Chapter 19
Balancing Conditions of Planar and Spatial Mechanisms in the Algebraic Form

Nguyen Van Khang and Nguyen Phong Dien

Abstract This chapter deals with an approach to formulate balancing conditions for the shaking force and shaking moment of planar mechanisms and spatial mechanisms. In the Mechanism Theory, every Mechanism has *p* moving members and a non-moving frame. According to tradition, a planar 8R-eightbar mechanism is a multibody system with 7 moving bodies.

Keywords Mechanism • Balancing condition • Shaking force • Shaking moment

19.1 Introduction

Dynamic balancing of mechanisms is a classical problem of machine dynamics [1–9]. Dynamic balancing of the moving links brings about a reduction of the variable dynamic loads on the mechanism frame. In effect, this minimizes the noise and wear, and improves the dynamic performance of the mechanism [3, 4]. The main objective of mass balancing is to completely eliminate or partially reduce the resultant inertia force (shaking force) and the resultant inertia moment with respect to the ground link (shaking moment) caused by all moving links of a mechanism. Although different methods and solutions have been proposed and reported, the balancing theory continues to develop and new approaches are regularly being published. Summaries of much of the past work are given in refs. [2–4]. Recently, the terminology "reactionless mechanism" has usually been used in design and dynamic synthesis of mechanisms, e.g., [19, 20]. A mechanism is said to be reactionless or dynamically balanced if the shaking force and the shaking moment are completely eliminated for any arbitrary motion of the mechanism.

N. Van Khang (✉) • N.P. Dien
Department of Applied Mechanics, Hanoi University of Science and Technology, 1. Dai Co Viet Road, Hanoi, Vietnam
e-mail: khang.nguyenvan2@hust.edu.vn; dien.nguyenphong@hust.edu.vn

D. Zhang, B. Wei (eds.), *Dynamic Balancing of Mechanisms and Synthesizing of Parallel Robots*, DOI 10.1007/978-3-319-17683-3_19

In other words, no dynamic reaction forces and no dynamic reaction moments are transmitted to the base during the motion.

In our opinion, the problem of shaking force and shaking moment balancing consists of two aspects. The first is to find all feasible design solutions (mass redistribution, using counterweights or adding supplementary members as cams, gears, parallelogram chains, planetary gears, etc.) in order to compensate the shaking force and shaking moment. For this purpose different approaches and solutions have been developed and reported. Berkof [11] presented a review of the methods based on the different movements of the counterweights for the shaking force balancing. Feng [30] used the concept of inertia counterweight proposed by Berkof [13] to carry out the dynamic balancing of a number of single degree freedom mechanisms. The publications by Lowen et al. [7] and Kochev [16] provide a critical review of the methods employing additional members for complete shaking moment balancing. Arakelian and Smith [9] investigated the dynamic balancing of single degree of freedom mechanisms by using the pantograph copying properties. A number of other solutions for the complete shaking force and shaking moment balancing can be found in the studies presented by Kochev [17], Wu and Gosselin [21], Dresig et al. [14, 15], Arakelian [26–28], and Moore [32].

The second aspect is related to the formulation of balancing conditions which are usually expressed in terms of the design variables (such as masses, moments of inertia, and geometrical parameters of the links) of the mechanism. There are several convenient ways to formulate balancing conditions of the shaking force. For instance, the *method of linearly independent vectors* was proposed by Berkof and Lowen [11] and later successfully employed by Kaufman and Sandor [12], Feng [31] to obtain full force balancing conditions for linkages, the *equivalence method* was proposed by Ye and Smith [18]. The *method of principal vectors* was used by Shchepetilnikov [10] to investigate the static balancing conditions of mechanisms. Because the shaking force is related to the first derivative of the total linear momentum with respect to time, the *linear momentum method* can also be used to establish balancing conditions of the shaking force [16, 31]. Conversely, research on efficient methods for deriving balancing conditions of the shaking moment has been less productive due to the complexity of the problem. It is well known that the shaking moment of a mechanism is related to the first derivative of the total angular momentum with respect to time. This relationship leads to an approach for the formulation of balancing conditions of the shaking moment, known as the *angular momentum method*. This method was used by several authors such as Kochev [9, 10], Feng [31], and Nguyen [22–25]. Arakelian and Dahan [27] formulated the moment balancing conditions of a multi-link planar mechanism by minimizing the root-mean-square value of the resultant inertia moment. Another recent approach to derive balancing conditions of planar multi-loop mechanisms using the equivalent method is investigated by Chaudhary and Saha [6, 29].

In contrast to the rapid progress in balancing theory of planar mechanisms, the development on the balancing theory of spatial mechanisms is still limited. Balancing methods of planar mechanisms cannot be directly applicable to spatial mechanisms since kinematic and dynamic properties of spatial mechanisms are much more complicated. The literature on this respect therefore is little [22, 24, 33–41]. One of the problems of the complete shaking force and shaking moment balancing of the mechanism consist of the deriving the so-called balancing conditions. These balancing conditions will be used to determine the size and location of counterweights or supplementary links which must be added to the initial mechanism, in order to eliminate the shaking force and the shaking moment.

Using the methods of multibody dynamics, this chapter deals with an approach to derive balancing conditions in the algebraic form for the shaking force and shaking moment of planar and spatial multi-loop mechanisms. The developed methods are suitable for the application of the widely accessible computer algebra systems such as MAPLE®. In the examples, the conditions for complete shaking force and shaking moment balancing of a planar multi-loop, multi-DOF mechanism and a spatial one-DOF mechanism are given.

19.2 Balancing Theory of Constrained Multibody Systems

We consider a multibody system with holonomic and rheonomic constraints as a set of p linked rigid bodies in a closed loop structure shown in Fig. 19.1.

The shaking force $\overrightarrow{F}^*$ and the shaking moment $\overrightarrow{M}^*_{\mathrm{O}}$ referred to a fixed point O of the considered system, which are caused by all moving bodies, can be expressed in the form [1, 2, 22, 24]

$$\overrightarrow{F}^* = -\frac{{}^{R_O}d}{\mathrm{d}t}\sum_{i=1}^{p} m_i \overrightarrow{v}_{Si}, \tag{19.1}$$

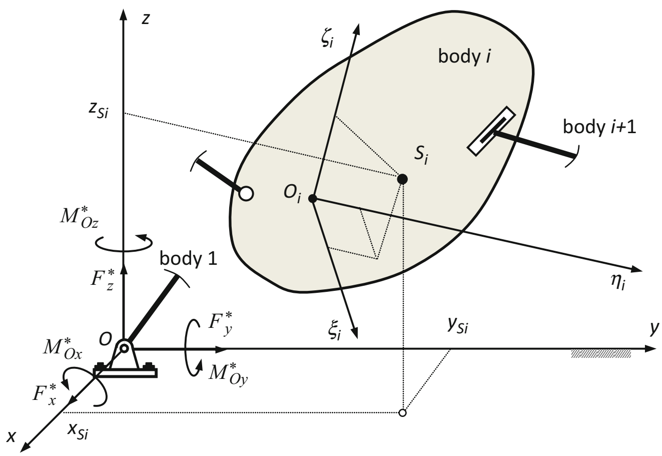


Fig. 19.1 Coordinate frames and the center of mass of body i

$$\overrightarrow{M}_{\mathrm{O}}^{*} = -\frac{{}^{R_O}d}{\mathrm{d}t}\sum_{i=1}^{p}\left(\overrightarrow{\overrightarrow{I}}_{Si}\cdot\overrightarrow{\omega}_{i}+\overrightarrow{r}_{Si}\times m_{i}\overrightarrow{v}_{Si}\right). \quad (19.2)$$

In Eqs. (19.1) and (19.2) the following symbols are used:

m_i mass of body i.
p number of bodies.
$\overrightarrow{r}_{Si}$ position vector of center of mass S_i of body i in the fixed coordinate frame $R_0\{x, y, z\}$.
$\overrightarrow{v}_{Si}$ velocity vector of center of mass S_i in the coordinate frame R_0.
$\overrightarrow{\overrightarrow{I}}_{Si}$ mass inertia tensor of body i referred to S_i.
$\overrightarrow{\omega}_i$ angular velocity of body i with respect to the coordinate frame R_0.

The multibody system is completely balanced if the shaking force and the shaking moment vanish at every position [1, 2]

$$\overrightarrow{F}^{*} = 0, \quad \overrightarrow{M}_{\mathrm{O}}^{*} = 0. \quad (19.3)$$

It follows that

$$\frac{{}^{R_O}d}{\mathrm{d}t}\sum_{i=1}^{p} m_i\overrightarrow{v}_{Si} = 0, \quad (19.4)$$

$$\frac{{}^{R_O}d}{\mathrm{d}t}\sum_{i=1}^{p}\left(\overrightarrow{\overrightarrow{I}}_{Si}\cdot\overrightarrow{\omega}_{i}+\overrightarrow{r}_{Si}\times m\overrightarrow{v}_{Si}\right) = 0. \quad (19.5)$$

Equations (19.4) and (19.5) can be rewritten in the matrix form as follows:

$$\frac{d}{\mathrm{d}t}\sum_{i=1}^{p} m_i\mathbf{v}_{Si} = 0, \quad (19.6)$$

$$\frac{d}{\mathrm{d}t}\left[\sum_{i=1}^{p}\left(\mathbf{I}_{Si}\boldsymbol{\omega}_{i}+m_{i}\tilde{\mathbf{r}}_{Si}\mathbf{v}_{Si}\right)\right] = 0, \quad (19.7)$$

where

$$\mathbf{r}_{Si} = \begin{bmatrix} x_{Si} \\ y_{Si} \\ z_{Si} \end{bmatrix}, \quad \tilde{\mathbf{r}}_{Si} = \begin{bmatrix} 0 & -z_{Si} & y_{Si} \\ z_{Si} & 0 & -x_{Si} \\ -y_{Si} & x_{Si} & 0 \end{bmatrix}. \quad (19.8)$$

For a f-DOF stationary multibody system described by n generalized coordinates q_1, q_2, …, q_n and $n \geq f$, position vector $\mathbf{r}_{Si}$ can be expressed in form of generalized coordinates

$$\mathbf{r}_{Si} = \mathbf{r}_{Si}\left(q_1,\ q_2, \ldots,\ q_n\right),\ \left(i = 1,\ 2, \ldots,\ p\right). \tag{19.9}$$

Differentiating Eq. (19.9) with respect to time in the coordinate frame R_0 yields

$$\mathbf{v}_{Si} = \frac{\mathrm{d}\mathbf{r}_{Si}}{\mathrm{d}t} = \frac{\partial \mathbf{r}_{Si}}{\partial \mathbf{q}}\dot{\mathbf{q}} = \mathbf{J}_{Ti}\left(\mathbf{q}\right)\dot{\mathbf{q}}, \tag{19.10}$$

where $\mathbf{J}_{Ti}(\mathbf{q})$ is the translation Jacobi matrix

$$\mathbf{J}_{Ti}\left(\mathbf{q}\right) = \frac{\partial \mathbf{r}_{Si}}{\partial \mathbf{q}} = \begin{bmatrix} \frac{\partial x_{Si}}{\partial q_1} & \frac{\partial x_{Si}}{\partial q_2} & \cdots & \frac{\partial x_{Si}}{\partial q_n} \\ \frac{\partial y_{Si}}{\partial q_1} & \frac{\partial y_{Si}}{\partial q_2} & \cdots & \frac{\partial y_{Si}}{\partial q_n} \\ \frac{\partial z_{Si}}{\partial q_1} & \frac{\partial z_{Si}}{\partial q_2} & \cdots & \frac{\partial z_{Si}}{\partial q_n} \end{bmatrix}, \tag{19.11}$$

and $\mathbf{q} = \left[q_1,\ q_2, \ldots,\ q_n\right]^T$. By introducing φ_i as the rotation vector of body i, the angular velocity $\boldsymbol{\omega}_i$ is defined by

$$\boldsymbol{\omega}_i = \frac{\mathrm{d}\boldsymbol{\varphi}_i}{\mathrm{d}t} = \frac{\partial \boldsymbol{\varphi}_i}{\partial \mathbf{q}}\dot{\mathbf{q}} = \mathbf{J}_{Ri}\left(\mathbf{q}\right)\dot{\mathbf{q}}, \tag{19.12}$$

where $\mathbf{J}_{Ri}(\mathbf{q})$ denotes the rotation Jacobi matrix

$$\mathbf{J}_{Ri} = \frac{\partial \boldsymbol{\varphi}_i}{\partial \mathbf{q}} = \frac{\partial \boldsymbol{\omega}_i}{\partial \dot{\mathbf{q}}} = \begin{bmatrix} \frac{\partial \omega_{ix}}{\partial \dot{q}_1} & \frac{\partial \omega_{ix}}{\partial \dot{q}_2} & \cdots & \frac{\partial \omega_{ix}}{\partial \dot{q}_n} \\ \frac{\partial \omega_{iy}}{\partial \dot{q}_1} & \frac{\partial \omega_{iy}}{\partial \dot{q}_2} & \cdots & \frac{\partial \omega_{iy}}{\partial \dot{q}_n} \\ \frac{\partial \omega_{iz}}{\partial \dot{q}_1} & \frac{\partial \omega_{iz}}{\partial \dot{q}_2} & \cdots & \frac{\partial \omega_{iz}}{\partial \dot{q}_n} \end{bmatrix}. \tag{19.13}$$

Substitution of Eq. (19.10) into Eq. (19.6) yields

$$\frac{d}{\mathrm{d}t}\left\{\left[\sum_{i=1}^{p} m_i \mathbf{J}_{Ti}\left(\mathbf{q}\right)\right]\dot{\mathbf{q}}\right\} = 0. \tag{19.14}$$

Substituting Eqs. (19.10) and (19.12) into Eq. (19.7), one obtains

$$\frac{d}{\mathrm{d}t}\left\{\left[\sum_{i=1}^{p}\left(\mathbf{I}_{Si}\mathbf{J}_{Ri}\left(\mathbf{q}\right) + m_i \tilde{\mathbf{r}}_{Si}\mathbf{J}_{Ti}\left(\mathbf{q}\right)\right)\right]\dot{\mathbf{q}}\right\} = 0. \tag{19.15}$$

Note that the inertia matrix $\mathbf{I}_{Si}$ with respect to the fixed frame R_0 can be written in term of the matrix $\mathbf{I}_{Si}^{(i)}$ using the formula

$$\mathbf{I}_{Si} = \mathbf{A}_i \mathbf{I}_{Si}^{(i)} \mathbf{A}_i^T, \tag{19.16}$$

where $\mathbf{A}_i$ denotes the direction cosine matrix of body i referred to the fixed frame R_0, $\mathbf{I}_{Si}^{(i)}$ is the matrix of the mass inertia tensor relative to the axes of the body-fixed coordinate system $R_i\{\xi_i, \eta_i, \zeta_i\}$ (see Fig. 19.1).

It follows that

$$\mathbf{I}_{Si}\mathbf{J}_{Ri}(\mathbf{q}) = \mathbf{A}_i \mathbf{I}_{Si}^{(i)} \mathbf{A}_i^T \mathbf{J}_{Ri}(\mathbf{q}). \tag{19.17}$$

Since $\mathbf{A}_i = \mathbf{A}_i(\mathbf{q})$ and $\frac{\partial \mathbf{A}_i}{\partial \dot{\mathbf{q}}} = 0$, it follows

$$\mathbf{J}_{Ri}(\mathbf{q}) = \frac{\partial \boldsymbol{\omega}_i}{\partial \dot{\mathbf{q}}} = \frac{\partial \left(\mathbf{A}_i \boldsymbol{\omega}_i^{(i)}\right)}{\partial \dot{\mathbf{q}}} = \mathbf{A}_i(\mathbf{q}) \frac{\partial \boldsymbol{\omega}_i^{(i)}}{\partial \dot{\mathbf{q}}} = \mathbf{A}_i \mathbf{J}_{Ri}^{(i)}, \tag{19.18}$$

where matrix $\mathbf{J}_{Ri}^{(i)}(\mathbf{q})$ is defined by

$$\mathbf{J}_{Ri}^{(i)}(\mathbf{q}) = \frac{\partial \boldsymbol{\omega}_i^{(i)}}{\partial \dot{\mathbf{q}}}. \tag{19.19}$$

Substitution of Eqs. (19.18) and (19.19) into Eq. (19.15) yields

$$\frac{d}{\mathrm{d}t}\left\{\left[\sum_{i=1}^{p} \mathbf{A}_i \mathbf{I}_{Si}^{(i)} \mathbf{J}_{Ri}^{(i)}(\mathbf{q}) + m_i \tilde{\mathbf{r}}_{Si} \mathbf{J}_{Ti}(\mathbf{q})\right] \dot{\mathbf{q}}\right\} = 0. \tag{19.20}$$

It follows from Eqs. (19.14) and (19.20) the general balancing conditions of a multibody system

$$\sum_{i=1}^{p} m_i \mathbf{J}_{Ti}(\mathbf{q}) = 0, \tag{19.21}$$

$$\sum_{i=1}^{p} \left[\mathbf{A}_i \mathbf{I}_{Si}^{(i)} \mathbf{J}_{Ri}^{(i)}(\mathbf{q}) + m_i \tilde{\mathbf{r}}_{Si} \mathbf{J}_{Ti}(\mathbf{q})\right] = 0. \tag{19.22}$$

19.3 Balancing Conditions of Planar Mechanisms

19.3.1 Theory and Procedure for Deriving Dynamic Balancing Conditions

19.3.1.1 General Balancing Conditions

We consider an arbitrary link of a multi DOF planar mechanism as depicted in Fig. 19.2. The mechanism consists of a set of p moving links in a closed loop structure with revolute joints. Parameters x_{Si}, y_{Si} are the coordinates of the center of mass S_i of link i in the ground-fixed coordinate frame $\{Oxy\}$, φ_i is the rotation angle, ξ_{Si}, η_{Si} are coordinates of S_i in the link-fixed coordinate frame $\{O_i\xi_i\eta_i\}$.

From Eqs. (19.1) and (19.2) the shaking force and the shaking moment transmitted to the base from all moving links can be expressed in the form [15]

$$F_x^* = -\frac{d}{\mathrm{d}t}\left(\sum_{i=1}^{p} m_i \dot{x}_{Si}\right),\ F_y^* = -\frac{d}{\mathrm{d}t}\left(\sum_{i=1}^{p} m_i \dot{y}_{Si}\right) \tag{19.23}$$

$$M_{\mathrm{O}}^* = \frac{d}{\mathrm{d}t}\left\{\sum_{i=1}^{p}\left[m_i\left(x_{Si}\dot{y}_{Si} - y_{Si}\dot{x}_{Si}\right) + J_{Si}\dot{\varphi}_i\right]\right\} \tag{19.24}$$

where m_i denotes the mass and I_{Si} the moment of inertia of the link about the axis passing through S_i and perpendicular to the plane of motion.

The planar mechanism can then be completely balanced if the shaking force and the shaking moment vanish. This yields the following sufficient conditions

$$\sum_{i=1}^{p} m_i \dot{\mathbf{r}}_i = 0, \tag{19.25}$$

$$\sum_{i=1}^{p}\left[m_i\left(x_{Si}\dot{y}_{Si} - y_{Si}\dot{x}_{Si}\right) + I_{Si}\dot{\varphi}_i\right] = 0, \tag{19.26}$$

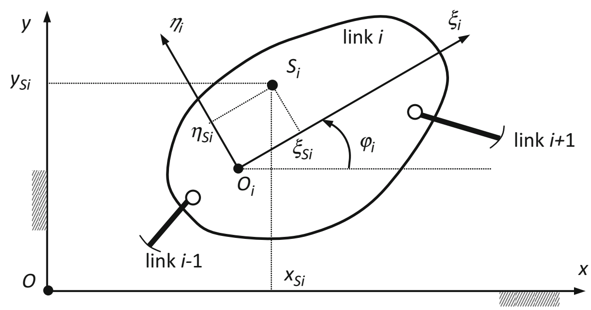


Fig. 19.2 Definition of parameters and coordinates

where $\mathbf{r}_i = [x_{Si},\ y_{Si}]^T$ and $\dot{\mathbf{r}}_i = [\dot{x}_{Si},\ \dot{y}_{Si}]^T$. Based on the general condition (19.25) for the shaking force balancing, there are some ways to derive the balancing conditions in form of algebraic expressions of parameters m_i, I_{Si}, ξ_{Si} and η_{Si} as mentioned in the previous section. Conversely, it is more difficult to formulate the dynamic balancing conditions of the shaking moment due to the presence of the term $I_{Si}\dot{\varphi}_i$ in Eq. (19.26).

19.3.1.2 Generalized Coordinates of the Second Type

Since the considered mechanism has only revolute joints, rotation angles φ_i $(i = 1,\ 2, \ldots p)$ can be chosen as generalized coordinates which describe the motion of particular links. Angle φ_i is known as "the generalized coordinates of the first type." Now we introduce vector $\mathbf{u}$

$$\mathbf{u} = \left[\cos\varphi_1, \sin\varphi_1, \ldots, \cos\varphi_p, \sin\varphi_p\right]^T, \tag{19.27}$$

where elements u_k $(k = 1,\ 2, \ldots, 2p)$ are trigonometric functions of φ_i. Logically, elements u_k are called "the generalized coordinates of the second type." As can be seen later, vector $\mathbf{u}$ can be used as the basis for developing a systematic procedure for deriving balancing conditions of the shaking force and moment.

19.3.1.3 Procedure to Derive Balancing Conditions of the Shaking Force

Generally, the position vector of the center of mass S_i can always be expressed in term of vector $\mathbf{u}$ as

$$\mathbf{r}_i = \mathbf{e}_i^u + \mathbf{C}_i\mathbf{u}, \tag{19.28}$$

where $i = 1, 2,\ \ldots, p$ and $\mathbf{e}_i^u$ is a vector of constants. The elements of matrix $\mathbf{C}_i$ $(2 \times 2p)$ are geometrical parameters and independent of $\mathbf{u}$. Similarly, the loop equations of the mechanism can be expressed in the compact matrix form

$$\mathbf{f}(\mathbf{u}) = \mathbf{d}. \tag{19.29}$$

In the cases of planar mechanisms articulated by revolute joints, Eq. (19.7) can be rewritten in linear form with vector $\mathbf{u}$

$$\mathbf{D}\,\mathbf{u} = \mathbf{d}, \tag{19.30}$$

where the elements of matrix $\mathbf{D}$ are geometrical parameters of the mechanism and independent of $\mathbf{u}$, vector $\mathbf{d}$ is constant. It follows that Eq. (19.28) can then be rewritten in term of a minimal set of elements u_k of $\mathbf{u}$. The following partitioning of $\mathbf{u}$ from Eq. (19.30)

$$\mathbf{u} = \begin{bmatrix} \mathbf{v} \\ \mathbf{w} \end{bmatrix}, \tag{19.31}$$

leads to the following relationship

$$\mathbf{D}^v\mathbf{v} + \mathbf{D}^w\mathbf{w} = \mathbf{d}. \tag{19.32}$$

where vector $\mathbf{v}$ consists of elements from this set, and the dimension of vector $\mathbf{w}$ is equal to the number of the loop equations. Matrix $\mathbf{D}^w$ is chosen so that it is a square and nonsingular matrix. When vectors $\mathbf{v}$ and $\mathbf{w}$ are assigned, an easily way to obtain matrices $\mathbf{D}^v$ and $\mathbf{D}^w$ is by taking the partial derivatives

$$\mathbf{D}^v = \frac{\partial \mathbf{f}}{\partial \mathbf{v}},\ \mathbf{D}^w = \frac{\partial \mathbf{f}}{\partial \mathbf{w}}. \tag{19.33}$$

From Eq. (19.32) we find

$$\mathbf{w} = \left(\mathbf{D}^w\right)^{-1}\left(\mathbf{d} - \mathbf{D}^v\mathbf{v}\right) = \mathbf{b} - \mathbf{G}\mathbf{v}, \tag{19.34}$$

where

$$\mathbf{G} = \left(\mathbf{D}^w\right)^{-1}\mathbf{D}^v,\ \mathbf{b} = \left(\mathbf{D}^w\right)^{-1}\mathbf{d}. \tag{19.35}$$

Differentiating Eq. (19.34) with respect to time yields

$$\dot{\mathbf{w}} = -\mathbf{G}\dot{\mathbf{v}}. \tag{19.36}$$

Using Eq. (19.31) one can rewrite Eq. (19.28) in the following form

$$\mathbf{r}_i = \mathbf{e}_i^u + \mathbf{C}_i^v\mathbf{v} + \mathbf{C}_i^w\mathbf{w}, \tag{19.37}$$

where matrices $\mathbf{C}_i^v$, $\mathbf{C}_i^w$ are given by

$$\mathbf{C}_i^v = \frac{\partial \mathbf{r}_i}{\partial \mathbf{v}},\ \mathbf{C}_i^w = \frac{\partial \mathbf{r}_i}{\partial \mathbf{w}}, \tag{19.38}$$

and the vector of constant parameters $\mathbf{e}_i^u$ is the remaining term from Eq. (19.15).

Substitution of Eq. (19.34) into Eq. (19.37) yields

$$\mathbf{r}_i = \mathbf{e}_i^u + \mathbf{C}_i^w\mathbf{b} + \left(\mathbf{C}_i^v - \mathbf{C}_i^w\mathbf{G}\right)\mathbf{v}. \tag{19.39}$$

This can be written as

$$\mathbf{r}_i = \mathbf{e}_i + \mathbf{B}_i \mathbf{v}, \tag{19.40}$$

where

$$\mathbf{e}_i = \mathbf{e}_i^u + \mathbf{C}_i^w \mathbf{b}, \tag{19.41}$$

$$\mathbf{B}_i = \mathbf{C}_i^v - \mathbf{C}_i^w \mathbf{G}. \tag{19.42}$$

Note that the elements of vector $\mathbf{e}_i$ and matrix $\mathbf{B}_i$ are geometrical parameters of the mechanism and independent of $\mathbf{v}$. Differentiating Eq. (19.40) with respect to time yields

$$\dot{\mathbf{r}}_i = \mathbf{B}_i \dot{\mathbf{v}}. \tag{19.43}$$

Substituting Eq. (19.43) into Eq. (19.25) leads to

$$\sum_{i=1}^{p} m_i \mathbf{B}_i \dot{\mathbf{v}} = 0. \tag{19.44}$$

As a result, the balancing conditions for the shaking force reduce to the algebraic form

$$\sum_{i=1}^{p} m_i \mathbf{B}_i = \mathbf{0}. \tag{19.45}$$

If the mechanism has p moving links and r loop equations, then vector $\mathbf{w}$ contains r elements whereas matrix $\mathbf{B}_i$ has the dimension of $2 \times (2p - r)$. From Eq. (19.45) we obtain $2\,(2p - r)$ balancing conditions in form of algebraic expressions of inertia and geometrical parameters.

19.3.1.4 Procedure to Derive Balancing Conditions of the Shaking Moment

The general balancing condition of the shaking moment according to Eq. (19.4) contains two terms. The first term is

$$h_1 = \sum_{i=1}^{p} m_i \left(x_{Si} \dot{y}_{Si} - y_{Si} \dot{x}_{Si} \right). \tag{19.46}$$

We note that

$$x_{Si} \dot{y}_{Si} - y_{Si} \dot{x}_{Si} = \mathbf{r}_i^T \mathbf{I}^* \ \dot{\mathbf{r}}_i, \tag{19.47}$$

where $\mathbf{I}^* = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. With the use of this relationship, Eq. (19.46) leads to

$$h_1 = \sum_{i=1}^{p} m_i \mathbf{r}_i^T \mathbf{I}^* \ \dot{\mathbf{r}}_i. \tag{19.48}$$

Substitution of Eqs. (19.40) and (19.43) into Eq. (19.48) yields

$$\begin{aligned} h_1 &= \sum_{i=1}^{p} m_i (\mathbf{e}_i + \mathbf{B}_i \mathbf{v})^T \mathbf{I}^* \ \mathbf{B}_i \dot{\mathbf{v}} \\ &= \mathbf{v}^T \left(\sum_{i=1}^{p} m_i \mathbf{B}_i^T \mathbf{I}^* \ \mathbf{B}_i \right) \dot{\mathbf{v}} + \left(\sum_{i=1}^{p} m_i \mathbf{e}_i^T \mathbf{I}^* \ \mathbf{B}_i \right) \dot{\mathbf{v}} \\ &= \mathbf{v}^T \mathbf{S}_1 \dot{\mathbf{v}} + \mathbf{k}_1^T \dot{\mathbf{v}}, \end{aligned} \tag{19.49}$$

where

$$\mathbf{S}_1 = \sum_{i=1}^{p} m_i \mathbf{B}_i^T \mathbf{I}^* \ \mathbf{B}_i, \quad \mathbf{k}_1^T = \sum_{i=1}^{p} m_i \mathbf{e}_i^T \mathbf{I}^* \ \mathbf{B}_i. \tag{19.50}$$

Now we consider the second term of Eq. (19.26)

$$h_2 = \sum_{i=1}^{p} I_{Si} \dot{\varphi}_i. \tag{19.51}$$

One can verify that

$$\dot{\varphi}_i = u_1^{(i)} \dot{u}_2^{(i)} - u_2^{(i)} \dot{u}_1^{(i)}, \tag{19.52}$$

where $u_1^{(i)} = \cos \varphi_i$ and $u_2^{(i)} = \sin \varphi_i$. Equation (19.52) can also rewritten in the matrix form as

$$\dot{\varphi}_i = \begin{bmatrix} u_1^{(i)} \\ u_2^{(i)} \end{bmatrix}^T \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \dot{u}_1^{(i)} \\ \dot{u}_2^{(i)} \end{bmatrix}. \tag{19.53}$$

Substitution of Eq. (19.53) into Eq. (19.51) yields

$$h_2 = \sum_{i=1}^{p} \begin{bmatrix} u_1^{(i)} \\ u_2^{(i)} \end{bmatrix}^T \begin{bmatrix} 0 & I_{Si} \\ -I_{Si} & 0 \end{bmatrix} \begin{bmatrix} \dot{u}_1^{(i)} \\ \dot{u}_2^{(i)} \end{bmatrix} = \mathbf{u}^T \mathbf{H} \dot{\mathbf{u}}, \tag{19.54}$$

where $\mathbf{H}$ is a $2p \times 2p$ matrix defined by

$$\mathbf{H} = \begin{bmatrix} 0 & I_{S1} & 0 & 0 & \dots & 0 & 0 \\ -I_{S1} & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & I_{S2} & \dots & 0 & 0 \\ 0 & 0 & -I_{S2} & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 & I_{Sp} \\ 0 & 0 & 0 & 0 & \dots & -I_{Sp} & 0 \end{bmatrix} \tag{19.55}$$

Matrix $\mathbf{H}$ can be partitioned in four sub-matrices corresponding to vectors $\mathbf{v}$ und $\mathbf{w}$ as follows:

$$\mathbf{H} = \left[\frac{\mathbf{H}_1\ \mathbf{H}_2}{\mathbf{H}_3\ \mathbf{H}_4},\right] \tag{19.56}$$

where $\mathbf{H}_1$ is a $(2p-r)\times(2p-r)$ matrix, $\mathbf{H}_2$ is a $(2p-r)\times r$ matrix of zero, $\mathbf{H}_3$ is a $r\times(2p-r)$ matrix of zeros and $\mathbf{H}_4$ a $r\times r$ matrix. Then Eq. (19.54) takes the form

$$h_2 = \begin{bmatrix}\mathbf{V}^T & \mathbf{W}^T\end{bmatrix}\begin{bmatrix}\frac{\mathbf{H}_1}{\mathbf{0}} & \frac{\mathbf{0}}{\mathbf{H}_4}\end{bmatrix}\begin{bmatrix}\frac{\dot{\mathbf{V}}}{\dot{\mathbf{W}}}\end{bmatrix} = \mathbf{V}^T\mathbf{H}_1\dot{\mathbf{V}} + W^T\mathbf{H}_4\dot{\mathbf{W}}. \tag{19.57}$$

Substitution of Eqs. (19.34) and (19.35) into Eq. (19.57) yields

$$\begin{aligned} h_2 &= \mathbf{v}^T\mathbf{H}_1\dot{\mathbf{v}} + (\mathbf{b}-\mathbf{G}\mathbf{v})^T\mathbf{H}_4\,(-\mathbf{G}\dot{\mathbf{v}}) \\ &= \mathbf{v}^T\left(\mathbf{H}_1 + \mathbf{G}^T\mathbf{H}_4\mathbf{G}\right)\dot{\mathbf{v}} - \left(\mathbf{b}^T\mathbf{H}_4\mathbf{G}\right)\dot{\mathbf{v}} \\ &= \mathbf{v}^T\mathbf{S}_2\dot{\mathbf{v}} + \mathbf{k}_2^T\dot{\mathbf{v}}. \end{aligned} \tag{19.58}$$

where matrix $\mathbf{S}_2$ and vector $\mathbf{k}_2$ are defined by

$$\mathbf{S}_2 = \mathbf{H}_1 + \mathbf{G}^T\mathbf{H}_4\mathbf{G}, \tag{19.59}$$

$$\mathbf{k}_2^T = -\mathbf{b}^T\mathbf{H}_4\mathbf{G}. \tag{19.60}$$

Using Eqs. (19.49) and (19.58), the general balancing condition of the shaking moment can be written in the matrix form as

$$\mathbf{v}^T(\mathbf{S}_1+\mathbf{S}_2)\,\dot{\mathbf{v}} + \left(\mathbf{k}_1^T + \mathbf{k}_2^T\right)\dot{\mathbf{v}} = \mathbf{0}. \tag{19.61}$$

Finally, the balancing conditions for the shaking moment reduce to the algebraic form

$$\mathbf{S}_1 + \mathbf{S}_2 = \mathbf{0}, \quad \mathbf{k}_1 + \mathbf{k}_2 = \mathbf{0}. \tag{19.62}$$

where matrices $\mathbf{S}_1$ and $\mathbf{S}_2$ have the dimension of $(2p - r) \times (2p - r)$ and $\mathbf{k}_1$, $\mathbf{k}_2$ are vectors of $2p - r$ elements. With the use of Eq. (19.40) we obtain a set of balancing conditions for the shaking moment in term of inertia and geometrical parameters of the mechanism, such as m_i, ξ_{Si}, η_{Si} and I_{Si}. In summary, the following steps are required to realize the proposed procedure:

– Formulating r loop equations and p position vectors of the mass centers of moving links according to Eqs. (19.28) and (19.30).
– Selecting the elements of vector $\mathbf{w}$ from elements of vector $\mathbf{u}$ based on the following rule: The number of elements in $\mathbf{w}$ is equal to r, and matrix $\mathbf{D}^w$ must be a square and nonsingular matrix.
– Calculating matrices $\mathbf{D}^v$ and $\mathbf{D}^w$ using Eq. (19.33), $\mathbf{C}_i^v$, $\mathbf{C}_i^w$ using Eq. (19.38), Matrix $\mathbf{G}$ and vector $\mathbf{b}$ using Eq. (19.35), matrices $\mathbf{B}_i$ and vectors $\mathbf{e}_i$ $(i = 1, 2, \ldots, p)$ using Eqs. (19.41) and (19.42).
– Substituting the expressions of matrices $\mathbf{B}_i$ into Eq. (19.45) to obtain the balancing conditions for the shaking force.
– Determining the elements of matrices $\mathbf{H}_1$ and $\mathbf{H}_4$ according to Eqs. (19.55) and (19.56).
– Calculating matrix $\mathbf{S}_1$ and vector $\mathbf{k}_1$ using Eq. (19.50), matrix $\mathbf{S}_2$ using Eq. (19.59) and vector $\mathbf{k}_2$ using Eq. (19.60).
– Substituting the expressions of $\mathbf{S}_1$,$\mathbf{S}_2$, $\mathbf{k}_1$ and $\mathbf{k}_2$ into Eq. (19.62) to get the balancing conditions for the shaking moment.

19.3.2 Application Example

A planar 8R-eightbar mechanism depicted in Fig. 19.3 is a multi degrees-of-freedom and multi-loop planar mechanism with seven moving links, where links 1, 4, and 6 are the driving links.

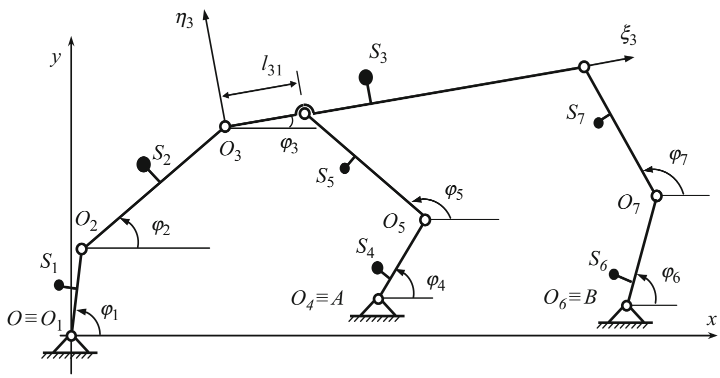


Fig. 19.3 Kinematic diagram of a planar 8R-eightbar mechanism

19.3.2.1 Formulation of Loop Equations

As shown in Fig. 19.2, the origin of the ground-fixed coordinate frame coincides with joint O of pivot link 1, and O_i denotes the origin of the link-fixed coordinate frame of link i. The loop equations of the mechanism can be written in the form

$$\begin{aligned} l_1\cos\varphi_1 + l_2\cos\varphi_2 + l_{31}\cos\varphi_3 - l_4\cos\varphi_4 - l_5\cos\varphi_5 &= x_A \\ l_1\sin\varphi_1 + l_2\sin\varphi_2 + l_{31}\sin\varphi_3 - l_4\sin\varphi_4 - l_5\sin\varphi_5 &= y_A \\ l_1\cos\varphi_1 + l_2\cos\varphi_2 + l_3\cos\varphi_3 - l_6\cos\varphi_6 - l_7\cos\varphi_7 &= x_B \\ l_1\sin\varphi_1 + l_2\sin\varphi_2 + l_3\sin\varphi_3 - l_6\sin\varphi_6 - l_7\sin\varphi_7 &= y_B \end{aligned} \tag{19.63}$$

where l_i denotes the length of link i, x_A, y_A and x_B, y_B are coordinates of the fixed points A and B in the fixed coordinate frame $\{Oxy\}$ respectively. According to Eq. (19.27), vector $\mathbf{u}$ is given by

$$\mathbf{u} = [\cos\varphi_1,\ \sin\varphi_1,\ \cos\varphi_2,\ \sin\varphi_2, \ldots,\ \cos\varphi_7,\ \sin\varphi_7]^T \tag{19.64}$$

According to Eq. (19.30), vector $\mathbf{d}$ are then determined from Eq. (19.63)

$$\mathbf{d} = [x_A,\ y_A,\ x_B,\ y_B]^T \tag{19.65}$$

Vector $\mathbf{w}$ and $\mathbf{v}$ is selected from the original vector $\mathbf{u}$ as follows:

$$\mathbf{w} = [\cos\varphi_4,\ \sin\varphi_4,\ \cos\varphi_6,\ \sin\varphi_6]^T \tag{19.66}$$

$$\mathbf{v} = [\cos\varphi_1,\ \ldots,\ \sin\varphi_3,\ \cos\varphi_5,\ \sin\varphi_5,\ \cos\varphi_7,\ \sin\varphi_7]^T \tag{19.67}$$

Note that there are other possibilities to choose the elements of $\mathbf{w}$ in order to obtain a nonsingular matrix $\mathbf{D}^w$. With vectors $\mathbf{v}$ and $\mathbf{w}$ given by Eqs. (19.66) and (19.67), matrices $\mathbf{D}^v$ and $\mathbf{D}^w$ are calculated from Eq. (19.63) by using Eq. (19.33)

$$\mathbf{D}^v = \begin{bmatrix} l_1 & 0 & l_2 & 0 & l_{31} & 0 & -l_5 & 0 & 0 & 0 \\ 0 & l_1 & 0 & l_2 & 0 & l_{31} & 0 & -l_5 & 0 & 0 \\ l_1 & 0 & l_2 & 0 & l_3 & 0 & 0 & 0 & -l_7 & 0 \\ 0 & l_1 & 0 & l_2 & 0 & l_3 & 0 & 0 & 0 & -l_7 \end{bmatrix} \tag{19.68}$$

$$\mathbf{D}^w = \begin{bmatrix} -l_4 & 0 & 0 & 0 \\ 0 & -l_4 & 0 & 0 \\ 0 & 0 & -l_6 & 0 \\ 0 & 0 & 0 & -l_6 \end{bmatrix}. \tag{19.69}$$

From Eq. (19.69) we get

$$(\mathbf{D}^w)^{-1} = \begin{bmatrix} -1/l_4 & 0 & 0 & 0 \\ 0 & -1/l_4 & 0 & 0 \\ 0 & 0 & -1/l_6 & 0 \\ 0 & 0 & 0 & -1/l_6 \end{bmatrix} \tag{19.70}$$

19.3.2.2 Balancing Conditions of the Shaking Force

Matrix $\mathbf{G}$ and vector $\mathbf{b}$ are calculated using the obtained matrices $\mathbf{D}^v$, $\mathbf{D}^w$ and vector $\mathbf{d}$ as follows:

$$\mathbf{G} = \begin{bmatrix} -\frac{l_1}{l_4} & 0 & -\frac{l_2}{l_4} & 0 & -\frac{l_{31}}{l_4} & 0 & \frac{l_5}{l_4} & 0 & 0 & 0 \\ 0 & -\frac{l_1}{l_4} & 0 & -\frac{l_2}{l_4} & 0 & -\frac{l_{31}}{l_4} & 0 & \frac{l_5}{l_4} & 0 & 0 \\ -\frac{l_1}{l_6} & 0 & -\frac{l_2}{l_6} & 0 & -\frac{l_3}{l_6} & 0 & 0 & 0 & \frac{l_7}{l_6} & 0 \\ 0 & -\frac{l_1}{l_6} & 0 & -\frac{l_2}{l_6} & 0 & -\frac{l_3}{l_6} & 0 & 0 & 0 & \frac{l_7}{l_6} \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} -\frac{x_A}{l_4} & -\frac{y_A}{l_4} & -\frac{x_B}{l_6} & -\frac{y_B}{l_6} \end{bmatrix}^T. \tag{19.71}$$

Now we can determine matrices $\mathbf{C}_i^v$, $\mathbf{C}_i^w$ and vector $\mathbf{e}_i^u$ related to vector $\mathbf{r}_i$ using Eq. (19.28). For example, for $i = 1$:

$$\mathbf{r}_1 = \begin{bmatrix} \xi_{S1}\cos\varphi_1 - \eta_{S1}\sin\varphi_1 \\ \xi_{S1}\sin\varphi_1 + \eta_{S1}\cos\varphi_1 \end{bmatrix}, \mathbf{C}_1^v = \begin{bmatrix} \xi_{S1} & -\eta_{S1} & 0\,0\,0\,0\,0\,0\,0\,0 \\ \eta_{S1} & \xi_{S1} & 0\,0\,0\,0\,0\,0\,0\,0 \end{bmatrix},$$

$$\mathbf{C}_1^w = \begin{bmatrix} 0\,0\,0\,0 \\ 0\,0\,0\,0 \end{bmatrix} \quad \mathbf{e}_1^u = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

For $i = 7$ we get

$$\mathbf{r}_7 = \begin{bmatrix} x_B + l_6\cos\varphi_6 + \xi_{S7}\cos\varphi_7 - \eta_{S7}\sin\varphi_7 \\ y_B + l_6\sin\varphi_6 + \xi_{S7}\sin\varphi_7 + \eta_{S7}\cos\varphi_7 \end{bmatrix},$$

$$\mathbf{C}_7^v = \begin{bmatrix} 0\,0\,0\,0\,0\,0\,0\,0 & \xi_{S7} & -\eta_{S7} \\ 0\,0\,0\,0\,0\,0\,0\,0 & \eta_{S7} & \xi_{S7} \end{bmatrix}, \ \mathbf{C}_7^w = \begin{bmatrix} 0 & 0 & l_6 & 0 \\ 0 & 0 & 0 & l_6 \end{bmatrix}, \ \mathbf{e}_7^u = \begin{bmatrix} x_B \\ y_B \end{bmatrix}.$$

Then, matrices $\mathbf{B}_i$ ($i = 1, 2, \ldots, 7$) are calculated using Eq. (19.42). Finally, by substituting matrices $\mathbf{B}_i$ into Eq. (19.45), we find balancing conditions of the shaking force as follows:

$$m_1\frac{\xi_{S1}}{l_1} + m_2 + m_3 + m_4\frac{\xi_{S4}}{l_4} + m_5 + m_6\frac{\xi_{S6}}{l_6} + m_7 = 0 \tag{19.72}$$

$$m_2\frac{\xi_{S2}}{l_2} + m_3 + m_4\frac{\xi_{S4}}{l_4} + m_5 + m_6\frac{\xi_{S6}}{l_6} + m_7 = 0 \tag{19.73}$$

$$m_3\frac{\xi_{S3}}{l_3} + m_4\frac{l_{31}}{l_3}\frac{\xi_{S4}}{l_4} + m_5\frac{l_{31}}{l_3} + m_6\frac{\xi_{S6}}{l_6} + m_7 = 0 \quad (19.74)$$

$$m_4\frac{\xi_{S4}}{l_4} + m_5\left(1 - \frac{\xi_{S5}}{l_5}\right) = 0 \quad (19.75)$$

$$m_6\frac{\xi_{S6}}{l_6} + m_7\left(1 - \frac{\xi_{S7}}{l_7}\right) = 0 \quad (19.76)$$

$$m_1\frac{\eta_{S1}}{l_1} + m_4\frac{\eta_{S4}}{l_4} + m_6\frac{\eta_{S6}}{l_6} = 0 \quad (19.77)$$

$$m_2\frac{\eta_{S2}}{l_2} + m_4\frac{\eta_{S4}}{l_4} + m_6\frac{\eta_{S6}}{l_6} = 0 \quad (19.78)$$

$$m_3\frac{\eta_{S3}}{l_3} + m_4\frac{l_{31}}{l_3}\frac{\eta_{S4}}{l_4} + m_6\frac{\eta_{S6}}{l_6} = 0 \quad (19.79)$$

$$m_4\frac{\eta_{S4}}{l_4} - m_5\frac{\eta_{S5}}{l_5} = 0 \quad (19.80)$$

$$m_6\frac{\eta_{S6}}{l_6} - m_7\frac{\eta_{S7}}{l_7} = 0. \quad (19.81)$$

19.3.2.3 Balancing Conditions of the Shaking Moment

Since matrices $\mathbf{B}_i$ ($i = 1, 2, \ldots, 7$) are known and vectors of constants $\mathbf{e}_i$ are given by Eq. (19.41), matrix $\mathbf{S}_1$ and vector $\mathbf{k}_1$ can be easily calculated using Eq. (19.50). Matrix $\mathbf{H}$ takes the same form as Eq. (19.55) for $p = 7$. By partitioning of matrix $\mathbf{H}$ related to Eq. (19.56), we obtain sub-matrices $\mathbf{H}_1$ and $\mathbf{H}_4$. Then, matrix $\mathbf{S}_2$ and vector $\mathbf{k}_2$ are calculated using Eqs. (19.59) and (19.60). By substituting the obtained expressions of $\mathbf{S}_1$,$\mathbf{S}_2$, $\mathbf{k}_1$,$\mathbf{k}_2$ into Eq. (19.62), the balancing conditions of the shaking moment are then derived in the following form

$$m_1\lambda_1^2 + \frac{I_{S1}}{l_1^2} + m_2 + m_3 + m_4\lambda_4^2 + \frac{I_{S4}}{l_4^2} + m_5 + m_6\lambda_6^2 + \frac{I_{S6}}{l_6^2} + m_7 = 0 \quad (19.82)$$

$$m_2\lambda_2 + m_3 + m_4\lambda_4^2 + \frac{I_{S4}}{l_4^2} + m_5 + m_6\lambda_6^2 + \frac{I_{S6}}{l_6^2} + m_7 = 0 \quad (19.83)$$

$$m_3\lambda_3 + \frac{l_{31}}{l_3}\left(m_4\lambda_4^2 + \frac{I_{S4}}{l_4^2} + m_5\right) + m_6\lambda_6^2 + \frac{I_{S6}}{l_6^2} + m_7 = 0 \quad (19.84)$$

$$m_4\lambda_4^2 + \frac{I_{S4}}{l_4^2} + m_5\,(1 - \lambda_5) = 0 \quad (19.85)$$

$$m_6\lambda_6^2 + \frac{I_{S6}}{l_6^2} + m_7(1-\lambda_7) = 0 \tag{19.86}$$

$$m_2\lambda_2^2 + \frac{I_{S2}}{l_2^2} + m_3 + m_4\lambda_4^2 + \frac{I_{S4}}{l_4^2} + m_5 + m_6\lambda_6^2 + \frac{I_{S6}}{l_6^2} + m_7 = 0 \tag{19.87}$$

$$m_3\lambda_3^2 + \frac{I_{S3}}{l_3^2} + \frac{l_{31}^2}{l_3^2}\left(m_4\lambda_4^2 + \frac{I_{S4}}{l_4^2} + m_5\right) + m_6\lambda_6^2 + \frac{I_{S6}}{l_6^2} + m_7 = 0 \tag{19.88}$$

$$m_4\lambda_4^2 + \frac{I_{S4}}{l_4^2} + m_5\lambda_5^2 + \frac{I_{S5}}{l_5^2} + m_5(1-2\lambda_5) = 0 \tag{19.89}$$

$$m_6\lambda_6^2 + \frac{I_{S6}}{l_6^2} + m_7\lambda_7^2 + \frac{I_{S7}}{l_7^2} + m_7(1-2\lambda_7) = 0 \tag{19.90}$$

$$m_4\lambda_4^2 + \frac{I_{S4}}{l_4^2} - m_4\lambda_4 = 0 \tag{19.91}$$

$$m_6\lambda_6^2 + \frac{I_{S6}}{l_6^2} - m_6\lambda_6 = 0 \tag{19.92}$$

$$\eta_{S1} = \eta_{S2} = \eta_{S3} = \eta_{S4} = \eta_{S5} = \eta_{S6} = \eta_{S7} = 0 \tag{19.93}$$

where $\lambda_i = \frac{\xi_{Si}}{l_i}$ for $i = 1, 2, \ldots, 7$.

In the case that S_i is positioned along the link line, that is, $\eta_{Si} = 0$ for $i = 1, 2, \ldots, 7$, the balancing conditions for the shaking force and shaking moment of the 8R-eightbar mechanism, Eqs. (19.72)–(19.93), are reduced into the following set of equations

$$m_1\lambda_1 + m_2(1-\lambda_2) = 0, \tag{19.94}$$

$$m_2\lambda_2 + m_3 + m_5\lambda_5 + m_7\lambda_7 = 0, \tag{19.95}$$

$$m_3\lambda_3 + m_5\lambda_5\frac{l_{31}}{l_3} + m_7\lambda_7 = 0, \tag{19.96}$$

$$m_4\lambda_4 + m_5(1-\lambda_5) = 0, \tag{19.97}$$

$$m_6\lambda_6 + m_7(1-\lambda_7) = 0, \tag{19.98}$$

$$m_i\lambda_i^2 + \frac{I_{Si}}{l_i^2} - m_i\lambda_i = 0 \ \text{ for } i = 1, 2, 4, 5, 6, 7, \tag{19.99}$$

Table 19.1 Initial parameters of the 8R-eightbar mechanism

Link i	l_i (m)	ξ_{Si}^0 (m)	η_{Si}^0 (m)	m_i^0 (kg)	λ_i^0
1	0.08	0.04	0.01	2.4	0.5
2	0.20	0.07	0.025	3.5	0.35
3	0.35	0.15	0.035	3.6	0.428
4	0.12	0.05	0.015	2.2	0.4167
5	0.15	0.08	0.01	2.4	0.5333
6	0.12	0.06	0.02	2.0	0.5
7	0.15	0.1	0.02	2.7	0.6667

$$m_3\lambda_3^2 + \frac{I_{S3}}{l_3^2} - m_3\lambda_3 - \frac{l_{31}}{l_3} m_5\lambda_5 \left(1 - \frac{l_{31}}{l_3}\right) = 0, \tag{19.100}$$

where Eqs. (19.94)–(19.98) are the balancing conditions of the shaking force and Eqs. (19.99)–(19.100) are the balancing conditions of the shaking moment of the fully force balanced mechanism.

19.3.2.4 Numerical Study

A numerical calculation is implemented to verify the correctness of the obtained balancing conditions. The geometry and mass distribution parameters of the links are given in Table 19.1, where m_i^0, ξ_{Si}^0, η_{Si}^0 and $\lambda_i^0 = \xi_{Si}^0/l_i$ denote the initial parameters. The other geometry parameters are: $x_A = 0.17$ (m), $x_B = 0.3$ (m), $y_A = y_B = 0.03$ (m)and $l_{31} = 0.07$ (m).

Upon assuming that parameter $\eta_{Si} = 0$ for $i = 1, 2, \ldots, 7$, the remaining five conditions (19.94)–(19.98) contain a set of 14 variables m_i and λ_i. We can establish a balancing scheme with counterweights by keeping the parameters of links 3 and 5, i.e., $m_3\lambda_3 = m_3^0\lambda_3^0$, $m_5\lambda_5 = m_5^0\lambda_5^0$, and solving parameters of the other links from these conditions as follows:

$$m_1\lambda_1 = -m_3^0\left(1 - \lambda_3^0\right) - m_5^0\lambda_5^0\left(1 - \frac{l_{31}}{l_3}\right) - m_2,$$

$$m_2\lambda_2 = -m_3^0\left(1 - \lambda_3^0\right) - m_5^0\lambda_5^0\left(1 - \frac{l_{31}}{l_3}\right),$$

$$m_4\lambda_4 = -m_5^0\left(1 - \lambda_5^0\right), \; m_6\lambda_6 = -\left(m_3^0\lambda_3^0 + m_5^0\lambda_5^0\frac{l_{31}}{l_3}\right) - m_7,$$

$$m_7\lambda_7 = -\left(m_3^0\lambda_3^0 + m_5^0\lambda_5^0\frac{l_{31}}{l_3}\right).$$

It follows that parameters λ_1, λ_2, λ_4, λ_6 and λ_7 will take negative values since $0 < \lambda_3^0 < 1$ and $0 < \lambda_5^0 < 1$. As a result, the centers of mass S_1, S_2, S_4, S_6 and S_7 must be positioned at the other side of joints O_1, O_2, O_4, O_6 and O_7 respectively.

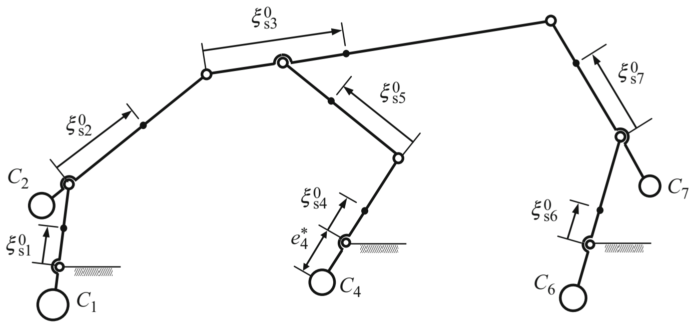


Fig. 19.4 A balancing scheme using five counterweights for the full force balancing

Table 19.2 Parameters of the force balanced the 8R-eightbar mechanism with counterweights

Link i	ξ_{Si} (m)	η_{Si} (m)	m_i (kg) (with counterweight)	Counterweights	
				e^*_i (m)	m^*_i (kg)
1	−0.0891	0.0	10.4	0.128	8.0
2	−0.0725	0.0	8.5	0.172	5.0
3	0.15	0.0	3.6	0.0	0.0
4	−0.0187	0.0	7.2	0.049	5.0
5	0.080	0.0	2.4	0.0	0.0
6	−0.1425	0.0	8.0	0.210	6.0
7	−0.0350	0.0	7.7	0.108	5.0

For this purpose, a balancing scheme with five counterweights attached to the corresponding links as shown in Fig. 19.4 is suggested. Using the same way, we can establish other force balancing schema by assigning the parameters of two arbitrary links and calculating parameters of the other links from Eqs. (19.94)–(19.98).

The mass m^*_i and the distance e^*_i of the counterweight C_i attached to link i can then be easily calculated by applying the relationship $m_i\lambda_i = m^0_i\lambda^0_i - m^*_i\frac{e^*_i}{l_i}$. The mass distribution parameters of the links and counterweights of the fully force balanced mechanism are given in Table 19.2.

Figure 19.5 shows two components of the shaking force produced by the initial mechanism and the force balanced mechanism. The numerical results verified that the shaking force is completely eliminated during the motion of the force balanced mechanism.

In the next step, the moment balancing conditions, Eqs. (19.99) and (19.100), will be taken into account for canceling the shaking moment of the fully force balanced mechanism. The moments of inertia of links of the full force balanced mechanism are as follows: $I^0_{S1} = 0.2$, $I^0_{S2} = 0.35$, $I^0_{S3} = 0.11$, $I^0_{S4} = 0.09$, $I^0_{S5} = 0.08$, $I^0_{S6} = 0.41$, $I^0_{S7} = 0.15$ $\left(\text{kg m}^2\right)$. A number of balancing schema using additional members were applied to balance the shaking moment at any rotating

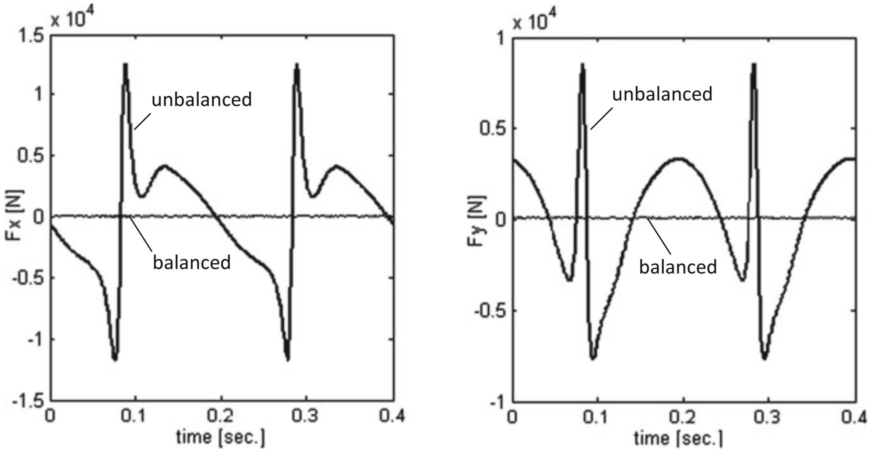


Fig. 19.5 Shaking forces of the unbalanced mechanism and the fully force balanced mechanism. (Rotating speeds of the cranks 1, 4 and 6 are assumed to be the same value of 300 rpm)

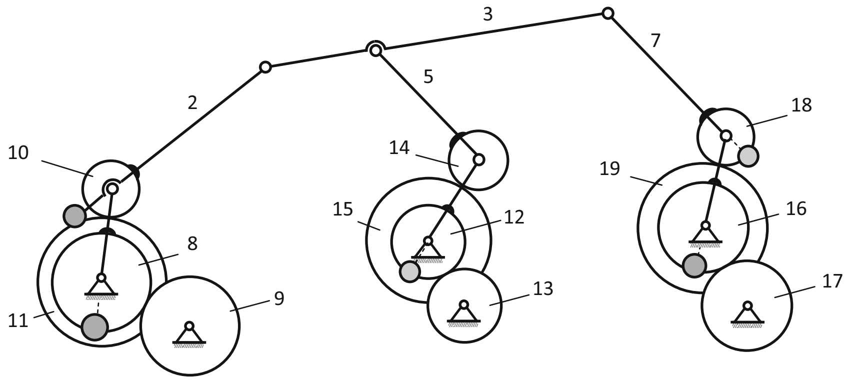


Fig. 19.6 A balancing scheme of the shaking moment of the fully force balanced mechanism

speed of the driving links, e.g., [12, 14, 15]. A well-known balancing scheme with counter-rotating balancers in Fig. 19.6 is used to verify the correctness of the conditions of moment balancing. The required moment of inertia of link 3 is calculated using Eq. (19.100) with the parameters given in Table 19.2, that yields $I_{S3} = 0.133\ (\text{kg m}^2)$. This value can be attained by mass redistribution for link 3.

As shown in Fig. 19.6, gears 11, 15, and 19 are mounted on the rotation axis of the input cranks 1, 4, and 6, respectively. They mesh with planetary gears 10, 14, and 18 mounted on links 2, 5 and 7 respectively. Using this balancing scheme, the additional balancing moments will be produced to balance correspondingly the inertia moments of all links. In other words, the shaking moment can be balanced, while the shaking force is still fully balanced. For brevity, the transmission ratios of the gear-pairs of the considered balancing scheme are chosen as follows:

$$\frac{r_8}{r_9} = \frac{r_{12}}{r_{13}} = \frac{r_{16}}{r_{17}} = 1, \ \frac{r_{10}}{r_{11}} = \frac{r_{14}}{r_{15}} = \frac{r_{18}}{r_{19}} = \frac{1}{2}.$$

According to Fig. 19.6, the kinematic relationship of gear-pair 10–11 is

$$r_{10}\dot{\varphi}_{10} + r_{11}\dot{\varphi}_{11} - (r_{10} + r_{11})\ \dot{\varphi}_1 = 0,\ \ \dot{\varphi}_{10} = \dot{\varphi}_2, \tag{19.101}$$

where r_i is the rolling circle radius of ith gear. Using Eq. (19.101) we obtain

$$I_{S11}\varphi_{11} = I_{S11}\left(\frac{r_{10}}{r_{11}} + 1\right)\varphi_1 - I_{S11}\frac{r_{10}}{r_{11}}\varphi_2, \tag{19.102}$$

$$I_{S10}\dot{\varphi}_{10} = I_{S10}\dot{\varphi}_2, \tag{19.103}$$

where I_{Si} is the moment of inertia of ith gear. Using Eqs. (19.102), (19.103) and the balancing condition (19.99) we obtain the following balancing condition for link 2 with the additional planetary gear

$$m_2\lambda_2^2 + \frac{1}{l_2^2}\left(I_{S2} + I_{S10} - \frac{r_{10}}{r_{11}}J_{S11}\right) - m_2\lambda_2 = 0. \tag{19.104}$$

By the same way, the balancing conditions with the additional gears for links 5 and 7 can be formulated as follows:

$$m_5\lambda_5^2 + \frac{1}{l_5^2}\left(I_{S5} + J_{S14} - \frac{r_{14}}{r_{15}}J_{S15}\right) - m_5\lambda_5 = 0, \tag{19.105}$$

$$m_7\lambda_7^2 + \frac{1}{l_7^2}\left(I_{S7} + J_{S18} - \frac{r_{18}}{r_{19}}J_{S19}\right) - m_7\lambda_7 = 0. \tag{19.106}$$

The moment of inertia of gear-pairs 10–11, 14–15, and 18–19 can then be chosen in order to satisfy Eqs. (19.104)–(19.106). Using Eq. (19.99), the moment of inertia of the gear-pairs 8–9, 12–13, and 16–17 can be determined by the similar way. Their values are given in Table 19.3.

Figure 19.7 shows the numerical results for the shaking moment of the fully moment balanced mechanism, where the input speeds of cranks 1, 4 and 6 are $\dot{\varphi}_1 =$

Table 19.3 Moments of inertia of the gears

Gear i	8	9	10	11	12	13	14	15	16	17	18	19
I_{Si} (kg m^2)	0.02	1.0	0.05	0.42	0.05	0.51	0.05	0.23	0.1	1.56	0.05	0.5

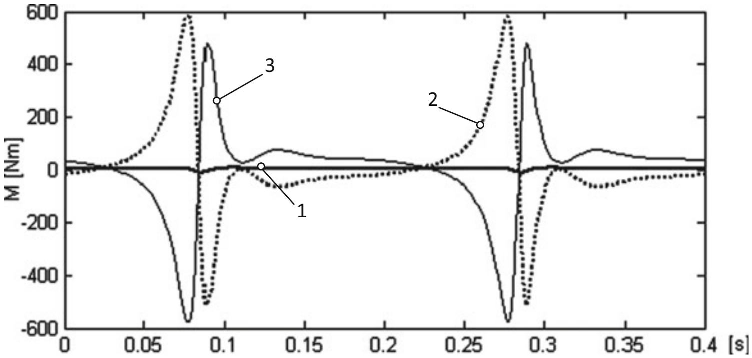


Fig. 19.7 The shaking moment (*curve 1*) of the fully moment balanced mechanism as a sum of the first term (*curve 2*) and the second term (*curve 3*)

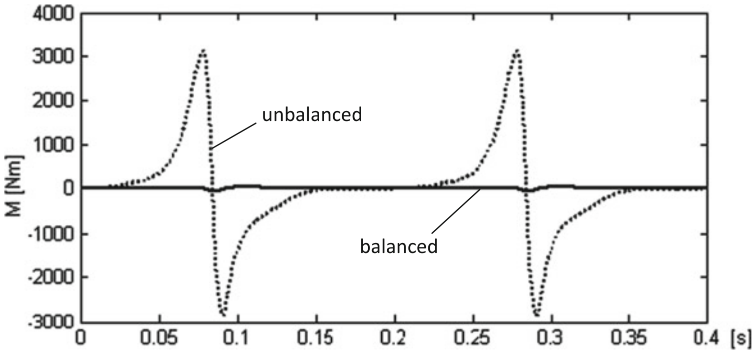


Fig. 19.8 Shaking moments of the unbalanced and the fully moment balanced mechanism

$\dot{\varphi}_4 = \dot{\varphi}_6 = 31.4$ (rad/s). The results shown in Fig. 19.8 demonstrated that the shaking moment of the 8R-eightbar mechanism is eliminated after balancing.

19.4 Balancing Conditions of Spatial One-DOF Mechanisms

19.4.1 Theory and Procedure for Deriving Balancing Conditions

19.4.1.1 The General Balancing Conditions of Spatial One-DOF Mechanisms

This section presents a method to algebraically derive the balancing conditions for shaking force and shaking moment of spatial one-degree-of freedom mechanisms. Let q be the independent generalized coordinate which describes the position of the mechanism. According to Eqs. (19.10) and (19.12), the velocity $\mathbf{v}_{Si}$ and the angular velocity $\boldsymbol{\omega}_i$ are given by

$$\mathbf{v}_{Si} = \mathbf{J}_{Ti}(q)\dot{q}(t), \tag{19.107}$$

$$\boldsymbol{\omega}_i = \mathbf{J}_{Ri}(q)\dot{q}(t), \tag{19.108}$$

where $\mathbf{J}_{Ti}(q)$ and $\mathbf{J}_{Ri}(q)$ are 3×1 Jacobian matrices and can be written in the form

$$\mathbf{J}_{Ti} = \left[x'_{Si}\ y'_{Si}\ z'_{Si}\right]^T, \quad \mathbf{J}_{Ri} = \left[s'_{ix}\ s'_{iy}\ s'_{iz}\right]^T, \tag{19.109}$$

where the prime represents the derivative with respect to the generalized coordinate q and s_{ix}, s_{iy}, s_{iz} are three components of rotational vector φ_i for link i (see Sect. 19.2).

We recall that the inertia matrix $\mathbf{I}_{Si}$ is defined with respect to the fixed coordinate frame $\{Oxyz\}$ as shown in Fig. 19.9. The elements of matrix $\mathbf{I}_{Si}$ are time dependent

$$\mathbf{I}_{Si} = \begin{bmatrix} I_{ixx} & I_{ixy} & I_{ixz} \\ I_{iyx} & I_{iyy} & I_{iyz} \\ I_{izx} & I_{izy} & I_{izz} \end{bmatrix}. \tag{19.110}$$

Using Eqs. (19.107) and (19.108), Eqs. (19.6) and (19.7) take the following form

$$\frac{d}{dt}\left\{\left[\sum_{i=1}^{p} m_i\,\mathbf{J}_{Ti}(q)\right]\dot{q}\right\} = \mathbf{0}, \tag{19.111}$$

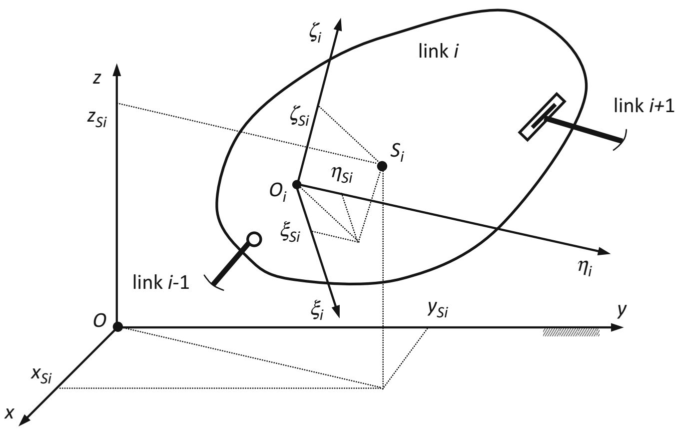


Fig. 19.9 Definition of coordinates

$$\frac{d}{\mathrm{d}t}\left\{\left[\sum_{i=1}^{p}\mathbf{I}_{Si}\boldsymbol{J}_{Ri}(q)+m_i\tilde{\mathbf{r}}_{Si}\;\mathbf{J}_{Ti}(q)\right]\dot{q}\right\}=\mathbf{0}. \tag{19.112}$$

The use of Eqs. (19.109)–(19.112) yields

$$\left(\ddot{q}+\dot{q}^2\frac{d}{dq}\right)\sum_{i=1}^{n}m_i x'_{Si}=0, \tag{19.113}$$

$$\left(\ddot{q}+\dot{q}^2\frac{d}{\mathrm{d}q}\right)\sum_{i=1}^{n}m_i y'_{Si}=0, \tag{19.114}$$

$$\left(\ddot{q}+\dot{q}^2\frac{d}{\mathrm{d}q}\right)\sum_{i=1}^{n}m_i z'_{Si}=0 \tag{19.115}$$

$$\left(\ddot{q}+\dot{q}^2\frac{d}{\mathrm{d}q}\right)\sum_{i=1}^{n}\left[m_i\left(y_{Si}z'_{Si}-z_{Si}y'_{Si}\right)+I_{ixx}s'_{ix}+I_{ixy}s'_{iy}+I_{ixz}s'_{iz}\right]=0, \tag{19.116}$$

$$\left(\ddot{q}+\dot{q}^2\frac{d}{\mathrm{d}q}\right)\sum_{i=1}^{n}\left[m_i\left(z_{Si}x'_{Si}-x_{Si}z'_{Si}\right)+I_{iyx}s'_{ix}+I_{iyy}s'_{iy}+I_{iyz}s'_{iz}\right]=0, \tag{19.117}$$

$$\left(\ddot{q}+\dot{q}^2\frac{d}{\mathrm{d}q}\right)\sum_{i=1}^{n}\left[m_i\left(x_{Si}y'_{Si}-y_{Si}x'_{Si}\right)+I_{izx}s'_{ix}+I_{izy}s'_{iy}+I_{izz}s'_{iz}\right]=0. \tag{19.118}$$

This yields the general conditions for complete balancing of spatial mechanisms

$$\sum_{i=1}^{n}m_i x'_{Si}=0,\;\sum_{i=1}^{n}m_i y'_{Si}=0,\;\sum_{i=1}^{n}m_i z'_{Si}=0, \tag{19.119}$$

$$\sum_{i=1}^{n}\left[m_i\left(y_{Si}z'_{Si}-z_{Si}y'_{Si}\right)+I_{ixx}s'_{ix}+I_{ixy}s'_{iy}+I_{ixz}s'_{iz}\right]=0 \tag{19.120}$$

$$\sum_{i=1}^{n}\left[m_i\left(z_{Si}x'_{Si}-x_{Si}z'_{Si}\right)+I_{iyx}s'_{ix}+I_{iyy}s'_{iy}+I_{iyz}s'_{iz}\right]=0 \tag{19.121}$$

$$\sum_{i=1}^{n}\left[m_i\left(x_{Si}y'_{Si}-y_{Si}x'_{Si}\right)+I_{izx}s'_{ix}+I_{izy}s'_{iy}+I_{izz}s'_{iz}\right]=0 \tag{19.122}$$

19.4.1.2 Algebraic Balancing Conditions of the Shaking Force

The position vector $\mathbf{r}_{Si}$ with respect to the fixed coordinate frame is given by

$$\mathbf{r}_{Si}=\mathbf{r}_{Oi}+\mathbf{A}_i\mathbf{r}_{Si}^{(i)}, \tag{19.123}$$

where $\mathbf{r}_{Oi}$ is position vector of origin O_i in the fixed coordinate frame $\{Oxyz\}$ and $\mathbf{r}_i^{(i)}$ is position vector of S_i in the moving coordinate frame $\{O_i\xi_i\eta_i\zeta_i\}$ shown in Fig. 19.9.

$$\mathbf{r}_{Si}^{(i)} = \left[\xi_{Si}\ \eta_{Si}\ \zeta_{Si}\right]^T. \tag{19.124}$$

The coordinates of the center of mass S_i, $\mathbf{r}_{Si} = \left[x_{Si}\ y_{Si}\ z_{Si}\right]^T$, can be rewritten as [14, 15]

$$x_{Si} = e_{xi}^* + \mathbf{a}_i^T\mathbf{z},\ y_{Si} = e_{yi}^* + \mathbf{b}_i^T\mathbf{z},\ z_{Si} = e_{zi}^* + \mathbf{c}_i^T\mathbf{z}, \quad i = 1,\ 2, .., n \tag{19.125}$$

where the vectors $\mathbf{a}_i$, $\mathbf{b}_i$ and $\mathbf{c}_i$ consist of components which are independent of q, the elements of vector $\mathbf{z}$ are functions of the generalized coordinates which describe the motion of particular links, e_{xi}^*, e_{yi}^* and e_{zi}^* are constant values.

Analog to Eq. (19.125), the loop equations of the mechanism may be written in the matrix form

$$\mathbf{D}\,\mathbf{z} = \mathbf{f},\ \ \mathbf{D} = [\mathbf{D}_{\mathrm{I}},\ \mathbf{D}_{\mathrm{II}}]\ . \tag{19.126}$$

Here the matrix $\mathbf{D}$ and the vector $\mathbf{f}$ include the components which are geometrical parameters and independent of q. A partitioning of vector $\mathbf{z}$ from Eq. (19.126)

$$\mathbf{z} = \begin{bmatrix} \mathbf{v} \\ \mathbf{w} \end{bmatrix}, \tag{19.127}$$

leads to the following relation

$$\mathbf{D}_{\mathrm{I}}\mathbf{v} + \mathbf{D}_{\mathrm{II}}\mathbf{w} = \mathbf{f}. \tag{19.128}$$

The matrix $\mathbf{D}_{\mathrm{II}}$ is chosen so that it must be a square matrix and nonsingular. The dimension of vector $\mathbf{w}$ and the number of the loop equations are equal. By solving Eq. (19.128) with the vector of variables $\mathbf{w}$, we get

$$\mathbf{w} = \mathbf{D}_{\mathrm{II}}^{-1}\left(\mathbf{f} - \mathbf{D}_{\mathrm{I}}\mathbf{v}\right). \tag{19.129}$$

Using Eqs. (19.127) and (19.129), the coordinates of the center of mass S_i and their derivatives can be expressed in terms of the reduced vector of variables $\mathbf{v}$ as

$$x_{Si} = e_{xi} + \mathbf{g}_i^T\mathbf{v},\ y_{Si} = e_{yi} + \mathbf{h}_i^T\mathbf{v},\ z_{Si} = e_{yi} + \mathbf{k}_i^T\mathbf{v} \tag{19.130}$$

$$x_{Si}' = \mathbf{g}_i^T\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}q}\ ,\ y_{Si}' = \mathbf{h}_i^T\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}q},\ z_{Si}' = \mathbf{k}_i^T\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}q}, \tag{19.131}$$

where

$$\mathbf{g}_i^T = \mathbf{a}_{i\,\mathrm{I}}^T - \mathbf{a}_{i\,\mathrm{II}}^T\,\mathbf{D}_{\mathrm{II}}^{-1}\,\mathbf{D}_{\mathrm{I}},\ \ \mathbf{h}_i^T = \mathbf{b}_{i\,\mathrm{I}}^T - \mathbf{b}_{i\,\mathrm{II}}^T\,\mathbf{D}_{\mathrm{II}}^{-1}\,\mathbf{D}_{\mathrm{I}},\ \ \mathbf{k}_i^T = \mathbf{c}_{i\,\mathrm{I}}^T - \mathbf{c}_{i\,\mathrm{II}}^T\,\mathbf{D}_{\mathrm{II}}^{-1}\,\mathbf{D}_{\mathrm{I}}$$
$$e_{xi} = e_{xi}^* + \mathbf{a}_{i\,\mathrm{II}}^T\,\mathbf{D}_{\mathrm{II}}^{-1}\,\mathbf{f},\ \ e_{yi} = e_{yi}^* + \mathbf{b}_{i\,\mathrm{II}}^T\,\mathbf{D}_{\mathrm{II}}^{-1}\,\mathbf{f},\ \ e_{zi} = e_{zi}^* + \mathbf{c}_{i\,\mathrm{II}}^T\,\mathbf{D}_{\mathrm{II}}^{-1}\,\mathbf{f}, \tag{19.132}$$

where vectors $\mathbf{a}_{i\mathrm{I}}, \mathbf{a}_{i\mathrm{II}}, \mathbf{b}_{i\mathrm{I}}, \mathbf{b}_{i\mathrm{II}}, \mathbf{c}_{i\mathrm{I}}, \mathbf{c}_{i\mathrm{II}}$ include elements which are independent of q. Substituting Eq. (19.131) into balancing conditions (19.119), we obtain

$$\left(\sum_{i=1}^{n} m_i \mathbf{g}_i^T\right)\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}q} = 0\,,\ \ \left(\sum_{i=1}^{n} m_i \mathbf{h}_i^T\right)\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}q} = 0,\ \ \left(\sum_{i=1}^{n} m_i \mathbf{k}_i^T\right)\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}q} = 0\,. \tag{19.133}$$

Finally, the algebraic balancing conditions for shaking force take the compact matrix form

$$\sum_{i=1}^{n} m_i \mathbf{g}_i^T = \mathbf{0},\ \ \sum_{i=1}^{n} m_i \mathbf{h}_i^T = \mathbf{0},\ \ \sum_{i=1}^{n} m_i \mathbf{k}_i^T = \mathbf{0}. \tag{19.134}$$

19.4.1.3 Algebraic Balancing Conditions of the Shaking Moment

To extract the conditions for the shaking moment balancing, some additional transformations are required. The substitution of Eqs. (19.130) and (19.131) into Eqs. (19.120)–(19.122) yields

$$\mathbf{u}_1^T\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}q} + \mathbf{v}^T\mathbf{S}_1\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}q} + \sum_{i=1}^{n}\left(I_{ixx}s'_{ix} + I_{ixy}s'_{iy} + I_{ixz}s'_{iz}\right) = 0, \tag{19.135}$$

$$\mathbf{u}_2^T\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}q} + \mathbf{v}^T\mathbf{S}_2\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}q} + \sum_{i=1}^{n}\left(I_{iyx}s'_{ix} + I_{iyy}s'_{iy} + I_{iyz}s'_{iz}\right) = 0, \tag{19.136}$$

$$\mathbf{u}_3^T\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}q} + \mathbf{v}^T\mathbf{S}_3\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}q} + \sum_{i=1}^{n}\left(I_{izx}s'_{ix} + I_{izy}s'_{iy} + I_{izz}s'_{iz}\right) = 0, \tag{19.137}$$

where

$$\mathbf{u}_1^T = \sum_{i=1}^{n} m_i\left(e_{yi}\mathbf{k}_i^T - e_{zi}\mathbf{h}_i^T\right),\ \ \mathbf{u}_2^T = \sum_{i=1}^{n} m_i\left(e_{zi}\mathbf{g}_i^T - e_{xi}\mathbf{k}_i^T\right),$$
$$\mathbf{u}_3^T = \sum_{i=1}^{n} m_i\left(e_{xi}\mathbf{h}_i^T - e_{yi}\mathbf{g}_i^T\right), \tag{19.138}$$

and skew-symmetric matrices

$$\mathbf{S}_1 = \sum_{i=1}^{n} m_i \left(\mathbf{h}_i\mathbf{k}_i^T - \mathbf{k}_i\mathbf{h}_i^T\right),\ \ \mathbf{S}_2 = \sum_{i=1}^{n} m_i \left(\mathbf{k}_i\mathbf{g}_i^T - \mathbf{g}_i\mathbf{k}_i^T\right),$$
$$\mathbf{S}_3 = \sum_{i=1}^{n} m_i \left(\mathbf{g}_i\mathbf{h}_i^T - \mathbf{h}_i\mathbf{g}_i^T\right). \tag{19.139}$$

Analog to Eq. (19.125), the elements of the rotational vector φ_i can be rewritten as [14, 15]

$$s_{ix} = s_{ix}^* + \mathbf{r}_{1i}^T\mathbf{z},\ \ s_{iy} = s_{iy}^* + \mathbf{r}_{2i}^T\mathbf{z},\ \ s_{iz} = s_{iz}^* + \mathbf{r}_{3i}^T\mathbf{z}, \tag{19.140}$$

where the vectors $\mathbf{r}_{1i}$, $\mathbf{r}_{2i}$ and $\mathbf{r}_{3i}$include components which are independent of q, the values s_{ix}^*, s_{iy}^* and s_{iz}^* are constant. The corresponding derivatives are given by

$$s'_{ix} = \mathbf{r}_{1i}^T\frac{\mathrm{d}\mathbf{z}}{\mathrm{d}q},\ \ s'_{iy} = \mathbf{r}_{2i}^T\frac{\mathrm{d}\mathbf{z}}{\mathrm{d}q},\ \ s'_{iz} = \mathbf{r}_{3i}^T\frac{\mathrm{d}\mathbf{z}}{\mathrm{d}q}. \tag{19.141}$$

With the vector of variables $\mathbf{z}$, the elements of the inertia matrix $\mathbf{I}_i$ may be rewritten in the matrix form as

$$\begin{aligned} I_{ixx} &= \mathbf{z}^T\mathbf{d}_{ixx},\ I_{ixy} = \mathbf{z}^T\mathbf{d}_{ixy},\ I_{ixz} = \mathbf{z}^T\mathbf{d}_{ixz},\\ I_{iyx} &= \mathbf{z}^T\mathbf{d}_{iyx},\ I_{iyy} = \mathbf{z}^T\mathbf{d}_{iyy},\ I_{iyz} = \mathbf{z}^T\mathbf{d}_{iyz},\\ I_{izx} &= \mathbf{z}^T\mathbf{d}_{izx},\ I_{izy} = \mathbf{z}^T\mathbf{d}_{izy},\ I_{izz} = \mathbf{z}^T\mathbf{d}_{izz}, \end{aligned} \tag{19.142}$$

where all elements in the vectors $\mathbf{d}_{ixx}, \mathbf{d}_{ixy}, \mathbf{d}_{ixz}, \mathbf{d}_{iyx}, \mathbf{d}_{iyy}, \mathbf{d}_{iyz}, \mathbf{d}_{izx}, \mathbf{d}_{izy}, \mathbf{d}_{izz}$ are independent of the generalized coordinate q. By using Eqs. (19.141), (19.142) and introducing the new matrices

$$\begin{aligned} \mathbf{H}_1 &= \mathbf{d}_{ixx}\mathbf{r}_{1i}^T + \mathbf{d}_{ixy}\mathbf{r}_{2i}^T + \mathbf{d}_{ixz}\mathbf{r}_{3i}^T,\\ \mathbf{H}_2 &= \mathbf{d}_{iyx}\mathbf{r}_{1i}^T + \mathbf{d}_{iyy}\mathbf{r}_{2i}^T + \mathbf{d}_{iyz}\mathbf{r}_{3i}^T,\\ \mathbf{H}_3 &= \mathbf{d}_{izx}\mathbf{r}_{1i}^T + \mathbf{d}_{izy}\mathbf{r}_{2i}^T + \mathbf{d}_{izz}\mathbf{r}_{3i}^T, \end{aligned} \tag{19.143}$$

the third term in Eqs. (19.135)–(19.137) may be expressed in the matrix form as

$$\sum_{i=1}^{n}\left(I_{ixx}s'_{ix} + I_{ixy}s'_{iy} + I_{ixz}s'_{iz}\right) = \mathbf{z}^T\mathbf{H}_1\frac{\mathrm{d}\mathbf{z}}{\mathrm{d}q}, \tag{19.144}$$

$$\sum_{i=1}^{n}\left(I_{iyx}s'_{ix} + I_{iyy}s'_{iy} + I_{iyz}s'_{iz}\right) = \mathbf{z}^T\mathbf{H}_2\frac{\mathrm{d}\mathbf{z}}{\mathrm{d}q}, \tag{19.145}$$

$$\sum_{i=1}^{n}\left(I_{izx}s'_{ix} + I_{izy}s'_{iy} + I_{izz}s'_{iz}\right) = \mathbf{z}^T\mathbf{H}_3\frac{\mathrm{d}\mathbf{z}}{\mathrm{d}q}. \tag{19.146}$$

The matrix $\mathbf{H}_j$ can be partitioned in four sub-matrices corresponding to the vector of variables $\mathbf{v}$ und $\mathbf{w}$ in Eq. (19.128) as follows:

$$\mathbf{H}_j = \begin{bmatrix} \mathbf{H}_{j1} & \mathbf{H}_{j2} \\ \mathbf{H}_{j3} & \mathbf{H}_{j4} \end{bmatrix},\ j = 1,\ 2,\ 3. \tag{19.147}$$

By using Eqs. (19.129) and (19.147), the following relation is found from Eqs. (19.144)–(19.146)

$$\begin{aligned} \mathbf{z}^T \mathbf{H}_j \tfrac{\mathrm{d}\mathbf{z}}{\mathrm{d}q} &= \begin{bmatrix} \mathbf{v} \\ \mathbf{w} \end{bmatrix}^T \begin{bmatrix} \mathbf{H}_{j1} & \mathbf{H}_{j2} \\ \mathbf{H}_{j3} & \mathbf{H}_{j4} \end{bmatrix} \begin{bmatrix} \tfrac{\mathrm{d}\mathbf{v}}{\mathrm{d}q} \\ \tfrac{\mathrm{d}\mathbf{w}}{\mathrm{d}q} \end{bmatrix} \\ &= \mathbf{v}^T \left[\mathbf{H}_{j1} + \left(\mathbf{D}_{\mathrm{II}}^{-1}\mathbf{D}_{\mathrm{I}}\right)^{\mathrm{T}} \left(\mathbf{H}_{j4}\mathbf{D}_{\mathrm{II}}^{-1}\mathbf{D}_{\mathrm{I}} - \mathbf{H}_{j3}\right) - \mathbf{H}_{j2}\mathbf{D}_{\mathrm{II}}^{-1}\mathbf{D}_{\mathrm{I}} \right] \tfrac{\mathrm{d}\mathbf{v}}{\mathrm{d}q} \\ &\quad + \left(\mathbf{D}_{\mathrm{II}}^{-1}\mathbf{f}\right)^{\mathrm{T}} \left(\mathbf{H}_{j3} - \mathbf{H}_{j4}\mathbf{D}_{\mathrm{II}}^{-1}\mathbf{D}_{\mathrm{I}}\right) \tfrac{\mathrm{d}\mathbf{v}}{\mathrm{d}q}. \end{aligned} \tag{19.148}$$

By introducing the vector $\mathbf{u}_j^*$

$$\left(\mathbf{u}_j^*\right)^T = \left(\mathbf{D}_{\mathrm{II}}^{-1}\mathbf{f}\right)^{\mathrm{T}} \left(\mathbf{H}_{j3} - \mathbf{H}_{j4}\mathbf{D}_{\mathrm{II}}^{-1}\mathbf{D}_{\mathrm{I}}\right), j = 1,\ 2,\ 3 \tag{19.149}$$

and the matrices $\mathbf{S}_j^*$ $(j = 1, 2, 3)$

$$\mathbf{S}_j^* = \mathbf{H}_{j1} + \left(\mathbf{D}_{\mathrm{II}}^{-1}\mathbf{D}_{\mathrm{I}}\right)^T \left(\mathbf{H}_{j4}\mathbf{D}_{\mathrm{II}}^{-1}\mathbf{D}_{\mathrm{I}} - \mathbf{H}_{j3}\right) - \mathbf{H}_{j2}\mathbf{D}_{\mathrm{II}}^{-1}\mathbf{D}_{\mathrm{I}}, \tag{19.150}$$

Equations (19.135)–(19.137) take the compact form

$$\left(\mathbf{u}_j + \mathbf{u}_j^*\right)^T \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}q} + \mathbf{v}^T \left(\mathbf{S}_j + \mathbf{S}_j^*\right) \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}q} = 0, j = 1,\ 2,\ 3. \tag{19.151}$$

Finally, the following algebraic balancing conditions for shaking moment are found from Eq. (19.151)

$$\mathbf{u}_j + \mathbf{u}_j^* = \mathbf{0}, \quad \mathbf{S}_j + \mathbf{S}_j^* = \mathbf{0}, j = 1,\ 2,\ 3. \tag{19.152}$$

Equations (19.134) and (19.152) can be used to derive the dynamic balancing conditions in form of algebraic expressions for spatial one-DOF mechanisms.

19.4.2 Application Example

In the following example we introduce the application of the balancing theory described above to a spatial slider crank mechanism shown in Fig. 19.10. The configuration of the mechanism is also prescribed by rotation angles ϕ, β and γ.

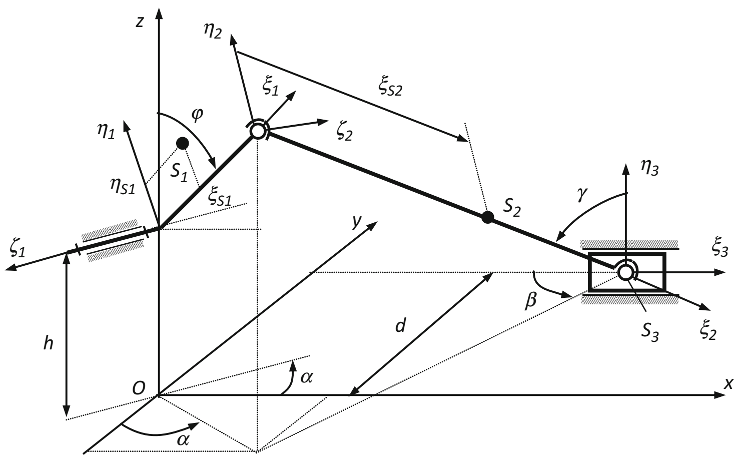


Fig. 19.10 A spatial slider crank mechanism

The angle φ is chosen as the independent generalized coordinates $q = \varphi$. The loop equations of the mechanism can be expressed in the form

$$\begin{aligned} h + l_1 \cos\phi - l_2 \cos\gamma &= 0, \\ -l_1 \sin\phi \cos\alpha + l_2 \sin\gamma \sin\beta - d &= 0, \end{aligned} \tag{19.153}$$

where l_i denotes the length of link i.

The direction cosine matrix $\mathbf{A}_i$ of link i referred to the fixed coordinate frame $\{Oxyz\}$ are given by

$$\mathbf{A}_1 = \begin{bmatrix} \sin\ \varphi\ \sin\ \alpha & -\cos\varphi\sin\alpha & -\cos\alpha \\ \sin\varphi\cos\alpha & \cos\varphi\cos\alpha & -\sin\alpha \\ \cos\varphi & \sin\varphi & 0 \end{bmatrix} \tag{19.154}$$

$$\mathbf{A}_2 = \begin{bmatrix} \sin\gamma\cos\beta & \cos\gamma\cos\beta & \sin\beta \\ \sin\gamma\sin\beta & \cos\gamma\sin\beta & -\cos\beta \\ -\cos\gamma & \sin\gamma & 0 \end{bmatrix} \tag{19.155}$$

According to the elements of matrices $\mathbf{A}_1$,$\mathbf{A}_2$, we choose the vector $\mathbf{z}$ with the following form

$$\begin{aligned} z = &\Big[\cos\varphi, \cos\beta, \cos\gamma, \sin\varphi, \sin\beta, \sin\gamma, \sin\gamma\cos\beta, \\ &\sin\gamma\sin\beta, \cos\gamma\cos\beta, \cos\gamma\sin\beta, 1\Big]^T \\ = &[z_1, z_2, z_3, z_4, z_5, z_6, z_7, z_8, z_9, z_{10}, z_{11}]^T \end{aligned} \tag{19.156}$$

For brevity, we assume that the center of mass S_2 of link 2 is positioned along the link line, the center of mass S_1 is positioned in the plane of axes ξ_2 and η_2. Then, $\eta_{S2} = 0$, $\zeta_{S1} = \zeta_{S2} = 0$. The coordinates of the center of mass S_i ($i = 1, 2, 3$) are expressed in term of the vector $\mathbf{z}$ as

$$\begin{aligned}
x_{S1} &= \left[-\eta_{S1}\sin\alpha \ 0\ 0\ \xi_{S1}\sin\alpha \ 0\ 0\ 0\ 0\ 0\ 0\ 0 \right]\mathbf{z} \\
y_{S1} &= \left[\eta_{S1}\cos\alpha \ 0\ 0\ -\xi_{S1}\cos\alpha \ 0\ 0\ 0\ 0\ 0\ 0\ 0 \right]\mathbf{z} \\
z_{S1} &= h + \left[\xi_{S1}\ 0\ 0\ \eta_{S1}\ 0\ 0\ 0\ 0\ 0\ 0\ 0 \right]\mathbf{z} \\
x_{S2} &= \left[0\ 0\ 0\ l_1\sin\alpha \ 0\ 0\ \xi_{S2}\ 0\ 0\ 0\ 0 \right]\mathbf{z} \\
y_{S2} &= \left[0\ 0\ 0\ -l_1\cos\alpha \ 0\ 0\ 0\ \xi_{S2}\ 0\ 0\ 0 \right]\mathbf{z} \\
z_{S2} &= h + \left[l_1\ 0\ -\xi_{S2}\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0 \right]\mathbf{z} \\
x_{S3} &= \left[0\ 0\ 0\ l_1\sin\alpha \ 0\ 0\ l_2\ 0\ 0\ 0\ 0 \right]\mathbf{z} \\
y_{S3} &= \left[0\ 0\ 0\ -l_1\cos\alpha \ 0\ 0\ 0\ l_2\ 0\ 0\ 0 \right]\mathbf{z} \\
z_{S3} &= h + \left[l_1\ 0\ -l_2\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0 \right]\mathbf{z}
\end{aligned} \tag{19.157}$$

It can be shown that the loop equations in Eq. (19.153) have the form

$$\begin{bmatrix} l_1 & 0 & -l_2 & 0 & 0\ 0\ 0 & 0 & 0\ 0\ 0 \\ 0 & 0 & 0 & l_1\cos\alpha & 0\ 0\ 0 & -l_2 & 0\ 0\ 0 \end{bmatrix} \mathbf{z} = -\begin{bmatrix} d \\ h \end{bmatrix} \tag{19.158}$$

The reduced vector of variables $\mathbf{v}$ and the vector of eliminated variables $\mathbf{w}$ are selected from the original vector $\mathbf{z}$ as follows:

$$\mathbf{v} = [\cos\phi,\ \cos\beta,\ \sin\phi,\ \sin\beta,\ \sin\gamma,\ \ \sin\gamma\cos\beta,\ \cos\gamma\cos\beta,\ \cos\gamma\sin\beta,\ 1]^T \tag{19.159}$$

$$\mathbf{w} = [\cos\gamma,\ \sin\gamma\sin\beta]^T \tag{19.160}$$

The matrices $\mathbf{D}_{\mathrm{I}}$, $\mathbf{D}_{\mathrm{II}}$ and $\mathbf{D}_{\mathrm{II}}^{-1}$ in Eq. (19.128) are given by

$$\mathrm{D}_{\mathrm{I}} = \begin{bmatrix} l_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & l_1\cos\alpha & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \mathrm{D}_{\mathrm{II}} = \begin{bmatrix} -l_2 & 0 \\ 0 & -l_2 \end{bmatrix}, D_{\mathrm{II}}^{-1} = -\frac{1}{l_2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{19.161}$$

19.4.2.1 Conditions of the Shaking Force Balancing

With the known coordinates of the center of masses from Eq. (19.157) and matrices $\mathbf{D}_{\mathrm{I}}$, $\mathbf{D}_{\mathrm{II}}^{-1}$ from Eq. (19.161), vectors $\mathbf{g}_i$, $\mathbf{h}_i$, and $\mathbf{k}_i$ can be determined according to Eq. (19.132) without any difficulty. Then, by substituting all these results into Eq. (19.134), we get the following conditions for the complete shaking force balancing

$$\begin{aligned}
&-m_1\eta_{S1}\sin\alpha = 0\\
&(m_1\xi_{S1} + m_2l_1 + m_3l_1)\sin\ \alpha = 0\\
&m_2\xi_{S2} + m_3l_2 = 0\\
&m_1\eta_{S1}\cos\ \alpha = 0\\
&m_1\xi_{S1} + m_2l_1\left(1 - \tfrac{\xi_{S2}}{l_2}\right) = 0\\
&m_1\xi_{S1}\cos\ \alpha + m_2l_1\left(1 - \tfrac{\xi_{S2}}{l_2}\right)\cos\ \alpha = 0
\end{aligned} \tag{19.162}$$

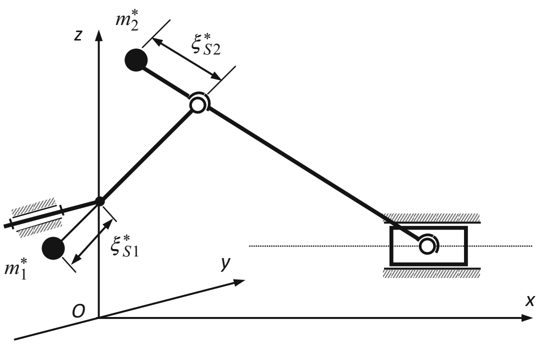
By simplifying the expressions in Eq. (19.162), the balancing conditions for shaking force of the mechanism are reduced into three equations $f_1 = 0$, $f_2 = 0$, $f_3 = 0$, in which

$$f_1 = \eta_{S1},\ f_2 = m_1\xi_{S1} + m_2l_1 + m_3l_1,\ f_3 = m_2\xi_{S2} + m_3l_2. \tag{19.163}$$

These conditions may be satisfied by internal mass redistribution or adding counterweights mounted on the links as shown in Fig. 19.11.

A simple numerical simulation is implemented in order to verify the correctness of these conditions for the static balancing. Parameters of the initial mechanism are given as follows: $m_1 = 7.0$ (kg), $m_2 = 12.5$ (kg), $m_3 = 10.5$ (kg), $l_1 = 0.1$ (m), $l_2 = 0.3$ (m), $h = 0.1$ (m), $d = 0.15$ (m), $\xi_{S1} = 0.01$ (m), $\eta_{S1} = 0.02$ (m), $\xi_{S2} = 0.05$ (m), $\eta_{S2} = 0$ (m). Using the conditions according to Eq. (19.163) we can determine the size and the location of the counterweights (see also Fig. 19.11): $m_1^*\xi_{S1}^* = 4.77$ (kg m), $m_2^*\xi_{S2}^* = 3.78$ (kg m). Figure 19.12 simultaneously shows three components of the shaking force produced by the unbalanced mechanism and the full force balanced mechanism. The results verified that the shaking force of the force balanced mechanism is completely eliminated and there is no forces transmitted to the base during the motion of the mechanism.

Fig. 19.11 Full force balanced mechanism with counterweights



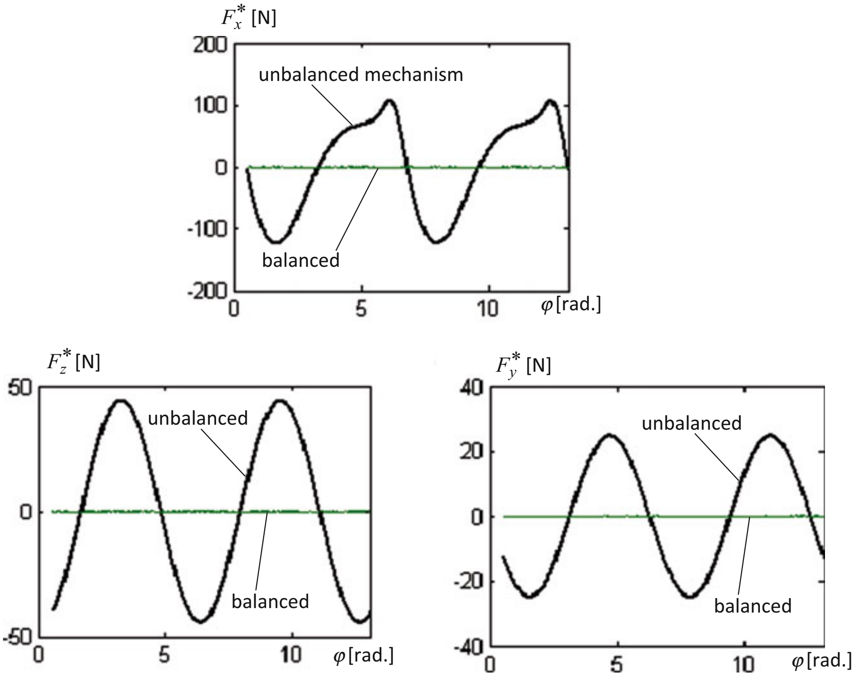


Fig. 19.12 Comparing shaking forces between the unbalanced mechanism and the full force balanced mechanism

19.4.2.2 Conditions of the Shaking Moment Balancing

To derive the conditions for the shaking moment balancing, the angular velocities of the links with respect to the fixed coordinate frame must be determined. Based on theory of multibody kinematics, these angular velocities can be calculated from the known matrices of the direction cosines $\mathbf{A}_i$ as follows:

$$\tilde{\boldsymbol{\omega}}_i = \dot{\mathbf{A}}_i \mathbf{A}_i^T, i = 1,\ 2,\ 3. \tag{19.164}$$

Using Eq. (19.164), we get

$$\boldsymbol{\omega}_1 = \begin{bmatrix} w_{1x} \\ w_{1y} \\ w_{1z} \end{bmatrix} = \begin{bmatrix} \cos\ \alpha \\ \sin\ \alpha \\ 0 \end{bmatrix} \dot{\varphi},\ \ \boldsymbol{\omega}_2 = \begin{bmatrix} \omega_{2x} \\ \omega_{2y} \\ \omega_{2z} \end{bmatrix} = \begin{bmatrix} \gamma'\ \sin\beta \\ \gamma'\ \cos\beta \\ \beta' \end{bmatrix} \dot{\varphi},$$

$$\boldsymbol{\omega}_2 = \begin{bmatrix} \omega_{3x} \\ \omega_{3y} \\ \omega_{3z} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \dot{\varphi} \tag{19.165}$$

where $\gamma' = \frac{\mathrm{d}\gamma}{\mathrm{d}\varphi},\ \ \beta' = \frac{\mathrm{d}\beta}{\mathrm{d}\varphi}$.

Upon assuming that axes ξ_i, η_i, ζ_i of the link-fixed coordinate frame are principal axes. The inertia matrix $\mathbf{I}_{Si}^{(i)}$ of link i about the center of mass S_i, referred to these principal axes, can be written in the simplified form

$$\mathbf{I}_{Si}^{(i)} = \begin{bmatrix} I_{i\xi\xi} & 0 & 0 \\ 0 & I_{i\eta\eta} & 0 \\ 0 & 0 & I_{i\zeta\zeta} \end{bmatrix}, i = 1,\ 2,\ 3 \tag{19.166}$$

By comparing elements of $\boldsymbol{\omega}_i$ in Eq. (19.165) with elements of $\mathbf{J}_{Ri}$ in Eq. (19.109), we obtain

$$\begin{array}{lll} s'_{1x} = \cos\alpha, & s'_{2x} = \gamma' \sin\beta, & s'_{3x} = 0, \\ s'_{1y} = \sin\alpha, & s'_{2y} = -\gamma' \cos\beta, & s'_{3y} = 0, \\ s'_{1z} = 0, & s'_{2z} = \beta'. & s'_{3z} = 0. \end{array} \tag{19.167}$$

So, the expressions in the left-hand side of Eqs. (19.144)–(19.146) can be established. With vector $\mathbf{z}$ according to Eq. (19.156) we find that

$$\begin{array}{ll} z_1 z'_4 - z_4 z'_1 = 1, & z_2 z'_5 - z_5 z'_2 = \beta', \\ z_{11} z'_1 = -\sin\phi, & z_{11} z'_4 = \cos\phi, \end{array} \tag{19.168}$$

Now we can determine matrices $\mathbf{H}_1 = \left[h_{i,j}^{(1)}\right]$, $\mathbf{H}_2 = \left[h_{i,j}^{(2)}\right]$, and $\mathbf{H}_3 = \left[h_{i,j}^{(3)}\right]$ as follows:

$$\begin{array}{l} h_{1,1}^{(1)} = -h_{4,1}^{(1)} = I_{1\zeta\zeta}\cos\alpha,\ h_{3,8}^{(1)} = -h_{8,3}^{(1)} = \frac{1}{2}\left(I_{2\eta\eta} + I_{2\zeta\zeta} - I_{2\xi\xi}\right), \\ h_{6,10}^{(1)} = -h_{10,6}^{(1)} = \frac{1}{2}\left(I_{2\eta\eta} - I_{2\xi\xi} - I_{2\zeta\zeta}\right),\ \text{every other } h_{i,j}^{(1)} = 0\ . \end{array}$$

$$\begin{array}{l} h_{1,1}^{(2)} = -h_{4,1}^{(2)} = I_{1\zeta\zeta}\sin\alpha,\ h_{3,7}^{(2)} = -h_{7,3}^{(2)} = \frac{1}{2}\left(I_{2\xi\xi} - I_{2\eta\eta} - I_{2\zeta\zeta}\right), \\ h_{6,9}^{(2)} = -h_{9,6}^{(2)} = \frac{1}{2}\left(I_{2\xi\xi} - I_{2\eta\eta} + I_{2\zeta\zeta}\right),\ \text{every other } h_{i,j}^{(2)} = 0 \end{array}$$

$$h_{2,5}^{(3)} = -h_{5,2}^{(3)} = I_{2\xi\xi},\ h_{7,8}^{(3)} = -h_{8,7}^{(2)} = I_{2\eta\eta} - I_{2\xi\xi},\ \text{every other } h_{i,j}^{(3)} = 0\ .$$

By partitioning of matrix $\mathbf{H}_j$ related to Eq. (19.147), we obtain sub-matrices $\mathbf{H}_{j1}$, $\mathbf{H}_{j2}$, $\mathbf{H}_{j3}$, $\mathbf{H}_{j4}$. Then, vectors $\mathbf{u}_j$ and $\mathbf{u}_j^*$, the matrices $\mathbf{S}_j$ and $\mathbf{S}_j^*$ can be formulated by using Eqs. (19.138), (19.139), (19.149), and (19.150), for example

$$u_2 = \left[-m_1 h \eta_{S1} \sin\alpha, 0, m_1 h \xi_{S1} \sin\alpha + m_2 h l_1 \left(1 - \frac{\xi_{S2}}{l_2}\right)\right.$$
$$\left. \sin\alpha, 0, 0,\ m_2 h \xi_2 \left(1 - \frac{\xi_{S2}}{l_2}\right),\ 0,\ 0,\ 0\right]^T$$

$$\mathbf{u}_2^* = \left[0,\ 0,\ 0,\ 0,\ 0,\ \frac{h}{2l_2}\left(I_{2\xi\xi} - I_{2\eta\eta} - I_{2\zeta\zeta}\right),\ 0,\ 0,\ 0\right]^T.$$

Finally, according to Eq. (19.152) we obtain the non-zero elements of vectors $\mathbf{u}_j + \mathbf{u}_j^*$ and matrices $\mathbf{S}_j + \mathbf{S}_j^*$ in the form

$$\begin{aligned}
k_1 &= m_1 h\eta_{S1}\ \cos\alpha - m_2 \mathrm{d}l_1\ \tfrac{\xi_{S2}}{l_2}\ \left(1 - \tfrac{\xi_{S2}}{l_2}\right) - \tfrac{\mathrm{d}l_1}{2l_2^2}\ \left(I_{2\xi\xi} - I_{2\eta\eta} - I_{2\zeta\zeta}\right)\\
k_2 &= \left[m_1 h\xi_{S1} + m_2 h l_1\left(1 - \tfrac{\xi_{S2}}{l_2}\right)^2 - \tfrac{hl_1}{2l_2^2}\left(I_{2\xi\xi} - I_{2\eta\eta} - I_{2\zeta\zeta}\right)\right]\cos\alpha\\
k_3 &= \left[m_1 h\xi_{S1} + m_2 h l_1\left(1 - \tfrac{\xi_{S2}}{l_2}\right)\right]\ \sin\alpha\\
k_4 &= m_2 h\xi_{S2}\left(1 - \tfrac{\xi_{S2}}{l_2}\right) + \tfrac{h}{2l_2}\left(I_{2\xi\xi} - I_{2\eta\eta} - I_{2\varsigma\varsigma}\right)\\
k_5 &= m_2 \mathrm{d}l_1 \tfrac{\xi_{S2}}{l_2}\sin\alpha + m_3 \mathrm{d}l\sin\alpha,\ \ k_6 = m_2 d\tfrac{\xi_{S2}^2}{l_2} + m_3 \mathrm{d}l_2 - \tfrac{d}{l_2}\left(I_{2\xi\xi} - I_{2\eta\eta}\right)\\
k_7 &= \left[m_1\left(\xi_{S1}^2 + \eta_{S1}^2\right) + m_2 l_1^2\left(1 - \tfrac{\xi_{S2}}{l_2}\right) - \tfrac{l_1^2}{2l_2^2}\left(I_{2\xi\xi} - I_{2\eta\eta} - I_{2\varsigma\varsigma}\right) + I_{1\varsigma\varsigma}\right]\cos\ \alpha\\
k_8 &= \left[m_1\left(\xi_{S1}^2 + \eta_{S1}^2\right) + m_2 l_1^2\left(1 - \tfrac{\xi_{S2}}{l_2}\right) + I_{1\varsigma\varsigma}\right]\sin\ \alpha\\
k_9 &= m_2 l_1\xi_{S2}\left(1 - \tfrac{\xi_{S2}}{l_2}\right) + \tfrac{l_1}{2l_2}\left(I_{2\xi\xi} - I_{2\eta\eta} - I_{2\varsigma\varsigma}\right)\\
k_{10} &= \left[m_2 l_1\xi_{S2}\left(1 - \tfrac{\xi_{S2}}{l_2}\right)\tfrac{l_1}{l_2}\left(I_{2\xi\xi} - I_{2\eta\eta}\right)\right]\cos\ \alpha\\
k_{11} &= I_{2\xi\xi} - I_{2\eta\eta} + I_{2\varsigma\varsigma},\ \ k_{12} = m_1 h\eta_{S1}\sin\ \alpha,\ \ k_{13} = I_{2\xi\xi}
\end{aligned} \tag{19.169}$$

Note that the above obtained expressions are original and can be further simplified. Now we choose $\alpha = \pi/2$, $\eta_{S1} = 0$ and let $I_{2\eta\eta} = I_{2\varsigma\varsigma}$, the expressions k_i in Eq. (19.169) are reduced as follows:

$$\begin{aligned}
&f_4 = m_2 l_2\xi_{S2} - m_2\xi_{S2}^2 - I_{2\varsigma\varsigma},\ f_5 = m_1\xi_{S1} + m_2 l_1\left(1 - \tfrac{\xi_{S2}}{l_2}\right)\\
&f_6 = \tfrac{m_2\xi_{S2}l_1}{l_2} + m_3 l_1,\ f_7 = m_1\xi_{S1}^2 + m_2 l_1^2\left(1 - \tfrac{\xi_{S2}}{l_2}\right) + I_{1\varsigma\varsigma},\ f_8 = I_{2\xi\xi}.
\end{aligned} \tag{19.170}$$

The shaking moment is completely balanced if the values of f_i $(i = 4,\ 5, \ldots, 8)$ in Eq. (19.170) vanish simultaneously. It is clearly shown that these conditions cannot be completely satisfied by adding counterweights, since the values of f_8 are not equal to zero in any case. These conditions are mainly of theoretical interest. However, Eq. (19.170) provide the necessary tool for the minimization of the shaking moment. Another way for solving the problem is the simultaneous minimization of the shaking force and shaking moment based on Eqs. (19.163) and (19.170). From the conditions

$$f_i \to \min\ \ (i = 1,\ 2, \ldots, 8)$$

one can choose a set of optimizing values for geometrical and inertia parameters of the links: m_1, m_2, m_3, $\xi_{S1}, \xi_{S2}, I_{1\varsigma\varsigma}, I_{2\xi\xi}, I_{2\eta\eta}$, $I_{2\varsigma\varsigma}$. This problem will be considered in the future investigation.

19.5 Conclusions

This chapter provided an approach to derive the dynamic balancing conditions of planar and spatial mechanisms. The following conclusions have been reached:

- Based on theory of multibody dynamics, the algebraic balancing conditions for the shaking force and shaking moment of planar and spatial mechanisms have been established.
- A specialized code has been developed on the MAPLE® environment for this study. It can be concluded that the proposed method is suitable for the application of the widely accessible computer algebra systems such as MAPLE®.
- The proposed method is illustrated for a planar 8R-eightbar mechanism having multi degrees-of-freedom and multi-links is an appropriate object to demonstrate the suggested procedure. Based on the obtained balancing conditions of the shaking force, a number of balancing schema with counterweights can be established by assigning the parameters of two arbitrary links and determining parameters of the other links.
- The proposed method is illustrated by using a spatial slider crank mechanism. In the application of balancing techniques using counterweights and supplementary links [35–37] for spatial mechanisms, the proposed method may provide a helpful tool to obtain exactly the balancing conditions and therefore we can get better balancing results. This will be the subject of future work.

Acknowledgment The work discussed in this chapter was completed with the financial support given by the National Foundation for Science and Technology Development of Vietnam.

References

1. Dresig, H, Vulfson, JI: Dynamik der Mechanismen. Deutscher Verlag der Wissenschaften, Berlin (1989)
2. Schiehlen, W, Eberhard, P: Technische Dynamik, 3rd edn. Vieweg + Teubner, Wiesbaden (2007)
3. Dresig, H, Holzweissig, F: Dynamics of Machinery. Springer, Berlin (2010)
4. Huston, RL: Multibody Dynamics. Butterworth-Heinemann, Boston (1990)
5. Haug, EJ: Computer Aided Kinematics and Dynamics of Mechanical Systems, vol. 1: Basic Methods. Allyn and Bacon, Boston, MA (1989)
6. Chaudhary, H, Saha, SK: Dynamics and Balancing of Multibody Systems. Springer, Berlin (2009)
7. Lowen, GG, Tepper, FR, Berkorf, RS: Balancing of linkages – an update. Mech Mach Theory **18**, 213–220 (1983)
8. Thümmel, T: Literaturbericht zum dynamischen Ausgleich schnellaufender Mechanismen. Wiss Schriftenreihe der TH Karl-Marx-Stadt. Mech Mater **7**, 57–92 (1983)
9. Arakelian, VH, Smith, MR: Shaking force and shaking moment balancing of mechanisms: a historical review with new examples. ASME J Mech Des **127**, 334–339 (2005)
10. Shchepetilnikov, VA: The determination of the mass centers of mechanisms in connection with the problem of mechanism balancing. J Mech **3**, 367–389 (1968)

11. Berkof, RS, Lowen, GG: A new method for completely force balancing simple linkage. Trans ASME J Eng Ind **91**(1), 21–26 (1969)
12. Kaufman, RE, Sandor, GN: Complete force balancing of spatial linkages. Trans ASME J Mech Des **93B**(2), 620–626 (1971)
13. Berkof, RS: Complete force and moment balancing of inline four-bar linkages. Mech Mach Theory **8**, 397–410 (1973)
14. Dresig, H, Rockhausen, L, Naake, S: Balancing conditions for planar mechanism. Flex Mech Dyn Anal ASME **47**, 67–73 (1992)
15. Dresig, H, Rockhausen, L, Naake, S: Vollständiger und harmonischer Ausgleich ebener Mechanismen, Fortschritt-Berichte VDI, Reihe 18, Nr. 155. VDI Verlag, Düsseldorf (1994)
16. Kochev, IS: General theory of complete shaking moment balancing of planar linkages: a critical review. Mech Mach Theory **35**, 1501–1514 (2000)
17. Kochev, IS: General method for active balancing of combined shaking moment and torque fluctuations in planar linkages. Mech Mach Theory **25**, 679–687 (1990)
18. Ye, Z, Smith, MR: Complete balancing of planar linkages by an equivalence method. Mech Mach Theory **29**(5), 701–712 (1994)
19. Esat, I, Bahai, H: A theory of complete force and moment balancing of planer linkage mechanisms. Mech Mach Theory **34**, 903–922 (1999)
20. Arakelian, VH, Smith, MR: Design of planar 3-DOF 3-RRR reactionless parallel manipulators. Mechatronics **18**(10), 601–606 (2008)
21. Wu, Y, Gosselin, CM: On the dynamic balancing of multi-DOF parallel mechanisms with multiple legs. ASME J Mech Des **129**(2), 234–238 (2007)
22. Nguyen, VK: Über den Massenausgleich in Mehrkörpersystemen. Tech Mech **14**(3–4), 231–238 (1994)
23. Nguyen, VK, Nguyen, PD, Pham, VS: Balancing conditions of planar mechanisms with multi-degree of freedom. Vietnam J Mech **27**, 204–212 (2005)
24. Nguyen, VK, Nguyen, PD: Balancing conditions for spatial mechanisms. Mech Mach Theory **42**, 1141–1152 (2007)
25. Nguyen, VK, Nguyen, PD: On the dynamic balancing conditions of planar multi-DOF parallel manipulators with revolute joints. In: Proceedings of the 1st IFToMM International Symposium on Robotics and Mechatronics, Hanoi, 21–23 Sept, 2009
26. Arakelian, VH, Smith, VH: Complete shaking force and shaking moment balancing of linkages. Mech Mach Theory **34**, 1141–1153 (1999)
27. Arakelian, V, Dahan, M: Partial shaking moment balancing of fully shaking force balanced linkages. Mech Mach Theory **36**, 1241–1252 (2001)
28. Arakelian, VH: Complete shaking force and shaking moment balancing of RSS'R spatial linkages. J Multibody Dyn **221**, 303–310 (2007)
29. Chaudhary, H, Saha, SK: Balancing of shaking forces and shaking moments for planar mechanisms using the equimomental systems. Mech Mach Theory **43**, 310–334 (2008)
30. Feng, G: Complete shaking force and shaking moment balancing of four types of six-bar linkages. Mech Mach Theory **24**(4), 275–287 (1989)
31. Feng, G: Complete shaking force and shaking moment balancing of 17 types of eight-bar linkages only with revolute pairs. Mech Mach Theory **26**, 197–206 (1991)
32. Moore, B, Schicho, J, Gosselin, CM: Determination of the complete set of shaking force and shaking moment balanced planar four-bar linkages. Mech Mach Theory **44**, 1338–1347 (2009)
33. Kaufman, RE, Sandor, GN: Complete force balancing of spatial linkages. Trans ASME J Eng Ind **93**, 620–626 (1971)
34. Bagci, C: Complete balancing of space mechanisms – shaking force balancing. J Mech Trans Automat Des **105**, 609–616 (1983)
35. Chen, N-X: The complete shaking force balancing of a spatial linkage. Mech Mach Theory **19**, 243–255 (1984)
36. Yue-Qing, Y: Complete shaking force and moment balancing of spatial irregular force transmission mechanisms using additional link. Mech Mach Theory **23**, 279–285 (1988)

37. Abdel-Rahman, TM, Elbestawi, MA: Synthesis and dynamics of statically balanced direct-drive manipulators with decoupled inertia tensors. Mech Mach Theory **26**, 389–402 (1991)
38. Wang, J, Gosselin, CM: Static balancing of spatial four-degree-of freedom parallel mechanisms. Mech Mach Theory **35**, 563–592 (2000)
39. Arakelian, V., Smith, M.R: Shaking moment minimization of fully force-balanced linkages. In: Proceedings of the 11th World Congress in Mechanism and Machine Science, Tianjin, China (2004)
40. Park, J: Principle of dynamical balance for multibody systems. Multibody Syst Dyn **14**, 269–299 (2005)
41. Russo, A, Sinatra, R, Xi, F: Static balancing of parallel robots. Mech Mach Theory **40**, 191–202 (2005)