right) e_1 + \frac{1}{M}\left(m_2 b_2 + m_3 a_2\right) e_2 + \frac{1}{M} m_3 b_3 e_3 \tag{2.7}$$

This equation can be rewritten to the general formula

$$r_t = \sum_{i=1}^{n} h_i = \sum_{i=1}^{n} h_i e_i \tag{2.8}$$

where the vectors h_i are the *principal vectors*. The absolute values h_i of the last and the k^{th} link are

$$h_{\mathrm{n}} = \frac{1}{M} m_{\mathrm{n}} b_{\mathrm{n}}, \quad h_{\mathrm{k}} = \frac{1}{M}\left(m_{\mathrm{k}} b_{\mathrm{k}} + \sum_{i=\mathrm{k}+1}^{n} m_i a_{\mathrm{k}}\right) \tag{2.9}$$

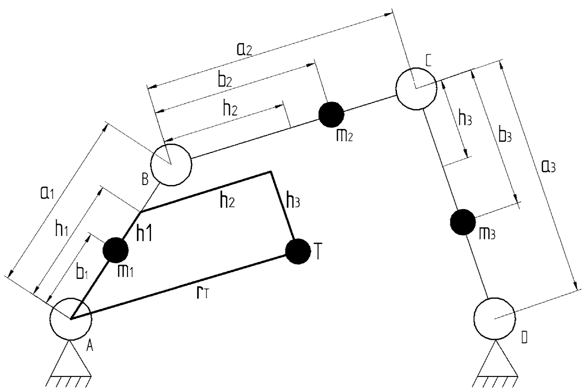


Fig. 2.1 The principle of principal vector method in the example of four-bar linkage

The end point T of the augmented mechanism performs motion of the center of mass of the origin mechanism. This point remains stationary for force balanced mechanisms. The conditions of force balancing are accomplished if the augmented mechanism is geometrically similar to the origin mechanism and if the end point of the augmented mechanism coincides with the fixed point of the origin mechanism. Let λ be the coefficient of geometrical similarity; then the mathematical expression of the first condition is equation

$$\lambda a_i = h_i \tag{2.10}$$

Equations for the four-bar linkage are

$$\lambda a_1 = \frac{1}{M}\left[m_1 b_1 + a_1\left(m_2 + m_3\right)\right] \tag{2.11}$$

$$\lambda a_2 = \frac{1}{M}\left(m_2 b_2 + m_3 a_2\right) \tag{2.12}$$

$$\lambda a_3 = \frac{1}{M} m_3 b_3 \tag{2.13}$$

Excluding λ, the system of equation is reduced to $n-1$ equation with *2n* variables which create the conditions of a force balanced mechanism. For the given geometry a_i, it is not possible to determine the variables m_i and b_i uniquely. This is an advantage because $n+1$ parameters can be generally chosen, for example, according to the design options. This rule is valid only for an open kinematic chain with revolute joints.

Finally, the conditions of a force balanced four-bar linkage are

$$m_1 \frac{b_1}{a_1} + \frac{m_2\left(a_2 - b_2\right)}{a_2} = 0 \tag{2.14}$$

$$m_2 \frac{b_2}{a_2} + \frac{m_3\left(a_3 - b_3\right)}{a_3} = 0 \tag{2.15}$$

The principal vector method is also usable for mechanisms with more degrees of freedom and loops if the links are connected with revolute joints.

Extensions: The previous example of the force balancing is based on redistribution of the mass of a mechanism. It is also possible to make the total center of mass stationary and to balance the mechanism without changing its properties, using counterweights or an additional mechanism. An augmented pantograph device is a good example of direct balancing [4].

Principal vectors are useful also for the shaking moment balancing [5]. The principal vector linkage is used as a tool for moment balance solutions. Balance conditions are derived from the equation of angular momentum about the center of mass and this equation is written with principal dimensions, total mass, and total inertia radii.

2.3 Linearly Independent Vector Method

An important method of force balance was published in 1969 [2] and it was named the method of linearly independent vectors. It is based on redistribution of link mass, so the total center of mass remains stationary. The total center of mass is stationary when the coefficients of time-dependent terms of the equation describing the position of the center of mass vanish. It is accomplished when the time-dependent unit vectors within the previous equation are linearly independent.

The principle of this method is shown on the four-bar linkage; see Fig. 2.2. The position of the total center of mass corresponds with Eq. (2.5). The position vectors of the individual link centers of mass are expressed in a complex plane using the unit vectors $e^{i\varphi_j}$ with the reference origin at point A as

$$r_{t1} = b_1 e^{i(\varphi_1+\alpha_1)} \tag{2.16}$$

$$r_{t2} = a_1 e^{i\varphi_1} + b_2 e^{i(\varphi_2+\alpha_2)} \tag{2.17}$$

$$r_{t3} = a_4 e^{i\alpha_4} + b_3 e^{i(\varphi_3+\alpha_3)} \tag{2.18}$$

The total center of mass of the four-bar linkage is then

$$\begin{aligned} r_t = \frac{1}{M}\Big[&\left(m_1 b_1 e^{i\alpha_1} + m_2 a_1\right) e^{i\varphi_1} + \left(m_2 b_2 e^{i\alpha_2}\right) e^{i\varphi_2} \\ &+ \left(m_3 b_3 e^{i\alpha_3}\right) e^{i\varphi_3} + m_3 a_4 e^{i\alpha_4}\Big] \end{aligned} \tag{2.19}$$

where M is the sum of link mass m_i. The unit vectors form the loop equation

$$a_1 e^{i\varphi_1} + a_2 e^{i\varphi_2} - a_3 e^{i\varphi_3} - a_4 e^{i\alpha_4} = 0 \tag{2.20}$$

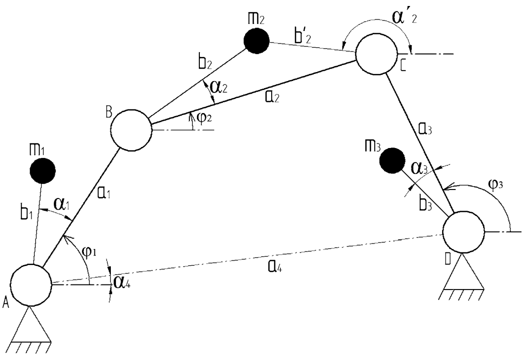


Fig. 2.2 The four-bar linkage

It means that the time-dependent terms in Eq. (2.19) are linearly dependent. If one of the unit vectors from Eq. (2.20) is derived and substituted to Eq. (2.19), then the equation of total center of mass position with linearly independent time-dependent terms is obtained:

$$r_t = \frac{1}{M}\left[\left(m_1 b_1 e^{i\alpha_1} + m_2 a_1 - m_2 \frac{a_1}{a_2} b_2 e^{i\alpha_2}\right) e^{i\varphi_1} + \left(m_3 b_3 e^{i\alpha_3} + m_2 \frac{a_3}{a_2} b_2 e^{i\alpha_2}\right) e^{i\varphi_3} + \left(m_3 a_4 + m_2 \frac{a_4}{a_2} b_2 e^{i\alpha_2}\right) e^{i\alpha_4}\right] \quad (2.21)$$

The third term of Eq. (2.21) is constant, so the total center of mass is stationary if the coefficient of the first two time-dependent terms vanishes. The first term can be simplified, using the kinematic identity

$$b_2 e^{i\alpha_2} = a_2 + b_2' e^{i\alpha_2'} \quad (2.22)$$

which finally leads to the conditions of force balancing

$$m_1 b_1 = m_2 b_2' \frac{a_1}{a_2}, \quad \alpha_1 = \alpha_2' \quad (2.23)$$

$$m_3 b_3 = m_2 b_2 \frac{a_3}{a_2}, \quad \alpha_3 = \alpha_2 + \pi \quad (2.24)$$

If another unit vector is substituted to Eq. (2.21), equivalent conditions of balancing would be found. These conditions are also similar to the conditions derived from the principal vector method (Eqs. (2.14) and (2.15)). If the geometry of links is prescribed and cannot be changed, the force balance can be achieved by addition of counterweights which can be mounted on any of the two links. If the condition of static replacement $m_i = m_i^0 + m_i^*$ is satisfied, then the equations for calculation of counterweight parameters are

$$m_i^* b_i^* = \sqrt{(m_i b_i)^2 + \left(m_i^0 b_i^0\right)^2 - 2 m_i m_i^0 b_i b_i^0 \cos\left(\alpha_i - \alpha_i^0\right)} \quad (2.25)$$

$$\tan \alpha_i^* = \frac{m_i b_i \sin \alpha_i - m_i^0 b_i^0 \sin \alpha_i^0}{m_i b_i \cos \alpha_i - m_i^0 b_i^0 \cos \alpha_i^0} \quad (2.26)$$

where m_i^*, b_i^*, α_i^* are the parameters of counterweights, m_i, b_i, α_i are the parameters of balanced linkage resulting from Eqs. (2.23) and (2.24), and m_i^0, b_i^0, α_i^0 are the parameters of the unbalanced linkage.

Extensions: Generalization of the method of linearly independent vectors involves deriving the equation for the position of the total center of mass, eliminating the time-dependent coefficients and equating these coefficients to zero. The solution yields a relation between the link masses and the link geometries which must be fulfilled to obtain force balance. It does not depend on the method of how the balancing conditions are satisfied if the counterweights are added or the links are redesigned. A drawback of counterweight addition is that the other dynamic properties of the mechanism—the input torque, bearing forces, and shaking moment—are greater.

The generalization of this method yields important outcomes known as a *contour theorem* [6]. The first states that a planar mechanism without axisymmetric link groupings can always be fully force balanced by internal mass redistribution or addition of counterweights if from each link there is a contour to the ground by way of revolute joints only. This theorem is equivalent to stating that each equation within a set of independent loop equations cannot contain more than one term with a time-dependent coefficient. The second statement said that n-linked mechanism with one degree of freedom can be fully force balanced with $n/2$ counterweights.

2.4 Complex Mass Method

The complex mass method is derived from the previous method of linearly independent vectors and it was first presented by Walker and Oldham in 1978 [3]. The complex mass method simplifies the theory of linearly independent vectors and it develops a set of general relationships for obtaining the force balancing conditions of multi degrees of freedom, multi bar, and planar linkages. These conditions can be written directly without extracting them from the kinematic equations.

Let a chain of n links connected by revolute joints be pivoted about a frame pivot at one end (Fig. 2.3). The force balance is achieved by adding counterweights to each link. The counterweight on the *k-th* link must satisfy the following condition:

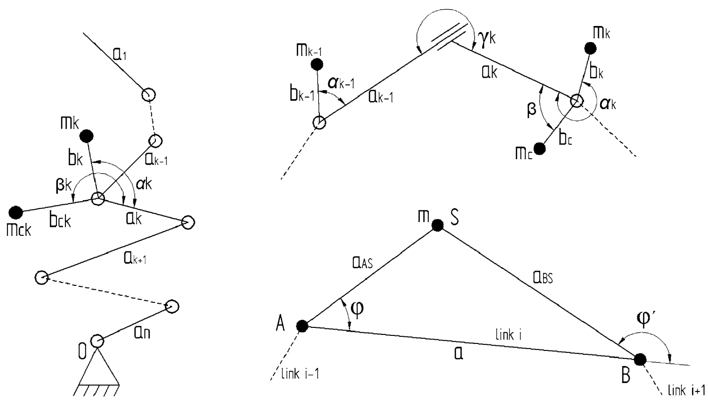


Fig. 2.3 Chain of n links with a counterweight attached to link k (*left*), prismatic joint with a counterweight attached to link k (*above*), and a dependent link i (*below*)

$$m_{\mathrm{ck}} b_{\mathrm{ck}} e^{i\beta_{\mathrm{k}}} + m_{\mathrm{k}} b_{\mathrm{k}} e^{i\alpha_{\mathrm{k}}} + a_{\mathrm{k}} \sum_{i=1}^{k-1} (m_i + m_{\mathrm{c}i}) \tag{2.27}$$

where i is a number of each link starting at the free end, m_i and m_{ci} are the masses of the links and counterweights, a_i the length of a link, and b_i, b_{ci}, α_i, and β_i are the radial and angular coordinates of the mass centers with respect to their lower joint and link.

Similarly, the condition for counterweight balancing of two links connected with prismatic joint is derived. One counterweight for the force balance is sufficient if the next condition is satisfied:

$$m_{\mathrm{c}} b_{\mathrm{c}} e^{i\beta} + m_{\mathrm{k}-1} b_{\mathrm{k}-1} e^{i\alpha_{\mathrm{k}-1}} + m_{\mathrm{k}} b_{\mathrm{k}} e^{i(\alpha_{\mathrm{k}}-\gamma)} = 0 \tag{2.28}$$

Only one prismatic joint in a loop can be contained if a linkage is to be force balanced.

It is not necessary to balance each link to obtain a force balance. If a mechanism contains a loop without prismatic joint, then one arbitrary link can be uncounterweighted. A loop containing one prismatic joint can have one link connected to the prismatic joint uncounterweighted. The loop is then divided into two counterweighted chains. For that reason, the uncounterweighted link, often called a dependent link, has to split its mass into both joints (see Fig. 2.3) according to the rules

$$m_{\mathrm{A}} = m \frac{a_{\mathrm{BS}}}{a} e^{i(\varphi' + \pi)} \tag{2.29}$$

$$m_{\mathrm{B}} = m \frac{a_{\mathrm{AS}}}{a} e^{i\varphi} \tag{2.30}$$

Generally a k-th link of a length a_k which lies in a chain between joints k-1/k and k/k + 1 can contain x masses m_t assigned to a joint as a result of a dependent link or another counterweighted chain being incident at or masses assigned to joint or links higher up the chain. This link can also have u revolute joints with y_{q} masses m_{d} assigned to the *q-th* joint. The *q-th* joint is in offset from the a_{k} link by an angle δ_{q} and in distance l_{k} from the joint k/k + 1. The link has v other links attached by prismatic joints. Masses m_{b} are assigned to the joint k/k + 1 and they are in offset from the a_{k} link by an angle η_{p}. The general force balance condition is

$$m_{\mathrm{k}} b_{\mathrm{k}} e^{i\alpha_{\mathrm{k}}} + m_{\mathrm{ck}} b_{\mathrm{ck}} e^{i\beta_{\mathrm{k}}} + a_{\mathrm{k}} \left[\sum_{t=1}^{x} m_t + \sum_{i=1}^{n} (m_i + m_{\mathrm{c}i}) \right] + \sum_{q=1}^{u} \left(l_{\mathrm{q}} e^{i\delta_{\mathrm{q}}} + \sum_{d=1}^{y_{\mathrm{q}}} m_{\mathrm{d}} \right) + \sum_{p=1}^{v} \sum_{b=1}^{z} m_b e^{i\eta_{\mathrm{p}}} = 0 \tag{2.31}$$

Eqs. (2.29) and (2.31) form a necessary and sufficient set for establishing the force balance conditions of planar linkages.

The extension of this method defines the minimum number of required counterweights and the most advantageous configuration of counterweights. In Ref. [7], the minimum number of counterweights c required to fully force balance linkages with n moving links and j simple joints is derived, which is given by

$$c = 2\,(n - 1) - j \tag{2.32}$$

This expression applies to any general planar linkage that can be force balanced irrespective of the number of degrees of freedom it has. However, some linkages with special geometries or mass distributions can be balanced with a smaller number of counterweights. If a planar linkage has only one degree of freedom, then Eq. (2.32) is simplified to the expression

$$c = \frac{n}{2} \tag{2.33}$$

which is in agreement with the conclusion in [6].

It was said that the addition of counterweights increases the dynamic parameters of mechanisms. For the given linkage, there can be many combinations of its links which can be counterweighted to give a force balance. The best results with respect to bearing forces, shaking moment, and driving torque are obtained if the chosen counterweighted links are as near as possible to the ground pivots.

The complex mass method was further extended to balance spatial linkages. Ye and Smith [8] developed an equivalence method for complete balancing of planar linkages. By this method, a complex planar linkage can be converted into a number of simple equivalent sub-linkages and cranks.

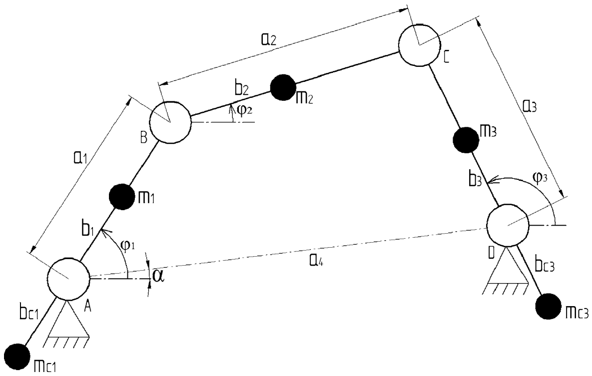
2.5 Linear Momentum Method

A very general method for deriving balancing conditions is based on equation of linear momentum which is for the force balanced mechanism constant (Eq. (2.3)). This method compared to the previous method requires calculation of derivatives and can be difficult and long, but the equations and conditions can be obtained for any linkage with a proposed balancing device. The principle of this method and derivation of balancing condition are shown again on the four-bar linkage.

The linear momentum of the four-bar linkage with two counterweights (Fig. 2.4) is given by

$$m_1\dot{\boldsymbol{r}}_{t1} + m_2\dot{\boldsymbol{r}}_{t2} + m_3\dot{\boldsymbol{r}}_{t3} + m_{c1}\dot{\boldsymbol{r}}_{tc1} + m_{c2}\dot{\boldsymbol{r}}_{tc2} = 0 \tag{2.34}$$

Fig. 2.4 The four-bar linkage with two counterweights attached to the crank and the rocker



where $\boldsymbol{r}_{ti}$ are the velocities of the masses m_i. The position vectors $\boldsymbol{r}_{ti}$ of individual centers of mass with reference origin at point A are

$$\boldsymbol{r}_{t1} = \begin{pmatrix} b_1 \cos\varphi_1 \\ b_1 \sin\varphi_1 \end{pmatrix} \tag{2.35}$$

$$\boldsymbol{r}_{c1} = \begin{pmatrix} -b_{c1} \cos\varphi_1 \\ -b_{c1} \sin\varphi_1 \end{pmatrix} \tag{2.36}$$

$$\boldsymbol{r}_{t2} = \begin{pmatrix} a_1 \cos\varphi_1 + b_2 \cos\varphi_2 \\ a_1 \sin\varphi_1 + b_2 \sin\varphi_2 \end{pmatrix} \tag{2.37}$$

$$\boldsymbol{r}_{t3} = \begin{pmatrix} a_4 \cos\alpha + b_3 \cos\varphi_3 \\ a_4 \sin\alpha + b_3 \sin\varphi_3 \end{pmatrix} \tag{2.38}$$

$$\boldsymbol{r}_{c3} = \begin{pmatrix} a_4 \cos\alpha - b_{c3} \cos\varphi_3 \\ a_4 \sin\alpha - b_{c3} \sin\varphi_3 \end{pmatrix} \tag{2.39}$$

Derivatives of these equations are substituted to Eq. (2.34). After some modifications, the first component of this equation is written as

$$\begin{aligned} &(m_1 b_1 - m_{c1} b_{c1}) \sin\varphi_1 \dot{\varphi}_1 + m_2 (a_1 \sin\varphi_1 \dot{\varphi}_1 + b_2 \sin\varphi_2 \dot{\varphi}_2) \\ &\quad + (m_3 b_3 - m_{c3} b_{c3}) \sin\varphi_3 \dot{\varphi}_3 = 0 \end{aligned} \tag{2.40}$$

The second component differs from the first only in terms with function *sin* where the second equation has function *cos*. The terms $\sin\varphi_2\dot{\varphi}_2$ are derived from the loop equation

$$\begin{pmatrix} a_1 \cos\varphi_1 + a_2 \cos\varphi_2 \\ a_1 \sin\varphi_1 + a_2 \sin\varphi_2 \end{pmatrix} = \begin{pmatrix} a_4 \cos\alpha + l_3 \cos\varphi_3 \\ a_4 \sin\alpha + l_3 \sin\varphi_3 \end{pmatrix} \tag{2.41}$$

Eq. (2.40) is then expressed as

$$\left(m_1 b_1 + m_2 a_1 - m_{c1} b_{c1} - \frac{m_2 b_2 a_1}{a_2}\right) \sin\varphi_1 \dot{\varphi}_1 + \left(m_3 b_3 - m_{c3} b_{c3} + \frac{m_2 b_2 a_3}{a_2}\right) \sin\varphi_3 \dot{\varphi}_3 = 0 \tag{2.42}$$

and the conditions of balancing are given by

$$\frac{m_1 b_1}{a_1} + \frac{m_2 (a_2 - b_2)}{a_2} - \frac{m_{c1} b_{c1}}{a_1} = 0 \tag{2.43}$$

$$\frac{m_3 b_3}{a_3} + \frac{m_2 b_2}{a_2} - \frac{m_{c3} b_{c3}}{a_3} = 0 \tag{2.44}$$

This method works well for other configurations of counterweights and other linkages. The same approach can be applied for moment balancing, only the angular momentum instead of linear momentum is used. Moment balancing usually requires addition of counter-rotations or other balancing methods which are described in the corresponding chapters.

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