Shape- and Topology Optimization for Passive Control of Crack Propagation

Günter Leugering, Jan Sokołowski and Antoni Zochowski

Abstract In this review article the theoretical foundations for shape-topological sensitivity analysis of elastic energy functional in bodies with nonlinear cracks and inclusions are presented. The results obtained can be used to determine the location and the shape of inclusions which influence in a desirable way the energy release at the crack tip. In contrast to the linear theory, where in principle, crack lips may mutually penetrate, here we employ nonlinear elliptic boundary value problems in non-smooth domains with cracks with non-penetration contact conditions across the crack lips or faces. A shape-topological sensitivity analysis of the associated variational inequalities is performed for the elastic energy functional. Topological derivatives of integral shape functionals for variational inequalities with unilateral boundary conditions are derived. The closed form results are obtained for the Laplacian and linear elasticity in two and three spatial dimensions. Singular geometrical perturbations in the form of cavities or inclusions are considered. In the variational context the singular perturbations are replaced by regular perturbations of bilinear forms. The obtained expressions for topological derivatives are useful in numerical method of shape optimization for contact problems as well as in passive control of crack propagation.

G. Leugerin (✉)
Institute of Applied Mathematics 2, Friedrich-Alexander University Erlangen-Nürnberg,
Cauerstrasse 11, 91058 Erlangen, Germany
e-mail: leugering@math.fau.de

J. Sokołowski
Institut Élie Cartan, UMR 7502 Nancy-Université-CNRS-INRIA,
Laboratoire de Mathématiques, Université Henri Poincaré Nancy 1, BP 239,
54506 Vandoeuvre Lès Nancy Cedex, France
e-mail: Jan.Sokolowski@univ-lorraine.fr

A. Zochowski
Systems Research Institute of the Polish Academy of Sciences, ul. Newelska 6,
01-447 Warszawa, Poland
e-mail: Antoni.Zochowski@ibspan.waw.pl

A. Pratelli and G. Leugering (eds.), *New Trends in Shape Optimization*,
International Series of Numerical Mathematics 166,
DOI 10.1007/978-3-319-17563-8_7

Keywords Frictionless contact · Elastic bodies with cracks · Signorini conditions on cracks · Variational inequality · Shape functional · Shape sensitivity · Topological sensitivity · Domain decomposition · Steklov-Poincaré operator · Contact problems

1 Introduction and Overview

Understanding of nucleation, growth and propagation of single cracks and crack patterns in the context of composite materials is a grand challenge in material sciences. This is even more true for the controlled interaction between shapes and geometries of material inclusions on the one side and defects leading to damage, cracks and finally failure on the other side. Within the last century there have been developed a number of theories describing the propagation of cracks in solids. From the point of view of mathematical rigour, the approaches by Griffith and Barenblat are by now well established and widely accepted. Given a particular distribution of inclusions in a matrix material of an elastic body, it is possible, using Griffith's theory, to evaluate the stress concentrations at the tip of an incipient crack. This provides a structure-property map from the shapes and geometries - and of course also the material properties - of the inclusions to the dissipated energy or the energy release rate. This mapping can be described in terms of the Griffith functional. Optimal design of composites with respect to influencing crack-properties is then a matter of 'inverting' that map, in the sense of inverse engineering. Mathematically, this inversion is at the heart of inverse problems and, more precisely in this context, of sensitivity based shape and topology optimization. To this end, directional derivatives of the Griffith functional play a major role in the context of control of crack propagation in brittle materials where the Griffith criterion applies. The idea of designing composites with this aim is not new, some attempts have been made in the literature. See [6] who initialized this field of research in studying a distributed control problem for the Laplacian with a linear crack, that is a crack where no non-penetration condition is assumed to hold. The goal of that paper was to stop the crack propagation under the action of the control. In [16] the problem of crack control has been treated with non-penetration condition along the crack and boundary controls. The authors of [37] consider the shape of inclusions with different material properties as controls but take a linear crack model for a problem in conductivity. See also [51] for examples in mechanical engineering, where sensitivities are typically based on FEM-models. All articles mentioned are concerned with the reduction of the energy release rate. There are only very few articles concerned with shape variations of rigid or elastic inclusions in order to influence the energy release rate associated with non-penetrating cracks. This leads to a problem of shape-optimization in the context of variational inequalities; see [18] for an approach involving obstacles. The maximization of the energy release rate, rather its reduction, is important in some cases, where one wants to release as much energy as possible such that the material does not undergo a global crack. A first attempt towards optimization of the shape of inclusion with respect to maximizing the energy release rate have been reported in [21, 29, 30, 34, 48–50].

However, a rigorous mathematical treatment on the infinite dimensional level is still in its infancy. This article aims at a self-contained description of sensitivity based crack-control in the particular sense that the design of composites is geared towards influencing the crack resistance and, finally, the crack propagation. The sensitivities used in order to optimize the crack propagation are topological and shape derivatives of the Griffith functional with respect to changes in the inclusions constituting the composite.

Topological derivatives of shape functionals are introduced in [54] for linear elliptic boundary value problems. The corresponding expressions depend on pointwise values of solutions as well as of its gradients [46]. Therefore, the expressions for topological derivatives are not well defined on the energy spaces associated with the boundary value problems. In this paper we propose equivalent expressions for the topological derivatives for variational inequalities which are derived by a domain decomposition technique. Such expressions are given by line integrals in two spatial dimensions, or by surface integrals in three spatial dimensions. In addition, the new expressions are well defined on the energy space. In order to derive the topological derivatives by an application of the domain decomposition technique an artificial interface $\Sigma \subset \Omega$ is introduced and $\Omega := \Omega_1 \cup \Sigma \cup \Omega_2$ is decomposed into two subdomains. The functional under consideration is the elastic energy $\mathcal{E}(\Omega)$ of the whole domain Ω. Mixed shape-topological or topological-shape second order derivatives of the energy are evaluated. While shape sensitivity analysis is performed in Ω_2, asymptotic analysis is performed in Ω_1. In the framework of shape-topological sensitivity analysis the velocity method is applied in order to determine the shape functional $J(\Omega) := d\mathcal{E}(\Omega; V)$, where V is the specific vector field in derivation of $V \to d\mathcal{E}(\Omega; V)$. Then an asymptotic expansion of $\epsilon \to J(\Omega_\epsilon)$ is obtained. In the framework of topological-shape sensitivity analysis, first the asymptotic expansion of $\epsilon \to \mathcal{E}(\Omega_\epsilon)$ is performed, and the first order term of such an expansion is called the topological derivative. It turns out [46, 54] that the topological derivative of the energy functional is unbounded in the energy space of the elasticity boundary value problems under considerations. Therefore, we study an equivalent representation of topological derivatives which are well defined in the energy space. These representations can be used as well to modify the state equations by replacing the singular domain perturbations by the regular perturbations of bilinear forms in variational setting.

The asymptotic analysis of the energy functional performed in one subdomain, e.g., Ω_1, can be used in the second subdomain Ω_2 by means of an asymptotic expansion of the Steklov-Poincaré operator on the interface. The method is justified by the fact that the first order expansion of the energy functional in the subdomain leads to the first order asymptotic expansion of the Dirichlet-to-Neumann mapping on the interface between subdomains. Thus, a first order expansion of the Steklov-Poincaré operator on the interface for the second subdomain is obtained. In this way, the first order expansion of the energy functional in the truncated domain Ω_2 is derived. The precision of the obtained expansion is sufficient [56, 58] to replace the original energy functional by its first order expansion, provided the obtained expression is well defined on the energy space. Furthermore, the first order approximation of the energy functional in Ω is established. We point out that another method of approximation of the state equation by using the so-called self-adjoint extensions of the elliptic operators can be considered [39, 40].

The proposed domain decomposition method is important for variational inequalities [2] related to crack problems with non-penetration conditions across the crack. The arguments can, however, be developed for general variational inequalities. In order to describe the methodology in a nut-shell, before going on to details for elasticity, we consider the following abstract set-up.

$$v \to I(v) = \frac{1}{2}a(v, v) - L(v) \tag{1}$$

over a convex, closed subset $K \subset H$ of the Hilbert space H called the energy space. The function space $H := H(\Omega)$ is a Sobolev space which contains the functions defined over a domain $\Omega \subset \mathbb{R}^d$, $d = 2, 3$. The singular geometrical perturbation ω_ϵ centered at $\widehat{x} \in \Omega$ of the domain Ω is denoted by Ω_ϵ, the size of the perturbation is governed by a small parameter $\epsilon \to 0$. The quadratic functional defined on $H := H(\Omega_\epsilon)$ becomes

$$v \to I_\epsilon(v) = \frac{1}{2}a_\epsilon(v, v) - L_\epsilon(v) \tag{2}$$

with the minimizers $u_\epsilon \in K := K(\Omega_\epsilon)$. The expansion of the associated energy functional

$$\epsilon \to \mathcal{E}(\Omega_\epsilon) := I_\epsilon(u_\epsilon) = \frac{1}{2}a_\epsilon(u_\epsilon, u_\epsilon) - L_\epsilon(u_\epsilon) \tag{3}$$

is considered at $\epsilon = 0$. Namely, we are looking for its asymptotic expansion

$$\mathcal{E}(\Omega_\epsilon) = \mathcal{E}(\Omega) + \epsilon^d \mathcal{T}(\widehat{x}) + o(\epsilon^d), \tag{4}$$

where $\widehat{x} \to \mathcal{T}(\widehat{x})$ is the topological derivative [46, 54]. We show that there are regular perturbations of the bilinear form defined on the energy space $H(\Omega)$,

$$v \to b(v, v)$$

such that the perturbed quadratic functional defined on the unperturbed function space $H(\Omega)$

$$v \to I^\epsilon(v) = \frac{1}{2}\left[a(v, v) + \epsilon^d b(v, v)\right] - L(v) \tag{5}$$

furnishes the first order expansion (4). In our applications to contact problems in linear elasticity, it turns out that the bilinear form $v \to b(v, v)$ is supported on $\Gamma_R := \{|x - \widehat{x}| = R\} \subset \Omega$ with $R > \epsilon > 0$.

Variational inequalities are used to model contact problems in elasticity. It is known that the solutions to variational inequalities are Lipschitz continuous with respect to the shape [52]. In general, however, the state governed by a variational inequality is not Fréchet differentiable with respect to the shape. For a class of variational inequalities described by unilateral constraints in Sobolev spaces of Dirichlet type, the metric projection onto the constraints turns out to be Hadamard differentiable [12]. This property is used in order to obtain the first order directional differentiability of the associated shape functionals.

In order to show second order shape differentiability for variational inequalities, we have to restrict ourselves to energy-type shape functionals. The energy functional is the so-called marginal function and it is Fréchet differentiable with respect to the shape [12]. The first order shape derivative of the energy functional in the direction of a specific velocity vector field is considered as the shape functional for topological optimization. Thus, its topological derivative is evaluated. The possible applications of shape-topological derivatives include the control of singularities of solutions to variational inequalities by insertion of elastic inclusions far from the singularities.

Example 1 We describe the shape-topological differentiability of the energy shape functional for the Signorini variational inequality in two spatial dimensions. The same idea can be used for the frictionless contact problems in linear elasticity.

Let us consider the Signorini problem posed in $\Omega \subset \mathbb{R}^2$, with boundary $\partial\Omega = \Gamma \cup \Gamma_0$, and $\Gamma_c \subset \Gamma$. Denote $H^1_{\Gamma_0}(\Omega) = \{v \in H^1(\Omega) \mid v = 0 \text{ on } \Gamma_0 \subset \partial\Omega\}$. The solution $u \in K$ minimizes the quadratic functional

$$I(v) = \frac{1}{2}a(\Omega; v, v) - (f, v)_\Omega$$

over the cone

$$K = \{v \in H^1_{\Gamma_0}(\Omega) \mid v \geqslant 0 \text{ on } \Gamma_c \subset \Gamma \subset \partial\Omega\}.$$

The shape functional is the energy

$$\mathcal{E}(\Omega) = \frac{1}{2}a(\Omega; u, u) - (f, u)_\Omega,$$

where

$$a(\Omega; u, u) = \int_\Omega \nabla u \cdot \nabla u dx,$$

$$(f, u)_\Omega = \int_\Omega f u dx.$$

We assume that $\overline{\Gamma} \cap \overline{\Gamma}_0 = \emptyset$. Let $\Gamma_0^t := T_t(V)(\Gamma_0)$ be the boundary variations [52] of the Dirichlet boundary Γ_0.

Let us consider the decomposition of $\Omega = \Omega_1 \cup \Sigma \cup \Omega_2$, $\overline{\Omega}_1 \cap \overline{\Omega}_2 = \Sigma$, such that $\Gamma_0 \subset \partial\Omega_1$ and $\Gamma_c \subset \partial\Omega_2$. It means that the boundary variations as well as the topological asymptotic analysis are performed in Ω_1, and the unilateral conditions are prescribed in the second subdomain Ω_2.

The shape derivative of the energy functional with respect to the boundary variations of Γ_0 can be written in distributed form [52]

$$d\mathcal{E}(\Omega; V) = \int\limits_{\Omega_1} \langle A'(0) \cdot \nabla u, \nabla u \rangle dx$$

where $A'(0) = \text{div}\, VI - DV - DV^\top$, under the assumption that the velocity field V is supported in a small neighborhood of Γ_0 and that supp $V \cap$ supp f $= \emptyset$.

The second shape functional for the purposes of topological optimization is simply defined by

$$J(\Omega) := \int\limits_{\Omega_1} \langle A'(0) \cdot \nabla u, \nabla u \rangle dx. \tag{6}$$

We are going to determine the topological derivatives of $\Omega \to J(\Omega)$ for insertion of small inclusions in Ω_1 far from Γ_0. In this way we can control the possible singularities on Γ_0 by topology optimization in Ω.

We consider the domain decomposition method for purposes of the shape-topological differentiability of energy shape functionals. First, the domain Ω is split into two subdomains Ω_1, Ω_2 and the interface Σ. See Fig. 1. The differentiability with respect to small parameter of the Dirichlet-to-Neumann map which lives on the boundary $\Sigma \subset \partial\Omega_1$ is established. This map is called the Steklov-Poincaré operator for subdomain Ω_2.

Once, the derivative of the energy functional is given, we can proceed with the subsequent topological optimization problem. For topological optimization another decomposition $\Omega := \Omega_R \cup \Gamma_R \cup \Omega_c$ is introduced. The small inclusion ω_ε centered at the origin $\widehat{x} := \mathcal{O}$ is located in subdomain $\Omega_R \subset \Omega$ with the interface $\Gamma_R \subset \partial\Omega_R$. See Fig. 2.

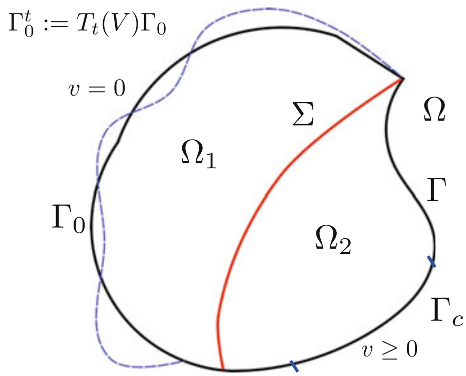
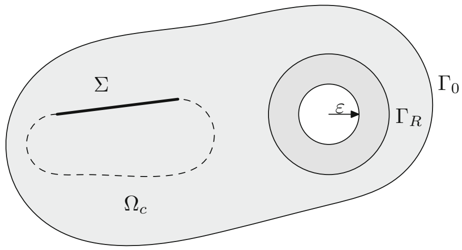


Fig. 1 Signorini problem with domain decomposition

Fig. 2 Domain Ω with crack



In particular, an elastic body weakened by small cracks is considered in the framework of unilateral variational problems in linearized elasticity. The frictionless contact conditions are prescribed on the crack lips in two spatial dimensions, or on the crack faces in three spatial dimensions. The weak solutions of the equilibrium boundary value problem for the elasticity problem are determined by minimization of the energy functional over the cone of admissible displacements. The associated elastic energy functional evaluated for the weak solutions is considered for the purpose of control of crack propagation. The singularities of the elastic displacement field at the crack front are characterized by the shape derivatives of the elastic energy with respect to the crack shape in the framework of Griffith's theory. For example, in two spatial dimensions, the first order shape derivative of the elastic energy functional evaluated in the direction of a velocity field supported in an open neighbourhood of one of crack tips is called the *Griffith functional*. The Griffith functional is minimized with respect to the shape and the location of small inclusions in the body. The inclusions are located far from the crack. In order to minimize the Griffith functional over an admissible family of inclusions, the second order directional, *mixed shape-topological derivatives* of the elastic energy functional are evaluated to determine the locations of inclusions. The boundary value problem for the elastic displacement field takes the form of a variational inequality over the positive cone in a fractional Sobolev space. The sensitivity analysis of variational inequalities under considerations lead to the property of directional differentiability of metric projection operator onto a polyhedric positive cone in fractional Sobolev spaces. Therefore, the concept of *conical or Hadamard differentiability* applies to shape and topological sensitivity analysis of variational inequalities under consideration.

In our framework of shape-topological differentiability we consider:

- Variational inequalities for cracks in solids, the associated Griffith functional is given by the shape derivative of the elastic energy;
- Conical differentiability of metric projection onto positive cone in the fractional Sobolev space of Dirichlet type equipped with natural order;
- Asymptotic analysis of the Dirichlet-to-Neumann map with applications to domain decomposition technique and Steklov-Poincaré nonlocal pseudodifferential boundary operators;
- The second order shape-topological derivatives of elastic energy for the purposes of passive control of crack propagation.

In linearized elasticity the Griffith criterion for crack propagation in two spatial dimensions uses the size of singularity coefficients at the crack tips, called the *stress intensity factors*, in order to forecast the crack propagation. In the pioneering paper [26] this criterion is extended to the nonlinear crack models with a mathematical proof which uses the *Griffith shape functional*, i.e. the shape derivative of the elastic energy with respect to the perturbations of positions of crack tips. The next step in the analysis of nonlinear crack models is the control of crack propagation. For such control the possible strategy is proposed in this paper with full proofs.

- Find the sensitivities of the Griffith functional with respect to the location of inclusions in elastic body;
- These sensitivities are called the topological derivatives [45] of Griffith's shape functional and can be determined by the asymptotic analysis in the singularly perturbed geometrical domains;
- Minimize the topological derivatives and in this way determine the possible locations of inclusions;
- Use the shape sensitivity analysis of Griffith's shape functional and determine optimal shape of inclusions.

The main difficulty of sensitivity analysis of solutions to nonlinear boundary value problems in non-smooth domains under consideration are the unilateral conditions prescribed on the crack lips which lead to the variational inequalities of the first kind. The asymptotic analysis of variational inequalities [57, 58] with respect to small parameter which governs the size of the singular domain perturbation is performed by a domain decomposition technique. In the present paper the mathematical foundation of the passive control strategy for crack propagation by means of shape-topological optimization is described in detail. First, the method of sensitivity analysis used in this paper is explained. The variational inequality in the perturbed domain $\Omega_\epsilon \subset \Omega$ is replaced by another variational inequality in the intact domain Ω. To this end the bilinear form $a(\Omega_\epsilon; \cdot, \cdot)$ is approximated by the bilinear form

$$a(\Omega; \cdot, \cdot) + \epsilon^d b(\Gamma_R; \cdot, \cdot), \tag{7}$$

where $d = 2, 3$ is the space dimension.

We apply the method of boundary variations [53] and the asymptotic analysis [45] in the subdomain Ω_R in order to obtain the expansions of the elastic energy with respect to an inclusion. These expansions are used in the subdomain Ω_c which contains the crack. As a surprising result expansion (7) of the bilinear form, which is well defined on the energy space in the intact domain Ω, is established. To our best knowledge, the bilinear form $b(\Gamma_R; \cdot, \cdot)$ has been employed in asymptotic analysis in singularly perturbed geometrical domains for the first time in [57, 58] for the Laplacian and the planar elasticity.

We now briefly describe the contents of paper, referring to the corresponding sections.

In Sect. 2, frictionless contact problems for the crack are introduced. The elastic energy of the elastic body is considered in the subdomain Ω_c. The contribution of the elastic energy from the subdomain Ω_R is given by the energy of the boundary Steklov-Poincaré operator. The Steklov-Poincaré operator depends on a small parameter $\epsilon \to 0$ which governs the size of singular geometrical perturbation in Ω_R.

In Sect. 3, general results on directional differentiability of metric projection are adapted to crack problems. The conical differentiability of solutions to the variational inequality in Ω_c leads to the main result of the paper, which is the directional differentiability of the Griffith functional with respect to the shape parameter. The abstract results on conical differentiability of the metric projection [12, 53] are adapted to the non-penetration conditions prescribed on the crack.

In Sect. 4, the representative case of cracks in two spatial dimensions are considered for shape-topological sensitivity.

In Sect. 5, the complete proof of shape and topological differentiability of the elastic energy in Ω_R is given. This implies the differentiability of the boundary bilinear form associated with the Steklov-Poincaré operator on Γ_R. Thus, the Griffith functional is differentiable. In this way we show that the main result of the paper applies to the crack control strategy.

In Sect. 6, the bounded perturbations of bilinear forms are presented for elliptic boundary value problems. In such a way the second order shape-topological derivatives of the energy functionals can be evaluated by easily implemented numerical methods.

The expansion of the Steklov-Poincaré operator involves a correction term **B**, an operator that is made explicit for ring-shaped regions in Sect. 7 for a number of situations.

Finally, in Sect. 8, an asymptotic analysis of the Steklov-Poincaré operator is considered for ring-type walled inclusions, where different material properties apply. This can be seen as an approach for coating of particles included in a matrix material.

In this article, some mathematical aspects of modeling and optimization for nonlinear partial differential equations are required, we refer the reader to the references which can be considered for the specific topics:

- potential theory in Dirichlet spaces and applications to unilateral problems [1, 5, 12, 14, 36, 53]
- mathematical theory of variational inequalities with applications to mechanics and contact problems [7, 8, 14, 15, 22, 23, 36, 53, 57, 58]
- shape optimization in domains with cracks and for variational inequalities [9–12, 18, 22, 53, 57, 58]
- asymptotic analysis and topological optimization for elasticity and variational inequalities [2, 35, 41–45, 57, 58]
- modeling and control of cracks [6, 16, 17, 19, 21, 24–28, 30–34, 37, 48–51]
- numerical methods for variational inequalities and crack problems [3, 4]
- optimization for nonlinear pde's [22, 47]

2 Unilateral Boundary Conditions in Isotropic Elasticity

We consider the following situation:
For the sake of simplicity, it is assumed that

- the crack in two spatial dimensions is given by the interval $\Gamma_c := \{0 < x_1 < 1, x_2 = 0\}$;
- the crack in three spatial dimensions is given by the disk $\Gamma_c := \{0 \leqslant x_1^2 + x_2^2 < 1, x_3 = 0\}$.

Therefore, the function spaces for the crack problem can be identified in Lipschitz domains (see Fig. 3)

- the traces $u^\pm$ on Σ of functions $u^\pm \in H^1(\Omega^\pm)$ live in the space $H^{1/2}(\Sigma)$;
- the traces $u^\pm$ on Γ_c of functions $u \in H^1(\Omega_c)$ are defined as the restrictions to Γ_c of functions from $H^{1/2}(\Sigma)$;
- the space of traces on the crack $H_{00}^{1/2}(\Gamma_c) \subset H^{1/2}(\Sigma)$ extended by zero outside the crack;
- the jump $[\![u]\!] := u^+ - u^-$ of a function $u \in H^1(\Omega_c)$ is well defined in $H_{00}^{1/2}(\Gamma_c)$;
- the convex constraints for the crack with nonpenetration condition are given by the positive cone in the space $H_{00}^{1/2}(\Gamma_c)$.

2.1 Isotropic Elasticity Boundary Value Problems

For a given displacement vector field $v = (v_1, v_2, v_3)^\top : \Omega \to \mathbb{R}^3$, we define the Jacobian $Dv = (\partial_{x_j} v_i)$ and the gradient is its transpose

$$\nabla v = Dv^\top = (\partial_{x_i} v_j) = [\nabla v_1, \nabla v_2, \nabla v_3]$$

The symmetrized gradient is denoted by

$$\nabla v^s = (\nabla v + \nabla v^\top)/2,$$

and it is called the linearized deformation tensor $\varepsilon(v) := \nabla v^s$.

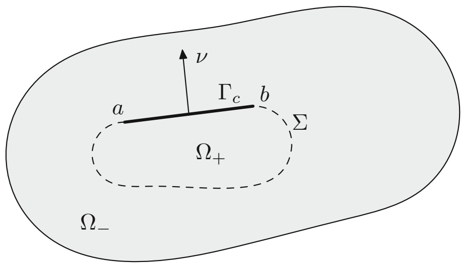


Fig. 3 Domain decomposition of elastic body Ω weakened by crack Γ_c

Given the symmetric and positive definite constitutive tensor $\mathbb{C}$ with the components c_{ijkl}, $i, j, k, l = 1, 2, 3$ and the inverse $\mathbb{S} := \mathbb{C}^{-1}$, the symmetric stress tensor is defined by

$$\sigma(v) = \mathbb{C}\nabla v^s, \quad \text{hence,} \quad \varepsilon(v) = \mathbb{S}\sigma(v)$$

or for the components $\sigma_{ij} = c_{ijrs}\varepsilon_{rs}$, where the summation convention over the repeated indices is used. In the case of the isotropic elasticity

$$c_{ijrs} = \lambda\delta_{ij}\delta_{rs} + \mu(\delta_{ir}\delta_{js} + \delta_{is}\delta_{jr}),$$

whence,

$$\sigma_{ij} = \lambda\delta_{ij}\varepsilon_{kk} + 2\mu\varepsilon_{ij},$$

where λ and μ are the Lame constants, μ is also known as the shear modulus.

Let us assume that the elastic body is given by a torus and let us consider the decomposition of the elastic body into two subdomains $\Omega := \Omega^+ \cup \Sigma \cup \Omega^-$ where Σ is a $C^{1,1}$ regular closed surface. Let $\Gamma_c \subset \Sigma$ be the regular subset of the surface with $C^{1,1}$-boundary given by the curve $\partial\Gamma_c$. We denote by ν the unit normal vector field on Σ which points out of Ω^+, and by n the unit normal vector field on $\partial\Gamma_c$ orthogonal to ν. See Fig. 3.

Given the displacement field $v \in H^1(\Omega)$, and $\sigma := \sigma(v)$, the associated stress field, we introduce the normal and tangential components of the stress field on Σ

$$\sigma_\nu = \sigma_{ij}\nu_j\nu_i, \quad \sigma_\tau = \sigma\nu - \sigma_\nu\nu, \quad \sigma_\tau = (\sigma_{\tau 1}, \sigma_{\tau 2}, \sigma_{\tau 3})^\top.$$

First, we recall the strong form of a general crack boundary value problem in two spatial dimensions.

Remark 2 For the sake of simplicity it is assumed that on the exterior boundary $\Gamma := \partial\Omega$ of the elastic body the homogeneous Dirichlet boundary conditions are prescribed. For a torus boundary conditions disappear. In the case of domain decomposition, the exterior boundary of the subdomain Ω_c is divided into two parts, $\Gamma_R = \partial\Omega_R$ and the exterior boundary $\partial\Omega$.

Problem 3 (*Equilibrium problem for a linear elastic body occupying* Ω_c) In the domain Ω_c with the boundary $\partial\Omega_c := \Gamma \cup \Gamma_c$ we have to find a displacement field $u = (u_1, u_2)$ and stress tensor components $\sigma = \{\sigma_{ij}\}$, $i, j = 1, 2$, such that

$$-\text{div}\sigma = f \quad \text{in} \quad \Omega_c, \tag{8}$$

$$\sigma = \mathbb{C}\varepsilon(u) \quad \text{in} \quad \Omega_c, \tag{9}$$

$$u = 0 \quad \text{on} \quad \Gamma, \tag{10}$$

$$[\![u]\!]\nu \geqslant 0, \quad [\![\sigma_\nu]\!] = 0, \quad \sigma_\nu \cdot [\![u]\!]\nu = 0 \quad \text{on} \quad \Gamma_c, \tag{11}$$

$$\sigma_\nu \leqslant 0, \quad \sigma_\tau = 0 \quad \text{on} \quad \Gamma_c^\pm. \tag{12}$$

Here $[\![v]\!] = v^+ - v^-$ is a jump of v on Γ_c, and signs $\pm$ correspond to the positive and negative crack faces with respect to ν, $f = (f_1, f_2) \in L^2(\Omega_c) := L^2(\Omega_c; \mathbb{R}^2)$ is a given function,

$$\sigma_\nu = \sigma_{ij}\nu_j\nu_i, \quad \sigma_\tau = \sigma\nu - \sigma_\nu \cdot \nu, \quad \sigma_\tau = (\sigma_\tau^1, \sigma_\tau^2),$$
$$\sigma\nu = (\sigma_{1j}\nu_j, \sigma_{2j}\nu_j),$$

the strain tensor components are denoted by $\varepsilon_{ij}(u)$,

$$\varepsilon_{ij}(u) = \frac{1}{2}(u_{i,j} + u_{j,i}), \quad \varepsilon(u) = \{\varepsilon_{ij}(u)\}, \quad i, j = 1, 2.$$

The elasticity tensor $\mathbb{C} = \{c_{ijkl}\}$, $i, j, k, l = 1, 2$, is given and satisfies the usual properties of symmetry and positive definiteness

$$c_{ijkl}\xi_{kl}\xi_{ij} \geqslant \alpha|\xi|^2, \quad \forall\, \xi_{ij},\ \xi_{ij} = \xi_{ji}, \quad \alpha > 0,$$

$c_{ijkl} = c_{klij} = c_{jikl}$, $c_{ijkl} \in L^\infty(\Omega_c)$.

Relations (8) are equilibrium equations, and (9) is the generalized Hooke's law, $u_{i,j} = \frac{\partial u_i}{\partial u_j}$, $(x_1, x_2) \in \Omega_c$. All functions with two lower indices are symmetric in those indices, i.e. $\sigma_{ij} = \sigma_{ji}$ etc.

In three spatial dimensions the strong form of the crack boundary value problem is completely analogous. The weak form is given by a variational inequality.

Problem 4 Introduce the Sobolev space

$$H^1_\Gamma(\Omega_c) = \{v = (v_1, v_2) \mid v_i \in H^1(\Omega_c),\ v_i = 0 \text{ on } \Gamma,\ i = 1, 2\}$$

and the closed convex set of admissible displacements

$$K = \{v \in H^1_\Gamma(\Omega_c) \mid [\![v]\!]\nu \geqslant 0 \quad \text{a.e. on} \quad \Gamma_c\}.$$

Find a solution $u \in K$ of the energy minimization problem

$$\min_{v \in K} \left\{ \frac{1}{2} \int_{\Omega_c} \sigma_{ij}(v)\varepsilon_{ij}(v) - \int_{\Omega_c} f_i v_i . \right\}$$

The solution satisfies the variational inequality

$$u \in K, \quad \int_{\Omega_c} \sigma_{ij}(u)\varepsilon_{ij}(v - u) \geqslant \int_{\Omega_c} f_i(v_i - u_i), \quad \forall v \in K, \tag{13}$$

where $\sigma_{ij}(u) = \sigma_{ij}$ are defined in (9).

Remark 5 1. Existence and uniqueness of solutions for the strong Problem 3 and the variational inequality (13) in Problem 4 as optimality conditions for the minimization Problem in 4 are given in e.g. [27].

2. The analysis of shape-topological differentiability of the Griffith functional can be reduced by the proposed domain decomposition approach to the differentiability property of the solution mapping for variational inequality (13)

$$f \to u$$

with respect to the input f. This will be investigated in Sect. 3. We claim that the mapping admits a conical differential. The proof of this claim follows by the Hadamard differentiability of metric projection onto positive cone in fractional Sobolev spaces.

2.2 Control of the Crack Front

The Griffith shape functional is an appropriate indicator in the framework of linear elasticity for the crack propagation scenario. In order to influence the crack propagation, we are going to design the elastic body in such a way that the Griffith functional assumes better properties. In order to improve the design, we consider a finite number of inclusions in the matrix material. Optimization in this context means the best choice of location and shape of inclusions, which can be complemented by optimization of material parameters for the inclusion. To this end, we employ the shape-topological sensitivity analysis [45, 53, 57]. Our analysis is performed for a single inclusion, the same approach works for a finite number of inclusions.

2.2.1 Elastic Body with a Crack and an Inclusion

The domain is divided in two parts as described above. The first part Ω_c which contains the crack, is built up from the matrix material (λ, μ), the second is Ω_R with an inclusion ω. The material properties of ω are denoted by $(\lambda_\omega, \mu_\omega)$. For simplicity, we can consider the inclusion in the form of a ball

$$\omega := B(y, r) = \{x \in \Omega_R \ : \ |x - y| < r\}, \quad \partial\omega = \{|x - y| = r\},$$

however a general shape of the inclusion can be treated in the same way. A finite number of inclusions, far from the crack, is also admissible for our approach.

2.2.2 The Griffith Shape Functional

For a given vector field $V := (V_1, V_2)^\top$ supported in Ω_c, denote $2E_{ij}(V; u) := u_{i,k}V_{k,j} + u_{j,k}V_{k,i}$, where $V_{k,j} := \dfrac{\partial V_k}{\partial x_j}$, $k, j = 1, 2$, and define the shape functional depending on ω,

$$J(\omega) := \frac{1}{2}\int_\Omega \left\{\operatorname{div} V \cdot \varepsilon_{ij}(u) - 2E_{ij}(V; u)\right\} \sigma_{ij}(u) - \int_\Omega \operatorname{div}(V f_i) u_i.$$

Problem 6 The problem is then to minimize $J(\omega)$ with respect to $\omega \subset \Omega_R$ and solutions u satisfying in the domain $\Omega := \Omega_c \cup \Gamma_R \cup \Omega_R$ the variational inequality

$$u \in K(\omega), \quad \int_\Omega \sigma_{ij}(u)\varepsilon_{ij}(v - u) \geqslant \int_\Omega f_i(v_i - u_i) \quad \forall v \in K(\omega),$$

where

$$K(\omega) = \{v \in H^1_\Gamma(\Omega) \mid [\![v]\!]\nu \geqslant 0 \quad \text{a.e. on} \quad \Gamma_c\}.$$

2.3 Main Results

The shape optimization problem under considerations depends on the *shape of the inclusions* exclusively via the characteristic functions of the inclusions. We are interested in the existence of the shape derivatives of $J(\omega)$ and also of the topological derivatives of this functional. In such a case we speak of the shape-topological differentiability of the Griffith functional.

Theorem 7 *The shape functional $\omega \to J(\omega)$ is directionally shape-topologically differentiable with respect to the inclusion ω in the cracked elastic body Ω.*

We precise the general result for the specific class of circular inclusions. First of all, the simplest choice of the admissible family $\mathcal{U}_{\text{ad}}$ of inclusions with the material properties $(\lambda_\omega, \mu_\omega)$ is

$$\mathcal{U}_{\text{ad}} := \{B(y, r) \subset \Omega_R\}.$$

Such a family, parametrized in a compact subset of $\mathbb{R}^{3+d}$ by $(\lambda_\omega, \mu_\omega, y, r) \in \mathbb{R}^{3+d}$, $d = 2, 3$, is compact with respect to the convergence of characteristic functions. Thus, the existence of an optimal inclusion within this family follows by standard arguments.

Theorem 8 *For given parameters $(\lambda_\omega, \mu_\omega)$ and $\omega = B(y, r) \subset \Omega_R$, the function*

$$r \to I(r) := \frac{1}{2}\int_\Omega \left\{\operatorname{div} V \cdot \varepsilon_{ij}(u) - 2E_{ij}(V; u)\right\} \sigma_{ij}(u) - \int_\Omega \operatorname{div}(V f_i) u_i$$

is Lipschitz continuous and admits the directional derivatives given by

- *the shape derivative of* $\omega \to J(\omega)$ *for* $r > 0$*,* r *small enough;*
- *the topological derivative of* $\omega \to J(\omega)$ *for* $r = 0^+$.

3 Applications of Directional Differentiability of Metric Projection in Fractional Sobolev Spaces

Our results on shape-topological sensitivities for the Griffith functional related to crack problems with non-penetration conditions across the crack interfaces depend crucially on the regularity properties of metric projections in Hilbert spaces. This is a classical issue that, due to its importance in this article, nevertheless, deserves a brief but self-contained description. The convex cone for the crack model with non-penetration conditions takes the form

$$\mathbb{K} := \{v \in H^1(\Omega_c) \ : \ [\![v]\!]\nu \in \mathcal{K}(\Gamma_c) \subset H^{1/2}_{00}(\Gamma_c)\},$$

where $\mathcal{K}(\Gamma_c)$ is the positive cone in the fractional Sobolev space $H^{1/2}_{00}(\Gamma_c)$. Therefore, we establish the *Hadamard differentiability* [15, 36] of the *metric projection* in the *Dirichlet space* $H^{1/2}_{00}(\Gamma_c)$ onto its *positive cone* [12]. Let us consider the directional differentiability of the metric projection onto the positive cone in the fractional Sobolev spaces $H^{1/2}(\Sigma)$. In the applications for the crack problem, we would like to have a $C^{1,1}$-surface in three spatial dimensions, and the $C^{1,1}$-curve in two spatial dimensions, selected in the interior of the elastic body Ω in such a way that the crack $\Gamma_c \subset \Sigma$. In order to present the results, we are going to consider a simple geometry of the crack Γ_c. In the general setting the results are obtained in a similar way. Therefore, we consider the subset $B = \{|x| < R\}$, $x = (x_1, \dots, x_d) \subset \Omega$, of the elastic body Ω, with the crack $\Gamma_c := \{x = (x', x_d) \in \mathbb{R}^d \ : \ x_d = 0, \quad |x'| < R/2\}$ and Σ defined by an extension of the subset $\widetilde{\Sigma} := \{x = (x', x_d) \in B \ : \ x_d = 0\}$. In such a case, the unit normal vector to the crack $\nu := (0, \dots, 0, 1)$ is constant on the crack, and the unit tangent vector orthogonal to ν on the boundary $\partial\Gamma_c$ of the crack $n := (n_1, \dots, n_{d-1}, 0)$. For the displacement field $u = (u_1, \dots, u_d)$ it follows that $u\nu = u_d$, hence, the unilateral constraints for the jump of the normal component over the crack $H^{1/2}_{00}(\Gamma_c) \ni [\![u]\!]\nu = u_d^- - u_d^+ \geqslant 0$. Thus, the convex cone of admissible displacements for the crack problem takes the form

$$\mathcal{U}_{\text{ad}} = \{v = (v_1, \dots, v_d) \in H^1(\Omega_c) \ : \ v_d^- - v_d^+ \geqslant 0 \text{ on } \ \Gamma_c\}$$

and our analysis of the metric projection is reduced to the positive cone in $H^{1/2}_{00}(\Gamma_c)$, hence, in $H^{1/2}(\Sigma)$.

Remark 9 We recall that in general for a domain Ω with the boundary Γ, the Sobolev spaces $H^1(\Omega)$ and $H^{1/2}(\Gamma)$ are [1, 14] examples of so-called Dirichlet spaces.

It means that for the scalar product $a(\cdot,\cdot)$, with $v^+ := \sup\{v, 0\}$ and $v^- := \sup\{-v, 0\}$, the property $a(v^+, v^-) \le 0$ holds for all elements of the Sobolev spaces.

Remark 10 The metric projection in Dirichlet spaces onto the cone of nonnegative elements is considered for the purpose of sensitivity analysis of solutions to frictionless contact problems in [53]. This result is extended to the crack problem. In order to avoid unnecessary technicalities, we restrict ourselves to a model problem. We consider the Hadamard differentiability of metric projection in Dirichlet space onto the cone of positive elements, and recall the result on its conical differentiability.

Consider the convex, closed cone

$$K = \{v \in H^{1/2}(\Sigma) \; : \; v \geqslant 0 \text{ on } \Sigma\}$$

and the metric projection $H^{1/2}(\Sigma) \ni f \to u = P_K(f) \in K$ onto K which is defined by the variational inequality

$$u \in K \; : \; (u - f, v - u)_{1/2,\Sigma} \geqslant 0 \quad \forall v \in K.$$

We denote $v^+ =: v \wedge 0 := \sup\{v, 0\}$ and $v^- =: -v \wedge 0 := \sup\{-v, 0\}$ in $H^{1/2}(\Sigma)$. With the element $u = P_K(f)$ we associate the convex cone

$$C_K(u) = \{v \in H^{1/2}(\Sigma) \; : \; u + tv \in K \text{ for some } t > 0\}$$

and denote by $T_K(u)$ the closure of $C_K(u)$ in $H^{1/2}(\Sigma)$. On the other hand [12] there is a nonnegative Radon measure m such that for all $v \in H^{1/2}(\Sigma)$ we have the equality $\int v\, dm = (u - f, v)_{1/2,\Sigma}$, hence, we denote

$$m[v] := (u - f, v)_{1/2,\Sigma}.$$

Definition 11 The convex cone K is polyhedric [15, 36] at $u \in K$ if

$$T_K(u) \cap m^\perp = \overline{C_K(u) \cap m^\perp}.$$

We recall the result on polyhedricity of the positive cone in a Dirichlet space [12].

Lemma 12 *The convex cone*

$$C_K(u) \cap m^\perp := \{v \in H^{1/2}(\Sigma) \; : \; v \in C_K(u) \textit{ such that } (u - f, v)_{1/2,\Sigma} = 0\}$$

is dense in the closed, convex cone

$$T_K(u) \cap m^\perp := \{v \in H^{1/2}(\Sigma) \; : \; v \in T_K(u) \textit{such that } (u - f, v)_{1/2,\Sigma} = 0\}.$$

Proof Using the property of the Dirichlet space

$$(v^+, v^-)_{1/2,\Sigma} \leqslant 0 \quad \text{for all } v \in H^{1/2}(\Sigma)$$

then

$$T_K(u) \cap m^\perp = \overline{C_K(u) \cap m^\perp}$$

follows easily. Indeed, let

$$w \in T_K(u) \cap m^\perp.$$

Then $w = 0$ m-a.e. Let $C_K(u) \ni v_n \to w$. Then $v_n^- \to w^-$, $v_n^+ \to w^+$ and $v_n^+ \wedge w^+ - v_n^- \to w$, here $v \wedge w = \inf\{v, w\}$. Now, if $v \in C_K(u)$ then $u + tv \geq 0$. We claim $v_n^+ \wedge w^+ - v_n^- \in C_K(u) \cap m^\perp$. Indeed, $u + t[v_n^+ \wedge w^+ - v_n^-] \geq 0$ so $v_n^+ \wedge w^+ - v_n^- \in C_K(u)$ and $m[v_n^+ \wedge w^+ - v_n^-] = m[v_n^+ \wedge w^+] = 0$, because of $m[w^+] = 0$. □

Remark 13 In [12] the tangent cone $T_K(u)$ is derived for $u \in K$, in the case of the positive cone $K = \{v \in \mathcal{H} \ : \ v \geqslant 0\}$ in the Dirichlet space $\mathcal{H}$ equipped with the scalar product $(u, v)_\mathcal{H}$. We have

$$T_K(u) = \{v \in \mathcal{H} \ : \ v \geqslant 0 \text{ on } \{u = 0\}\}.$$

The convex cone $S := T_K(u) \cap m^\perp$ is important for our applications. It is obtained in [12]

$$T_K(u) \cap m^\perp = \{v \in \mathcal{H} \ : \ v \geqslant 0 \text{on } \{u = 0\} \quad \text{and } v = 0 \ m - \text{a.e.}\}.$$

The following result on the directional differentiability of metric projection holds for polyhedric convex sets [15, 36].

Lemma 14 *Let K be a polyhedric cone. For $t > 0$, t small enough,*

$$P_K(u + th) = P_K(u) + t P_S(h) + o(t; h) \textit{in } H^{1/2}(\Sigma)$$

where

$$S := T_K(u) \cap m^\perp$$

and the remainder $o(t; h)$ is uniform on compact subsets of $H^{1/2}(\Sigma)$. Hence, the directional derivative of the metric projection is uniquely determined by the variational inequality

$$q := P_S(h) \in S \ : \ (q - h, v - q)_{1/2,\Sigma} \geqslant 0 \quad \forall v \in S.$$

For a crack $\Gamma_c \subset \Sigma$ we introduce the following convex cones

$$\mathcal{K}(\Sigma) := \{v \in H^{1/2}(\Sigma) \ : \ v = 0 \quad \text{on } \Sigma \setminus \overline{\Gamma}_c, \quad v \geqslant 0 \text{ on } \Gamma_c\},$$

and

$$\mathcal{K}(\Gamma_c) := \{v \in H^{1/2}_{00}(\Gamma_c) \ : \ v \geqslant 0 \text{ on } \Gamma_c\}.$$

For the variational problems with unilateral conditions for the jump of normal component of the displacement vector field over the crack, the convex cones $\mathcal{K}(\Gamma_c)$ and $\mathcal{K}(\Sigma)$ are employed in order to show the polyhedricity of the cone of admissible displacements.

Remark 15 The proof of Lemma 12 applies as well to the convex cone $\mathcal{K}(\Gamma_c) \subset H^{1/2}_{00}(\Gamma_c)$ since the space $C^\infty_0(\Gamma_c)$ is dense in $H^{1/2}_{00}(\Gamma_c)$, hence, a nonnegative distribution is a Radon measure. In addition, *contraction operates* [5] for the scalar product (16) in $H^{1/2}_{00}(\Gamma_c)$. Let us note that the scalar products in $H^{1/2}(\Sigma)$ and in $H^{1/2}_{00}(\Gamma_c)$ are not the same, the latter is a weighted space.

We recall an abstract result on shape sensitivity analysis of variational inequalities. The conical differentiability of solutions to variational inequalities for the crack problem follows from the abstract result given by Theorem 17. The general result [53] is adapted here to our setting within the domain decomposition framework. Thus, the bilinear form $a(\cdot,\cdot) + b_t(\cdot,\cdot)$ defined in the subdomain Ω_c is introduced, where $b_t(\cdot,\cdot)$ is the contribution from the Steklov-Poincaré operator on $\Gamma_R = \partial\Omega_R$. The real parameter $t > 0$ governs the shape perturbations of the inclusion $t \to \omega_t$ in Ω_R, where $t \to 0$ governs the topological changes of Ω_R in the framework of asymptotic analysis. The two boundary value problems in two subdomains are coupled by the transmission conditions on the interface Γ_R. The linear boundary value problem in Ω_R furnishes the expansions of the Steklov-Poincaré operators resulting from perturbations of the inclusion in the interior of the subdomain. The sensitivity analysis of solutions to variational inequality in Ω_c is performed for compact perturbations of nonlocal boundary conditions on the interface. As a result, the weak solution to the unilateral elasticity boundary value problem under considerations is directionally differentiable with respect to the parameter $t \to 0$ which governs the perturbations of the inclusion far from the crack. We provide the precise result on the conical differentiability of solutions to variational inequalities [15, 36, 53] (see also [12]) which is given here without proof.

Let $\mathcal{K} \subset \mathcal{H}$ be a convex and closed subset of a Hilbert space $\mathcal{H}$, and let $\langle\cdot,\cdot\rangle$ denote the duality pairing between $\mathcal{H}'$ and $\mathcal{H}$, where $\mathcal{H}'$ denotes the dual of $\mathcal{H}$. Let us assume that there are given symmetric bilinear forms $a(\cdot,\cdot)+b_t(\cdot,\cdot) \ : \ \mathcal{H}\times\mathcal{H} \to \mathbb{R}$ parametrized by $t \geqslant 0$, and the linear form $f \in \mathcal{H}'$, such that

Condition 16 *1. There are* $0 < \alpha \leqslant M$ *such that*

$$|a(u,v) + b_t(u,v)| \leqslant M\|u\|\|v\|, \quad \alpha\|u\|^2 \leqslant a(v,v) + b_t(v,v) \quad \forall u, v \in \mathcal{H}$$

uniformly with respect to $t \in [0, t_0)$. *Furthermore, there exists* $\mathcal{Q}' \in \mathcal{L}(\mathcal{H};\mathcal{H}')$ *such that*

$$\mathcal{Q}_t = \mathcal{Q} + t\mathcal{Q}' + o(t) \quad in \quad \mathcal{L}(\mathcal{H};\mathcal{H}'),$$

where $\mathcal{Q}_t \in \mathcal{L}(\mathcal{H}; \mathcal{H}')$

$$a(\phi, \varphi) + b_t(\phi, \varphi) = \langle \mathcal{Q}_t(\phi), \varphi \rangle \quad \forall \phi, \varphi \in \mathcal{H}.$$

2. *The set* $\mathcal{K} \subset \mathcal{H}$ *is convex and closed, and the solution operator* $\mathcal{H}' \ni f \to \mathcal{P}(f) \in \mathcal{H}$ *for* (15)

$$\mathcal{P}(f) \in \mathcal{K} : \quad a(\mathcal{P}(f), \varphi - \mathcal{P}(f)) \geq \langle f, \varphi - \mathcal{P}(f) \rangle \quad \forall \varphi \in \mathcal{K}$$

is differentiable in the sense that

$$\forall h \in \mathcal{H}' : \quad \mathcal{P}(f + sh) = \mathcal{P}(f) + s\mathcal{P}'(h) + o(s) \quad in \quad \mathcal{H}$$

for $s > 0$, *s small enough, where the mapping* $\mathcal{P}' : \mathcal{H}' \to \mathcal{H}$ *is continuous and positively homogeneous, in addition, the remainder* $o(s)$ *is uniform with respect to the direction* $h \in \mathcal{H}'$ *on compact subsets of* $\mathcal{H}'$.

Let us consider the unique solutions $u_t = \mathcal{P}_t(f)$ to variational inequalities depending on a parameter $t \in [0, t_0)$, $t_0 > 0$,

$$u_t \in \mathcal{K} \; : \; a(u_t, \varphi - u_t) + b_t(u_t, \varphi - u_t) \geqslant \langle f, \varphi - u_t \rangle \qquad \forall \varphi \in \mathcal{K}. \tag{14}$$

In particular, for $t = 0$

$$u \in \mathcal{K} \; : \; a(u, \varphi - u) + b(u, \varphi - u) \geq \langle f, \varphi - u \rangle \qquad \forall \varphi \in \mathcal{K}, \tag{15}$$

with $u = \mathcal{P}(f)$ a unique solution to (15). The mapping $t \to u_t$ is strongly differentiable in the sense of Hadamard at 0^+, and its derivative is given by a unique solution of the auxiliary variational inequality [53].

Theorem 17 *Assume that Condition 16 is satisfied. Then the solutions to the variational inequality (14) are right-differentiable with respect to t at* $t = 0$, *i.e. for* $t > 0$, *t small enough,*

$$u_t = u + tu' + o(t) \quad in \quad \mathcal{H},$$

where

$$u' = \mathcal{P}'(-\mathcal{Q}'u).$$

3.1 Metric Projection onto Positive Cone in $H^{1/2}_{00}(\Gamma_c)$

For boundary value problems in domains with cracks, unilateral conditions are prescribed on the crack for the normal component of the displacement field. Hence, the normal component of the displacement field belongs to the positive cone in the fractional Sobolev space $H^{1/2}_{00}(\Gamma_c)$. The sensitivity analysis of variational inequalities for Signorini problems was reduced in [53] to the directional differentiability of the metric projection onto the positive cone in a fractional space which is the Dirichlet space. This result is further extended in [12] to some crack problem. The method is also used in the present paper, however for the other purposes.

Sensitivity analysis of the crack problem. We are going to explain how the results obtained in [53] for the Signorini problem in linear elasticity can be extended to the crack problems with unilateral constraints. To this end, the abstract analysis performed in [12] for the differentiability of the metric projection onto the cone of nonnegative elements in the Dirichlet space is employed. The framework for analysis is established in function spaces over $\Omega := \Omega^+ \cup \Sigma \cup \Omega^-$, where Σ is a $C^{1,1}$ regular curve without intersections. The regularity assumption can be weakened, if necessary. Let $\Gamma_c \subset \Sigma$ be the segment $\{(x_1, 0) \ : \ 0 < x_1 < 1\}$ included in the curve Σ. We denote by ν the unit normal vector field on Σ which points out of Ω^+, and by n the unit normal vector field on $\partial\Gamma_c$ orthogonal to ν. We consider deformations of the crack in the direction of the vector field V colinear with n in the neighbourhood of the crack tip $A = (1, 0) \in \Omega_c \subset \mathbb{R}^2$. In the Sobolev space defined on the cracked domain Ω_c, the elements enjoy jumps over the crack which are denoted by $[\![v]\!] := v^+ - v^-$, and we have the regularity property of traces $[\![v]\!] \in H^{1/2}_{00}(\Gamma_c)$. In our geometry of Ω_c, the Sobolev space $H^{1/2}_{00}(\Gamma_c)$ coincides with the linear subspace of $H^{1/2}(\Sigma)$

$$H^{1/2}_{00}(\Gamma_c) = \{\varphi \in H^{1/2}(\Sigma) \ : \ \varphi = 0 \quad \text{q.e. on } \Sigma \setminus \Gamma_c\},$$

where q.e. means *quasi-everywhere* with respect to the capacity, see e.g. [47] for the definition and elementary properties of the capacity useful for the existence of optimal shapes in shape optimization problems with nonlinear PDE's constraints. In order to investigate the properties of the metric projection in the space of admissible displacement fields onto the convex cone

$$K := \{v \in H^1(\Omega_c) \ : \ [\![v]\!]\nu \geqslant 0\},$$

where $H^1(\Omega_c) := H^1(\Omega_c; \mathbb{R}^2)$, we need to show that the positive convex cone

$$\mathcal{K} = \{\varphi \in H^{1/2}_{00}(\Gamma_c) \ : \ \varphi \geqslant 0 \quad \text{on} \quad \Gamma_c\}.$$

is polyhedric in the sense of [12, 15, 36]. We consider here the rectilinear crack Γ_c in two spatial dimensions. The scalar product in $H^{1/2}_{00}(\Gamma_c) := H^{1/2}_{00}(0, 1)$ is defined

$$\langle \varphi, \psi \rangle_c = \int_{\Gamma_c} \int_{\Gamma_c} \frac{(\varphi(x) - \varphi(y))(\psi(x) - \psi(y))}{|x - y|^2} dxdy + \int_{\Gamma_c} \left[\varphi(x)\psi(x) + \frac{\varphi(x)\psi(x)}{\text{dist}(x, \partial\Gamma_c)} \right] dx \tag{16}$$

Polyhedricity of the positive cone in $\mathbf{H^{1/2}_{00}(\Gamma_c)}$. In order to show the polyhedricity of the nonnegative cone $\mathcal{K}$ in $\mathcal{H} := H^{1/2}_{00}(0, 1)$, it is enough to check the property

$$\langle \varphi^+, \varphi^- \rangle_c \leqslant 0 \quad \forall v \in H^{1/2}_{00}(0, 1)$$

which is straightforward, here $\varphi^+(x) = \max\{v(x), 0\}$. The full proof of polyhedricity in such a case is provided in [12]. It is easy to check that the polyhedricity with respect to the scalar product implies the polyhedricity with respect to a bilinear form which is equivalent to the scalar product.

Theorem 18 *Let us consider the variational inequality for the metric projection of $f + th \in \mathcal{H}$ onto $\mathcal{K}$*

$$u_t \in \mathcal{K} \ : \ \langle u_t - f - th, v - u_t \rangle \geqslant 0 \quad \forall v \in \mathcal{K},$$

where $f, h \in \mathcal{H}$ are given, denote by $\Xi\{u\} = \{x \in \Gamma_c \ : \ u(x) = 0\}$. Then

$$u_t = u + tq(h) + o(t; h) \ in \quad \mathcal{H},$$

where the remainder $o(t; h)$ is uniform on compact subsets of $\mathcal{H}$, and the conical diffferential of the metric projection $q := q(h)$ is given by the unique solution to the variational inequality

$$q \in \mathcal{S}(u) \ : \ \langle q - h, v - q \rangle \geqslant 0 \quad \forall v \in \mathcal{S}(u)$$

and the closed convex cone

$$\mathcal{S}(u) = \{\varphi \in \mathcal{H} \ : \ \varphi \geqslant 0 q.e. \ on \ \Xi\{u\}, \quad \langle u - f, \varphi \rangle = 0\}.$$

4 Rectilinear Crack in Two Spatial Dimensions

In this section the general method of shape-topological sensitivity analysis is presented in the domain $\Omega := \Omega_c \cup \Gamma_R \cup \Omega_R$, where the first subdomain Ω_c contains the rectilinear crack Γ_c and the second subdomain Ω_R contains the inclusion ω. We denote by $\Omega_{\rm in} := \Omega_c \cup \overline{\Gamma}_c$, the first subdomain in the elastic body without the crack. We assume that there is a regular $C^{1,1}$-curve $\Sigma \subset \Omega_{\rm in}$, without intersections, which contains the rectilinear crack $\Gamma_c := \{(x_1, 0) \ : \ 0 \leqslant x_1 \leqslant 1\}$. To simplify the presentation, let us consider a torus $\Omega := \mathbb{T} := \mathbb{T}^2$ with 2π-periodic coordinates $x = (x_1, x_2)$. The deformations of the subdomain Ω_c are defined by the vector field $(x, t) \to V(x, t) = (v(x, t), 0)$, where the $C^\infty_0(\Omega^+)$ function $(x, t) \to v(x, t)$ is supported in $[1 - \delta, 1 + \delta]^2 \times [-t_0, t_0] \subset \Omega^+ \subset \mathbb{R}^2 \times \mathbb{R}$ and $v(x, t) \equiv 1$ on $[1 - \delta/2, 1 + \delta/2]^2 \times [-t_0/2, t_0/2]$. In our notation, the real variable $t \in \mathbb{R}$ is a parameter. It means that the vector field V deforms the reference domain Ω^+_c to

$t \to T_t(V)(\Omega_c^+)$ just by moving the tip of the crack $X = (1, 0) \to x(t) = (x_1(t), 0)$ in the direction of the x_1-axis. The mapping $T_t \ : \ X \to x(t)$ is given by the system of equations

$$\frac{dx}{dt}(t) = V(x(t), t), \quad x(0) = X.$$

The boundary value problem of linear isotropic elasticity in Ω_c is defined by the variational inequality

$$u \in K \ : \ a(u, v - u) \geqslant (f, v - u) \quad \forall u \in K,$$

where

$$K = \{v \in H^1(\Omega_c) \ : \ [\![v]\!] \cdot \nu := (v^+ - v^-) \cdot \nu \geqslant 0 \quad \text{on} \quad \Gamma_c\}.$$

The bilinear form

$$a(u, v) = \int_{\Omega_c} \left[\frac{\mu}{2} \sum_{j,k=1}^{2} (\partial_j u_k + \partial_k u_j)(\partial_j v_k + \partial_k v_j) + \lambda \operatorname{div} u \operatorname{div} v \right] dx$$

is associated with the operator

$$Lu := -\mu \Delta u - (\lambda + \mu)\mathbf{grad} \operatorname{div} u.$$

The deformation tensor $2\varepsilon(u) = \partial_j u_k + \partial_k u_j$ as well as the stress tensor $\sigma(u) =$ associated with the displacement field u are useful in the description of the boundary value problems in linear elasticity. The energy functional $\mathcal{E}(\Omega_c) = 1/2a(u, u) - (f, u)_{\Omega_c}$ is twice differentiable [12] in the direction of a vector field V, for the specific choice of the field $V = (v, 0)$. The first order shape derivative

$$V \to d\mathcal{E}(\Omega_c; V) = \frac{1}{t} \lim_{t \to 0} (\mathcal{E}(T_t(\Omega_c)) - (\mathcal{E}(\Omega_c))$$

can be interpreted as the derivative of the elastic energy with respect to the crack length, we refer the reader to [26] for the proof, the same result for the Laplacian is given in [24, 25].

Theorem 19 *We have*

$$d\mathcal{E}(\Omega_c; V) = \frac{1}{2} \int_{\Omega_c} \left\{ \operatorname{div} V \cdot \varepsilon_{ij}(u) - 2E_{ij}(V; u) \right\} \sigma_{ij}(u) - \int_{\Omega_c} \operatorname{div}(V f_i) u_i. \quad (17)$$

Now we restrict our consideration to the perturbation of the crack tip only in the direction which coincides with the crack direction. The derivative is evaluated in the framework of the velocity method [53] for a specific velocity vector field V

selected in such a way that the result $d\mathcal{E}(\Omega_c; V)$ is independent of the field V and it depends only on the perturbation of the crack tip. That is why, this derivative is called the *Griffith functional* $J(\Omega_c) := d\mathcal{E}(\Omega_c; V)$ defined for the elastic energy in a domain with crack. We are interested in the dependence of this functional on domain perturbations far from the crack. As a result, shape and topological derivatives of the nonsmooth Griffith shape functional are obtained with respect to the boundary variations of an inclusion.

4.1 Green Formulae and Steklov-Poincaré Operators

The Steklov-Poincaré operator on the interface for the domain $\Omega_c \cup \Gamma_R \cup \Omega_R$ is defined by the Green formula, first as the Dirichlet-to-Neumann map in Ω_R, then it is used on the interface as nonlocal boundary operator. Therefore, we recall here the Green formula for linear elasticity operators in two and three spatial dimensions. We start with analysis in two spatial dimensions. To simplify the presentation let us consider the reference domain without a crack in the form of the torus $\mathbb{T} := \mathbb{T}^2$ with 2π-periodic coordinates $x = (x_1, x_2)$. For the purpose of shape-topological sensitivity analysis we assume that the elastic body without the crack is decomposed into two subdomains, Ω_{in} and Ω_R, separated from each other by the interface Γ_R. Thus, the elastic body with the crack Γ_c is written as

$$\Omega := \Omega_c \cup \Gamma_R \cup \Omega_R.$$

The rectilinear crack $\Gamma_c \subset \Sigma \subset \Omega_{in}$ is an open set, where the fictitious interface $\Sigma \subset \Omega_{in}$ is a closed $C^{1,1}$-curve without intersections. In our notation $\Omega_c = \Omega_{\text{in}} \setminus \overline{\Gamma}_c$. The bilinear form of the linear isotropic elasticity is associated with the operator

$$Lu := -\mu\Delta u - (\lambda + \mu)\,\mathbf{grad}\ \mathrm{div}\, u$$

for given Lame coefficients $\mu > 0, \lambda \geqslant 0$. The displacement field u in the elastic body Ω is given by the unique solution of the variational inequality

$$u \in K \ :\ a(u, v - u) \geqslant (f, v - u) \quad \forall u \in K,$$

where

$$K = \{v \in H^1(\Omega_c)\ :\ [\![v]\!] \cdot \nu := (v^+ - v^-) \cdot \nu \geqslant 0 \text{ on } \quad \Gamma_c\}.$$

Given the unique solution $u \in K$ of the variational inequality and the admissible vector field V compactly supported in Ω_c, we consider the associated shape functional (17) evaluated in Ω_c, which is called the Griffith functional

$$J(\Omega_c) := d\mathcal{E}(\Omega_c; V). \tag{18}$$

Let $\omega \subset \Omega_R$ be an elastic inclusion. Introduce the family of inclusions $t \to \omega_t \subset \Omega_R$ governed by the velocity field W compactly supported in Ω_R. The elastic energy in Ω_R with the inclusion ω_t is denoted by

$$\omega_t \to \mathcal{E}_t(\Omega_R) := \frac{1}{2} a_t(\Omega_R\ ;\ u, u) - (f, u)_{\Omega_R}.$$

Its shape derivative $d\mathcal{E}(\Omega_R; W)$ in the direction W is obtained by differentiation at $t = 0$ of the function

$$t \to \mathcal{E}_t(\Omega_R) := \frac{1}{2} a_t(\Omega_R\ ;\ u, u) - (f, u)_{\Omega_R}.$$

Proposition 20 *Assume that the energy shape functional in the subdomain Ω_R,*

$$\omega \to \mathcal{E}(\Omega_R) := \frac{1}{2} a(\Omega_R\ ;\ u, u) - (f, u)_{\Omega_R}$$

is differentiable in the direction of the velocity field W compactly supported in a neighbourhood of the inclusion $\overline{\omega} \subset \Omega_R$, then the Griffith functional (18) is directionally differentiable in the direction of the velocity field W. Therefore, the second order directional shape derivative $d\mathcal{E}(\Omega; V, W)$ of the energy functional in Ω in the direction of fields V, W is obtained.

This result can be proved by the domain decomposition technique:

- the shape differentiability of the energy functional in the subdomain Ω_R implies the differentiability of the associated Steklov-Poincaré operator defined on the Lipschitz curve given by the interface $\overline{\Omega}_R \cap \overline{\Omega}_c$ with respect to the scalar parameter $t \to 0$ which governs the boundary variations of the inclusion ω;
- the expansion of the Steklov-Poincaré nonlocal boundary pseudodifferential operator obtained in the subdomain Ω_R is used in the boundary conditions for the variational inequality defined in the cracked subdomain Ω_c and leads to the conical differential of the solution to the unilateral problem in the subdomain;
- the one term expansion of the solution to the unilateral problem is used in the Griffith functional in order to obtain the directional derivative with respect to the boundary variations of the inclusion.

Remark 21 For the circular inclusion $\omega := \{x \in \Omega_R\ :\ |x - y| < r_0\}$, $r_0 > 0$, the scalar parameter $t \to 0$ which governs the shape perturbations of $\partial\omega$ in the direction of a field W [53] can be replaced by the parameter $r \to r_0$. Thus, the moving domain $t \to \omega_t$ is replaced by the moving domain $r \to \{x \in \Omega_R\ :\ |x - y| < r\}$. In this way the shape sensitivity analysis [53] for $r_0 > 0$ and the topological sensitivity analysis [45] for $r_0 = 0^+$ are performed in the same framework for the simple case of circular inclusion.

5 Shape and Topological Derivatives of Elastic Energy in Two Spatial Dimensions for an Inclusion

In the subdomain Ω_c the unique weak solutions

$$\varepsilon \to u := u_\varepsilon$$

of the elasticity boundary value subproblem are given by the variational inequality

$$u \in K \ : \ a(\Omega_c; u, v - u) + b_\varepsilon(\Gamma_R; u, v - u) \geqslant (f, v - u)_{\Omega_c} \quad \forall v \in K.$$

In order to differentiate the solution mapping for this variational inequality, it is required to differentiate the bilinear form $\varepsilon \to b_\varepsilon(\Gamma_R; u, v)$, which is performed in this section.

5.1 Shape and Topological Derivatives of the Energy Functional in Ω_R with Respect to the Inclusion ω

In order to evaluate the topological derivative of energy functional in isotropic elasticity, the shape sensitivity analysis is combined with the asymptotic analysis [45]. In this section the small parameter is denoted by $\varepsilon \to 0$, and the circular inclusion $\varepsilon \to \omega_\varepsilon := B_\varepsilon$ is considered. The general shape of the inclusion $\varepsilon \to \omega_\varepsilon$ can be considered in the same way for shape sensitivity analysis [53] and asymptotic analysis [45]. For the sake of simplicity, the subscript R is omitted, thus, we denote $\Omega := \Omega_R$, since the inclusion is located in the subdomain Ω_R. We also allow for the Neumann Γ_N and Dirichlet Γ_D pieces of the boundary $\partial\Omega := \partial\Omega_R$, thus, $\partial\Omega_R := \Gamma_N \cup \Gamma_D \cup \Gamma$. Thus, we evaluate the shape and topological derivative [45] of the total potential energy associated to the plane stress linear elasticity problem, considering the nucleation of a small inclusion, represented by $B_\varepsilon \subset \Omega$, as the topological perturbation. In this way the expansion of the Steklov-Poincaré operator on the interface $\Gamma := \Gamma_R$ is obtained.

5.1.1 Steklov-Poincaré Operator

Let us consider the nonhomogeneous Dirichlet linear elasticity boundary value problem in the domain Ω with the boundary $\partial\Omega := \Gamma_N \cup \Gamma_D \cup \Gamma$.

$$\begin{cases} \text{Find } u, \text{ such that} \\ \operatorname{div}\sigma(u) = 0 & \text{in } \Omega, \\ \quad \sigma(u) = \mathbb{C}\nabla u^s, \\ \qquad u = 0 & \text{on } \Gamma_D, \\ \qquad u = \overline{u} & \text{on } \Gamma, \\ \sigma(u)n = 0 & \text{on } \Gamma_N, \end{cases}$$

where the only columnists term is the Dirichlet condition $u = \overline{u}$ on the interface Γ. Let

$$a(u, u) := \int_\Omega \sigma(u) \cdot \nabla u^s$$

stands for the associated bilinear form, thus the elastic energy of the solution u is given by

$$\mathcal{E}(\Omega; u) = \frac{1}{2} a(u, u).$$

Then by Green's formula

$$\mathcal{E}(\Omega; u) = \langle \mathcal{T}(\overline{u}), \overline{u} \rangle_\Gamma.$$

In the case of an inclusion $\omega_\varepsilon \subset \Omega$, the formula becomes

$$\mathcal{E}_\varepsilon(\Omega; u) = \langle \mathcal{T}_\varepsilon(\overline{u}), \overline{u} \rangle_\Gamma. \tag{19}$$

Hence, the expansion of the energy functional in Ω, on the left hand side of (19) with respect to the parameter $\varepsilon \to 0$ can be used in order to determine the associated expansion of the Steklov-Poincaré operator $\overline{u} \to \mathcal{T}(\overline{u})$ on the right hand side of (19). Therefore, let us consider the smooth domain Ω with the boundary $\partial\Omega := \Gamma_N \cup \Gamma_D \cup \Gamma$, here Γ is the interface on which the Steklov-Poincaré operator introduced in our domain decomposition method is defined.

5.2 Shape and Topological Differentiability of the Energy Functional for Expansion of Steklov-Poincaré Operator

The notation of monograph [45] is used in this section. We recall the known results [45, 53] on the shape gradient of the energy functional $\varepsilon \to \mathcal{E}_\varepsilon(\Omega)$ with respect to moving interface $\varepsilon \to \partial\omega_\varepsilon$ which is the boundary of inclusion $\omega_\varepsilon \subset \Omega$. Finally, the topological derivative of the energy functional with respect to $\varepsilon \to 0^+$ is obtained [45]. In this way, the shape and topological differentiability of the Steklov-Poincaré operator on the fictitious interface Γ is established. Let us consider the subdomain

Ω_R with the interface $\Gamma_R \subset \partial\Omega_R$, which are denoted by Ω and Γ, respectively. Let us consider a circular inclusion in Ω. The inclusion $\omega_\varepsilon := B_\varepsilon(y) \subset \Omega_R$ depends on the parameter $\varepsilon \in [0, \varepsilon_0)$ $\varepsilon_0 \gg 0$. The energy functional $\varepsilon \to \mathcal{E}_\varepsilon(\Omega)$ is *shape differentiable* for $\varepsilon > 0$ and *topologically differentiable* for $\varepsilon = 0^+$. In this way the expansion of the Steklov-Poincaré operator is obtained on the interface Γ_R. The energy shape functional associated to the unperturbed domain with $\varepsilon = 0$, i.e., without inclusion, which we are dealing with is defined as

$$\psi(\chi) = \frac{1}{2}\int_\Omega \sigma(u) \cdot \nabla u^s,$$

where χ stands for the characteristic function of Ω, and the vector function u is the solution to the variational problem:

$$\begin{cases} \text{Find } u \in \mathcal{U}, \text{ such that} \\ \displaystyle\int_\Omega \sigma(u) \cdot \nabla\eta^s = 0 \quad \forall \eta \in \mathcal{V}, \\ \qquad\qquad \text{with} \quad \sigma(u) = \mathbb{C}\nabla u^s. \end{cases} \tag{20}$$

In the above equation, $\mathbb{C}$ is the constitutive tensor given by

$$\mathbb{C} = \frac{E}{1-\nu^2}((1-\nu)\mathbb{I} + \nu \mathrm{I} \otimes \mathrm{I}),$$

where I and $\mathbb{I}$ are the second and fourth order identity tensors, respectively, E is the Young modulus and ν the Poisson ratio, both considered constants everywhere. For the sake of simplicity, we also assume that the thickness of the elastic body is constant and equal to one. The convex set $\mathcal{U}$ written for the columnists Dirichlet boundary condition on the interface and the associated space of test functions $\mathcal{V}$ are respectively defined as

$$\mathcal{U} := \{\varphi \in H^1(\Omega; \mathbb{R}^2) : \varphi_{|\Gamma_D} = 0, \quad \varphi_{|\Gamma} = \overline{u}\},$$
$$\mathcal{V} := \{\varphi \in H^1(\Omega; \mathbb{R}^2) : \varphi_{|\Gamma_D} = 0 \quad \varphi_{|\Gamma} = 0\}.$$

In addition, $\partial\Omega = \Gamma \cup \Gamma_D \cup \Gamma_N$ with $\Gamma_D \cap \Gamma_N = \varnothing$, $\Gamma \cap \Gamma_N = \varnothing$, and $\Gamma_D \cap \Gamma = \varnothing$, where Γ_D and Γ_N are Dirichlet and Neumann boundaries, respectively. Thus, $\overline{u}$ is a Dirichlet data on Γ, and there are homogeneous Dirichlet data on Γ_D and Neumann data on Γ_N. The strong system associated to the variational problem (20) reads:

$$\begin{cases} \text{Find } u, \text{ such that} \\ \operatorname{div}\sigma(u) = 0 & \text{in } \Omega, \\ \quad \sigma(u) = \mathbb{C}\nabla u^s, \\ \qquad u = \overline{u} & \text{on } \Gamma, \\ \qquad u = 0 & \text{on } \Gamma_D, \\ \sigma(u)n = 0 & \text{on } \Gamma_N. \end{cases}$$

Remark 22 Since the Young modulus E and the Poisson ratio ν are constants, the above boundary value problem reduces to the well-known Navier system, namely

$$-\mu\Delta u - (\lambda + \mu)\nabla(\mathrm{div} u) = 0 \quad \text{in} \quad \Omega,$$

with the Lame's coefficients μ and λ respectively given by

$$\mu = \frac{E}{2(1+\nu)} \quad \text{and} \quad \lambda = \frac{\nu E}{1-\nu^2}.$$

Now, let us state the same problem in the perturbed domain which contains the inclusion B_ε. More precisely, the perturbed domain is obtained if a circular hole $B_\varepsilon(y)$ is introduced inside $\Omega \subset \mathbb{R}^2$, where $\overline{B_\varepsilon(y)} \Subset \Omega$ denotes a ball of radius ε and center at $y \in \Omega$. Then, $B_\varepsilon(y)$ is filled by an inclusion with different material property compared to the unperturbed domain Ω. The material properties are characterized by a piecewise constant function γ_ε of the form

$$\gamma_\varepsilon = \gamma_\varepsilon(x) := \begin{cases} 1 \text{ if } x \in \Omega \setminus \overline{B_\varepsilon}, \\ \gamma \text{ if } x \in B_\varepsilon, \end{cases} \tag{21}$$

where $\gamma \in \mathbb{R}_+$ is the contrast coefficient. In this case, the shape functional reads

$$\psi(\chi_\varepsilon) := \frac{1}{2}\int_\Omega \sigma_\varepsilon(u_\varepsilon) \cdot \nabla u_\varepsilon^s, \tag{22}$$

where the vector function u_ε solves the variational problem:

$$\begin{cases} \text{Find } u_\varepsilon \in \mathcal{U}_\varepsilon, \text{ such that} \\ \displaystyle\int_\Omega \sigma_\varepsilon(u_\varepsilon) \cdot \nabla \eta^s = 0 \quad \forall \eta \in \mathcal{V}_\varepsilon, \\ \qquad \text{with} \quad \sigma_\varepsilon(u_\varepsilon) = \gamma_\varepsilon \mathbb{C} \nabla u_\varepsilon^s. \end{cases} \tag{23}$$

with γ_ε given by (21). The set $\mathcal{U}_\varepsilon$ and the space $\mathcal{V}_\varepsilon$ are defined as

$$\mathcal{U}_\varepsilon := \{\varphi \in \mathcal{U} : [\![\varphi]\!] = 0 \text{ on } \partial B_\varepsilon\},$$
$$\mathcal{V}_\varepsilon := \{\varphi \in \mathcal{V} : [\![\varphi]\!] = 0 \text{ on } \partial B_\varepsilon\},$$

where the operator $[\![\varphi]\!]$ is used to denote the jump of function φ on the boundary of the inclusion ∂B_ε, namely $[\![\varphi]\!] := \varphi_{|_{\Omega\setminus\overline{B_\varepsilon}}} - \varphi_{|_{B_\varepsilon}}$ on ∂B_ε. The *strong system* associated to the variational problem (23) reads:

$$\begin{cases} \text{Find } u_\varepsilon, \text{ such that} \\ \operatorname{div}\sigma_\varepsilon(u_\varepsilon) = 0 & \text{in } \Omega, \\ \sigma_\varepsilon(u_\varepsilon) = \gamma_\varepsilon \mathbb{C}\nabla u_\varepsilon^s, \\ u_\varepsilon = \overline{u} & \text{on } \Gamma, \\ u_\varepsilon = 0 & \text{on } \Gamma_D, \\ \sigma(u_\varepsilon)n = 0 & \text{on } \Gamma_N, \\ \left.\begin{array}{r} [\![u_\varepsilon]\!] = 0 \\ [\![\sigma_\varepsilon(u_\varepsilon)]\!]n = 0 \end{array}\right\} & \text{on } \partial B_\varepsilon. \end{cases} \tag{24}$$

The *transmission condition* on the boundary of the inclusion ∂B_ε comes out from the variation formulation (23).

5.3 Shape Derivative of Steklov-Poincaré Operator

The next step consists in evaluating the shape derivative of functional $\psi(\chi_\varepsilon)$ with respect to an uniform expansion of the inclusion B_ε. In the particular case of circular inclusions, for a given $y \in \Omega$ and $0 < \varepsilon < \ell$, with $\ell := \operatorname{dist}(y, \partial\Omega)$, we can construct a shape change velocity field $\mathfrak{V}$ that represents uniform expansion of $B_\varepsilon(y)$. In fact, it is sufficient to define $\mathfrak{V}$ on the boundary ∂B_ε i.e., $\mathfrak{V}_{|\partial B_\varepsilon(y)} = -n$, where $n = -(x - y)/\varepsilon$, with $x \in \partial B_\varepsilon$, is the normal unit vector field pointing toward the center of the circular inclusion B_ε Let us introduce the *Eshelby energy-momentum tensor* [45], namely

$$\mathbb{E}_\varepsilon = \frac{1}{2}(\sigma_\varepsilon(u_\varepsilon) \cdot \nabla u_\varepsilon^s)\mathrm{I} - \nabla u_\varepsilon^\top \sigma_\varepsilon(u_\varepsilon). \tag{25}$$

In addition, we note that after considering the constitutive relation $\sigma_\varepsilon(u_\varepsilon) = \gamma_\varepsilon \mathbb{C}\nabla u_\varepsilon^s$ in (22), with the contrast γ_ε given by (21), the shape functional $\psi(\chi_\varepsilon)$ can be written as follows

$$\psi(\chi_\varepsilon) = \frac{1}{2}\left(\int_{\Omega\setminus\overline{B_\varepsilon}} \sigma(u_\varepsilon) \cdot \nabla u_\varepsilon^s + \int_{B_\varepsilon} \gamma\sigma(u_\varepsilon) \cdot \nabla u_\varepsilon^s\right), \tag{26}$$

where $\sigma(u_\varepsilon) = \mathbb{C}\nabla u_\varepsilon^s$. Therefore, the explicit dependence with respect to the parameter ε arises, and we recall the following result [45]

Proposition 23 *Let $\psi(\chi_\varepsilon)$ be the energy shape functional defined by (22). Then, the shape derivative of $\psi(\chi_\varepsilon)$ with respect to the small parameter $\varepsilon > 0$ is given by*

$$\dot{\psi}(\chi_\varepsilon) = \int_\Omega \mathbb{E}_\varepsilon \cdot \nabla\mathfrak{V},$$

where $\mathfrak{V}$ is the shape change velocity field defined by an extension of the normal vector field $n = -(x - y)/\varepsilon$, with $x \in \partial B_\varepsilon$, and $\mathbb{E}_\varepsilon$ is the Eshelby energy-momentum tensor given by (25).

Proof Before starting, let us recall that the constitutive operator is defined as $\sigma_\varepsilon(\varphi) = \gamma_\varepsilon \mathbb{C}\nabla\varphi^s$. Thus, by making use of the Reynolds' transport theorem and the concept of material derivative of spatial fields [45], the derivative with respect to ε of the shape functional (26) is given by

$$\begin{aligned}
\dot{\psi}(\chi_\varepsilon) &= \frac{1}{2}\left(\int_{\Omega\setminus\overline{B_\varepsilon}} \sigma(u_\varepsilon)\cdot\nabla u_\varepsilon^s + \int_{B_\varepsilon} \gamma\sigma(u_\varepsilon)\cdot\nabla u_\varepsilon^s\right) \\
&= \int_{\Omega\setminus\overline{B_\varepsilon}} \sigma(u_\varepsilon)\cdot\nabla\dot{u}_\varepsilon^s + \int_{B_\varepsilon} \gamma\sigma(u_\varepsilon)\cdot\nabla\dot{u}_\varepsilon^s \\
&\quad + \frac{1}{2}\int_{\Omega\setminus\overline{B_\varepsilon}} ((\sigma(u_\varepsilon)\cdot\nabla u_\varepsilon^s)\mathrm{I} - 2\nabla u_\varepsilon^\top\sigma(u_\varepsilon))\cdot\nabla\mathfrak{V} \\
&\quad + \frac{1}{2}\int_{B_\varepsilon} \gamma((\sigma(u_\varepsilon)\cdot\nabla u_\varepsilon^s)\mathrm{I} - 2\nabla u_\varepsilon^\top\sigma(u_\varepsilon))\cdot\nabla\mathfrak{V}.
\end{aligned}$$

Then,

$$\begin{aligned}
\dot{\psi}(\chi_\varepsilon) &= \frac{1}{2}\int_{\Omega} ((\sigma_\varepsilon(u_\varepsilon)\cdot\nabla u_\varepsilon^s)\mathrm{I} - 2\nabla u_\varepsilon^\top\sigma_\varepsilon(u_\varepsilon))\cdot\nabla\mathfrak{V} \\
&\quad + \int_{\Omega} \sigma_\varepsilon(u)\cdot\nabla\dot{u}_\varepsilon^s.
\end{aligned}$$

Since $\dot{u}_\varepsilon$ is a variation of u_ε in the direction of the velocity field $\mathfrak{V}$, then $\dot{u}_\varepsilon \in \mathcal{V}_\varepsilon$ [53]. Finally, by taking $\dot{u}_\varepsilon$ as test function in the variational problem (23), we have that the last two terms of the above equation vanish. □

The shape gradient of energy functional is supported on the moving interface $\varepsilon \to \partial B_\varepsilon$ as it is predicted by the structure theorem of the shape gradient [45, 53].

Proposition 24 *Let $\psi(\chi_\varepsilon)$ be the shape functional defined by (22). Then, its derivative with respect to the small parameter ε is given by*

$$\dot{\psi}(\chi_\varepsilon) = \int_{\partial B_\varepsilon} [\![\mathbb{E}_\varepsilon]\!] n\cdot\mathfrak{V}, \tag{27}$$

with $\mathfrak{V}$ standing for the shape change velocity field compactly supported in a neighbourhood of ∂B_ε and tensor $\mathbb{E}_\varepsilon$ given by (25).

Proof Before starting, let us recall the constitutive operator $\sigma_\varepsilon(\varphi) = \gamma_\varepsilon\mathbb{C}\nabla\varphi^s$ and the relation between material and spatial derivatives of vector fields $\dot{\varphi} = \varphi' + (\nabla\varphi)\mathfrak{V}$. By making use of the Reynolds' transport theorem [45], the shape derivative of the functional (22) results in

$$
\begin{aligned}
\dot{\psi}(\chi_\varepsilon) &= \left(\frac{1}{2}\int_\Omega \sigma_\varepsilon(u_\varepsilon)\cdot\nabla u_\varepsilon^s\right) \\
&= \int_\Omega \sigma_\varepsilon(u_\varepsilon)\cdot(\nabla u_\varepsilon')^s + \frac{1}{2}\int_{\partial\Omega}(\sigma_\varepsilon(u_\varepsilon)\cdot\nabla u_\varepsilon^s)n\cdot\mathfrak{V} \\
&\quad + \frac{1}{2}\int_{\partial B_\varepsilon}[\![\sigma_\varepsilon(u_\varepsilon)\cdot\nabla u_\varepsilon^s]\!]n\cdot\mathfrak{V}.
\end{aligned}
$$

In addition, we have

$$
\begin{aligned}
\dot{\psi}(\chi_\varepsilon) = \frac{1}{2}\int_{\partial\Omega}(\sigma_\varepsilon(u_\varepsilon)\cdot\nabla u_\varepsilon^s)n\cdot\mathfrak{V} + \frac{1}{2}\int_{\partial B_\varepsilon}[\![\sigma_\varepsilon(u_\varepsilon)\cdot\nabla u_\varepsilon^s]\!]n\cdot\mathfrak{V} \\
- \int_\Omega \sigma_\varepsilon(u_\varepsilon)\cdot\nabla((\nabla u_\varepsilon)\mathfrak{V})^s + \int_\Omega \sigma_\varepsilon(u)\cdot\nabla\dot{u}_\varepsilon^s.
\end{aligned}
$$

Since $\dot{u}_\varepsilon$ is a variation of u_ε in the direction of the velocity field $\mathfrak{V}$, then $\dot{u}_\varepsilon \in \mathcal{V}_\varepsilon$ [53]. Now, by taking into account that u_ε is the solution to the variational problem (23), we have that the last two terms of the above equation vanish. From integration by parts

$$
\begin{aligned}
\dot{\psi}(\chi_\varepsilon) = \frac{1}{2}\int_{\partial\Omega}(\sigma_\varepsilon(u_\varepsilon)\cdot\nabla u_\varepsilon^s)n\cdot\mathfrak{V} + \frac{1}{2}\int_{\partial B_\varepsilon}[\![\sigma_\varepsilon(u_\varepsilon)\cdot\nabla u_\varepsilon^s]\!]n\cdot\mathfrak{V} \\
- \int_{\partial\Omega}(\nabla u_\varepsilon^\top\sigma_\varepsilon(u_\varepsilon))n\cdot\mathfrak{V} - \int_{\partial B_\varepsilon}[\![\nabla u_\varepsilon^\top\sigma_\varepsilon(u_\varepsilon)]\!]n\cdot\mathfrak{V} \\
+ \int_\Omega \operatorname{div}(\sigma_\varepsilon(u_\varepsilon))\cdot(\nabla u_\varepsilon)\mathfrak{V},
\end{aligned}
$$

and rewriting the above equation in the compact form, we obtain

$$
\dot{\psi}(\chi_\varepsilon) = \int_{\partial\Omega}\mathbb{E}_\varepsilon n\cdot\mathfrak{V} + \int_{\partial B_\varepsilon}[\![\mathbb{E}_\varepsilon]\!]n\cdot\mathfrak{V} + \int_\Omega \operatorname{div}(\sigma_\varepsilon(u_\varepsilon))\cdot(\nabla u_\varepsilon)\mathfrak{V}.
$$

Finally, taking into account that u_ε is the solution to the state equation (24), namely $\operatorname{div}\sigma_\varepsilon(u_\varepsilon) = 0$, we have that the last term in the above equation vanishes, which leads to the result. □

Corollary 25 *We have*

$$
\dot{\psi}(\chi_\varepsilon) = \int_{\partial B_\varepsilon}[\![\mathbb{E}_\varepsilon]\!]n\cdot\mathfrak{V} - \int_\Omega \operatorname{div}\mathbb{E}_\varepsilon\cdot\mathfrak{V}.
$$

Since the above equation and (27) remain valid for all velocity fields $\mathfrak{V}$, we have that the last term of the above equation must satisfy

$$
\int_\Omega \operatorname{div}\mathbb{E}_\varepsilon\cdot\mathfrak{V} = 0 \quad \forall\mathfrak{V} \quad \Rightarrow \quad \operatorname{div}\mathbb{E}_\varepsilon = 0.
$$

hence

$$\frac{d}{d\varepsilon}\psi(\chi_\varepsilon) = \dot{\psi}(\chi_\varepsilon) = -\int_{\partial B_\varepsilon} [\![\mathbb{E}_\varepsilon]\!] n \cdot n. \tag{28}$$

5.4 Application to Conical Differentiability for Model Problem

We return to the variational inequality with regularly perturbed bilinear form, see (32) for an application. Therefore, for given $\varepsilon > 0$ we consider the variational inequality in Ω_c,

$$u_t \in K \ : \ a(u_t, v - u_t) + b_t(u_t, v - u_t) = (f, v - u_t) \quad \forall v \in K,$$

where for $t > 0$, t small enough, the symmetric, boundary bilinear form b_t is defined on $\Gamma := \Gamma_R$ by the elastic energy in Ω_R,

$$t \to b_t(u, u) := \langle \mathcal{T}_{\varepsilon+t}(\overline{u}), \overline{u} \rangle_\Gamma.$$

Here, $\overline{u}$ stands for the trace of u on Γ. Thus, the shape derivative of this bilinear form with respect to the deformations of interface ∂B_ε governed by $t \to 0$ is given by

$$b'(u, u) := \dot{\psi}(\chi_\varepsilon) = -\int_{\partial B_\varepsilon} [\![\mathbb{E}_\varepsilon]\!] n \cdot n.$$

In this case Lemma 14 applies and we have

Proposition 26 *For $t > 0$, t small enough,*

$$u_t = u + tq + o(t),$$

where

$$q \in S \ : \ a(q, v - q) + b(q, v - q) + b'(q, v - q) \geqslant 0 \quad \forall v \in S.$$

Remark 27 For $\varepsilon = 0^+$ the result remain valid with the modification that

$$u_\varepsilon = u + \varepsilon^2 q + o(\varepsilon^2),$$

and with the shape derivative of the Steklov-Poincaré replaced by the topological derivative which is evaluated in the section below.

Remark 28 Given the one term expansion of the solution to variational inequality in Ω_c with respect to ε, it is straightforward to obtain the directional derivative of the Griffith functional.

5.5 Topological Derivative of the Steklov-Poincaré Operator

We recall known results on topological sensitivity analysis given in [45] which are adapted to our setting. The shape derivative of functional $\dot{\psi}(\chi_\varepsilon)$ is given in terms of an integral over the boundary of the inclusion ∂B_ε (28). The formula for the topological derivative $\mathcal{J}_\psi$ of the shape functional ψ is obtained by asymptotic analysis of u_ε with respect to ε. The asymptotic expansion of the solution u_ε is associated to the transmission condition on the inclusion. We start with an *ansatz* for u_ε

$$u_\varepsilon(x) = u(x) + w_\varepsilon(x) + \widetilde{u}_\varepsilon(x).$$

After applying the operator σ_ε, we have

$$\begin{aligned}\sigma_\varepsilon(u_\varepsilon(x)) &= \sigma_\varepsilon(u(x)) + \sigma_\varepsilon(w_\varepsilon(x)) + \sigma_\varepsilon(\widetilde{u}_\varepsilon(x)) \\ &= \sigma_\varepsilon(u(y)) + \nabla\sigma_\varepsilon(u(\widehat{y}))(x-y) + \sigma_\varepsilon(w_\varepsilon(x)) + \sigma_\varepsilon(\widetilde{u}_\varepsilon(x)),\end{aligned}$$

where $\widehat{y}$ is an intermediate point between x and y. On the boundary of the inclusion ∂B_ε we have

$$[\![\sigma_\varepsilon(u_\varepsilon)]\!]n = 0 \quad \Rightarrow \quad (\sigma(u_\varepsilon)_{|_{\Omega\setminus\overline{B_\varepsilon}}} - \gamma\sigma(u_\varepsilon)_{|_{B_\varepsilon}})n = 0,$$

with $\sigma_\varepsilon(\varphi) = \gamma_\varepsilon \mathbb{C}\nabla\varphi^s$ and $\sigma(\varphi) = \mathbb{C}\nabla\varphi^s$. The above expansion, evaluated on ∂B_ε, leads to

$$(1-\gamma)\sigma(u(y))n - \varepsilon(1-\gamma)(\nabla\sigma(u(y))n)n + [\![\sigma_\varepsilon(w_\varepsilon(x))]\!]n + [\![\sigma_\varepsilon(\widetilde{u}_\varepsilon(x))]\!]n = 0.$$

Thus, we can choose $\sigma_\varepsilon(w_\varepsilon)$ such that

$$[\![\sigma_\varepsilon(w_\varepsilon(x))]\!]n = -(1-\gamma)\sigma(u(y))n \quad \text{on} \quad \partial B_\varepsilon.$$

Now, the following exterior problem is considered, and formally obtained as $\varepsilon \to 0$:

$$\begin{cases} \text{Find } \sigma_\varepsilon(w_\varepsilon), \text{ such that} \\ \operatorname{div}\sigma_\varepsilon(w_\varepsilon) = 0 \text{ in } \mathbb{R}^2, \\ \sigma_\varepsilon(w_\varepsilon) \to 0 \text{ at } \infty, \\ [\![\sigma_\varepsilon(w_\varepsilon)]\!]n = \widehat{u} \text{ on } \partial B_\varepsilon, \end{cases}$$

with $\widehat{u} = -(1-\gamma)\sigma(u(y))n$. The above boundary value problem admits an explicit solution, which will be used later to construct the expansion for $\sigma_\varepsilon(u_\varepsilon)$. Now we can construct $\sigma_\varepsilon(\widetilde{u}_\varepsilon)$ in such a way that it compensates the discrepancies introduced by the higher order terms in ε as well as by the boundary layer $\sigma_\varepsilon(w_\varepsilon)$ on the exterior boundary $\partial\Omega$. It means that the remainder $\widetilde{u}_\varepsilon$ must be solution to the following boundary value problem:

$$
\begin{cases}
\text{Find } \widetilde{u}_\varepsilon, \text{ such that} \\
\operatorname{div}\sigma_\varepsilon(\widetilde{u}_\varepsilon) = 0 & \text{in } \Omega, \\
\sigma_\varepsilon(\widetilde{u}_\varepsilon) = \gamma_\varepsilon \mathbb{C}\nabla \widetilde{u}^s_\varepsilon, \\
\widetilde{u}_\varepsilon = -w_\varepsilon & \text{on } \Gamma_D, \\
\sigma(\widetilde{u}_\varepsilon)n = -\sigma(w_\varepsilon)n & \text{on } \Gamma_N, \\
\left.\begin{array}{r} [\![\widetilde{u}_\varepsilon]\!] = 0 \\ [\![\sigma_\varepsilon(\widetilde{u}_\varepsilon)]\!]n = \varepsilon h \end{array}\right\} & \text{on } \partial B_\varepsilon,
\end{cases}
\tag{29}
$$

with $h = (1-\gamma)(\nabla\sigma(u(y))n)n$. The following lemma is proved in [45]:

Lemma 29 *Let $\widetilde{u}_\varepsilon$ be the solution to (29) or equivalently the solution to the following variational problem:*

$$
\begin{cases}
\textit{Find } \widetilde{u}_\varepsilon \in \widetilde{\mathcal{U}}_\varepsilon, \textit{ such that} \\
\displaystyle\int_\Omega \sigma_\varepsilon(\widetilde{u}_\varepsilon)\cdot\nabla\eta^s = \varepsilon^2 \int_{\Gamma_N} \sigma(g)n\cdot\eta + \varepsilon\int_{\partial B_\varepsilon} h\cdot\eta \quad \forall \eta \in \widetilde{\mathcal{V}}_\varepsilon, \\
\qquad\qquad\qquad\qquad\qquad \textit{with } \ \sigma_\varepsilon(\widetilde{u}_\varepsilon) = \gamma_\varepsilon \mathbb{C}\nabla\widetilde{u}^s_\varepsilon,
\end{cases}
$$

where the set $\widetilde{\mathcal{U}}_\varepsilon$ and the space $\widetilde{\mathcal{V}}_\varepsilon$ are defined as

$$
\begin{aligned}
\widetilde{\mathcal{U}}_\varepsilon &:= \{\varphi \in H^1(\Omega;\mathbb{R}^2) : [\![\varphi]\!] = 0 \textit{ on } \partial B_\varepsilon,\ \varphi_{|\Gamma_D} = \varepsilon^2 g\}, \\
\widetilde{\mathcal{V}}_\varepsilon &:= \{\varphi \in H^1(\Omega;\mathbb{R}^2) : [\![\varphi]\!] = 0 \textit{ on } \partial B_\varepsilon,\ \varphi_{|\Gamma_D} = 0\},
\end{aligned}
$$

with functions $g = -\varepsilon^{-2}w_\varepsilon$ and $h = (1-\gamma)(\nabla\sigma(u(y))n)n$ independent of the small parameter ε. Then, we have the estimate $\|\widetilde{u}_\varepsilon\|_{H^1(\Omega;\mathbb{R}^2)} = O(\varepsilon^2)$ for the remainder.

Therefore, the *expansion* for $\sigma_\varepsilon(u_\varepsilon)$ can be written [45] in a polar coordinate system (r,θ) centered at the point y as:

- For $r \geqslant \varepsilon$ (outside the inclusion)

$$
\begin{aligned}
\sigma^{rr}_\varepsilon(u_\varepsilon(r,\theta)) &= \varphi_1\left(1 - \tfrac{1-\gamma}{1+\gamma\alpha}\tfrac{\varepsilon^2}{r^2}\right) \\
&\quad + \varphi_2\left(1 - 4\tfrac{1-\gamma}{1+\gamma\beta}\tfrac{\varepsilon^2}{r^2} + 3\tfrac{1-\gamma}{1+\gamma\beta}\tfrac{\varepsilon^4}{r^4}\right)\cos 2\theta + O(\varepsilon^2), \\
\sigma^{\theta\theta}_\varepsilon(u_\varepsilon(r,\theta)) &= \varphi_1\left(1 + \tfrac{1-\gamma}{1+\gamma\alpha}\tfrac{\varepsilon^2}{r^2}\right) \\
&\quad - \varphi_2\left(1 + 3\tfrac{1-\gamma}{1+\gamma\beta}\tfrac{\varepsilon^4}{r^4}\right)\cos 2\theta + O(\varepsilon^2), \\
\sigma^{r\theta}_\varepsilon(u_\varepsilon(r,\theta)) &= -\varphi_2\left(1 + 2\tfrac{1-\gamma}{1+\gamma\beta}\tfrac{\varepsilon^2}{r^2} - 3\tfrac{1-\gamma}{1+\gamma\beta}\tfrac{\varepsilon^4}{r^4}\right)\sin 2\theta + O(\varepsilon^2).
\end{aligned}
$$

- For $0 < r < \varepsilon$ (inside the inclusion)

$$\sigma_\varepsilon^{rr}(u_\varepsilon(r,\theta)) = \varphi_1\left(\tfrac{2}{1-\nu}\tfrac{\gamma}{1+\gamma\alpha}\right) + \varphi_2\left(\tfrac{4}{1+\nu}\tfrac{\gamma}{1+\gamma\beta}\right)\cos 2\theta + O(\varepsilon^2),$$
$$\sigma_\varepsilon^{\theta\theta}(u_\varepsilon(r,\theta)) = \varphi_1\left(\tfrac{2}{1-\nu}\tfrac{\gamma}{1+\gamma\alpha}\right) - \varphi_2\left(\tfrac{4}{1+\nu}\tfrac{\gamma}{1+\gamma\beta}\right)\cos 2\theta + O(\varepsilon^2),$$
$$\sigma_\varepsilon^{r\theta}(u_\varepsilon(r,\theta)) = -\varphi_2\left(\tfrac{4}{1+\nu}\tfrac{\gamma}{1+\gamma\beta}\right)\sin 2\theta + O(\varepsilon^2).$$

Some terms in the above formulae require explanations. The coefficients φ_1 and φ_2 are given by

$$\varphi_1 = \frac{1}{2}(\sigma_1(u(y)) + \sigma_2(u(y))), \quad \varphi_2 = \frac{1}{2}(\sigma_1(u(y)) - \sigma_2(u(y))),$$

where $\sigma_1(u(y))$ and $\sigma_2(u(y))$ are the eigenvalues of tensor $\sigma(u(y))$, which can be expressed as

$$\sigma_{1,2}(u(y)) = \frac{1}{2}\left(\operatorname{tr}\sigma(u(y)) \pm \sqrt{2\sigma^D(u(y))\cdot\sigma^D(u(y))}\right),$$

with $\sigma^D(u(y))$ standing for the deviatory part of the stress tensor $\sigma(u(y))$, namely

$$\sigma^D(u(y)) = \sigma(u(y)) - \frac{1}{2}\operatorname{tr}\sigma(u(y))\mathrm{I}.$$

In addition, the constants α and β are given by

$$\alpha = \frac{1+\nu}{1-\nu} \quad \text{and} \quad \beta = \frac{3-\nu}{1+\nu}. \tag{30}$$

Finally, $\sigma_\varepsilon^{rr}(u_\varepsilon)$, $\sigma_\varepsilon^{\theta\theta}(u_\varepsilon)$ and $\sigma_\varepsilon^{r\theta}(u_\varepsilon)$ are the components of tensor $\sigma_\varepsilon(u_\varepsilon)$ in the polar coordinate system, namely $\sigma_\varepsilon^{rr}(u_\varepsilon) = e^r\cdot\sigma_\varepsilon(u_\varepsilon)e^r$, $\sigma_\varepsilon^{\theta\theta}(u_\varepsilon) = e^\theta\cdot\sigma_\varepsilon(u_\varepsilon)e^\theta$ and $\sigma_\varepsilon^{r\theta}(u_\varepsilon) = \sigma_\varepsilon^{\theta r}(u_\varepsilon) = e^r\cdot\sigma_\varepsilon(u_\varepsilon)e^\theta$, with e^r and e^θ used to denote the canonical basis associated to the polar coordinate system (r,θ), such that, $\|e^r\| = \|e^\theta\| = 1$ and $e^r\cdot e^\theta = 0$.

5.6 Formulae for Topological Derivative

Now, we can evaluate the integral in formula (28). With this result, we can perform the limit passage $\varepsilon\to 0$. The integral in (28) can be evaluated by using the expansion for $\sigma_\varepsilon(u_\varepsilon)$ given by (30). The idea is to introduce a polar coordinate system (r,θ) with center at y. Then, we can write u_ε in this coordinate system to evaluate the integral explicitly. In particular, the integral in (28) yields

$$\int_{\partial B_\varepsilon} [\![\mathbb{E}_\varepsilon]\!] n \cdot n = 2\pi\varepsilon \mathbb{P}_\gamma \sigma(u(y)) \cdot \nabla u^s(y) + o(\varepsilon).$$

Finally,

$$\mathcal{J}_\psi(y) = -\lim_{\varepsilon \to 0} \frac{1}{f'(\varepsilon)} \left(2\pi\varepsilon \mathbb{P}_\gamma \sigma(u(y)) \cdot \nabla u^s(y) + o(\varepsilon) \right),$$

where the *polarization tensor* $\mathbb{P}_\gamma$ is given by the following fourth order isotropic tensor

$$\mathbb{P}_\gamma = \frac{1}{2} \frac{1-\gamma}{1+\gamma\beta} \left((1+\beta)\mathbb{I} + \frac{1}{2}(\alpha - \beta) \frac{1-\gamma}{1+\gamma\alpha} \mathrm{I} \otimes \mathrm{I} \right),$$

with the parameters α and β given by (30). Now, in order to extract the leading term of the above expansion, we choose

$$f(\varepsilon) = \pi\varepsilon^2,$$

which leads to the final formula for the *topological derivative*, namely

$$\mathcal{J}_\psi(y) = -\mathbb{P}_\gamma \sigma(u(y)) \cdot \nabla u^s(y).$$

Remark 30 Polarization tensors for cracks are considered e.g., in [41–44].

Finally, the topological asymptotic expansion of the energy shape functional takes the form

$$\psi(\chi_\varepsilon(y)) = \psi(\chi) - \pi\varepsilon^2 \mathbb{P}_\gamma \sigma(u(y)) \cdot \nabla u^s(y) + o(\varepsilon^2),$$

whose mathematical justification is given in [45].

Remark 31 We note that the obtained polarization tensor is isotropic because we are dealing with circular inclusions. Some results on the polarization tensor associated to arbitrary shaped inclusions can be found in [35, 43].

Remark 32 Formally, we can consider the limit cases $\gamma \to 0$ and $\gamma \to \infty$. For $\gamma \to 0$, the inclusion leads to a void and the transmission condition on the boundary of the inclusion degenerates to homogeneous Neumann boundary condition. In fact, in this case the polarization tensor is given by

$$\mathbb{P}_0 = \frac{1}{2}(1+\beta)\mathbb{I} + \frac{1}{4}(\alpha - \beta)\mathrm{I} \otimes \mathrm{I} = \frac{2}{1+\nu}\mathbb{I} - \frac{1-3\nu}{2(1-\nu^2)} \mathrm{I} \otimes \mathrm{I}.$$

In addition, for $\gamma \to \infty$, the elastic inclusion leads to a rigid one and the polarization tensor is given by

$$\mathbb{P}_\infty = -\frac{1+\beta}{2\beta}\mathbb{I} + \frac{\alpha - \beta}{4\alpha\beta}\mathrm{I} \otimes \mathrm{I} = -\frac{2}{3-\nu}\mathbb{I} - \frac{1-3\nu}{2(1+\nu)(3-\nu)} \mathrm{I} \otimes \mathrm{I}.$$

6 Asymptotic Analysis with Bounded Perturbations of Variational Inequalities

The bounded perturbations of bilinear forms in variational inequalities resulting from the approximation of the energy by (5) are employed in asymptotic analysis of variational inequalities in singularly perturbed geometrical domains. The proposed method of asymptotic analysis is sufficiently precise for the first order topological differentiability [56].

6.1 Applications of Steklov-Poincaré Operators in Asymptotic Analysis

We analyse the precision of the proposed method of approximation for variational inequalities in singularly perturbed geometrical domains. We assume for simplicity that the singular perturbation is a disc $B_\epsilon = \{|x| < \epsilon\}$. The Signorini variational inequality in $\Omega_\epsilon := \Omega \setminus \overline{B}_\epsilon$,

$$u_\epsilon \in K(\Omega_\epsilon) \ : \ a(\Omega_\epsilon; u_\epsilon, v - u_\epsilon) - L(\Omega_\epsilon; v - u_\epsilon) \geqslant 0 \quad \forall v \in u_\epsilon \in K(\Omega_\epsilon),$$

can be considered in the truncated domain $\Omega_c := \Omega \setminus \overline{B}_R$ for $R > \epsilon > 0$, R small enough. It is assumed that the source or linear form $v \to L(\Omega; v)$ is supported in Ω_c. Hence the restriction $u_\epsilon \in K(\Omega_c)$ of $u_\epsilon \in K(\Omega_\epsilon)$ to the truncated domain is given by the solution to variational inequality

$$u_\epsilon \in K(\Omega_c) \ : \ a(\Omega_c; u_\epsilon, v-u_\epsilon)+\langle \mathcal{A}_\epsilon(u_\epsilon), v-u_\epsilon\rangle - L(\Omega_c; v-u_\epsilon) \geqslant 0 \ \forall v \in K(\Omega_\epsilon), \quad (31)$$

where $\mathcal{A}_\epsilon$ stands for the Steklov-Poincaré operator which *replaces* the portion of bilinear form over the ring $C(R, \epsilon) := \{R > |x| > \epsilon\}$.

Proposition 33 *Assume that the Steklov-Poincaré operator admits the one-term expansion*

$$\langle \mathcal{A}_\epsilon(v), v\rangle = \langle \mathcal{A}(v), v\rangle + \epsilon^2 \langle \mathcal{B}(v), v\rangle + o(\epsilon^2; v, v)$$

with the compact remainder $o(\epsilon^2; v, v)$, then we can replace in (31) the Steklov-Poincaré operator by its one term approximation

$$\begin{aligned} &\widetilde{u}_\epsilon \in K(\Omega_c) \ : \\ &a(\Omega_c; \widetilde{u}_\epsilon, v - \widetilde{u}_\epsilon) + \langle \mathcal{A}(\widetilde{u}_\epsilon), v - \widetilde{u}_\epsilon\rangle + \\ &\qquad \epsilon^2 \langle \mathcal{B}(\widetilde{u}_\epsilon), v - \widetilde{u}_\epsilon\rangle - L(\Omega_c; v - \widetilde{u}_\epsilon) \geqslant 0 \quad \forall v \in K(\Omega_\epsilon), \end{aligned}$$

with the estimate

$$\|\widetilde{u}_\epsilon - u_\epsilon\| = o(\epsilon^2).$$

Remark 34 From Proposition 33 it follows that for the shape-topological differentiability of the energy functional we can consider the variational inequality

$$\widehat{u}_\epsilon \in K(\Omega) \; : \; a(\Omega; \widehat{u}_\epsilon, v - \widehat{u}_\epsilon) + \epsilon^2 \langle \mathcal{B}(\widehat{u}_\epsilon), v - \widehat{u}_\epsilon \rangle - L(\Omega; v - \widehat{u}_\epsilon) \geqslant 0 \quad \forall v \in K(\Omega), \tag{32}$$

since $\|\widehat{u}_\epsilon - u_\epsilon\| = o(\epsilon^2)$ in Ω_c. In this way, the approximation (5) of quadratic functional (2) is justified for the first order topological derivatives of variational inequalities in truncated domains.

For the quadratic functional (1) and the associated boundary value problem, the bilinear form

$$v \to b(\Gamma_R; v, v) := \langle \mathcal{B}(v), v \rangle$$

is determined. The linear operator $\mathcal{B}$ is obtained from the one term expansion of the Steklov-Poincaré operator $\mathcal{A}_\epsilon$, the expansion results from the energy expansion in the subdomain Ω_R. Therefore, the perturbed quadratic functional (3) can be replaced by its approximation given by (5). For the Signorini problem in two spatial dimensions it means that the variational inequality is obtained for minimization of perturbed functional (3) over the energy space in unperturbed domain Ω, and the associated energy functional

$$\mathcal{E}_\epsilon(\Omega) = \frac{1}{2} a(\Omega; u_\epsilon, u_\epsilon) + \frac{\epsilon^2}{2} b(\Gamma_R; u_\epsilon, u_\epsilon) - (f, u_\epsilon)_\Omega,$$

is evaluated for the solution of variational inequality

$$u_\epsilon \in K(\Omega) \; : \; a(\Omega; u_\epsilon, v - u_\epsilon) + \epsilon^2 b(\Gamma_R; u_\epsilon, v - u_\epsilon) - (f, v - u_\epsilon)_\Omega \geqslant 0 \quad \forall v \in K(\Omega).$$

6.2 Asymptotic Analysis by Domain Decomposition Method

In order to apply the domain decomposition technique to topological differentiability $\omega_\epsilon \to J_\epsilon(\Omega)$ in topologically perturbed domains $\Omega := \Omega_\epsilon$ for the shape functionals $\Omega \to J(\Omega)$ we need the appropriate results on topological differentiability $\epsilon \to \mathcal{B}_\epsilon$ of the Steklov-Poincaré pseudodifferential boundary operators defined on the artificial interface Σ. In the particular case of holes $\epsilon \to \omega_\epsilon$ the notation is straighforward, with the singularly perturbed domain $\Omega_\epsilon := \Omega \setminus \overline{\omega}_\epsilon$ and with the shape functional to be analysed with respect to small parameter $\epsilon \to J_\epsilon(\Omega) := J(\Omega \setminus \overline{\omega}_\epsilon)$. In the case of inclusions $\epsilon \to \omega_\epsilon$ the shape functional depends on the characteristic functions

$\epsilon \to \chi_\epsilon$ of the domain perturbation ω_ϵ. For inclusions the state solution $\epsilon \to u_\epsilon \in H(\Omega)$ is obtained by solving boundary value problems with operator coefficients depending on the small parameter $\epsilon \to 0$. In both cases the asymptotics of Steklov-Poincaré operators are obtained by asymptotic analysis of the energy functional for linear elliptic boundary value problems in subdomains Ω_2 which contains the perturbations $\epsilon \to \omega_\epsilon$. Let us consider the direct method of sensitivity analysis in subdomain Ω_1 which contains the contact subset $\Gamma_c \subset \partial\Omega$. This is possible due to the conical differentiability of metric projection onto the convex set K which is valid under some assumptions (e.g., the convex, closed cone K is polyhedric in the Dirichlet space $H(\Omega)$ [12]).

Example 35 In the case of the Signorini problem in two spatial dimensions the direct method of asymptotic analysis for the shape functional (6)

$$J_\epsilon(\Omega_\epsilon) := \int_{\Omega_1} \langle A'(0) \cdot u_\epsilon, u_\epsilon \rangle dx$$

can be described as follows for the disc $\omega_\epsilon := B(\epsilon) = \{|x| < \epsilon\}$ located at the origin.

1. We solve the variational inequality in Ω_1 : determine $u \in K$ and its coincidence set $\Xi := \{x \in \Gamma_c \ : \ u(x) = 0\}$. Thus, the convex cone

$$S = \{v \in H^1_{\Gamma_0}(\Omega) \ : \ v \geqslant 0 \text{ on } \Xi \quad a(\Omega; u, v) = (f, v)_\Omega\}$$

used in conical differentiability of the element u with respect to the shape can be determined.

2. The asymptotic analysis of solutions to variational inequality in singularly perturbed domain $\Omega(\epsilon) : \Omega \setminus \overline{B(\epsilon)}$ with respect to small parameter $\epsilon \to 0$ which governs the size of the hole $B(\epsilon)$ leads to the expansion

$$u_\epsilon = u + \epsilon^2 q + o(\epsilon^2)$$

obtained by the domain decomposition method with the Steklov-Poincaré boundary operators, where

$$q \in S \ : \ a(\Omega; q, v - q) + \epsilon^2 \langle \mathcal{B}q, v - q \rangle_R \geqslant \quad \forall v \in S.$$

3. The shape functional

$$J_\epsilon(\Omega_\epsilon) := \int_{\Omega_1} \langle A'(0) \cdot u_\epsilon, u_\epsilon \rangle dx$$

can be expanded in Ω_1, the expansion is valid in the whole domain Ω,

$$J_\epsilon(\Omega_\epsilon) = \int_\Omega \langle A'(0)\cdot u, u\rangle dx + 2\epsilon^2 \int_\Omega \langle A'(0)\cdot q, u\rangle dx + o(\epsilon^2),$$

however the obtained expression for the topological derivative may not be constructive in numerical methods. We want to obtain an equivalent expression, when possible, which replaces the topological derivative

$$\mathcal{T}(\mathcal{O}) = 2\int_\Omega \langle A'(0)\cdot q, u\rangle dx$$

in the first order expansion of the energy functional for Signorini problem. In the linear boundary value problems such an expression can always be obtained by the introduction of an appropriate adjoint state. We point out that for variational inequalities the existence of an adjoint state in general cannot be expected.

7 Asymptotic Analysis of Boundary Value Problems in Rings or Spherical Shells

In this section we shall consider asymptotic corrections to the energy functional for the elasticity boundary value problems or the Laplace equation in $\mathbb{R}^d$, where $d = 2, 3$. The dependence of the energy on small parameter is caused by creating a small ball-like void of variable radius ϵ in the interior of the domain Ω, with the homogeneous Neumann boundary conditions for the boundary value problems on its surface. We assume that this void has its centre at the origin $\mathcal{O}$. In order to eliminate the variability of the domain, we take as Ω_R the open ball $B(\mathcal{O}, R) = B(R)$ with fixed R. In this way the void $B(\epsilon)$ is surrounded by $B(R) \subset \operatorname{int}\Omega$. We denote also the ring or spherical shell as $C(R,\epsilon) = B(R)\setminus \overline{B}(\epsilon)$, $\Omega(R) = \Omega \setminus \overline{B}(R)$ and $\Gamma_R = \partial B(R)$. Using these notations we define our main tool, namely the Dirichlet-to-Neumann mapping for linear elasticity or the Steklov-Poincaré operator

$$\mathcal{A}_\epsilon : \mathbf{H}^{1/2}(\Gamma_R) \longmapsto \mathbf{H}^{-1/2}(\Gamma_R)$$

by means of the boundary value problem:

$$\begin{aligned}
&(1-2\nu)\Delta\mathbf{w} + \mathbf{grad}\,\mathrm{div}\,\mathbf{w} = 0, \quad \text{in } \; C(R,\epsilon),\\
&\mathbf{w} = \mathbf{v} \;\text{ on }\; \Gamma_R,\\
&\boldsymbol{\sigma}(\mathbf{w}).\mathbf{n} = 0 \;\text{ on }\; \Gamma_\epsilon
\end{aligned}$$

so that

$$\mathcal{A}_\epsilon \mathbf{v} = \boldsymbol{\sigma}(\mathbf{w}).\mathbf{n} \;\text{ on }\; \Gamma_R.$$

Domain decomposition—Steklov-Poincaré operator. Let $\mathbf{u}^R$ be the restriction of $\mathbf{u}$ to $\Omega(R)$ and $\gamma^R\varphi$ the projection of φ on Γ_R. We may then define the functional

$$I_\epsilon^R(\varphi_\epsilon) = \frac{1}{2}\int_{\Omega(R)} \boldsymbol{\sigma}(\varphi_\epsilon) : \varepsilon(\varphi_\epsilon)\,dx - \int_{\Gamma_N} \mathbf{h}.\varphi_\epsilon\,ds \\ + \frac{1}{2}\int_{\Gamma_R} (\mathcal{A}_\epsilon\gamma^R\varphi_\epsilon).\gamma^R\varphi_\epsilon\,ds$$

and the solution $\mathbf{u}_\epsilon^R$ as a minimal argument for

$$I_\epsilon^R(\mathbf{u}_\epsilon^R) = \inf_{\varphi_\epsilon \in K \subset V_\epsilon} I_\epsilon^R(\varphi_\epsilon),$$

Here lies the essence of the domain decomposition concept: we have replaced the the variable domain by a fixed one, at the price of introducing variable boundary operator $\mathcal{A}_\epsilon$. The above expressions have even simpler form in case of a single Laplace equation. It is enough to replace the displacement by the scalar function u, elasticity operator by $-\Delta$, and

$$\boldsymbol{\sigma}(\mathbf{u}) := \mathbf{grad}\,u, \qquad \varepsilon(\mathbf{u}) := \mathbf{grad}\,u, \qquad \boldsymbol{\sigma}(\mathbf{u}).\mathbf{n} := \partial u/\partial\mathbf{n}.$$

The goal is to find the expansion

$$\mathcal{A}_\epsilon = \mathcal{A} + \epsilon^d\mathcal{B} + \mathcal{R}_\epsilon, \tag{33}$$

where the remainder $\mathcal{R}_\epsilon$ is of order $o(\epsilon^d)$ in the operator norm in the space $L(\mathbf{H}^{1/2}(\Gamma_R), \mathbf{H}^{-1/2}(\Gamma_R))$, and the operator $\mathcal{B}$ is regular enough, namely it is bounded and linear:

$$\mathcal{B} \in L\big(\mathbf{L}_2(\Gamma_R), \mathbf{L}_2(\Gamma_R)\big).$$

Under this assumption the following propositions hold.

Proposition 36 *Assume that (33) holds in the operator norm. Then strong convergence takes place*

$$\mathbf{u}_\epsilon^R \to \mathbf{u}^R$$

in the norm of $\mathbf{H}^1(\Omega(R))$.

Proposition 37 *The energy functional has the representation*

$$I_\epsilon^R(\mathbf{u}_\epsilon^R) = I^R(\mathbf{u}^R) + \epsilon^d\langle\mathcal{B}(\mathbf{u}^R), \mathbf{u}^R\rangle_R + o(\epsilon^3),$$

where $o(\epsilon^d)/\epsilon^d \to 0$ *with* $\epsilon \to 0$ *in the same energy norm.*

Here $I^R(\mathbf{u}^R)$ denotes the functional I_ϵ^R on the intact domain, i.e. $\epsilon := 0$ and $\mathcal{A}_\epsilon := \mathcal{A}$, applied to truncation of $\mathbf{u}$. Generally, the energy correction for both the elasticity system and the Laplace operator has the form

$$\langle \mathcal{B}(\mathbf{u}^R), \mathbf{u}^R \rangle_R = -c_d e_u(\mathcal{O}),$$

where $c_d = \mathrm{vol}(B(1))$. The energy-like density function $e_u(\mathcal{O})$ has the form:

- In case of the Laplace operator

$$e_u(\mathcal{O}) = \frac{1}{2}\|\nabla u^R(\mathcal{O})\|^2$$

  for both $d = 2$ and $d = 3$, see [56].
- In case of the elasticity system

$$e_u(\mathcal{O}) = \frac{1}{2}\mathbb{P}\boldsymbol{\sigma}(\mathbf{u}^R(\mathcal{O}) : \varepsilon(\mathbf{u}^R(\mathcal{O}),$$

  where for $d = 2$ and plain stress

$$\mathbb{P} = \frac{1}{1-\nu}(4\mathbb{I} - \mathrm{I} \otimes \mathrm{I})$$

  and for $d = 3$

$$\mathbb{P} = \frac{1-\nu}{7-5\nu}\left(10\mathbb{I} - \frac{1-5\nu}{1-2\nu}\mathrm{I} \otimes \mathrm{I}\right)$$

  see [46, 55]. Here $\mathbb{I}$ is the fourth order identity tensor, and I is the second order identity tensor.

This approach is important for variational inequalities since it allows us to derive the formulas for topological derivatives which are similar to the expressions obtained for the corresponding linear boundary value problems.

Explicit form of the operator $\mathcal{B}$—the Laplace operator in two spatial dimensions. If the function u is harmonic in a ball $B(R) \subset \mathbb{R}^2$, of radius $R > 0$ and centre at $\mathbf{x}_0 = \mathcal{O}$, then the exact expressions for the first order derivatives of u take on the following form [56]

$$u_{/1}(\mathcal{O}) = \frac{1}{\pi R^3}\int_{\Gamma_R} u \cdot x_1 \, ds,$$
$$u_{/2}(\mathcal{O}) = \frac{1}{\pi R^3}\int_{\Gamma_R} u \cdot x_2 \, ds.$$

Since the line integrals on Γ_R are well defined for functions in $L_2(\Gamma_R)$, it follows that the operator $\mathcal{B}$ can be extended to a bounded operator on $L_2(\Gamma_R)$,

$$\mathcal{B} \in \mathcal{L}(L_2(\Gamma_R) \to L_2(\Gamma_R)).$$

The symmetric bilinear form for this operator, given by

$$\langle \mathcal{B}u, v\rangle_R = -\tfrac{1}{2\pi R^6}\left[\left(\int_{\Gamma_R} ux_1\, ds\right)\left(\int_{\Gamma_R} vx_1\, ds\right) + \left(\int_{\Gamma_R} ux_2\, ds\right)\left(\int_{\Gamma_R} vx_2\, ds\right)\right],$$

is continuous for all $u, v \in L_2(\Gamma_R)$. In fact, the bilinear form

$$L_2(\Gamma_R) \times L_2(\Gamma_R) \ni (u, v) \mapsto b(\Gamma_R; u, v) \in \mathbb{R}$$

is continuous with respect to the weak convergence because of the simple structure

$$b(\Gamma_R; u, v) = l_1(u)l_1(v) + l_2(u)l_2(v) \quad u, v \in L_1(\Gamma_R)$$

with two linear forms $v \to l_i(v)$, $i = 1, 2$,

$$l_i(u) \;=\; \frac{1}{\sqrt{2\pi}} R^{-3} \int_{\Gamma_R} ux_i\, ds$$

defined as line integrals on Γ_R. This gives an additional regularity for the regular non-local perturbation $\mathcal{B}$ of the pseudo-differential Steklov-Poincaré boundary operator $\mathcal{A}_\epsilon$.

Explicit form of the operator $\mathcal{B}$—the Laplace operator in three spatial dimensions. Similarly as in two spatial dimensions, for harmonic functions in $\mathbb{R}^3$ it may be proved [56] that

$$u_{/1}(\mathcal{O}) = \frac{3}{4\pi R^4}\int_{S(R)} ux_1\, ds,$$
$$u_{/2}(\mathcal{O}) = \frac{3}{4\pi R^4}\int_{S(R)} ux_2\, ds,$$
$$u_{/3}(\mathcal{O}) = \frac{3}{4\pi R^4}\int_{S(R)} ux_3\, ds.$$

Using this one can easily write down the bilinear form

$$b(\Gamma_R; u, v) = \langle \mathcal{B}u, v\rangle_R = l_1(u)l_1(v) + l_2(u)l_2(v) + l_3(u)l_3(v)$$

where

$$l_i(u, v) \;=\; \sqrt{\frac{3}{8\pi}} R^{-4} \int_{S(R)} ux_i\, ds.$$

From the computational point of view, the effort in comparison to two spatial dimensions grows similarly as the difficulty of computing integrals over circle versus integrals over sphere.

Explicit form of the operator $\mathcal{B}$—elasticity in two spatial dimensions. Let us denote for the plain stress case

$$k = \frac{\lambda + \mu}{\lambda + 3\mu}.$$

It has been proved in [56] that the following exact formulae hold

$$\varepsilon_{11}(\mathcal{O}) + \varepsilon_{22}(\mathcal{O}) = \frac{1}{\pi R^3} \int_{\Gamma_R} (u_1 x_1 + u_2 x_2)\, ds,$$
$$\varepsilon_{11}(\mathcal{O}) - \varepsilon_{22}(\mathcal{O}) = \frac{1}{\pi R^3} \int_{\Gamma_R} \left[(1 - 9k)(u_1 x_1 - u_2 x_2) + \frac{12k}{R^2}(u_1 x_1^3 - u_2 x_2^3) \right] ds,$$
$$2\varepsilon_{12}(\mathcal{O}) = \frac{1}{\pi R^3} \int_{\Gamma_R} \left[(1 + 9k)(u_1 x_2 + u_2 x_1) - \frac{12k}{R^2}(u_1 x_2^3 + u_2 x_1^3) \right] ds.$$

These expressions are easy to compute numerically, but contain additional integrals of third powers of x_i. Therefore, strains $\varepsilon_{ij}(\mathcal{O})$ may be expressed as linear combinations of integrals over circle which have the form

$$\int_{\Gamma_R} u_i x_j\, ds, \qquad \int_{\Gamma_R} u_i x_j^3\, ds.$$

The same is true, due to Hooke's law, for stresses $\sigma_{ij}(\mathcal{O})$. They may then be substituted into expression for the operator B, yielding

$$\langle \mathcal{B}(\mathbf{u}^R), \mathbf{v}^R \rangle_R = -\frac{1}{2} c_2 \mathbb{P} \boldsymbol{\sigma}(\mathbf{u}) : \boldsymbol{\varepsilon}(\mathbf{v}).$$

These formulas are quite similar to the ones obtained for Laplace operator and easy to compute numerically.

Explicit form of the operator $\mathcal{B}$ for elasticity in three spatial dimensions. It turns out that similar situation holds in three spatial dimensions, but obtaining the formulas is more difficult. Assuming given values of $\mathbf{u}$ on Γ_R, the solution of elasticity system in $B(R)$ may be expressed as

$$\mathbf{u} = \sum_{n=0}^{\infty} [\mathbf{U}_n + (R^2 - r^2) k_n(\nu) \,\mathbf{grad}\, \mathrm{div}\, \mathbf{U}_n].$$

where $k_n(\nu) = 1/2[(3 - 2\nu)n - 2(1 - \nu)]$ and $r = \|\mathbf{x}\|$. In addition

$$\mathbf{U}_n = \frac{1}{R^n} [\mathbf{a}_{n0} d_n(\mathbf{x}) + \sum_{m=1}^{n} (\mathbf{a}_{nm} c_n^m(\mathbf{x}) + \mathbf{b}_{nm} s_n^m(\mathbf{x}))].$$

The vectors

$$\mathbf{a}_{n0} = (a_{n0}^1, a_{n0}^2, a_{n0}^3)^\top,$$

$$\mathbf{a}_{nm} = (a_{nm}^1, a_{nm}^2, a_{nm}^3)^\top,$$

$$\mathbf{b}_{nm} = (b_{nm}^1, b_{nm}^2, b_{nm}^3)^\top$$

are constant and the set of functions

$$\{d_0;\ d_1, c_1^1, s_1^1;\ d_2, c_2^1, s_2^1, c_2^2, s_2^2;\ d_3, c_3^1, s_3^1, c_3^2, s_3^2, c_3^3, s_3^3;\ \ldots\}$$

constitutes the complete system of orthonormal harmonic polynomials on Γ_R, related to Laplace spherical functions, see next paragraph. Specifically,

$$c_k^l(\mathbf{x}) = \frac{\hat{P}_k^{l,c}(\mathbf{x})}{\|\hat{P}_k^{l,c}\|_R}, \quad s_k^l(\mathbf{x}) = \frac{\hat{P}_k^{l,s}(\mathbf{x})}{\|\hat{P}_k^{l,s}\|_R}, \quad d_k = \frac{P_k((\mathbf{x})}{\|\hat{P}_k\|_R}.$$

For example,

$$c_3^2(\mathbf{x}) = \frac{1}{R^4}\sqrt{\frac{7}{240\pi}}(15x_1^2x_3 - 15x_2^2x_3),$$

If the value of $\mathbf{u}$ on Γ_R is assumed as given, then, denoting

$$\langle\phi, \psi\rangle_R = \int_{\Gamma_R} \phi\psi\, ds,$$

we have for $n \geqslant 0$, $m = 1..n$, $i = 1, 2, 3$:

$$\begin{aligned} a_{n0}^i &= R^n\langle u_i, d_n(\mathbf{x})\rangle_R, \\ a_{nm}^i &= R^n\langle u_i, c_n^m(\mathbf{x})\rangle_R, \\ b_{nm}^i &= R^n\langle u_i, s_n^m(\mathbf{x})\rangle_R. \end{aligned} \tag{34}$$

Since we are looking for $\varepsilon_{ij}(\mathcal{O})$, only the part of $\mathbf{u}$ which is linear in $\mathbf{x}$ is relevant. It contains two terms:

$$\hat{\mathbf{u}} = \mathbf{U}_1 + R^2k_3(\nu)\,\mathbf{grad}\,\mathrm{div}\,\mathbf{U}_3.$$

For any $f(\mathbf{x})$, $\mathbf{grad}\,\mathrm{div}\,(\mathbf{a}f) = H(f)\cdot\mathbf{a}$, where $\mathbf{a}$ – constant vector and $H(f)$ is the Hessian matrix of f. Therefore

$$\hat{\mathbf{u}} = \frac{1}{R}[\mathbf{a}_{10}d_1(\mathbf{x}) + \mathbf{a}_{11}c_1^1(\mathbf{x}) + \mathbf{b}_{11}s_1^1(\mathbf{x}))]$$
$$+ R^2 k_3(\nu)\frac{1}{R^3}\left[H(d_3)(\mathbf{x})\mathbf{a}_{30} + \sum_{m=1}^{3}\left(H(c_3^m)(\mathbf{x})\mathbf{a}_{3m} + H(s_3^m)(\mathbf{x})\mathbf{b}_{3m}\right)\right]$$

From the above we may single out the coefficients standing at x_1, x_2, x_3 in u_1, u_2, u_3. For example,

$$\varepsilon_{11}(\mathcal{O}) = \frac{1}{R^3}\sqrt{\frac{3}{4\pi}}a_{11}^1 + \frac{1}{R^5}k_3(\nu)\left[-3\sqrt{\frac{7}{4\pi}}a_{30}^3 - 9\sqrt{\frac{7}{24\pi}}a_{31}^1\right.$$
$$\left.-3\sqrt{\frac{7}{24\pi}}b_{31}^2 + 30\sqrt{\frac{7}{240\pi}}a_{32}^3 + 90\sqrt{\frac{7}{1440\pi}}a_{33}^1 + 90\sqrt{\frac{7}{1440\pi}}b_{33}^2\right],$$

$$\varepsilon_{12}(\mathcal{O}) = \frac{1}{R^3}\sqrt{\frac{3}{4\pi}}(b_{11}^1 + a_{11}^2) + \frac{1}{R^5}k_3(\nu)\left[-3\sqrt{\frac{7}{24\pi}}a_{31}^2 - \sqrt{\frac{7}{24\pi}}b_{31}^1\right.$$
$$\left.+15\sqrt{\frac{7}{60\pi}}b_{32}^3 - 90\sqrt{\frac{7}{1440\pi}}a_{33}^2 + 90\sqrt{\frac{7}{1440\pi}}b_{33}^1\right].$$

Observe that

$$\varepsilon_{11}(\mathcal{O}) + \varepsilon_{22}(\mathcal{O}) + \varepsilon_{33}(\mathcal{O}) = \frac{1}{R^3}\sqrt{\frac{3}{4\pi}}\left(R\langle u_1, c_1^1\rangle_R + R\langle u_2, s_1^1\rangle_R + R\langle u_3, d_1\rangle_R\right)$$

and $c_1^1 = \frac{1}{R^2}\sqrt{\frac{3}{4\pi}}x_1, s_1^1 = \frac{1}{R^2}\sqrt{\frac{3}{4\pi}}x_2, d_1 = \frac{1}{R^2}\sqrt{\frac{3}{4\pi}}x_3$, exactly the same as for the case of Laplace equation. This should be expected, since tr $\boldsymbol{\varepsilon}$ is a harmonic function.

As a result, the operator **B** may be defined by the formula

$$\langle \mathcal{B}\mathbf{u}, \mathbf{u}\rangle_R = -c_3\mathbb{P}\boldsymbol{\sigma}(\mathbf{u}((\mathcal{O}))) : \boldsymbol{\varepsilon}(\mathbf{u}((\mathcal{O})))$$

but the right-hand side consists of integrals of **u** multiplied by first and third order polynomials in x_i over Γ_R resulting from (34). This is a very similar situation as in two spatial dimensions. Thus, the new expressions for strains make possible to rewrite $\mathcal{B}$ in the form possessing the desired regularity.

Laplace spherical polynomials. For $n = 1$:

$$\hat{P}_1(\mathbf{x}) = x_3, \quad \hat{P}_1^{1,c}(\mathbf{x}) = x_1, \quad \hat{P}_1^{1,s}(\mathbf{x}) = x_2,$$
$$\|\hat{P}_1\|_R = \|\hat{P}_1^{1,c}\|_R = \|\hat{P}_1^{1,s}\|_R = R^2\sqrt{\frac{4\pi}{3}},$$

and for $n = 3$:

$$\hat{P}_3(\mathbf{x}) = x_3^3 - \frac{3}{2}x_2^2 x_3 - \frac{3}{2}x_1^2 x_3, \qquad \|\hat{P}_3\|_R = R^4\sqrt{\frac{4\pi}{7}},$$

$$\hat{P}_3^{1,c}(\mathbf{x}) = 6x_1 x_3^2 - \frac{3}{2}x_1^3 - \frac{3}{2}x_1 x_2^2, \qquad \|\hat{P}_3^{1,c}\|_R = R^4\sqrt{\frac{24\pi}{7}},$$

$$\hat{P}_3^{1,s}(\mathbf{x}) = 6x_2 x_3^2 - \frac{3}{2}x_2^3 - \frac{3}{2}x_1^2 x_2, \qquad \|\hat{P}_3^{1,s}\|_R = R^4\sqrt{\frac{24\pi}{7}},$$

$$\hat{P}_3^{2,c}(\mathbf{x}) = 15x_1^2 x_3 - 15x_2^2 x_3, \qquad \|\hat{P}_3^{2,c}\|_R = R^4\sqrt{\frac{240\pi}{7}},$$

$$\hat{P}_3^{2,s}(\mathbf{x}) = 15x_1 x_2 x_3, \qquad \|\hat{P}_3^{2,s}\|_R = R^4\sqrt{\frac{60\pi}{7}},$$

$$\hat{P}_3^{3,c}(\mathbf{x}) = 15x_1^3 - 45x_1 x_2^2, \qquad \|\hat{P}_3^{3,c}\|_R = R^4\sqrt{\frac{1440\pi}{7}},$$

$$\hat{P}_3^{3,s}(\mathbf{x}) = 45x_1^2 x_2 - 15x_2^3, \qquad \|\hat{P}_3^{3,s}\|_R = R^4\sqrt{\frac{1440\pi}{7}},$$

8 Asymptotic Analysis of Steklov-Poincaré Operators in Reinforced Rings in Two Spatial Dimensions

In this section the similar asymptotic analysis of elliptic boundary value problems in subdomain $\Omega_R \in \mathbb{R}^2$ is performed, but we modify the situation, assuming that the hole is filled only partially, different material constituting a fixed part of it. In this way, we may consider double asymptotic transition, where both the size of the hole, as well as the proportion of the different material contained in it can vary. Mechanically this situation corresponds e.g. to the hole with hardened walls.

The analysis is based again on exact representation of solutions and allows to obtain the perturbation of solutions, using the fact that these solutions may be considered as minimizers of energy functional. The method is also suitable for double asymptotic expansions of solutions as well as energy form. The ultimate goal is to use obtained formulas in the evaluation of topological derivatives for elliptic boundary value problems.

8.1 Model Problem

Let us consider the the domain Ω containing the hole with boundary made of modified material. For simplicity the hole is located at the origin of the coordinate system. In order to write down the model problem, we introduce some notations.

$$\begin{aligned} B_s &= \{ x \in \mathbb{R}^2 \mid \|x\| < s \} \\ C_{s,t} &= \{ x \in \mathbb{R}^2 \mid s < \|x\| < t \} \\ \Gamma_s &= \{ x \in \mathbb{R}^2 \mid \|x\| = s \} \\ \Omega_s &= \Omega \setminus B_s \end{aligned}$$

Then the problem in the intact domain Ω has the form

$$\begin{aligned} k_1 \Delta w_0 &= 0 \quad \text{in} \;\; \Omega \\ w_0 &= g_0 \quad \text{on} \;\; \partial\Omega \end{aligned} \tag{35}$$

The model problem in the modified domain reads:

$$\begin{aligned} k_1 \Delta w_\rho &= 0 \quad \text{in} \;\; \Omega_\rho \\ w_\rho &= g_0 \quad \text{on} \;\; \partial\Omega \\ w_\rho &= v_\rho \quad \text{on} \;\; \Gamma_\rho \\ k_2 \Delta v_\rho &= 0 \quad \text{in} \;\; C_{\lambda\rho,\rho} \end{aligned} \tag{36}$$

$$\begin{aligned} k_2 \frac{\partial v_\rho}{\partial n_2} &= 0 \quad \text{on} \;\; \Gamma_{\lambda\rho} \\ k_1 \frac{\partial w_\rho}{\partial n_1} + k_2 \frac{\partial v_\rho}{\partial n_2} &= 0 \quad \text{on} \;\; \Gamma_\rho, \end{aligned}$$

where n_1—exterior normal vector to Ω_ρ, n_2—exterior normal vector to $C_{\lambda\rho,\rho}$, and $0 < \lambda < 1$. We want to investigate the influence of the small ring-like inclusion made of another material on the difference $w_\rho - w_0$ in Ω_R, where Γ_R surrounds $C_{\lambda\rho,\rho}$ and R is fixed. We assume that $\rho \to 0+$ and λ is considered temporarily constant.

If we define

$$u_\rho = \begin{cases} w_\rho & \text{in} \;\; \Omega_\rho \\ v_\rho & \text{in} \;\; C_{\lambda\rho,\rho} \end{cases}$$

then the problem (36) reduces to finding minimum of the energy functional

$$\mathcal{E}_1(u_\rho) = \frac{1}{2} \int_{\Omega_\rho} k_1 \nabla u_\rho \cdot \nabla u_\rho \, dx + \frac{1}{2} \int_{C_{\lambda\rho,\rho}} k_2 \nabla u_\rho \cdot \nabla u_\rho \, dx$$

for $u_\rho \in H^1(\Omega_\rho)$, $u_\rho = g_0$ on $\partial\Omega$. This expression may be rewritten as

$$\begin{aligned}\mathcal{E}_1(u_\rho) = \frac{1}{2}\int_{\Omega_R} k_1 \nabla w_\rho \cdot \nabla w_\rho \, dx \\ + \frac{1}{2}\int_{C_{\rho,R}} k_1 \nabla w_\rho \cdot \nabla w_\rho \, dx \\ + \frac{1}{2}\int_{C_{\lambda\rho,\rho}} k_2 \nabla v_\rho \cdot \nabla v_\rho \, dx.\end{aligned}$$

Using integration by parts we obtain

$$\begin{aligned}\mathcal{E}_1(u_\rho) = \frac{1}{2}\int_{\Omega_R} k_1 \nabla w_\rho \cdot \nabla w_\rho \, dx \\ + \frac{1}{2}\int_{\Gamma\rho} \left(w_\rho k_1 \frac{\partial w_\rho}{\partial n_1} + v_\rho k_2 \frac{\partial v_\rho}{\partial n_2} \right) ds \\ + \frac{1}{2}\int_{\Gamma_R} k_1 w_\rho \frac{\partial w_\rho}{\partial n_3} \, ds,\end{aligned}$$

where n_3—exterior normal to Ω_R. Hence, due to boundary and transmission condition,

$$\mathcal{E}_1(u_\rho) = \frac{1}{2}\int_{\Omega_R} k_1 \nabla w_\rho \cdot \nabla w_\rho \, dx + \frac{1}{2}\int_{\Gamma_R} k_1 w_\rho \frac{\partial w_\rho}{\partial n_3} \, ds \tag{37}$$

8.2 Steklov-Poincaré Operator

Observe that $\mathcal{E}_1(w_0)$ corresponds to the problem (35). Therefore the main goal is to find the Steklov-Poincaré operator

$$\mathcal{A}_{\lambda,\rho} \; : \; w \in H^{1/2}(\Gamma_R) \;\longmapsto\; \frac{\partial w_\rho}{\partial n_3} \in H^{-1/2}(\Gamma_R)$$

where the normal derivative is computed from auxiliary problem

$$\begin{aligned} k_1 \Delta w_\rho &= 0 \quad \text{in} \;\; C_{\rho,R} \\ w_\rho &= w \quad \text{on} \;\; \Gamma_R \\ w_\rho &= v_\rho \quad \text{on} \;\; \Gamma_\rho \\ k_2 \Delta v_\rho &= 0 \quad \text{in} \;\; C_{\lambda\rho,\rho} \\ k_2 \frac{\partial v_\rho}{\partial n_2} &= 0 \quad \text{on} \;\; \Gamma_{\lambda\rho} \\ k_1 \frac{\partial w_\rho}{\partial n_1} + k_2 \frac{\partial v_\rho}{\partial n_2} &= 0 \quad \text{on} \;\; \Gamma_\rho \end{aligned}$$

The geometry of domains of definition for functions is shown in Fig. 2. Now let us adopt the polar coordinate system around origin and assume the Fourier series form for w on Γ_R.

$$w = C_0 + \sum_{k=1}^{\infty} (A_k \cos k\varphi + B_k \sin k\varphi)$$

The general form of the solution w_ρ is

$$w_\rho = A^w + B^w \log r + \sum_{k=1}^{\infty} \left(w_k^c(r) \cos k\varphi + w_k^s(r) \sin k\varphi \right),$$

where

$$w_k^c(r) = A_k^c r^k + B_k^c \frac{1}{r^k}, \quad w_k^s(r) = A_k^s r^k + B_k^s \frac{1}{r^k}.$$

Similarly for v_ρ:

$$v_\rho = A^v + B^v \log r + \sum_{k=1}^{\infty} \left(v_k^c(r) \cos k\varphi + v_k^s(r) \sin k\varphi \right),$$

where

$$v_k^c(r) = a_k^c r^k + b_k^c \frac{1}{r^k}, \quad v_k^s(r) = a_k^s r^k + b_k^s \frac{1}{r^k}.$$

Additionally, we denote the Fourier expansion of v_ρ on Γ_ρ by

$$v_\rho = c_0 + \sum_{k=1}^{\infty} (a_k \cos k\varphi + b_k \sin k\varphi)$$

From boundary conditions on $\Gamma_{\lambda\rho}$ it follows easily $B^v = 0$, $A^v = c_0$, and then $B^w = 0$, $A^w = A^v = c_0 = C_0$. There remains to find a_k, b_k, a_k^c, b_k^c, a_k^s, b_k^s, A_k^c, B_k^c, A_k^s, B_k^s assuming A_k, B_k as given.

8.3 Asymptotic Expansion

In order to eliminate the above mentioned coefficients we consider first the terms at $\cos k\varphi$. From boundary and transmission conditions we have for $k = 1, 2, \ldots$

$$A_k^c R^k + B_k^c \frac{1}{R^k} = A_k$$
$$A_k^c \rho^k + B_k^c \frac{1}{\rho^k} - a_k = 0$$
$$a_k^c \rho^k + b_k^c \frac{1}{\rho^k} - a_k = 0$$
$$a_k^c (\lambda\rho)^{k-1} - b_k^c \frac{1}{(\lambda\rho)^{k+1}} = 0$$
$$k_1 A_k^c \rho^{k-1} - k_1 B_k^c \frac{1}{\rho^{k+1}} - k_2 a_k^c \rho^{k-1} + k_2 b_k^c \frac{1}{\rho^{k+1}} = 0$$

This may be rewritten in the matrix form: grouping unknown parameters into a vector $\mathbf{p}_k = [A_k^c, B_k^c, a_k^c, b_k^c, a_k]^\top$ we obtain

$$T(k_1, k_2, R, \lambda, \rho)\mathbf{p}_k = R^k A_k \mathbf{e}_1$$

where

$$T = \begin{bmatrix} R^{2k} & 1 & 0 & 0 & 0 \\ \rho^{2k} & 1 & 0 & 0 & -\rho^k \\ 0 & 0 & (\lambda\rho)^{2k} & 1 & -\rho^k \\ 0 & 0 & (\lambda\rho)^{2k} & -1 & 0 \\ k_1\rho^{2k} & -k_1 & -k_2\rho^{2k} & k_2 & 0 \end{bmatrix}$$

where $\mathbf{e}_1 = [0, 0, 0, 0, 1]^\top$. It is easy to see that

$$\mathbf{p}_k = \mathbf{p}_k^0 A_k + \rho^{2k} \mathbf{p}_k^1 A_k + o(\rho^{2k})$$

where

$$\mathbf{p}_k^0 = \lim_{\rho\to 0+} \lim_{\lambda\to 0+} \frac{\mathbf{p}_k(k_1, k_2, R, \lambda, \rho)}{A_k}$$

and $\mathbf{p}_k^0 = [1/R^k, 0, 0, 0, 0]^\top$, which corresponds to the ball B_R filled completely with material k_1. Similar reasoning may be conducted for terms containing $\sin k\varphi$.

As a result,

$$\mathcal{A}_{\lambda,\rho} = \mathcal{A}_{0,0} + \rho^2 \mathcal{A}^1_{\lambda,\rho}(k_1, k_2, R, \lambda, \rho, A_1, B_1) + o(\rho^2).$$

The exact form of $\mathcal{A}^1_{\lambda,\rho}(k_1, k_2, R, \lambda, \rho, A_1, B_1)$ is obtained from inversion of matrix T, but, what is crucial, it is linear in both A_1 and B_1. They in turn are computed as line integrals

$$A_1(w) = \frac{1}{\pi R^2} \int_{\Gamma_R} w x_1 \, ds, \quad B_1(w) = \frac{1}{\pi R^2} \int_{\Gamma_R} w x_2 \, ds.$$

As a result, for computing u_ρ we may use the following energy form

$$\mathcal{E}(u_\rho) = \frac{1}{2} \int_\Omega k_1 \nabla u_\rho \cdot \nabla u_\rho \, dx + \rho^2 Q(k_1, k_2, R, \lambda, \rho, A_1, B_1) + o(\rho^2),$$

where $A_1 = A_1(u_\rho)$, $B_1 = B_1(u_\rho)$ and Q is a quadratic function of A_1, B_1. This constitutes a regular perturbation of the energy functional which allows computing perturbations of any functional depending on this solution and caused by small inclusion of the described above form.

8.4 Extension to Linear Elasticity

Let us consider the plane elasticity problem in the ring $C_{R,\rho}$. We use polar coordinates (r, θ) with $\mathbf{e}_r$ pointing outwards and $\mathbf{e}_\theta$ perpendicularly in the counter-clockwise direction. Then there exists an exact representation of both solutions, using the complex variable series. It has the form [20, 38]

$$\begin{aligned} \sigma_{rr} - i\sigma_{r\theta} &= 2\Re\phi' - e^{2i\theta}(\bar{z}\phi'' + \psi') \\ \sigma_{rr} + i\sigma_{\theta\theta} &= 4\Re\phi' \\ 2\mu(u_r + iu_\theta) &= e^{-i\theta}(\kappa\phi - z\bar{\phi}' - \bar{\psi}). \end{aligned} \tag{38}$$

The functions ϕ, ψ are given by complex series

$$\begin{aligned} \phi &= A\log(z) + \sum_{k=-\infty}^{k=+\infty} a_k z^k \\ \psi &= -\kappa\bar{A}\log(z) + \sum_{k=-\infty}^{k=+\infty} b_k z^k. \end{aligned} \tag{39}$$

Here μ—the Lame constant, ν—the Poisson ratio, $\kappa = 3 - 4\nu$ in the plain strain case, and $\kappa = (3 - \nu)/(1 + \nu)$ for plane stress.

Similarly as in the simple case described in former sections, the displacement data may be given in the form of Fourier series,

$$2\mu(u_r + iu_\theta) = \sum_{k=-\infty}^{k=+\infty} A_k e^{ik\theta}$$

The traction-free condition on some circle means $\sigma_{rr} = \sigma_{r\theta} = 0$. From (38) and (39) we get for displacements the formula

$$2\mu(u_r + iu_\theta) = 2\kappa Ar\log(r)\frac{1}{z} - \bar{A}\frac{1}{r}z + \sum_{p=-\infty}^{p=+\infty}\left[\kappa r a_{p+1} - (1-p)\bar{a}_{1-p}r^{-2p+1} - \bar{b}_{-(p+1)}r^{-2p-1}\right]z^p.$$

Similarly we obtain representation of tractions on some circle

$$\sigma_{rr} - i\sigma_{r\theta} = 2A\frac{1}{z} + (\kappa+1)\frac{1}{r^2}\bar{A}z + \sum_{p=-\infty}^{p=+\infty}(1-p)\left[(1+p)a_{p+1} + \bar{a}_{1-p}r^{-2p} + \frac{1}{r^2}b_{p-1}\right]z^p.$$

As we see, in principle it is possible to repeat the same procedure again, glueing solutions in two rings together and eliminating the intermediary Dirichlet data on the interface. The only difference lies in considerably more complicated calculations, see e.g. [13]. This could be applied for making double asymptotic expansion, in term of both ρ and λ. However, in our case λ does not need to be small in comparison to ρ.

8.5 Summary of Results for Particular Cases

The explicit form of solutions in B_R allows us to conclude that for

$$\|w_\rho\|_{H^{1/2}(\Gamma_R)} \leqslant \Lambda_0$$

the correction to the energy functional contains part proportional to ρ^d and the remainder of order $o(\rho^d)$. This in turn [56, 58] implies the possibility of representation

$$w_\rho = w_0 + \rho^2 q + o(\rho^2) \quad \text{in} \quad H^1(\Omega_R)$$

for both standard and contact problems, justifying computations of topological derivatives. It is well known that the singularities of solutions to Partial Differential Equations due to the singularities of geometrical domains can be characterized by specific shape derivatives of the associated energy shape functionals [12]. Therefore, the influence of topological changes in domains on the singularities can be measured by the appropriate second-order topological derivatives of the energy functionals. It means that we evaluate the shape derivatives of the energy functional by using the velocity field method, and subsequently the second order topological derivatives of the functionals by an application of the domain decomposition method,

- the portion Γ_0 of the boundary with the homogeneous Dirichlet boundary conditions is deformed to obtain $t \to T_t(V)(\Gamma_0)$ as well as $t \to \mathcal{E}(\Omega_t)$ for the energy shape functional; as a result the first order shape derivative $J(\Omega) := d\mathcal{E}(\Omega; V)$ is obtained in the distributed form as a volume integral.
- the second order derivative of the energy functional is evaluated with respect to small parameter $\varepsilon \to 0$, the parameter governs the size of small inclusion with the material defined by a contrast parameter $\gamma \in [0, \infty)$.

We consider the energy shape functional $\Omega \to \mathcal{E}(\Omega)$ for Signorini problems for the Laplacian as well for the frictionless contact. The shape derivative $J(\Omega) := d\mathcal{E}(\Omega; V)$ of this functional is evaluated with respect to the boundary variations of the portion $\Gamma_0 \subset \partial\Omega$. In another words the velocity vector field V is supported in a small neighbourhood of Γ_0. The topological derivatives of $J(\Omega)$ are evaluated with respect to nucleation of small inclusions far from Γ_0. The domain decomposition method is applied in order to obtain the robust expressions for topological derivatives.

9 Conclusions

In the paper the review of mathematical techniques required for shape-topological sensitivity analysis for variational inequalities is presented. The singular geometrical perturbations depending on small parameter $\epsilon \to 0$ are considered. It is shown that the singular geometrical domain perturbations can be replaced, without any loss of precision, by the regular perturbations of bilinear forms depending on the small parameter. Non-smooth analysis is employed in order to obtain the second order topological derivatives. The proposed method can be now used in numerical methods of topology optimization as well in passive control of crack propagation.

Acknowledgments This work has been supported by the DFG EC315 'Engineering of Advanced Materials' and by the ANR-12- BS01-0007 Optiform.

References

1. A. Ancona, Sur les espaces de Dirichlet: principes, fonction de Green. J. Math. Pures Appl. **54**, 75–124 (1975)
2. I.I. Argatov, J. Sokołowski, Asymptotics of the energy functional in the Signorini problem under small singular perturbation of the domain. Zh. Vychisl. Mat. Mat. Fiz. **43**, 744–758, translation in Comput. Math. Math. Phys. **43**, 710–724 (2003)
3. Z. Belhachmi, J.-M. Sac-Epée, J. Sokołowski, Approximation par la méthode des élement finit de la formulation en domaine régulière de problems de fissures. Comptes Rendus Acad. Sci. Paris, Ser. I **338**, 499–504 (2004)

4. Z. Belhachmi, J.M. Sac-Epée, J. Sokołowski, Mixed finite element methods for smooth domain formulation of crack problems. SIAM J. Numer. Anal. **43**, 1295–1320 (2005)
5. A. Beurling, J. Deny, Dirichlet spaces. Proc. Natl. Acad. Sci. USA **45**, 208–215 (1959)
6. P. Destuynder, Remarques sur le contrôle de la propagation des fissures en régime stationnaire. Comptes Rendus Acad. Sci. Paris Sèr. II Méc. Phys. Chim. Sci. Univers Sci. Terre **308**, 697–701 (1989)
7. G. Fichera, Boundary value problems of elasticity with unilateral constraints, in *Festkörpermechanik/Mechanics of Solids, Handbuch der Physik*, ed. by S. Flügge, C.A. Truesdell (Springer, New York, 1984), pp. 391–424
8. G. Fichera, Existence theorems in elasticity, in *Festkörpermechanik/Mechanics of Solids, Handbuch der Physik*, ed. by S. Flügge, C.A. Truesdell (Springer, Berlin, 1984), pp. 347–389
9. G. Frémiot, Eulerian semiderivatives of the eigenvalues for Laplacian in domains with cracks. Adv. Math. Sci. Appl. **12**, 115–134 (2002)
10. G. Frémiot, J. Sokołowski, Hadamard formula in nonsmooth domains and applications, in *Partial Differential Equations on Multistructures (Luminy, 1999)*. Lecture Notes in Pure and Applied Mathematics, vol. 219 (Dekker, New York, 2001), pp. 99–120
11. G. Frémiot, J. Sokołowski, Shape sensitivity analysis of problems with singularities, in *Shape Optimization and Optimal Design (Cambridge, 1999)*. Lecture Notes in Pure and Applied Mathematics, vol. 216 (Dekker, New York, 2001), pp. 255–276
12. G. Frémiot, W. Horn, A. Laurain, M. Rao, J. Sokołowski, On the analysis of boundary value problems in nonsmooth domains. Diss. Math. **462**, 149 (2009)
13. W.A. Gross, The second fundamental problem of elasticity applied to a plane circular ring. Z. für Angew. Math. Phys. **8**, 71–73 (1957)
14. B. Hanouzet, J.-L. Joly, Méthodes d'ordre dans l'interprétation de certaines inéquations variationnelles et applications. J. Funct. Anal. **34**, 217–249 (1979)
15. A. Haraux, How to differentiate the projection on a convex set in Hilbert space. Some applications to variational inequalities. J. Math. Soc. Jpn. **29**, 615–631 (1977)
16. P. Hild, A. Münch, Y. Ousset, On the active control of crack growth in elastic media. Comput. Methods Appl. Mech. Eng. **198**, 407–419 (2008)
17. M. Hintermüller, V.A. Kovtunenko, From shape variation to topology changes in constrained minimization: a velocity method based concept, in Special issue on advances in shape and topology optimization: theory, numerics and new applications areas, ed. by C. Elliott, M. Hintermüller, G. Leugering, J. Sokołowski, *Optimization Methods and Software*, vol. 26 (2011), pp. 513–532
18. M. Hintermüller, A. Laurain, Optimal shape design subject to variational inequalities. SIAM J. Control Optim. **49**, 1015–1047 (2011)
19. D. Hömberg, A.M. Khludnev, J. Sokołowski, Quasistationary problem for a cracked body with electrothermoconductivity. Interfaces Free Bound. **3**, 129–142 (2001)
20. M. Kachanov, B. Shafiro, I. Tsukrov, *Handbook of Elasticity Solutions* (Kluwer Academic Publishers, Dordrecht, 2003)
21. A.M. Khludnev, Optimal control of crack growth in elastic body with inclusions. Eur. J. Mech. A Solids **29**, 392–399 (2010)
22. A.M. Khludnev, J. Sokołowski, *Modelling and Control in Solid Mechanics* (Birkhäuser, Basel 1997), reprinted by Springer in 2012
23. A.M. Khludnev, J. Sokołowski, On solvability of boundary value problems in elastoplasicity. Control Cybern. **27**, 311–330 (1998)
24. A.M. Khludnev, J. Sokołowski, The Griffith formula and the Rice-Cherepanov integral for crack problems with unilateral conditions in nonsmooth domains. Eur. J. Appl. Math. **10**, 379–394 (1999)
25. A.M. Khludnev, J. Sokołowski, Griffith's formula and Rice-Cherepanov's integral for elliptic equations with unilateral conditions in nonsmooth domains, in *Optimal Control of Partial Differential Equations (Chemnitz, 1998)*. International Series of Numerical Mathematics, vol. 133 (Birkhäuser, Basel, 1999), pp. 211–219

26. A.M. Khludnev, J. Sokołowski, Griffith's formulae for elasticity systems with unilateral conditions in domains with cracks. Eur. J. Mech. A Solids **19**, 105–119 (2000)
27. A.M. Khludnev, V.A. Kovtunenko, *Analysis of Cracks in Solids* (WIT Press, Southampton, 2000)
28. A.M. Khludnev, J. Sokołowski, Smooth domain method for crack problems. Q. Appl. Math. **62**, 401–422 (2004)
29. A.M. Khludnev, G. Leugering, Optimal control of cracks in elastic bodies with thin rigid inclusions. Z. Angew. Math. Mech. **91**, 125–137 (2011)
30. A.M. Khludnev, G. Leugering, M. Specovius-Neugebauer, Optimal control of inclusion and crack shapes in elastic bodies. J. Optim. Theory Appl. **155**, 54–78 (2012)
31. A.M. Khludnev, K. Ohtsuka, J. Sokołowski, On derivative of energy functional for elastic bodies with a crack and unilateral conditions. Q. Appl. Math. **60**, 99–109 (2002)
32. A.M. Khludnev, A.A. Novotny, J. Sokołowski, A. Żochowski, Shape and topology sensitivity analysis for cracks in elastic bodies on boundaries of rigid inclusions. J. Mech. Phys. Solids **57**, 1718–1732 (2009)
33. A.M. Khludnev, J. Sokołowski, K. Szulc, Shape and topological sensitivity analysis in domains with cracks. Appl. Math. **55**, 433–469 (2010)
34. G. Leugering, M. Prechtel, P. Steinmann, M. Stingl, A cohesive crack propagation model: mathematical theory and numerical solution. Commun. Pure Appl. Anal. **12**, 1705–1729 (2013)
35. T. Lewiński, J. Sokołowski, Energy change due to the appearance of cavities in elastic solids. Int. J. Solids Struct. **40**, 1765–1803 (2003)
36. F. Mignot, Contrôle dans les inéquations variationelles elliptiques. J. Funct. Anal. **22**, 130–185 (1976)
37. A. Münch, P. Pedregal, Relaxation of an optimal design problem in fracture mechanics: the anti-plane case. ESAIM Control Optim. Calc. Var. **16**, 719–743 (2010)
38. N.I. Muskhelishvili, *Some Basic Problems on the Mathematical Theory of Elasticity* (Noordhoff, Groningen, 1952)
39. S.A. Nazarov, J. Sokołowski, Asymptotic analysis of shape functionals. J. Math. Pures Appl. **82**(9), no. 2, pp. 125–196 (2003)
40. S.A. Nazarov, J. Sokołowski, Self-adjoint extensions for the Neumann Laplacian and applications. Acta Math. Sin. (Engl. Ser.) **22**(3), pp. 879–906 (2006)
41. S.A. Nazarov, J. Sokołowski, Shape sensitivity analysis of eigenvalues revisited. Control Cybern. **37**, 999–1012 (2008)
42. S.A. Nazarov, J. Sokołowski, Spectral problems in the shape optimisation. Singular boundary perturbations. Asymptot. Anal. **56**, 159–204 (2008)
43. S.A. Nazarov, J. Sokołowski, M. Specovius-Neugebauer, Polarization matrices in anisotropic heterogeneous elasticity. Asymptot. Anal. **68**, 189–221 (2010)
44. S.A. Nazarov, J. Sokołowski, On asymptotic analysis of spectral problems in elasticity. Lat. Am. J. Solids Struct. **8**, 27–54 (2011)
45. A.A. Novotny, J. Sokołowski, *Topological Derivatives in Shape Optimization*. Interaction of Mechanics and Mathematics (Springer, New York, 2013)
46. A.A. Novotny, J. Sokołowski, *Topological Derivative in Shape Optimization* (Springer, Berlin, 2013)
47. P. Plotnikov, J. Sokołowski, *Compressible Navier-Stokes Equations. Theory and Shape Optimization*. Mathematics Institute of the Polish Academy of Sciences. Mathematical Monographs (New Series), vol. 73 (Birkhäuser/Springer Basel AG, Basel, 2012)
48. M. Prechtel, P. Leiva Ronda, R. Janisch, G. Leugering, A. Hartmaier, P. Steinmann, *Cohesive Element Model for Simulation of Crack Growth in Composite Materials, International Conference on Crack Paths*, Vicenza (2009)
49. M. Prechtel, P. Leiva Ronda, R. Janisch, A. Hartmaier, G. Leugering, P. Steinmann, M. Stingl, Simulation of fracture in heterogeneous elastic materials with cohesive zone models. Int. J. Fract. **168**, 15–29 (2010)
50. M. Prechtel, G. Leugering, P. Steinmann, M. Stingl, Towards optimization of crack resistance of composite materials by adjusting of fiber shapes. Eng. Fract. Mech. **78**, 944–960 (2011)

51. V.V. Saurin, Shape design sensitivity analysis for fracture conditions. Comput. Struct. **76**, 399–405 (2001)
52. J. Sokołowski, J.-P. Zolésio, *Introduction to Shape Optimization, Shape Sensitivity Analysis*. Springer Series in Computational Mathematics, vol. 16 (Springer, Berlin, 1992)
53. J. Sokołowski, J.-P. Zolésio, *Introduction to Shape Optimization. Shape Sensitivity Analysis*, Springer Series in Computational Mathematics, vol. 16 (Springer, Berlin, 1992), reprinted in 2013
54. J. Sokołowski, A. Żochowski, On the topological derivative in shape optimization. SIAM J. Control Optim. **37**(4), 1251–1272 (1999)
55. J. Sokołowski, A. Żochowski, Optimality conditions for simultaneous topology and shape optimization. SIAM J. Control Optim. **42**(4), 1198–1221 (2003)
56. J. Sokołowski, A. Zochowski, Modelling of topological derivatives for contact problems. Numer. Math. **102**(1), 145–179 (2005)
57. J. Sokołowski, A. Żochowski, Modelling of topological derivatives for contact problems. Numer. Math. **102**, 145–179 (2005)
58. J. Sokołowski, A. Żochowski, Topological derivatives for optimization of plane elasticity contact problems. Eng. Anal. Bound. Elem. **32**, 900–908 (2008)