A Phase Field Approach for Shape and Topology Optimization in Stokes Flow

Harald Garcke and Claudia Hecht

Abstract A new formulation for shape and topology optimization in a Stokes flow is introduced. The investigated problem minimizes the total potential power of the flow. By combining a porous medium and a phase field approach we obtain a well-posed problem in a diffuse interface setting that can be reformulated into a problem without state equations. We can derive a sharp interface problem with zero permeability outside the fluid region as a Γ-limit of this porous medium—phase field problem.

Keywords Shape and topology optimization · Phase field method · Diffuse interfaces · Stokes flow · Fictitious domain · Γ-convergence

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1 Introduction

By shape optimization in fluids one generally refers to the problem of finding a shape of a fluid region, or of an obstacle inside a fluid, respectively, such that a certain objective functional is minimal. Often, one does not want to prescribe the topology of this region in advance, as one may not know how many connected components or holes of the shape are optimal for instance. There are several well-developed approaches for shape and topology optimization when it comes to finding the optimal configuration in a mixture of several conducting or elastic materials, see [4]. But even though there are numerous applications in the field of shape optimization in fluids, such as optimizing airplanes and cars, biomechanical design or several problems in the machine industry, the mathematical theory is not yet so elaborated than in other areas of shape and topology design. In industry, like in aerospace

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engineering, practical methods are quite sophisticated and one can find many mathematical contributions to those numerical methods, for instance in the field of shape sensitivity analysis. However, even basic mathematical questions like the existence of a minimizer remain open. In general, shape optimization problems are known to be not well-posed, see for instance [5], and hence several ideas have been developed in different areas to overcome this issue. Important contributions for this can be found in the field of finding optimal material configurations. Mentionable are certainly the ideas of using a perimeter penalization in optimal shape design and considering this problem in the framework of Caccioppoli sets, see [2], and of introducing a so-called ersatz material approach, see [8]. The latter replaces the void regions by a fictitious material which may be very weak for instance, see [1]. A comparable idea in a fluid dynamical setting has been proposed by [7], where the non-fluid region is replaced by a porous medium with small permeability. In this work, we extend this porous medium approach by including an additional perimeter penalization in order to arrive in a problem that can be generalised to nonlinear state equations and a large class of objective functionals. Anyhow, in this work we introduce this formulation by means of the well-known problem of minimizing the total potential power in a Stokes flow. This yields in particular a special structure of the problem where the state equations can be dropped. This is the best understood setting in shape optimization problems, see also comparable settings in material design [2, 8]. The design variable in the porous medium approach does not only take two discrete values for material and fluid, but can also have values in between and hence we obtain a diffuse interface. Consequently, also the perimeter functional is replaced by a functional, here the Ginzburg-Landau energy, corresponding to the perimeter on the diffuse interface level. The resulting porous medium—phase field problem will be introduced and discussed in more detail in Sect. 2 and can be roughly outlined as

$$\min_{(\varphi,\boldsymbol{u})} \int_\Omega \frac{1}{2}\alpha_\epsilon(\varphi)|\boldsymbol{u}|^2 + \frac{\mu}{2}|\nabla \boldsymbol{u}|^2 - \boldsymbol{f}\cdot\boldsymbol{u} + \frac{\gamma\epsilon}{2}|\nabla\varphi|^2 + \frac{\gamma}{\epsilon}\psi(\varphi)\,\mathrm{d}x$$
$$\text{subject to} \int_\Omega \alpha_\epsilon(\varphi)\,\boldsymbol{u}\cdot\boldsymbol{v} + \mu\nabla\boldsymbol{u}\cdot\nabla\boldsymbol{v}\,\mathrm{d}x = \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v}\,\mathrm{d}x \quad \forall \boldsymbol{v}.$$

This problem is formulated in more detail in (7) and we will show that it admits a minimizer. Additionally, we prove that the objective functional Γ-converges as the interfacial thickness tends to zero to a perimeter penalized sharp interface shape optimization problem where in particular the permeability of the non-fluid region is zero. The sharp interface problem, which is described in more detail in (11) and (12) in Sect. 3, is in a simplified form given as

$$\min_{(\varphi,\boldsymbol{u})} \int_{\{\varphi=1\}} \frac{\mu}{2}|\nabla\boldsymbol{u}|^2 - \boldsymbol{f}\cdot\boldsymbol{u}\,\mathrm{d}x + \gamma c_0 P_\Omega(\{\varphi=1\})$$
$$\text{subject to} \int_{\{\varphi=1\}} \mu\nabla\boldsymbol{u}\cdot\nabla\boldsymbol{v}\,\mathrm{d}x = \int_{\{\varphi=1\}} \boldsymbol{f}\cdot\boldsymbol{v}\,\mathrm{d}x \quad \forall \boldsymbol{v}.$$

2 Porous Medium—Phase Field Formulation

The investigated problem in this work is to minimize a certain objective functional, depending on the velocity of some fluid, by optimizing the shape, geometry and topology of the region which is filled with this fluid. This region can be chosen in a large class of admissible shapes but has to stay inside a given, fixed holdall container $\Omega \subset \mathbb{R}^d$ which is chosen such that

(A1) $\Omega \subseteq \mathbb{R}^d$, $d \in \{2, 3\}$, is a bounded Lipschitz domain with outer unit normal $\boldsymbol{n}$.

The velocity $\boldsymbol{u}$ and the pressure p of the fluid, whose viscosity is denoted by $\mu > 0$, are described by the Stokes equations

$$-\mu\Delta\boldsymbol{u} + \nabla p = \boldsymbol{f}, \qquad \operatorname{div}\boldsymbol{u} = 0 \tag{1}$$

inside the fluid region. We use Dirichlet boundary conditions on the boundary of Ω, hence we may prescribe some in-or outflow region on the boundary. Additionally, we allow here external forces $\boldsymbol{f}$ to act on the whole of Ω.

(A2) Let $\boldsymbol{f} \in \boldsymbol{L}^2(\Omega)$ denote the applied body force and $\boldsymbol{g} \in \boldsymbol{H}^{\frac{1}{2}}(\partial\Omega)$ the given boundary function such that $\int_{\partial\Omega} \boldsymbol{g} \cdot \boldsymbol{n}\,\mathrm{d}x = 0$.

We remark, that throughout this work $\mathbb{R}^d$-valued functions of function spaces of $\mathbb{R}^d$-valued functions are denoted by boldface letters.

The design variable describing the regions filled with fluid and the ones not filled with fluid is in general denoted by φ and is chosen in $H^1(\Omega)$. As already indicated in the introduction, we do not only allow φ to take the values that correspond to fluid regions (which means $\varphi = 1$) and non-fluid regions (hence $\varphi = -1$), but also values in between (i.e. $|\varphi| < 1$) and so we arrive in a diffuse interface setting. Additionally, we want to include a volume constraint on the design variable, so we only optimize over all $\varphi \in H^1(\Omega)$ fulfilling $\int_\Omega \varphi\,\mathrm{d}x \leq \beta|\Omega|$. The constant $\beta \in (-1, 1)$ is fixed but arbitrary and can be chosen dependent on the application. Including this constraint yields an additional upper bound on the amount of fluid that can be used during the optimization process. Hence, the admissible shapes in the optimization problem are described by all design functions inside

$$\Phi_{ad} := \left\{\varphi \in H^1(\Omega) \mid |\varphi| \leq 1 \text{ a.e. in } \Omega, \int_\Omega \varphi\,\mathrm{d}x \leq \beta\,|\Omega|\right\}. \tag{2}$$

It is a known fact, that shape optimization problems lack in general existence of a minimizer, compare for instance the discussions in [15]. One approach to overcome this problem is the so called perimeter penalization, where a multiple of the perimeter of the fluid region is added to the objective functional. This excludes oscillations and microscopic perforations, see for instance [5], and hence realizes simultaneously certain engineering constraints. As we work in a diffuse interface setting, i.e. the design variable does not only take discrete values, we do not add a multiple of perimeter functional to the objective functional but merely a multiple of the Ginzburg-Landau energy, namely

$$\gamma \int_{\Omega} \frac{\epsilon}{2} |\nabla\varphi|^2 + \frac{1}{\epsilon}\psi(\varphi)\,\mathrm{d}x,$$

since this energy is known to be a diffuse interface approximation of a multiple of the perimeter functional, see for instance [16]. Here, $\gamma > 0$ is an arbitrary constant which can be considered as a weighting parameter for the perimeter penalization and $\psi : \mathbb{R} \to \overline{\mathbb{R}} := \mathbb{R} \cup \{+\infty\}$ is a potential with two global minima at ± 1. In this work we choose a double obstacle potential, hence

$$\psi(\varphi) := \begin{cases} \frac{1}{2}\left(1-\varphi^2\right), & \text{if } |\varphi| \le 1, \\ +\infty, & \text{otherwise.} \end{cases}$$

This gives rise to a so-called phase field problem where the phase field variable is given by the design function φ and the phase field parameter $\epsilon > 0$ describes the interface thickness. To be precise, the thickness of the interface is proportional to the small parameter $\epsilon > 0$.

Similar to [7], we replace the region outside the fluid by a porous medium with small permeability $(\overline{\alpha}_\epsilon)^{-1} > 0$. Thus we couple the permeability to the phase field parameter $\epsilon > 0$. The velocity $\boldsymbol{u}$ of the fluid in this porous medium is then, due to Darcy's law, described by

$$\overline{\alpha}_\epsilon \boldsymbol{u} - \mu\Delta\boldsymbol{u} + \nabla p = \boldsymbol{f}, \qquad \operatorname{div} \boldsymbol{u} = 0, \tag{3}$$

where p denotes the corresponding pressure. In the interfacial region we interpolate between the equations of flow through porous medium (3) and the Stokes equations (1) by using an interpolation function $\alpha_\epsilon : [-1, 1] \to [0, \overline{\alpha}_\epsilon]$ fulfilling the following assumptions:

(A3) Let $\alpha_\epsilon : [-1, 1] \to [0, \overline{\alpha}_\epsilon]$ be decreasing, surjective and continuous for every $\epsilon > 0$.
It is required that $\overline{\alpha}_\epsilon > 0$ is chosen such that $\lim_{\epsilon \searrow 0} \overline{\alpha}_\epsilon = +\infty$ and α_ϵ converges pointwise to some function $\alpha_0 : [-1, 1] \to [0, +\infty]$. Additionally, we impose $\alpha_\delta(x) \ge \alpha_\epsilon(x)$ if $\delta \le \epsilon$ for all $x \in [-1, 1]$, $\lim_{\epsilon \searrow 0} \alpha_\epsilon(0) < \infty$ and a growth condition of the form $\overline{\alpha}_\epsilon = \mathrm{o}\left(\epsilon^{-\frac{2}{3}}\right)$.

Remark 1 For space dimension $d = 2$ we can even choose $\overline{\alpha}_\epsilon = \mathrm{o}\left(\epsilon^{-\kappa}\right)$ for any $\kappa \in (0, 1)$, compare also the proof of Theorem 2.

The complete state equations for our problem can be written in its strong form as

$$\alpha_\epsilon(\varphi)\boldsymbol{u} - \mu\Delta\boldsymbol{u} + \nabla p = \boldsymbol{f} \qquad \text{in } \Omega, \tag{4a}$$

$$\operatorname{div}\boldsymbol{u} = 0 \qquad \text{in } \Omega, \tag{4b}$$

$$\boldsymbol{u} = \boldsymbol{g} \qquad \text{on } \partial\Omega. \tag{4c}$$

The weak formulation of this system is given as follows: find $\boldsymbol{u} \in \boldsymbol{U} := \{\boldsymbol{v} \in \boldsymbol{H}^1(\Omega) \mid \operatorname{div}\boldsymbol{v} = 0, \boldsymbol{v}|_{\partial\Omega} = \boldsymbol{g}\}$ such that

$$\int_\Omega \alpha_\epsilon(\varphi)\,\boldsymbol{u}\cdot\boldsymbol{v} + \mu\nabla\boldsymbol{u}\cdot\nabla\boldsymbol{v}\,\mathrm{d}x = \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v}\,\mathrm{d}x \quad \forall\boldsymbol{v}\in\boldsymbol{V} \tag{5}$$

with $\boldsymbol{V} := \{\boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega) \mid \operatorname{div}\boldsymbol{v} = 0\}$.

As mentioned above, our goal is to minimize the total potential power

$$\int_\Omega \frac{1}{2}\alpha_\epsilon(\varphi)\,|\boldsymbol{u}|^2 + \frac{\mu}{2}\,|\nabla\boldsymbol{u}|^2 - \boldsymbol{f}\cdot\boldsymbol{u}\,\mathrm{d}x \tag{6}$$

of the fluid. The first term in (6) can also be considered as a penalization term ensuring that $|\boldsymbol{u}|$ is small enough outside the fluid region (i.e. $\varphi = -1$), and vanishes in the limit $\epsilon \searrow 0$. In the sharp interface problem (hence "$\epsilon = 0$") the fact of $\boldsymbol{u}$ vanishing outside the fluid region is essential, compare Sect. 3.

We finally arrive in a porous medium—phase field formulation of the shape optimization problem:

$$\begin{aligned}\min_{(\varphi,\boldsymbol{u})} J_\epsilon(\varphi,\boldsymbol{u}) := &\int_\Omega \frac{1}{2}\alpha_\epsilon(\varphi)\,|\boldsymbol{u}|^2\,\mathrm{d}x + \int_\Omega \frac{\mu}{2}\,|\nabla\boldsymbol{u}|^2 - \boldsymbol{f}\cdot\boldsymbol{u}\,\mathrm{d}x \\ &+ \gamma\int_\Omega \frac{\epsilon}{2}\,|\nabla\varphi|^2 + \frac{1}{\epsilon}\psi(\varphi)\,\mathrm{d}x\end{aligned} \tag{7}$$

subject to $(\varphi,\boldsymbol{u}) \in \Phi_{ad}\times\boldsymbol{U}$ and (5).

We notice, that for fixed $\varphi \in \Phi_{ad}$ the weak formulated state equations (5) correspond exactly to the necessary and sufficient first order optimality conditions of the convex optimization problem

$$\min_{\boldsymbol{u}\in\boldsymbol{U}} J_\epsilon(\varphi,\boldsymbol{u}).$$

Therefore, the optimization problem (7) is in this case equivalent to

$$\begin{aligned}\min_{(\varphi,\boldsymbol{u})\in\Phi_{ad}\times\boldsymbol{U}} J_\epsilon(\varphi,\boldsymbol{u}) := &\int_\Omega \frac{1}{2}\alpha_\epsilon(\varphi)\,|\boldsymbol{u}|^2\,\mathrm{d}x + \int_\Omega \frac{\mu}{2}\,|\nabla\boldsymbol{u}|^2 - \boldsymbol{f}\cdot\boldsymbol{u}\,\mathrm{d}x \\ &+ \gamma\int_\Omega \frac{\epsilon}{2}\,|\nabla\varphi|^2 + \frac{1}{\epsilon}\psi(\varphi)\,\mathrm{d}x.\end{aligned} \tag{8}$$

In this formulation, no explicit state equations as constraint are necessary any more.

One major advantage of this porous medium—phase field formulation for shape optimization problems in fluids is the existence of a minimizer, as the following theorems shows:

Theorem 1 *For every* $\epsilon > 0$ *there exists a minimizer* $(\varphi_\epsilon, \boldsymbol{u}_\epsilon) \in \Phi_{ad} \times \boldsymbol{U}$ *of the optimization problem* (8).

Proof This can be established quite easily by using the direct method in the calculus of variations. For details we refer to [14]. □

Remark 2 We introduced the porous medium—phase field approach for the problem of minimizing the total potential power in a Stokes flow here. But this approach can also be applied to a larger class of objective functionals and also to different state equations like the stationary Navier-Stokes equations, see [14]. We could also include a term in the objective functional including the pressure of the fluid.

3 Sharp Interface Problem

The optimization problem (8) introduced in the previous section depends on the phase field parameter $\epsilon > 0$, which describes both the interfacial thickness and the permeability of the porous medium outside the fluid region. The natural question arising is what happens if ϵ tends to zero. We expect to arrive in a perimeter penalized sharp interface problem, whose solutions can be considered as so-called black-and-white solutions (see for instance [13]), which means that there exists only pure fluid regions and pure non-fluid regions with zero permeability. And actually, it can be verified in the framework of Γ-convergence that problem (8) has a sharp interface analogue. For a detailed introduction to the notion of Γ-convergence and its applications we refer here for instance to [11].

The resulting problem in the limit will be a shape optimization problem formulated in the setting of Caccioppoli sets. In order to formulate this problem in the right manner we briefly introduce some notation. However, for a detailed introduction into the theory of Caccioppoli sets and functions of bounded variation we refer here to [3, 12]. We call a function $\varphi \in L^1(\Omega)$ a function of bounded variation if its distributional derivative is a vector-valued finite Radon measure. The space of a functions of bounded variation in Ω is denoted by $BV(\Omega)$, and by $BV(\Omega, \{\pm 1\})$ we denote functions in $BV(\Omega)$ having only the values ± 1 a.e. in Ω. We then call a measurable set $E \subset \Omega$ Caccioppoli set, if $\chi_E \in BV(\Omega)$. For any Caccioppoli set E, one can hence define the total variation $|\mathrm{D}\chi_E|(\Omega)$ of $\mathrm{D}\chi_E$, as $\mathrm{D}\chi_E$ is a finite measure. This value is then called the perimeter of E in Ω and is denoted by $P_\Omega(E) := |\mathrm{D}\chi_E|(\Omega)$.

An important point in the formulation of the sharp interface problem is that the velocity $\boldsymbol{u}$ of the fluid is still defined on the whole of Ω, even though we have black-and-white solutions and there are only certain regions inside of Ω filled with fluid.

This is done by defining $\boldsymbol{u}$ to be zero if no fluid is present, which is the case if $\varphi = -1$. And hence the velocity is here an element in $\boldsymbol{U}^{\varphi} := \{\boldsymbol{u} \in \boldsymbol{U} \mid \boldsymbol{u} = \boldsymbol{0}$ a.e. in $\{\varphi = -1\}\}$ if $\varphi \in L^1(\Omega)$. And correspondingly, we introduce the space $\boldsymbol{V}^{\varphi} := \{\boldsymbol{u} \in \boldsymbol{V} \mid \boldsymbol{u} = \boldsymbol{0}$ a.e. in $\{\varphi = -1\}\}$.

The design space for the sharp interface problem is given as

$$\Phi^0_{ad} := \left\{\varphi \in BV(\Omega, \{\pm 1\}) \mid \int_\Omega \varphi \,\mathrm{d}x \leq \beta|\Omega|,\ \boldsymbol{U}^{\varphi} \neq \emptyset\right\}.$$

The constraint $\boldsymbol{U}^{\varphi} \neq \emptyset$ is a necessary condition in order to obtain at least one admissible velocity field for this case, since the two conditions of $\boldsymbol{u} = \boldsymbol{0}$ if $\varphi = -1$ and $\boldsymbol{u}|_{\partial\Omega} = \boldsymbol{g}$ may be conflicting.

We extend J_ϵ to the whole space $L^1(\Omega) \times \boldsymbol{H}^1(\Omega)$ by defining $J_\epsilon : L^1(\Omega) \times \boldsymbol{H}^1(\Omega) \to \overline{\mathbb{R}}$ as

$$J_\epsilon(\varphi, \boldsymbol{u}) := \begin{cases} \int_\Omega \frac{1}{2}\alpha_\epsilon(\varphi)\,|\boldsymbol{u}|^2 \,\mathrm{d}x + \int_\Omega \frac{\mu}{2}|\nabla \boldsymbol{u}|^2 - \boldsymbol{f}\cdot\boldsymbol{u}\,\mathrm{d}x + & \\ +\gamma\int_\Omega \frac{\epsilon}{2}|\nabla\varphi|^2 + \frac{1}{\epsilon}\psi(\varphi)\,\mathrm{d}x, & \text{if } \varphi \in \Phi_{ad}, \boldsymbol{u} \in \boldsymbol{U}, \\ +\infty, & \text{else.} \end{cases} \tag{9}$$

We will show in Sect. 4 that the Γ-limit of $(J_\epsilon)_{\epsilon>0}$ for $\epsilon \searrow 0$ is given by $J_0 : L^1(\Omega) \times \boldsymbol{H}^1(\Omega) \to \overline{\mathbb{R}}$, where

$$J_0(\varphi, \boldsymbol{u}) := \begin{cases} \int_\Omega \frac{\mu}{2}|\nabla \boldsymbol{u}|^2 - \boldsymbol{f}\cdot\boldsymbol{u}\,\mathrm{d}x + c_0\gamma P_\Omega(\{\varphi = 1\}), & \text{if } \varphi \in \Phi^0_{ad}, \boldsymbol{u} \in \boldsymbol{U}^{\varphi}, \\ +\infty, & \text{else.} \end{cases}$$

The constant $c_0 := \int_{-1}^{1}\sqrt{2\psi(s)}\,\mathrm{d}s = \frac{\pi}{2}$ arises due to technical reasons, compare Sect. 4.

We find as in the previous section, that the optimization problem

$$\min_{(\varphi,\boldsymbol{u})\in L^1(\Omega)\times \boldsymbol{H}^1(\Omega)} J_0(\varphi, \boldsymbol{u}) \tag{10}$$

is equivalent to the optimization problem with state constraints given by

$$\min_{(\varphi,\boldsymbol{u})} J_0(\varphi, \boldsymbol{u}) := \int_\Omega \frac{\mu}{2}|\nabla\boldsymbol{u}|^2 - \boldsymbol{f}\cdot\boldsymbol{u}\,\mathrm{d}x + \gamma c_0 P_\Omega(\{\varphi = 1\}) \tag{11}$$

subject to $\varphi \in \Phi^0_{ad}$, $\boldsymbol{u} \in \boldsymbol{U}^{\varphi}$ and

$$-\mu\Delta\boldsymbol{u} + \nabla p = \boldsymbol{f} \quad \text{in } \{\varphi = 1\}, \tag{12a}$$

$$\operatorname{div}\boldsymbol{u} = 0 \quad \text{in } \Omega, \tag{12b}$$

$$\boldsymbol{u} = \boldsymbol{g} \quad \text{on } \partial\Omega. \tag{12c}$$

The strong formulation (12) of the state equations are to be understood in the following weak sense: find $\boldsymbol{u} \in \boldsymbol{U}^{\varphi}$ such that

$$\int_{\Omega} \mu \nabla \boldsymbol{u} \cdot \nabla \boldsymbol{v} \, \mathrm{d}x = \int_{\Omega} \boldsymbol{f} \cdot \boldsymbol{v} \, \mathrm{d}x \quad \forall \boldsymbol{v} \in \boldsymbol{V}^{\varphi}.$$

The shape optimization problem (10) allows in particular every Caccioppoli set as admissible shape, which yields that no geometric properties are prescribed. Additionally, no boundary regularity of the shapes is necessary and even the topology can change during the optimization process. Hence this yields a very large class of possible solutions, in contrast to existing formulations in shape optimization, see for instance [9, 10, 18]. Additionally, we will see in the next section, that there exists a minimizer for J_0, compare Remark 3, which is, as already mentioned above, not a trivial fact in shape optimization problems.

4 Sharp Interface Limit

Let X denote the topological space $L^1(\Omega) \times \boldsymbol{H}^1(\Omega)$ equipped with the strong $L^1(\Omega)$ and weak $\boldsymbol{H}^1(\Omega)$ topology. In this section we will show the already announced result that $(J_\epsilon)_{\epsilon>0}$ converges in the sense of Γ-convergence to J_0 as $\epsilon \searrow 0$ in X, hence in $L^1(\Omega) \times \boldsymbol{H}^1(\Omega)$ with respect to the topology of X. One important ingredient here is the special structure of the objective functional, hence that no state equations are necessary to be stated explicitly. The proof is based on the result of [16] which ensures that the Ginzburg-Landau energy Γ-converges in $L^1(\Omega)$ to a multiple of the perimeter functional as the phase field parameter ϵ tends to zero. We directly state the main result:

Theorem 2 *The functionals $(J_\epsilon)_{\epsilon>0}$ converge in the sense of Γ-convergence in X to J_0 as $\epsilon \searrow 0$.*

A direct and important consequence of this theorem is given by the following corollary:

Corollary 1 *Let $(\varphi_\epsilon, \boldsymbol{u}_\epsilon)$ be a minimizer of J_ϵ for every $\epsilon > 0$, whose existence is guaranteed by Theorem 1. Then there exists a subsequence, which will be denoted by the same, such that $(\varphi_\epsilon, \boldsymbol{u}_\epsilon)_{\epsilon>0}$ converges (strongly) in $L^1(\Omega) \times \boldsymbol{H}^1(\Omega)$ to some limit $(\varphi_0, \boldsymbol{u}_0)$. Additionally, $(\varphi_0, \boldsymbol{u}_0)$ is a minimizer of J_0 and $\lim_{\epsilon \searrow 0} J_\epsilon(\varphi_\epsilon, \boldsymbol{u}_\epsilon) = J_0(\varphi_0, \boldsymbol{u}_0)$.*

Remark 3 Corollary 1 ensures in particular the existence of a minimizer of J_0 and hence also the existence of a minimizer of the constrained optimization problem (11) and (12).

We start by proving the Γ-convergence result of Theorem 2:

Proof of Theorem 2 We use the sequential characterization of the Γ-limit, see [11]. Hence we have to prove two properties in order to deduce the theorem. First we show that for every $(\varphi, \boldsymbol{u}) \in L^1(\Omega) \times \boldsymbol{H}^1(\Omega)$, there exists a sequence $(\varphi_\epsilon, \boldsymbol{u}_\epsilon)_{\epsilon>0} \subset L^1(\Omega) \times \boldsymbol{H}^1(\Omega)$ converging to $(\varphi, \boldsymbol{u})$ in X such that

$$\limsup_{\epsilon \searrow 0} J_\epsilon(\varphi_\epsilon, \boldsymbol{u}_\epsilon) \leq J_0(\varphi, \boldsymbol{u}).$$

This sequence is often called recovery sequence. The second step is to show that J_0 provides a lower bound, i.e. we have to show that for every sequence $(\varphi_\epsilon, \boldsymbol{u}_\epsilon)_{\epsilon>0} \subset L^1(\Omega) \times \boldsymbol{H}^1(\Omega)$ converging to some element $(\varphi, \boldsymbol{u})$ in X it holds

$$J_0(\varphi, \boldsymbol{u}) \leq \liminf_{\epsilon \searrow 0} J_\epsilon(\varphi_\epsilon, \boldsymbol{u}_\epsilon). \tag{13}$$

For this purpose, we start with a preparatory observation. Let $(\varphi_\epsilon)_{\epsilon>0}$ be any sequence converging pointwise almost everywhere in Ω to some $\varphi \in L^1(\Omega)$. As it holds $\alpha_\delta \leq \alpha_\epsilon$ for all $\epsilon \leq \delta$ we obtain for fixed $\delta > 0$ that

$$\alpha_\delta(\varphi(x)) = \lim_{\epsilon \searrow 0} \alpha_\delta\left(\varphi_\epsilon\left(x\right)\right) \leq \liminf_{\epsilon \searrow 0} \alpha_\epsilon(\varphi_\epsilon(x))$$

and thus, as $\delta \searrow 0$,

$$\alpha_0(\varphi(x)) = \lim_{\delta \searrow 0} \left(\alpha_\delta\left(\varphi\left(x\right)\right)\right) \leq \liminf_{\epsilon \searrow 0} \alpha_\epsilon(\varphi_\epsilon(x))$$

for almost every $x \in \Omega$. On the other hand we have, as $\alpha_\epsilon \leq \alpha_0$, also

$$\limsup_{\epsilon \searrow 0} \alpha_\epsilon(\varphi_\epsilon(x)) \leq \limsup_{\epsilon \searrow 0} \alpha_0(\varphi_\epsilon(x)) = \alpha_0(\varphi(x)).$$

Altogether we thus find

$$\lim_{\epsilon \searrow 0} \alpha_\epsilon\left(\varphi_\epsilon\left(x\right)\right) = \alpha_0(\varphi(x)) \quad \text{for a.e. } x \in \Omega. \tag{14}$$

We next construct the recovery sequence and choose some $(\varphi, \boldsymbol{u}) \in \Phi_{ad}^0 \times \boldsymbol{U}^\varphi$ with $J_0(\varphi, \boldsymbol{u}) < \infty$. To this end, we use the construction of [16], see also [6, 17], which ensures the existence of a sequence $(\varphi_\epsilon)_{\epsilon>0}$ converging strongly in $L^1(\Omega)$ to φ such that

$$\int_\Omega \varphi_\epsilon \, \mathrm{d}x \leq \int_\Omega \varphi \, \mathrm{d}x \leq \beta|\Omega| \quad \forall \epsilon \ll 1$$

and

$$\limsup_{\epsilon \searrow 0} \int_\Omega \frac{\epsilon}{2} |\nabla \varphi_\epsilon|^2 + \frac{1}{\epsilon} \psi(\varphi_\epsilon) \,\mathrm{d}x \leq c_0 P_\Omega(\{\varphi = 1\}).$$

The construction yields additionally the convergence rate

$$\|\varphi_\epsilon - \varphi\|_{L^1(\Omega)} = \mathcal{O}(\epsilon). \tag{15}$$

For details on this construction, in particular on the convergence rate, we refer also to [14]. From $\boldsymbol{u}|_{\{\varphi=-1\}} = \boldsymbol{0}$ and (14) we find $\lim_{\epsilon \searrow 0} \alpha_\epsilon(\varphi_\epsilon(x))|\boldsymbol{u}|^2(x) = 0$ for almost every $x \in \Omega$. This gives us in view of Lebesgue's dominated convergence theorem and by using the pointwise estimate

$$\alpha_\epsilon(\varphi_\epsilon)|\boldsymbol{u}|^2 \leq \alpha_\epsilon(0)|\boldsymbol{u}|^2 \leq \alpha_0(0)|\boldsymbol{u}|^2 \quad \text{a.e. in } \{\varphi_\epsilon \geq 0\}$$

that

$$\lim_{\epsilon \searrow 0} \int_{\{\varphi_\epsilon \geq 0\}} \alpha_\epsilon(\varphi_\epsilon)|\boldsymbol{u}|^2 \,\mathrm{d}x = 0.$$

We can use the pointwise estimates $|\varphi_\epsilon| \leq 1$, $|\varphi| \leq 1$ and the inclusion $\{\boldsymbol{u} \neq \boldsymbol{0}\} \subset \{\varphi = 1\}$ to obtain

$$\begin{aligned} \int_{\{\varphi_\epsilon \leq 0\}} \alpha_\epsilon(\varphi_\epsilon)|\boldsymbol{u}|^2 \,\mathrm{d}x &\leq \overline{\alpha}_\epsilon \int_\Omega \chi_{\{\varphi_\epsilon \leq 0, \varphi = 1\}} \underbrace{|\varphi_\epsilon - \varphi|}_{\geq 1} |\boldsymbol{u}|^2 \,\mathrm{d}x \\ &\leq C \overline{\alpha}_\epsilon \|\varphi - \varphi_\epsilon\|_{L^1(\Omega)}^{\frac{2}{3}} \|\boldsymbol{v}\|_{\boldsymbol{L}^6(\Omega)}^2. \end{aligned}$$

Combining the convergence rate (15) and $\overline{\alpha}_\epsilon = o\left(\epsilon^{-\frac{2}{3}}\right)$, see Assumption (A3), we hence deduce $\lim_{\epsilon \searrow 0} \int_{\{\varphi_\epsilon \leq 0\}} \alpha_\epsilon(\varphi_\epsilon)|\boldsymbol{u}|^2 \,\mathrm{d}x = 0$ and so we end up with

$$\lim_{\epsilon \searrow 0} \int_\Omega \alpha_\epsilon(\varphi_\epsilon)|\boldsymbol{u}|^2 \,\mathrm{d}x = 0.$$

Altogether, this yields

$$\limsup_{\epsilon \searrow 0} J_\epsilon(\varphi_\epsilon, \boldsymbol{u}) \leq J_0(\varphi, \boldsymbol{u})$$

and finishes the first step in this proof.

It remains to show that J_0 is a lower bound on $(J_\epsilon)_{\epsilon>0}$ as described above. For this purpose, we choose an arbitrary sequence $(\varphi_\epsilon, \boldsymbol{u}_\epsilon)_{\epsilon>0} \subset L^1(\Omega) \times \boldsymbol{H}^1(\Omega)$ converging to some element $(\varphi, \boldsymbol{u})$ in X. Without loss of generality we assume $\liminf_{\epsilon \searrow 0} J_\epsilon(\varphi_\epsilon, \boldsymbol{u}_\epsilon) < \infty$, otherwise (13) is trivial. We use again the results of [16] to observe that for an arbitrary sequence $(\varphi_\epsilon, \boldsymbol{u}_\epsilon)_{\epsilon>0} \subset L^1(\Omega) \times \boldsymbol{H}^1(\Omega)$ converging to some element $(\varphi, \boldsymbol{u})$ in X it holds

$$c_0 P_\Omega(\{\varphi = 1\}) \leq \liminf_{\epsilon \searrow 0} \int_\Omega \frac{\epsilon}{2} |\nabla \varphi_\epsilon|^2 + \frac{1}{\epsilon}\psi(\varphi_\epsilon)\, \mathrm{d}x.$$

Besides, we obtain with the pointwise considerations (14) and Fatou's lemma

$$\begin{aligned} \int_\Omega \alpha_0(\varphi) |\boldsymbol{u}|^2 \, \mathrm{d}x &= \int_\Omega \liminf_{\epsilon \searrow 0} (\alpha_\epsilon(\varphi_\epsilon)) \left(\liminf_{\epsilon \searrow 0} |\boldsymbol{u}_\epsilon|^2 \right) \mathrm{d}x \\ &\leq \int_\Omega \liminf_{\epsilon \searrow 0} \left(\alpha_\epsilon(\varphi_\epsilon) |\boldsymbol{u}_\epsilon|^2 \right) \mathrm{d}x \leq \liminf_{\epsilon \searrow 0} \int_\Omega \alpha_\epsilon(\varphi_\epsilon) |\boldsymbol{u}_\epsilon|^2 \, \mathrm{d}x. \end{aligned}$$

This yields in particular $\boldsymbol{u} = \boldsymbol{0}$ a.e. in $\{\varphi = -1\}$ and hence $\boldsymbol{u} \in \boldsymbol{U}^\varphi$. Additionally,

$$\boldsymbol{H}^1(\Omega) \ni \boldsymbol{u} \mapsto \int_\Omega \frac{\mu}{2} |\nabla \boldsymbol{u}|^2 - \boldsymbol{f} \cdot \boldsymbol{u} \, \mathrm{d}x$$

is a continuous, convex and thus weakly lower semicontinuous functional. And hence we obtain

$$J_0(\varphi, \boldsymbol{u}) \leq \liminf_{\epsilon \searrow 0} J_\epsilon(\varphi_\epsilon, \boldsymbol{u}_\epsilon)$$

and can hence finish the proof. For some additional technical details and generalizations we refer to [14]. □

Proof of Corollary 1 Similar as in the proof of Theorem 2 we construct for some arbitrary element $(\varphi, \boldsymbol{u}) \in L^1(\Omega) \times \boldsymbol{H}^1(\Omega)$ with $J_0(\varphi, \boldsymbol{u}) < \infty$ a sequence $(\widetilde{\varphi}_\epsilon, \widetilde{\boldsymbol{u}}_\epsilon)_{\epsilon>0} \subset L^1(\Omega) \times \boldsymbol{H}^1(\Omega)$ such that $\limsup_{\epsilon \searrow 0} J_\epsilon(\widetilde{\varphi}_\epsilon, \widetilde{\boldsymbol{u}}_\epsilon) \leq J_0(\varphi, \boldsymbol{u})$. Using the minimizing property of $(\varphi_\epsilon, \boldsymbol{u}_\epsilon)$ for each $\epsilon > 0$ this implies that there is some constant $C > 0$ such that

$$J_\epsilon(\varphi_\epsilon, \boldsymbol{u}_\epsilon) \leq J_\epsilon(\widetilde{\varphi}_\epsilon, \widetilde{\boldsymbol{u}}_\epsilon) < C \quad \forall \epsilon \ll 1. \tag{16}$$

Therefrom, we find directly that $\int_\Omega \left(\frac{\epsilon}{2} |\nabla \varphi_\epsilon|^2 + \frac{1}{\epsilon}\psi(\varphi_\epsilon) \right) \mathrm{d}x \leq C$. As in [16] we can hence estimate $\int_\Omega |\nabla \phi(\varphi_\epsilon)| \, \mathrm{d}x$ with $\phi(t) = \int_0^t \sqrt{2\psi(s)}\, \mathrm{d}s$ and with the help of the compactness argument in [16, Proposition 3] we get thus the existence of a subsequence of $(\varphi_\epsilon)_{\epsilon>0}$, which we will denote by the same, converging in $L^1(\Omega)$ to some limit element φ_0. Additionally, we obtain thanks to (16) a subsequence of $(\boldsymbol{u}_\epsilon)_{\epsilon>0}$, which is again denoted by the same, that converges weakly in $\boldsymbol{H}^1(\Omega)$ to some limit element $\boldsymbol{u}_0$. This gives us in view of standard results for Γ-convergence, see [11], and the Γ-convergence result of Theorem 2 that the limit point $(\varphi_0, \boldsymbol{u}_0)$ is a minimizer of J_0 and

$$\lim_{\epsilon \searrow 0} J_\epsilon(\varphi_\epsilon, \boldsymbol{u}_\epsilon) = J_0(\varphi_0, \boldsymbol{u}_0). \tag{17}$$

Finally we combine

$$0 \leq \liminf_{\epsilon \searrow 0} \int_\Omega \alpha_\epsilon(\varphi_\epsilon) |\boldsymbol{u}_\epsilon|^2 \, \mathrm{d}x, \quad c_0 P_\Omega(\{\varphi_0 = 1\})$$
$$\leq \liminf_{\epsilon \searrow 0} \int_\Omega \frac{\epsilon}{2} |\nabla \varphi_\epsilon|^2 + \frac{1}{\epsilon}\psi(\varphi_\epsilon) \, \mathrm{d}x,$$

see [16], and

$$\int_\Omega \frac{\mu}{2} |\nabla \boldsymbol{u}_0|^2 - \boldsymbol{f} \cdot \boldsymbol{u}_0 \, \mathrm{d}x \leq \liminf_{\epsilon \searrow 0} \left(\int_\Omega \frac{\mu}{2} |\nabla \boldsymbol{u}_\epsilon|^2 - \boldsymbol{f} \cdot \boldsymbol{u}_\epsilon \, \mathrm{d}x \right)$$

to deduce from (17) that

$$\int_\Omega \frac{\mu}{2} |\nabla \boldsymbol{u}_0|^2 - \boldsymbol{f} \cdot \boldsymbol{u}_0 \, \mathrm{d}x = \lim_{\epsilon \searrow 0} \left(\int_\Omega \frac{\mu}{2} |\nabla \boldsymbol{u}_\epsilon|^2 - \boldsymbol{f} \cdot \boldsymbol{u}_\epsilon \, \mathrm{d}x \right).$$

And hence we can conclude the strong convergence of $(\boldsymbol{u}_\epsilon)_{\epsilon>0}$ to $\boldsymbol{u}_0$ in $\boldsymbol{H}^1(\Omega)$. For more details we refer to [14]. □

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