Recent Existence Results for Spectral Problems

Dario Mazzoleni

Abstract In this survey we present the new techniques developed for proving existence of optimal sets when one minimizes functionals depending on the eigenvalues of the Dirichlet Laplacian with a measure constraint, the most important being:

$$\min\left\{\lambda_k(\Omega) : \Omega \subset \mathbb{R}^N,\ |\Omega| = 1\right\}.$$

In particular we sketch the main ideas of some recent works, which allow to extend the now classic result by Buttazzo and Dal Maso to $\mathbb{R}^N$.

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1 Introduction

The aim of this note is to report some recent existence results for classical shape optimization problems involving eigenvalues of the Dirichlet Laplacian. More precisely, we consider minimization problems of the following form:

$$\min\{\lambda_k(\Omega) : \Omega \in \mathcal{A}\}, \tag{1.1}$$

where $k \in \mathbb{N}$, λ_i denotes the ith eigenvalue of the Dirichlet Laplacian (counted with multiplicity) and $\mathcal{A}$ is the class of admissible shapes. A natural choice for this class, that we use in Sects. 3 and 4, is:

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D. Mazzoleni (✉)
Dipartimento di Matematica “G. Peano”, Via Carlo Alberto, 10, 10123 Torino, Italy
e-mail: dmazzole@unito.it

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$$\mathcal{A} := \left\{\Omega \subset \mathbb{R}^N,\ \text{quasi-open},\ |\Omega| \leq 1\right\}, \tag{1.2}$$

where $|\cdot|$ denotes the Lebesgue measure in $\mathbb{R}^N$, $N \in \mathbb{N}$. We need to have a bound on the measure of admissible sets, otherwise the monotonicity of Dirichlet eigenvalues would trivialize the problem; moreover the bound on the measure is taken less or equal to 1 only for simplicity: with every other positive constant the setting is unchanged. Then, since eigenvalues are decreasing with respect to set inclusion, it is equivalent to consider the problem with the equality constraint. An alternative (less common) choice, instead of the measure constraint, is a bound on the perimeter, which was studied only recently in [18]. The choice of *quasi-open*[1] sets is made in order to get compactness with a suitable topology and will be enlightened in Sect. 2. At last, one can consider also shapes contained in (see Sect. 2) or containing (see Sect. 5) a (quasi-)open bounded set.

Optimization problems like (1.1) naturally arise in the study of many physical phenomena, e.g. heat diffusion or wave propagation inside a domain $\Omega \subset \mathbb{R}^N$, and the literature is very wide (see [8, 13, 21, 22] for an overview), with many works in the last few years. Problem (1.1) in the class (1.2) was studied first by Lord Rayleigh in his treatise *The theory of sound* of 1877 (see [28]) and he conjectured the ball to be the optimal set when $k = 1$. This was proved by Faber [19] and Krahn [23, 24] in the 1920s, using techniques based on spherical decreasing rearrangements. From that result the case $k = 2$ follows with little additional effort: Krahn [23, 24] and Szegö [29] proved two disjoint equal balls of half measure each to be optimal. The situation for $k \geq 3$ becomes more complicate and it is not known what are the optimal shapes, yet. The only other functionals of eigenvalues for which the optimal shape is known are λ_1/λ_2 and λ_2/λ_3; Ashbaugh and Benguria (see [2]) proved that the minimizers are the unit ball and two equal disjoint balls of half measure each respectively.

Since the search for explicit optimal shapes did not give other results, it is natural to study at least whether a minimizer for (1.1) exists, and this subject turns out to be a difficult one, too. It is natural to attack an existence problem using the direct method of the Calculus of Variations. One first difficulty in order to apply it in this setting consists in finding a suitable notion of convergence for sets, which "behaves well" with respect to eigenvalues of Dirichlet Laplacian. More important, one needs also to find out how to suitably choose the class of admissible sets. It is immediately clear that the convergence in measure (or L^1 convergence of the characteristic functions) does not fit well, since it neglects sets of positive capacity: as an example one can consider a ball and the same ball minus a radius (in $\mathbb{R}^2$), which are the same set for this topology, but have different Dirichlet eigenvalues.

The search for a "right" notion of convergence in this setting was a main problem for many years. In the 1980s Dal Maso and Mosco (see [16, 17]) proposed the notion of γ-convergence, which has the "good" property that Dirichlet eigenvalues are continuous with respect to it. This was the main tool used by Buttazzo and Dal

[1] Quasi-open sets are superlevels of Sobolev functions.

Maso in 1993 (see [14]) for proving a fundamental existence result for a very general class of functions of eigenvalues, in the class of *quasi-open* sets inside a fixed bounded box. More precisely, they fix $D \subset \mathbb{R}^N$ bounded and open, and consider $F : \mathbb{R}^k \to \mathbb{R}$ a functional increasing in each variable and lower semicontinuous (l.s.c.). Then there exists a minimizer for the problem

$$\min \{F(\lambda_1(\Omega), \ldots, \lambda_k(\Omega)) : \Omega \subset D, \text{ quasi-open}, |\Omega| \leq 1\}. \tag{1.3}$$

The above result gives a definitive answer to the existence problem for a very general class of spectral functionals in a bounded ambient space (actually it is sufficient to suppose D to have finite measure). We give the main ideas of the proof of this result in Sect. 2, together with some preliminaries about γ-convergence. The extension of the result by Buttazzo and Dal Maso to generic domains in $\mathbb{R}^N$ is a non trivial topic, because minimizing sequences, in principle, could have a significant portion of volume moving to infinity.

A first partial result in the direction of an extension to unbounded domains was obtained by Bucur and Henrot in 2000 (see [11]); they proved the existence of a minimizer for λ_3, using a concentration-compactness argument (see [5]). Moreover they showed that, given $k \geq 1$, if there exists a bounded minimizer for λ_j for all $j = 1, \ldots, k-1$, then there exists a minimizer for λ_k (and more in general for Lipschitz functionals of the first k eigenvalues). Unfortunately this boundedness hypothesis was not known even for λ_3, till Dorin Bucur in a very recent paper (see [7]) was able to study the regularity of *energy shape subsolutions*. Employing techniques coming from the theory of free boundaries, it is possible to prove boundedness and finiteness of the perimeter for this class of sets, stable with respect to internal perturbations. Since optimal sets for (1.1) can be proved to be energy shape subsolutions, the existence of a minimizer for λ_k for all $k \in \mathbb{N}$ follows easily from the result by Bucur and Henrot. We present the ideas behind the proof of these results in Sect. 3.

In the same period another independent proof of existence of a solution for problem (1.3) in $\mathbb{R}^N$, with F satisfying the same hypotheses as in the result by Buttazzo and Dal Maso, was given by Mazzoleni and Pratelli (see [27]). Their idea consists in showing that, given a minimizing sequence for the problem

$$\min \left\{F(\lambda_1(\Omega), \ldots, \lambda_k(\Omega)) : \Omega \subset \mathbb{R}^N, \text{ quasi-open}, |\Omega| \leq 1\right\}, \tag{1.4}$$

it is then possible to find a new one made of sets with diameter bounded by a constant depending only on k, N (but not on the particular functional) and with all the first k eigenvalues not increased. This argument, roughly speaking, works because sets with long "tails" can not have the first k eigenvalues very small. Moreover, with minor changes in the proof, it is also possible to deduce that every minimizer for (1.4) is bounded, provided that F is *weakly strictly increasing* (see [26]). This more "direct" method is presented in Sect. 4.

In recent years the existence of optimal sets was studied also for another kind of shape optimization problem (among sets with a measure constraint) involving eigenvalues of Dirichlet Laplacian: when there is an internal *obstacle*, that is,

$$\min\left\{\lambda_k(\Omega) : D \subset \Omega \subset \mathbb{R}^N,\ \text{quasi-open},\ |\Omega| \leq 1\right\}, \tag{1.5}$$

where D is a fixed quasi-open box with $|D| \leq 1$. Bucur et al. in [10], using a concentration-compactness argument similar to the one in [5], proved existence of a solution for $k = 1$, gave a characterization of the cases when $k \geq 2$ and provided a partial regularity result for the solutions. In Sect. 5 we deal with the main ideas of their work.

The results exposed above give a quite complete understanding for the problem of existence of minimizers for spectral functionals involving eigenvalues of the Dirichlet Laplacian with a measure constraint. On the other hand the study of the regularity of solutions is still a main subject of research, both in the bounded (see [3]) and in the unbounded case (see the recent work [12]). In particular it is not known in general whether the minimizers for λ_k are open sets and not only quasi-open. This is one major open problem in spectral shape optimization.

It is also possible to consider minimization problems like (1.1) with perimeter constraint instead of volume constraint. This kind of problem was studied in the recent paper by De Philippis and Velichkov [18], where they prove that there exists a minimizer for

$$\min\left\{\lambda_k(\Omega) : \Omega \subset \mathbb{R}^N,\ \text{measurable},\ P(\Omega) \leq 1\right\}.$$

They use techniques to some extent analogous to those used by Bucur in [7], combining a concentration compactness argument and the study of the regularity for *perimeter* shape subsolutions. The perimeter constraint turns out to have a better *regularizing effect* than the volume constraint. In fact De Philippis and Velichkov are able to give many informations about regularity of optimal shapes: first of all the optimal shapes are open, so the above problem can be formulated among open sets. Moreover every optimal set Ω is bounded, has finite perimeter and its boundary $\partial\Omega$ is $C^{1,\alpha}$ for all $\alpha \in (0, 1)$, outside a closed set of Hausdorff dimension at most $N-8$.

2 Preliminaries and Existence in a Bounded Box

First of all we need to recall some basic tools, which you can find in more detail in the books [8, 21, 22]. We define the Sobolev space $H^1_0(\Omega)$ as

$$H^1_0(\Omega) = \left\{u \in H^1(\mathbb{R}^N) :\ \text{cap}(\{u \neq 0\} \setminus \Omega) = 0\right\}, \tag{2.1}$$

where for every $E \subset \mathbb{R}^N$ the capacity of E is defined as

$$\text{cap}(E) := \min \quad \left\{ \|v\|^2_{H^1(\mathbb{R}^N)} : v \in H^1(\mathbb{R}^N),\ v \geq 1 \text{ a.e. in a neighborhood of } E \right\}.$$

Then, given a function $u \in H^1_0(\Omega)$, its *quasi-continuous* representative is defined as

$$\widetilde{u}(x) := \lim_{r \to 0} \fint_{B_r(x)} u(y)\, dy.$$

Since outside a set of zero capacity every point is Lebesgue for u (see [22] for example), then the quasi-continuous representative is defined up to zero capacity and we identify every H^1 function with its quasi-continuous representative.

A set Ω is called *quasi-open* if for all $\varepsilon > 0$ there exists an open set Ω_ε such that cap $(\Omega_\varepsilon \Delta \Omega) < \varepsilon$; for example superlevels of H^1 functions are quasi-open sets. Moreover, given a bounded open box D, we call $R_\Omega : L^2(D) \to L^2(D)$ the resolvent operator for the Dirichlet Laplacian, that is,

$$R_\Omega(f) := \arg\min \left\{ \frac{1}{2} \int_D |Du|^2 - \int_D uf,\ u \in H^1_0(\Omega) \right\},$$

for all $f \in L^2(D)$. The definition above can be extended also to capacitary[2] measures:

$$R_\mu(f) := \arg\min \left\{ \frac{1}{2} \int_D |Du|^2 + \int_D u^2\, d\mu - \int_D uf,\ u \in H^1_0(\Omega) \cap L^2_\mu(D) \right\}.$$

When $f = 1$, $R_\Omega(1) =: w_\Omega$ is called *torsion function* and it is an important tool for proving existence results. In particular w_Ω is the solution of

$$\begin{cases} -\Delta w = 1 & \text{in } \Omega, \\ \quad w \in H^1_0(\Omega), \end{cases}$$

and hence a minimizer for the so called *torsion energy functional*

$$E(\Omega) := \min_{u \in H^1_0(\Omega)} \left\{ \frac{1}{2} \int_D |Du|^2 - \int_D u \right\}.$$

After that, given a sequence of quasi-open sets contained in D, $(\Omega_n)_{n \in \mathbb{N}}$, we say that Ω_n *γ-converge* to a quasi-open set $\Omega \subset D$ as $n \to \infty$ when $w_{\Omega_n} \rightharpoonup w_\Omega$ in $H^1_0(D)$. Moreover Dal Maso and Mosco proved (see [16, 17]) that the convergence above implies for all $f \in L^2(D)$ $R_{\Omega_n}(f) \to R_\Omega(f)$ in $L^2(D)$, hence also $R_{\Omega_n} \to R_\Omega$ in the operator norm $\mathcal{L}(L^2(D))$ and hence the full spectrum converges. Thus eigenvalues of the Dirichlet Laplacian are continuous with respect to γ-convergence. Unfortunately, γ is a rather strong convergence and it is not compact in the class $\mathcal{A}(D) = \{\Omega \subset D,\ \text{quasi-open},\ |\Omega| \leq 1\}$; it is then necessary to weaken it, in order

[2] A Borel measure μ is called capacitary if, for every set E, cap $(E) = 0$ implies $\mu(E) = 0$.

to apply the direct method of the calculus of variations to problem (1.3). A natural choice is the following.

A sequence $\Omega_n \in \mathcal{A}(D)$ is said to *weak γ-converge* to a domain $\Omega \in \mathcal{A}(D)$ if $w_{\Omega_n} \rightharpoonup w$ in $H_0^1(D)$ as $n \to \infty$, with $\Omega := \{w > 0\}$. Note that w coincide with $w_\Omega = R_\Omega(1)$ if and only if the convergence is γ and not only weak γ. More precisely, for some capacitary measure μ, $w = R_\mu(1)$: in fact we can say that the γ-convergence is compact in the class of capacitary measures, where a set Ω corresponds to the following measure:

$$\infty_\Omega(E) = \begin{cases} +\infty & \text{if } \operatorname{cap}(E \setminus \Omega) > 0, \\ 0 & \text{if } \operatorname{cap}(E \setminus \Omega) = 0. \end{cases}$$

A well known example of a sequence of quasi-open sets γ-converging to a measure which is not a quasi-open set is due to Cioranescu and Murat [15].

Buttazzo and Dal Maso used the compactness properties of the weak γ-convergence and the lower semicontinuity of Dirichlet eigenvalues with respect to it for proving a very general existence result.

Theorem 2.1 (Buttazzo–Dal Maso) *Let $D \subset \mathbb{R}^N$ be a bounded, open set and $F: \mathbb{R}^k \to \mathbb{R}$ be a functional increasing in each variable and lower semicontinuous (l.s.c.). Then there exists a minimizer for the problem*

$$\min\{F(\lambda_1(\Omega), \ldots, \lambda_k(\Omega)) : \Omega \subset D,\ \textit{quasi-open},\ |\Omega| \leq 1\}. \tag{2.2}$$

First of all, the weak γ-convergence is built in order to be compact in the class $\mathcal{A}(D)$ and so a minimizing sequence converges, up to subsequences. Then it is easy to see that the weak γ-convergence is l.s.c. with respect to the Lebesgue measure, so the constraint $|\Omega| \leq 1$ is satisfied by the limit of a weak γ-converging sequence of sets. It is then necessary to study the lower semicontinuity of the weak γ-convergence with respect to eigenvalues and this turns out to be a crucial point in the argument by Buttazzo and Dal Maso for proving Theorem 2.1. The following proposition gives a positive answer, for a (quite large) class of functionals.

Proposition 2.2 *A functional $J: \mathcal{A}(D) \to \mathbb{R}$ non decreasing with respect to set inclusion is γ l.s.c if and only if it is weak γ l.s.c.*

The hypothesis on the functional J to be nondecreasing with respect to set inclusion is quite strong, but it is satisfied by eigenvalues of the Dirichlet Laplacian and hence also by increasing functions of them. Thus the above Proposition can be applied in the hypothesis of Theorem 2.1.

The proof of Proposition 2.2 is based on the following (non trivial) key points, whose proof relies also on the maximum principle for the Dirichlet Laplacian.

(a) If w_{Ω_n} converge weakly in $H_0^1(D)$ to w and $v_n \in H_0^1(\Omega_n)$ converge weakly in $H_0^1(D)$ to v, then $v \in H_0^1(\{w > 0\})$.
(b) Let $\Omega_n \subset D$ be quasi-open sets such that w_{Ω_n} converge weakly in $H_0^1(D)$ to $w \in H_0^1(\Omega)$ for some quasi-open set $\Omega \subset D$. Then there exist a subsequence (not relabeled) and a sequence of quasi-open sets $\widetilde{\Omega}_n$ that γ-converge to Ω satisfying $\Omega_n \subset \widetilde{\Omega}_n \subset D$.

Then the Buttazzo and Dal Maso Theorem follows easily from Proposition 2.2 using the direct method of the Calculus of Variations. Given a minimizing sequence (Ω_n) of quasi-open sets for problem (1.3), by the compactness of the weak γ-convergence we can extract a subsequence (not relabeled) that weak γ-converges to a quasi-open set $\Omega \in \mathcal{A}(D)$. Using the properties of the weak γ-convergence highlighted above, the hypotheses on F and Proposition 2.2, we have that

$$|\Omega| \leq \liminf_{n\to\infty} |\Omega_n| \leq 1,$$
$$F(\lambda_1(\Omega), \dots, \lambda_k(\Omega)) \leq \liminf_{n\to\infty} F(\lambda_1(\Omega_n), \dots, \lambda_k(\Omega_n)),$$

thus Ω is an optimal set for (1.3).

Remark 2.3 In the hypotheses of Theorem 2.1, it is sufficient to suppose that $D \subset \mathbb{R}^N$ has finite measure, so that the embedding $H^1(D) \hookrightarrow L^2(D)$ remains compact (see [9]).

3 Concentration Compactness and Subsolutions

The main problem in extending the result by Buttazzo and Dal Maso to (quasi-)open sets of $\mathbb{R}^N$ is the lack of compactness of the embedding $H^1(\mathbb{R}^N) \hookrightarrow L^2(\mathbb{R}^N)$. The concentration-compactness principle by P.L. Lions (see [25]) tries to focus on "how" the embedding $H^1(\mathbb{R}^N) \hookrightarrow L^2(\mathbb{R}^N)$ can be non compact. In the case of sets Bucur (see [5]) rearranged the principle in the following way, ruling out the *vanishing* case.

Theorem 3.1 (Lions, Bucur) *Let $(\Omega_n)_n \subset \mathbb{R}^N$ be a sequence of quasi-open sets with $|\Omega_n| \leq 1$ for all $n \geq 1$. Then there exists a subsequence (not relabeled) such that one of the following situations occur:*

(1) **Compactness**. *There exist $y_n \in \mathbb{R}^N$ and a capacitary measure μ such that $R_{y_n+\Omega_n} \to R_\mu$ in $\mathcal{L}(L^2(\mathbb{R}^N))$.*
(2) **Dichotomy**. *There exist Ω_n^i, $i = 1, 2$ such that $|\Omega_n^i| > 0$, $d(\Omega_n^1, \Omega_n^2) \to \infty$ and $\|R_{\Omega_n} - R_{\Omega_n^1 \cup \Omega_n^2}\|_{L^2(\mathbb{R}^N)} \to 0$ as $n \to \infty$.*

Thanks to the concentration compactness argument above, it is easy to prove the following partial existence result (see [11]) for the unbounded case.

Theorem 3.2 (Bucur–Henrot) *For $k \geq 2$ if there exists a bounded minimizer for $\lambda_1, \dots, \lambda_k$ in the class $\mathcal{A}(\mathbb{R}^N)$, then there exists at least a minimizer for λ_{k+1} in $\mathcal{A}(\mathbb{R}^N)$.*

In particular this provides existence of a minimizer for the problem:

$$\min\left\{\lambda_3(\Omega) : \Omega \subset \mathbb{R}^N,\ \text{quasi-open},\ |\Omega| \leq 1\right\}, \tag{3.1}$$

since the minimizers for λ_1 and λ_2 are respectively a ball and two balls, which are bounded. The idea of the proof of Theorem 3.2 is quite simple. Given a minimizing sequence for λ_{k+1} in $\mathcal{A}(\mathbb{R}^N)$, made of bounded sets Ω_n, if compactness occur, existence follows directly considering the regular set[3] Ω_μ of the limit measure (see [21, Theorem 5.3.3]). On the other hand, if dichotomy happens, then $\Omega_n^1 \cup \Omega_n^2$ is also a minimizing sequence. But it is thus possible to see that the sequence $(\Omega_n^i)_n$ must be minimizing for some lower eigenvalue in the class $\mathcal{A}(\mathbb{R}^N)$, with different measure constraints: $c_1, c_2 > 0$ such that $c_1 + c_2 \leq 1$. Hence, up to translations, a minimizer for λ_{k+1} will be the union of the two minimizers corresponding to some lower eigenvalues. Note that if we do not know that there exists a bounded minimizer for every lower eigenvalue, it is not possible to consider the union of two of them, since in principle one can be dense in $\mathbb{R}^N$.

Since not even the boundedness of a minimizer for λ_3 was known, Bucur studied the link between this kind of shape optimization problems and free boundary problems, in order to be able to apply also in this framework the powerful techniques developed by Alt and Caffarelli (see [1]) and later implemented in the study of the energy of the Dirichlet Laplacian by Briançon, Hayouni and Pierre (see [4]).

First of all we need to be able to deal with measurable sets A, with $|A| < \infty$ (we call $\mathcal{M}$ the class of such sets), so we define the *Sobolev-like* space

$$\tilde{H}_0^1(A) := \left\{u \in H^1(\mathbb{R}^N) : u = 0 \ \text{ a.e. on } \mathbb{R}^N \setminus A\right\}. \tag{3.2}$$

It is well known (see [18] for a more detailed discussion of those spaces) that there exists a quasi-open set $\omega_A \subseteq A$ such that

$$H_0^1(\omega_A) = \tilde{H}_0^1(A),$$

hence for functionals decreasing with respect to set inclusion (e.g. single eigenvalues) it is equivalent to solve problem (1.4) in the class of quasi-open sets with the classical definition of Sobolev space (2.1), or in the family of measurable sets associated to $\tilde{H}_0^1$.

Then it is possible to endow the family of measurable sets with a distance induced by γ-convergence:

[3]The regular set Ω_μ of a measure μ is the largest (in the sense of inclusion q.e.) countable union of sets of finite (μ-)measure.

$$d_\gamma(A, B) := \int_{\mathbb{R}^N} |w_A - w_B|, \qquad A, B \in \mathcal{M},$$

where we considered the torsion functions in $H^1(\mathbb{R}^N)$ extended to zero: $w_\Omega = 0$ in $\mathbb{R}^N \setminus \Omega$.

The most important notion in order to link shape optimization problems with free boundary problems is the one of *shape subsolution*.

Definition 3.3 We say that a set $A \in \mathcal{M}$ is a *local shape subsolution* for a functional $\mathcal{F}\colon \mathcal{M} \to \mathbb{R}$ if there exist $\delta > 0$ and $\Lambda > 0$ such that

$$\mathcal{F}(A) + \Lambda|A| \le \mathcal{F}(\tilde{A}) + \Lambda|\tilde{A}|, \qquad \forall\, \tilde{A} \subset A,\ d_\gamma(A, \tilde{A}) < \delta.$$

Roughly speaking, a shape subsolution is a set that is optimal with respect to internal perturbations. Bucur (see [7]) proved a very powerful regularity result for shape subsolution of the *torsion energy* functional

$$E(A) := \min_{u \in \tilde{H}^1_0(A)} \left\{ \frac{1}{2} \int_{\mathbb{R}^N} |Du|^2 - \int_{\mathbb{R}^N} u \right\}.$$

Theorem 3.4 *Let A be a local shape subsolution (with constants δ, Λ) for the torsion energy E. Then it is bounded, with $diam(A) \le C(|A|, \delta, \Lambda)$, has finite perimeter and its fine interior has the same measure of A.*

The proof of the theorem for the finite perimeter part is based on controlling the term $\int_{\{0 \le w_A \le \varepsilon\}} |Dw_A|^2$, while the boundedness and the inner density estimate come from the following Alt–Caffarelli type estimate: there exist $r_0, C_0 > 0$ such that for all $r \le r_0$

$$\sup_{B_{2r}(x)} w_A \le C_0 r \qquad \text{implies} \qquad u = 0 \text{ in } B_r(x).$$

The next key point in Bucur's argument consists in linking the minimizers of eigenvalues of Dirichlet Laplacian with shape subsolution of the energy. We consider the minimization problem, equivalent to (1.4) up to choose $\Lambda > 0$ small enough (for a detailed discussion about this equivalence, see [4]),

$$\min \left\{ F(\lambda_1(A), \ldots, \lambda_k(A)) + \Lambda|A| : A \subset \mathbb{R}^N,\ \text{quasi-open} \right\}, \tag{3.3}$$

for a functional $F\colon \mathbb{R}^k \to \mathbb{R}$ which satisfies the following Lipschitz-like condition for some positive α_i, $i = 1, \ldots, k$:

$$F(x_1, \ldots, x_k) - F(y_1, \ldots, y_k) \le \sum_{i=1}^{k} \alpha_i (x_i - y_i), \qquad \forall x_i \ge y_i,\ i = 1, \ldots, k. \tag{3.4}$$

Theorem 3.5 *Assume that A is a solution of* (3.3)*, then it is a local shape subsolution for the energy problem.*

The proof is based on [6, Theorem 3.4], which assures, for all $k \in \mathbb{N}$, the existence of a constant $c_k(A)$ such that:

$$\left| \frac{1}{\lambda_k(\tilde{A})} - \frac{1}{\lambda_k(A)} \right| \leq c_k(A) d_\gamma(A, \tilde{A}).$$

Then, up to choose δ small enough and $\tilde{A} \subseteq A$ with $d_\gamma(\tilde{A}, A) < \delta$, it follows

$$\begin{aligned}\Lambda(|A| - |\tilde{A}|) &\leq F(\lambda_1(\tilde{A}), \dots, \lambda_k(\tilde{A})) - F(\lambda_1(A), \dots, \lambda_k(A)) \\ &\leq \sum_i \alpha_i (\lambda_i(\tilde{A}) - \lambda_i(A)) \\ &\leq \sum_i \alpha_i c_i' (E(\tilde{A}) - E(A)) \leq K(E(\tilde{A}) - E(A)),\end{aligned}$$

with a constant K depending on $c_i' = c_i'(A, \delta, i)$ and α_i, for $i = 1, \dots, k$.

Now a straightforward application of Theorem 3.4, together with Theorem 3.2, gives the main existence result.

Theorem 3.6 (Bucur) *If the functional F satisfies the Lipschitz-like condition* (3.4)*, then problem* (3.3) *has at least a solution for every* $k \in \mathbb{N}$*. Moreover every optimal set is bounded and has finite perimeter.*

In particular there exists a solution for the problem

$$\min \left\{ \lambda_k(\Omega) : \Omega \subset \mathbb{R}^N \text{ quasi-open}, |\Omega| \leq 1 \right\},$$

for all $k \in \mathbb{N}$. We highlight here that, to our knowledge, it is not known yet if the above problem admits solutions in the class of *open* sets. It is possible to give slightly different proof of Theorem 3.6 that does not use the concentration-compactness principle, but only the regularity of energy shape subsolutions. This proof is due to Bozhidar Velichkov and it has never appeared on a published paper, to our knowledge.

Remark 3.7 (Velichkov) Let $(\Omega_n)_{n \geq 1}$ be a minimizing sequence for problem (3.3), with $|\Omega_n| < \infty$ for all $n \in \mathbb{N}$, and then we consider, for all $n \in \mathbb{N}$, the minimum problem

$$\min \{ F(\lambda_1(\Omega), \dots, \lambda_k(\Omega)) + \Lambda |\Omega| : \Omega \subset \Omega_n \},$$

for some $\Lambda > 0$ sufficiently small. Theorem 2.1 by Buttazzo and Dal Maso assures that there exists a solution Ω_n^*, but this is also a subsolution and hence by Theorem 3.4 it has diameter uniformly bounded, depending only on k, N. Hence we have a new minimizing sequence Ω_n^* uniformly bounded to which it is possible to apply again Theorem 2.1, thus obtaining existence for problem (3.3).

4 How to Choose an Uniformly Bounded Minimizing Sequence

In this section we aim to provide the main ideas of the proof of the existence theorem presented by Mazzoleni and Pratelli in [27], which uses an "elementary" method that requires neither a concentration-compactness argument nor regularity of shape subsolutions.

Theorem 4.1 *Let $k, N \in \mathbb{N}$ and $F\colon \mathbb{R}^k \to \mathbb{R}$ be a functional increasing in each variable and l.s.c., then there exists a (bounded) minimizer for the problem*

$$\min\left\{F(\lambda_1(\Omega), \ldots, \lambda_k(\Omega)) : \Omega \subset \mathbb{R}^N,\ \textit{quasi-open},\ |\Omega| \le 1\right\}. \qquad (4.1)$$

More precisely the diameter of the optimal set is controlled by a constant depending only on k, N (and not on the particular functional F).

The proof is based on the following Proposition, which gives the possibility to consider minimizing sequences for (4.1) with uniformly bounded diameters, which means that we can employ Buttazzo–Dal Maso Theorem.

Proposition 4.2 *If $\Omega \subset \mathbb{R}^N$ is an open set with unit volume, there exists another open set of unit volume, $\widehat{\Omega}$, contained in cube of side $R = R(k, N)$ and such that*

$$\lambda_i(\widehat{\Omega}) \le \lambda_i(\Omega), \qquad \forall\, i = 1, \ldots, k.$$

From Proposition 4.2, Theorem 4.1 follows easily: in fact, given a minimizing sequence $(\Omega_n)_{n\in\mathbb{N}}$ made of open sets with unit volume, it is sufficient to take the corresponding sequence $(\widehat{\Omega}_n)_{n\in\mathbb{N}}$, which is again minimizing and then to apply Theorem 2.1 by Buttazzo and Dal Maso to it.

On the other hand the proof of Proposition 4.2 is quite delicate: we give here below the main ideas of how it is carried on. In particular, given Ω open and with unit volume, we focus on its left "tail", that is, the set

$$\Omega^l_{\bar{t}} := \left\{x \in \Omega : x_1 < \bar{t}\right\},$$

for a $\bar{t}$ such that $\left|\Omega^l_{\bar{t}}\right| = \widehat{m}$, for a suitably choosen $\widehat{m}$, very small but fixed (depending only on k, N). Then it is possible to find a new set $\widehat{\Omega}$ with bounded tail and the first k eigenvalues lowered. We need some notations: for all $t \le \bar{t}$ we define:

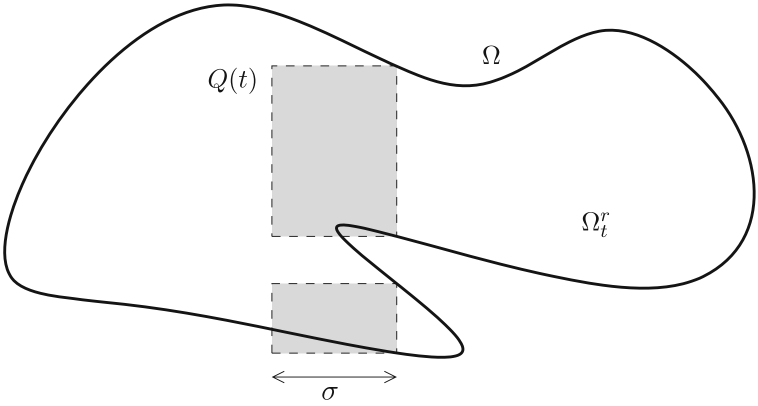

Fig. 1 A set Ω with the cylinder $Q(t)$ (*shaded*)

$$\Omega_t^r := \{x \in \Omega : x_1 > t\},$$
$$\Omega_t := \Big\{(x_2, \ldots, x_N) \in \mathbb{R}^{N-1} : (t, x_2, \ldots, x_N) \in \Omega\Big\},$$
$$\varepsilon(t) := \mathcal{H}^{N-1}(\Omega_t), \quad m(t) = \int_{-\infty}^{t} \varepsilon(s)\, ds,$$
$$\delta(t) := \sum_{i=1}^{k} \int_{\Omega_t} |Du_i(t, x_2, \ldots, x_N)|^2 \, d\mathcal{H}^{N-1}.$$

For all $t \le \bar{t}$ it is possible to compare the first k eigenvalues of Ω with those of $\widetilde{\Omega}(t) := \Omega_t^r \cup Q(t)$, which is obtained by "cutting" the "tail" at level t and adding a suitable small cylinder $Q(t)$ of height $\sigma(t) = \varepsilon(t)^{\frac{1}{N-1}}$ (see Fig. 1).

Using the min-max principle for eigenvalues one obtains

$$\lambda_i(\widetilde{\Omega}(t)) \le \lambda_i(\Omega) + C(k, N)\varepsilon(t)^{\frac{1}{N-1}}\delta(t), \qquad \forall i = 1, \ldots, k,$$

if $\varepsilon(t), \delta(t) \le \nu$, for some constant $\nu = \nu(k, N)$. After rescaling $\widetilde{\Omega}(t)$ to unit volume, being $\widehat{\Omega}(t) := |\widetilde{\Omega}(t)|^{-\frac{1}{N}}\widetilde{\Omega}(t)$, it is possible to prove that for a suitable constant $\overline{C} = \overline{C}(k, N)$, exactly one of the following conditions hold.

(1) $\max\{\varepsilon(t), \delta(t)\} > \nu$.
(2) (1) does not hold and $m(t) \le \overline{C}(\varepsilon(t) + \delta(t))\varepsilon(t)^{\frac{1}{N-1}}$.
(3) (1) and (2) do not hold and for all $i = 1, \ldots, k$, $\lambda_i(\widehat{\Omega}(t)) < \lambda_i(\Omega)$. Moreover if $m(t) \ge \widehat{m}$, then there exist $\eta = \eta(k, N)$ such that $\lambda_i(\widehat{\Omega}(t)) < \lambda_i(\Omega) - \eta$ for all $i = 1, \ldots, k$.

In order to conclude the boundedness of the "tail", we define

$$\hat{t} := \sup\big\{t \le \bar{t} : \text{condition (3) holds for } t\big\},$$

with the usual convention that $\hat{t} = -\infty$ if condition (3) is false for every $t \leq \bar{t}$. We consider the following subsets of $(\hat{t}, \bar{t})$

$$A := \left\{t \in (\hat{t}, \bar{t}) : \text{ condition (1) holds for } t\right\},$$
$$B := \left\{t \in (\hat{t}, \bar{t}) : \text{ condition (2) holds for } t \text{ and } m(t) > 0\right\},$$

and it is possible to prove that $|A| + |B| \leq C(k, N)$, since in this case we obtain a differential equation, regarding the measure of the "tail", since for a.e. $t \in \mathbb{R}$ $m'(t) = \varepsilon(t)$,

$$m'(t) \geq \frac{1}{C} m(t)^{\frac{N-1}{N}},$$

and an analogous one about $\int_{-\infty}^{t} \delta(s)\, ds$.

Then, if $\hat{t} = -\infty$, that is, only case (1) or (2) happen, the set Ω has itself a bounded "tail".

On the other hand, if $\hat{t} > -\infty$, we pick a $t^{\star} \in [\hat{t} - 1, \hat{t}]$ for which condition (3) holds and consider $U_1 := \widehat{\Omega}(t^{\star})$.

If $m(t^{\star}) < \widehat{m}$, then we have that $\lambda_i(U_1) < \lambda_i(\Omega)$ for all $i = 1, \ldots, k$ and U_1 has a bounded "tail", so we have concluded. Instead, if $m(t^{\star}) \geq \widehat{m}$, the stronger estimate $\lambda_i(U_1) < \lambda_i(\Omega) - \eta$ holds for all $i = 1, \ldots, k$, but possibly U_1 has not bounded "tail". Hence we iterate the procedure, by applying the whole construction to U_1 and thus finding U_2 which either has bounded "tail", or it satisfies $\lambda_i(U_2) < \lambda_i(\Omega) - 2\eta$ for all $i = 1, \ldots, k$. After l steps, if we have not concluded yet, there is U_l such that

$$\lambda_i(U_l) < \lambda_i(\Omega) - l\eta, \qquad \forall\, i = 1, \ldots, k.$$

Since we can reduce to consider sets with $\lambda_k(\Omega) \leq M$ (see [27, Appendix]), the inequality above is impossible if $l\eta \geq M$: as a consequence, the iteration must stop after less than M/η steps, thus finding a set with bounded "tail" and with the first k eigenvalues lowered.

The same procedure can be performed with small changes also for the "inner" part of the set, that is, $\Omega^i := \left\{(x, y) \in \Omega : \widehat{m} \leq |\Omega^-(x)| \leq 1 - \widehat{m}\right\}$ and to the right tail. Then one can apply the same arguments in all the other coordinate directions. This concludes Proposition 4.2.

At this point, Theorem 4.1 does not guarantee that *every* minimizer is bounded, in fact a constant functional satisfies the hypothesis of the Theorem, but it can not have minimizers uniformly bounded! With a necessary additional assumption on the functional F, in [26] was proved the following.

Theorem 4.3 *In the hypotheses of Theorem 4.1, if the functional F is also* weakly strictly increasing, *that is,*

$$\forall x_i < y_i,\ \forall i = 1, \ldots, k, \qquad F(x_1, \ldots, x_k) < F(y_1, \ldots, y_k),$$

then all the minimizers for problem (4.1) *have diameter bounded by a constant depending only on* k, N.

The proof is carried out improving Proposition 4.2. More precisely, given a sequence of open sets with unit measure that γ-converge to a minimum Ω for problem (4.1), then either, up to pass to a subsequence, $diam(\Omega_n) \leq C(k, N)$, or there exist $(\widehat{\Omega}(t_n))_n$ open sets with unit measure (obtained with a "cutting" procedure as above) such that

$$\lambda_i(\widehat{\Omega}(t_n)) < \lambda_i(\Omega) - \eta(k, N), \qquad \forall\, i = 1, \dots, k.$$

Hence in this last case,

$$\inf_n \left\{ F\Big(\lambda_1(\Omega(t_n)), \dots, \lambda_k(\Omega(t_n))\Big) \right\} < F(\lambda_1(\Omega), \dots, \lambda_k(\Omega)),$$

which is absurd.

Remark 4.4 The main differences in the existence results in $\mathbb{R}^N$ described in this section and in the previous one are the following:

- Bucur's proof gives the important information that all minimizers have finite perimeter, while this property can not be easily deduced from the approach by Mazzoleni and Pratelli;
- On the other hand, the result by Bucur applies to "Lipschitz" functionals of the first k eigenvalues (more precisely satisfying condition (3.4)), while the method by Mazzoleni and Pratelli requires the functionals only to be increasing in each variables and l.s.c.

Remark 4.5 As we have already highlighted, the regularity issue for problem (4.1) is a difficult one and it is not completely understood yet, to our knowledge. In the recent work [12] it is proved that optimal sets are open for a very special class of functional, among which $\lambda_1(\cdot) + \cdots + \lambda_k(\cdot)$. Moreover it is shown that an optimal set for $\lambda_k(\cdot)$ admits an eigenfunction, corresponding to the k th eigenvalue, which is Lipschitz continuous in $\mathbb{R}^N$, but this does not assure the openness.

5 The Case of Internal Constraint

In this section we present the approach used in [10] in order to give some existence results for problem (1.5) with an internal constraint, where D is a quasi-open set with $|D| \leq 1$, possibly unbounded. The main point in order to prove existence results for such problems is the following concentration-compactness principle, inspired by Theorem 3.1, in the case of inner constraints. We remark that the main changes are in the "compactness" case, where there are no more translations.

Theorem 5.1 *Let $(\Omega_n)_n$ be a sequence of quasi-open sets in $\mathbb{R}^N$, each of them containing a given quasi-open set D, with $|\Omega_n| \le 1$ for all $n \ge 1$. Then there exists a subsequence (not relabeled) such that one of the following situations occur:*

(1) **Compactness**. *There exists a capacitary measure μ such that $R_{\Omega_n} \to R_\mu$ in $\mathcal{L}(L^2(\mathbb{R}^N))$ and moreover $D \subset \Omega_\mu$.*
(2) **Dichotomy**. *There exist Ω_n^i, $i = 1, 2$ such that $\liminf_{n\to\infty} |\Omega_n^i| > 0$, $d(\Omega_n^1, \Omega_n^2) \to \infty$ and $\|R_{\Omega_n} - R_{\Omega_n^1 \cup \Omega_n^2}\|_{L^2(\mathbb{R}^N)} \to 0$ as $n \to \infty$. Moreover $\limsup_{n\to\infty} |\Omega_n^1 \cap D| = 0$ or $\limsup_{n\to\infty} |\Omega_n^2 \cap D| = 0$.*

From the above concentration-compactness principle it is possible to prove the following existence result (see [10]). First of all we need to introduce, for $m \ge 0$, the value

$$\lambda_k^*(m) := \inf\{\lambda_k(\Omega) : \Omega \text{ quasi-open}, |\Omega| \le m\}.$$

Theorem 5.2 *Let D be a quasi-open set with $|D| \le 1$. For $k \in \mathbb{N}$ we define*

$$\alpha_k := \inf\left\{\lambda_k(\Omega) : D \subset \Omega \subset \mathbb{R}^N,\ \textit{quasi-open},\ |\Omega| \le 1\right\}. \tag{5.1}$$

If $k = 1$ the problem has at least a solution.
For $k \ge 2$ one of the following assertions holds:

(a) Problem (1.5) has a solution;
(b) There exists $l \in \{1, \dots, k-1\}$ and an admissible quasi-open set Ω such that $\alpha_k = \lambda_{k-l}(\Omega) = \lambda_l^*(1 - |\Omega|)$;
(c) There exists $l \in \{1, \dots, k-1\}$ such that $\alpha_k = \lambda_l^*(1 - |D|) > \lambda_{k-l}(D)$.

Clearly in case (b) and (c) we do not have existence of a solution in general. Something more can be said with stronger hypotheses on D and it will be stated later. Now we sketch the proof of Theorem 5.2 for the case $k = 1$. Let $(\Omega_n)_{n\ge1}$ be a minimizing sequence such that $\liminf_{n\to\infty} |\Omega_n|$ is minimal (clearly the value must be strictly positive). Following Theorem 5.1, if we are in the compactness situation, there is a subsequence (not relabeled) that γ-converges to a capacitary measure μ. The set $\Omega_\mu := \{R_\mu(1) > 0\}$ is admissible and thus it is a solution.

On the other hand, if dichotomy occurs, we get a contradiction. We may assume that $\lambda_1(\Omega_n^1 \cup \Omega_n^2) = \lambda_1(\Omega_n^1)$, since the two sets have positive distance, and clearly the sequence $(\Omega_n^1 \cup D)_n$ is also minimizing. Then only two situations can happen:

1. Either $\liminf_{n\to\infty} |\Omega_n^1 \cup D| < \liminf_{n\to\infty} |\Omega_n|$;
2. Or $\lim_{n\to\infty} |\Omega_n^2 \setminus D| = 0$.

Case (1) contradicts the fact that $(\Omega_n)_n$ is the minimal minimizing sequence. Also case (2) is impossible, since it implies $d(\Omega_n^1, \{0\}) \to \infty$, otherwise the measure of D would be infinite. Hence $|\Omega_n^1 \cap D| \to 0$ and consider the ball B with measure equal to $\limsup_n |\Omega_n^1|$: $B \cup D$ is a solution for every position of B, and when B intersects (but not cover) some connected component of D we have the contradiction.

The proof of the case $k \geq 2$ follows from similar ideas. One takes again $(\Omega_n)_n$ a minimizing sequence with minimal $\liminf |\Omega_n|$ and if there is compactness one gets immediately the existence of a solution. If dichotomy happens, then we can suppose

$$|\Omega_n^1| \to \alpha^1, \qquad |\Omega_n^2| \to \alpha^2, \qquad |\Omega_n^1 \cap D| \to 0,$$

and (up to subsequences) we can take the maximal $l \in \{1, \ldots, k-1\}$ for which one of the following holds:

- $|\lambda_k(\Omega_n) - \lambda_{k-l}(\Omega_n^2)| \to 0$ and $\lambda_l(\Omega_n^1) \leq \lambda_{k-l}(\Omega_n^2) \leq \lambda_{l+1}(\Omega_n^1)$,
- $|\lambda_k(\Omega_n) - \lambda_l(\Omega_n^1)| \to 0$ and $\lambda_{k-l}(\Omega_n^2) \leq \lambda_l(\Omega_n^1) \leq \lambda_{k-l+1}(\Omega_n^2)$.

It is clear that case (b) of the thesis follows from the first one and case (c) follows from the second one. With an easy induction argument one can now conclude.

The next result highlight that stronger hypotheses lead to a good improvement.

Theorem 5.3 *In the hypotheses of Theorem 5.2, if moreover we ask the set D to be bounded, then also the cases (b) and (c) of Theorem 5.1 lead to the existence of a solution.*

Moreover in [10] are proved also some regularity properties of solutions of (5.1). In particular, if $k = 1$, $|D| < 1$ and D is *quasi-connected*,[4] all the minimizers are open sets even if D is only quasi-open.

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[4] A quasi-open set D is called quasi-connected if for all open and nonempty set A_1, A_2 such that cap $(A_i \cap D) > 0$ for $i = 1, 2$ and with $D \subset A_1 \cup A_2$, we have $\mathrm{cap}(A_1 \cap A_2) > 0$.

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