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Analysis and Geometry

MIMS-GGTM, Tunis, Tunisia, March 2014. In Honour of Mohammed Salah Baouendi

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Preface

The Mediterranean Institute for the Mathematical Sciences (MIMS) and the Geometry and Topology Grouping for the Maghreb (GGTM) held an international conference at the Cité des Sciences in Tunis, Tunisia during March 24–27, 2014. This event was held in memory of Mohammed Salah Baouendi, one the finest mathematicians of the past century. It brought together geometers and topologists from around the world, many of whom were distinguished collaborators of Baouendi.

A special lecture series on CR-Geometry was given by Peter Ebenfelt and was attended by a sizable group of graduate students from Algeria and Tunisia. Invited speakers included Abbès Bahri, Hajer Bahouri, Ali Baklouti, Oliver Baues, Mehdi Belraouti, Jean-Michel Bony, Makhlouf Derridj, Ali Maalaoui, Hamid Meziani, Nordine Mir, Mohameden Ould Ahmedou, Ludovic Rifford, Linda Rothschild, Dmitri Zaitsev and Ghani Zeghib.

MIMS was founded in January 2012 in Tunis (Tunisia), and joins a long list of mathematical institutes around the world. Its multifaceted mission is to be a rallying point for the growing community of mathematicians across the MENA and the Mediterranean regions, to raise public awareness of the mathematical sciences in this part of the world, and to be a springboard for the young generation, making it easier for prospective students to connect to the vast world of mathematical research.

GGTM was founded at the initiative of a group of North-African geometers in 2003 in Marrakech (Morocco). Its objective is to promote mathematical research in geometry and topology in the Maghreb. This is done through increased collaboration between mathematicians north and south of the Mediterranean and across the Maghreb, by organizing regular meetings and by advising students.

Mohammed Salah Baouendi was born in Tunis and held one of its early academic positions there. After holding positions in Paris (France) and in Purdue (Indiana), where he headed the department for 8 years, he spent the rest of his life at UCSD (California). His influence on mathematics at large and CR-geometry in particular has been deep and long-lasting. This volume is a tribute to his work. It contains fine contributions by collaborators, friends and students of M.S. Baouendi.

This conference was sponsored by CIMPA (Centre International des Mathématiques Pures et Appliquées), LEM2I (Laboratoire Euro-Maghrébin de Mathématiques et de leur Interactions), TAFA (Tunisian-American Friendship Association), MedTech (Mediterranean Institute of Technology) and the Laboratoires LAMHA and LR13ES21 of the University of Sfax (Tunisia).

We warmly thank all contributors to this volume and all participants who made this conference a real success.

Tunis
December 2014

Ali Baklouti
Aziz El Kacimi
Sadok Kallel
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Contents

Salah Baouendi 1937–2011: A Mathematical Life on Three Continents

Linda P. Rothschild

Abstract In this brief account, I have tried to weave together some of the many strands of Salah Baouendi's remarkable mathematical life.

Keywords Partial differential equations · Several complex variables

2010 Mathematics Subject Classification 32V05 · 35H10

1 Early Life: Tunis to Paris

Salah Baouendi was born in Tunis on October 15, 1937. Mathematics was Salah's real passion, and it was his mathematical talent that took him from the Sadiki Lycée in Tunis (Fig. 1) to Paris, where he spent the last two years of high school.

Paris was, and still is, one of the world's greatest centers of mathematics. After entering the University of Paris, his abilities and enthusiasm were recognized by such mathematical greats as Laurent Schwartz. Laurent Schwartz introduced him to Jacques-Louis Lions and Bernard Malgrange, who were also great influences on his future research. At that time there was an explosion of new ideas in the area of partial differential equations (PDEs), giving a new foundation to linear PDEs. Applications of functional analysis and duality became crucial tools, beginning with the fundamental work of Schwartz on distributions and the role of operators between various spaces of functions.

Early in his studies Salah obtained an important result concerning nonsolvability of a PDE with constant coefficients in a certain functions space. Roughly speaking, application of a constant coefficient PDO to a function corresponds to multiplication of the Fourier transform of the function by a polynomial. Through this technique,

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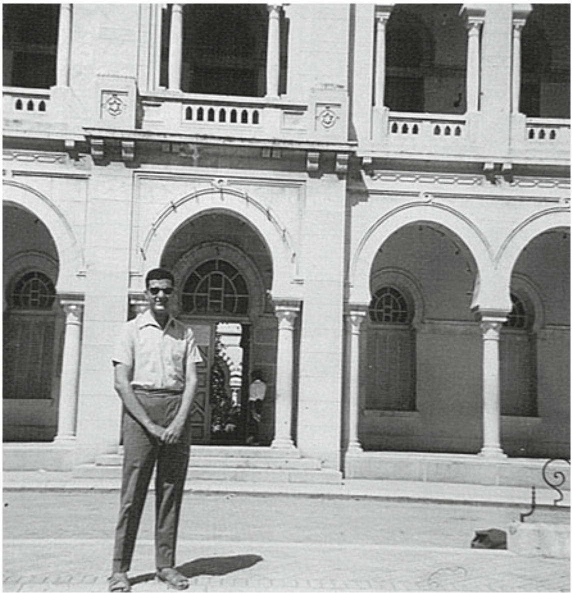


Fig. 1 Salah Baouendi as a young man in front of his lycée, Sadiki College, in Tunis

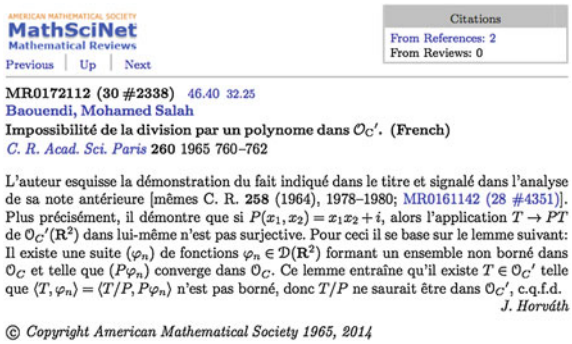
solvability was known in many function spaces, but Salah showed it was false in the space of slow-growing functions.

The setting may be easily described. Let $\mathcal{S}$ be the space of all Schwartz functions in $\mathbb{R}^n$ and $\mathcal{S}'$ the dual space. The Schwartz space is self-dual, i.e. $\mathcal{S} = \mathcal{S}'$. If P is a polynomial, there is a unique PDO, $P(D)$, with constant coefficients such that for any $f \in \mathcal{S}$ such that $Pf = P(D)$, where $\hat{f}$ is the Fourier transform of f. Furthermore the mapping

$$\mathcal{S} \ni f \mapsto P\hat{f} \in \mathcal{S}'$$

is surjective. Salah proved that the corresponding result does not hold in the space $\mathcal{C}$ of slow-growing functions, where a polynomial P maps $\mathcal{C}$ into the dual $\mathcal{C}'$. That is, he showed that the mapping is not surjective, as some had conjectured.

Salah went on to write a brilliant thesis in which he studied the boundary behavior of solutions of elliptic PDEs. I will describe the context only briefly here. He considered a class of second order differential operators in an open subset of $\mathbb{R}^n$, which are elliptic in the open set, but degenerate on the boundary, and studied solutions of the equations near and on the boundary. The results of his thesis were published in a long paper in the Bulletin of the French Mathematical Society in 1967. In the review in MathSci Net (Fig. 2), the reviewer recognized the importance of this work and also praised the clarity of the writing. Throughout his life, Salah displayed a clarity of thinking and writing, both in mathematics and in his world view.



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MR0172112 (30 #2338) 46.40 32.25
Baouendi, Mohamed Salah
Impossibilité de la division par un polynome dans $\mathcal{O}_C{}'$. (French)
C. R. Acad. Sci. Paris **260** 1965 760–762

L'auteur esquisse la démonstration du fait indiqué dans le titre et signalé dans l'analyse de sa note antérieure [mêmes C. R. **258** (1964), 1978–1980; MR0161142 (28 #4351)]. Plus précisément, il démontre que si $P(x_1, x_2) = x_1x_2 + i$, alors l'application $T \to PT$ de $\mathcal{O}_C{}'(\mathbf{R}^2)$ dans lui-même n'est pas surjective. Pour ceci il se base sur le lemme suivant: Il existe une suite (φ_n) de fonctions $\varphi_n \in \mathcal{D}(\mathbf{R}^2)$ formant un ensemble non borné dans $\mathcal{O}_C$ et telle que $(P\varphi_n)$ converge dans $\mathcal{O}_C$. Ce lemme entraîne qu'il existe $T \in \mathcal{O}_C{}'$ telle que $\langle T, \varphi_n \rangle = \langle T/P, P\varphi_n \rangle$ n'est pas borné, donc T/P ne saurait être dans $\mathcal{O}_C{}'$, c.q.f.d.

J. Horváth

Fig. 2 Review of one of Salah's very first papers (*Source* http://www.ams.org/mathscinet-getitem?mr=172112, reproduced with permission from MathSciNet, © American Mathematical Society)

2 Paris to Tunis and Back

Now let's get back in time to the story of the continents. After earning his first university degree in Paris supported by a Tunisian scholarship, Salah was required to return Tunis to teach high school and continue his studies there, as he did. However, with the help of Laurent Schwartz (written in Schwartz's memoirs, "A mathematician and his century"), the Tunisian Ministry allowed Salah to return to Paris for his doctoral work after a year of teaching in Tunis. Following the completion of his doctorate in Paris in 1967, Salah returned with his young family to the University of Tunis to take a faculty position.

A number of prominent French mathematicians, including Malgrange, also came to Tunis for extended stays during the years that Salah spent there. However, this was a troubled and turbulent time in that part of the world. Tunisia was newly independent, and the Algerian-French war had left many wounds. Always a person of action, Salah did what he could to promote mathematics and research at the University of Tunis. He wrote a four page document on the need for support of mathematics in Tunisia.

In December 1969, while Salah was a professor in Tunis, the French Academy awarded him the "Prix d'Aumale," a high honor. Although he was well respected and admired at the University of Tunis, Salah was not comfortable in his home country. He chafed under the repressive political atmosphere in Tunisia and returned to France, where he could devote himself to mathematics without interference from politics. He spent a year at the University of Nice and several years at the University of Paris. Through his lectures and publications, he gained an international reputation as a research mathematician.

One cannot mention Salah's early mathematical career without discussing his collaboration with Charles Goulaouic, from the early 60s as fellow students in Paris until Goulaouic's tragic death from cancer at the age of 46 in 1983. Much of Salah's later work in applications of PDEs to several complex variables had its roots in his joint work with Goulaouic. Among these were:

- Boundary behavior of solutions of PDEs.
- Approximation of functions by complex polynomials.

- Characterizations of analyticity by growth estimates.
- Analyticity of solutions of systems of first-order PDEs.
- Nonanalytic hypoellipticity for Høormander "sum of squares" operator.

In addition to their longtime collaboration, Baouendi and Goulaouic mentored a whole generation of students in PDEs and organized conferences of mathematicians from around the world. Among the younger mathematicians Salah met during the Paris years were Louis Boutet de Monvel, François Treves, Jean-Michel Bony. Among Salah's doctoral students were the North African transplants Makhlouf Derridj and Claude Zuily, who became prominent mathematicians.

In the mid-seventies there were three students officially working with Goulaouic: Bernard Helffer, Serge Alinhac, and Guy Métiver who also regarded Salah as their mentor and spent time at Purdue University when Salah was a professor there. Both Alinhac and Métivier published joint research with Salah conducted in the U.S. All three became prominent mathematicians in France and leaders in the profession. Like Salah, Goulaouic had a natural sense of leadership, and their students were mentored in more than mathematics. Helffer became president of the French Mathematical Society and Métivier director of the mathematical division of the CNRS.

3 Paris to America (Purdue University)

As a foreigner Salah could not, in those years, be appointed as a professor in France, so he accepted a tenured faculty position in the mathematics department at Purdue University, a leading American university in the Midwest. After Salah began his career at Purdue, a new change in French laws allowed non citizens to be appointed as professors in French universities. Salah accepted a professorship at the University of Paris VI (Jussieu) and returned to France in 1974. In the same year he was chosen as a speaker at the International Congress of Mathematicians in Vancouver. However, despite his reputation as a prominent scientist, in Paris Salah was often regarded as a North African who would always be less than a real French citizen. In 1976 he returned with his family to Purdue, where he was destined to leave his mark on the university and on the American mathematical community. His children, Moungi and Meriem would grow up in the Midwest as Americans. Moungi is now a famous American scientist, a professor of chemistry at MIT, and the father of an 11 year old daughter. Meriem lives in California with her husband and two teenaged children.

Immediately recognized not only for his mathematical research, but also for his scientific leadership, Salah was appointed Head of the Mathematics Department at Purdue in 1980. His accomplishments there include the recruitment of numerous top mathematicians as well as the establishment of a center for applied mathematics. His eight years as Head at Purdue are still fondly remembered as the "Baouendi years." As his national reputation grew in the U.S., Salah received many inquiries about his willingness to be considered for higher administrative positions, such as deanships, at other universities and also for a high level position at the National

Science Foundation. He was also sought by the American Mathematical Society (AMS), where he ultimately chaired many committees and served on the Board of Trustees. Among his most remarkable accomplishments, in the 1990s he led a successful AMS campaign to prevent the University of Rochester from abolishing its research program in mathematics.

Refusing to sacrifice his research career to become a high-level university administrator, Salah remained devoted to mathematics until his death in 2011. His mathematical investigations turned to applications of PDEs to geometric problems in complex analysis. His early work in boundary values of PDEs was very relevant to extensions of functions and mappings in the complex domain in several variables. With collaborators Goulaouic and François Treves, Salah succeeded in a breakthrough in this new field known as "several complex variables." He also succeeded in hiring promising young mathematicians at Purdue, including Steven Bell, David Catlin, and Laszlo Lempert, all of whom went on to impressive research careers. Among his students and postdocs there were Mei-Chi Shaw and Alexander Himonas, now both professors at Notre Dame.

There was a natural transition from Salah's research in PDEs to several complex variables. A complex analytic function is just a solution to a system of homogenous first order PDEs, namely the Cauchy-Riemann equations in several complex variables. Salah had studied solutions of systems of first order in past work with Goulaouic and Treves. In 1981, Treves and Salah published their celebrated joint paper whose main result has become known as the Baouendi-Treves Approximation Theorem. Recall that a CR function is a solution of the tangential Cauchy-Riemann equations.

Theorem 3.1 (Baouendi-Treves Approximation Theorem (1981)) *Any continuous solution of a homogeneous system of a locally integrable vector fields can be uniformly approximated (locally) by polynomials in a set of fundamental solutions of that system.*

Corollary 3.2 *Any CR function on a smooth hypersurface M is locally approximable by restrictions of holomorphic functions to M.*

Further work of Baouendi-Treves (early 80s):

- Criteria for extension of CR functions on a hypersurface.
- Extension of holomorphic functions from one side of a hypersurface.
- Holomorphic extension of mappings of one bounded domain into another.

Salah continued his groundbreaking work in several complex variables in the 1980s, attracting the attention of mathematicians worldwide.

He received offers of high-level professorships at the Johns Hopkins University, Rutgers University (with a State of New Jersey Chair), and the University of California, San Diego (UCSD). In 1988 he moved to San Diego to accept a Distinguished Professorship at UCSD.

4 Mathematical Life in California

After moving to San Diego, Salah continued to open new frontiers of research in several complex analysis (Fig. 3). His research, joint with several other mathematicians included major breakthroughs in the following.

A sample of major progress by Baouendi and coauthors since 1990:

- Holomorphic extension of smooth CR mappings from one smooth submanifold of $\mathbb{C}^n$ into another.
- Development of the method of Segre sets and mappings to analyze real submanifolds of higher codimension.
- Extension of holomorphic mappings between smoothly bounded domains.
- Multiplicity and transversality of holomorphic mappings between given real submanifolds in complex spaces.
- Existence and rigidity of mappings into hyperquadrics in complex spaces of different dimensions.

In 2003 Salah and I received the Stefan Bergman Award for our work in several complex variables, much of it joint. We were both elected to the American Academy of Arts and Sciences in 2006. In the mid 1990s a young Swedish mathematician, Peter Ebenfelt, came to UCSD. Salah and I did much joint research with Peter, and the three of us wrote a research monograph, which was published by Princeton University Press and is regarded as a fundamental book in the area. Peter is now Chair of the UCSD Mathematics Department and after Salah's death established a graduate fellowship

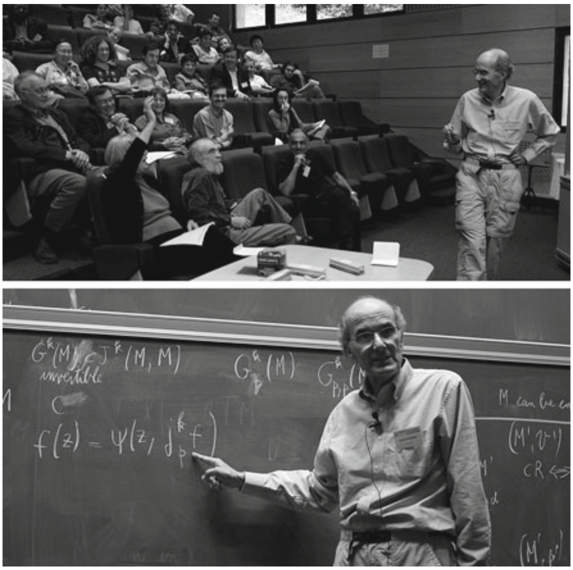

Fig. 3 Salah Baouendi lecturing at IHES (Paris) in August 2007

in his memory. Among Salah's other collaborators and students after moving to UCSD were Xiaojun Huang, Dmitri Zaitsev, Bernhard Lamel, Francine Meylan, and Nordine Mir.

5 Contributions to the Profession

Few scientists have contributed so much to their profession as Salah did. He was a fabulous teacher, earning high praise from students of all levels. He cofounded two mathematical journals, "Communications in Partial Differential Equations" and "Mathematical Research Letters," both of which became top venues and still continue decades later. He co-organized a number of conference series including "Journées equations aux derivées partielles" in France, "Midwest Partial Differential Equations Seminar" in the U.S. Midwest, and "Southern California Analysis and PDE Seminar," all of which are still active.

In his later years Salah played an important role in the international mathematical community. He served as U.S. representative on the Executive Committee of the International Mathematical Union (IMU) and worked on many projects for the IMU, which organizes the "International Conference of Mathematicians" (ICM) every four years. Two decades ago when Salah attended an ICM as a member of the official U.S. delegation, he encountered his old mentor J.-P. Lions, who was part of the French delegation. Seeing Salah seemed to make Lions sad. He said, "You should be representing France, but France did not appreciate you when it had the chance." Tunisia should be proud of its native son.

Compactness of the $\bar{\partial}$-Neumann Operator on the Intersection of Two Domains

Mustafa Ayyürü and Emil J. Straube

In memory of M. Salah Baouendi

Abstract Assume that Ω_1 and Ω_2 are two smooth bounded pseudoconvex domains in $\mathbb{C}^2$ that intersect (real) transversely, and that $\Omega_1 \cap \Omega_2$ is a domain (i.e. is connected). If the $\bar{\partial}$-Neumann operators on Ω_1 and on Ω_2 are compact, then so is the $\bar{\partial}$-Neumann operator on $\Omega_1 \cap \Omega_2$. The corresponding result holds for the $\bar{\partial}$-Neumann operators on $(0, n-1)$-forms on domains in $\mathbb{C}^n$.

Keywords $\bar{\partial}$-Neumann operator · Compactness · Pseudoconvex domains · Intersections of domains

2000 Mathematics Subject Classification 32W05 · 35N15

1 Introduction

Let Ω be a bounded domain in $\mathbb{C}^n$. For $0 \leq q \leq n$, the space of $(0, q)$-forms $u = \sum'_{|J|=q} u_J d\bar{z}_J$, where $u_J \in \mathcal{L}^2(\Omega)$ for each strictly increasing q-tuple J, is denoted by $\mathcal{L}^2_{(0,q)}(\Omega)$. The inner product on $\mathcal{L}^2_{(0,q)}(\Omega)$ is given by

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$$(u, v)_{\mathcal{L}^2_{(0,q)}(\Omega)} := {\sum_{|J|=q}}' \int_\Omega u_J \overline{v_J} dV, \tag{1.1}$$

where the prime denotes summation over strictly increasing q-tuples. The Cauchy-Riemann operator $\bar{\partial}_q$ acting on $(0, q)$-forms is defined as follows:

$$\bar{\partial}_q u = \bar{\partial}_q \left({\sum_{|J|=q}}' u_J d\bar{z}_J \right) = \sum_{k=1}^{n} {\sum_{|J|=q}}' \frac{\partial u_J}{\partial \bar{z}_k} d\bar{z}_k \wedge d\bar{z}_J. \tag{1.2}$$

Here, the derivatives are taken in the distributional sense and the domain of $\bar{\partial}_q$, which we denote by $\mathrm{dom}(\bar{\partial}_q)$, consists of those $(0, q)$-forms with $\bar{\partial}_q u \in \mathcal{L}^2_{(0,q+1)}(\Omega)$. Then $\overline{\partial}_{q+1}\overline{\partial}_q = 0$; the resulting complex is referred to as the $\bar{\partial}$ (or Dolbeault)-complex. $\bar{\partial}_q$ is a linear, densely defined, closed operator, and as such has a Hilbert space adjoint $\bar{\partial}_q^* : \mathcal{L}^2_{(0,q+1)}(\Omega) \to \mathcal{L}^2_{(0,q)}(\Omega)$. The complex Laplacian is then the unbounded operator $\Box_q := \bar{\partial}_{q-1}\bar{\partial}^*_{q-1} + \bar{\partial}^*_q\bar{\partial}_q$, with domain so that the compositions are defined (this imposes a boundary condition not only on a form u, but on $\overline{\partial} u$ as well; these are the $\bar{\partial}$-Neumann boundary conditions). It is a deep result of Hörmander [12] that when Ω is bounded and pseudoconvex, $\Box_q$ is injective and onto, and so has a bounded inverse. This inverse is the $\bar{\partial}$-Neumann operator N_q. Regularity properties of N_q in various function spaces are of great importance, both in several complex variables and in partial differential equations. We refer the reader to [17, 18] for details and historical developments.

One of the properties of interest is compactness of N_q. That is, N_q is not just bounded on $\mathcal{L}^2_{(0,q)}(\Omega)$, but is compact. Compactness is interesting for a number of reasons: it implies $\mathcal{L}^2$-Sobolev estimates for N_q (with all their ramifications) [13], and there are applications to the Fredholm theory of Toeplitz operators [5, 10], to existence or non-existence of Henkin-Ramirez type kernels for solving $\overline{\partial}$ [9], and to certain C^* algebras naturally associated to a domain in $\mathbb{C}^n$ [15]. Details may be found in [7, 18], and in their references.

In this note, we address the question of compactness of the $\bar{\partial}$-Neumann operator on the intersection of two domains, given that the $\bar{\partial}$-Neumann operator on each domain is compact. In addition to its intrinsic interest, the question serves as a test for how well compactness of the $\bar{\partial}$-Neumann operator is understood, in particular with respect to identifying 'the obstruction to compactness'. Namely, if the obstruction is absent from the boundary of both domains, it ought to be absent from the boundary of the intersection. For example, for a convex domain it is known that N_q is compact if and only if the boundary does not contain complex varieties of dimension at least q [6, 7, 18]; that is, these varieties in the boundary form the obstruction to compactness. Clearly, if the boundaries of both domains do not contain these varieties, then neither does the boundary of the intersection. Likewise, if both boundaries repel 'q-dimensional analytic structure' in the sense of the potential theoretic sufficient condition known as Property (P_q) ([2, 7, 18]; see [14] for a variant), then so does the boundary of the intersection (and the $\bar{\partial}$-Neumann operator on the intersection is compact). However, in general, Property (P_q) is not known to be equivalent to compactness, so that its failure does not (more precisely, is not known to) constitute an obstruction to compactness.

We mention that it is straightforward to identify abstractly the obstruction to compactness of the $\bar{\partial}$-Neumann operator via the zero set of the ideal of compactness multipliers introduced in [4]: the $\bar{\partial}$-Neumann operator is compact if and only if this common zero set is empty. However, so far, it is only possible to identify this set in cases where compactness is understood (i.e. convex domains and Hartogs domains in $\mathbb{C}^2$, see [4]). In particular, we do not understand how the ideal of compactness multipliers on the intersection of two domains arises from the respective ideals on the domains.

The material presented here is mainly from the Ph.D. dissertation [1] of the first author, written under the supervision of the second author at Texas A&M University.

2 The $\bar{\partial}$-Neumann Operators on the Intersection of Two Domains

Assume now that $\Omega_1$and Ω_2 are two bounded pseudoconvex domains in $\mathbb{C}^n$ whose intersection $\Omega_1 \cap \Omega_2$ is also a domain (i.e. is connected), and whose $\bar{\partial}$-Neumann operators $N_q^{\Omega_1}$ and $N_q^{\Omega_2}$ are compact, for some q with $1 \leq q \leq n$. Because compactness of N_q is a local property ([18], Proposition 4.4), one can obtain compactness results for $N_q^{\Omega_1 \cap \Omega_2}$ by imposing conditions on $b\Omega_1 \cap b\Omega_2$. In particular, if one assumes that $b\Omega_1 \cap b\Omega_2$ satisfies Property (P_q) mentioned above, or the variant $(\widetilde{P}_q)$ from [14], then $N_q^{\Omega_1 \cap \Omega_2}$ is compact ([1], Theorem 4.1.2). We do not pursue this direction here; instead, we focus on the question discussed in the introduction: obtain compactness on the intersection assuming only compactness on the two domains.

The following result, although formulated for domains in $\mathbb{C}^n$, is most relevant in dimension $n = 2$, as $q = 1$ is the case of most interest.

Theorem 2.1 *Let Ω_1 and Ω_2 be smooth bounded pseudoconvex domains in $\mathbb{C}^n$ which intersect (real) transversely, and assume that $\Omega_1 \cap \Omega_2$ is a domain (i.e. is connected). If the $\bar{\partial}$-Neumann operators $N_{(n-1)}^{\Omega_1}$ and $N_{(n-1)}^{\Omega_2}$ are compact, then so is $N_{(n-1)}^{\Omega_1 \cap \Omega_2}$.*

Proof For economy of notation, we set $\Omega := \Omega_1 \cap \Omega_2$, and $S := b\Omega_1 \cap b\Omega_2$. S is a smooth oriented submanifold of $\mathbb{C}^n$ of real codimension two, and Ω is smooth except at the points of S, where it is only Lipschitz. We also omit subscripts from $\overline{\partial}$, as the form level q is clear from the context.

We first note that compactness of $N_{(n-1)}$ is equivalent to the compactness of the canonical solution operators $\bar{\partial}^* N_{(n-1)}$ and $\bar{\partial}^* N_n$ (see Proposition 4.2 in [18]; compare also Lemma 3 in [3]). $\bar{\partial}^* N_n$ is always compact, because N_n maps $W^{-1}_{(0,n)}(\Omega)$ continuously to $W^1_{(0,n)}(\Omega)$. That is because for $(0, n)$-forms, the $\bar{\partial}$-Neumann problem reduces to the Dirichlet problem (see for example the discussion following estimate (2.94) on p. 36 of [18]), and $\Delta : W_0^1(\Omega) \to W^{-1}(\Omega)$ is an isomorphism (see for instance Theorem 23.1 in [19]). $\bar{\partial}^* N_n$ thus maps $W^{-1}_{(0,n)}(\Omega)$ continuously to $\mathcal{L}^2_{(0,n-1)}(\Omega)$, hence is compact as an operator from $\mathcal{L}^2_{(0,n)}(\Omega) \to \mathcal{L}^2_{(0,n-1)}(\Omega)$ (since $\mathcal{L}^2_{(0,n)}(\Omega)$ embeds compactly into $W^{-1}_{(0,n)}(\Omega)$). Therefore, to show that $N_{(n-1)}$ is compact, it suffices to show that $\bar{\partial}^* N_{(n-1)}$ is compact. This, in turn, will follow if we can show that there is some compact solution operator for $\bar{\partial}$: composing it with the projection onto $\ker(\bar{\partial})^\perp$ (which preserves compactness) gives $\overline{\partial}^* N_{(n-1)}$. That is, it suffices to find a linear compact operator $T : \mathcal{L}^2_{(0,n-1)}(\Omega) \cap \ker(\bar{\partial}) \to \mathcal{L}^2_{(0,n-2)}(\Omega)$ such that $\bar{\partial} T u = u$ for all $u \in \ker(\bar{\partial}) \cap \mathcal{L}^2_{(0,n-1)}(\Omega)$.

The strategy for constructing T is to write a form $\alpha \in \ker(\bar{\partial}_{n-1}) \cap \mathcal{L}^2_{(0,n-1)}(\Omega)$ as

$$\alpha = \beta_1|_\Omega + \beta_2|_\Omega \ , \quad \beta_j \in \ker(\bar{\partial}_{n-1}) \cap \mathcal{L}^2_{(0,n-1)}(\Omega_j), \ j = 1, 2, \tag{2.1}$$

with

$$\|\beta_1\|_{\mathcal{L}^2_{(0,n-1)}(\Omega_1)} + \|\beta_2\|_{\mathcal{L}^2_{(0,n-1)}(\Omega_2)} \lesssim \|\alpha\|_{\mathcal{L}^2_{(0,n-1)}(\Omega)}, \tag{2.2}$$

and β_1 and β_2 depending linearly on α. Then setting $T\alpha := \overline{\partial}^* N^{\Omega_1}_{(n-1)}\beta_1 + \overline{\partial}^* N^{\Omega_2}_{(n-1)}\beta_2$ on Ω gives the desired compact solution operator T. We use here that compactness of $N^{\Omega_1}_{(n-1)}$ and $N^{\Omega_2}_{(n-1)}$ imply compactness of the canonical solution operators $\overline{\partial}^* N^{\Omega_1}_{(n-1)}$ and $\overline{\partial}^* N^{\Omega_2}_{(n-1)}$, respectively (see again [18], Proposition 4.2).

The situation in (2.1) is reminiscent of that in a Cousin problem. We proceed accordingly; extra care is needed because we need to control $\mathcal{L}^2$-norms. Because Ω_1 and Ω_2 intersect transversely, we can choose a partition of unity $\{\varphi, 1-\varphi\}$ of $\Omega_1 \cup \Omega_2$, subordinate to the cover $\{\Omega_1, \Omega_2\}$, with $|\nabla\varphi(z)| \lesssim 1/d_S(z)$; here, d_S denotes the distance to S. We will give details in the appendix (Sect. 3). Now set

$$\widetilde{\beta_1} := (1-\varphi)\alpha \ , \quad \widetilde{\beta_2} := \varphi\alpha. \tag{2.3}$$

We can think of $\widetilde{\beta_1}$ and $\widetilde{\beta_2}$ as forms in $\mathcal{L}^2_{(0,n-1)}(\Omega_1)$ and $\mathcal{L}^2_{(0,n-1)}(\Omega_2)$, respectively, by setting them zero outside Ω. Of course, the forms need not be $\overline{\partial}$-closed. We have

$$\overline{\partial}\widetilde{\beta_1} = -(\overline{\partial}\varphi \wedge \alpha) \ , \ \overline{\partial}\widetilde{\beta_2} = \overline{\partial}\varphi \wedge \alpha \tag{2.4}$$

on Ω_1 and Ω_2 respectively. Now $\overline{\partial}\varphi \wedge \alpha$ is a form on $\Omega_1 \cup \Omega_2$, by setting it equal to zero outside the support of $\nabla\varphi$. If we can write it as $\overline{\partial}\gamma$ on $\Omega_1 \cup \Omega_2$, then setting $\beta_1 := \widetilde{\beta_1} + \gamma$ on Ω_1, and $\beta_2 := \widetilde{\beta_2} - \gamma$ on Ω_2, produces forms that satisfy (2.1) (as the two corrections will cancel in the sum). Of course, we also need to preserve the estimates (2.2) (which are satisfied by $\widetilde{\beta_1}$ and $\widetilde{\beta_2}$).

Because $\overline{\partial}\varphi\wedge\alpha$ is a $(0, n)$-form, we can solve the equation $\overline{\partial}\gamma = \overline{\partial}\varphi\wedge\alpha$ explicitly on $\Omega_1 \cup \Omega_2$, using again that for $(0, n)$-forms, the $\bar{\partial}$-Neumann problem reduces to the Dirichlet problem for the Laplacian. Define g by $\overline{\partial}\varphi \wedge \alpha = g\, d\overline{z_1} \wedge \cdots \wedge d\overline{z_n}$. We use again that $\Delta : W^1_0(\Omega_1 \cup \Omega_2) \to W^{-1}(\Omega_1 \cup \Omega_2)$ is an isomorphism. If we set

$$\gamma = \overline{\partial}^* \left(-4(\Delta^{-1}g)d\overline{z_1} \wedge \cdots \wedge d\overline{z_n}\right), \tag{2.5}$$

then, on $\Omega_1 \cup \Omega_2$,

$$\overline{\partial}\gamma = \overline{\partial}\overline{\partial}^* \left(-4(\Delta^{-1}g)d\overline{z_1} \wedge \cdots \wedge d\overline{z_n}\right) = \left(\Delta(\Delta^{-1}g)\right) d\overline{z_1} \wedge \cdots \wedge d\overline{z_n} = \overline{\partial}\varphi \wedge \alpha. \tag{2.6}$$

We have used here that $\Box$ (which equals $\overline{\partial}\overline{\partial}^*$ on $(0, n)$-forms) acts diagonally as $(-1/4)\Delta$, see for example [18], Lemma 2.11. From (2.5) we immediately obtain

$$\|\gamma\|_{\mathcal{L}^2_{(0,n-1)}(\Omega_1\cup\Omega_2)} \lesssim \|\Delta^{-1}g\|_{W^1_0(\Omega_1\cup\Omega_2)} \lesssim \|g\|_{W^{-1}(\Omega_1\cup\Omega_2)} \simeq \|\overline{\partial}\varphi \wedge \alpha\|_{W^{-1}_{(0,n)}(\Omega_1\cup\Omega_2)}. \tag{2.7}$$

In order to estimate the right hand side of (2.7), we recall that $|\nabla\varphi| \lesssim 1/d_S \lesssim 1/d_{b(\Omega_1\cup\Omega_2)}$, where $d_{b(\Omega_1\cup\Omega_2)}$ denotes the distance to the boundary of $\Omega_1 \cup \Omega_2$. This implies that multiplication by a derivative of φ maps $W^1_0(\Omega_1 \cup \Omega_2)$ continuously into $\mathcal{L}^2(\Omega_1 \cup \Omega_2)$ (see for example [8], Theorem 1.4.4.4; $\Omega_1 \cup \Omega_2$ has a Lipschitz boundary). By duality, this multiplication maps $\mathcal{L}^2(\Omega_1 \cup \Omega_2)$ continuously into $W^{-1}(\Omega_1 \cup \Omega_2)$. As a result, the right hand side of (2.7) is dominated by $\|\widetilde{\alpha}\|_{\mathcal{L}^2_{(0,n-1)}(\Omega_1\cup\Omega_2)} = \|\alpha\|_{\mathcal{L}^2_{(0,n-1)}(\Omega)}$, where $\widetilde{\alpha} = \alpha$ on Ω, and zero otherwise.

Now we set

$$\beta_1 := (1 - \varphi)\alpha + \gamma\ ;\ \ \beta_2 := \varphi\alpha - \gamma. \tag{2.8}$$

Then β_1 and β_2 are $\overline{\partial}$-closed, so that we have (2.1). The estimates above imply that (2.2) also holds. The discussion following (2.2) shows that the proof of Theorem 2.1 is now complete.

Remark One's first tendency would probably be to take the decomposition (2.3) and apply the compactness estimates on Ω_1 and Ω_2 to $\widetilde{\beta}_1$ and $\widetilde{\beta}_2$, respectively. However, derivatives of φ blow up at S; as a result, $\widetilde{\beta}_1$ and $\widetilde{\beta}_2$ are not known to be in $\mathrm{dom}(\overline{\partial}) \cap \mathrm{dom}(\overline{\partial}^*)$ on the respective domains. By contrast, our approach above only requires the estimation of $\|\overline{\partial}\varphi \wedge \alpha\|_{W^{-1}_{(0,n)}(\Omega_1\cup\Omega_2)}$, as in (2.7), rather than $\|\overline{\partial}\varphi \wedge \alpha\|_{\mathcal{L}^2_{(0,n)}(\Omega_j)}$, $j = 1, 2$. This weaker estimate suffices because we can exploit the elliptic gain of the $\bar{\partial}$-Neumann operator on $(0, n)$-forms (which is essentially $\Delta^{-1} : W^{-1}(\Omega_1 \cup \Omega_2) \to W^1_0(\Omega_1 \cup \Omega_2)$, as explained above) to recover the loss that derivatives of φ introduce. It is this part of the argument, more than anything else, that confines us to consider only $(0, n-1)$-forms. For example, the fact that $\Omega_1 \cup \Omega_2$ is not pseudoconvex should be less of an issue. In order to prove Theorem 2.1 via our approach for $(0, q)$-forms, one has to solve $\overline{\partial}$ on $\Omega_1 \cup \Omega_2$ at the level of $(q + 1)$-forms. At least in the context of smooth $\overline{\partial}$-cohomology on $\Omega_1 \cup \Omega_2$, the cohomology groups are trivial at levels $q \geq 2$ (this follows from a Mayer-Vietoris sequence argument; see for example [16], Proposition 3.7; this reference contains a systematic discussion of 'cohomological q-completeness').

3 Appendix

In this section, we show how to construct a partition of unity $\{\varphi, 1-\varphi\}$ on $\Omega_1 \cup \Omega_2$ subordinate to the cover $\{\Omega_1, \Omega_2\}$, such that we have the estimate $|\nabla\varphi| \lesssim 1/d_S$.

Define the unit vector fields X and Y on S as follows. For $\zeta \in S$, $X(\zeta)$ is the unique unit vector perpendicular to S and tangential to Ω_2, such that $X\rho_1(\zeta) < 0$ (i.e. $X(\zeta)$ points inside Ω_1). The latter is possible because derivatives of ρ_1 tangential to Ω_2 and transverse to S do not vanish, by transversality of the intersection of Ω_1 and Ω_2. Y is defined analogously, with the roles of Ω_1 and Ω_2 interchanged. Then the vector $X+Y$ points inside Ω at points of S. Indeed, for $\zeta \in S$, we have

$$(X+Y)\rho_1(\zeta) = X\rho_1(\zeta) + Y\rho_1(\zeta) = X\rho_1(\zeta) < 0. \tag{3.1}$$

We have used that $Y\rho_1 = 0$ (Y is tangential to Ω_1). Equation (3.1) says that $X+Y$ points inside Ω_1 at points of S. Similarly (or by symmetry), this vector also points inside Ω_2, hence inside Ω.

Denote by $D_r \subset \mathbb{C}$ the disc of radius r, centered at 0. We consider a diffeomorphism h from $S \times D_r$, for r sufficiently small, onto a tubular neighborhood V of S (see e.g. [11], Chap. 4), defined as follows:

$$h(\zeta, w) = \zeta + Re(w)X(\zeta) + Im(w)Y(\zeta)\ ,\ \zeta \in S\ ,\ w \in D_r. \tag{3.2}$$

By continuity, there is $\alpha > 0$ such that the sector of D_r where $\pi/4 - \alpha \leq \arg(w) \leq \pi/4 + \alpha$, less the origin, is mapped into Ω, and the opposite sector is mapped into the complement of $\overline{\Omega_1} \cup \overline{\Omega_2}$. Here, $\arg(w)$ denotes the branch of the argument with values between $-3\pi/4$ and $5\pi/4$. Choose a function $\sigma \in C^\infty(\mathbb{R})$ with $0 \leq \sigma \leq 1$, $\sigma \equiv 1$ on $(-\infty, \pi/4 - \alpha]$, and $\sigma \equiv 0$ on $[\pi/4 + \alpha, \infty)$. On $V \cap (\Omega_1 \cup \Omega_2)$, we define φ as follows:

$$\varphi(z) = \sigma(\arg(w))\ ,\ z = h(\zeta, w). \tag{3.3}$$

For points $(\zeta, w) \in h^{-1}(b\Omega_2 \cap \Omega_1 \cap V)$, $\arg(w)$ takes values in the sectors $(-3\pi/4 + \alpha, \pi/4 - \alpha)$, so that for these points, $\sigma(\arg(w)) = 1$ (possibly after shrinking V). This is because S and $X(\zeta)$ span the tangent space to $b\Omega_2$ at ζ, and $X(\zeta)$ points inside Ω_1. Similarly, $\sigma(\arg(w)) = 0$ for points $(\zeta, w) \in h^{-1}(b\Omega_1 \cap \Omega_2 \cap V)$.

Arguing geometrically or directly computing one finds that $|\nabla\sigma(\arg(w))| \lesssim 1/|w|$. Because h is a diffeomorphism and $|w|$ is comparable to d_S, $|\nabla\varphi|$ has the desired upper bound near S.

It remains to extend φ to $\Omega_1 \cup \Omega_2$. First, we extend φ by 0 into a (small enough) neighborhood in Ω_2 of $\Omega_2 \backslash \Omega_1$. Similarly, we extend φ by 1 into a neighborhood in Ω_1 of $\Omega_1 \backslash \Omega_2$. Using a suitable cutoff function, φ so defined on these neighborhoods and V can be extended from a slightly smaller set via a suitable cutoff function to obtain the function we need on $\Omega_1 \cup \Omega_2$.

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CR Complexity and Hyperquadric Maps

John P. D'Angelo

In memory of M. Salah Baouendi

Abstract We survey aspects of CR complexity for maps between spheres and hyperquadrics, provide some new interpretations of the maps found by Lebl and Reiter, and indicate how group-invariance fit into the story.

Keywords CR Geometry · CR complexity · Proper holomorphic mappings · Homotopy equivalence · Spherical equivalence · Hyperquadrics · Unit sphere · Blaschke product · Group-invariant CR maps

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1 Introduction

The author and Peter Ebenfelt have twice (2006, 2010) co-organized meetings at AIM whose titles included the words *CR complexity theory*. As with most developing areas of mathematics, the precise meaning and scope of these words is not yet clear. One thing, however, is certain. Salah Baouendi's work in CR Geometry has been a major part of the development of this story. Let us say roughly that CR complexity theory considers how complicated CR maps between CR manifolds M and M' can be, based on geometric information about M and M'. The book [2] provides a wealth of information about mappings between CR manifolds. This paper will consider some newer and more specific situations.

The first observation is trivial. In a sense one could make precise, but which is not worth doing, for *most* CR manifolds M and M', the only CR maps between them are

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constant. It is therefore natural to consider situations in which non-constant maps exist, and then to find restrictions on the maps based on information about the domain and target manifolds. Hyperquadrics in $\mathbb{C}^n$ provide examples of real hypersurfaces which contain the images of many CR maps. For example, if M is a real hypersurface defined by an arbitrary polynomial equation, then there is a non-constant polynomial mapping from M to a hyperquadric. See [5] for many uses of this idea. See Sect. 7 for additional related remarks.

We begin by considering maps between hyperquadrics. Write variables in $\mathbb{C}^n$ as (z, w), where $w \in \mathbb{C}$. Let $\mathbb{H}_l^n$ denote the real submanifold of $\mathbb{C}^n$ defined by

$$\operatorname{Re}(w) = -\sum_{j=1}^{l} |z_j|^2 + \sum_{j=l+1}^{n-1} |z_j|^2. \tag{1}$$

The case $l = 0$ corresponds to the Heisenberg group, which is locally biholomorphically equivalent to the unit sphere.

Given (l, n) and (L, N), one basic problem is to determine the holomorphic maps from $\mathbb{H}_l^n$ to $\mathbb{H}_L^N$. For $L = l$, Baouendi and Huang [3] proved a remarkable rigidity result in 2005. Assume $n - 1 > 2l > 0$ and $1 < n < N$. If f is a holomorphic map from $\mathbb{H}_l^n$ to $\mathbb{H}_l^N$, and f preserves sides, then f is either constant or, up to composition with automorphisms preserving the origin, the standard linear embedding from $\mathbb{C}^n \to \mathbb{C}^N$. It is possible (see Theorem 6.1) for additional polynomial maps to exist, but they do not preserve sides.

Baouendi et al. [1] considered the case where the number l of negative eigenvalues in the domain hyperquadric and the number l' in the target hyperquadric need not be the same. Assume that this difference is small. Normalize by assuming $0 < 2l \leq n-1$ and $2l' \leq N - 1$. Thus there as many positive eigenvalues as negative eigenvalues in the Hermitian form in the z space used to define the hyperquadrics, and $n \geq 3$. For any holomorphic map ψ, the map

$$(z, w) \to (z, \psi(z, w), 0, \psi(z, w), 0, w) \tag{2}$$

takes $\mathbb{H}_l^n$ to $\mathbb{H}_{l'}^N$, simply because of cancellation. (The zeroes are there to ensure that the number of components is right.) The main result in [1] is that, under fairly general circumstances, the only possible non-constant maps are the compositions of the maps in (2) with automorphisms. This result typifies CR complexity. Maps exist, and with appropriate geometric assumptions, must be of a restricted form.

We pause to note an easy way to pass from (1) to the equation

$$1 = -\sum_{j=1}^{l} |\zeta_j|^2 + \sum_{j=l+1}^{n-1} |\zeta_j|^2 + |\zeta_n|^2. \tag{3}$$

First use $\operatorname{Re}(w) = |w + \frac{1}{4}|^2 - |w - \frac{1}{4}|^2$ in (1). Then divide the equation by the unit $|w + \frac{1}{4}|^2$ and rename coordinates to obtain (3). While the defining equation (1) nicely illustrates the CR aspects of its zero set, the defining equation (3) has certain advantages as well. We write $Q(n-l,l)$ for the real submanifold defined by (3). In particular, $Q(n,0)$ denotes the unit sphere S^{2n-1}.

Lebl [21] and Reiter [25] found all the the holomorphic maps from S^3 to $Q(2,1)$. We discuss this result in detail in Sect. 5. The analogue of the map (2) appears in this setting also, but other rational maps of degree two and three arise as well. Faran [12] had earlier found all the maps from S^3 to S^5. See also [13, 16–19, 22], for various related rigidity results.

Maps from spheres to hyperquadrics arise also in the following situation. Let Γ be a finite subgroup of $U(n)$. Then there is a canonical nonconstant Γ-invariant polynomial mapping $p : \mathbb{C}^n \to \mathbb{C}^N$ such that the image of the unit sphere under p is a hyperquadric. Unless the group is cyclic and represented in one of two particular ways, the target cannot be a sphere. For a given subgroup Γ, the target hyperquadric requires sufficiently many eigenvalues of both signs. Hence there is an interplay between the values of N, l' and representation theory. See [9] and its references for an introduction to this topic. See also Sect. 6 in this paper.

This paper has the modest aim of illustrating CR complexity in several interesting situations. In Sect. 2 we survey results about CR complexity for sphere maps. Section 3 summarizes results from [10] about homotopy equivalence for sphere maps. Section 4 discusses Whitney maps and tensor products, in both the sphere and hyperquadric cases. Section 5 considers the classification due to Lebl [21] and Reiter [25] of maps from $Q(2,0)$ to $Q(2,1)$. We show how to interpret these maps via certain elementary constructions. Of course these authors have already done the hard work. Our discussion indicates how these maps fit into a general framework and how to simplify certain computations. Section 6 uses group-invariance to find a class of polynomial maps between hyperquadrics which do not increase the number of negative eigenvalues. Theorem 6.1 comes from [9].

Section 7 discusses a few older results that could be thought of as prequels to the subject of CR complexity and connects these results to positivity conditions. Thus CR complexity and Hermitian analogues of Hilbert's 17th problem become part of the same story.

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2 Sphere Maps

We will use the term **sphere map** to mean a (holomorphic) rational function f such that $f : S^{2n-1} \to S^{2N-1}$. When $n \geq 2$ such maps are either constant or extend to be proper holomorphic maps from the unit ball $\mathbf{B}_n$ to $\mathbf{B}_N$. A well-known theorem ([14]) of Forstnerič states, when $n \geq 2$, that a proper holomorphic map $f : \mathbf{B}_n \to \mathbf{B}_N$, assumed (sufficiently) smooth on the sphere, must be a rational function. In this setting (see [4]), the map extends holomorphically past the sphere.

The automorphism group of the unit ball, which consists of linear fractional transformations, is transitive. It is therefore natural to define spherical equivalence: sphere maps f, g are *spherically equivalent* if there are automorphisms ϕ in the domain and χ in the target such that $f = \chi \circ g \circ \phi$. The study of sphere maps already leads to CR complexity theory. For $n \geq 3$ and $N \leq 2n-2$, Faran [13] showed that a nonconstant sphere map $f : S^{2n-1} \to S^{2N-1}$ is spherically equivalent to the map $z \mapsto (z, 0) = z \oplus 0$. In [12] he found the four spherical equivalence classes of maps from S^3 to S^5. See Corollary 3.2 for these maps and an additional conclusion.

We mention some additional facts about sphere maps. Assume $n \geq 2$. If $N < n$, then each sphere map is constant. If $N = n \geq 2$, then a non-constant sphere map is an automorphism, and hence of degree 1. If $N \geq 2n$, then there are uncountably many spherical equivalence classes of sphere maps. See [6], and for a stronger result from [10], see Corollary 3.1.

For each n, N (with $n \geq 2$) there is a smallest number $c(n, N)$ such that the degree d of every sphere map $f : S^{2n-1} \to S^{2N-1}$ is at most $c(n, N)$. The sharp value of $c(n, N)$ is not known, but see [11] for the inequality

$$d \leq \frac{N(N-1)}{2(2n-3)}. \tag{4}$$

For $n = 1$, there is no bound, as the maps $z \to z^d$ illustrate. The smallest value of $c(2, N)$ is unknown; in this case there are examples of degree $2N - 3$, and this value is known to be sharp for monomial maps. For $n \geq 3$, the smallest value of $c(n, N)$ is unknown; in this case there are examples of degree

$$d = \frac{N-1}{n-1}.$$

By [22] this bound is sharp for monomial maps. The case $n = 2$ is the most interesting, as phenomena from lower and higher dimensions clash.

It is also interesting that a family of group-invariant *sharp polynomials* exists. See [9] for an entry to the literature in this situation.

Let us return to sphere maps in general. By allowing the target dimension to be sufficiently large, there is essentially no restriction on such maps. More precisely, consider the following result (see [7]).

Theorem 2.1 *Let $\frac{p}{q} : \mathbb{C}^n \to \mathbb{C}^N$ be a rational function such that $||\frac{p}{q}||^2 < 1$ on the closed unit ball. Then there is an integer k and a polynomial mapping $g : \mathbb{C}^n \to \mathbb{C}^k$ such that $\frac{p \oplus g}{q}$ is a sphere map.*

Proof The key point in the proof is that a polynomial $r(z, \overline{z})$ whose values on the sphere are strictly positive agrees with a squared norm there. Thus there is a holomorphic polynomial mapping g for which $r(z, \overline{z}) = ||g(z)||^2$ on the sphere. See [8] for this result and related ideas. We therefore consider $|q|^2 - ||p||^2$, which is strictly positive on the sphere. Hence there is a g with

$$|q|^2 - ||p||^2 = ||g||^2$$

on the sphere. □

We discuss analogues of the key point in Sect. 7. Several of the results discussed there of are of the following flavor: if a function is positive on a set, does it agree with the squared norm of a holomorphic mapping there?

It is useful to place Theorem 2.1 in the context of CR complexity. It is not possible to bound either the degree of g or the dimension k in terms of n and the degrees of p and q alone. In order to achieve sphere maps of arbitrary complexity, one must allow the target dimension to be arbitrarily high. We give one simple example to further illustrate the depth of this result.

Example 2.1 Consider the family of polynomials given by $q_a(z) = 1 - az_1z_2$. For $|a| < 2$, the polynomial q has no zeroes on the closed ball. If we seek a sphere map, one of whose components is $\frac{cz_1}{q(z)}$ (here $c \neq 0$ is a constant), then the minimum possible target dimension for this map tends to infinity as a tends to 2.

3 Homotopy Equivalence for Sphere Maps

Let $U(n)$ denote the group of unitary transformation of $\mathbf{C}^n$. Such transformations are the simplest examples of automorphisms of $\mathbf{B}_n$. We note that $U(n)$, as a connected Lie group, is path connected.

Following [10], we consider two versions of *homotopy equivalence* between rational sphere maps. In one version we assume that f and g have the same target dimension, whereas in the other we allow the target dimensions to differ. We motivate these notion of homotopy for sphere maps by first considering (in Proposition 3.1) maps of the circle. This one-dimensional situation is precise, beautiful, and easy to describe; it therefore tempts us to seek generalizations to higher dimensions.

Proposition 3.3 shows that any pair of proper maps from the same ball are *homotopy equivalent in target dimension M* when M is sufficiently large. Placing restrictions on M then fits nicely into the general framework of CR complexity theory. In particular, given proper maps f and g with the same domain ball, there is a minimal M for which f and g are homotopy equivalent in target dimension M. Computing this dimension for explicit rational maps seems to be difficult.

Example 3.1 is striking; it shows when $n \geq 2$ that the degree of a family of rational proper maps between balls is *not* a homotopy invariant. Theorem 3.1 gives a finiteness result: for $n \geq 2$ and N fixed, the set of homotopy classes of sphere maps from S^{2n-1} to S^{2N-1} is finite. By contrast, the number of distinct spherical equivalence classes is infinite when $N \geq 2n$. Theorem 3.2 and Corollary 3.1 also illustrate the distinction between homotopy and spherical equivalence. Given two rational but spherically inequivalent maps, a homotopy between them must contain *uncountably many* spherically inequivalent maps. It follows that the four maps of Faran from $\mathbf{B}_2$ to $\mathbf{B}_3$ are not homotopic through rational maps in target dimension three, although they are homotopic in target dimension five.

Definition 3.1 Let $f, g : S^{2n-1} \to S^{2N-1}$ be rational holomorphic maps. Then f and g are *homotopic* if, for each $t \in [0, 1]$ there is a rational holomorphic mapping $H_t : S^{2n-1} \to S^{2N-1}$ such that:

- $H_0 = f$ and $H_1 = g$.
- The Taylor coefficients of H_t depend continuously on t.

We sometimes identify a sphere map $f : S^{2n-1} \to S^{2N-1}$ with the map

$$f \oplus 0 = (f, 0) : S^{2n-1} \to S^{2N'-1}.$$

The maps f and $f \oplus 0$ have the same norm, but they are not spherically equivalent because their target dimensions differ. We write $f \sim h$ when $||f||^2 = ||h||^2$.

Definition 3.2 Let $f : S^{2n-1} \to S^{2N_1-1}$ and $g : S^{2n-1} \to S^{2N_2-1}$ be sphere maps. Then f and g are *homotopic in target dimension k* if, for each $t \in [0, 1]$ there is a sphere map $H_t : S^{2n-1} \to s^{2k-1}$ such that:

- $H_0 \sim f \oplus 0$ and $H_1 \sim g \oplus 0$.
- The Taylor coefficients of H_t depend continuously on t.

In these definitions,
the following decisive result in one dimension holds:

Proposition 3.1 *Suppose $f : S^1 \to S^1$ is rational. Then there is a unique integer m such that f is homotopic in dimension 1 to the map $z \mapsto z^m$.*

Proof Each nonconstant such map can be written

$$f(z) = e^{i\theta} \prod_{j=1}^{d} \frac{z - a_j}{1 - \overline{a}_j z}. \tag{5}$$

None of the points a_j, which need not be distinct, lie on the circle. If a_j satisfies $|a_j| < 1$, we replace it by $(1-t)a_j$. If a_j satisfies $|a_j| > 1$, we replace it by $\frac{a_j}{1-t}$. Call the resulting rational function H_t. Each H_t maps the circle to itself, $H_0 = f$ and $H_1 = cz^m$, where $|c| = 1$ and m is the number of zeroes in the disk minus the number of poles in the disk. We can then also deform c into 1. The uniqueness follows because the number of zeroes minus the number of poles of H_t is given by the line integral:

$$\frac{1}{2\pi i} \int_{|z|=1} \frac{H_t'(z)}{H_t(z)} dz.$$

As usual, an integer-valued continuous function is locally constant. □

The situation differs for higher dimensional sphere maps. First we have

Proposition 3.2 *Assume $n \geq 2$. Each sphere map $f : S^{2n-1} \to S^{2n-1}$ is homotopic to the identity.*

Proof By a well-known result of Pinchuk, f must be an automorphism of the ball. We claim that each automorphism ϕ is homotopic to the identity: ϕ is a composition of a unitary transformation U and a linear fractional automorphism of the form

$$z \to \frac{L_a(z) - a}{1 - \langle z, a \rangle}.$$

Here L_a is a linear map depending continuously on a, and a is a point in the unit ball. By multiplying a by $1-t$, and deforming U into the identity, we obtain a family H_t where $H_0(z) = \phi(z)$ and $H_1(z) = z$. □

In case the product in (5) is complex analytic in the disk, the number m is the degree of f. It is a surprising result from [10] that the degree of a rational sphere map in higher dimensions is not a homotopy invariant.

Example 3.1 (10) We define proper polynomial maps f, g from $\mathbb{C}^2$ to $\mathbb{C}^5$; each maps the sphere to the sphere. Both maps have embedding dimension 5. They are of different degree but they are homotopic in target dimension 5.

$$f(z, w) = (z, zw, zw^2, zw^3, w^4).$$

$$g(z, w) = (-w^2, zw, -zw^2, z^2w, z^2).$$

Since each of f and g is a monomial map with five distinct monomials, the embedding dimension in each case is 5. They are endpoints of a one-parameter family of proper maps. Put $t = \cos(\theta) = c$. Write $s = \sin(\theta)$.

$$H_t(z, w) = (cz-sw^2, zw, (cz-sw^2)(sz+cw^2), zw(sz+cw^2), (sz+cw^2)^2). \quad (6)$$

When $t = 0$ in (6) we obtain f and when $t = 1$ in (6) we obtain g. Furthermore, the reader can check that each H_t is a sphere map.

Spherical equivalence implies homotopy equivalence and the converse is obviously false. We note that if f and g are homotopy equivalent in target dimension M_0, then they are homotopy equivalent in target dimension M if $M \geq M_0$.

We continue with the following results from [10].

Proposition 3.3 *Let $f : S^{2n-1} \to S^{2N-1}$ and $g : S^{2n-1} \to S^{2K-1}$ be sphere maps. Then f and g are homotopic in target dimension M if $M \geq n + \max(N, K)$.*

Theorem 3.1 *Let S denote the set of homotopy classes (of rational maps and in target dimension N) of proper rational maps $f : \mathbf{B}_n \to \mathbf{B}_N$. Assume that $n \geq 2$. Then S is a finite set. (For $n = 1$, S is countable by Proposition 3.1.)*

Theorem 3.2 *Assume $n \geq 2$. Let $H_t : \mathbf{B}_n \to \mathbf{B}_N$ be a homotopy of rational proper maps. The set of t in* $[0, 1]$ *such that H_t is spherically equivalent to H_0 is closed in* $[0, 1]$.

Corollary 3.1 *Suppose H_t is a homotopy of proper rational maps between balls in dimension N. If H_0 and H_1 are not spherically equivalent, then H_t contains uncountably many spherically inequivalent maps.*

Example 3.2 It was shown in [6] that the sphere maps

$$z \mapsto H_t(z) = (z_1, \ldots, z_{n-1}, tz_n, \sqrt{1-t^2}z_1 z_n, \ldots, \sqrt{1-t^2}z_n^2) \tag{7}$$

are spherically inequivalent for distinct t in $[0, 1]$. Corollary 3.1 is more general.

Corollary 3.2 *All four Faran maps (8a)–(8d) from S^3 to S^5 are homotopically inequivalent in target dimension* 3 *through rational maps.*

$$f(z, w) = (z, w, 0) \tag{8a}$$

$$g(z, w) = (z^2, zw, w) \tag{8b}$$

$$h(z, w) = (z^2, \sqrt{2}zw, w^2) \tag{8c}$$

$$\phi(z, w) = (z^3, \sqrt{3}zw, w^3). \tag{8d}$$

These maps are in four distinct homotopy classes in target dimension 3, but they are in the same homotopy class in target dimension 5. It is unknown whether the map ϕ is homotopically equivalent to the others in target dimension 4.

4 Whitney Sequences for Sphere and Hyperquadric Maps

First we consider the case of sphere maps. Then we extend some of the techniques to hyperquadric maps.

Let $f : \mathbf{B}_n \to \mathbf{B}_N$ be a proper rational mapping. Let A be a subspace of $\mathbf{C}^N$, and let π_A denote orthogonal projection onto A. Following [5], we may form the new proper mapping $E_A(f)$, defined by

$$E_A(f) = (\pi_A f \otimes z) \oplus (1 - \pi_A)(f).$$

Suppose that B is another subspace of $\mathbf{C}^N$ of the same dimension d as A, and $A \cap B = \{0\}$. Then there is a unitary mapping $U \in U(N)$ such that $U(A) = B$.

Since the unitary group is path connected, we can find a one-parameter family of unitary mappings connecting U to the identity. It follows that the maps $E_A(f)$ and $E_B(f)$ are homotopic in dimension K, where $K = N + d(n-1)$.

Definition 4.1 A *Whitney sequence* is a collection $F_0, F_1, \ldots$ of rational proper maps from $\mathbf{B}_n$ to $\mathbf{B}_{N_k}$ defined as follows. Put $F_0(z) = \phi_0$, where ϕ_0 is an automorphism of $\mathbf{B}_n$. Given $F_k : \mathbf{B}_n \to \mathbf{B}_{N_k}$, let A_k be a non-zero subspace of $\mathbf{C}^{n_k}$, and let π_k denote orthogonal projection onto A_k. Choose an automorphism ϕ_k of $\mathbf{B}_n$. Choose a linear, norm-preserving injection j_k to whatever target dimension we wish. Define $F_{k+1}(z)$ by

$$F_{k+1} = j_k \circ \big((\pi_k F_k \otimes \phi_k) \oplus (1 - \pi_k)F_k\big).$$

The degree of the rational function F_k is at most $k+1$, but it can be smaller. By [10], each F_k in a Whitney sequence is homotopic to a monomial proper mapping of degree $k+1$.

The maps H_t in Example 3.1 are each part of a Whitney sequence. The degree is not a homotopy invariant because the tensor products are taken on different subspaces, and hence the tensor product need not increase the degree.

Not every proper rational mapping is a term of a Whitney sequence. For example, even the monomial map $(z, w) \to (z^3, \sqrt{3}zw, w^3)$ cannot be obtained in this fashion. One must allow also the inverse operation of replacing F_{k+1} with F_k in the procedure above. See the last paragraph of this section.

More information on the tensor product operation for sphere maps appears, for example, in [5, 7]. A similar idea applies for hyperquadric maps. The key point is the following essentially trivial result. Given the set up in the proposition, we define a polynomial mapping from $\mathbb{C}^{n-l} \times \mathbb{C}^l$ to some $\mathbb{C}^K \times \mathbb{C}^L$ by

$$E_{A,B}(f, g) =$$

$$\big((\pi_A f \otimes z) \oplus (\pi_B g \otimes w) \oplus (1 - \pi_A)f, (\pi_B g \otimes z) \oplus (\pi_A f \otimes w) \oplus (1 - \pi_B)g\big).$$

Proposition 4.1 *Let $(f, g) : \mathbb{C}^{n-l} \times \mathbb{C}^l \to \mathbb{C}^{N_1} \times \mathbb{C}^{N_2}$ be a polynomial mapping whose components are linearly independent. Write (z, w) for the variables in the domain. Let A be a subspace of $\mathbb{C}^{N_1}$ and let B be a subspace of $\mathbb{C}^{N_2}$. Let $\pi_A : \mathbb{C}^{N_1} \to A$ and $\pi_B : \mathbb{C}^{N_2} \to B$ denote orthogonal projection onto these spaces. Define $E_{A,B}(f, g)$ as above. If $(f, g) : Q(n-l, l) \to Q(N_1, N_2)$, then $E_{A,B}(f, g) : Q(n-l, l) \to Q(K, L)$. The integers (K, L) are determined by linear algebra.*

Proof Let $Q(n-l, l)$ denote the set where $||z||^2 - ||w||^2 = 1$. We start with $||f||^2 - ||g||^2 = 1$ on $Q(n-l, l)$. We write $f = \pi_A f \oplus (1 - \pi_A f)$ and similarly for g to obtain

$$1 = ||\pi_A f||^2 + ||(1 - \pi_A)f||^2 - ||\pi_B g||^2 - ||(1 - \pi_B)g||^2 \tag{9}$$

on the set $Q(n-l, l)$. We multiply the first and third terms on the right-hand side of (9) by $||z||^2 - ||w||^2$, which is 1 on $Q(n-l, l)$.

We then use the two formal identities

$$||h \otimes H||^2 = ||h||^2\ ||H||^2$$

$$||h \oplus H||^2 = ||h||^2 + ||H||^2$$

each several times to obtain the result. □

Proposition 4.1 allows us to construct many polynomial and rational maps via iterating the tensor product operation. We obtain analogues of Whitney sequences by also allowing automorphisms. This operation does not produce all rational sphere maps; even in the polynomial case one must also allow a kind of *tensor division*. Here is the simplest example.

Consider the map ϕ from (8d). It is obtained from the homogeneous map

$$H_3(z, w) = (z^3, \sqrt{3}z^2 w, \sqrt{3}zw^2, w^3)$$

by replacing the middle two components with the single component $\sqrt{3}zw$. This procedure is the tensor division. The homogeneous map H_3 itself is essentially a tensor product. To obtain it, first tensor the identity map with itself on the full space three times. The result maps to $\mathbb{C}^8$. After a unitary change of coordinates, we obtain

$$(z^3, \sqrt{3}z^2 w, \sqrt{3}zw^2, w^3, 0, 0, 0, 0).$$

Finally H_3 is obtained by dropping the zeroes.

5 Hyperquadric Maps

We first state the result of Lebl and Reiter which finds all the equivalence classes of maps from the sphere S^3 to the hyperquadric $Q(2, 1)$. We write variables in $\mathbb{C}^2$ as (z, w), and hence $Q(2, 0)$ (the sphere) is defined by $|z|^2+|w|^2 = 1$. The hyperquadric $Q(2, 1) \subset \mathbb{C}^3$ will be defined by

$$|\zeta_1|^2 + |\zeta_2|^2 - |\zeta_3|^2 = 1.$$

Theorem 5.1 (Lebl, Reiter) *Fix $p \in S^3$. Let Ω be a connected open neighborhood of p. Assume $f : \Omega \to \mathbb{C}^3$ is holomorphic and $f(\Omega \cap Q(2, 0)) \subset Q(2, 1)$. Then up to equivalence via automorphisms, f is equivalent to one of the following maps:*

$$(z, w, 0) \tag{10a}$$

$$(z^2, \sqrt{2}w, w^2) \tag{10b}$$

$$\frac{1}{z}(z^2, w, w^2) \tag{10c}$$

$$\frac{1}{6w^2-2}\left(4z^3, (6-2w^2)w, \sqrt{3}z(2+2w^2)\right) \tag{10d}$$

$$\frac{1}{1+\sqrt{2}z+w}\left((2+\sqrt{2}z)z, (1+\sqrt{2}z+w)w, (1+\sqrt{2}z-w)z\right) \tag{10e}$$

$$\frac{1}{1-w-w^2}\left((1-w)z, 1+w-w^2, (1+w)z\right) \tag{10f}$$

$$(1, h(z, w), h(z, w)). \tag{10g}$$

In (10g), h is an arbitrary holomorphic function.

This result beautifully illustrates CR complexity. The example (10g) arises because of cancellation. The other cases are all of degree at most 3. In other words, once one eliminates arbitrary maps which arise through cancellation, the degree is bounded. We next discuss properties of the specific maps in Theorem 5.1.

The map (10b) arises from the simplest special case of a general construction (see [5, 9]) of group-invariant CR maps from spheres to hyperquadrics. Given a finite subgroup Γ of $U(n)$, we define Φ_Γ by

$$\Phi_\Gamma(z, \overline{z}) = 1 - \prod_{\gamma \in \Gamma}(1 - \langle \gamma z, z \rangle). \tag{11}$$

We write $\Phi_\Gamma = ||f||^2 - ||g||^2$ for Γ-invariant holomorphic polynomials maps f, g; we obtain a canonical Γ-invariant map $f \oplus g$ from a sphere to a hyperquadric. We note in passing that the Faran maps (8c) and (8d) are also group-invariant. Furthermore, there are many relationships between group-invariant maps, CR complexity, and algebraic combinatorics. See Sect. 6 and [9] for a glimpse of these ideas.

To obtain (10b), let Γ be the cyclic subgroup of order 2 of $U(2)$ generated by $(z, w) \to (-z, w)$. Here w is not changed. Note that the representation is reducible. The corresponding Γ-invariant CR Map ζ is determined from (11) by writing

$$\Phi_\Gamma(z, w, \overline{z}, \overline{w}) = 1 - (1 - |w|^2 - |z|^2)(1 - |w|^2 + |z|^2)$$

$$= |z|^4 + 2|w|^2 - |w|^4 = |\zeta_1(z, w)|^2 + |\zeta_2(z, w)|^2 - |\zeta_3(z, w)|^2.$$

Thus ζ is the map in (10b).

The map in (10b) also arises from the technique implicit in Proposition 4.1. Here is the computation on the sphere:

$$1 = (|z|^2 + |w|^2)^2 = |z|^4 + 2|z|^2|w|^2 + |w|^4$$
$$= |z|^4 + 2(1 - |w|^2)|w|^2 + |w|^4 = |z|^4 + 2|w|^2 - |w|^4. \tag{12}$$

We start with the identity. Via two tensor products we form the sphere map (8c) to $Q(3, 0)$ and look at its squared norm. We find a term including $|z|^2$ as a factor, and replace $|z|^2$ with $(1 - |w|^2)$. The result creates a negative eigenvalue and a new map. Given a map to $Q(A, B)$, this method creates a new map going to $Q(A, B + 1)$. In (12), we were lucky; the newly created negative term canceled one of the positive terms, and hence our map had target $Q(2, 1)$ rather than $Q(3, 1)$.

The map (10c) can also be understood in several ways. For $z \neq 0$, note that

$$\frac{1 - |w|^2}{|z|^2} = 1$$

on the unit sphere. Hence, on the unit sphere, we have

$$1 = |z|^2 + |w|^2 = |z|^2 + |w|^2 (\frac{1 - |w|^2}{|z|^2}) =$$

$$\frac{|z|^4 + |w|^2 - |w|^4}{|z|^2} = |\zeta_1(z, w)|^2 + |\zeta_2(z, w)|^2 - |\zeta_3(z, w)|^2.$$

We obtain (10c).

Another way to obtain (10c) uses the symmetry of interchanging the variables, a unitary change in the domain. Consider the map $(z, w) \mapsto (z^2, w) = h(z, w)$. On the sphere, we have

$$|z|^4 + |w|^2 = ||h(z, w)||^2 = ||h(w, z)||^2 = |w|^4 + |z|^2. \tag{13}$$

Dividing both sides of (13) by $|z|^2$ and moving the $|\frac{w^2}{z}|^2$ term over yields (10c). We could have divided instead by $|w|^2$ and found an equivalent map. More interesting is to divide by $|z|^4$. Then we obtain the map

$$\frac{1}{z^2}(w^2, z, w)$$

from Lebl's list [21], which (by [25]) is also equivalent to (10c).

We note the following abstraction of the discussion of (10c). Suppose f and g are polynomial maps such the components of f and g are linearly independent, f has rank A, and g has rank B. Suppose that $||f||^2 = ||g||^2$ on $Q(a, b)$, in analogy with (13) on the sphere. Let g^* denote the first $B - 1$ components of g. Then

$$||\frac{f}{g_B}||^2 - ||\frac{g^*}{g_B}||^2 = 1$$

on $Q(a, b)$, and we obtain a rational map to $Q(A, B - 1)$. We can then generate more examples by using Proposition 4.1.

The map (10f) can be verified and understood in several ways. One way starts with the map $h : \mathbb{C} \to \mathbb{C}^3$ defined by

$$w \mapsto h(w) = (1 + w, w - w^2, 1 - w - w^2). \tag{14}$$

Although h is not invariant under $w \mapsto -w$, its squared norm $||h||^2$ is invariant. One checks easily that

$$||h(w)||^2 = |1 - w^2|^2 + |w|^4 + 3|w|^2 + 1 = ||h(-w)||^2.$$

Hence, there is a unitary U such that $h(-w) = Uh(w)$. On the sphere, we have

$$|\pm w - w^2|^2 = |w|^2|\pm 1 - w|^2 = (1 - |z|^2)|\pm 1 - w|^2 = |\pm 1 - w|^2 - |z(\pm 1 - w)|^2. \tag{15}$$

Plugging these expressions into $||h(w)||^2 = ||h(-w)||^2$ and canceling like terms verifies that (10f) maps the sphere into the hyperquadric $Q(2, 1)$.

We next offer a rather different derivation of the mapping in (10f). Consider polynomials f_1, f_2, f_3, f_4 in a single complex variable w and assume they have no common factor. The polynomial f_4 will be the denominator of our desired map. We want the identity (16) to hold on the sphere:

$$|zf_1(w)|^2 + |f_2(w)|^2 = |zf_3(w)|^2 + |f_4(w)|^2. \tag{16}$$

Replacing $|z|^2$ by $1 - |w|^2$ shows that (16) holds if and only if

$$|f_1(w)|^2 + |f_2(w)|^2 + |wf_3(w)|^2 = |wf_1(w)|^2 + |f_3(w)|^2 + |f_4(w)|^2 \tag{17}$$

holds for all w. But this equality can be rewritten in the following manner, which stays within the framework of polynomials in a single variable w. Equation (17) holds if and only if there is a 3-by-3 unitary matrix $U = (u_{jk})$ such that

$$\begin{pmatrix} f_1 \\ f_2 \\ wf_3 \end{pmatrix} = U \begin{pmatrix} wf_1 \\ f_3 \\ f_4 \end{pmatrix}. \tag{18}$$

We can regard (18) as a system of 3 equations in 4 unknowns. Finding all solutions to such a system is tedious but possible. The polynomials $f_1 = 1 - w$, $f_2 = 1 + w - w^2$, $f_3 = 1 + w$, and $f_4 = 1 - w - w^2$ satisfy the three equations in (18).

A similar analysis applies to (10e). It is convenient to write $\beta = 1 + \sqrt{2}z$. Then $|\beta - 1|^2 = 2(1 - |w|^2)$ on the sphere. Put $f_1 = (1 + \beta)z$, $f_2 = (\beta + w)w$, $f_3 = (\beta - w)z$, and $f_4 = \beta + w$. The identity (16) is changed only by allowing the f_j to depend on β. The analogues of (17) and (18) are changed in the same way.

We verify that (10e) works by using the parallelogram law. That law gives both of the following identities:

$$|\beta + w|^2 + |\beta - w|^2 = 2|w|^2 + 2|\beta|^2$$

$$|\beta + 1|^2 + |\beta - 1|^2 = 2 + 2|\beta|^2.$$

Subtract the first identity from the second. Using $|\beta - 1|^2 = 2(1 - |w|^2)$ we get

$$|\beta + 1|^2 - |\beta - w|^2 = |\beta + w|^2 \tag{19}$$

and therefore

$$\left(|\beta + 1|^2 - |\beta - w|^2\right)(1 - |w|^2) = |\beta + w|^2(1 - |w|^2). \tag{20}$$

But (19) and (20) together give

$$|f_1|^2 - |f_3|^2 = |f_4|^2 - |f_2|^2$$

on the sphere. Hence (10e) maps the sphere to $Q(2, 1)$.

The map (10d) can be found in a similar manner as for (10f), except that we replace (16) with

$$|z^3 f_1(w)|^2 + |f_2(w)|^2 = |zf_3(w)|^2 + |f_4(w)|^2. \tag{21}$$

Now we must replace $|z|^6$ with $(1 - |w|^2)^3$. Doing so leads to the following analogue of (17).

$$\begin{aligned} &|f_1(w)|^2 + |\sqrt{3}w^2 f_1(w)|^2 + |f_2(w)|^2 + |wf_3(w)|^2 = \\ &\quad |\sqrt{3}wf_1(w)|^2 + |w^3 f_1(w)|^2 + |f_3(w)|^2 + |f_4(w)|^2. \end{aligned} \tag{22}$$

We must find a 4-by-4 unitary matrix, but solving the equations is possible.

The crucial point involves CR complexity. Finding all the maps as in [21] or [25] involves massive computations. The maps have fairly simple structure and can be found and understood by elementary means. We can eliminate the map (10g) by seeking only maps with linearly independent components. If we knew *a priori* that the degrees of all other maps were bounded, then we could reduce to a tractable problem.

Things differ when the domain manifold is a hyperquadric rather than a sphere. The paper [16] establishes a rigidity theorem for CR maps between hyperquadrics in the spirit of the results of [1, 3]. Given a real-analytic CR mapping of a hyperquadric (but not a sphere) to another hyperquadric $Q(A, B)$, there are two distinct possibilities:

- The image of the mapping is contained in a complex affine subspace. (The analogue of (10g)).

- The number of positive eigenvalues A is bounded by a constant depending only on B.

Furthermore, when both A and B are sufficiently large and comparable, there exist CR maps whose image is not contained in a hyperplane.

6 Preserving the Number of Negative Eigenvalues

The following result from [9] illustrates a failure of rigidity. It also points out a subtle difference between sphere maps and hyperquadric maps. For a sphere map f, the function $||f||^2$ is plurisubharmonic and hence satisfies the maximum principle. Thus f maps the ball to the ball (it preserves sides) if it is not constant. For a map $f \oplus g$ whose target is a hyperquadric, however, the analogous function $||f||^2 - ||g||^2$ will not in general be plurisubharmonic. Hence, the hypothesis in [3] about preserving sides is meaningful. One can interpret the following result as providing a counterexample when this hypothesis is omitted.

Theorem 6.1 *For each odd positive number $2p+1$, there is a positive integer $N(p)$ and a polynomial g_p of degree $2p$ such that*

$$g_p : Q(2, 2p+1) \rightarrow Q(N(p), 2p+1),$$

and such that g_p maps to no hyperquadric $Q(a, b)$ with $a < N(p)$ or $b < 2p+1$.

This result does not contradict the theorem of [3]. This mapping g_p does not preserve sides of the hyperquadric. By their result, if g_p also preserved sides, then it would have to be linear.

The mapping in Theorem 6.1 is constructed as follows. First one considers a cyclic group of order $2p$, represented as the subgroup of $U(2)$ generated by a diagonal matrix whose eigenvalues are ω and ω^2, where ω is a primitive pth root of unity. The invariant polynomial Φ_Γ has the following formula:

$$\Phi_\Gamma(z, \overline{z}) = 1 - \prod_{j=0}^{2p-1} (1 - \omega^j |z_1|^2 - \omega^{2j} |z_2|^2). \tag{23}$$

Putting $x = |z_1|^2$ and $y = |z_2|^2$ in (23) yields a polynomial $f(x, y)$ in two real variables x, y with these properties:

- $f(x, y) = 1$ on $x + y = 1$.
- $f(\omega x, \omega^2 y) = f(x, y)$.
- f has $p + 1$ positive terms and 1 negative term, namely $-y^{2p}$.
- $f(-x, y) = f(x, y)$.

In fact, f has the explicit formula

$$(\frac{x+\sqrt{x^2+4y}}{2})^{2p}+(\frac{x-\sqrt{x^2+4y}}{2})^{2p}-y^{2p}.$$

Consider the polynomial F in $2p+3$ variables defined by

$$F(x_1,\ldots,x_{2p+1},y_1,y_2)=f(-\sum_{j=1}^{2p+1}x_j,y_1+y_2).$$

The polynomial F is 1 on the set

$$y_1+y_2-\sum_{j=1}^{2p+1}x_j=1.$$

Replace each x_j by $|z_j|^2$. Replace y_j by $|\zeta_j|^2$. This set becomes the hyperquadric $Q(2,2p+1)$. The polynomial F has $2p+1$ negative terms, arising from expanding $(y_1+y_2)^{2p}$. Hence the number of negative terms is preserved. We define g_p by $g_p=u\oplus v$, where $||u(\zeta,z)||^2$ equals the positive terms in F and $-||v(\zeta,z)||^2$ equals the negative terms in F. The claimed properties for the polynomial map g_p follow.

7 Embeddings and Positivity Conditions

We conclude by discussing several earlier results that helped set the stage for CR complexity and create links with the study of positivity conditions. In 1985, Løw [23] showed that a strongly pseudoconvex domain D in $\mathbb{C}^n$ with twice differentiable boundary can be embedded as a closed submanifold of the unit ball in $\mathbb{C}^N$ for N sufficiently large. The embedding can be made continuous at the boundary. Furthermore, a positive function on bD agrees there with the squared-norm of a holomorphic map $f:D\to\mathbb{C}^N$; here f is holomorphic on D and continuous on the boundary.

In 1986, Forstnerič [15] showed that there exist strongly pseudoconvex domains D in $\mathbb{C}^n$ with real-analytic boundary such that no proper mapping $f:D\to\mathbf{B}_N$ extends smoothly to the closure of D. In fact, for most real-analytic hypersurfaces no *formal* embedding into a sphere exists. Hence one must consider embeddings into infinite-dimensional spaces.

Lempert [20] considered the possibility of embedding strongly pseudoconvex hypersurfaces into Hilbert spaces. He established the following result.

Theorem 7.1 (Lempert) *Any compact, real-analytic, strictly pseudoconvex hypersurface $M\subseteq\mathbb{C}^n$ admits a real-analytic CR embedding into the unit sphere of the Hilbert space l^2.*

Furthermore, a real-analytic function positive on M agrees there with a convergent sum $\sum_j |f_j(z)|^2$ (the squared norm of a holomorphic map to l^2). Here the functions are holomorphic on the strongly pseudoconvex domain inside M.

Given a real-analytic real-valued function r, defined near a point $p \in \mathbb{C}^n$, there is a neighborhood of p and sequences of holomorphic functions f_j and g_j such that

$$r(z, \overline{z}) = \sum |f_j(z)|^2 - \sum |g_j(z)|^2.$$

These sums converge near p. We can regard this decomposition as determining a local holomorphic embedding into the unit ball of an *indefinite* Hilbert space. An indefinite Hilbert space is a complex vector space with a continuous bilinear form that is not positive definite. This approach is analogous to regarding the hyperquadric in $\mathbb{C}^n$ as the unit sphere in an indefinite finite-dimensional Hilbert space. Lempert showed however that it is **not** always possible to find proper holomorphic embeddings from domains with real-analytic boundary to the unit ball in an indefinite Hilbert space.

Theorem 7.2 (Lempert) *There is a bounded domain $D \subseteq \mathbb{C}^2$ with real-analytic boundary that does not admit a proper holomorphic embedding f into the unit ball of any indefinite Hilbert space.*

These results are closely connected to the author's study (see [8] for a survey and many references) of Hermitian analogues of Hilbert's 17th problem. Under what circumstances is a non-negative polynomial (or real-analytic function) a squared norm of a holomorphic mapping, or more generally, the quotient of squared norms of holomorphic mappings? For example, a polynomial that is strictly positive on the unit sphere agrees with a squared norm there. By a result of Putinar-Scheiderer (see [24] and also [8]), the conclusion fails for general strongly pseudoconvex hypersurfaces, even when the defining equation is a polynomial. In Løw's result, a positive function r agrees with a squared norm $||f||^2$ of a holomorphic map f to a finite-dimensional space, but the resulting f cannot be chosen to be a polynomial even when r is a polynomial. In Lempert's result, a positive function agrees with a squared norm of a holomorphic map to a Hilbert space, but the resulting f cannot in general be chosen to map to a finite-dimensional space.

Theorem 2.1 of this paper also illustrates how results about proper mappings between balls are linked with squared norms. Recall the key point used in its proof: a polynomial that is strictly positive on the unit sphere agrees with the squared norm of a holomorphic polynomial mapping there. The conclusion fails for non-negative polynomials and can also be interpreted as a statement about CR complexity.

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Logarithmic Littlewood-Paley Decomposition and Applications to Orlicz Spaces

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In memory of M. Salah Baouendi

Abstract This paper is devoted to the construction of a logarithmic Littlewood-Paley decomposition. The approach we adopted to carry out this construction is based on the notion introduced in (Bahouri, Trends Math pp 1–15 (2013), [3]) of being log-oscillating with respect to a scale. The relevance of this theory is illustrated on several examples related to Orlicz spaces.

Keywords Orlicz · log-oscillating · Littlewood-Paley decomposition

2010 Mathematics Subject Classification Primary 35L70 · 35B33; Secondary 35B40

1 Introduction and Statement of the Results

1.1 Setting of the Problem

The aim of this paper is to construct a logarithmic Littlewood-Paley decomposition taking advantage of the notion introduced in [3] of being log-oscillating with respect to a scale. Our main motivation to carry out this decomposition is that it provides a new point of view in the understanding of the Orlicz spaces $\mathcal{L}(\mathbb{R}^{2N})$. Recall that

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generally the Orlicz spaces are defined as follows (for a complete presentation and more details, we refer the reader to [24]):

Definition 1.1 Let $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ be a convex increasing function such that

$$\phi(0) = 0 = \lim_{s\to 0^+} \phi(s) \quad \text{and} \quad \lim_{s\to\infty} \phi(s) = \infty\,.$$

We say that a measurable function $u : \mathbb{R}^d \to \mathbb{C}$ belongs to L^ϕ if there exists $\lambda > 0$ such that

$$\int_{\mathbb{R}^d} \phi\left(\frac{|u(x)|}{\lambda}\right) dx < \infty\,.$$

We denote then

$$\|u\|_{L^\phi} = \inf\left\{\lambda > 0, \quad \int_{\mathbb{R}^d} \phi\left(\frac{|u(x)|}{\lambda}\right) dx \leq 1\right\}. \tag{1.1}$$

The space $\mathcal{L}(\mathbb{R}^{2N})$ it will be question in this paper is the Orlicz space associated to the function $\phi(s) = \mathrm{e}^{s^2} - 1$. This space intervenes via the following sharp Moser-Trudinger type inequalities (see [1, 2, 25, 26] for further details):

Proposition 1.2

$$\sup_{\|u\|_{H^N(\mathbb{R}^{2N})}\leq 1} \int_{\mathbb{R}^{2N}} \left(\mathrm{e}^{\beta_N |u(x)|^2} - 1\right) dx < \infty\,, \tag{1.2}$$

where $\beta_N = \dfrac{2N\pi^{2N}2^{2N}}{\omega_{2N-1}}$, *with* $\omega_{2N-1} = \dfrac{2\,\pi^N}{(N-1)!}$ *the measure of the unit sphere* $\mathbb{S}^{2N-1}$.

Indeed, Estimate (1.2) leads obviously to the Sobolev embedding

$$H^N(\mathbb{R}^{2N}) \hookrightarrow \mathcal{L}(\mathbb{R}^{2N})\,, \tag{1.3}$$

whose lack of compactness has been investigated by several authors (for further details, we refer to [7, 9, 12, 21, 22, 27]). Since the works of P.-L. Lions ([21, 22]), it is well understood that the defect of compactness of the Sobolev embedding (1.3) in 2D is due to two reasons. The first reason is the lack of compactness at infinity that can be illustrated by the sequence $u_n(x) = \varphi(x + x_n)$, where $0 \neq \varphi \in \mathcal{D}$ and $|x_n| \to \infty$, and the second reason is of concentration-type and can be highlighted by the example by Moser (see [21–23]) defined by:

$$f_{\alpha_n}(x) = \begin{cases} \sqrt{\frac{\alpha_n}{2\pi}} & \text{if} \quad |x| \leq \mathrm{e}^{-\alpha_n}, \\ -\frac{\log|x|}{\sqrt{2\alpha_n\pi}} & \text{if} \quad \mathrm{e}^{-\alpha_n} \leq |x| \leq 1, \\ 0 & \text{if} \quad |x| \geq 1, \end{cases}$$

where $\underline{\alpha} := (\alpha_n)$ is a sequence of positive real numbers going to infinity. Recall that by straightforward computations (detailed for instance in [7]):

$$f_{\alpha_n} \rightharpoonup 0 \quad \text{in} \quad H^1(\mathbb{R}^2) \quad \text{and} \quad \|f_{\alpha_n}\|_{\mathcal{L}(\mathbb{R}^2)} \to \frac{1}{\sqrt{4\pi}}, \quad \text{as} \quad n \to \infty \cdot \tag{1.4}$$

Contrary to the case of the elementary concentrations involved in the framework studied by P. Gérard in [18] (see also [5, 15, 19]) concerning the critical Sobolev embedding

$$\dot{H}^s(\mathbb{R}^d) \hookrightarrow L^p(\mathbb{R}^d)\,, \tag{1.5}$$

with $0 \leq s < d/2$ and $p = 2d/(d-2s)$, the frequencies of the sequence $(f_{\alpha_n})_{n\geq 0}$ are spread. More precisely with the vocabulary of [18] (see also Definition 1.5 in this paper), the sequence $(\nabla f_{\alpha_n})_{n\geq 0}$ is "unrelated" to any scale. As it has been emphasized in [18] that the characteristic of being unrelated to any scale is measured using the Besov norm $\dot{B}^0_{2,\infty}$ (see for example [6] for a detailed exposition on Besov spaces), this gives rise to

$$\|\nabla f_{\alpha_n}\|_{\dot{B}^0_{2,\infty}(\mathbb{R}^2)} \longrightarrow 0, \quad \text{as} \quad n \to \infty\,. \tag{1.6}$$

Actually in [11], we have generalized this phenomenon to the 2ND case, which implies that the classical Besov space $B^N_{2,\infty}(\mathbb{R}^{2N})$ does not embed into the Orlicz space $\mathcal{L}(\mathbb{R}^{2N})$. We will rather see in Paragraph 3.1 that a more suitable Besov space built up from the logarithmic Littlewood-Paley decomposition embeds in the Orlicz space.

Let us end this paragraph by noting that in general it can be inferred from Moser-Trudinger inequalities (1.2) that

$$H^N(\mathbb{R}^{2N}) \hookrightarrow L^{\phi_p}(\mathbb{R}^{2N}), \quad \forall p \geq 1\,, \tag{1.7}$$

where $L^{\phi_p}(\mathbb{R}^{2N})$ denotes the Orlicz space associated to the function

$$\phi_p(s) = \mathrm{e}^{s^2} - \sum_{\mathrm{k}=0}^{\mathrm{p}-1} \frac{\mathrm{s}^{2\mathrm{k}}}{\mathrm{k}!}\,\cdot$$

1.2 Background Material

The notion of log-oscillating sequences have proved to be very efficient in the characterization of the lack of compactness of the critical Sobolev embedding (1.3) in the 2ND general case (see [11] for further details). Let us then start by recalling this notion and some basic related facts:

Definition 1.3 Let $v := (v_n)_{n\geq 0}$ be a bounded sequence in $L^2(\mathbb{R}^d)$ and $\underline{\alpha} := (\alpha_n)_{n\geq 0}$ be a sequence of positive real numbers going to infinity.

- The sequence v is said $\underline{\alpha}$ log-oscillating if[1]

$$\limsup_{n\to\infty} \left(\int_{|\xi|\leq \mathrm{e}^{\frac{\alpha_n}{R}}} |\widehat{v_n}(\xi)|^2\, d\xi + \int_{|\xi|\geq \mathrm{e}^{R\alpha_n}} |\widehat{v_n}(\xi)|^2\, d\xi\right) \stackrel{R\to\infty}{\longrightarrow} 0\,. \tag{1.8}$$

- The sequence v is said log-unrelated to the scale $\underline{\alpha}$ if for any real numbers $b > a > 0$

$$\int_{\mathrm{e}^{a\alpha_n}\leq|\xi|\leq \mathrm{e}^{b\alpha_n}} |\widehat{v_n}(\xi)|^2\, d\xi \stackrel{n\to\infty}{\longrightarrow} 0\,. \tag{1.9}$$

Remark 1.4

- Clearly the notion of log-oscillating is only relevant for scales $(\alpha_n)_{n\geq 0}$ converging towards infinity.
- Inspired by the counter-example of P. Gérard in [18], one can prove the existence of sequences log-unrelated to any scale which nevertheless do not converge strongly to 0 in $L^2(\mathbb{R}^d)$. To be convinced, let us consider in $L^2(\mathbb{R}^d)$ the sequence $(v_n)_{n\geq 3}$ defined by:

$$\widehat{v_n}(\xi) := \frac{1}{\sqrt{\log(\log n)}}\, \frac{\widehat{\psi}(\frac{\xi}{n})}{\sqrt{(1+|\xi|^d|\log|\xi||)}}\,, \tag{1.10}$$

where ψ is a function in $\mathcal{S}(\mathbb{R}^d)$ satisfying $\displaystyle\int_{\mathbb{R}^d} \psi(x)dx \neq 0$.

On the one hand by straightforward computations, we get for any sequence (α_n) tending to infinity and any real numbers $b > a > 0$

$$\int_{\mathrm{e}^{a\,\alpha_n}\leq|\xi|\leq \mathrm{e}^{b\,\alpha_n}} |\widehat{v_n}(\xi)|^2\, d\xi \leq \frac{C_\psi}{\log(\log n)} \int_{\mathrm{e}^{a\,\alpha_n}}^{\mathrm{e}^{b\,\alpha_n}} \frac{d\rho}{\rho\log(\rho)}\,\cdot$$

Performing the change of variables $\rho = \mathrm{e}^u$, we easily deduce that

$$\int_{\mathrm{e}^{a\,\alpha_n}\leq|\xi|\leq \mathrm{e}^{b\,\alpha_n}} |\widehat{v_n}(\xi)|^2\, d\xi \leq \frac{C_\psi \log(\frac{b}{a})}{\log(\log n)} \stackrel{n\to\infty}{\longrightarrow} 0\,,$$

which ensures that the sequence (v_n) is log-unrelated to any scale tending to infinity.

In other respects for any fixed M, we have

$$\int_{|\xi|\leq M} |\widehat{v_n}(\xi)|^2\, d\xi \lesssim \frac{\|\widehat{\psi}\|^2_{L^\infty(\mathbb{R}^d)}\, M^d}{\log(\log n)} \stackrel{n\to\infty}{\longrightarrow} 0\,,$$

---

[1] Where $\widehat{u}$ denotes the Fourier transform of u defined by: $\widehat{u}(\xi) = \displaystyle\int_{\mathbb{R}^{2N}} \mathrm{e}^{-i\,x\cdot\xi}\, u(x)\, dx\,.$

which implies that the sequence (v_n) is log-unrelated to any bounded scale.
On the other hand by Fourier-Plancherel formula, we have

$$\|v_n\|^2_{L^2(\mathbb{R}^d)} = \frac{(2\pi)^{-d}}{\log(\log n)} \int_{\mathbb{R}^d} \frac{|\widehat{\psi}(\frac{\xi}{n})|^2 \, d\xi}{(1 + |\xi|^d | \log |\xi||)} = \mathcal{I}^1_n + \mathcal{I}^2_n ,$$

with

$$\mathcal{I}^1_n := \frac{(2\pi)^{-d}}{\log(\log n)} \int_{|\xi| \leq n} \frac{|\widehat{\psi}(\frac{\xi}{n})|^2 \, d\xi}{(1 + |\xi|^d | \log |\xi||)} \cdot$$

Performing the change of variables $\xi = n\eta$, we get by applying Lebesgue theorem

$$\mathcal{I}^2_n = \frac{(2\pi)^{-d}}{\log(\log n)} \int_{|\eta| \geq 1} \frac{|\widehat{\psi}(\eta)|^2 \, d\eta}{(\frac{1}{n^d} + |\eta|^d | \log |n\,\eta||)} \overset{n\to\infty}{\longrightarrow} 0 \cdot \tag{1.11}$$

Since $\widehat{\psi}(0) = \displaystyle\int_{\mathbb{R}^d} \psi(x)dx \neq 0$, we obtain making use again of the change of variables $\xi = n\eta$

$$\mathcal{I}^1_n = \frac{(2\pi)^{-d}}{\log(\log n)} \int_{|\eta| \leq 1} \frac{|\widehat{\psi}(\eta)|^2 \, d\eta}{(\frac{1}{n^d} + |\eta|^d | \log |n\,\eta||)} = \frac{(2\pi)^{-d}}{\log(\log n)} \int_{|\eta| \leq 1} \frac{(|\widehat{\psi}(0)|^2 + \mathcal{O}(|\eta|)) \, d\eta}{(\frac{1}{n^d} + |\eta|^d | \log |n\,\eta||)},$$

which, by straightforward computations, implies that

$$\mathcal{I}^1_n = \frac{(2\pi)^{-d} \, |\widehat{\psi}(0)|^2}{\log(\log n)} \int_{\frac{\mathrm{e}}{n} \leq |\eta| \leq 1} \frac{d\eta}{(\frac{1}{n^d} + |\eta|^d | \log |n\,\eta||)} + \circ(1) \cdot$$

We deduce that

$$\mathcal{I}^1_n \overset{n\to\infty}{\longrightarrow} (2\pi)^{-d} \omega_{d-1} \Big| \int_{\mathbb{R}^d} \psi(x)dx \Big|^2 , \tag{1.12}$$

where ω_{d-1} denotes the measure of the unit sphere $\mathbb{S}^{d-1}$.
Invoking (1.11) and (1.12), we infer that

$$\|v_n\|^2_{L^2(\mathbb{R}^d)} \overset{n\to\infty}{\longrightarrow} (2\pi)^{-d} \omega_{d-1} \Big| \int_{\mathbb{R}^d} \psi(x)dx \Big|^2 ,$$

which ends the proof of the claim.

These notions of being log-oscillating with respect to a scale and of being log-unrelated to any scale are a natural adaptation to Orlicz spaces setting of the vocabulary of P. Gérard introduced in [18] as follows:

Definition 1.5 Let $v := (v_n)_{n \geq 0}$ be a bounded sequence in $L^2(\mathbb{R}^d)$ and $\underline{h} := (h_n)_{n \geq 0}$ be a sequence of positive real numbers.

- The sequence v is said $\underline{h}$-oscillating if

$$\limsup_{n\to\infty}\left(\int_{h_n|\xi|\leq\frac{1}{R}}|\widehat{v_n}(\xi)|^2\,d\xi+\int_{h_n|\xi|\geq R}|\widehat{v_n}(\xi)|^2\,d\xi\right)\overset{R\to\infty}{\longrightarrow}0\,. \tag{1.13}$$

- The sequence v is said unrelated to the scale $\underline{h}$ if for any reals $b > a > 0$

$$\int_{a\leq h_n|\xi|\leq b}|\widehat{v_n}(\xi)|^2\,d\xi\overset{n\to\infty}{\longrightarrow}0\,. \tag{1.14}$$

Since our first aim in this paper is to construct a logarithmic Littlewood-Paley decomposition, let us recall the definition of the classical dyadic partition of unity on $\mathbb{R}^d$ (we refer for instance to [6, 13, 14, 28] and the references therein for more details).

Definition 1.6 Let $\mathcal{C}$ be the annulus $\{\xi\in\mathbb{R}^d\,/\,3/4\leq|\xi|\leq 8/3\}$. There exist two radial functions χ and φ valued in the interval [0, 1], belonging respectively to $\mathcal{D}(B(0,4/3))$ and to $\mathcal{D}(\mathcal{C})$, and such that

$$\forall\xi\in\mathbb{R}^d\,,\ \chi(\xi)+\sum_{j\geq 0}\varphi(2^{-j}\xi)=1\ \textit{and} \tag{1.15}$$

$$\forall\xi\in\mathbb{R}^d\setminus\{0\}\,,\ \sum_{j\in\mathbf{Z}}\varphi(2^{-j}\xi)=1\,. \tag{1.16}$$

Remark 1.7

- For all u in $\mathcal{S}'(\mathbb{R}^d)$, we have[2]

$$u=\sum_j\Delta_j u\,, \tag{1.17}$$

where the nonhomogeneous dyadic blocks Δ_j are defined by

$$\Delta_j u=0 \text{ if } j\leq -2,\ \ \Delta_{-1}u=\chi(D)u \text{ and } \Delta_j u=\varphi(2^{-j}D)u \text{ if } j\geq 0\,.$$

Note that Identity (1.17) also assumes the form

$$u=\lim_{j\to\infty}\sum_{j'\leq j-1}\Delta_{j'}u=\lim_{j\to\infty}S_j u \quad\text{in}\quad \mathcal{S}'(\mathbb{R}^d)\,,$$

where the nonhomogeneous low frequency cut-off operator S_j writes

[2]We recall that $\mathcal{F}(\Theta(D)u)(\xi)=\Theta(\xi)\mathcal{F}(u)(\xi)$, with $\mathcal{F}$ the Fourier transform.

$$S_j u = \sum_{-1 \le j' \le j-1} \Delta_{j'} u = \chi(2^{-j} D) u \quad \text{for} \quad j \ge 0\,.$$

- Equality (1.17) is not valid for all u in $\mathcal{S}'(\mathbb{R}^d)$ for homogeneous dyadic blocks $\dot{\Delta}_j$ defined by

$$\dot{\Delta}_j u = \varphi(2^{-j} D) u \quad \text{for} \quad j \in \mathbf{Z}\,.$$

  It clearly fails for nonzero polynomials. However, it holds true in $\mathcal{S}'_{\rm h}(\mathbb{R}^d)$ the subspace of tempered distributions u satisfying $\|\dot{S}_j u\|_{L^\infty(\mathbb{R}^d)} \stackrel{j \to -\infty}{\longrightarrow} 0$, where $\dot{S}_j$ designates the homogeneous low frequency cut-off operator defined by

$$\dot{S}_j u = \sum_{j' \le j-1} \dot{\Delta}_{j'} u \quad \text{for} \quad j \in \mathbf{Z}\,.$$

- Decomposition (1.17), which supplies an elementary device for splitting a possibly rough function into a sequence of spectrally localized smooth functions, allows among other to define a wide class of function spaces like Besov spaces and to provide elementary and elegant proofs of various inequalities such as refined Sobolev and Hardy inequalities. We can consult Chapter 2 in [6] for an overview of this theory in the classical case.
- Recall that for $s \in \mathbb{R}$ and $1 \le p, r \le \infty$, the nonhomogeneous Besov space $B^s_{p,r}(\mathbb{R}^d)$ is the set of all tempered distributions u so that

$$\|u\|_{B^s_{p,r}(\mathbb{R}^d)} := \Big\| (2^{js} \|\Delta_j u\|_{L^p})_{j \in \mathbf{Z}} \Big\|_{\ell^r(\mathbf{Z})} < \infty\,,$$

  and the homogeneous Besov space $\dot{B}^s_{p,r}(\mathbb{R}^d)$ is the set of tempered distributions belonging to $\mathcal{S}'_{\rm h}(\mathbb{R}^d)$ such that

$$\|u\|_{\dot{B}^s_{p,r}(\mathbb{R}^d)} := \Big\| (2^{js} \|\dot{\Delta}_j u\|_{L^p})_{j \in \mathbf{Z}} \Big\|_{\ell^r(\mathbf{Z})} < \infty\,.$$

  Thus Estimate (1.6) also reads

$$\sup_{j \in \mathbf{Z}} \|\dot{\Delta}_j \nabla f_{\alpha_n}\|_{L^2(\mathbb{R}^2)} \longrightarrow 0, \quad \text{as} \quad n \to \infty\,. \tag{1.18}$$

- Property (1.18) highlights the fact that the example by Moser is spread in frequency. Note that it was proved in [3] that the sequence (f_{α_n}) can be written under the form:

$$f_{\alpha_n}(x) = \widetilde{f_{\alpha_n}}(x) + {\rm r}_n(x)\,,$$

  with $\|{\rm r}_n\|_{H^1(\mathbb{R}^2)} \stackrel{n \to \infty}{\longrightarrow} 0$ and

$$\widetilde{f_{\alpha_n}}(x) = \frac{1}{(2\pi)^2}\sqrt{\frac{2\pi}{\alpha_n}} \int_{\mathbb{R}^2} e^{i\,x\cdot\xi} \frac{1}{|\xi|^2}\,\varphi\Big(\frac{\log|\xi|}{\alpha_n}\Big) d\xi\,,$$

where $\varphi(\eta) = \mathbf{1}_{[0,1]}(\eta)$.

- Obviously, we have

$$\mathcal{F}(\widetilde{f_{\alpha_n}})(\xi) = \sqrt{\frac{2\pi}{\alpha_n}}\,\frac{\mathbf{1}_{[1,e^{\alpha_n}]}(|\xi|)}{|\xi|^2}\,, \tag{1.19}$$

which shows that the spectrum of f_{α_n} is spread over the disk of radius e^{α_n} centered at the origin.

- More generally, it has been emphasized in [11] that the lack of compactness of the Sobolev embedding (1.3) is displayed by sequences under the form:

$$g_{\alpha_n}(x) = \frac{C_N}{\sqrt{\alpha_n}} \int_{|\xi|\geq 1} \frac{e^{i\,x\cdot\xi}}{|\xi|^{2N}}\,\varphi\Big(\frac{\log|\xi|}{\alpha_n}\Big)\, d\xi\,,$$

with $C_N = \dfrac{1}{(2\pi)^N\sqrt{\omega_{2N-1}}}$, $(\alpha_n)_{n\geq 0}$ a sequence of positive real numbers going to infinity and $\varphi \not\equiv 0$ in $L^2(\mathbb{R}_+)$. It was also proved that $(|D|^N\, g_{\alpha_n})_{n\geq 0}$ is $\underline{\alpha}$ log-oscillating and that

$$\lim_{n\to\infty} \|g_{\alpha_n}\|_{\mathcal{L}(\mathbb{R}^{2N})} = \frac{1}{\sqrt{\beta_N}}\,\max_{s>0}\,\frac{|\psi(s)|}{\sqrt{s}}\,, \tag{1.20}$$

where $\psi(s) := \displaystyle\int_0^s \varphi(t)\,dt$.

The basic idea of Littlewood-Paley theory is contained in two fundamental inequalities known as Bernstein inequalities. The first one says that, for a function whose Fourier transform is supported in an annulus of size λ, differentiate and then take the L^p norm amounts to do a dilation of ratio λ on the L^p norm. The second one specifies that, for such functions, the passage from the L^p norm to the L^q norm, for $q \geq p \geq 1$, costs $\lambda^{d\left(\frac{1}{p}-\frac{1}{q}\right)}$, which should be understood as a Sobolev embedding. More precisely, we have the following lemma the proof of which can be for instance found in [6]:

Lemma 1.8 *Let $\mathcal{C}$ be an annulus and B a ball of $\mathbb{R}^d$ centered at the origin. A constant C exists so that, for any nonnegative integer k, any couple (p,q) in $[1,\infty]^2$ with $q \geq p \geq 1$ and any function u of $L^p(\mathbb{R}^d)$, we have*

$$\text{Supp}\,\widehat{u} \subset \lambda B \Longrightarrow \|D^k u\|_{L^q(\mathbb{R}^d)} := \sup_{|\alpha|=k} \|\partial^\alpha u\|_{L^q(\mathbb{R}^d)} \leq C^{k+1}\lambda^{k+d(\frac{1}{p}-\frac{1}{q})}\|u\|_{L^p(\mathbb{R}^d)} \quad \text{and}$$

$$\text{Supp}\,\widehat{u} \subset \lambda\mathcal{C} \Longrightarrow C^{-k-1}\lambda^k\|u\|_{L^p(\mathbb{R}^d)} \leq \|D^k u\|_{L^p(\mathbb{R}^d)} \leq C^{k+1}\lambda^k\|u\|_{L^p(\mathbb{R}^d)}\,.$$

1.3 Main Results

The development of microlocal tools adapted to the framework of problems at hand is an important issue: we refer for instance to the articles [4, 8, 10, 16, 17] where was constructed respectively Littlewood-Paley decompositions on the Heisenberg group, on graded Lie groups and on Lie groups of polynomial growth satisfying properties as Bony's decomposition ([13]) in the euclidean case, which enabled to transpose many classical results to these general settings. One can also mention the work [20] where the construction of an adapted Littlewood-Paley theory to the geometric situation of the Einstein equations allows to reach optimal regularity indexes for the initial data.

Our main goal in this paper is to develop a logarithmic Littlewood-Paley theory taking advantage of the notion introduced in [3] of being log-oscillating with respect to a scale which has already proved to be efficacious in [11] in the framework of Orlicz spaces. For that purpose, let us start by introducing the following definition:

Definition 1.9 Under the notations of Definition 1.6, we define for all u in $\mathcal{S}'(\mathbb{R}^d)$ the logarithmic dyadic blocks $\Delta_j^{\log}$ by

$$\Delta_j^{\log} u = \varphi(2^{-j}\log|D|)(1-\widetilde{\chi}(D))u \quad \text{for} \quad j \in \mathbf{Z}\,,$$

where $\widetilde{\chi}$ is a radial function belonging to $\mathcal{D}(B(0,2))$ and satisfying $\widetilde{\chi}(\xi) = 1$ for $|\xi| \leq \frac{3}{2}$, and the low logarithmic frequency cut-off operator $S_j^{\log}$ by

$$S_j^{\log} u = \widetilde{\chi}(D)u + \sum_{j' \leq j-1} \Delta_{j'}^{\log} u = \widetilde{\chi}(D)u + \chi(2^{-j}\log|D|)(1-\widetilde{\chi}(D))u \quad \text{for} \quad j \in \mathbf{Z}\,.$$

Formally, we have

$$\mathrm{Id} = \widetilde{\chi}(D) + \sum_{j' \in \mathbf{Z}} \Delta_{j'}^{\log} u\,. \tag{1.21}$$

Actually as in the usual case, we have the following result which ensures that the logarithmic Littlewood-Paley decomposition (1.21) makes sense in $\mathcal{S}'(\mathbb{R}^d)$:

Proposition 1.10 *Let u be in $\mathcal{S}'(\mathbb{R}^d)$. Then*

$$u = \lim_{j\to\infty} S_j^{\log} u \ \textit{in}\ \mathcal{S}'(\mathbb{R}^d)\,. \tag{1.22}$$

Proof By classical arguments, one can reduce to the proof of (1.22) in $\mathcal{S}(\mathbb{R}^d)$. Because the Fourier transform is an automorphism of $\mathcal{S}(\mathbb{R}^d)$, we can alternatively prove that for any multi-index α of length n, we have

$$N_{n,\alpha}\Big(\big(1-\chi(2^{-j}\log|\cdot|)\big)(1-\widetilde{\chi})\,\widehat{u}\Big) \overset{j\to\infty}{\longrightarrow} 0\,,$$

where $N_{n,\alpha}(f) = \sup_{\mathbb{R}^d}(1+|\xi|)^n \big|\partial^\alpha f(\xi)\big|$.

By virtue of Leibnitz formula, we get (making use of the fact that the function $(1-\widetilde{\chi})\,\widehat{u}$ is supported on the set $\big\{\xi \in \mathbb{R}^d \,/\, |\xi| \geq 1\big\}\big)$

$$N_{n,\alpha}\Big(\big(1-\chi(2^{-j}\log|\cdot|)\big)(1-\widetilde{\chi})\widehat{u}\Big) \leq \sup_{\mathbb{R}^d}(1+|\xi|)^n\Big\{\big(1-\chi(2^{-j}\log|\xi|)\big)\big|\partial^\alpha\widehat{v}(\xi)\big| + \sum_{\beta<\alpha} C_\alpha^\beta\big|\partial^{\alpha-\beta}\big(\chi(2^{-j}\log|\xi|)\big)\,\partial^\beta\widehat{v}(\xi)\big|\Big\},$$

with $\widehat{v}(\xi) := (1-\widetilde{\chi}(\xi))\widehat{u}(\xi)$.
Since $\widetilde{\chi}(\xi) \equiv 1$ in the ball centered at the origin and of radius $\frac{3}{2}$, we deduce that

$$N_{n,\alpha}\Big(\big(1-\chi(2^{-j}\log|\cdot|)\big)(1-\widetilde{\chi})\widehat{u}\Big) \lesssim 2^{-j}\sup_{|\beta|\leq|\alpha|} N_{n+1,\beta}\big((1-\widetilde{\chi})\widehat{u}\big)\,,$$

which ends the proof of the result. □

Remark 1.11

- As emphasized in Remarks 1.4, the notion of log-oscillating with respect to a scale is only relevant for scales tending to infinity. This justifies the fact that the definition of the logarithmic dyadic blocks $\Delta_j^{\log}$ does not take into account the low frequencies.
- Clearly there is $j_0 \in \mathbf{Z}$ such that for any function u in $\mathcal{S}'(\mathbb{R}^d)$, we have $\Delta_j^{\log}u \equiv 0$ for $j \leq j_0$. Furthermore for $j \geq j_0$, the function $\Delta_j^{\log}u$ is spectrally localized in $\mathrm{e}^{2^j}\mathcal{C}$, where $\mathcal{C}$ is the annulus introduced in Definition 1.6. This obviously ensures that the sequence $(\Delta_j^{\log}u)_{j\geq j_0}$ is $(2^j)_{j\geq j_0}$ log-oscillating.
- Since for applications to the Orlicz space, the logarithmic Littlewood-Paley theory is mostly relevant in 2ND case, we shall limit ourselves in what follows to this case.
- Finally, let us point out that

$$\limsup_{j\to+\infty}\|\Delta_j^{\log}u\|_{\mathcal{L}(\mathbb{R}^{2N})} = K_N \limsup_{j\to+\infty}\max_{s>0}\frac{\Big|\int_{1\leq|\xi|\leq \mathrm{e}^{2^j s}}\widehat{\Delta}_j^{\log}u(\xi)\,d\xi\Big|}{\sqrt{2^j s}}\,, \tag{1.23}$$

with $K_N = \dfrac{1}{\sqrt{2N}\,(2\pi)^{2N}}\cdot$ Indeed, by definition[3]

[3]Where obviously $\xi = |\xi|\cdot\omega$, with $\omega \in \mathbb{S}^{2N-1}$.

$$\Delta_j^{\log} u(x) = \frac{1}{(2\pi)^{2N}} \int_{|\xi|\geq 1} \frac{e^{i\,x\cdot\xi}}{|\xi|^{2N}}\, \widetilde{\Delta}_j^{\log} u\big(\log|\xi|, \omega\big)\, d\xi\,,$$

with $\widetilde{\Delta}_j^{\log} u\Big(\log|\xi|, \omega\Big) := |\xi|^{2N}\, \widehat{\Delta}_j^{\log} u(\xi)\cdot$ Setting

$$\widetilde{\Delta}_j^{\log} u\Big(\log|\xi|, \omega\Big) = \frac{1}{\sqrt{2^j}}\, \Delta_j^{\log,\flat} u\Big(\frac{\log|\xi|}{2^j}, \omega\Big)\,,$$

we deduce that

$$\Delta_j^{\log} u(x) = \frac{1}{(2\pi)^{2N}\sqrt{2^j}} \int_{|\xi|\geq 1} \frac{e^{i\,x\cdot\xi}}{|\xi|^{2N}}\, \Delta_j^{\log,\flat} u\Big(\frac{\log|\xi|}{2^j}, \omega\Big)\, d\xi\,.$$

This ensures in view of Lemma 3.4 in [11] that

$$\Delta_j^{\log} u(x) = \frac{1}{(2\pi)^{2N}\sqrt{2^j}} \int_{|\xi|\geq 1} \frac{e^{i\,x\cdot\xi}}{|\xi|^{2N}}\, \Delta_j^{\log,\sharp} u\Big(\frac{\log|\xi|}{2^j}\Big)\, d\xi + \mathrm{r}_j(x)\,,$$

with $\|\mathrm{r}_j\|_{\mathcal{L}(\mathbb{R}^{2N})} \overset{j\to\infty}{\longrightarrow} 0$ and

$$\Delta_j^{\log,\sharp}(t) = \frac{1}{\omega_{2N-1}} \int_{\mathbb{S}^{2N-1}} \Delta_j^{\log,\flat} u(t,\omega)\, d\omega\,.$$

In light of (1.20), this gives rise to

$$\|\Delta_j^{\log} u\|_{\mathcal{L}(\mathbb{R}^{2N})} = \frac{1}{\sqrt{2N}} \max_{s>0} \frac{\Big|\displaystyle\int_0^s \Delta_j^{\log,\sharp} u(t)\, dt\Big|}{\sqrt{s}} + \circ(1)\,,$$

which by straightforward computations leads to

$$\|\Delta_j^{\log} u\|_{\mathcal{L}(\mathbb{R}^{2N})} = \frac{1}{\sqrt{2N}\,(2\pi)^{2N}} \max_{s>0} \frac{\Big|\displaystyle\int_{1\leq|\xi|\leq e^{2^j s}} \widehat{\Delta}_j^{\log} u(\xi)\, d\xi\Big|}{\sqrt{2^j s}} + \circ(1)\cdot$$

This ends the proof of Claim (1.23).

In order to state the logarithmic Bernstein inequalities in a clear way, let us define the notion of annulus of exponential size. Given $\mathcal{C} = \Big\{\xi \in \mathbb{R}^{2N} \,/\, r \leq |\xi| \leq R\Big\}$ an annulus of $\mathbb{R}^{2N}$, we shall denote by $e^{\mathcal{C}}$ the annulus defined as follows:

$$e^{\mathcal{C}} := \Big\{\eta \in \mathbb{R}^{2N} \,/\, e^r \leq |\eta| \leq e^R\Big\}\,. \tag{1.24}$$

As in the euclidean case, the first interest of this logarithmic localization procedure in frequency space is that the “logarithmic derivatives” act almost as homothety on distributions the Fourier transform of which is supported in an annulus of exponential size. More precisely, we have the following inequalities which are the counterpart of the second Bernstein inequality stated in Lemma 1.8:

Lemma 1.12 *Let $\mathcal{C}$ be an annulus of $\mathbb{R}^{2N}$ included in the set $\Big\{\xi \in \mathbb{R}^{2N} \,/\, |\xi| > 1\Big\}$. For any nonnegative integer k, there exist positive constants C_k and $\widetilde{C}_k$ so that, for any real number $p \geq 1$ and any function u belonging to $L^p(\mathbb{R}^{2N})$ satisfying Supp $\widehat{u} \subset e^{\lambda\mathcal{C}}$, with $\lambda \geq 1$, we have*

$$\widetilde{C}_k \lambda^k \|u\|_{L^p(\mathbb{R}^{2N})} \leq \|(\log|D|)^k u\|_{L^p(\mathbb{R}^{2N})} \leq C_k \lambda^k \|u\|_{L^p(\mathbb{R}^{2N})}\,. \tag{1.25}$$

The generalization of the first part of classical Bernstein inequalities stated in Lemma 1.8 is more challenging. The result we obtain in the logarithmic frame reads differently from that of the classical case, but expresses the same phenomenon, namely that we lose $2N\left(\frac{1}{p}-\frac{1}{q}\right)$ derivatives in the passage from the L^p norm to the L^q norm, for $q \geq p \geq 1$. More precisely, our result formulates as follows:

Lemma 1.13 *Let $\mathcal{C}$ be an annulus of $\mathbb{R}^{2N}$ included in the set $\Big\{\xi \in \mathbb{R}^{2N} \,/\, |\xi| > 1\Big\}$. There is a positive real number b so that the following holds. For any nonnegative integer k, there exists a positive constant C_k such that, for any real number $\lambda \geq 1$, any couple (p,q) in $[1,\infty]^2$ with $q \geq p \geq 1$ and any function u belonging to $L^p(\mathbb{R}^{2N})$ whose spectrum is included in $e^{\lambda\mathcal{C}}$, we have*

$$\|(\log|D|)^k u\|_{L^q(\mathbb{R}^{2N})} \leq C_k\, \lambda^k\, e^{2N\,\lambda\, b\,(\frac{1}{p}-\frac{1}{q})}\, \|u\|_{L^p(\mathbb{R}^{2N})}\,.$$

Remark 1.14 Contrary to the classical case, the cost of the passage from the L^p norm to the L^q norm is exponential. This is justified by the fact that the spectrum of the functions considered is of exponential size.

As mentioned above, among the objectives of the Littlewood-Paley theory is to construct functional spaces like Besov spaces and to study their properties. As in the euclidean case, let us define the logarithmic Besov spaces.

Definition 1.15 Let s be a real number, and (p,r) be in $[1,\infty]^2$. The logarithmic Besov space $B^{s,\log}_{p,r}(\mathbb{R}^{2N})$ is the subset of tempered distributions u of $\mathcal{S}'(\mathbb{R}^{2N})$ such that

$$\|u\|_{B^{s,\log}_{p,r}(\mathbb{R}^{2N})} := \|\widetilde{\chi}(D)u\|_{L^p(\mathbb{R}^{2N})} + \Big\|\big(2^{js}\|\Delta^{\log}_j u\|_{L^p(\mathbb{R}^{2N})}\big)_{j\in\mathbf{Z}}\Big\|_{\ell^r(\mathbf{Z})} < \infty\,.$$

Remark 1.16

- Clearly $L^2(\mathbb{R}^{2N})$ coincides with $B^{0,\log}_{2,2}(\mathbb{R}^{2N})$ which implies that

$$L^2(\mathbb{R}^{2N}) \hookrightarrow B^{0,\log}_{2,\infty}(\mathbb{R}^{2N})\,. \tag{1.26}$$

- Note also that in view of logarithmic Bernstein inequalities, $(\log|D|)^k$ operate on logarithmic Besov spaces $B^{s,\log}_{p,r}(\mathbb{R}^{2N})$, namely that for any real number s and any (p,q) in $[1,\infty]^2$, the map

$$(\log|D|)^k : B^{s,\log}_{p,r}(\mathbb{R}^{2N}) \longrightarrow B^{s-k,\log}_{p,r}(\mathbb{R}^{2N})$$

  defines a continuous linear functional.
- As in the classical case, the logarithmic cut-off operators $\Delta^{\log}_j$ and $S^{\log}_j$ are convolution operators on $\mathbb{R}^{2N}$ defined for $j \in \mathbf{Z}$ by:

$$\Delta^{\log}_j u = h_j \star (1-\widetilde{\chi}(D))u\,,$$

  and

$$S^{\log}_j u = \widetilde{h}_j \star (1-\widetilde{\chi}(D))u\,.$$

  Contrary to the classical case, the functions h_j and $\widetilde{h}_j$ are not dilated of integrable functions. However, they belong to $L^1(\mathbb{R}^{2N})$ with norms independent of j. This property is more challenging than the classical case. Its proof will be given in Sect. 2.1. In view of Hölder inequalities, it implies that the operators $\Delta^{\log}_j$ and $S^{\log}_j$ map $L^p(\mathbb{R}^{2N})$ into $L^p(\mathbb{R}^{2N})$ with norms independent of j and p.
- Let us finally note that as in the classical case the definition of the Besov space $B^{s,\log}_{p,r}$ is independent of the functions $\widetilde{\chi}$ and φ used for defining the logarithmic dyadic bloks, and changing these functions yields an equivalent norm.

1.4 Layout of the Paper

The paper is organized as follows. In Sect. 2, we prove Bernstein inequalities in the framework of the logarithmic Littlewood-Paley decomposition which are the subject of Lemmas 1.12 and 1.13. Then in Sect. 3, we state and establish some logarithmic Sobolev embeddings that occur in Orlicz spaces.

We mention that the letter C will be used to denote an absolute constant which may vary from line to line. We also use $A \lesssim B$ to denote an estimate of the form $A \le CB$ for some absolute constant C.

2 Proof of Bernstein Inequalities

This section is devoted to the proof of Bernstein inequalities in the framework of the logarithmic Littlewood-Paley decomposition. Adapting these fundamental inequalities provides various functional inequalities such as Sobolev embeddings and their refined versions. We will dedicate Sect. 3 to some functional inequalities which does not arise immediately from an adaptation of the classical framework.

2.1 Proof of Lemma 1.12

Assuming that $\mathcal{C} = \Big\{\xi \in \mathbb{R}^{2N} \,/\, 1 < r \leq |\xi| \leq R < b\Big\}$, let ϕ be a function in $\mathcal{D}(\mathbb{R})$ such that $\mathrm{supp}(\phi) \subset]1, b[$ and $\phi \equiv 1$ near $]r, R[$. Since the spectrum of u is included in $\mathrm{e}^{\lambda \mathcal{C}}$ with $\lambda \geq 1$, we have $\widehat{u}(\xi) = \phi(\lambda^{-1} \log|\xi|)\, \widehat{u}(\xi)$. Thus

$$(\log|D|)^k\, u = (\log|D|)^k\, g_\lambda \star u\,,$$

where $\widehat{g_\lambda}(\xi) = \phi(\lambda^{-1}\log|\xi|)$. Applying Young's inequality, we get

$$\|(\log|D|)^k u\|_{L^p} \leq \|(\log|D|)^k\, g_\lambda\|_{L^1} \|u\|_{L^p}\,.$$

But

$$\mathcal{F}((\log|D|)^k\, g_\lambda)(\xi) = \lambda^k (\lambda^{-1}\log|\xi|)^k \phi(\lambda^{-1}\log|\xi|)\,.$$

We are then reduced to estimate the L^1-norm of the function

$$g_{k,\lambda}(x) := (2\pi)^{-2N} \int_{\mathbb{R}^{2N}} \mathrm{e}^{i\,x\cdot\xi}\, (\lambda^{-1}\log|\xi|)^k \phi(\lambda^{-1}\log|\xi|)\, d\xi\,.$$

On the one hand

$$\big|g_{k,\lambda}(x)\big| \lesssim \int_{\mathbb{R}^{2N}} \big|\phi_k(\lambda^{-1}\log|\xi|)\big|\, d\xi\,,$$

with $\phi_k(\rho) := \rho^k\, \phi(\rho)$. According to the fact that $\phi_k \in \mathcal{D}(]1, b[)$, this gives rise to

$$\big|g_{k,\lambda}(x)\big| \lesssim \mathrm{e}^{2N\,\lambda\, b}\,.$$

We deduce that for any positive real number δ, we have

$$\int_{|x|\leq\delta} \big|g_{k,\lambda}(x)\big|\, dx \lesssim (\delta\, \mathrm{e}^{\lambda\, b})^{2N}\,. \tag{2.1}$$

On the other hand by straightforward integrations by parts, we get for any $x \neq 0$

$$|x|^2 \, g_{k,\lambda}(x) = -(2\pi)^{-2N} \int_{\mathbb{R}^{2N}} \mathrm{e}^{i\, x\cdot\xi} \; \Delta_\xi(\phi_k(\lambda^{-1} \log|\xi|)) \, d\xi \,,$$

where $\Delta_\xi := \sum_{j=1}^{2N} \partial^2_{\xi_j}$. Observing that

$$\Delta_\xi(\phi_k(\lambda^{-1}\log|\xi|)) = \lambda^{-2}\phi_k''(\lambda^{-1}\log|\xi|)\,\frac{1}{|\xi|^2} + \lambda^{-1}\phi_k'(\lambda^{-1}\log|\xi|)\,\frac{2N-2}{|\xi|^2}\,, \tag{2.2}$$

we infer that in the 2D case

$$\big||x|^2\, g_{k,\lambda}(x)\big| \lesssim \lambda^{-2} \int_1^\infty \big|\phi_k''(\lambda^{-1}\log\rho)\big| \,\frac{d\rho}{\rho} \lesssim \lambda^{-1} \|\phi_k''\|_{L^1}\,.$$

Through a second integration by parts, we easily find that for any $\lambda \geq 1$

$$\big||x|^3\, g_{k,\lambda}(x)\big| \lesssim \lambda^{-2} \int_1^\infty \big|H(\lambda^{-1}\log\rho)\big| \,\frac{d\rho}{\rho^2} \lesssim \lambda^{-2}\,,$$

where the function H (depending on k) is a function of $\mathcal{D}(\mathbb{R})$.

Taking advantage of Formula (2.2) and the fact that $\Delta_\xi^{(N-1)}\big(|\xi|^{-2}\big) = c_N\,\delta_0$ in $\mathbb{R}^{2N}$ when $N > 1$, we get by repeated integrations by parts for any $\lambda \geq 1$

$$\big||x|^{2N}\, g_{k,\lambda}(x)\big| \lesssim \lambda^{-2} \int_{|\xi|\geq 1} H_{2N}(\lambda^{-1}\log|\xi|)\,\frac{d\xi}{|\xi|^{2N}} \lesssim \lambda^{-1}\,, \tag{2.3}$$

and

$$\big||x|^{2N+1}\, g_{k,\lambda}(x)\big| \lesssim \lambda^{-2} \int_{|\xi|\geq 1} H_{2N+1}(\lambda^{-1}\log|\xi|)\,\frac{d\xi}{|\xi|^{2N+1}} \lesssim \lambda^{-2} \lesssim 1\,, \tag{2.4}$$

where the functions H_{2N} and H_{2N+1} (also depending on k) are functions belonging to $\mathcal{D}(\mathbb{R})$. This implies that

$$\int_{|x|\geq 1} \big|g_{k,\lambda}(x)\big|\, dx \lesssim 1\,, \tag{2.5}$$

and for any $0 < \delta \leq 1$

$$\int_{\delta\leq|x|\leq 1} \big|g_{k,\lambda}(x)\big|\, dx \lesssim \lambda^{-1} \int_\delta^1 \frac{d\rho}{\rho} \lesssim -\lambda^{-1}\log(\delta)\,. \tag{2.6}$$

Choosing $\delta = e^{-\lambda b}$ and invoking (2.1), (2.5) and (2.6), we infer that

$$\|g_{k,\lambda}\|_{L^1(\mathbb{R}^{2N})} \lesssim 1\,,$$

which ends the proof of the right hand side of the assertion.

Once observed that the function u can be recast under the form

$$u = (\log|D|)^{-k}\, g_\lambda \star (\log|D|)^k\, u\,,$$

we end the proof of the result.

2.2 Proof of Lemma 1.13

The proof of this lemma goes the same lines as the proof of Lemma 1.12. Taking advantage of the fact that the spectrum of the function u is included in $e^{\lambda\mathcal{C}}$ with $\lambda \geq 1$, we find that $\widehat{u}(\xi) = \phi(\lambda^{-1}\log|\xi|)\,\widehat{u}(\xi)$, where ϕ is a function of $\mathcal{D}(\mathbb{R})$ chosen as above.

Therefore

$$(\log|D|)^k\, u = (\log|D|)^k\;\, g_\lambda \star u\,,$$

where $\widehat{g_\lambda}(\xi) = \phi(\lambda^{-1}\log|\xi|)$. Thanks to Young inequalities, we obtain

$$\|(\log|D|)^k\;\, g_\lambda \star u\|_{L^q(\mathbb{R}^{2N})} \leq \|(\log|D|)^k\, g_\lambda\|_{L^r(\mathbb{R}^{2N})}\|u\|_{L^p(\mathbb{R}^{2N})}\;,$$

with $\dfrac{1}{r} = \dfrac{1}{q} - \dfrac{1}{p} + 1\cdot$ We are then reduced to prove that

$$\|(\log|D|)^k\, g_\lambda\|_{L^r(\mathbb{R}^{2N})} \lesssim \lambda^k\, e^{2N\,\lambda\, b\,(\frac{1}{p}-\frac{1}{q})}\,. \tag{2.7}$$

Obviously

$$\mathcal{F}((\log|D|)^k\;\, g_\lambda)(\xi) = \lambda^k(\lambda^{-1}\log|\xi|)^k\phi(\lambda^{-1}\log|\xi|)\,,$$

thus in view of the relation $\dfrac{1}{r} = \dfrac{1}{q} - \dfrac{1}{p} + 1\cdot$ our purpose is to establish that the function

$$g_{k,\lambda}(x) := (2\pi)^{-2N}\int_{\mathbb{R}^{2N}} e^{i\,x\cdot\xi}\;\phi_k(\lambda^{-1}\log|\xi|)\,d\xi$$

with $\phi_k(\rho) = \rho^k\,\phi(\rho)$, satisfies $\|g_{k,\lambda}\|_{L^r(\mathbb{R}^{2N})} \lesssim e^{2N\,\lambda\, b\,(1-\frac{1}{r})}$.

To this end, we shall follow the strategy adopted in the proof of Lemma 1.12. Firstly since the function ϕ_k belongs to $\mathcal{D}(]1, b[)$, we infer that

$$\big|g_{k,\lambda}(x)\big| \lesssim \int_1^{e^{\lambda b}} |\phi_k(\lambda^{-1}\log\rho)|\,\rho^{2N-1}\,d\rho\,, \tag{2.8}$$

which gives rise to

$$\|g_{k,\lambda}\|_{L^\infty(\mathbb{R}^{2N})} \lesssim e^{2N\lambda b}\,. \tag{2.9}$$

This ends the proof of the result in the case when $r=\infty$.

Recall that the case when $r=1$ corresponds to the case studied in 1.12. Thus to achieve our goal, it suffices to consider the case when $1<r<\infty$. Taking advantage of (2.9), we deduce that, for any positive real number δ, the following estimate holds

$$\int_{|x|\le\delta} \big|g_{k,\lambda}(x)\big|^r\,dx \lesssim (\delta\, e^{\lambda b r})^{2N}\,. \tag{2.10}$$

Moreover according to (2.3) and (2.4), we have

$$\big||x|^{2N}\,g_{k,\lambda}(x)\big| \lesssim \lambda^{-1} \text{ and } \big||x|^{2N+1}\,g_{k,\lambda}(x)\big| \lesssim \lambda^{-2}\,.$$

We deduce that

$$\begin{aligned}\int_{|x|\ge1} \big|g_{k,\lambda}(x)\big|^r\,dx &= \int_{|x|\ge1} \big||x|^{2N+1}\,g_{k,\lambda}(x)\big|\,\big|g_{k,\lambda}(x)\big|^{r-1}\,\frac{dx}{|x|^{2N+1}}\\ &\lesssim \lambda^{-2}\,e^{2N\lambda b(r-1)},\end{aligned}$$

and for any $0<\delta\le1$

$$\begin{aligned}\int_{\delta\le|x|\le1} \big|g_{k,\lambda}(x)\big|^r\,dx &= \int_{\delta\le|x|\le1} \big||x|^{2N}\,g_{k,\lambda}(x)\big|\,\big|g_{k,\lambda}(x)\big|^{r-1}\,\frac{dx}{|x|^{2N}}\\ &\lesssim \lambda^{-1}\,e^{2N\lambda b(r-1)}\int_\delta^1 \frac{d\rho}{\rho} \lesssim -\lambda^{-1}\,e^{2N\lambda b(r-1)}\log(\delta)\,.\end{aligned}$$

Selecting $\delta = e^{-\lambda b r}$ achieves the proof of the lemma.

3 Logarithmic Sobolev Embeddings

3.1 Sobolev Embedding of Logarithmic Besov Spaces into the Orlicz Spaces

The following result, which is an immediate consequence of Proposition 4.1 in [11], improves the Sobolev embedding (1.3). We sketch its proof here for the reader's convenience.

Proposition 3.1 *There is a positive constant C such that*

$$\|w\|_{\mathcal{L}(\mathbb{R}^{2N})} \leq C \, \||D|^N w\|_{B^{0,\log}_{2,\infty}(\mathbb{R}^{2N})} \, .$$

Remark 3.2 As shown by estimates (1.4) and (1.6), the classical Besov space $B^N_{2,\infty}(\mathbb{R}^{2N})$ does not embed in $\mathcal{L}(\mathbb{R}^{2N})$.

Proof To go to the proof of Proposition 3.1, let us split the function w into two parts as follows:

$$w = w_1 + w_2$$

where $w_1 := \Theta(D)w$, with Θ a function of $\mathcal{D}(\mathbb{R}^{2N})$ identically equal to 1 near the unit ball. Since for functions in $L^2(\mathbb{R}^{2N}) \cap L^\infty(\mathbb{R}^{2N})$, $\mathcal{L}(\mathbb{R}^{2N})$ behaves like $L^2(\mathbb{R}^{2N})$ (see for instance [7] for further details), we infer that

$$\|w_1\|_{\mathcal{L}(\mathbb{R}^{2N})} \leq C \|w_1\|_{L^2(\mathbb{R}^{2N})} \, . \tag{3.1}$$

To handle w_2, let us for fixed $\lambda > 0$ estimate the integral:

$$\int_{\mathbb{R}^{2N}} \left(\mathrm{e}^{|\frac{w_2(x)}{\lambda}|^2} - 1 \right) dx \, .$$

Obviously

$$\int_{\mathbb{R}^{2N}} \left(\mathrm{e}^{|\frac{w_2(x)}{\lambda}|^2} - 1 \right) dx = \sum_{p \geq 1} \frac{\|w_2\|^{2p}_{L^{2p}}}{\lambda^{2p} p!} \, .$$

Firstly let us investigate $\|w_2\|^{2p}_{L^{2p}}$. Knowing that for any $p \geq 1$

$$\|w_2\|^{2p}_{L^{2p}} \leq C^{2p} \|\widehat{w_2}\|^{2p}_{L^{\frac{2p}{2p-1}}} \, ,$$

we are led to estimate $\|\widehat{w_2}\|_{L^{\frac{2p}{2p-1}}}$. For that purpose, let us write

$$\widehat{w_2}(\xi) = \frac{1}{|\xi|^{2N}} \, \widetilde{w}(\log |\xi|, \omega) \, .$$

Observing that

$$\||D|^N w_2\|_{B^{0,\log}_{2,\infty}(\mathbb{R}^{2N})} \sim \sup_{j \in \mathbf{Z}} \int_{2^{j-1}}^{2^{j+1}} \int_{\mathbb{S}^{2N-1}} \left| \widetilde{w}(t, \omega) \right|^2 dt \, d\omega \, , \tag{3.2}$$

and

$$\||D|^N w_2\|_{L^2(\mathbb{R}^{2N})} \sim \int_0^\infty \int_{\mathbb{S}^{2N-1}} \left| \widetilde{w}(t, \omega) \right|^2 dt \, d\omega \, , \tag{3.3}$$

we deduce from Hölder inequality that for any $p \geq 2$

$$\|\widehat{w_2}\|^{\frac{2p}{2p-1}}_{L^{\frac{2p}{2p-1}}} \lesssim \||D|^N w_2\|^{\frac{2p}{2p-1}}_{B^{0,\log}_{2,\infty}(\mathbb{R}^{2N})} \sum_{j\in\mathbf{Z}} \Big(\int_{2^{j-1}}^{2^{j+1}} \mathrm{e}^{-\frac{2Nt}{p-1}} dt\Big)^{\frac{p-1}{2p-1}} .$$

By straightforward computations, we find that

$$\Big(\int_{2^{j-1}}^{2^{j+1}} \mathrm{e}^{-\frac{2Nt}{p-1}} dt\Big)^{\frac{p-1}{2p-1}} = \Big(\frac{p-1}{2N}\Big)^{\frac{p-1}{2p-1}} \Big(\mathrm{e}^{\frac{-2N2^{j-1}}{p-1}} - \mathrm{e}^{\frac{-2N2^{j+1}}{p-1}}\Big)^{\frac{p-1}{2p-1}} .$$

Taking advantage of the fact that the ratio $\dfrac{p-1}{2p-1}$ is uniformly bounded with respect to $p \geq 2$, we deduce that

$$\Big(\int_{2^{j-1}}^{2^{j+1}} \mathrm{e}^{-\frac{2Nt}{p-1}} dt\Big)^{\frac{p-1}{2p-1}} \lesssim \big(2p-1\big)^{\frac{p-1}{2p-1}} \mathrm{e}^{\frac{-2N2^{j-1}}{2p-1}} .$$

Choosing j_0 so that $\dfrac{1}{2} \leq \dfrac{2^{j_0}}{2p-1} \leq 1$, we infer that

$$\begin{aligned}\sum_{j\geq j_0} \Big(\int_{2^{j-1}}^{2^{j+1}} \mathrm{e}^{-\frac{2Nt}{p-1}} dt\Big)^{\frac{p-1}{2p-1}} &\lesssim \big(2p-1\big)^{\frac{p-1}{2p-1}} \sum_{j\geq j_0} \mathrm{e}^{-\frac{2N2^j}{2p-1}} \\ &\lesssim \big(2p-1\big)^{\frac{p-1}{2p-1}} \sum_{j\geq j_0} 2^{-j}(2p-1) \\ &\lesssim \big(2p-1\big)^{\frac{p-1}{2p-1}} 2^{-j_0}(2p-1) \lesssim \big(2p-1\big)^{\frac{p-1}{2p-1}} .\end{aligned}$$

Making use again of the fact that the ratio $\dfrac{p-1}{2p-1}$ is uniformly bounded with respect to $p \geq 2$, we obtain for $j \leq j_0$

$$\begin{aligned}\Big(\int_{2^{j-1}}^{2^{j+1}} \mathrm{e}^{-\frac{2Nt}{p-1}} dt\Big)^{\frac{p-1}{2p-1}} &= \Big(\frac{p-1}{2N}\Big)^{\frac{p-1}{2p-1}} \mathrm{e}^{-\frac{2N2^{j+1}}{2p-1}} \Big(\mathrm{e}^{\frac{3N2^j}{p-1}} - 1\Big)^{\frac{p-1}{2p-1}} \\ &\lesssim \Big(2p-1\Big)^{\frac{p-1}{2p-1}} \Big(\frac{2^j}{2p-1}\Big)^{\frac{p-1}{2p-1}} \lesssim \big(2^j\big)^{\frac{p-1}{2p-1}} ,\end{aligned}$$

which gives rise to

$$\sum_{j\leq j_0} \Big(\int_{2^{j-1}}^{2^{j+1}} \mathrm{e}^{-\frac{2Nt}{p-1}} dt\Big)^{\frac{p-1}{2p-1}} \lesssim \sum_{j\leq j_0} \big(2^{\frac{p-1}{2p-1}}\big)^j \lesssim 2^{(j_0+1)\frac{p-1}{2p-1}} \lesssim \big(2p-1\big)^{\frac{p-1}{2p-1}} .$$

We deduce that for any $p \geq 2$

$$\|w_2\|_{L^{2p}}^{2p} \lesssim C^{2p} \, \||D|^N w_2\|_{B^{0,\log}_{2,\infty}(\mathbb{R}^{2N})}^{2p} \, \big(2p-1\big)^{p-1} .$$

Along the same lines, we obtain

$$\|w_2\|_{L^2}^2 \lesssim \||D|^N w_2\|_{B^{0,\log}_{2,\infty}(\mathbb{R}^{2N})}^2 ,$$

which leads to

$$\int_{\mathbb{R}^{2N}} \Big(e^{|\frac{w_2(x)}{\lambda}|^2} - 1\Big) \, dx \lesssim \sum_{p\geq 1} \frac{C^{2p} \, \||D|^N w_2\|_{B^{0,\log}_{2,\infty}(\mathbb{R}^{2N})}^{2p} \, \big(2p-1\big)^{p-1}}{\lambda^{2p} \, p!} .$$

In view of Stirling formula, we infer that

$$\|w_2\|_{\mathcal{L}(\mathbb{R}^{2N})} \leq C \, \||D|^N w_2\|_{B^{0,\log}_{2,\infty}(\mathbb{R}^{2N})} , \tag{3.4}$$

which in view of (3.1) achieves the proof of the result. □

3.2 A Refined Radial Estimate

The following result is the counterpart of the well-known radial estimate away from the origin available for any radial function in $H^1(\mathbb{R}^{2N})$:

$$|u(x)| \leq \frac{C_2}{\sqrt{|x|^{2N-1}}} \, \|u\|_{L^2(\mathbb{R}^{2N})}^{\frac{1}{2}} \|\nabla u\|_{L^2(\mathbb{R}^{2N})}^{\frac{1}{2}} . \tag{3.5}$$

Proposition 3.3 *There is a positive constant C such that for any radial function w, we have*

$$\sup_{0<|x|\leq e^{-1}} \frac{|w(x)|}{\sqrt{-\log|x|}} \leq C\|w\|_{\dot{H}^N(\mathbb{R}^{2N})}^{\frac{1}{2}} \, \||D|^N w\|_{B^{0,\log}_{2,\infty}(\mathbb{R}^{2N})}^{\frac{1}{2}} . \tag{3.6}$$

Proof To go to the proof of the radial estimate (3.6), let us as in the proof of Proposition 3.1, split the function w into two parts as follows:

$$w = w_1 + w_2$$

where $w_1 = \Theta(D)w$. Obviously

$$\|w_1\|_{L^\infty(\mathbb{R}^{2N})} \lesssim \|w_1\|_{L^2(\mathbb{R}^{2N})} . \tag{3.7}$$

Now arguing as in the proof of Proposition 3.1, write

$$\widehat{w_2}(\xi) = \frac{1}{|\xi|^{2N}}\ \widetilde{w}(\log|\xi|) .$$

Observing that for any $0 < |x| \leq \mathrm{e}^{-1}$, there exists $p \in \mathbf{N}$ such $\mathrm{e}^{-2^{p+1}} \leq |x| \leq \mathrm{e}^{-2^p}$, let us decompose $w_2(x)$ as follows

$$w_2(x) = W^{(1)}(x) + W^{(2)}(x),$$

with $W^{(1)}(x) := \displaystyle\frac{1}{(2\pi)^{2N}} \int_{1\leq|\xi|\leq \mathrm{e}^{2^{p+1}}} \frac{\mathrm{e}^{i\,x\cdot\xi}}{|\xi|^{2N}}\ \widetilde{w}(\log|\xi|)\, d\xi.$

To estimate the part $W^{(1)}$, we shall perform the change of variable $t = \log|\xi|$ and make use of Cauchy-Schwarz inequality which give rise to

$$\begin{aligned}|W^{(1)}(x)| &\leq \frac{1}{(2\pi)^{2N}} \int_{1\leq|\xi|\leq \mathrm{e}^{2^{p+1}}} \frac{\left|\widetilde{w}(\log|\xi|)\right|}{|\xi|^{2N}}\, d\xi\\ &\lesssim \sum_{j=-\infty}^{j=p} \int_{2^j}^{2^{j+1}} \left|\widetilde{w}(t)\right| dt\\ &\lesssim \sum_{j=-\infty}^{j=p} 2^{\frac{j}{2}} \Big(\int_{2^j}^{2^{j+1}} \left|\widetilde{w}(t)\right|^2 dt\Big)^{\frac{1}{2}} \lesssim 2^{\frac{p}{2}} \sup_{j\in\mathbf{Z}} \Big(\int_{2^j}^{2^{j+1}} \left|\widetilde{w}(t)\right|^2 dt\Big)^{\frac{1}{2}} .\end{aligned}$$

This implies in view of (3.2) that

$$|W^{(1)}(x)| \leq 2^{\frac{p}{2}}\ \||D|^N w_2\|_{B^{0,\log}_{2,\infty}(\mathbb{R}^{2N})} .$$

Consequently, we deduce that for $\mathrm{e}^{-2^{p+1}} \leq |x| \leq \mathrm{e}^{-2^p}$

$$\frac{|W^{(1)}(x)|}{\sqrt{-\log|x|}} \lesssim \frac{2^{\frac{p}{2}}}{\sqrt{-\log|x|}}\ \||D|^N w_2\|_{B^{0,\log}_{2,\infty}(\mathbb{R}^{2N})} \lesssim \||D|^N w_2\|_{B^{0,\log}_{2,\infty}(\mathbb{R}^{2N})} ,$$

which achieves the proof of the result for the part $W^{(1)}$ according to the Sobolev embedding

$$L^2(\mathbb{R}^{2N}) \hookrightarrow B^{0,\log}_{2,\infty}(\mathbb{R}^{2N}) .$$

To address the part $W^{(2)}$, we shall make advantage of the radial estimate (3.5) which ensures that

$$|W^{(2)}(x)| \leq \frac{C}{\sqrt{|x|^{2N-1}}} \|W^{(2)}\|^{\frac{1}{2}}_{L^2(\mathbb{R}^{2N})} \|\nabla W^{(2)}\|^{\frac{1}{2}}_{L^2(\mathbb{R}^{2N})} \cdot$$

Obviously, we have

$$\begin{aligned}
\|W^{(2)}\|^2_{L^2(\mathbb{R}^{2N})} &= \frac{1}{(2\pi)^{2N}} \int_{|\xi|\geq e^{2^{p+1}}} \frac{\left|\widetilde{w}(\log|\xi|)\right|^2}{|\xi|^{4N}} d\xi \\
&\lesssim \sum_{q\geq p+1} e^{-2N2^q} \int_{e^{2^q} \leq|\xi|\leq e^{2^{q+1}}} \frac{\left|\widetilde{w}(\log|\xi|)\right|^2}{|\xi|^{2N}} d\xi \\
&\lesssim \sum_{q\geq p+1} e^{-2N2^q} \sup_{q\in\mathbf{Z}} \int_{2^q}^{2^{q+1}} \left|\widetilde{w}(t)\right|^2 dt \lesssim e^{-2N2^{p+1}} \||D|^N w_2\|^2_{B^{0,\log}_{2,\infty}(\mathbb{R}^{2N})} \cdot
\end{aligned}$$

Along the same lines

$$\begin{aligned}
\|\nabla W^{(2)}\|^2_{L^2(\mathbb{R}^{2N})} &= \frac{1}{(2\pi)^{2N}} \int_{|\xi|\geq e^{2^{p+1}}} \frac{\left|\widetilde{w}(\log|\xi|)\right|^2}{|\xi|^{4N-2}} d\xi \\
&\lesssim e^{-2(N-1)2^{p+1}} \int_{|\xi|\geq e^{2^{p+1}}} \frac{\left|\widetilde{w}(\log|\xi|)\right|^2}{|\xi|^{2N}} d\xi \\
&\lesssim e^{-2(N-1)2^{p+1}} \|\widetilde{w}\|^2_{L^2} \cdot
\end{aligned}$$

Taking advantage of (3.3), we infer that

$$\|\nabla W^{(2)}\|_{L^2(\mathbb{R}^{2N})} \lesssim e^{-(N-1)2^{p+1}} \||D|^N w_2\|_{L^2(\mathbb{R}^{2N})} \cdot$$

This leads to the following estimate for $e^{-2^{p+1}} \leq |x| \leq e^{-2^p}$

$$\begin{aligned}
|W^{(2)}(x)| &\lesssim \frac{e^{-\frac{(2N-1)\,2^{p+1}}{2}}}{\sqrt{|x|^{2N-1}}} \||D|^N w_2\|^{\frac{1}{2}}_{L^2(\mathbb{R}^{2N})} \||D|^N w_2\|^{\frac{1}{2}}_{B^{0,\log}_{2,\infty}(\mathbb{R}^{2N})} \\
&\lesssim \||D|^N w_2\|^{\frac{1}{2}}_{L^2(\mathbb{R}^{2N})} \||D|^N w_2\|^{\frac{1}{2}}_{B^{0,\log}_{2,\infty}(\mathbb{R}^{2N})} ,
\end{aligned}$$

which ends the proof of the proposition. □

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A Cauchy-Kovalevsky Theorem for Nonlinear and Nonlocal Equations

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In memory of M. Salah Baouendi

Abstract For a generalized Camassa-Holm equation it is shown that the solution to the Cauchy problem with analytic initial data is analytic in both variables, locally in time and globally in space. Furthemore, an estimate for the analytic lifespan is provided. To prove these results, the equation is written as a nonlocal autonomous differential equation on a scale of Banach spaces and then a version of the abstract Cauchy-Kovalevsky theorem is applied, which is derived by the power series method in these spaces. Similar abstract versions of the nonlinear Cauchy-Kovalevsky theorem have been proved by Ovsyannikov, Treves, Baouendi and Goulaouic, Nirenberg, and Nishida.

Keywords Cauchy problem · Nonlinear evolution equations · Analytic spaces · Sobolev spaces · Well-posedness · Generalized Camassa-Holm equation · Nonlinear Cauchy-Kovalevsky theorem · Ovsyannikov theorem · Degasperis-Procesi equation · Novikov equation · Integrable equations

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1 Introduction and Results

For k any positive integer and b any real number, we consider the Cauchy problem for the following generalized Camassa-Holm equation (g-kbCH)

$$u_t = (1-\partial_x^2)^{-1}\big[u^k u_{xxx} + bu^{k-1}u_x u_{xx} - (b+1)u^k u_x\big], \qquad u(0) = u_0, \quad (1.1)$$

and prove that if the initial datum u_0 is analytic on the line or the torus, then the solution is analytic in both variables, globally in x and locally in t. This should be contrasted with the KdV equation, whose solution is analytic in x but not in t when the initial data are analytic (see [14, 43]). Well-posedness in the sense of Hadamard of the initial value problem for this equation in Sobolev spaces has been proved in [20]. More precisely,there it was proved that if $s > 3/2$ and $u_0 \in H^s$ then there exists $T > 0$ and a unique solution $u \in C([0, T]; H^s)$ of the initial value problem for g-kbCH which depends continuously on the initial data u_0.

Furthermore, we have the estimate

$$\|u(t)\|_{H^s} \le 2\|u_0\|_{H^s}, \quad \text{for} \quad 0 \le t \le T \le \frac{1}{2kc_s\|u_0\|_{H^s}^k}, \quad (1.2)$$

where $c_s > 0$ is a constant depending on s. Also, the data-to-solution map is not uniformly continuous from any bounded subset in H^s into $C([0, T]; H^s)$. Concerning global solutions, it was shown in [25] that if $u_0 \in H^s$, $s > 3/2$, and $m_0 = (1-\partial_x^2)u_0$ does not change sign on $\mathbb{R}$, then the solution to the Cauchy problem for g-kbCH persists for all time in the case $b = k + 1$.

Furthermore, in the cases that $b = k$ with k a positive odd number or $k = 1$ and $b \in [0, 3]$ this equation exhibits unique continuation properties.

The g-kbCH equation, besides having interesting analytic properties, it also contains two integrable equations with quadratic nonlinearities. The first is the well known Camassa-Holm equation(see [4, 12, 13])

$$u_t = (1-\partial_x^2)^{-1}\big[uu_{xxx} + 2u_x u_{xx} - 3uu_x\big], \quad (1.3)$$

which is obtained from (1.1) by letting $k = 1$ and $b = 2$, and the second is the Degasperis-Procesi equation [10]

$$u_t = (1-\partial_x^2)^{-1}\big[uu_{xxx} + 3u_x u_{xx} - 4uu_x)\big], \quad (1.4)$$

which is obtained from (1.1) by letting $k = 1$ and $b = 3$. Also, for $k = 2$ and $b = 3$ it gives the Novikov equation [34]

$$u_t = (1-\partial_x^2)^{-1}\big[u^2 u_{xxx} + 3uu_x u_{xx} - 4u^2 u_x\big], \quad (1.5)$$

which is an integrable equation with cubic nonlinearities.

Integrable equations possess many special properties including a Lax pair, a bi-Hamiltonian formulation, and they can be solved by the Inverse Scattering Method. Also, they possess infinitely many conserved quantities. The H^1-norm of a solution u is such a quantity for the Camassa-Holm and the Novikov equations, since it can be shown that

$$\frac{d}{dt}\|u(t)\|_{H^1}^2 = \frac{d}{dt}\int_{\mathbb{R} \text{ or } \mathbb{T}} \left[u^2(t) + u_x^2(t)\right] dx = 0. \tag{1.6}$$

In fact, this quantity is conserved for all members of g-kbCH with $b = k + 1$. Another interesting property of the g-kbCH equation is that it possesses peakon-type solitary wave solutions [14]. On the line, these solutions are of the form

$$u(x,t) = c^{1/k} e^{-|x-ct|},$$

where $c > 0$ is the wave speed. On the circle, these solutions take the form

$$u(x,t) = \frac{c^{1/k}}{\cosh(\pi)} \cosh([x-ct]_p - \pi),$$

where

$$[x-ct]_p \doteq x - ct - 2\pi \left[\frac{x-ct}{2\pi}\right].$$

In this work we study the Cauchy problem for the g-kbCH equation for initial data in spaces of analytic functions. More precisely, the initial data belong in the following scale of decreasing Banach spaces. For $\delta > 0$ and $s \geq 0$, in the periodic case they are defined by

$$G^{\delta,s}(\mathbb{T}) = \{\varphi \in L^2(\mathbb{T}) : ||\varphi||^2_{G^{\delta,s}(\mathbb{T})} \doteq ||\varphi||^2_{\delta,s} = \sum_{k\in\mathbb{Z}} (1+|k|)^{2s} e^{2\delta|k|} |\widehat{\varphi}(k)|^2 < \infty\}, \tag{1.7}$$

while in the nonperiodic case they are defined by

$$G^{\delta,s}(\mathbb{R}) = \{\varphi \in L^2(\mathbb{R}) : ||\varphi||^2_{G^{\delta,s}(\mathbb{R})} \doteq ||\varphi||^2_{\delta,s} = \int_{\mathbb{R}} (1+|\xi|)^{2s} e^{2\delta|\xi|} |\widehat{\varphi}(\xi)|^2 d\xi < \infty\}. \tag{1.8}$$

Here, when a result holds for both the periodic and non-periodic case then we use the notation $||\cdot||_{\delta,s}$ for the norm and $G^{\delta,s}$ for the space in both cases. We observe that a function φ in $G^{\delta,s}(\mathbb{T})$ has an analytic extension to a symmetric strip around the real axis with width δ (see Lemma 1).This $delta$ is called the *radius of analyticity* of φ.

Next, we state the main result of this work. For the sake of simplicity we shall assume that our initial data u_0 belong in $G^{1,s+2}$.

Theorem 1 *Let $s > \frac{1}{2}$. If $u_0 \in G^{1,s+2}$ on the circle or the line, then there exists a positive time T, which depends on the initial data u_0 and s, such that for every $\delta \in (0, 1)$, the Cauchy problem (1.1) has a unique solution u which is a holomorphic function in $D(0, T(1-\delta))$ valued in $G^{\delta,s+2}$. Furthermore, the analytic lifespan T satisfies the estimate*

$$T \approx \frac{1}{||u_0||^k_{1,s+2}}. \tag{1.9}$$

A more precise statement of estimate (1.9) is provided in Sect. 4 (see (4.6)). For the Camassa-Holm equation on the circle, a result similar to Theorem 1 but without an analytic lifespan estimate like (1.9) was proved in [23]. Furthermore, for the Camassa-Holm, the Degasperis-Procesi and the Novikov equations Theorem 1 was proved in [3]. The present research note generalizes this result to the g-kbCH equation using very similar techniques. We mention here that all this work was motivated by the Cauchy-Kovalevsky type result for the Euler equations that was proved by Baouendi and Goulaouic in [1] as an application of a more general theory about analytic pseudo-differential operators. For more information about nonlinear versions of the Cauchy-Kovalevsky theorem, we refer the reader to Ovsyannikov [35–37], Treves [40, 42], Baouendi and Goulaouic [2], Nirenberg [32], and Nishida [33]. Finally, we mention that there is an extensive literature about Camassa-Holm type equations. For results about well-posedness, continuity properties and traveling wave solutions for these and related evolution equations, we refer the reader to [5–9, 11, 15–24, 26–31, 38, 39, 44], and the references therein.

The paper is organized as follows. In Sect. 2, we state the basic properties of the $G^{\delta,s}$ spaces and their norms. Then, in Sect. 3 we use the power series method to provide a version of an autonomous Ovsyannikov theorem. Finally, in Sect. 4 we prove Theorem 1 by using the Ovsyannikov theorem.

2 Properties of $G^{\delta,s}$ spaces

Recall that a family of Banach spaces $\{X_\delta\}_{0<\delta\leq 1}$ is said to be a scale of decreasing Banach spaces if for any $0 < \delta' < \delta \leq 1$ we have

$$X_\delta \subset X_{\delta'}, \ \ ||.||_{\delta'} \leq ||.||_\delta. \tag{2.1}$$

In the following lemmas, whose proof can be found in [3], we summarize the basic properties of the $G^{\delta,s}$ spaces and their norms. Lemma 1 provides an alternative description of the $G^{\delta,s}$ spaces, while Lemmas 2 and 3 show that the $G^{\delta,s}$ spaces form a scale of decreasing Banach spaces and provide the tools for estimating the right hand-side of the g-kbCH Eq. (1.1).

Lemma 1 *Let $\varphi \in G^{\delta,s}$. Then, φ has an analytic extension to a symmetric strip around the real axis of width δ, for $s \geq 0$ in the periodic case and $s > \frac{1}{2}$ in the non-periodic case.*

Lemma 2 *If $0 < \delta' < \delta \leq 1$, $s \geq 0$ and $\varphi \in G^{\delta,s}$ on the circle or the line, then*

$$||\partial_x \varphi||_{\delta',s} \leq \frac{e^{-1}}{\delta - \delta'}||\varphi||_{\delta,s} \tag{2.2}$$

$$||\partial_x \varphi||_{\delta,s} \leq ||\varphi||_{\delta,s+1} \tag{2.3}$$

$$||(1-\partial_x^2)^{-1}\varphi||_{\delta,s+2} \leq 2||\varphi||_{\delta,s} \tag{2.4}$$

$$||(1-\partial_x^2)^{-1}\varphi||_{\delta,s} \leq ||\varphi||_{\delta,s} \tag{2.5}$$

$$||\partial_x(1-\partial_x^2)^{-1}\varphi||_{\delta,s} \leq ||\varphi||_{\delta,s}. \tag{2.6}$$

Lemma 3 *For $\varphi \in G^{\delta,s}$ on the circle or the line the following properties hold true:*
(1) If $0 < \delta' < \delta$ and $s \geq 0$, then $||\cdot||^2_{\delta',s} \leq ||\cdot||^2_{\delta,s}$; i.e. $G^{\delta,s} \hookrightarrow G^{\delta',s}$.
(2) If $0 < s' < s$ and $\delta > 0$, then $||\cdot||^2_{\delta,s'} \leq ||\cdot||^2_{\delta,s}$; i.e. $G^{\delta,s} \hookrightarrow G^{\delta,s'}$.
(3) For $s > 1/2$ and $\varphi, \psi \in G^{\delta,s}$ we have

$$||\varphi\psi||_{\delta,s} \leq c_s||\varphi||_{\delta,s}||\psi||_{\delta,s}, \tag{2.7}$$

where $c_s = \sqrt{2(1+2^{2s})\sum_{k=0}^{\infty}\frac{1}{(1+|k|)^{2s}}}$ in the periodic case and $c_s = \sqrt{\frac{2(1+2^{2s})}{2s-1}}$ in the non-periodic case.

Remark For $s = 1$ we obtain, in the periodic case, $c_1 = \sqrt{2(1+4)\sum_{\ell=1}^{\infty}\frac{1}{\ell^2}} = \sqrt{\frac{5\pi^2}{3}}$, and, in the non-periodic case, we have $c_1 = \sqrt{10}$.

Lemma 4 *If $u_0 \in C^{\omega}(\mathbb{T})$, there exists $\delta_0 > 0$ such that $u_0 \in G^{\delta_0,s}(\mathbb{T})$ for any $s \geq 0$.*

From now on we fix $s > 1/2$, and without loss of generality we assume that $\delta_0 = 1$.

3 The Power Series Method for the Autonomous Ovsyannikov Theorem

Next, following Treves [40–42] we provide a brief description of an autonomous Ovsyannikov theorem that we will use for the proof of Theorem 1. A more detailed exposition is contained in [3].

Given a decreasing scale of Banach spaces $\{X_\delta\}_{0<\delta\leq 1}$ and initial data $u_0 \in X_1$ we consider the Cauchy problem

$$\frac{du}{dt} = F(u), \quad u(0) = u_0, \tag{3.1}$$

where $F : X_0 \to X_0$ is Ovsyannikov analytic at u_0 and $X_0 = \bigcup_{0<\delta<1} X_\delta$. We recall that $F(u)$ is Ovsyannikov analytic at u_0 if there exist positive constants R, A and C_0 such that for all $k \in \mathbb{Z}_+$ and $0 < \delta' < \delta < 1$ we have

$$||D^k F(u)(v_1, \ldots, v_k)||_{\delta'} \leq \frac{AC_0^k k!}{\delta - \delta'} ||v_1||_\delta \ldots ||v_k||_\delta, \tag{3.2}$$

for all $u \in \{u \in X_\delta : ||u - u_0||_\delta < R\}$ and $(v_1, \ldots, v_k) \in X_\delta^k$, where $D^k F$ is the Frechet derivative of F of order k. Such a function can be represented by its Taylor series near u_0. More precisely, given any pair (δ, δ'), $0 < \delta' < \delta < 1$ and any $u \in B_\delta(u_0; R)$ the Taylor series

$$\sum_{k=0}^{\infty} \frac{1}{k!} D^k F(u_0) \underbrace{(u - u_0, \ldots, u - u_0)}_{k}$$

converges absolutely to $F(u)$ in $X_{\delta'}$.

The fundamental result, which we shall need for the proof of Theorem 1, reads as follows.

Theorem 2 *If $u_0 \in X_1$ and F is Ovsyannikov analytic, then there exists $T > 0$ such that the Cauchy problem (3.1) has a unique solution which, for every $\delta \in (0, 1)$ is a holomorphic function in $D(0, T(1-\delta))$ valued in X_δ satisfying*

$$\sup_{|t|<T(1-\delta)} \|u(t) - u_0\|_\delta < R, \quad 0 < \delta < 1. \tag{3.3}$$

Moreover, the lifespan T is given by

$$T = \frac{1}{2e^2 AC_0}, \tag{3.4}$$

where the constants R, A and C_0 come from the definition of Ovsyannikov analytic function.

The proof of this result uses the power series method and it can be found in [3].

4 Proof of Theorem 1

Next, we shall use Theorem 2 in order to prove Theorem 1 for the Cauchy problem of the g-kbCH equation (1.1). In this situation the function $F(u)$ has the following nonlocal form

$$F(u) = (1-\partial_x^2)^{-1}\big[u^k u_{xxx} + bu^{k-1}u_x u_{xx} - (b+1)u^k u_x\big]. \tag{4.1}$$

Also, the scale of decreasing Banach spaces is given by

$$\{G^{\delta,s+2}\}_{0<\delta\le 1}, \quad \text{with norm} \quad ||\cdot||_{\delta,s}. \tag{4.2}$$

In order to prove the existence and uniqueness of a holomorphic solution to our Cauchy problem (1.1), by using Theorem 2, it suffices to estimate $||D^k F(u_0)(v_1,\dots,v_k)||_{\delta'}$ for all $(v_1,\dots,v_k)\in X_\delta^k$. This, in combination with formula (3.4) in Sect. 3, will also provide the desired estimate (1.9) for the analytic lifespan of the solution in terms of the norm of the initial data.

Next, we shall provide an estimate for $||D^k F(u_0)(v_1,\dots,v_k)||_{\delta'}$, only for the first term of the right-hand side of F Eq. (4.1), that is

$$F_1(u) \doteq (1-\partial_x^2)^{-1}[u^k\partial_x^3 u]. \tag{4.3}$$

The estimate for the other two terms is analogous. By using the following formula for the Frechet derivative of F_1 of order j, $1\le j\le k$, at the point u_0,

$$D^j F_1(u_0)(v_1,\dots,v_j) = \frac{d}{d\tau_j}\cdots\frac{d}{d\tau_1}\Big\{F_1(u_0+\sum_{i=1}^{j}\tau_i v_i)\Big\}\Big|_{\tau_1=\cdots=\tau_j=0},$$

we obtain

$$\begin{aligned} D^j F_1(u_0)(v_1,\dots,v_j) = (1-\partial_x^2)^{-1}\Bigg[&\frac{k!}{(k-j)!}u_0^{k-j}(\partial_x^3 u_0)v_1 v_2\cdots v_j \\ &+ \frac{k!}{(k-j+1)!}u_0^{k-j+1}\big((\partial_x^3 v_1)v_2\cdots v_j+\cdots+v_1\cdots v_{j-1}(\partial_x^3 v_j)\big)\Bigg], \end{aligned}$$

where $v_\ell\in G^{\delta,s+2}$, $j=1,\dots,k$. We also have that

$$D^{k+1}F_1(u_0)(v_1,\dots,v_{k+1}) = (1-\partial_x^2)^{-1}\big[k!\big((\partial_x^3 v_1)v_2\cdots v_{k+1}+\cdots+v_1\cdots v_k(\partial_x^3 v_{k+1})\big)\big],$$

and $D^j F_1(u_0)=0$ for all $j>k+1$.

By using Lemmas 2 and 3, for $0<\delta'<\delta\le 1$, $1\le j\le k$ and $v_1,\dots,v_j\in G^{\delta,s+2}$ and assuming that $s>1/2$ we can estimate

$$
\begin{aligned}
\|D^jF_1(u_0)(v_1,\ldots,v_j)\|_{\delta',s+2} &\le 2\frac{k!}{(k-j)!}\|u_0^{k-j}(\partial_x^3u_0)v_1v_2\cdots v_j\|_{\delta',s}\\
&+2\frac{k!}{(k-j+1)!}\|u_0^{k-j+1}\left((\partial_x^3v_1)v_2\cdots v_j+\cdots+v_1\cdots v_{j-1}(\partial_x^3v_j)\right)\|_{\delta',s}\\
&\le 2c_s^k\frac{k!}{(k-j)!}\|u_0\|_{\delta',s}^{k-j}\|\partial_x^3u_0\|_{\delta',s}\|v_1\|_{\delta',s}\|v_2\|_{\delta',s}\cdots\|v_j\|_{\delta',s}\\
&+2c_s^k\frac{k!}{(k-j+1)!}\|u_0\|_{\delta',s}^{k-j+1}\|\partial_x^3v_1\|_{\delta',s}\|v_2\|_{\delta',s}\cdots\|v_j\|_{\delta',s}\\
&+\cdots+2c_s^k\frac{k!}{(k-j+1)!}\|u_0\|_{\delta',s}^{k-j+1}\|v_1\|_{\delta',s}\cdots\|v_{j-1}\|_{\delta',s}\|\partial_x^3v_j\|_{\delta',s}\\
&\le\frac{2c_s^ke^{-1}}{\delta-\delta'}\frac{k!}{(k-j)!}\|u_0\|_{1,s+2}^{k-j+1}\|v_1\|_{\delta,s+2}\|v_2\|_{\delta,s+2}\cdots\|v_j\|_{\delta,s+2}\\
&+\frac{2c_s^ke^{-1}}{\delta-\delta'}\frac{k!j}{(k-j+1)!}\|u_0\|_{1,s+2}^{k-j+1}\|v_1\|_{\delta,s+2}\|v_2\|_{\delta,s+2}\cdots\|v_j\|_{\delta,s+2}.
\end{aligned}
$$

Notice now that

$$
\begin{aligned}
\frac{k!}{(k-j)!}+\frac{k!j}{(k-j+1)!} &= j!\left(\frac{k!}{j!(k-j)!}+\frac{k!}{(j-1)!(k-j+1)!}\right)\\
&= j!\left(\frac{k!}{j(j-1)!(k-j)!}+\frac{k!}{(j-1)!(k-j+1)(k-j)!}\right)\\
&= j!\left(\frac{k!}{(j-1)!(k-j)!}\left(\frac{1}{j}+\frac{1}{k-j+1}\right)\right)\\
&= j!\frac{k!}{(j-1)!(k-j)!}\frac{k+1}{j(k-j+1)}\\
&= j!\frac{(k+1)!}{j!(k-j+1)!}\\
&= j!\binom{k+1}{k-j+1}\\
&\le j!2^{k+1},\ \forall\ 1\le j\le k.
\end{aligned}
$$

Hence, if we take $C_0=\dfrac{1}{\|u_0\|_{1,s+2}}$ and $A_1=c_s^ke^{-1}2^{k+2}\|u_0\|_{1,s+2}^{k+1}$ then we have that

$$
\|D^jF_1(u_0)(v_1,\ldots,v_j)\|_{\delta',s+2}\le\frac{A_1C_0^jj!}{\delta-\delta'}\|v_1\|_{\delta,s+2}\|v_2\|_{\delta,s+2}\cdots\|v_j\|_{\delta,s+2}. \tag{4.4}
$$

By proceeding analogously with the other two terms in (4.1), we have that

$$
\|D^jF(u_0)(v_1,\ldots,v_j)\|_{\delta',s+2}\le\frac{AC_0^jj!}{\delta-\delta'}\|v_1\|_{\delta,s+2}\|v_2\|_{\delta,s+2}\cdots\|v_j\|_{\delta,s+2}, \tag{4.5}
$$

where $A=(1+|b|)c_s^k2^{k+3}e^{-1}\|u_0\|_{1,s+2}^{k+1}$.

Therefore, by Theorem 2 we conclude that the problem (1.1) has a unique solution, which for $0 < \delta < 1$ is a holomorphic function in the disc $D(0, T(1-\delta))$ valued in $G^{\delta,s+2}$. Moreover, the lifespan T is given by

$$T = \frac{1}{2e^2 AC_0} = \frac{1}{c\|u_0\|_{1,s+2}^k}, \tag{4.6}$$

where $c = e(1+|b|)c_s^k 2^{k+4}$. The proof of Theorem 1 is now complete. □

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Dirichlet Eigenfunctions of the Square Membrane: Courant's Property, and A. Stern's and Å. Pleijel's Analyses

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In memory of M. Salah Baouendi

Abstract In this paper, we revisit Courant's nodal domain theorem for the Dirichlet eigenfunctions of a square membrane, and the analyses of A. Stern and Å. Pleijel.

Keywords Nodal lines · Nodal domains · Courant theorem

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1 Introduction

Courant's celebrated nodal domain theorem [6] says that the number of nodal domains of an eigenfunction associated with a kth eigenvalue of the Dirichlet Laplacian is less than or equal to k. Here, the eigenvalues are chosen to be positive, and listed in increasing order. It follows from a theorem of Pleijel [17] that equality in Courant's theorem only occurs for finitely many values of k. In this case, we speak of the Courant sharp situation. We refer to [11, 12] for the connection of this property with the question of minimal spectral partitions.

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In the case of the square, it is immediate that the first, second and fourth eigenvalues are Courant sharp. In the first part of this note, Sects. 2 and 3, we provide some missing arguments in Pleijel's paper leading to the conclusion that there are no other cases.

In the second part of this paper, we discuss some results of Antonie Stern. She was a Ph.D. student of R. Courant, and defended her Ph.D. in 1924, see [19, 20], and [21, p. 180]. In her thesis, she in particular provides an infinite sequence of Dirichlet eigenfunctions for the square, as well as an infinite sequence of spherical harmonics on the 2-sphere, which have exactly two nodal domains. In this paper, we focus on her results concerning the square, and refer to [3] for an analysis of the spherical case. In Sect. 4, we analyze Stern's argument, leading to the conclusion that her proofs are not quite complete. In Sect. 6, we provide a detailed proof of Stern's main result for the square, Theorem 4.1.

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2 Pleijel's Analysis

Consider the rectangle $\mathcal{R}(a,b) =]0, a\pi[\times]0, b\pi[$. The Dirichlet eigenvalues for $-\Delta$ are given by

$$\hat{\lambda}_{m,n} = \left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right), \quad m, n \geq 1,$$

with a corresponding basis of eigenfunctions given by

$$\phi_{m,n}(x, y) = \sin\frac{mx}{a}\,\sin\frac{ny}{b}\,.$$

It is easy to determine the Courant sharp eigenvalues when b^2/a^2 is irrational (see for example [12]). The rational case is more difficult. Let us analyze the zero set of the Dirichlet eigenfunctions for the square. If we normalize the square as $]0, \pi[\times]0, \pi[$, we have,

$$\phi_{m,n}(x, y) = \phi_m(x)\phi_n(y)\,, \quad \text{with } \phi_m(t) = \sin(mt)\,.$$

Due to multiplicities, we have (at least) to consider the family of eigenfunctions,

$$(x, y) \mapsto \Phi_{m,n}(x, y, \theta) := \cos\theta\,\phi_{m,n}(x, y) + \sin\theta\,\phi_{n,m}(x, y)\,,$$

with $m, n \geq 1$, and $\theta \in [0, \pi[$.

In [17], Pleijel claims that the Dirichlet eigenvalue λ_k of the square is Courant sharp if and only if $k = 1, 2, 4$. The key point in his proof is to exclude the eigenvalues λ_5, λ_7 and λ_9 which correspond respectively to the pairs $(m, n) = (1, 3)$, $(m, n) = (2, 3)$ and $(m, n) = (1, 4)$.

Let us briefly recall Pleijel's argument. Let $N(\lambda) := \#\{n \mid \lambda_n < \lambda\}$ be the counting function. Using a covering of $\mathbb{R}^2 \cap \{x \geq 1, y \geq 1\}$ by the squares $[k, k+1] \times [\ell, \ell+1]$, he first establishes the estimate

$$\text{for } \lambda \geq 2, \quad N(\lambda) > \frac{\pi}{4}\lambda - 2\sqrt{\lambda} + 1\,. \tag{2.1}$$

If λ_n is Courant sharp, then $\lambda_{n-1} < \lambda_n$, hence $N(\lambda_n) = n - 1$, and

$$n > \frac{\pi}{4}\lambda_n - 2\sqrt{\lambda_n} + 2\,. \tag{2.2}$$

On the other hand, if λ_n is Courant sharp, the Faber-Krahn inequality [1, 9] gives the necessary condition

$$\frac{\lambda_n}{n} \geq \frac{{j_{0,1}}^2}{\pi}$$

or

$$\frac{n}{\lambda_n} \leq \pi\, {j_{0,1}}^{-2} < 0.54323\,. \tag{2.3}$$

Recall that $j_{0,1}$ is the smallest positive zero of the Bessel function of order 0, and that $\pi\, {j_{0,1}}^2$ is the ground state energy of the disk of area 1.

Combining (2.2) and (2.3), leads to the inequality

$$\lambda_n < 51\,. \tag{2.4}$$

After re-ordering the values $m^2 + n^2$, we get the following spectral sequence for $\lambda_n \leq 52$,

$$\begin{array}{llll}
\lambda_1 = 2\,, & \lambda_2 = \lambda_3 = 5\,, & \lambda_4 = 8\,, & \lambda_5 = \lambda_6 = 10\,,\\
\lambda_7 = \lambda_8 = 13\,, & \lambda_9 = \lambda_{10} = 17\,, & \lambda_{11} = 18\,, & \lambda_{12} = \lambda_{13} = 20\,,\\
\lambda_{14} = \lambda_{15} = 25\,, & \lambda_{16} = \lambda_{17} = 26\,, & \lambda_{18} = \lambda_{19} = 29\,, & \lambda_{20} = 32\,,\\
\lambda_{21} = \lambda_{22} = 34\,, & \lambda_{23} = \lambda_{24} = 37\,, & \lambda_{25} = \lambda_{26} = 40\,, & \lambda_{27} = \lambda_{28} = 41\,,\\
\lambda_{29} = \lambda_{30} = 45\,, & \lambda_{31} = \lambda_{32} = \lambda_{33} = 50\,, & \lambda_{34} = \lambda_{35} = 52\,. &
\end{array} \tag{2.5}$$

It follows that it remains to analyze, among the eigenvalues which are less than or equal to 50, those which satisfy (2.3), and hence which can be Courant sharp. Computing the quotients $\frac{n}{\lambda_n}$ in the list (2.5), leaves us with the eigenvalues λ_5, λ_7 and λ_9. For these last three cases, Pleijel refers to pictures in Courant-Hilbert [7], Sect. V.5.3, p. 302, actually reproduced from [18], Sect. II.B.6, p. 80, see Fig. 1 in which $u_{mn}(x, y)$ stands for $\sin(mx)\sin(ny)$. Although the details are not provided in these textbooks, the choice of the parameter values in the pictures suggests that

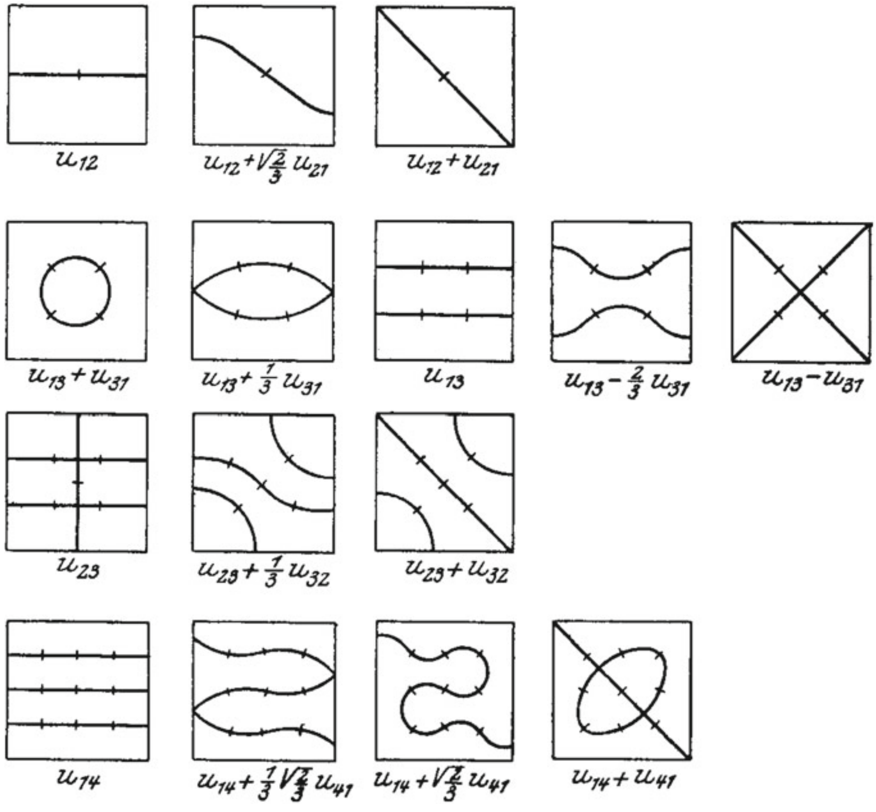


Fig. 1 Nodal sets for the Dirichlet eigenvalues λ_2, λ_5, λ_7 and λ_9 (reproduced with permission from [8])

some theoretical analysis is involved. It is however not clear whether the displayed nodal patterns represent all possible shapes, up to deformation or symmetries. The difficulty being that we have to analyze the nodal sets of eigenfunctions living in two-dimensional eigenspaces. Clearly, $\phi_{m,n}$ has mn nodal components. This corresponds to the "product" situation with $\theta = 0$ or $\theta = \frac{\pi}{2}$. The figures illustrate the fact that the number of nodal domains for a linear combination of two given independent eigenfunctions can be smaller or larger than the number of nodal domains of the given eigenfunctions. For these reasons, Pleijel's argument does not appear fully convincing. In Sect. 3, we give a detailed proof that eigenvalues λ_5, λ_7 and λ_9 are not Courant sharp.

Remark As pointed out to us by M. Persson-Sundqvist, the last two cases can be easily dealt with using the following adaptation of an observation due to Leydold [14]. For any (m, n),

$$\Phi_{m,n}(\pi - x, \pi - y, \theta) = (-1)^{m+n} \Phi_{m,n}(x, y, \theta) .$$

When $(m+n)$ is odd, the corresponding eigenfunction function is odd under the symmetry with respect to the center of the square, and hence must have an even number of nodal domains. Our results are actually stronger, and describe the variation of the nodal sets.

3 The Three Remaining Cases of Pleijel

Behind all the computations, we have the property that, for $x \in]0,\pi[$,

$$\sin mx = \sqrt{1-u^2}\, U_{m-1}(u)\,, \tag{3.1}$$

where U_{m-1} is the Chebyshev polynomial of second type and $u = \cos x$, see [15].

3.1 First Case: Eigenvalue λ_5, or $(m,n) = (1,3)$

We look at the zeroes of $\Phi_{1,3}(x, y, \theta)$, see Fig. 1, 2nd row. Since $\Phi_{1,3}(x, y, \frac{\pi}{2}-\theta) = \Phi_{1,3}(y, x, \theta)$, we can reduce the analysis of the nodal patterns to $\theta \in [\frac{\pi}{4}, \frac{3\pi}{4}]$. Let,

$$\cos x = u\,,\ \cos y = v\,. \tag{3.2}$$

This is a C^∞ change of variables from the square $]0,\pi[\times]0,\pi[$ onto $]-1,+1[\times]-1,+1[$. In these coordinates, the zero set of $\Phi_{1,3}(x, y, \theta)$ inside the square is given by

$$\cos\theta\,(4v^2-1) + \sin\theta\,(4u^2-1) = 0\,. \tag{3.3}$$

To completely determine the nodal set, we have to take the closure in $[-1,1]\times[-1,1]$ of the zero set (3.3). The curve defined by (3.3) is an ellipse, when $\theta \in [\frac{\pi}{4}, \frac{\pi}{2}[$, a hyperbola, when $\theta \in]\frac{\pi}{2}, \frac{3\pi}{4}[$, the union of two vertical lines, when $\theta = \frac{\pi}{2}$, the diagonals $\{x-y=0\}\cup\{x+y=0\}$, when $\theta = \frac{3\pi}{4}$. We only have to analyze how the ellipses and the hyperbolas are situated within the square.

Boundary points. At the boundary, for example on $u = \pm 1$, (3.3) gives:

$$\cos\theta\,(4v^2-1) + 3\sin\theta = 0\,.$$

Depending on the value of θ, we have no boundary point, the zero set (3.3) is an ellipse contained in the open square; one double boundary point, the zero set (3.3) is an ellipse, contained in the closed square, which touches the boundary at two points; or two boundary points, the zero set (3.3) is an ellipse or a hyperbola meeting the boundary of the square at four points.

Interior critical points. We now look at the critical points of the zero set of the function

$$\Psi_{1,3}(u, v, \theta) := \cos\theta\,(4v^2 - 1) + \sin\theta\,(4u^2 - 1)\,.$$

We get two equations:

$$v\cos\theta = 0\,,\ u\sin\theta = 0\,.$$

Except for the two easy cases when $\cos\theta = 0$ or $\sin\theta = 0$, which can be analyzed directly (product situation), we immediately get that the only possible critical point is $(u, v) = (0, 0)$, i.e., $(x, y) = (\frac{\pi}{2}, \frac{\pi}{2})$, and that this can only occur for $\cos\theta + \sin\theta = 0$, i.e., for $\theta = \frac{3\pi}{4}$.

The possible nodal patterns for an eigenfunction associated with λ_5 all appear in Fig. 1, second row.

This analysis shows rigorously that the number of nodal domains is 2, 3 or 4 as claimed in [17], and numerically observed in Fig. 1. Observe that we have a rather complete description of the situation by analyzing the points at the boundary, and the critical points of the zero set inside the square. When no critical point or no change of multiplicity is observed at the boundary, the number of nodal domains remains constant. Hence, the complete computation could be done by analyzing the "critical" values of θ, i.e., those for which there is a critical point on the zero set of $\Psi_{1,3}$ in the interior, or a change of multiplicity at the boundary, and one "regular" value of θ in each non critical interval. To explore all possible nodal patterns, and number of nodal domains, of the eigenfunctions $\Phi^{\theta}_{1,3}$, $\theta \in [\frac{\pi}{4}, \frac{3\pi}{4}]$, associated with the eigenvalue λ_5, it is consequently sufficient to consider the values $\theta = \frac{\pi}{4}$, $\theta = \arctan 3$, $\theta = \frac{\pi}{2}$, $\theta = \frac{3\pi}{4}$.

3.2 Second Case: Eigenvalue λ_7, or $(m, n) = (2, 3)$

We look at the zeros of $\Phi_{2,3}(x, y, \theta)$. We first observe that

$$\Phi_{2,3}(x, y, \theta) = \sin x \sin y \Big(2\cos\theta\cos x(\cos 2y + 2\cos^2 y) + 2\sin\theta\cos y(\cos 2x + 2\cos^2 x)\Big)\,.$$

In the coordinates (3.2), this reads:

$$\Phi_{2,3}(x, y, \theta) = 2\sqrt{1-u^2}\sqrt{1-v^2}\,\Big(u\cos\theta(4v^2-1) + v\sin\theta(4u^2-1)\Big)\,. \tag{3.4}$$

We have to look at the solutions of

$$\Psi_{2,3}(u, v, \theta) := u(4v^2-1)\,\cos\theta + v(4u^2-1)\,\sin\theta = 0\,, \tag{3.5}$$

inside $[-1, +1] \times [-1, +1]$.

Analysis at the boundary. The function $\Psi_{2,3}$ is anti-invariant under the change $(u, v) \to (-u, -v)$. Changing u into $-u$ amounts to changing θ in $\pi-\theta$. Exchanging (x, y) into (y, x) amounts to changing θ into $\frac{\pi}{2} - \theta$. This implies that it suffices to consider the values $\theta \in [0, \frac{\pi}{2}]$, and the boundaries $u = -1$ and $v = -1$. At the boundary $u = -1$, we get:

$$-\cos\theta\,(4v^2 - 1) + 3v\sin\theta = 0\,, \tag{3.6}$$

with the condition that $v \in [-1, +1]$.

We note that the product of the roots is $-\frac{1}{4}$, and that there are always two distinct solutions in $\mathbb{R}$. For $\theta = 0$, we have two solutions given by $v = \pm\frac{1}{2}$. When θ increases we still have two solutions in $[-1, 1]$ till the largest one is equal to 1 (the other one being equal to $-\frac{1}{4}$). This is obtained for $\theta = \frac{\pi}{4}$. For $\theta > \frac{\pi}{4}$, there is only one negative solution in $[-1, 1]$, tending to zero as $\theta \to \frac{\pi}{2}$.

At the boundary $v = -1$, we have for $\theta = 0$, $u = 0$ as unique solution. When θ increases, there is only one solution in $[-1, 1]$, till $\frac{\pi}{4}$, where we get two solutions $u = -\frac{1}{4}$ and $u = 1$. For this value of θ, the zero set is given by $(u+v)(4uv-1) = 0$. For $\theta \in]\frac{\pi}{4}, \frac{\pi}{2}]$, we have two solutions.

We conclude that the zero set of $\Psi_{2,3}$ always hits the boundary at six points.

Critical points. We now look at the critical points of $\Psi_{2,3}$. We get two equations:

$$(4v^2 - 1)\cos\theta + 8uv\sin\theta = 0\,, \tag{3.7}$$

and

$$8uv\,\cos\theta + (4u^2 - 1)\,\sin\theta = 0\,. \tag{3.8}$$

The critical points on the zero set of $\Psi_{2,3}$ are the common solutions of (3.5), (3.7), and (3.8).

If $\cos\theta\sin\theta \neq 0$, we immediately obtain that $u = v = 0$, and these equations have no common solution. It follows that the eigenfunctions associated with λ_7 have no interior critical point on their nodal sets.

One can give the following expressions for the partial derivatives of $\Psi_{2,3}$,

$$\partial_u\Psi_{2,3}(u, v, \theta) = \frac{v}{u}(4u^2 + 1)\sin\theta \text{ and } \partial_v\Psi_{2,3}(u, v, \theta) = \frac{u}{v}(4v^2 + 1)\cos\theta\,,$$

for u, *resp.* v, different from 0. Since a regular closed curve contains points with vertical or horizontal tangents, it follows that the zero set of $\Psi_{2,3}$ cannot contain any closed component (necessarily without self-intersections, otherwise the nodal set of $\Phi_{2,3}$ would have a critical point). The components of this zero set are lines joining two boundary points which are decreasing from the left to the right. These lines cannot intersect each other (for the same reason as before).

The possible nodal patterns for an eigenfunction associated with λ_7 all appear in Fig. 1, third row.

The number of nodal domains is four (delimited by three non intersecting lines) or six in the product case. Hence the maximal number of nodal domains is six.

3.3 Third Case: Eigenvalue λ_9, or $(m, n) = (1, 4)$

We look at the zeros of $\Phi_{1,4}(\cdot, \cdot, \theta)$. Here we can write

$$\Phi_{1,4}(x, y, \theta) = 4 \sin x \sin y \, \Psi_{1,4}(u, v, \theta)$$

with

$$\Psi_{1,4}(u, v, \theta) := \cos\theta \, v(2v^2 - 1) + \sin\theta \, u(2u^2 - 1) \,.$$

Hence, we have to analyze the equation

$$\cos\theta \, v(2v^2 - 1) + \sin\theta \, u(2u^2 - 1) = 0 \,. \tag{3.9}$$

Notice that the functions $\Psi_{1,4}(u, v, \theta)$ are anti-invariant under the symmetry $(u, v) \to (-u, -v)$, and that one can reduce from $\theta \in [0, \pi[$ to the case $\theta \in [0, \frac{\pi}{2}]$ by making use of the symmetries with respect to the lines $\{u = 0\}$, $\{v = 0\}$ and $\{u = v\}$. One can even reduce the analysis to $\theta \in [0, \frac{\pi}{4}]$ by changing θ into $\frac{\pi}{2} - \theta$, and (x, y) into (y, x).

Boundary points. Due to the symmetries, the zero set of $\Psi_{1,4}$ hits parallel boundaries at symmetrical points. For $u = \pm 1$ these points are given by:

$$v(2v^2 - 1) \pm \tan\theta = 0 \,.$$

If we start from $\theta = 0$, we first have three zeroes, corresponding to points at which the zero set of $\Psi_{1,4}$ arrives at the boundary: $0, \pm\frac{1}{\sqrt{2}}$. Looking at the derivative, we have a double point when $v = \pm\frac{1}{\sqrt{6}}$, which corresponds to $\tan\theta = \frac{\sqrt{2}}{3\sqrt{3}}$. For larger values of θ, we have only one point till $\tan\theta = 1$.

Hence, there are 3, 2, 1 or 0 solutions satisfying $v \in [-1, +1]$. The analogous equation for $v = \pm 1$ appears with $\cot\theta$ instead of $\tan\theta$, so that the boundary analysis depends on the comparison of $|\tan\theta|$ with $\frac{\sqrt{2}}{3\sqrt{3}}$, 1 and $\frac{3\sqrt{3}}{\sqrt{2}}$. When the points disappear on $u = \pm 1$, they appear on $v = \pm 1$. Notice that the value $\frac{\sqrt{2}}{3\sqrt{3}}$ appears in Fig. 1 and in Courant-Hilbert's book [7], Sect. V.5.3, p. 302. Finally, the maximal number of points along the boundary is six, counting multiplicities.

Critical points. The critical points of $\Psi_{1,4}$ satisfy:

$$\cos\theta \, (6v^2 - 1) = 0 \,, \tag{3.10}$$

and

$$\sin\theta\,(6u^2 - 1) = 0\,. \tag{3.11}$$

If we exclude the "product" case, the only critical points are determined by $u^2 = \frac{1}{6}$ and $v^2 = \frac{1}{6}$. Plugging these values in (3.9), we obtain that interior critical points on the zero set of $\Psi_{1,4}$ can only appear when:

$$\cos\theta\, \pm \sin\theta\, = 0\,. \tag{3.12}$$

Hence, we only have to look at $\theta = \frac{\pi}{4}$ and $\theta = \frac{3\pi}{4}$. Because of symmetries, it suffices to consider the case $\theta = \frac{\pi}{4}$:

$$\Psi_{1,4}\left(u, v, \frac{\pi}{4}\right) := \frac{1}{\sqrt{2}}(v(2v^2-1)+u(2u^2-1)) = \frac{1}{\sqrt{2}}(u+v)\left(2\left(u-\frac{v}{2}\right)^2+\frac{3}{2}v^2-1\right).$$

The zero set is the union of an ellipse contained in the square, and the anti-diagonal, with two intersection points. It follows that the function $\Phi_{1,4}(x, y, \frac{\pi}{4})$ has four nodal domains. Figure 2 shows the deformation of the nodal set of $\Phi_{1,4}(x, y, \theta)$ for $\theta \leq \frac{\pi}{4}$ close to $\frac{\pi}{4}$, as well as the grid $\{\sin(4x)\sin(4y) = 0\}$. Figure 3 shows the deformation for $\Phi_{1,6}(x, y, \theta)$.

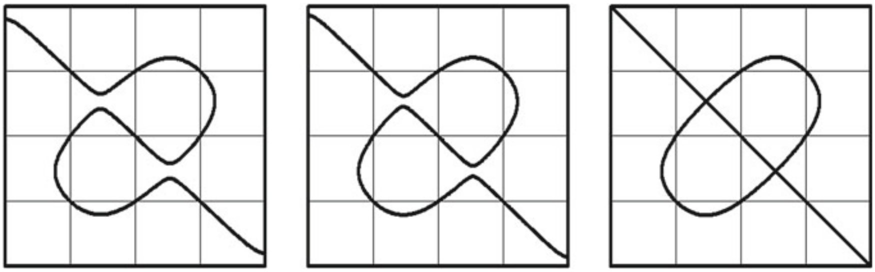


Fig. 2 Eigenvalue λ_9, deformation of the nodal set near $\theta = \frac{\pi}{4}$

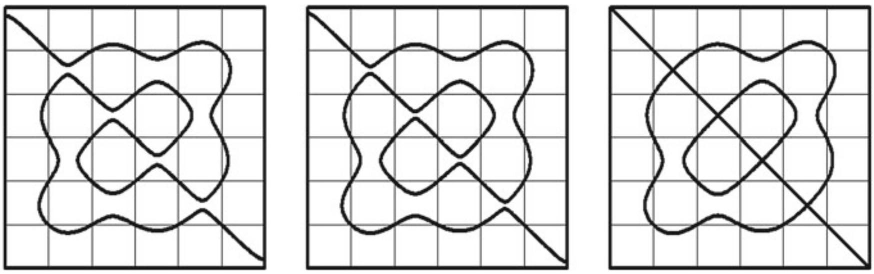


Fig. 3 Eigenvalue λ_{23}, deformation of the nodal set near $\theta = \frac{\pi}{4}$

Let us summarize what we have so far obtained for the eigenfunctions associated with λ_9.

- We have determined the shape of the nodal set of $\Phi_{1,4}$ when $\theta = \frac{\pi}{4}$ or $\frac{3\pi}{4}$, and we can easily see that these are the only cases in which the interior part of the nodal set hits the boundary at the vertices.
- When $\theta \neq \frac{\pi}{4}$ or $\frac{3\pi}{4}$, we have proved that the nodal set of $\Phi_{1,4}$ has no interior critical point, hence no self-intersection, and that it hits the boundary at 2 or 6 points, counting multiplicities.
- We can also observe that all nodal sets must contain the lattice points $(i\frac{\pi}{4}, j\frac{\pi}{4})$ for $1 \leq i, j \leq 3$, and that these points are always regular points of the nodal sets. This implies, by energy considerations, that the nodal sets cannot contain any closed component avoiding these lattice points. The lattice points are indicated in Fig. 1. They appear in Figs. 2 and 3 as the vertices of the grids.

We still need to prove that all the possible nodal patterns for an eigenfunction associated with the eigenvalue λ_9 appear in Fig. 1, fourth row, and hence that the maximal number of nodal domains is 4, so that λ_9 is not Courant sharp. When $\theta \in [0, \frac{\pi}{4}]$, this can be done by looking at the intersections of the nodal sets with the horizontal lines $\{y = \arccos(\pm\sqrt{\frac{1}{6}})\}$. We leave the details to the reader, and refer to Sect. 6 for general arguments.

Remark Figures 1 and 2 indicate that for some values of θ, the function $\Phi_{1,4}(x, y, \theta)$ has exactly two nodal domains. This phenomenon has been studied by Antonie Stern [19] who claims that for any $k \geq 2$, there exists an eigenfunction associated with the Dirichlet eigenvalue $(1 + 4k^2)$ of the square $[0, \pi]^2$, with exactly two nodal domains. In Sects. 4–6, we look at Stern's thesis more carefully.

4 The Observations of A. Stern

The general topic of A. Stern's thesis [19] is the asymptotic behaviour of eigenvalues and eigenfunctions. In Part I, she studies the nodal sets of eigenfunctions of the Laplacian in the square with Dirichlet boundary condition, and on the 2-sphere. The eigenvalues are chosen to be positive, and listed in increasing order, with multiplicities. In [20], we propose extracts from Stern's thesis, with annotations and highlighting to point out the main results and ideas.

As we have seen in the previous sections, Pleijel's theorem [17] states that for a plane domain, there are only finitely many Courant sharp Dirichlet eigenvalues. For the Dirichlet Laplacian in the square, Stern claims [20, tags E1, Q1] that there are actually infinitely many eigenfunctions having exactly two nodal domains.[1]

> [E1] … Im eindimensionalen Fall wird nach den Sätzen von Sturm[2] das Intervall durch die Knoten der $n$ten Eigenfunktion in n Teilgebiete zerlegt. Dies Gesetz verliert seine Gültigkeit bei mehrdimensionalen Eigenwertproblemen, … es läßt sich beispielsweise leicht zeigen, daß auf der Kugel bei jedem Eigenwert die Gebietszahlen 2 oder 3 auftreten, und daß bei Ordnung nach wachsenden Eigenwerten auch beim Quadrat die Gebietszahl 2 immer wieder vorkommt.

> [Q1] … Wir wollen nun zeigen, daß beim Quadrat die Gebietszahl zwei immer wieder auftritt.

The second statement is mentioned in the book of Courant-Hilbert [7], Sect. VI.6, p. 455.

Pleijel's theorem has been generalized to surfaces by Peetre [16], see also [4]. As a consequence, only finitely many eigenvalues of the sphere are Courant sharp. Stern claims [20, tags E1, K1, K2] that, for any $\ell \geq 2$, there exists a spherical harmonic of degree ℓ with exactly three nodal domains when ℓ is odd, *resp.* with exactly two nodal domains when ℓ is even,

> [K1] … Zunächst wollen wir zeigen, daß es zu jedem Eigenwert Eigenfunktionen gibt, deren Nullinien die Kugelfläche nur in zwei oder drei Gebiete teilen.

> [K2] … Die Gebietszahl zwei tritt somit bei allen Eigenwerten $\lambda_n = (2r+1)(2r+2) \quad r = 1, 2, \ldots$ auf; ebenso wollen wir jetzt zeigen, daß die Gebietszahl drei bei allen Eigenwerten
>
> $$\lambda_n = 2r(2r+1) \qquad r = 1, 2, \ldots$$
>
> immer wieder vorkommt.

These two statements are usually attributed to Lewy [13].

[1] English translation from German citation:

> [E1] … In dimension one, according to Sturm's theorem (see footnote 2), the interval is divided into n subsets by the nodes of the nth eigenfunction. This rule no longer holds for multidimensional problems, … It can be easily shown that on the sphere, for each eigenvalue, two or three appear as numbers of nodal domains, and that when ordering the eigenvalues in nondecreasing order the number of nodal domains 2 always reappears.

> [Q1] … We now want to show that for the square the number of nodal domains two always reappears.

> [K1] … We next want to show that for each eigenvalue there exists an eigenfunction whose nodal lines divide the sphere into two or three domains.

> [K2] … the number of nodal domains two appears for all eigenvalues $\lambda_n = (2r+1)(2r+2) \quad r =, 1, 2, \ldots$ and we now want to show that the number of nodal domains three always reappears for all eigenvalues
>
> $$\lambda_n = 2r(2r+1) \qquad r = 1, 2, \ldots$$

[2] Journal de Mathématiques, T.1, 1836, pp. 106–186, 269–277, 375–444.

In this paper, we shall only deal with the case of the square, leaving the case of the sphere for [3]. First, we quote the main statements made by Stern [20, tags Q1–Q3], and summarize them in Theorem 4.1.[3]

> [Q2] … Wir betrachten die Eigenwerte
>
> $$\lambda_n = \lambda_{2r,1} = 4r^2 + 1\,,\ r = 1, 2, \dots$$
>
> und die Knotenlinie der zugehörige Eigenfunktion
>
> $$u_{2r,1} + u_{1,2r} = 0\,,$$
>
> für die sich, wie leicht mittels graphischer Bilder nachgewiesen werden kann, die Figur 7 ergibt.
>
> [Q3] … Laßen wir nur μ von $\mu = 1$ aus abnehmen, so lösen sich die Doppelpunkte der Knotenlinie alle gleichzeitig und im gleichem Sinne auf, und es ergibt sich die Figur 8. Da die Knotenlinie aus einem Doppelpunktlosen Zuge besteht, teilt sich das Quadrat in zwei Gebiete und zwar geschieht dies für alle Werte $r = 1, 2, \dots$, also Eigenwerte $\lambda_n = \lambda_{2r,1} = 4r^2 + 1$.

Theorem 4.1 *For any $r \in \mathbb{N}$, consider the family $\Phi_{1,2r}(x, y, \theta)$ of eigenfunctions of the Laplacian in the square $[0, \pi]^2$, associated with the Dirichlet eigenvalue $1 + 4r^2$,*

$$\Phi^{\theta}_{1,2r}(x, y) := \Phi_{1,2r}(x, y, \theta) := \cos\theta \sin x \sin(2ry) + \sin\theta \sin(2rx) \sin y\,.$$

Then:

(i) For $\theta = \frac{\pi}{4}$, the nodal pattern of Φ is as shown in Fig. 4, left, [19, Fig. 7].
(ii) For $\theta < \frac{\pi}{4}$, and θ sufficiently close to $\frac{\pi}{4}$, the double points all disappear at the same time and in a similar manner ('im gleichem Sinne') as in Fig. 4, right, [19, Fig. 8]. The nodal set consists of a connected line ('aus einem Zuge') with no double point. It divides the square into two domains.

Remark Although this is not stated explicitly, one can infer from Stern's thesis, (i) that the eigenfunction $\Phi_{1,2r}(x, y, \frac{\pi}{4})$ has $2r$ nodal domains and $(2r - 2)$ double points,

---

[3] English translation from German citation:

> [Q2] … We consider the eigenvalues
>
> $$\lambda_n = \lambda_{2r,1} = 4r^2 + 1\,,\ r = 1, 2, \dots$$
>
> and the nodal lines of the associated eigenfunction
>
> $$u_{2r,1} + u_{1,2r} = 0\,,$$
>
> which, as can be easily proved by using graphic images, gives Fig. 7.
>
> [Q3] … If starting from $\mu = 1$ we decrease μ, then the double points of the nodal lines disappear simultaneously and in the same way as shown in Fig. 8. As the nodal set consists of one line without double point, the square becomes divided into two domains and this occurs for all eigenvalues $\lambda_n = \lambda_{2r,1} = 4r^2 + 1$ $(r = 1, 2, \dots)$.

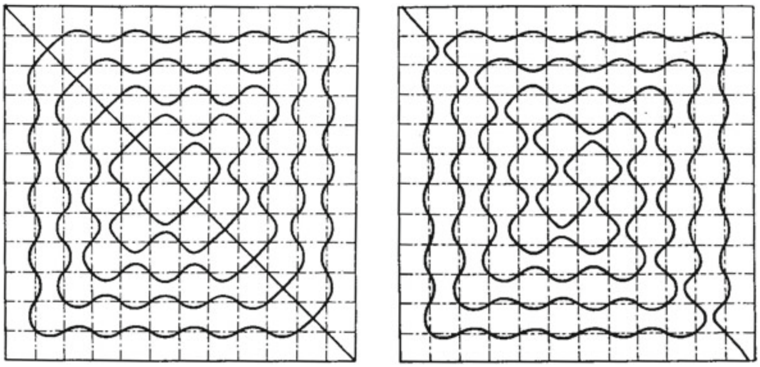


Fig. 4 Case $r = 6$, nodal sets for $\theta = \frac{\pi}{4}$ and θ close to $\frac{\pi}{4}$ (reproduced from with permission from [8])

and (ii) that for θ close to and different from $\frac{\pi}{4}$, the nodal set of $\Phi_{1,2r}(x, y, \theta)$ consists of the boundary of the square and a *connected* simple curve from one point of the boundary to a symmetric point. This curve divides the domain into two connected components (Fig. 5).

A. Stern states two simple properties which play a key role in the proofs [20, tags I1, I2]. These statements are formalized in Property 4.2 below.[4]

> [I1] … Um den typischen Verlauf der Knotenlinie zu bestimmen, haben wir ähnliche Anhaltspunkte wie auf der Kugelfläche. Legen wir die Knotenliniensysteme von $u_{\ell,m}$ ($\ell - 1$ Parallelen zur y-Achse, $m - 1$ zur x-Achse) und $u_{m,\ell}$ ($m - 1$ Parallelen zur y-Achse, $\ell - 1$ zur x-Achse) übereinander, so kann für $\mu > 0$ (< 0) die Knotenlinie nur in den Gebieten verlaufen, in denen beide Funktionen verschiedenes (gleiches) Vorzeichen haben.

> [I2] … Weiter müssen alle zum Eigenwert $\lambda_{\ell,m}$ gehörigen Knotenlinien durch Schnittpunkte der Liniensysteme $u_{\ell,m} = 0$ und $u_{m,\ell} = 0$, also durch $(\ell - 1)^2 + (m - 1)^2$ feste Punkte hindurchgehen …

Property 4.2 *Let ϕ and ψ be two linearly independent eigenfunctions associated with the same eigenvalue for the square $\mathcal{S}$. Let μ be a real parameter, and consider the family of eigenfunctions $\phi_\mu = \psi + \mu\phi$. Let $N(\phi)$ denote the nodal set of the eigenfunction ϕ.*

---

[4] English translation from German citation:

> [I1 … In order to determine the typical course of the nodal lines, we have similar key points as for the sphere. If we superimpose the systems of nodal lines of $u_{\ell,m}$ and $u_{m,\ell}$, then for $\mu > 0$ (< 0) the nodal lines can only visit the domains for which the two functions have opposite (same) signs.

> [I2] … Moreover all the nodal lines associated with the eigenvalue $\lambda_{\ell,m}$ should meet all the crossing points between the nodal sets $u_{\ell,m} = 0$ and $u_{m,\ell} = 0$, hence going through $(\ell - 1)^2 + (m - 1)^2$ fixed points …

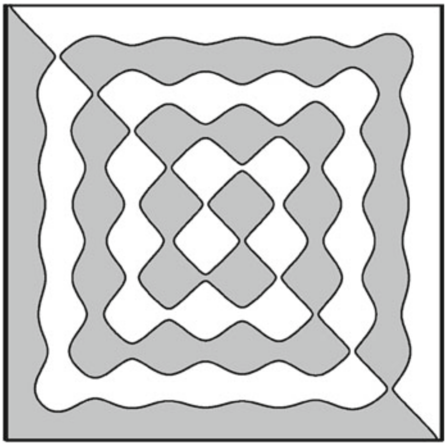


Fig. 5 Nodal domains [5] (reproduced with permission of Virginie Bonnaillie-Noël)

(i) *Consider the domains in* $\mathcal{S} \setminus N(\phi) \cup N(\psi)$ *in which* $\mu\,\phi\,\psi > 0$ *and hatch them.*[5] *Then the nodal set* $N(\phi_\mu)$ *avoids the hatched domains.*
(ii) *The points in* $N(\phi) \cap N(\psi)$ *belong to the nodal set* $N(\phi_\mu)$ *for all* μ.

Property 4.3 *The nodal set* $N(\phi_\mu)$ *depends continuously on* μ.

Remark As a matter of fact, A. Stern uses Property 4.2 in both cases (square and sphere), and only mentions Property 4.3 in the case of the sphere. She says nothing on the proof of this second property which is more or less clear near regular points, but not so clear near multiple points. H. Lewy provides a full proof in the case of the sphere [13, Lemmas 2–4].

Finally, A. Stern mentions that she uses a graphical method ('mittels graphischer Bilder' and 'unter Zuhilfenahme graphischer Bilder' [20, tags Q2, Q4]) which may have been classical at her time, and could explain the amazing quality of her pictures. On this occasion, she mentions a useful idea in her Sect. I.3, namely looking at the intersections of the nodal set $N(\phi_\mu)$ with horizontal or vertical lines.

All in all, the arguments given by A. Stern seem rather sketchy to us, and we do not think that they are quite sufficient to conclude the proof of Theorem 4.1.

In our opinion, taking care of the following items is missing in Stern's thesis.

(i) Complete determination of the multiple points of $N(\Phi^{\frac{\pi}{4}})$.
(ii) Absence of multiple points in $N(\Phi^{\theta})$, when θ is different from $\frac{\pi}{4}$, and close to $\frac{\pi}{4}$.
(iii) Connectedness of the nodal set $N(\Phi^{\theta})$, or why there are no other components, e.g.closed inner components, in the nodal set.

The aim of this paper is to complete the proofs of A. Stern in the case of the square.

[5] 'Schraffieren', see [20, tag I1], in the spherical case.

Remark In [13], H. Lewy gives a complete proof of Stern's results[6] in the case of the sphere. In the unpublished preprint [10], the authors provide partial answers to the above items in the case of the square.

The key steps to better understand the possible nodal patterns for the eigenvalues $1 + 4r^2$ (and other eigenvalues as well), and to answer the above items, are the following.

- Section 6.3, in which we study the points which are both zeroes and critical points of the functions $\Phi^{\theta}_{1,R}$.
- Section 6.4, in which we study the possible local nodal patterns of the functions $\Phi^{\theta}_{1,R}$.
- Section 6.5, in which we determine the nodal sets of the functions $\Phi^{\theta}_{1,R}$ for $\theta = \frac{\pi}{4}$ or $\frac{3\pi}{4}$.

In the subsequent subsections, we study the deformation of the nodal set of $\Phi^{\theta}_{1,R}$ when θ varies close to $\frac{\pi}{4}$ or $\frac{3\pi}{4}$, and conclude the proof of Theorem 4.1. As a matter of fact, our approach gives the maximal interval in which the nodal set of $\Phi^{\theta}_{1,R}$ remains connected, without critical points, see Lemma 6.10(i).

Sketch of the proof of Theorem 4.1. Consider the eigenvalue $\hat{\lambda}_{1,R} := 1 + R^2$ for the square $\mathcal{S} :=]0, \pi[^2$ with Dirichlet boundary conditions, and consider the eigenfunction

$$\Phi^{\theta}(x, y) := \Phi(x, y, \theta) := \cos\theta \, \sin x \, \sin(Ry) + \sin\theta \, \sin(Rx) \, \sin y \,, \quad \theta \in [0, \pi[\,.$$

Let us introduce the Q-squares,

$$Q_{i,j} :=]\frac{i\pi}{R}, \frac{(i+1)\pi}{R}[\times]\frac{j\pi}{R}, \frac{(j+1)\pi}{R}[\,, \text{ for } 0 \le i, j \le R-1 \,,$$

and the lattice,

$$\mathcal{L} := \left\{ \left(\frac{i\pi}{R}, \frac{j\pi}{R} \right) \mid 1 \le i, j \le R-1 \right\} .$$

The basic idea is to start from the analysis of a given nodal set, e.g. from the nodal set $N(\Phi^{\frac{\pi}{4}})$, and then to use some kind of perturbation argument.

[6] A. Stern and H. Lewy were both students of R. Courant at about the same time, 1925. H. Lewy does however not refer to A. Stern's Thesis in his paper. We refer to [3] for a further discussion.

Here are the key points.

(i) Use Property 4.2: Assertion(i) defines checkerboards by Q-squares (depending on the sign of $\cos\theta$), whose grey squares do not contain any nodal point of Φ^θ. Assertion(ii) says that the lattice $\mathcal{L}$ is contained in the nodal set $N(\Phi^\theta)$ for all θ.

(ii) Determine the possible *critical zeroes* of the eigenfunction Φ^θ, i.e., the zeroes which are also critical points, both in the interior of the square or on the boundary. They indeed correspond to multiple points in the nodal set. Note that the points in $\mathcal{L}$ are not critical zeroes, see Sect. 6.3.

(iii) Determine whether critical zeroes are degenerate or not and their order when they are degenerate.

(iv) Determine how critical zeroes appear or disappear when θ varies, and how the nodal set $N(\Phi^\theta)$ evolves. For this purpose, make a local analysis in the square $Q_{i,j}$, depending on whether it is contained in $\mathcal{S}$ or touches the boundary, see Sect. 6.4.

(v) Determine the nodal sets of the eigenfunctions $Z_\pm$ associated with the eigenvalue $\hat{\lambda}_{1,R}$. For this purpose, determine precisely the critical zeroes of Φ^θ for $\theta = \frac{\pi}{4}$ and $\frac{3\pi}{4}$, and prove a separation lemma in the $Q_{i,j}$ to determine whether the medians of this Q-square meet the nodal set of Φ^θ when $\theta = \frac{\pi}{4}$ or $\frac{3\pi}{4}$, see Sect. 6.4.

(vi) Prove that the nodal set $N(\Phi^\theta)$ does not contain any closed component.

Take $R = 2r$ and $0 < \frac{\pi}{4} - \theta \ll 1$. Using the above analysis one can actually give a complete proof of Theorem 4.1. The analysis of the local possible nodal patterns shows that the nodal set $N(\Phi^{\frac{\pi}{4}})$ is indeed as stated. For $0 < \frac{\pi}{4} - \theta \ll 1$, the eigenfunction Φ^θ has no critical zero in $\mathcal{S}$, and exactly two critical zeroes on the boundary, symmetric with respect to the center of the square. This proves in particular that the critical zeroes of $\Phi^{\frac{\pi}{4}}$ all disappear at once when θ changes, $\theta < \frac{\pi}{4}$. Starting from one of the critical zeroes on the boundary, and using the above analysis, one can actually follow a connected nodal simple curve passing through all the points in $\mathcal{L}$ and going from on of the critical zeroes on the boundary to the second one. To finish the proof it suffices to show that there are no other component of $N(\Phi^\theta)$ in $\mathcal{S}$.

5 Notation and Definitions. General Properties of the Nodal Sets

5.1 Notation and Definitions, I

Let $\mathcal{S}$ be the open square $]0, \pi[^2$ in the plane. We denote by $\partial\mathcal{S}$ boundary, by $\mathcal{D}_+$ the diagonal, by $\mathcal{D}_-$ the anti-diagonal, and by $O := (\frac{\pi}{2}, \frac{\pi}{2})$ the center of $\mathcal{S}$.

Let Φ be an eigenfunction for the Dirichlet Laplacian in $\mathcal{S}$. We let

$$N(\Phi) := \left\{(x, y) \in \overline{\mathcal{S}} \mid \Phi(x, y) = 0\right\} \tag{5.1}$$

denote the nodal set of Φ, and

$$N_i(\Phi) := N(\Phi) \cap \mathcal{S} \tag{5.2}$$

denote the interior part of $N(\Phi)$.

Given two integers $m, n \geq 1$, we consider the one-parameter family of eigenfunctions,

$$\Phi^{\theta}_{m,n}(x, y) := \Phi_{m,n}(x, y, \theta) := \cos\theta \sin(mx) \sin(ny) + \sin\theta \sin(nx) \sin(my) \,, \tag{5.3}$$

with $x, y \in [0, \pi]$ and $\theta \in [0, \pi[$.

Unless necessary, we skip the index (m, n). These eigenfunctions are associated with the eigenvalue

$$\hat{\lambda}_{m,n} := m^2 + n^2 \,. \tag{5.4}$$

The following eigenfunctions are of particular interest.

$$\begin{aligned} &X := \Phi^0 \,, \quad Y := \Phi^{\frac{\pi}{2}} \,, \\ &Z_+ := \Phi^{\frac{\pi}{4}} \,, \; Z_- := \Phi^{\frac{3\pi}{4}} \,. \end{aligned} \tag{5.5}$$

Denote by

$$\mathcal{L} := N_i(X) \cap N_i(Y) \,, \tag{5.6}$$

the set of zeroes which are common to all eigenfunctions Φ^{θ}, $\theta \in [0, \pi[$.

Definition 5.1 A *critical zero* of Φ is a point $(x, y) \in \overline{\mathcal{S}}$ such that both Φ and $\nabla\Phi$ vanish at (x, y).

5.2 General Properties of Nodal Sets

Although stated in the case of the square, the following properties are quite general (see [2] and references therein) for eigenfunctions of the Dirichlet realization of the Laplacian in a regular domain of $\mathbb{R}^2$.

Property 5.2 *Let (x, y) be a point in $\mathcal{S}$ (an interior point).*

(i) A non-zero eigenfunction Φ cannot vanish to infinite order at (x, y).
(ii) If the non-zero eigenfunction Φ vanishes at (x, y), then the leading part of its Taylor expansion at (x, y) is a harmonic homogeneous polynomial.
(iii) If the point (x, y) is a critical zero of the eigenfunction Φ, then the nodal set $N(\Phi)$ at the point (x, y) consists of finitely many regular arcs which form an equi-angular system.

(iv) *The nodal set can only have self-intersections at critical zeroes, and the number of arcs which meet at a self-intersection is determined by the order of vanishing of the eigenfunction. Nodal curves cannot meet tangentially.*

(v) *The nodal set cannot have an end point in the interior of $\mathcal{S}$, and consists of finitely many analytic arcs.*

(vi) *Let the eigenfunction Φ be associated with the eigenvalue λ. If ω is a* nodal domain*, i.e., a connected component of $\mathcal{S} \setminus N(\Phi)$, then the first Dirichlet eigenvalue of ω is equal to λ.*

(vii) *Similar properties hold at boundary points, in particular property (iii).*

Remark Since the eigenfunctions of $\mathcal{S}$ are defined over the whole plane, the analysis of the critical zeroes at interior points easily extends to the boundary.

Property 5.3 *Let Φ be an eigenfunction $\Phi^{\theta}_{m,n}$ of the square $\mathcal{S} =]0,\pi[^2$, with $\theta \in [0,\pi[$.*

(i) *For $\theta \neq \frac{\pi}{2}$, the nodal set $N(\Phi)$ satisfies*

$$\mathcal{L} \cup \partial\mathcal{S} \subset N(\Phi) \subset \mathcal{L} \cup \partial\mathcal{S} \cup \left\{(x,y) \in [0,\pi]^2 \mid \cos\theta\, X(x,y)\, Y(x,y) < 0\right\}. \tag{5.7}$$

(ii) *If* $\gcd(m,n) = 1$*, then all the points in $\mathcal{L}$ are regular points of the nodal set.*

(iii) *The nodal set $N(\Phi)$ can only hit the boundary of the square at critical zeroes (either in the interior of the edges or at the vertices).*

(iv) *The nodal set $N(\Phi)$ can only pass from one connected component of the set*

$$\mathcal{W}^{\theta}_{m,n} := \left\{(x,y) \in [0,\pi]^2 \mid \cos\theta\, X(x,y)\, Y(x,y) < 0\right\}$$

to another through one of the points in $\mathcal{L}$.

(v) *No closed connected component of $N(\Phi)$ can be contained in the closure of one of the connected components of $\mathcal{W}^{\theta}_{m,n}$. Equivalently, any connected component of $N_i(\Phi)$ must contain at least one point in $\mathcal{L}$.*

Proof (i) We have $\sin\theta > 0$, so that for $\cos\theta\, X(x,y)\, Y(x,y) \geq 0$ the function Φ is either positive or negative, it cannot vanish unless $(x,y) \in \mathcal{L}$. (ii) Follows by direct analysis. (iii) Follows from Property 5.2. (iv) Clear. (v) Any connected component of $N(\Phi)$ which does not meet $\mathcal{L}$ would be strictly strictly contained in one of the nodal domains of the eigenfunctions X or Y, a contradiction with Property 5.2(vi). □

Figure 6 illustrates property (i) when $(m,n) = (1,3), (1,4)$ or $(2,3)$. When $\cos\theta > 0$, the nodal set is contained in the white sub-squares; when $\cos\theta < 0$ it is contained in the grey sub-squares. The points in $\mathcal{L}$ are the points labelled $a, b, \ldots$ in the figures.

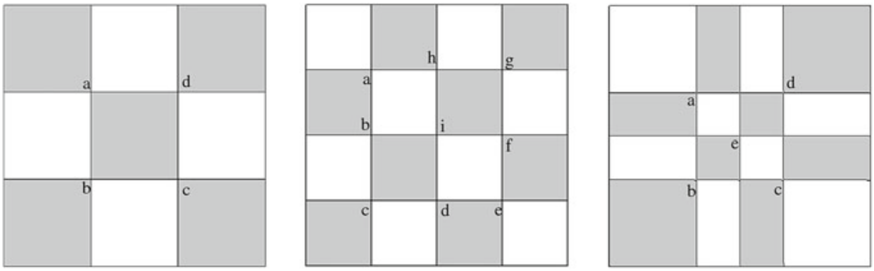


Fig. 6 Checkerboards for eigenvalues $\hat{\lambda}_{1,3}$, $\hat{\lambda}_{1,4}$ and $\hat{\lambda}_{2,3}$

5.3 *Notation and Definitions, II*

We now consider the case of the eigenvalue $\hat{\lambda}_{1,R} = 1 + R^2$, for some integer $R \geq 1$.
We introduce:

- The numbers

$$p_i := i\frac{\pi}{R}\,, \text{ for } 0 \leq i \leq R\,, \tag{5.8}$$

$$m_i := \left(i + \frac{1}{2}\right)\frac{\pi}{R}\,, \text{ for } 0 \leq i \leq R-1\,. \tag{5.9}$$

- The collection of squares

$$Q_{i,j} := \;]p_i, p_{i+1}[\times]p_j, p_{j+1}[\,, \text{ for } 0 \leq i, j \leq R-1, \tag{5.10}$$

whose centers are the points (m_i, m_j).
- The lattice

$$\mathcal{L} := \left\{(p_i, p_j) \mid 1 \leq i, j \leq R-1\right\}. \tag{5.11}$$

Coloring the squares. Assume that $\theta \neq 0$ and $\frac{\pi}{2}$. If $(-1)^{i+j}\cos\theta < 0$, we color the square $Q_{i,j}$ in white, otherwise we color it in grey. The collection of squares $\{Q_{i,j}\}$ becomes a grey/white checkerboard (which depends on R and on the sign of $\cos\theta$). Depending on the sign of $\cos\theta$, the white part of the checkerboard is given by,

$$\begin{aligned} \mathcal{W}(+) &:= \textstyle\bigcup_{(-1)^{i+j}=-1} Q_{i,j}\,, \text{ when } \cos\theta > 0\,,\\ \mathcal{W}(-) &:= \textstyle\bigcup_{(-1)^{i+j}=1} Q_{i,j}\,, \text{ when } \cos\theta < 0\,. \end{aligned} \tag{5.12}$$

For the eigenfunction Φ^θ, we have,

$$\mathcal{L} \cup \partial\mathcal{S} \subset N(\Phi^\theta) \subset \mathcal{W}(\pm) \cup \mathcal{L} \cup \partial\mathcal{S}\,, \tag{5.13}$$

if $(\pm\cos\theta > 0)$.

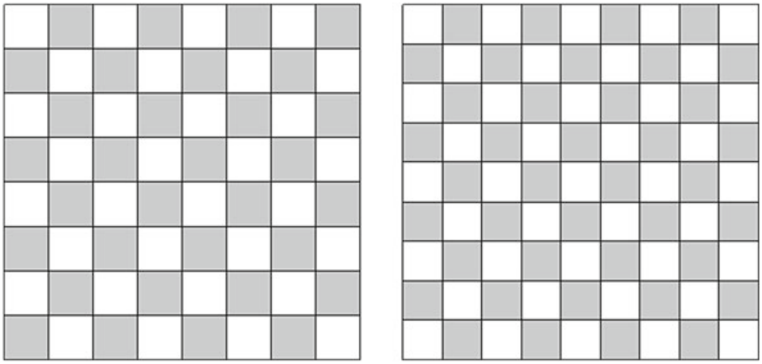


Fig. 7 Checkerboards $\mathcal{W}(+)$ for the eigenvalue $\hat{\lambda}_{1,8}$, and $\mathcal{W}(-)$ for the eigenvalue $\hat{\lambda}_{1,9}$

Remark Observe that the squares $Q_{i,j}$ are open, the sets $\mathcal{W}(\pm)$ do not contain the segments $\{x = p_i\} \cap \mathcal{S}$ and $\{y = p_j\} \cap \mathcal{S}$.

Figure 7 shows the checkerboards for the eigenvalue $\hat{\lambda}_{1,8}$, when $\cos\theta > 0$, *resp.* for $\hat{\lambda}_{1,9}$, when $\cos\theta < 0$.

To describe the global aspect of the nodal sets, we will also use the following squares.

Denote by

$$r := \left[\frac{R}{2}\right], \tag{5.14}$$

the integer part of $R/2$. For $0 \le i \le r$, define the square

$$\mathcal{S}_i := \,]p_i, p_{R-i}[\times]p_i, p_{R-i}[\,. \tag{5.15}$$

With this notation, we have

$$\mathcal{S}_r \subset \mathcal{S}_{r-1} \subset \cdots \subset \mathcal{S}_0 = \mathcal{S}.$$

Furthermore, when $R = 2r$, $\mathcal{S}_{r-1} =]p_{r-1}, p_{r+1}[^2$ consists of four Q-squares, while $\mathcal{S}_r$ is empty; when $R = 2r + 1$, $\mathcal{S}_r$ is a single Q-square. All these squares have the same center $O = (\pi/2, \pi/2)$.

6 Eigenfunctions Associated with the Eigenvalue $\hat{\lambda}_{1,R}$

In this section, we consider the eigenfunctions associated with the eigenvalue $\hat{\lambda}_{1,R}$, for an integer $R \ge 1$. More precisely, we consider the 1-parameter family of eigenfunctions,

$$\Phi^{\theta}(x,y) := \Phi(x,y,\theta) := \cos\theta\, \sin x\, \sin(Ry) + \sin\theta \sin(Rx)\, \sin y\,, \tag{6.1}$$

where $x, y \in [0,\pi]^2$ and $\theta \in [0,\pi[$.

This eigenfunction can be written as

$$\Phi(x,y,\theta) = \sin x\, \sin y\, \phi(x,y,\theta)\,, \tag{6.2}$$

with

$$\phi(x,y,\theta) := \cos\theta\, U_{R-1}(\cos y) + \sin\theta\, U_{R-1}(\cos x)\,, \tag{6.3}$$

where $U_n(t)$ is the nth Chebyshev polynomial of second type defined by the relation,

$$\sin t\; U_n(\cos t) := \sin\big((n+1)t\big)\,. \tag{6.4}$$

6.1 Chebyshev Polynomials and Special Values of θ

In this section, we list some properties of the Chebyshev polynomials to be used later on.

Property 6.1 *For $R \in \mathbb{N}\setminus\{0\}$, the Chebyshev polynomial $U_{R-1}(t)$ has the following properties.*

(i) *The polynomial U_{R-1} has degree $R-1$ and the same parity as $R-1$. Its zeroes are the points $\cos p_j$, $1 \le j \le R-1$, see (5.8). Furthermore, $U_{R-1}(1) = R$, $U_{R-1}(-1) = (-1)^{R-1}R$, and $-R \le U_{R-1}(t) \le R$ for all $t \in [-1,1]$.*

(ii) *The polynomial U'_{R-1} has exactly $R-2$ simple zeroes $\cos q_j$, $1 \le j \le R-2$, with $q_j \in]p_j, p_{j+1}[$.*

(iii) *When R is even, $R = 2r$, the values q_j satisfy,*

$$\begin{aligned} 0 < q_1 < q_2 \cdots < q_{r-1} < \tfrac{\pi}{2} < q_r < \cdots < q_{2r-2} < \pi\,,\\ q_{2r-1-j} = \pi - q_j\,,\ 1 \le j \le r-1\,. \end{aligned} \tag{6.5}$$

(iv) *When R is odd, $R = 2r+1$, the values q_j satisfy,*

$$\begin{aligned} 0 < q_1 < q_2 \cdots < q_{r-1} < q_r = \tfrac{\pi}{2} < q_{r+1} < \cdots < q_{2r-1} < \pi\,,\\ q_{2r-j} = \pi - q_j,\ 1 \le j \le r-1\,. \end{aligned} \tag{6.6}$$

(v) *Let $M_j := U_{R-1}(\cos q_j)$, for $1 \le j \le R-2$, denote the local extrema of U_{R-1}. Then,*

$$\begin{aligned} &(-1)^j M_j > 0 \text{ and } (-1)^j U_{R-1}(\cos t) > 0 \text{ in }]p_j, p_{j+1}[\,,\\ &(-1)^{j+1}\big(U_{R-1}(\cos t) - M_j\big) \ge 0 \text{ in }]p_j, p_{j+1}[\,. \end{aligned} \tag{6.7}$$

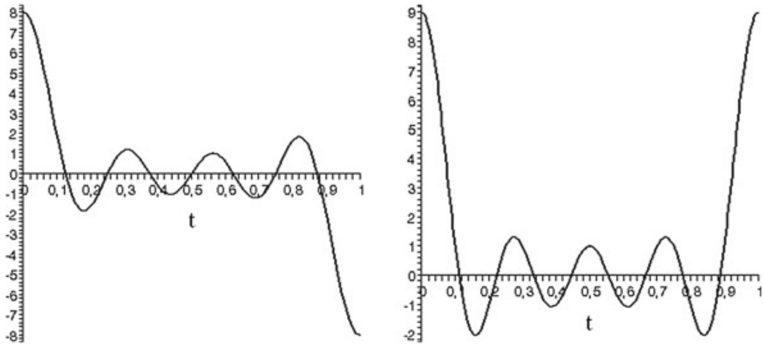


Fig. 8 Functions $U_7(\cos(\pi t))$ and $U_8(\cos(\pi t))$

Proof The above properties are easy to prove, and illustrated by the graph of the function $t \to U_{R-1}(\cos t)$ in the interval $[0, \pi]$, see Fig. 8, for the cases $R = 8$ and $R = 9$. □

Special values of the parameter θ. We shall now associate some special values of the parameter θ with the zeroes

$$\mathcal{Q} := \{q_j \mid 1 \le j \le R - 2\} \tag{6.8}$$

of the function $t \to U'_{R-1}(\cos t)$. As we shall see later on, they are related to changes in the nodal patterns of the eigenfunctions Φ^θ when θ varies from 0 to π.

The values of θ to be introduced below are well defined because the polynomial U_{R-1} does not vanish at the points $\cos q_k$, $1 \le k \le R - 2$. These values of θ will clearly depend on R, although we do not indicate the dependence in the notations.

• For $1 \le i, j \le R - 2$, define $\theta(q_i, q_j)$, alias $\theta_{i,j}$, to be the unique angle in the interval $[0, \pi[$ such that

$$\cos\theta_{i,j}\, U_{R-1}(\cos q_j) + \sin\theta_{i,j}\, U_{R-1}(\cos q_i) = 0\,. \tag{6.9}$$

Let $\mathcal{T}_o$ denote the corresponding set,

$$\mathcal{T}_o := \{\theta_{i,j} \mid 1 \le i, j \le R - 2\}\,. \tag{6.10}$$

• For $* \in \{0, \pi\}$, and $1 \le j \le R - 2$, define $\theta(*, q_j)$, alias $\theta_{*,j}$, to be the unique angle in the interval $[0, \pi[$ such that

$$\cos\theta_{*,j}\, U_{R-1}(\cos q_j) + \sin\theta_{*,j}\, U_{R-1}(\cos *) = 0\,. \tag{6.11}$$

Let $\mathcal{T}_x$ denote the corresponding set,

$$\mathcal{T}_x := \left\{\theta_{*,j} \mid * \in \{0, \pi\},\ 1 \le j \le R-2\right\}. \tag{6.12}$$

• For $1 \le i \le R-2$, and $* \in \{0, \pi\}$, define $\theta(q_i, *)$, alias $\theta_{i,*}$, to be the unique angle in the interval $[0, \pi[$ such that

$$\cos\theta_{i,*}\, U_{R-1}(\cos *) + \sin\theta_{i,*} U_{R-1}(\cos q_i) = 0\,. \tag{6.13}$$

Let $\mathcal{T}_y$ denote the corresponding set,

$$\mathcal{T}_y := \left\{\theta_{i,*} \mid * \in \{0, \pi\}\,,\ 1 \le i \le R-2\right\}. \tag{6.14}$$

Observe the following relations between the above values of θ,

$$\theta(q_j, q_i) = \frac{\pi}{2} - \theta(q_i, q_j)\,. \tag{6.15}$$

When $R = 2r + 1$ is odd, we have

$$\begin{aligned} \theta(q_i, q_j) = \theta(\pi - q_i, \pi - q_j) = \theta(\pi - q_i, q_j) &= \theta(q_i, \pi - q_j)\,, \\ \theta(0, q_j) &= \theta(\pi, q_j)\,, \\ \theta(q_i, 0) &= \theta(q_i, \pi)\,. \end{aligned} \tag{6.16}$$

When $R = 2r$ is even, we have

$$\begin{aligned} \theta(q_i, q_j) &= \theta(\pi - q_i, \pi - q_j)\,, \\ \theta(\pi - q_i, q_j) &= \pi - \theta(q_i, q_j)\,, \\ \theta(q_i, \pi - q_j) &= \pi - \theta(q_i, q_j)\,, \\ \theta(\pi, q_j) &= \pi - \theta(0, q_j)\,, \\ \theta(q_i, \pi) &= \pi - \theta(q_i, 0)\,. \end{aligned} \tag{6.17}$$

Finally, define the number θ_- to be,

$$\theta_- := \arctan\left(\frac{1}{R}|\inf_{[-1,1]} U_{R-1}|\right). \tag{6.18}$$

We have $0 < \theta_- \le \pi/4$, with $\theta_- = \pi/4$ when R is even, and $\theta_- < \pi/4$ when R is odd.

Remark The pictures and numerical computations seem to indicate that the infimum is achieved at $\cos q_1$.

Example Numerical computations give the following approximate data when $R = 8$ or $R = 9$. The indication π after the set means that the values in the set should be multiplied by π.

• Special values of θ when $R = 8$.

$$\begin{aligned}
\mathcal{Q} &= \{0.179749, 0.309108, 0.436495, 0.563505, 0.690892, 0.820251\}\,\pi,\\
\mathcal{T}_o &= \{0.161605, 0.185335, 0.223323, 0.25, 0.276677, 0.314665, 0.338395,\\
&\qquad 0.661605, 0.685335, 0.723323, 0.75, 0.776677, 0.814665, 0.838395\}\,\pi,\\
\mathcal{T}_x &= \{0.040363, 0.047665, 0.071705, 0.928295, 0.952335, 0.959636\}\,\pi,\\
\mathcal{T}_y &= \{0.428295, 0.452335, 0.459636, 0.540363, 0.547665, 0.571705\}\,\pi\,.
\end{aligned} \tag{6.19}$$

• Special values of θ when $R = 9$.

$$\begin{aligned}
\mathcal{Q} &= \{0.159593, 0.274419, 0.387439, 0.500000, 0.612561, 0.725581, 0.840407\}\,\pi\,,\\
\mathcal{T}_o &= \{0.145132, 0.181901, 0.217145, 0.239975, 0.260025\,, 0.282855,\\
&\qquad 0.318099, 0.354868, 0.653215, 0.707395, 0.75, 0.792605, 0.846785\}\,\pi\,,\\
\mathcal{T}_x &= \{0.037494, 0.070922, 0.953949, 0.964777\}\,\pi\,,\\
\mathcal{T}_y &= \{0.429078, 0.462505, 0.535223, 0.546050\}\,\pi\,.
\end{aligned} \tag{6.20}$$

Up to symmetries, one can actually reduce the range of the parameter θ to $[0, \pi/4]$ when R is even, and to $[\pi/4, 3\pi/4]$ when R is odd, see Sect. 6.2. Up to this reduction, the above values correspond to the values which appear in the figures showing the nodal patterns for the eigenvalues $\hat{\lambda}_{1,8}$ and $\hat{\lambda}_{1,9}$, see Figs. 16, 17, and 18 at the end of the paper).

6.2 Symmetries of the Eigenfunctions Associated with $\hat{\lambda}_{1,R}$

When studying the family of eigenfunctions $\{\Phi^\theta\}$ associated with the eigenvalue $\hat{\lambda}_{1,R}$, it is useful to take symmetries into account.

Property 6.2 *The following relations hold for any* $(x, y) \in [0, \pi] \times [0, \pi]$ *and* $\theta \in [0, \pi[$.

(i) For any $R \in \mathbb{N} \setminus \{0\}$,

$$\Phi(\pi - x, \pi - y, \theta) = (-1)^{R+1}\Phi(x, y, \theta)\,. \tag{6.21}$$

This relation implies that the nodal set $N(\Phi^\theta)$ *is symmetrical with respect to the center* O *of the square* $\mathcal{S}$. *Furthermore,*

$$\Phi(x, y, \frac{\pi}{2} - \theta) = \Phi(y, x, \theta)\,. \tag{6.22}$$

(ii) When R *is odd, the function* Φ *has more symmetries, namely,*

$$\Phi(\pi - x, y, \theta) = \Phi(x, \pi - y, \theta) = \Phi(x, y, \theta)\,. \tag{6.23}$$

This means that the nodal set $N(\Phi^\theta)$ is symmetrical with respect to the lines $\{x = \pi/2\}$ and $\{y = \pi/2\}$.

(iii) When R is even, we have

$$\Phi(x, \pi - y, \theta) = \Phi(x, y, \pi - \theta) = -\Phi(\pi - x, y, \theta)\,. \tag{6.24}$$

(iv) Up to symmetries with respect to the first diagonal, or to the lines $\{x = \pi/2\}$ and $\{y = \pi/2\}$, the nodal patterns of the family of eigenfunctions $\{\Phi^\theta\}$, are those displayed by the sub-families $\theta \in [0, \pi/4]$ when R is even, and $\theta \in [\pi/4, 3\pi/4]$ when R is odd.

6.3 Critical Zeroes of the Eigenfunctions Associated With $\hat{\lambda}_{1,R}$

Recall that a *critical zero* of the eigenfunction Φ^θ is a point $(x, y) \in \overline{\mathcal{S}}$ such that

$$\Phi(x, y, \theta) = \Phi_x(x, y, \theta) = \Phi_y(x, y, \theta) = 0\,. \tag{6.25}$$

At a critical zero, the nodal set $N(\Phi^\theta)$ consists of several arcs (or semi-arcs when the point is on $\partial\mathcal{S}$) which form an equi-angular system, see Property 5.2. Away from the critical zeroes, the nodal set consists of smooth embedded arcs. To determine the possible critical zeroes of Φ^θ is the key to describing the global aspect of the nodal set $N(\Phi^\theta)$.

We classify the critical zeroes into three (possibly empty) categories: (i) the *vertices* of the square $\mathcal{S}$; (ii) the *edge critical zeroes* located in the interior of the edges, typically a point of the form $(0, y)$, with $y \in]0, \pi[$; (iii) the *interior critical zeroes* of the form $(x, y) \in \mathcal{S}$.

6.3.1 Behaviour at the Vertices

Using the symmetry of $N(\Phi)$ with respect to the point O, see (6.21), it suffices to consider the vertices $(0, 0)$ and $(0, \pi)$. Recalling (6.1) and (6.2), the Taylor expansion at $(0, 0)$ of $\phi(x, y, \theta)$ is given by,

$$\begin{aligned}\phi(x, y, \theta) = R\,(\cos\theta + \sin\theta) \\ + \frac{R(1-R^2)}{6}\,\left(\cos\theta\, y^2 + \sin\theta\, x^2\right) \\ + O(x^4 + y^4)\,.\end{aligned} \tag{6.26}$$

When R is odd, the behaviour is the same at the four vertices and given by (6.26), due to the symmetries (6.23).

When R is even, the Taylor expansion of $\phi(x, y, \theta)$ at $(0, \pi)$, follows from the previous one and relation (6.24). In the variables x and z such that $y = \pi - z$, we have,

$$\begin{aligned}\phi(x, \pi - z, \theta) = & R\,(-\cos\theta + \sin\theta) \\ & -\tfrac{R(1-R^2)}{6}\,\left(-\cos\theta\, z^2 + \sin\theta\, x^2\right) \\ & + O(x^4 + z^4)\,.\end{aligned} \tag{6.27}$$

With the link between Φ and ϕ in mind, we obtain:

Property 6.3 *The vertices of the square $\mathcal{S}$ are critical zeroes for the eigenfunction Φ^θ for all θ.*

(i) ***Case R even****. The vertices $(0, \pi)$ and $(\pi, 0)$ are non degenerate critical zeroes of Φ^θ if and only if $\theta \neq \pi/4$. When $\theta = \pi/4$, they are degenerate critical zeroes of order 4. The vertices $(0, 0)$ and (π, π) are non-degenerate critical zeroes of Φ if and only if $\theta \neq 3\pi/4$. When $\theta = 3\pi/4$, they are degenerate critical zeroes of order 4. The nodal patterns at the vertices are shown in Fig. 9.*

(ii) ***Case R odd****. The four vertices are non-degenerate critical zeroes of Φ^θ if and only if $\theta \neq 3\pi/4$. When $\theta = 3\pi/4$, they are degenerate critical zeroes of order 4. The nodal patterns at the vertices are shown in Fig. 10.*

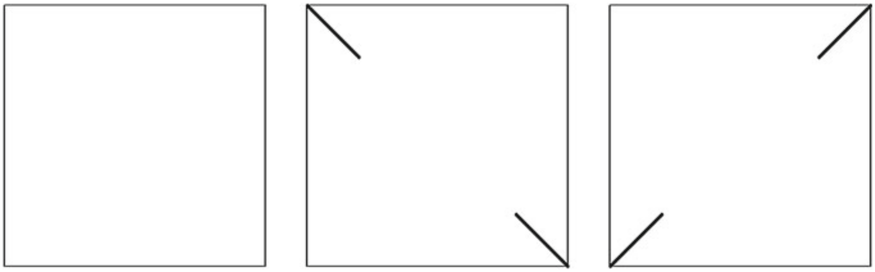
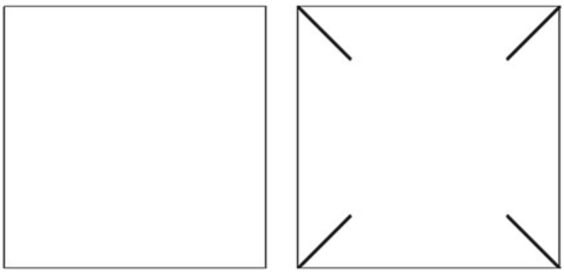


Fig. 9 R even, nodal patterns at the vertices. From *left* to *right*: $\theta \neq \pi/4$ and $3\pi/4$; $\theta = \pi/4$; $\theta = 3\pi/4$

Fig. 10 R odd, nodal patterns at the vertices. From *left* to *right*: $\theta \neq 3\pi/4$; $\theta = 3\pi/4$



6.3.2 Critical Zeroes, Formulas

To determine the critical zeroes of the eigenfunction Φ, we recall our notation at the beginning of the section. The first partial derivatives with respect to x and y are given by,

$$\begin{aligned}\Phi_x(x, y, \theta) &= \cos x\ \sin y\ \phi(x, y, \theta)\\ &\quad - \sin\theta \sin^2 x\ \sin y\ U'_{R-1}(\cos x)\,,\\ \Phi_y(x, y, \theta) &= \sin x\ \cos y\ \phi(x, y, \theta)\\ &\quad - \cos\theta\ \sin x\ \sin^2 y\ U'_{R-1}(\cos y)\,.\end{aligned} \tag{6.28}$$

The second partial derivatives are given by,

$$\begin{aligned}\Phi_{xx}(x, y, \theta) &= -\sin x\ \sin y\ \phi(x, y, \theta)\\ &\quad -3\sin\theta\ \cos x\ \sin x\ \sin y\ U'_{R-1}(\cos x)\\ &\quad + \sin\theta \sin^3 x\ \sin y\ U''_{R-1}(\cos x)\,,\\ \Phi_{xy}(x, y, \theta) &= \cos x\ \cos y\ \phi(x, y, \theta)\\ &\quad - \cos\theta\ \cos x\ \sin^2 y\ U'_{R-1}(\cos y)\\ &\quad - \sin\theta\ \sin^2 x\ \cos y\ U'_{R-1}(\cos x)\,,\\ \Phi_{yy}(x, y, \theta) &= -\sin x\ \sin y\ \phi(x, y, \theta)\\ &\quad -3\cos\theta\ \sin x\ \cos y\ \sin y\ U'_{R-1}(\cos y)\\ &\quad + \cos\theta\ \sin x\ \sin^3 y\ U''_{R-1}(\cos y)\,.\end{aligned} \tag{6.29}$$

6.3.3 Behaviour Along the Edges

Recall the notation of Sect. 6.1. Due (6.21), the symmetry with respect to the center O of the square $\mathcal{S}$, it suffices to consider the open edges $\{0\}\times]0, \pi[$ and $]0, \pi[\times\{0\}$.

• **Critical zeroes on the edge** $\{0\}\times]0, \pi[$. Using formulas (6.28) and (6.29), as well as Property 6.1, we infer that the point $(0, y)$ is a critical zero for Φ^θ if and only if,

$$\cos\theta\, U_{R-1}(\cos y) + R\sin\theta = 0\,, \tag{6.30}$$

with second derivatives at $(0, y)$, $\Phi_{xx} = \Phi_{yy} = 0$, and $\Phi_{xy} = -\cos\theta\ \sin^2 y\ U'_{R-1}(\cos y)$.

The point $(0, y)$ is a non degenerate critical zero, unless $y = q_j \in \mathcal{Q}$ for some $j, 1 \le j \le R-2$. This can only occur when $\theta = \theta(0, q_j)$. In this case, the third derivative Φ_{xy^2} at the degenerate critical zero $(0, q_j)$ is equal to

$$\cos\big(\theta(0, q_j)\big)\ \sin^3 q_j\ U''_{R-1}(\cos q_j) \neq 0\,,$$

and the critical zero has order 3.

• **Critical zeroes on the edge** $]0, \pi[\times\{0\}$. Similarly, the point $(x, 0)$ is a critical zero for Φ^θ if and only if,

$$R\ \cos\theta + \sin\theta\, U_{R-1}(\cos x) = 0\,, \tag{6.31}$$

with second derivatives at $(x, 0)$, $\Phi_{xx} = \Phi_{yy} = 0$, and

$$\Phi_{xy} = -\sin\theta\ \sin^2 x\ U'_{R-1}(\cos x)\,.$$

The point $(x, 0)$ is a non degenerate critical zero unless $x = q_i \in \mathcal{Q}$ for some i, $1 \le i \le R-2$. This can only occur when $\theta = \theta(q_i, 0)$. In this case, the third derivative Φ_{x^2y} at the degenerate critical zero $(q_i, 0)$ is equal to

$$\sin(\theta(q_i, 0))\ \sin^3 q_i\ U''_{R-1}(\cos q_i) \neq 0\,,$$

and the critical zero has order 3.
Remark At an edge critical zero which is non degenerate, an arc from the nodal set hits the edge orthogonally. At a degenerate edge critical zero, two arcs from the nodal set hit the edge with equal angle $\pi/3$. See Figs. 12 and 13.

The following properties summarize the analysis of the above equations.

Property 6.4 *The critical zeroes on the open edges, if any, appear in pairs of points which are symmetrical with respect to the center O of the square $\mathcal{S}$.*

Case R even.

- *(i)* *For $\theta \in [0, \pi/4[\cup]3\pi/4, \pi[$, there are critical zeroes on the vertical edges $\{0, \pi\}\times]0, \pi[$, and no critical zero on the horizontal edges $]0, \pi[\times\{0, \pi\}$.*
- *(ii)* *For $\theta \in]\pi/4, 3\pi/4[$, there are critical zeroes on the horizontal edges $]0, \pi[\times\{0, \pi\}$, and no critical zero on the vertical edges $\{0, \pi\}\times]0, \pi[$.*
- *(iii)* *The number of critical zeroes depends on θ, more precisely on the number of solutions of (6.30) or (6.31).*
- *(iv)* *When $\theta = \pi/4$ or $3\pi/4$, the only boundary critical zeroes are vertices, see Property 6.3.*

Case R odd.

- *(i)* *Recall the value $0 < \theta_- < \pi/4$ defined in Sect. 6.1. For $\theta \in [0, \theta_-]\cup]3\pi/4, \pi[$, there are critical zeroes on the vertical edges $\{0, \pi\}\times]0, \pi[$, and no critical zero on the horizontal edges $]0, \pi[\times\{0, \pi\}$.*
- *(ii)* *For $\theta \in [\pi/2 - \theta_-, 3\pi/4[$, there are critical zeroes on the horizontal edges $]0, \pi[\times\{0, \pi\}$, and no critical zero on the vertical edges $\{0, \pi\}\times]0, \pi[$.*
- *(iii)* *For $\theta \in]\theta_-, \pi/2 - \theta_-[$, there is no critical zero on the open edges.*
- *(iv)* *The number of critical zeroes depends on θ, more precisely on the number of solutions of (6.30) or (6.31).*
- *(v)* *The critical zeroes have order at most 3. Degenerate critical zeroes can only occur for finitely many values of θ and x or y.*
- *(vi)* *When $\theta = 3\pi/4$, the only boundary critical zeroes are the vertices, see Property 6.3.*

In both cases, the edge critical zeroes are non degenerate unless they occur on a horizontal edge for some $x = q_i \in \mathcal{Q}$, resp. *on a vertical edge for some $y = q_j \in \mathcal{Q}$, in which case θ must be equal to $\theta(q_i, 0)$,* resp. *to $\theta(0, q_j)$. Degenerate critical zeroes have order* 3. *If $\theta \neq 0$ or $\pi/2$, the points $(*, p_j)$ and $(p_j, *)$ with $1 \leq j \leq R-1$ and $* = 0$ or π are not critical zeroes of the eigenfunction Φ^θ.*

Remark A more detailed description of the localization of the edge critical zeroes is given in Sect. 6.4.

6.3.4 Interior Critical Zeroes

Recall the notations of Sect. 6.1. The following properties follow from (6.28) and (6.29).

Property 6.5 *Let $(x, y) \in \mathcal{S}$ be an interior point.*

(i) *The functions Φ and Φ_x vanish at (x, y) if and only if*

$$\begin{aligned} &\cos\theta\, U_{R-1}(\cos y) + \sin\theta\, U_{R-1}(\cos x) = 0\,, \text{ and} \\ &U'_{R-1}(\cos x) = 0\,. \end{aligned} \tag{6.32}$$

This happens in particular at regular points of the nodal set with a horizontal tangent.

(ii) *The functions Φ and Φ_y vanish at (x, y) if and only if*

$$\begin{aligned} &\cos\theta\, U_{R-1}(\cos y) + \sin\theta\, U_{R-1}(\cos x) = 0\,, \text{ and} \\ &U'_{R-1}(\cos y) = 0\,. \end{aligned} \tag{6.33}$$

This happens in particular at regular points of the nodal set with a vertical tangent.

(iii) *The point (x, y) is an interior critical zero of Φ, if and only if*

$$\begin{aligned} &\cos\theta\, U_{R-1}(\cos y) + \sin\theta\, U_{R-1}(\cos x) = 0\,, \text{ and} \\ &U'_{R-1}(\cos x) = 0 \text{ and } U'_{R-1}(\cos y) = 0\,. \end{aligned} \tag{6.34}$$

The only possible interior critical zeroes for the family of eigenfunctions $\{\Phi^\theta\}$ are the points (q_i, q_j), $1 \leq i, j \leq R-2$. The point (q_i, q_j) can only occur as a critical zero of the eigenfunction $\Phi^{\theta(q_i, q_j)}$. When θ is not one of the values $\theta(q_i, q_j)$, $1 \leq i, j \leq R-2$, the eigenfunction Φ^θ does not have any interior critical zero.

(iv) *When (x, y) is an interior critical zero of Φ, the Hessian of Φ at (x, y) is given by,*

$$\sin x\ \sin y \begin{pmatrix} \sin\theta\ \sin^2 x\ U''_{R-1}(\cos x) & 0 \\ 0 & \cos\theta\ \sin^2 y\ U''_{R-1}(\cos y) \end{pmatrix},$$

so that the interior critical zeroes, if any, are always non degenerate.

(v) The lattice points $\mathcal{L} = \{(\frac{i\pi}{R}, \frac{j\pi}{R}),\ 1 \le i, j \le R-1\}$ *(see Sect. 5.3) are common zeroes to all the eigenfunctions* Φ^θ *when* $\theta \in [0, \pi[$*. They are not interior critical zeroes.*

6.4 Q-Nodal Patterns of Eigenfunctions Associated with $\hat{\lambda}_{1,R}$

The purpose of this section is to list all the possible patterns of the nodal set $N(\Phi^\theta)$ inside the Q-squares $Q_{i,j}$, $0 \le i, j \le R-1$, see Sect. 5.3.

The following properties are derived from the previous sections and from Property 5.3.

(i) The nodal set $N(\Phi^\theta)$ is contained in $\mathcal{W}(\pm) \cup \mathcal{L} \cup \partial\mathcal{S}$.
(ii) If a white square $Q_{i,j}$ does not touch the boundary $\partial\mathcal{S}$, the nodal set $N(\Phi^\theta)$ cannot cross nor hit the boundary of $Q_{i,j}$, except at the vertices which belong to $\mathcal{L}$.
(iii) If a white square $Q_{i,j}$ touches the boundary $\partial\mathcal{S}$, the nodal set $N(\Phi^\theta)$ cannot intersect the boundary of $Q_{i,j}$, except at the vertices which belong to $\mathcal{L}$, or at an edge contained in $\partial\mathcal{S}$.
(iv) Since the points in $\mathcal{L}$ are not critical zeroes, the nodal set consists of a single regular arc at such a point.
(v) Inside a white square $Q_{i,j}$, the nodal set $N(\Phi^\theta)$ can have at most one self intersection at (q_i, q_j) if this point is a critical zero for Φ^θ. In this case, the critical zero is non degenerate, and the nodal set at (q_i, q_j) consists locally of two regular arcs meeting orthogonally.
(vi) Inside a square $Q_{i,j}$, the nodal set cannot stop at a point, and consists of at most finitely many arcs.
(vii) The nodal set $N(\Phi^\theta)$ cannot contain a closed curve contained in the closure of $Q_{i,j}$ (energy reasons).

• **Inner** Q-square. Figure 11 shows all the possible nodal patterns inside a square $Q_{i,j}$ which does not touch the boundary. The patterns A and B occur when the eigenfunction Φ^θ does not have any interior critical zero inside the Q-square. Pattern C occurs when Φ^θ admits (q_i, q_j) as interior critical zero (necessarily unique and non degenerate), in which case θ must be equal to $\theta(q_i, q_j)$. The properties recalled above show that there are no other possible nodal patterns.
• **Boundary Q-square, R even**. As stated in Property 6.2(iv), it suffices to consider the case $\theta \in [0, \pi/4]$. The only boundary critical zeroes of Φ^θ are the vertices $(0, \pi)$ and $(\pi, 0)$, this case occurs if and only if $\theta = \pi/4$, and points on the vertical edges $\{0, \pi\}\times]0, \pi[$ if and only if $0 \le \theta < \pi/4$. These points come in pairs of symmetric points with respect to the center O of the square $\mathcal{S}$. It suffices to describe the points located on the edge $\{0\}\times]0, \pi[$, i.e., the points $(0, y)$ satisfying Eq. (6.30),

$$U_{R-1}(\cos y) + R\tan\theta = 0\,,\ \text{ for some } \theta\,,\quad 0 \le \theta < \pi/4\,.$$

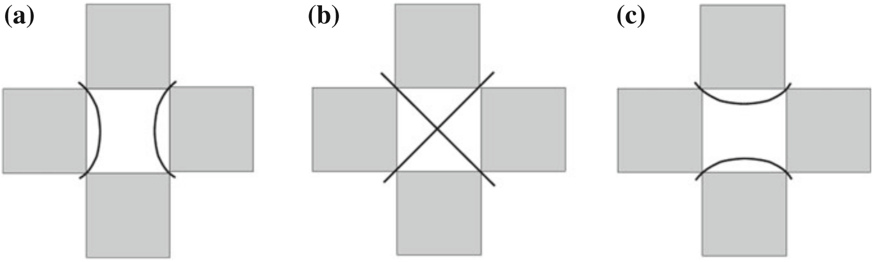


Fig. 11 Nodal pattern in an inner Q-square

When $\theta = 0$, this equation provides exactly $(R-1)$ non degenerate critical zeroes, the points p_j, $1 \le j \le R-1$. When $0 < \theta < \pi/4$ the equation has at least one solution located in the interval $]p_{R-1}, \pi[$. This is the sole solution when θ is close enough to $\pi/4$, and it corresponds to a non degenerate critical zero. The other solutions, if any, are located in the intervals $]p_j, p_{j+1}[$, with j odd. Each such interval contains at most two solutions, which correspond to non degenerate critical zeroes. When an interval contains only one point, this is a double solution, and it corresponds to a degenerate critical zero q_j. This can only occur for the special value $\theta(0, q_j)$.

Figure 12 gives all the possible nodal patterns of the eigenfunction Φ^θ in a square $Q_{0,j}$ which touches the edge $\{0\}\times]0, \pi[$. The base point of the square $(0, p_j)$ is the black dot in the figures. The other dot is the vertex $(0, \pi)$.

Figure (Ae) shows the nodal pattern in the square $Q_{0,R-1}$ which touches the vertex $(0, \pi)$, and contains the persistent edge non degenerate critical zero. Figure (Be) shows the nodal pattern when there is a degenerate edge critical zero $(0, q_j)$; it is always of order three, with two arcs hitting the boundary with equal angles $\pi/3$. Figure (Ce) shows the nodal pattern when there are two non degenerate critical zeroes in the interval $\{0\}\times]p_j, p_{j+1}[$. There are two arcs hitting the boundary orthogonally. Figure (De) shows the nodal pattern when the interval $\{0\}\times]p_j, p_{j+1}[$ contains no critical zero. The properties recalled above show that there are no other possible nodal patterns.

• **Boundary Q-square, R odd**. The description of the critical zeroes of Φ^θ on the boundary $\partial\mathcal{S}$ in the case R odd is similar to the case R even, with some changes. As stated in Property 6.2(iv), up to symmetries, we can restrict ourselves to $\theta \in [\pi/4, 3\pi/4]$. Recall the value $0 < \theta_- < \pi/4$ defined in Sect. 6.1. The vertices are critical zeroes, and they are non degenerate unless $\theta = 3\pi/4$. For $\theta \in [\pi/4, \pi/2 - \theta_-[$, there is no critical zero on the edges. For $\theta \in]\pi/2-\theta_-, 3\pi/4[$, there are critical zeroes on the horizontal edges, and none on the vertical edges. Since the nodal sets are symmetrical with respect to $\{y = \pi/2\}$, it suffices to describe the critical zeroes on the horizontal edge $]0, \pi[\times\{0\}$. For $\theta = \pi/2 - \theta_-$, there are at least two order 3 critical zeroes (except when $R = 3$ in which case there is only one). As a matter of fact, it seems that there are exactly two critical zeroes for $R \ge 5$ because the local extrema of U_{R-1} decrease in absolute value on $[-1, 0]$. For $\theta \in]\pi/2-\theta_-, \pi/2]$, the

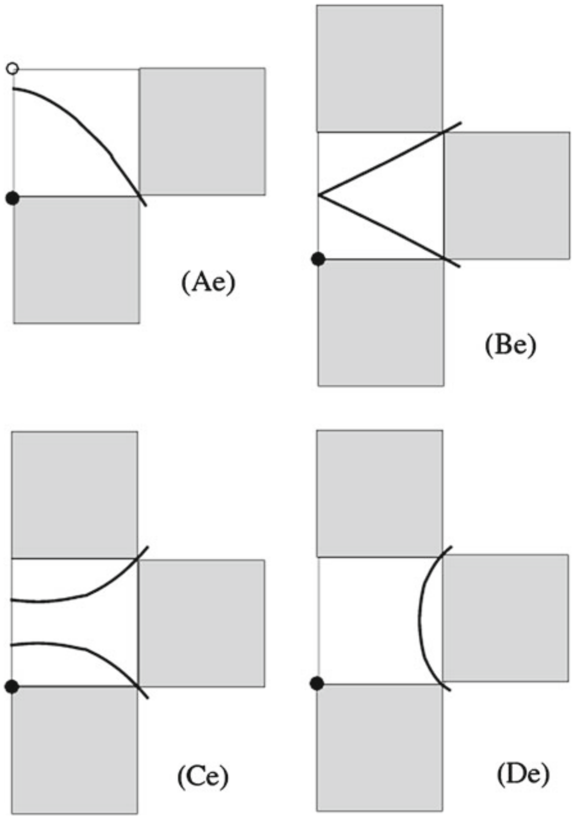


Fig. 12 Local nodal patterns at the boundary, R even

number of critical zeroes depends on the number of solutions of Eq. (6.31),

$$R \cot\theta + U_{R-1}(\cos x) = 0\,,$$

with 0 or 2 solutions in each interval $]p_i, p_{i+1}[\times\{0\}$, or a degenerate critical zero at some $(q_i, 0)$, when $\theta = \theta(q_i, 0)$. For $\theta \in]\pi/2, 3\pi/4[$, there are two non degenerate critical zeroes, one in each interval $]0, p_1[\times\{0\}$ and $]p_{R-1}, \pi[\times\{0\}$, near the vertices. For $\theta = 3\pi/4$, there is no critical zero on the open edge, and only critical zeroes of order 4 at the vertices. Using the properties listed at the beginning of Sect. 6.4, one can show that Fig. 13 contains all the possible nodal patterns in a Q-square touching the edge $]0, \pi[\times\{0\}$.

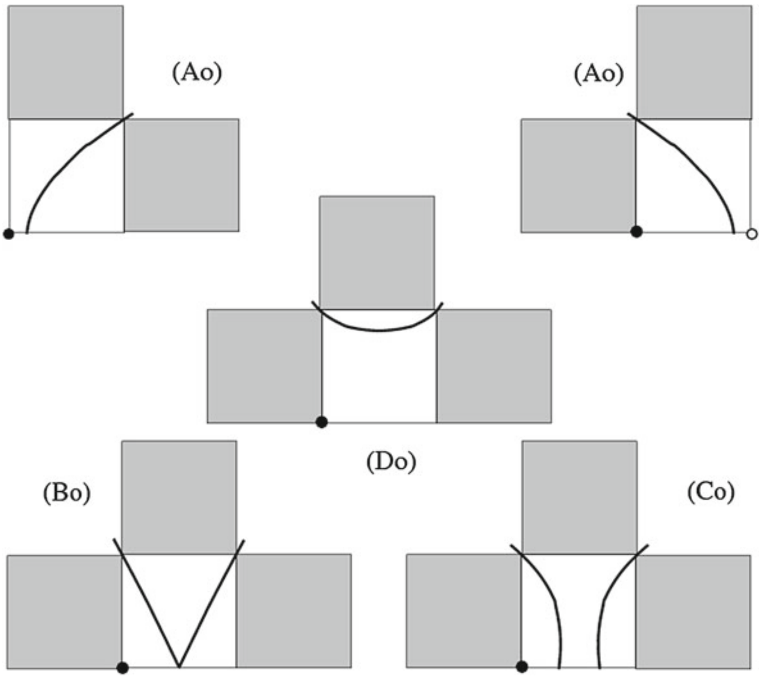


Fig. 13 Local nodal patterns at the boundary, R odd

6.5 Nodal Sets of the Functions $Z_\pm$ Associated with $\hat{\lambda}_{1,R}$

Recall the notation in Sect. 5.3, (5.15), $Z_+ = \Phi^{\frac{\pi}{4}}$ and $Z_- = \Phi^{\frac{3\pi}{4}}$. The purpose of this section is to prove the following proposition.

Proposition 6.6 *Nodal sets of the eigenfunctions $Z_\pm$ associated with the eigenvalue $\hat{\lambda}_{1,R}$.*

(i) ***Case** R **even,*** $R = 2r$. *The symmetries with respect to the lines $\{x = \pi/2\}$ and $\{y = \pi/2\}$ send the nodal set $N(Z_+)$ to the nodal set $N(Z_-)$. Both nodal sets $N(Z_+)$ and $N(Z_-)$ are invariant under the symmetry with respect to the center O of the square. The nodal set $N(Z_+)$ consists of the boundary $\partial\mathcal{S}$, the anti-diagonal, and a collection of $(r-1)$ simple closed curves γ_i. The curve γ_i winds around $\partial\mathcal{S}_i$, passing through the points in $\mathcal{L} \cap \partial\mathcal{S}_i$, and crosses the anti-diagonal $\mathcal{D}_-$ orthogonally. The curves γ_i do not intersect each other.*

(ii) ***Case** R **odd,*** $R = 2r + 1$. *The nodal sets $N(Z_\pm)$ are invariant under the symmetries with respect to the lines $\{x = \pi/2\}$ and $\{y = \pi/2\}$. The nodal set $N(Z_+)$ consists of the boundary $\partial\mathcal{S}$, and a collection of r simple closed curves α_i. The curve α_i winds around $\partial\mathcal{S}_i$, passing through the points in $\mathcal{L}\cap\partial\mathcal{S}_i$. The curves α_i do not intersect each other. The nodal set $N(Z_-)$ consists of the boundary $\partial\mathcal{S}$,*

the two diagonals $\mathcal{D}_\pm$, and a collection of $(r-1)$ simple closed curves β_i. The curve β_i winds around $\partial\mathcal{S}_i$, passing through the points in $\mathcal{L}\cap\partial\mathcal{S}_i$, and crosses the diagonals $\mathcal{D}_\pm$ orthogonally. The curves β_i do not intersect each other.

The proof of this proposition relies on three lemmas which we prove below. Lemma 6.7 determines precisely the interior critical zeroes of the nodal sets $N(Z_\pm)$. Lemmas 6.8 and 6.9 are "separation" lemmas.

Lemma 6.7 *When R is even, the diagonal,* resp. *the anti-diagonal, is contained in the nodal set $N(Z_-)$,* resp. *in the nodal set $N(Z_+)$. When R is odd, the diagonal and anti-diagonal meet the nodal set $N(Z_+)$ at finitely many points. They are both contained in the nodal set $N(Z_-)$. The critical zeroes of the functions $Z_\pm$ are as follows.*

(i) ***Case** R **even**, $R=2r$. The interior critical zeroes of the function Z_+ are exactly the $(R-2)$ points, $(q_i,\pi-q_i)$, for $1\le i\le R-2$, located on the anti-diagonal. The interior critical zeroes of the function Z_- are precisely the $(R-2)$ points, (q_i,q_i), for $1\le i\le R-2$, located on the diagonal.*

(ii) ***Case** R **odd**, $R=2r+1$. The function Z_+ has no interior critical zero. The interior critical zeroes of the function Z_- are precisely the $(2R-5)$ points, (q_i,q_i), $(q_i,\pi-q_i)$, for $1\le i\le R-2$, located on the diagonal and anti-diagonal.*

Remark Note that Lemma 6.7, Properties 6.4 and 6.3 provide a complete description of the critical zeroes of the functions $Z_\pm$.

Proof The first assertions are clear. We concentrate on the determination of the interior critical zeroes. Let $\epsilon=\pm1$. The point $(x,y)\in\mathcal{S}$ is a critical zero of the function $Z_\pm$ if and only if (x,y) is a common solution to the following three equations.

$$\sin x\ \sin(Ry)+\epsilon\ \sin(Rx)\ \sin y=0\,. \tag{6.35}$$

$$\cos x\ \sin(Ry)+\epsilon R\ \cos(Rx)\ \sin y=0\,. \tag{6.36}$$

$$R\ \sin x\ \cos(Ry)+\epsilon\ \sin(Rx)\ \cos y=0\,. \tag{6.37}$$

Since (x,y) is an interior critical zero, (6.35) and (6.36) imply Eq. (6.38) below; (6.35) and (6.37) imply Eq. (6.39) below.

$$\cos x\ \sin(Rx)-R\ \sin x\ \cos(Rx)=0\,. \tag{6.38}$$

$$\cos y\ \sin(Ry)-R\ \sin y\ \cos(Ry)=0\,. \tag{6.39}$$

Substituting $\sin(Rx)$ and $\sin(Ry)$ in (6.35) using (6.38) and (6.39), we obtain the equation

$$\cos x\ \cos(Ry)+\epsilon\ \cos(Rx)\ \cos y=0\,. \tag{6.40}$$

Adding and substracting (6.35) to/from (6.40), we obtain that an interior critical zero (x, y) of $Z_{\pm}$ satisfies the system,

$$\begin{aligned} \cos(x - Ry) + \epsilon \cos(Rx - y) = 0\,, \\ \cos(x + Ry) + \epsilon \cos(Rx + y) = 0\,. \end{aligned} \tag{6.41}$$

• Case $\epsilon = 1$. Modulo 2π, the system (6.41) is equivalent to

$$\begin{gathered} (a1)\ x - Ry = \pi - Rx + y\ [2\pi] \text{ or } (b1)\ x - Ry = \pi + Rx - y\ [2\pi]\,, \\ \text{and} \\ (a2)\ x + Ry = \pi - Rx - y\ [2\pi] \text{ or } (b2)\ x + Ry = \pi + Rx + y\ [2\pi]\,. \end{gathered} \tag{6.42}$$

We have to consider four cases.

✓ (a1) and (a2) These conditions imply that $Rx = -x + k\pi$ for some integer k. Using (6.38), we find that

$$(-1)^{k+1}(1 + R)\ \sin x\ \cos x = 0\,.$$

This implies that $x = \pi/2$. Similarly, using (6.39), we find that $y = \pi/2$. Since (x, y) is a critical zero, using Sect. 6.1, this can only occur when R is odd. On the other hand, using (6.35), we find that $\sin(R\pi/2) = 0$ which implies that R is even. The conditions (a1) and (a2) cannot occur simultaneously.
✓ (a1) and (b2) These conditions imply that $x = y$. Using (6.35), we find that $\sin(Rx) = 0$ and hence, by (6.38), $\cos(Rx) = 0$. The conditions (a1) and (b2) cannot occur simultaneously.
✓ (b1) and (a2) These conditions imply that $y = \pi - x$. Using (6.35), we see that

$$\big((-1)^{R+1} + 1\big)\ \sin x\ \sin(Rx) = 0\,.$$

If R were odd, we would have a contradiction with (6.38). This case can only occur when R is even.
✓ (b1) and (b2) These conditions imply that $Rx = x + k\pi$ for some integer k. By (6.38), this implies that $x = \pi/2$. Similarly, we find that $y = \pi/2$. As above, this implies that R is odd. On the other-hand, (6.35), implies that $\sin(R\pi/2)$ which implies that R is even. The conditions (b1) and (b2) cannot occur simultaneously.

We conclude that the function Z_+ has no interior critical zero when R is odd, and that its only critical zeroes are the points $(q_i, \pi - q_i)$, for $1 \le i \le R - 2$ when R is even.

• Case $\epsilon = -1$. The system (6.41) is equivalent to

$$\begin{gathered} (a1)\ x - Ry = Rx - y\ [2\pi] \text{ or } (b1)\ x - Ry = -Rx + y\ [2\pi]\,, \\ \text{and} \\ (a2)\ x + Ry = Rx + y\ [2\pi] \text{ or } (b2)\ x + Ry = -Rx - y\ [2\pi]\,. \end{gathered} \tag{6.43}$$

We have to consider four cases.

✓ (a1) and (a2) These conditions imply that $Rx = x + k\pi$ for some integer k. Using (6.38), we find that

$$(-1)^{k+1}(1-R)\, \sin x\, \cos x = 0\,.$$

This implies that $x = \pi/2$. Similarly, using (6.39), we find that $y = \pi/2$. Since (x, y) is a critical zero, using Sect. 6.1, this case can only occur when R is odd.
✓ (a1) and (b2) These conditions imply that $\pi - x = y$. Using (6.35), we find that

$$\big((-1)^{R+1} - 1\big)\, \sin(Rx) = 0\,.$$

Since (x, y) is a critical zero, $\sin(Rx) \neq 0$ and this case can only occur when R is odd.
✓ (b1) and (a2) These conditions imply that $y = x$. This case occurs for both R even and R odd.
✓ (b1) & (b2) These conditions imply that $Rx = -x + k\pi$ for some integer k. By (6.38), this implies that $x = \pi/2$. Similarly, we find that $y = \pi/2$. This case can only occur when R is odd.

We conclude that the only critical zeroes of the function Z_- are the points (q_i, q_i), for $1 \leq i \leq R-2$ when R is even, and are the points, (q_i, q_i), $(q_i, \pi - q_i)$, for $1 \leq i \leq R-2$ when R is odd. □

Recall that the function $Z_+(x, y)$ satisfies the relations

$$Z_+(y, x) = Z_+(x, y) \text{ and } Z_+(\pi - x, \pi - y) = (-1)^{R+1}\, Z_+(x, y)$$

which imply that the nodal set $N(Z_+)$ is invariant under the symmetry with respect to the diagonal $\mathcal{D}_+$, and under the symmetry with respect to the centre O of the square $\mathcal{S}$. Consider the subsets

$$\begin{aligned}
\mathcal{F}_1 &:= \{\mathcal{S} \cap \{x > y\} \cap \{x + y < \pi\}\,,\\
\mathcal{F}_2 &:= \{\mathcal{S} \cap \{x > y\} \cap \{x + y > \pi\}\,,\\
\mathcal{F}_3 &:= \{\mathcal{S} \cap \{x < y\} \cap \{x + y > \pi\}\,,\\
\mathcal{F}_4 &:= \{\mathcal{S} \cap \{x < y\} \cap \{x + y < \pi\}\,.
\end{aligned} \tag{6.44}$$

Due to the symmetries mentioned above, it suffices to understand the nodal set into one of theses domains. Since Z_+ corresponds to the value $\theta = \pi/4$, the diagonal $\mathcal{D}_+$ is covered by grey Q-squares. When R is even, the anti-diagonal $\mathcal{D}_-$ is covered by white squares which either contain a unique critical zero of Z_+, or have as vertex one of the vertices $(0, \pi)$ or $(\pi, 0)$. In either situations, the structure of $N(Z_+)$ inside these diagonal white Q-squares is known, see Sect. 6.4. When R is odd, both diagonals $\mathcal{D}_+$ and $\mathcal{D}_-$ are covered by grey squares, and the white Q-squares meeting a given $\mathcal{F}_i$ are actually contained in $\mathcal{F}_i$. In summary, it suffices to understand the nodal pattern of Z_+ inside the white squares contained into the $\mathcal{F}_i$, and it suffices to look at the case $i = 1$, and use the symmetries.

Claim *In each white square $Q_{i,j} \subset \mathcal{W}(+) \cap \mathcal{F}_1$, the horizontal segment $]p_i, p_{i+1}[\times\{m_j\}$ does not meet the nodal set $N(Z_+)$. More precisely, for $R = 2r$ and $j \le r - 1$,*

$$(-1)^j Z_+(x, m_j) > 0 \text{ on the interval }]p_{j+1}, \pi - p_{j+1}[\,.$$

Proof Since $Q_{i,j} \in \mathcal{W}(+)$, we must have $i + j$ odd i.e., $(-1)^{i+j} = -1$. Since $Q_{i,j} \subset \mathcal{F}_1$, we must have the inequalities $j \le i - 1$ and $i + j \le R - 2$. Up to the positive factor $1/\sqrt{2}$, we have, for any $x \in]p_i, p_{i+1}[$,

$$\begin{aligned} Z_+(x, m_j) &= \sin x \ \sin(Rm_j) + \sin(Rx) \ \sin m_j \\ &= (-1)^j \big(\sin x + (-1)^j \sin m_j \ \sin(Rx)\big) \\ &= (-1)^j \big(\sin x - \sin m_j \,|\sin(Rx)|\big)\,, \end{aligned} \tag{6.45}$$

where the last equality follows from the equalities $\sin(Rx) = (-1)^i |\sin(Rx)|$ on the interval $]p_i, p_{i+1}[$, and $(-1)^{i+j} = -1$. On the other-hand, the inequalities $j \le i-1$ and $i + j \le R - 2$ imply that $m_j < p_i < p_{i+1} < \pi - m_j$, so that $\sin x > \sin m_j$ on $]p_i, p_{i+1}[$. This proves that $Z_+(x, m_j) \neq 0$ on $]p_i, p_{i+1}[$, hence the claim. □

Taking into account the preceding discussion, we have obtained the following lemma.

Lemma 6.8 *The horizontal segments (medians) through the points (m_i, m_j) which are contained in the white squares $Q_{i,j} \subset \mathcal{W}(+) \cap \mathcal{F}_1$ or $\mathcal{W}(+) \cap \mathcal{F}_3$ do not meet the nodal set $N(Z_+)$. The vertical segments (medians) through (m_i, m_j) which are contained in the white squares $Q_{i,j} \subset \mathcal{W}(+) \cap \mathcal{F}_2$ or $\mathcal{W}(+) \cap \mathcal{F}_4$ do not meet the nodal set $N(Z_+)$.*

We have a similar lemma for the eigenfunction Z_-.

Lemma 6.9 *The horizontal segments (medians) through (m_i, m_j) which are contained in the white squares $Q_{i,j} \subset \mathcal{W}(+) \cap \mathcal{F}_1$ or $\mathcal{W}(-) \cap \mathcal{F}_3$ do not meet the nodal set $N(Z_-)$. The vertical segments (medians) through (m_i, m_j) which are contained in the white squares $Q_{i,j} \subset \mathcal{W}(+) \cap \mathcal{F}_2$ or $\mathcal{W}(+) \cap \mathcal{F}_4$ do not meet the nodal set $N(Z_-)$.*

Proof We sketch the proof of the lemma. Since $N(Z_+)$ and $N(Z_-)$ are symmetrical to each other with respect to $\{x = \pi/2\}$ when R is even, it suffices to study $N(Z_-)$ for R odd. The nodal set is contained in $\mathcal{W}(-)$. The diagonal and the anti-diagonal are covered by white Q-squares, and in these squares the nodal pattern is known since they either contain a critical zero or touch a vertex of the square $\mathcal{S}$. As above, we look at the white squares inside $\mathcal{F}_1$. The indices of these squares satisfy

$$(-1)^{i+j} = 1\,, \quad j \le i - 1\,, \quad i + j \le R - 2\,.$$

As in the previous proof, we can write,

$$\begin{aligned} Z_{-}(x, m_j) &= \sin x \ \sin(Rm_j) - \sin(Rx) \ \sin m_j \\ &= (-1)^j \left(\sin x + (-1)^j \sin m_j \ \sin(Rx) \right) \\ &= (-1)^j \left(\sin x - \sin m_j |\sin(Rx)| \right), \end{aligned} \tag{6.46}$$

because $\sin(Rx) = (-1)^i |\sin(Rx)|$ in the interval $]p_i, p_{i+1}[$, and $(-1)^{i+j} = 1$. The same argument as above gives that the horizontal median $]p_i, p_{i+1}[\times\{m_j\}$ does not meet $N(Z_-)$. □

Remark Similar lemmas hold with the horizontal and vertical segments $]p_i, p_{i+1}[\times\{q_j\}$ and $\{q_i\}\times]p_j, p_{j+1}[$.

Proof of Proposition 6.6 The idea of the proof is to follow the nodal set along the boundary of each square $\partial\mathcal{S}_i$, with $i = 1, 2, \ldots, r$ (say from the point (p_i, p_i) anti-clockwise, through (p_{R-i}, p_i), (p_{R-i}, p_{R-i}), (p_i, p_{R-i}), and back to (p_i, p_i)), and to use the properties of the functions $Z_\pm$ (no critical zeroes on the open edges, known nodal patterns at the vertices, known interior critical zeroes, and their localization together with Lemmas 6.8 and 6.9).

When $R = 2r$ is even, it suffices to prove the result for Z_+. We already know that the nodal set of Z_+ contains the anti-diagonal and $\partial\mathcal{S}$. Start from (p_1, p_1) horizontally. The absence of critical zero on the edge $]0, \pi[\times\{0\}$ and Lemma 6.8 tell us that the nodal line can only intertwine the edge of $\mathcal{S}_1$ untill it enters the square $Q_{1,2r-2}$ at the point (p_1, p_{2r-2}). Due to the nodal pattern in this square which contains the critical zero $(q_1, \pi - q_1)$, the nodal line exits the square at the point (p_{R-1}, p_2). By Lemma 6.8, and the absence of critical zero on the edge $\{\pi\}\times]0, \pi[$, the nodal line has to follow upwards along $x = p_{R-1}$, till the point (p_{R-1}, p_{R-1}) where there is no choice but to get along the horizontal edge at this point, backwards until the nodal line enters $Q_{1,R-2}$ at the point (p_2, p_{R-1}). In this square, the nodal pattern in known, and the nodal line has to leave throught the point (p_1, p_{R-2}) downwards along the last edge of $\mathcal{S}_1$ back to the starting point. This is the first closed curve γ_1. We can now iterate the procedure along $\mathcal{S}_2$, using Lemma 6.8 to constrain the nodal set from both sides. After $(r-1)$ iterations, we end up with $(r-1)$ closed curves, and we have visited every point in $\mathcal{L}$ (if we take the diagonal into account). The curves γ_i cannot meet because an intersection point would be a critical zero, and we know that the only critical zeroes of Z_+ are on the anti-diagonal. The nodal set cannot contain any other connected component, otherwise such a component would be entirely contained in a white Q-square, and we know that this is not possible for energy reasons. This proves Assertion(i).

We obtain the other assertions by similar arguments. □

Remark We have just proved that the nodal patterns of $Z_\pm$ are as suggested by the pictures (see Figs. 16, 17 and 18).

6.6 Deformation of Nodal Patterns

In this subsection, we investigate how nodal patterns of the family of eigenfunctions $\{\Phi^{\theta}\}$ evolve when the parameter θ varies.

Lemma 6.10 *Assume θ is not a critical value of the parameter, i.e., does not belong to the set $\mathcal{T} := \mathcal{T}_o \cup \mathcal{T}_x \cup \mathcal{T}_y$.*

(i) The patterns (A) and (B) in Fig. 11, and the patterns (A), (C) and (D) in Figs. 12 or 13, are stable in any interval $]\theta - \epsilon, \theta + \epsilon[\subset \mathcal{T}$.
*(ii) Let $\theta_k \in \mathcal{T}_o$ be some critical value of θ. When θ is close to θ_k and $\theta > \theta_k$ (*resp. *$\theta < \theta_k$), the pattern (C) in Fig. 11 changes to one of the patterns (A) or (B) (*resp. *(B) or (A)).*
*(iii) Let $\theta_k \in \mathcal{T}_x \cup \mathcal{T}_y$ be some critical value of θ. When θ is close to θ_k and $\theta > \theta_k$ (*resp. *$\theta < \theta_k$), the patterns (B) in Figs. 12 or 13 change to one of the patterns (C) or (D) (*resp. *(D) or (C)).*

Remark The proof provides more information than the above statement.
Proof The proofs are similar for R even and R odd. We only sketch the proofs for R even.

Assertion (i) Assume we are in a square $Q_{i,j}$ which does not touch the boundary of the square. In order to prove the first assertion, we consider the segment $]0, \pi[\times\{q_j\} \cap Q_{i,j}$ for the patterns Fig. 11a, and the segment $\{q_i\}\times]0, \pi[\cap Q_{i,j}$ for the pattern Fig. 11b. The argument is the same in the two cases. Let us consider the last one. The function $y \to \Phi(q_i, y, \theta)$ has precisely two simple zeroes in the interval $]p_j, p_{j+1}[$. The function $y \to \Phi(q_i, y, \theta')$ will still have two simple zeroes for θ' close to θ. As a matter of fact, when θ' varies, the two arcs of nodal set become closer and closer, and eventually touch, which occurs precisely when θ' reaches a critical value in $\mathcal{T}_o$.

Assertion (ii) We consider some critical value θ_k, and use the same segments as in the proof of the first assertion. There are two cases for $\Phi(x, y, \theta_k)$ in the square $Q_{i,j}$: it is non negative on the vertical segment and non positive on the horizontal one, or vice and versa. Both cases are dealt with in the same manner. For symmetry reasons, we can also assume that $0 < \theta < \pi/4$, so that the nodal set meets $Q_{i,j}$ if and only if $(-1)^{i+j} = -1$. For θ close to and different from θ_k, we write,

$$\begin{aligned}\Phi(q_i, y, \theta) = \Phi(q_i, y, \theta_k) & \\ +(-1)^j \Big((\cos\theta - \cos\theta_k) \sin q_i \, (-1)^j \sin(Ry) & \\ -(\sin\theta - \sin\theta_k)(-1)^i \sin(Rq_i) \sin y\Big) . &\end{aligned} \tag{6.47}$$

Assuming that $\Phi(q_i, y, \theta_k) \geq 0$ inside the square $Q_{i,j}$ and looking at signs, we then see that $\Phi(q_i, y, \theta) > 0$ inside $Q_{i,j}$ if either j is even and $\theta < \theta_k$, or j is odd and $\theta > \theta_k$. This means that the nodal pattern Fig. 11c evolves to the nodal pattern Fig. 11a in these cases. Similarly, we write

$$\begin{aligned}\Phi(x, q_j, \theta) = \Phi(x, q_j, \theta_k) \\ +(-1)^j \Big((\cos\theta - \cos\theta_k) \sin x \, (-1)^j \sin(Rq_j) \\ -(\sin\theta - \sin\theta_k)(-1)^i \sin(Rx) \, \sin q_j \Big) .\end{aligned} \tag{6.48}$$

Assuming that $\Phi(x, q_j, \theta_k) \leq 0$ inside the square $Q_{i,j}$, and looking at signs, we see that $\Phi(x, q_j, \theta) < 0$ inside $Q_{i,j}$ if either j is even and $\theta > \theta_k$, or j is odd and $\theta < \theta_k$. This means that the nodal pattern Fig. 11c evolves to the nodal pattern Fig. 11b in these cases.

Assertion (iii) The proof is similar to the proof of Assertion(ii). □

6.7 Desingularization of Z_+

In this subsection, we study how the nodal set of the eigenfunction $\{\Phi^\theta\}$ changes when θ varies in a small neighborhood of $\pi/4$, while (x, y) lies in the neighborhood of a critical zero of the eigenfunction Z_+. Since Z_+ has no critical zero when $R = 2r+1$, we only consider the case $R = 2r$.

Recall the results and notations of Sect. 6.1. By Lemma 6.7, the critical zeros of Z_+ are the points $(q_i, \pi - q_i)$, $1 \leq i \leq R-2$. Take into account the fact that R is even, and hence that U_{R-1} is odd. For (x, y) in the square $Q(i) := Q_{i,2r-1-i}$ which contains the point $(q_i, \pi - q_i)$, write

$$\begin{aligned}\sqrt{2}\, Z_+(q_i, y) &= \sin q_i \, \sin y \, (-1)^i |U_{R-1}(\cos y) + M_i| , \\ \sqrt{2}\, Z_+(x, \pi - q_i) &= \sin q_i \, \sin x (-1)^{i+1} |U_{R-1}(\cos x) - M_i| ,\end{aligned} \tag{6.49}$$

where M_i is defined in Property 6.1(v). These equations give the local nodal pattern for the eigenfunction Z_+ in the square $Q(i)$. When i is odd, *resp.* even, the nodal pattern is given by Fig. 14(i), *resp.* by Fig. 14(ii).

On the other hand, we can write,

$$\begin{aligned}\Phi(q_i, y, \theta) &= \sin q_i \, \sin y \, \Big\{ \cos\theta \sqrt{2}\, Z_+(q_i, y) + \big(\sin\theta - \cos\theta \big) M_i \Big\} \\ &= (-1)^i \Big\{ \cos\theta \sqrt{2}\, |Z_+(q_i, y)| - \sin q_i \, \sin y \, \big(\sin\theta - \cos\theta \big) |M_i| \Big\} .\end{aligned} \tag{6.50}$$

The last factor in the second line of (6.50) is positive when $0 < \pi/4 - \theta \ll 1$. It follows that the local pattern of the nodal set $N(\Phi)$ inside $Q(i)$, is given by Fig. 15(II).

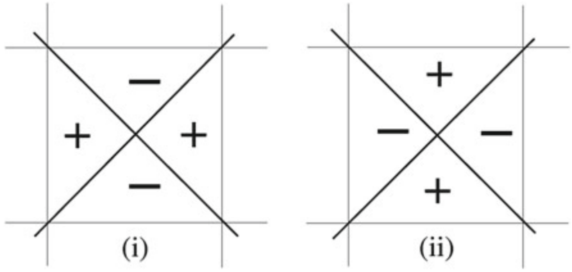


Fig. 14 Local nodal patterns at an interior critical zero

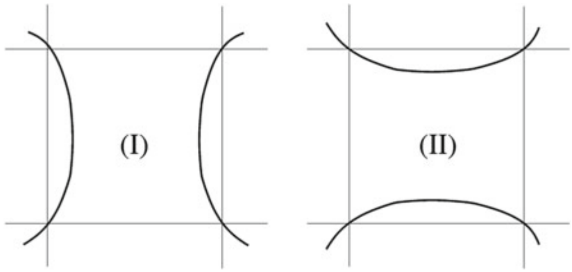


Fig. 15 Local nodal patterns in the absence of critical zero

Similarly, we can write

$$\begin{aligned}\Phi(x, \pi - q_i, \theta) &= \sin q_i \ \sin x \left\{\sin\theta \sqrt{2}\, Z_+(x, \pi - q_i) + \left(\sin\theta - \cos\theta\right) M_i\right\} \\ &= (-1)^{i+1} \left\{\sin\theta \sqrt{2}\, |Z_+(x, \pi - q_i)| + \sin q_i \ \sin y \left(\sin\theta - \cos\theta\right) |M_i|\right\}.\end{aligned} \tag{6.51}$$

The last factor in the second line of (6.51) is positive when $0 < \theta - \pi/4 \ll 1$. It follows that the local nodal pattern of the nodal set $N(\Phi^\theta)$ inside $Q(i)$, is given by Fig. 15 (I).

Remark Notice that the pattern is independent of i. When θ leaves the value $\pi/4$, all the critical zeroes of the eigenfunction Z_+ disappear at once, and the local nodal patterns of Φ^θ in the Q-squares containing the critical zeroes of Z_+ look alike, opening "horizontally" as in Fig. 15(II), when $\theta < \pi/4$; *resp.* opening "vertically" as in Fig. 15(I), when $\theta > \pi/4$, as stated in Theorem 4.1(ii).

6.8 Desingularization of Z_-

Since Z_+ and Z_- are symmetrical with respect to $\{x = \pi/2\}$ when $R = 2r$, we only have to consider the case $R = 2r+1$. When $R = 2r+1$, the interior critical zeroes of Z_- are the point $(\pi/2, \pi/2)$ and the points (q_i, q_i), $(q_i, \pi - q_i)$, for $1 \le i \le R-2$. Due to the symmetries with respect to $\{x = \pi/2\}$ and $\{y = \pi/2\}$, it suffices to consider the points (q_i, q_i), $1 \le i \le r$, and the square $Q_{i,i}$.

We can write

$$\begin{aligned}\sqrt{2}\, Z_-(q_i, y) &= \sin q_i \ \sin y\, (-1)^{i+1}\ |U_{R-1}(\cos y) - M_i|\,,\\ \sqrt{2}\, Z_-(x, q_i) &= \sin q_i \ \sin x\, (-1)^{i}\ |U_{R-1}(\cos x) - M_i|\,.\end{aligned} \tag{6.52}$$

These equations give the local nodal pattern for the eigenfunction Z_- in the square $Q_{i,i}$. When i is even, *resp.* odd, the pattern is given by Fig. 14(i), *resp.* by Fig. 14(ii).

On the other hand, we can write,

$$\begin{aligned}\Phi(q_i, y, \theta) &= \cos\theta\,\sqrt{2}\, Z_-(q_i, y) + \sin q_i \ \sin y \left(\sin\theta + \cos\theta\right) M_i\\ &= (-1)^i \left\{|\cos\theta\,\sqrt{2}\, Z_+(q_i, y)| + \sin q_i \ \sin y \left(\sin\theta + \cos\theta\right) |M_i|\right\}\,.\end{aligned} \tag{6.53}$$

The last factor in the second line of (6.53) is positive when $0 < 3\pi/4 - \theta \ll 1$. It follows that the local pattern of the nodal set $N(\Phi^\theta)$, is given by Fig. 15(I).

Similarly, we can write

$$\begin{aligned}\Phi(x, q_i, \theta) &= -\sin\theta\,\sqrt{2}\, Z_-(x, q_i) + \sin q_i \ \sin x \left(\sin\theta + \cos\theta\right) M_i\\ &= (-1)^{i+1} \left\{\sin\theta\,\sqrt{2}\,|Z_-(q_i, y)| - \sin q_i \ \sin y \left(\sin\theta + \cos\theta\right) |M_i|\right\}\,.\end{aligned} \tag{6.54}$$

The last factor in the second line of (6.54) is positive when $0 < \theta - 3\pi/4 \ll 1$. It follows that the local pattern of the nodal set $N(\Phi^\theta)$, is given by Fig. 15(II).

We point out that the pattern is independent of the sign of i. When θ leaves the value $3\pi/4$, all the critical zeroes of the eigenfunction Z_- disappear at once, and the local nodal patterns of Φ^θ in the squares containing the critical zeroes of Z_- look alike, opening "vertically" as in Fig. 15(I), when $\theta < 3\pi/4$; *resp.* opening "horizontally" as in Fig. 15(II), when $\theta > 3\pi/4$.

6.9 The Nodal Pattern of Φ^θ for θ Close to $\pi/4$ and R Even

By Property 6.2(iv), we can assume that $\theta \in [0, \pi/4]$. We know from Properties 6.3–6.5 that for $R = 2r$ and $0 < \pi/4 - \theta \ll 1$, the eigenfunction Φ^θ has no interior critical zero, two non degenerate edge critical zeroes, respectively in the intervals $\{0\}\times]p_{R-1}, \pi[$ and $\{\pi\}\times]0, p_1[$, and that the vertices are non degenerate critical zeroes.

Using Lemma 6.10, the nodal pattern of $N(\Phi^\theta)$ for θ close to $\pi/4$, is the same as the nodal pattern of Z_+ in the Q-squares without critical zero, namely the white Q-squares which do not meet the anti-diagonal. To determine the nodal set of $N(\Phi^\theta)$, it suffices to know the local nodal patterns in the white Q-squares covering the anti-diagonal. This is given by Sect. 6.7, *the crosses at an interior critical zero open up horizontally*, and by the description of the critical zeroes on the vertical edges near

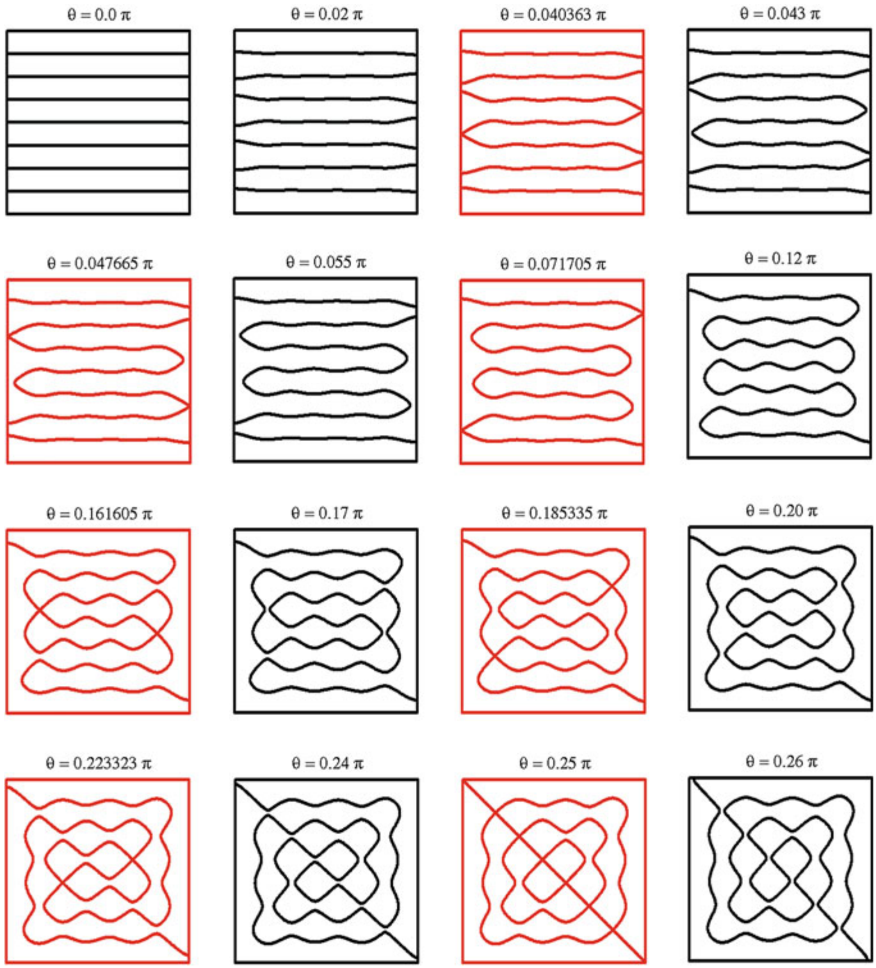


Fig. 16 Typical nodal patterns for the eigenvalue $(1, 8)$

the vertices $(0, \pi)$ and $(\pi, 0)$. It is now clear that the nodal set of Φ^{θ} for θ close enough to $\pi/4$ is connected, and divides the square into two connected components.

Remark As the diaporama of subfigures in Fig. 16 shows, there is another way to obtain examples of eigenfunctions with exactly two domains, using a deformation of one of the simplest product eigenfunctions. The following figures display the nodal sets for the eigenfunctions $\Phi^{\theta}_{1,8}$ and $\Phi^{\theta}_{1,9}$. The values of θ appear in the title. The values with more than two digits correspond to the critical values of the parameter, i.e., the values of θ for which critical zeroes or equivalently multiple points appear/disappear in the nodal set, see (6.19) and (6.20). The values with two digits are intermediate values between two consecutive critical values. The topology of the nodal set does not change in such intervals.

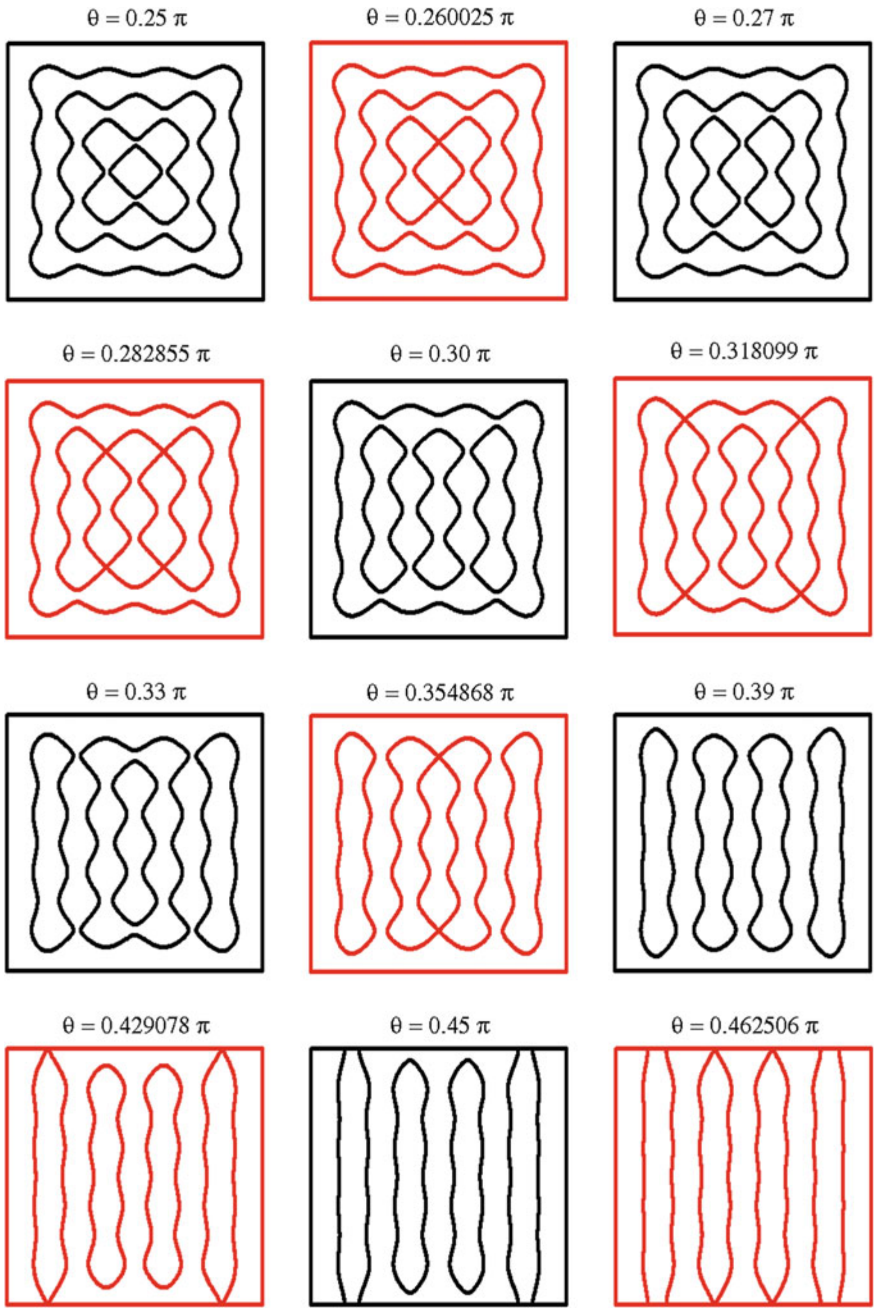


Fig. 17 Typical nodal patterns for the eigenvalue (1, 9)

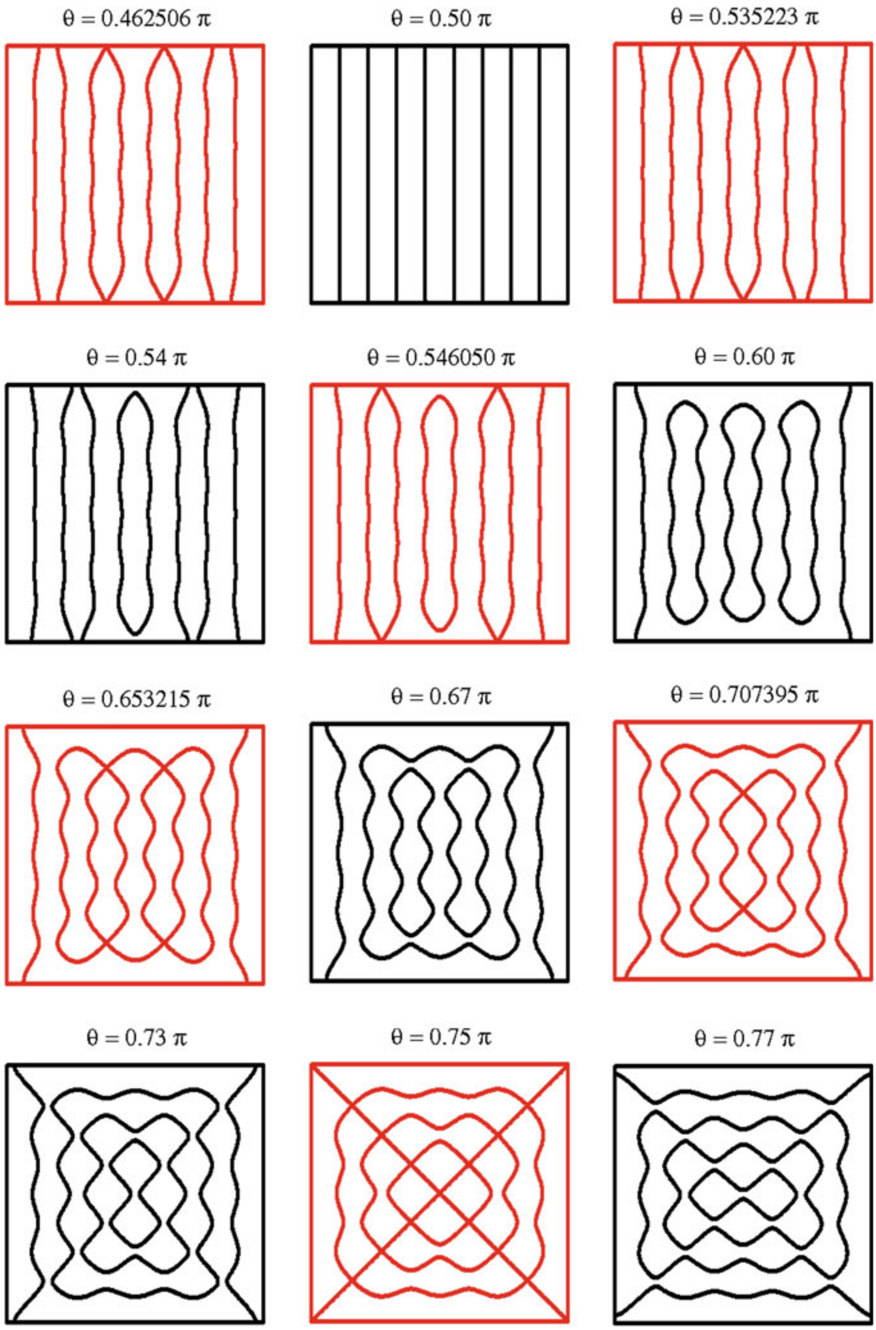


Fig. 18 Typical nodal patterns for the eigenvalue (1, 9), continued

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Analyticity and Smoothness for a Class of First Order Nonlinear PDEs

Shiferaw Berhanu

In memory of M. Salah Baouendi

Abstract We study the microlocal analyticity and smoothness for the solutions of a class of first order complex nonlinear partial differential equations of the form $u_t = f(x, t, u, u_x)$.

Keywords Microlocal analyticity · Microlocal smoothness · FBI transform

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1 Introduction

Consider the first order fully nonlinear partial differential equation

$$u_t = f(x, t, u, u_x)$$

where $f = f(x, t, \zeta_0, \zeta)$ is a real analytic function on an open subset $\Omega \times (-T, T) \times V_1 \times V_2$ of $\mathbb{R}^N \times \mathbb{R} \times \mathbb{C} \times \mathbb{C}^N$, holomorphic in the variables (ζ_0, ζ). In our recent paper [1], we proved microlocal analyticity hypoellipticity results for the trace $u(x, 0)$ of a solution of the Cauchy problem

$$\begin{cases} u_t = f(x, t, u, u_x), & 0 < t < T,\ x \in \Omega \\ u(x, 0) = \omega(x), & x \in \Omega \end{cases}$$

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under some assumptions on the repeated brackets of the linearized operator and its conjugate. When $f(x,t,\zeta_0,\zeta)$ is C^∞ in all the variables and holomorphic in (ζ_0,ζ), we proved a C^∞ microlocal regularity result for the trace $u(x,0)$.

In this paper, we study the regularity of the solution $u(x,t)$ itself instead of the trace $u(x,0)$. Under the same bracket assumptions as in [1], we prove analytic and C^∞ regularity results for the solution $u = u(x,t)$. The bracket assumptions are expressed in terms of the vector field $\mathcal{L}^v$ and its complex conjugate $\overline{\mathcal{L}^v}$ where $\mathcal{L}^v$ is the linearization of the equation $u_t = f(x,t,u(x,t),u_x(x,t))$ at u given by

$$\mathcal{L}^v = \frac{\partial}{\partial t} - \sum_{j=1}^{N} f^v_{\zeta_j}(x,t)\frac{\partial}{\partial x_j}$$

where $f^v_{\zeta_j}(x,t) = f_{\zeta_j}(x,t,u(x,t),u_x(x,t))$ for $1 \le j \le N$. Section 5 contains some examples that apply our results. The approach to the fully nonlinear case by using the Holomorphic Hamiltonian is motivated by [3].

2 Statement of the Results

The work [1] generalized to the fully nonlinear case the main result in [12] where the authors studied the microlocal analyticity and strong instability (with respect to a C^∞ perturbation) of the Cauchy problem for quasi-linear equations of the type

$$\begin{cases} \frac{\partial u}{\partial t} + \sum_{j=1}^{N} a_j(x,t,u)\frac{\partial u}{\partial x_j} = b(x,t,u), & 0 < t < T,\ x \in \Omega \\ u(x,0) = \omega(x), & x \in \Omega \end{cases} \tag{2.1}$$

where $\Omega \subseteq \mathbb{R}^N$ is an open subset, $T > 0$. The functions a_j, b, $j = 1,\ldots,N$ are the restrictions to $\Omega \times [0,T] \times V_3$ of some holomorphic functions defined on a domain $V = V_1 \times V_2 \times V_3 \subseteq \mathbb{C}^{N+2}$. Let

$$\mathcal{M} = \frac{\partial}{\partial t} + \sum_{j=1}^{N} a_j(x,t,v)\frac{\partial}{\partial x_j} + b(x,t,v)\frac{\partial}{\partial v}.$$

We recall from [12] the vectors ν_k for $k \in \mathbb{N}$ defined by

$$\nu_0 = (a_1,\ldots,a_N),\ \nu_1 = (\mathcal{M}(a_1),\ldots,\mathcal{M}(a_N)) = \mathcal{M}(\nu_0),\ldots,\nu_k = \mathcal{M}(\nu_{k-1}) = \mathcal{M}^k(\nu_0).$$

The main result in [12] is as follows:

Theorem 2.1 *Let $k \in \mathbb{N}$. If the Cauchy Problem for* (2.1) *has a C^{k+1} solution for $t \ge 0$ on a neighborhood of $(x_0, 0)$, and $\forall x \in \Omega$, $\forall j$ with $0 \le$*

$j < k$, $\Im\nu_j(x, 0, \omega(x)) = 0$, $\Im\nu_k(x_0, 0, \omega(x_0)) \neq 0$, then $\forall\xi^0 \in \mathbb{R}^N$ such that $\Im\nu_k(x_0, 0, \omega(x_0)) \cdot \xi^0 > 0$, the point $(x_0, \xi^0) \notin WF_a(\omega)$.

Here $WF_a(\omega)$ denotes the analytic wave front set of $\omega(x)$. The reader is referred to [14] for the definition of the analytic wave front set.

In the paper [1], Theorem 2.1 was extended to the fully nonlinear equation $u_t = f(x, t, u, u_x)$ by first generalizing the vectors $\nu_k(x, 0, \omega(x))$.

Let u be a sufficiently smooth solution of the nonlinear pde

$$u_t = f(x, t, u, u_x)$$

where $f(x, t, \zeta_0, \zeta)$ is a smooth function, holomorphic in (ζ_0, ζ). Let

$$\mathcal{L} = \frac{\partial}{\partial t} - \sum_{j=1}^{N} f_{\zeta_j}(x, t, \zeta_0, \zeta) \frac{\partial}{\partial x_j}.$$

Set $\upsilon = (u, u_x)$. If $\psi = \psi(x, t, \zeta_0, \zeta)$ is a smooth function, holomorphic in (ζ_0, ζ), we will use the notation

$$\psi^{\upsilon}(x, t) = \psi(x, t, u, u_x).$$

With this notation, the linearized operator of $u_t = f(x, t, u, u_x)$ can be written as

$$\mathcal{L}^{\upsilon} = \frac{\partial}{\partial t} - \sum_{j=1}^{N} f^{\upsilon}_{\zeta_j}(x, t)\, \partial_{x_j}.$$

Set $g = (g_0, \ldots, g_N)$,

$$g_0(x, t, \zeta_0, \zeta) = f(x, t, \zeta_0, \zeta) - \sum_{j=1}^{N} \zeta_j f_{\zeta_j}(x, t, \zeta_0, \zeta), \text{ and}$$

$$g_i(x, t, \zeta_0, \zeta) = f_{x_i}(x, t, \zeta_0, \zeta) - \zeta_i f_{\zeta_0}(x, t, \zeta_0, \zeta) \quad (1 \le i \le N).$$

Consider now the principal part of the holomorphic Hamiltonian:

$$\mathcal{H} = \mathcal{L} + g_0 \partial_{\zeta_0} + \sum_{j=1}^{N} g_j \partial_{\zeta_j}. \tag{2.2}$$

We can now state the generalizations of Theorem 2.1 proved in [1]:

Theorem 2.2 *Let $f(x, t, \zeta_0, \zeta)$ be a real analytic function that is holomorphic in (ζ_0, ζ). Let $k \in \mathbb{N}$. If the nonlinear first order equation*

$$\partial_t u = f(x, t, u(x, t), u_x(x, t)) \tag{2.3}$$

has a C^{k+1} *solution for* $t \geq 0$ *on a neighborhood of* $(x_0, 0)$*, and*

$$\forall x \in \Omega,\ \forall 0 \leq j < k,\ \Im\left(\mathcal{H}^j f_\zeta\right)^{\upsilon}(x,0) = 0,\ \Im\left(\mathcal{H}^k f_\zeta\right)^{\upsilon}(x_0,0) \neq 0, \tag{2.4}$$

then for all $\xi^0 \in \mathbb{R}^N$ *such that*

$$\Im\left(\mathcal{H}^k f_\zeta\right)^{\upsilon}(x_0,0) \cdot \xi^0 < 0, \tag{2.5}$$

the point $\left(x_0, \xi^0\right)$ *does not belong to the analytic wave front set of the trace* $u(x,0)$*.*

When $f(x,t,\zeta_0,\zeta)$ is C^∞, we also proved

Theorem 2.3 *Let* $f(x,t,\zeta_0,\zeta)$ *be a* C^∞ *function that is holomorphic in* (ζ_0,ζ)*. Let* $k \in \mathbb{N}$*. If the nonlinear first order equation*

$$\partial_t u = f\left(x,t,u(x,t),u_x(x,t)\right) \tag{2.6}$$

has a C^{k+1} *solution for* $t \geq 0$ *on a neighborhood of* $(x_0, 0)$*, and*

$$\forall x \in \Omega,\ \forall 0 \leq j < k,\ \Im\left(\mathcal{H}^j f_\zeta\right)^{\upsilon}(x,0) = 0,\ \Im\left(\mathcal{H}^k f_\zeta\right)^{\upsilon}(x_0,0) \neq 0, \tag{2.7}$$

then for all $\xi^0 \in \mathbb{R}^N$ *such that*

$$\Im\left(\mathcal{H}^k f_\zeta\right)^{\upsilon}(x_0,0) \cdot \xi^0 < 0, \tag{2.8}$$

the point $\left(x_0, \xi^0\right)$ *does not belong to the* C^∞ *wave front set of the trace* $u(x,0)$*.*

The main results of this paper are as follows:

Theorem 2.4 *Let* $f(x,t,\zeta_0,\zeta)$ *be a real analytic function that is holomorphic in* (ζ_0,ζ)*. Let* $k \in \mathbb{N}$*. If the nonlinear first order equation*

$$\partial_t u = f\left(x,t,u(x,t),u_x(x,t)\right) \tag{2.9}$$

has a C^{k+1} *solution on a neighborhood of* $(x_0, 0)$*, and*

$$\forall x \in \Omega,\ \forall 0 \leq j < k,\ \Im\left(\mathcal{H}^j f_\zeta\right)^{\upsilon}(x,0) = 0,\ \Im\left(\mathcal{H}^k f_\zeta\right)^{\upsilon}(x_0,0) \neq 0, \tag{2.10}$$

then

(1) If k is even, for all $\xi^0 \in \mathbb{R}^N$ *such that*

$$\Im\left(\mathcal{H}^k f_\zeta\right)^{\upsilon}(x_0,0) \cdot \xi^0 \neq 0, \tag{2.11}$$

the point $\left(x_0, 0, \xi^0, 0\right)$ *does not belong to the analytic wave front set of the solution* $u(x, t)$.

(2) *If k is odd, for all* $\xi^0 \in \mathbb{R}^N$ *such that*

$$\Im\left(\mathcal{H}^k f_\zeta\right)^{\upsilon}(x_0, 0) \cdot \xi^0 < 0, \tag{2.12}$$

the point $\left(x_0, 0, \xi^0, 0\right)$ *does not belong to the analytic wave front set of the solution* $u(x, t)$.

Theorem 2.5 *Let* $f(x, t, \zeta_0, \zeta)$ *be a smooth function that is holomorphic in* (ζ_0, ζ). *Let* $k \in \mathbb{N}$. *If the nonlinear first order equation*

$$\partial_t u = f\left(x, t, u\left(x, t\right), u_x\left(x, t\right)\right) \tag{2.13}$$

has a C^{k+1} *solution on a neighborhood of* $(x_0, 0)$, *and*

$$\forall x \in \Omega,\ \forall 0 \leq j < k,\ \Im\left(\mathcal{H}^j f_\zeta\right)^{\upsilon}(x, 0) = 0,\ \Im\left(\mathcal{H}^k f_\zeta\right)^{\upsilon}(x_0, 0) \neq 0, \tag{2.14}$$

then

(1) *If k is even, for all* $\xi^0 \in \mathbb{R}^N$ *such that*

$$\Im\left(\mathcal{H}^k f_\zeta\right)^{\upsilon}(x_0, 0) \cdot \xi^0 \neq 0, \tag{2.15}$$

the point $\left(x_0, 0, \xi^0, 0\right)$ *does not belong to the* C^∞ *wave front set of the solution* $u(x, t)$.

(2) *If k is odd, for all* $\xi^0 \in \mathbb{R}^N$ *such that*

$$\Im\left(\mathcal{H}^k f_\zeta\right)^{\upsilon}(x_0, 0) \cdot \xi^0 < 0, \tag{2.16}$$

the point $\left(x_0, 0, \xi^0, 0\right)$ *does not belong to the* C^∞ *wave front set of the solution* $u(x, t)$.

3 A Bracket Characterization

We next indicate how the assumptions in the quasi-linear case (Theorem 2.1) involving the vectors ν_j $(0 \leq j \leq k)$ get generalized in terms of the vectors $\mathcal{H}^j f_\zeta$, $f_\zeta = (f_{\zeta_1}, \ldots, f_{\zeta_N})$.

Write the quasi-linear equation as

$$\frac{\partial u}{\partial t} = -\sum_{j=1}^{N} a_j(x, t, u)\frac{\partial u}{\partial x_j} + b(x, t, u) = f(x, t, u, u_x),$$

where $f(x,t,\zeta_0,\zeta) = -\sum_{j=1}^N a_j(x,t,\zeta_0)\zeta_j + b(x,t,\zeta_0)$. In this case, we have

$$g_0(x,t,\zeta_0,\zeta) = f(x,t,\zeta_0,\zeta) - \sum_{j=1}^N \zeta_j f_{\zeta_j}(x,t,\zeta_0,\zeta) = b(x,t,\zeta_0)$$

and

$$\mathcal{L} = \frac{\partial}{\partial t} - \sum_{j=1}^N f_{\zeta_j}(x,t,\zeta_0,\zeta)\frac{\partial}{\partial x_j} = \frac{\partial}{\partial t} + \sum_{j=1}^N a_j(x,t,\zeta_0)\frac{\partial}{\partial x_j}.$$

The holomorphic Hamiltonian in the quasi-linear case thus becomes

$$\begin{aligned}
\mathcal{H} &= \mathcal{L} + g_0\frac{\partial}{\partial\zeta_0} + \sum_{j=1}^N g_j\frac{\partial}{\partial\zeta_j} \\
&= \frac{\partial}{\partial t} + \sum_{j=1}^N a_j(x,t,\zeta_0)\frac{\partial}{\partial x_j} + b(x,t,\zeta_0)\frac{\partial}{\partial\zeta_0} + \sum_{j=1}^N g_j\frac{\partial}{\partial\zeta_j} \\
&= \mathcal{M} + \sum_{j=1}^N g_j\frac{\partial}{\partial\zeta_j}.
\end{aligned}$$

It follows that since $f_\zeta = -(a_1(x,t,\zeta_0),\ldots,a_N(x,t,\zeta_0)) = -\nu_0(x,t,\zeta_0)$,

$$\mathcal{H}^j(f_\zeta) = -\mathcal{M}^j(\nu_0) = \nu_j, \quad \forall j$$

and hence the assumptions in Theorems 2.4 and 2.5 on $\mathcal{H}^j(f_\zeta)$ become the conditions on ν_j in the quasi-linear case.

We recall from Proposition 11 in [1] that the conditions

$$\Im(\mathcal{H}^j f_\zeta)^v(x_0,0) = 0 \text{ for } 0 \le j \le k-1 \text{ and } \Im(\mathcal{H}^k f_\zeta)^v(x_0,0) \ne 0$$

hold if and only if all the brackets of $\mathcal{L}^v$ and $\overline{\mathcal{L}^v}$ of order $< k$ vanish at $(x_0,0)$ and the $k-$ bracket

$$\frac{1}{2i}\left[\mathcal{L}^v,\left[\mathcal{L}^v,\ldots,\left[\mathcal{L}^v,\overline{\mathcal{L}^v}\right]\right]\right](x_0,0) \ne 0. \tag{3.1}$$

Here if $X_0, X_1, \ldots, X_k$ are vector fields, the bracket $[X_k,[X_{k-1},\ldots,[X_1,X_0]]]$ is considered a bracket of order k.

4 Proofs of Theorems 2.4 and 2.5

Proof of Theorem 2.4 Let $f = f(x, t, \zeta_0, \zeta)$ be a real analytic function holomorphic in the variables (ζ_0, ζ). Recall the notation $f^v(x, t) = f(x, t, u(x, t), u_x(x, t))$.

Consider then the function $u = u(x, t)$ which is a solution of $u_t = f(x, t, u, u_x)$. We introduce a new real variable s and note that $w(x, t, s) = u(x, t)$ is a solution of

$$w_s = f(x, t, w, w_x) - w_t = h(x, t, s, w, w_x, w_t)$$

where $h(x, t, s, \zeta_0, \zeta, \tau) = f(x, t, \zeta_0, \zeta) - \tau$. The vector field $\mathcal{L}_1$ (the analogue of $\mathcal{L}$) associated to this equation is

$$\mathcal{L}_1 = \frac{\partial}{\partial s} + \frac{\partial}{\partial t} - \sum_{j=1}^{N} f_{\zeta_j}(x, t, \zeta_0, \zeta)\frac{\partial}{\partial x_j}$$

and the corresponding Hamiltonian is given by

$$\mathcal{H}_1 = \mathcal{L}_1 + \sum_{j=0}^{N} g_j \frac{\partial}{\zeta_j} + g_{N+1}\frac{\partial}{\partial \tau}$$

where for $0 \leq j \leq N$, the g_j are the same as before and $g_{N+1} = f_t + \tau f_{\zeta_0}$.

If $\psi = \psi(x, t, s, \zeta_0, \zeta, \tau)$, we will use the notation

$$\psi^v(x, t, s) = \psi(x, t, s, w(x, t, s), w_x(x, t, s), w_t(x, t, s))$$

where we recall that $w(x, t, s) = u(x, t)$.

Consider the Hamiltonian $\mathcal{H}_1$ on a neighborhood $V = V_1 \times V_2 \times V_3 \times V_4 \subset \mathbb{C}^N \times \mathbb{C} \times \mathbb{C} \times \mathbb{C}^N$ of the point $(x_0, 0, u(x_0, 0), u_x(x_0, 0))$, and we assume, with $r_0, T_0, \rho_0, \rho > 0$, that

$$\begin{aligned}
V_1 &= \left\{x \in \mathbb{C}^N : |x - x_0| < r_0\right\}, \\
V_2 &= \{t \in \mathbb{C} : |t| < T_0\}, \\
V_3 &= \{\zeta_0 \in \mathbb{C} : |\zeta_0 - u(x_0, 0)| < \rho_0\}, \text{ and} \\
V_4 &= \left\{\zeta \in \mathbb{C}^N : |\zeta - u_x(x_0, 0)| < \rho\right\}.
\end{aligned}$$

Let $\Xi_0(x, t, s, \zeta_0, \zeta, \tau)$, and for $1 \leq j \leq N+1$, let $Z_j(x, t, s, \zeta_0, \zeta, \tau)$ be real analytic functions in V (after shrinking V), holomorphic in (ζ_0, ζ, τ), such that

$$\mathcal{H}_1 Z_j = 0 \text{ and } Z_j(x, t, 0, \zeta_0, \zeta, \tau) = x_j \;\; (1 \leq j \leq N),\; Z_{N+1}(x, t, 0, \zeta_0, \zeta, \tau) = t$$
$$\mathcal{H}_1 \Xi_0 = 0 \text{ and } \Xi_0(x, t, 0, \zeta_0, \zeta, \tau) = \zeta_0.$$

Note that $Z_{N+1} = t - s$. Let

$$\mathcal{L}_1^{\upsilon} = \partial_s + \partial_t - \sum_{j=1}^{N} f_{\zeta_j}^{\upsilon}(x,t)\,\partial_{x_j}.$$

Recall from Lemma 2 in [3] that for any smooth $h = h(x, t, s, \zeta_0, \zeta, \tau)$, holomorphic in (ζ_0, ζ, τ), $\mathcal{L}_1^{\upsilon} h^{\upsilon} = (\mathcal{H}_1 h)^{\upsilon}$. This implies that the Z_j's and Ξ_0 (when we restrict them to $(x, t, s, u(x,t), u_x(x,t), u_t(x,t))$) are solutions of $\mathcal{L}_1^{\upsilon}$, that is,

$$\mathcal{L}_1^{\upsilon}(Z_j^{\upsilon}) = 0, \ \text{ and } \ \mathcal{L}_1^{\upsilon}(\Xi_0^{\upsilon}) = 0.$$

By (25) and (26) in [1], we have:

$$0 = \Im\left(\mathcal{H}^j f_{\zeta_i}\right)^{\upsilon}(x,0) = \left(\mathcal{L}^{\upsilon}\right)^j \Im f_{\zeta_i}^{\upsilon}(x,0) = \partial_t^j \Im f_{\zeta_i}^{\upsilon}(x,0)$$

where $\mathcal{H}$ is as in (2.2) and

$$\mathcal{L}^{\upsilon} = \partial_t - \sum_{j=1}^{N} f_{\zeta_j}^{\upsilon}(x,t)\partial_{x_j}.$$

Thus by (2.10),

$$\partial_t^j \Im f_{\zeta_i}^{\upsilon}(x,0) = 0 \text{ for } 0 \le j \le k-1. \tag{4.1}$$

and

$$0 \neq \Im\left(\mathcal{H}^k f_{\zeta_i}\right)^{\upsilon}(x_0,0) = \left(\mathcal{L}^{\upsilon}\right)^k \Im f_{\zeta_i}^{\upsilon}(x_0,0) = \partial_t^k \Im f_{\zeta_i}^{\upsilon}(x_0,0). \tag{4.2}$$

We can write the solutions Z_i^{υ} for $1 \le i \le N$ as

$$Z_i^{\upsilon}(x,t,s) = x_i + s\psi_i(x,t,s)$$

Since $\mathcal{L}_1^{\upsilon} Z_i^{\upsilon}(x,t,s) = 0$, we have:

$$\psi_i + s\partial_s\psi_i + s\partial_t\psi_i - f_{\zeta_i}^{\upsilon}(x,t) - s\sum_{j=1}^{N} f_{\zeta_j}^{\upsilon}(x,t)\partial_{x_j}\psi_i = 0. \tag{4.3}$$

Setting $s = 0$ the latter leads to

$$\psi_i(x,t,0) = f_{\zeta_i}^{\upsilon}(x,t), \ \ 1 \le i \le N. \tag{4.4}$$

Observe that

$$\partial_s^m (s\partial_s\psi_i) = s\partial_s^{m+1}\psi_i + m\partial_s^m\psi_i,$$

$$\partial_s^m (s\partial_t \psi_i) = s\left(\partial_s^m \partial_t \psi_i\right) + m\left(\partial_s^{m-1}\partial_t \psi_i\right),$$

and

$$\partial_s^m \left(s \sum_{j=1}^{N} f_{\zeta_j}^{v} \partial_{x_j} \psi_i \right) = s \left(\sum_{j=1}^{N} f_{\zeta_j}^{v} \partial_s^m \partial_{x_j} \psi_i \right) + m \sum_{j=1}^{N} f_{\zeta_j}^{v} \partial_s^{m-1} \partial_{x_j} \psi_i .$$

Apply ∂_s^m to Eq. (4.3) and use the preceding equations to get:

$$(m+1)\partial_s^m \psi_i + s\partial_s^{m+1}\psi_i + s\left(\partial_s^m \partial_t \psi_i\right) + m\left(\partial_s^{m-1}\partial_t \psi_i\right) - s\left(\sum_{j=1}^{N} f_{\zeta_j}^{v} \partial_s^m \partial_{x_j}\psi_i\right) \\ + m\sum_{j=1}^{N} f_{\zeta_j}^{v} \partial_s^{m-1}\partial_{x_j}\psi_i = 0. \tag{4.5}$$

We will show that for $0 \leq l \leq k$,

$$\partial_s^l \Im \psi_i(x,t,0) = -\frac{(-1)^{l+1}}{l+1} \partial_t^l \Im f_{\zeta_i}^{v}(x,t) + O(t^{k-l+1}). \tag{4.6}$$

This equation holds for $l = 0$ by (4.4). Assume it holds for some $l \geq 1$. Using (4.5) with $m = l + 1$ and plugging $s = 0$ we get:

$$(l+2)\partial_s^{l+1}\psi_i + (l+1)\partial_s^l \partial_t \psi_i - m\sum_{j=1}^{N} f_{\zeta_j}^{v} \partial_s^l \partial_{x_j}\psi_i = 0.$$

Using the inductive assumption for l, it follows that the equation holds for $l + 1$. Thus (4.6) holds for all l.

Using the assumptions in Theorem 2.4, (4.1) and (4.2), for some functions b_i, we can write

$$\Im f_{\zeta_i}^{v}(x,t) = b_i(x,t)t^k,\ 1 \leq i \leq N,\ \ b(x_0,0) = (b_1(x_0,0),\ldots,b_N(x_0,0)) \neq 0.$$

Taylor expanding in s and using (4.6) we therefore get:

$$\Im \psi_i(x,t,s) = -b_i(x,t)\left(\sum_{j=0}^{k} \frac{(-1)^{j+1}}{j+1}\binom{k}{j} t^{k-j} s^j\right) + O(t^{k+1}) + O(s^{k+1}). \tag{4.7}$$

We may assume that $x_0 = 0$. We will use the FBI transform in (x, t) space. For $h = h(x, t, s)$ at level s', we have:

$$\mathcal{F}h(x,t,\xi,\tau,s') = \int_{|x'|^2+|t'|^2\le 2r^2} e^{Q(x,t,x',t',\xi,\tau,s')}\,\eta(x',t')h(x',t',s')\,dZ_1\wedge\cdots dZ_{N+1}$$

where $\eta \in C_0^\infty(\mathbb{R}^{N+1})$, $\eta(x,t) \equiv 1$ when $|x|^2 + t^2 \le r^2$, and $\eta(x,t) \equiv 0$ for $|x|^2 + t^2 \ge 2r^2$ for some $r > 0$ to be fixed. Here

$$\begin{aligned} Q(x,t,x',t',\xi,\tau,s') &= \sqrt{-1}\langle(\xi,\tau),(x - Z'(x',t',s'), t - Z_{N+1}(x',t',s'))\rangle \\ &\quad - |(\xi,\tau)|\left((x - Z'(x',t',s'))^2 + (t - Z_{N+1}(x',t',s'))^2\right) \end{aligned}$$

where $Z' = (Z_1, \ldots, Z_N)$ and $(x - Z'(x',t',s'))^2 = \sum_{j=1}^N (x_j - Z_j(x',t',s'))^2$. Let

$$M_i = \sum_{j=1}^N b_{ij}(x,t,s)\frac{\partial}{\partial x_j},\ 1 \le i \le N \text{ and } M_{N+1} = \frac{\partial}{\partial t} + \sum_{j=1}^N c_j(x,t,s)\frac{\partial}{\partial x_j}$$

be vector fields that satisfy $M_i Z_l^\upsilon = \delta_{il}$ for $1 \le i, l \le N+1$ near $x = 0$, $t = 0$, and $s = 0$. For any C^1 function $h = h(x,t,s)$,

$$dh = \sum_{i=1}^{N+1} M_i(h)\,dZ_i^\upsilon + \mathcal{L}_1^\upsilon(h)\,ds$$

as can be seen by applying both sides of the equation to the basis of vector fields $\{\mathcal{L}_1^\upsilon, M_1, \ldots, M_{N+1}\}$. The latter implies that

$$d(h\,dZ_1^\upsilon \wedge\cdots\wedge dZ_{N+1}^\upsilon) = \mathcal{L}_1^\upsilon(h)\,ds \wedge dZ_1^\upsilon \wedge\cdots\wedge dZ_{N+1}^\upsilon. \tag{4.8}$$

Let

$$q(x,t,x',t',\xi,\tau,s') = \eta(x')\,\Xi_0^\upsilon(x',t',s')e^{Q(x,t,x',t',\xi,\tau,s')}.$$

Denoting $dZ_1^\upsilon \wedge\cdots\wedge dZ_{N+1}^\upsilon$ by dZ and using (4.8), we get:

$$d(q\,dZ) = \mathcal{L}_1^\upsilon(\eta)\,\Xi_0^\upsilon\,e^Q\,ds \wedge dZ. \tag{4.9}$$

By Stokes theorem we have, for small s_0:

$$\begin{aligned} &\int_{\mathbb{R}^{N+1}} q(x,t,x',t',\xi,\tau,0)\,dx'dt' \\ &= \int_{\mathbb{R}^{N+1}} q(x,t,x',t',\xi,\tau,s_0)\,dZ(x',t',s_0) + \int_0^{s_0}\int_{\mathbb{R}^{N+1}} d(qdZ). \end{aligned} \tag{4.10}$$

We will estimate the two integrals on the right in (4.10). Note that

$$\Re Q(x,t,x',t',\xi,\tau,s') = -s'p(t',s')\langle \xi, b(x',t')\rangle - |(\xi,\tau)|(|x-x'-s'\Re\psi(x',t',s')|^2 \tag{4.11}$$

$$+|t-(t'-s')|^2 - |s'\Im\psi(x',t',s')|^2) + O(|t'|^{k+2}+|s'|^{k+2}),$$

where

$$p(t,s) = \sum_{m=0}^{k} \frac{(-1)^{m+1}}{m+1}\binom{k}{m} t^{k-m}s^m, \text{ and } b=(b_1,\ldots,b_N).$$

Case (i): Assume k **is even**. Consider the expression for $\Re Q(x,t,x',t',\xi,\tau,s')$ in (4.11). Let $a \in \mathbb{R}$, $a \neq 0$. Then for any $t \neq 0$,

$$\begin{aligned}\frac{p(t,at)}{t^k} &= \sum_{m=0}^{k}\frac{(-1)^{m+1}}{m+1}a^m\binom{k}{m} = -\sum_{m=0}^{k}\binom{k}{m}a^m\int_{-1}^{0}x^m\,dx = -\int_{-1}^{0}(ax+1)^k\,dx \\ &= \frac{(1-a)^{k+1}-1}{a(k+1)} \\ &< 0 \end{aligned} \tag{4.12}$$

since k is even. We also have

$$p(0,s) = \frac{-1}{k+1}s^k \le 0 \text{ and } p(t,0) = -t^k \le 0. \tag{4.13}$$

It follows from (4.12) and (4.13) that for some $C > 0$,

$$p(t,s) \le -C(|t|^k + |s|^k)\ \forall (t,s). \tag{4.14}$$

Note that for (x,t,s) near $(0,0,0)$,

$$|s\Im\psi(x,t,s)|^2 \le C|s|^2(|s|^{2k}+|t|^{2k}) \quad \text{for some } C>0. \tag{4.15}$$

Suppose now $\langle b(0,0),\xi^0\rangle > 0$. Then in the region $s' \le 0$, for some $C > 0$, (4.14) and (4.15) lead to

$$\Re Q(x,t,x',t',\xi,\tau,s') \le -C(|t'|^{k+1}+|s'|^{k+1})|\xi| - |x-x'-s'\Re\psi(x',t',s')|^2|\xi| \tag{4.16}$$

for (ξ,τ) in a conic neighborhood Γ of $(\xi^0,0)$, and (x,t) in a neighborhood W of $(0,0)$ in $\mathbb{R}^{N+1}$. We choose $r > 0$ so that (4.16) holds for $|x'|^2 + |t'|^2 \le 2r^2$. If $\langle b(0,0),\xi^0\rangle < 0$, we work in the region $s' \ge 0$ and arrive at (4.16). For $(x,t) \in W$

and $(\xi,\tau)\in\Gamma$, inequality (4.16) leads to the estimate for the first integral on the right in (4.10):

$$\left|\int_{\mathbb{R}^{N+1}} q(x,t,x',t',\xi,\tau,s_0)\,dZ(x',t',s_0)\right| \le e^{-C|\xi|},\quad C>0 \tag{4.17}$$

where s_0 is a small nonzero number whose sign depends on that of $\langle b(0,0),\xi^0\rangle$. To estimate the second integral on the right in (4.10), by (4.9), we may assume that $|x'|^2+|t'|^2\ge r^2$. We first assume that $x=0$, $t=0$. If $|t'|\ge\frac{r}{2}$, then (4.16) implies that $\Re Q\le -C|\xi|$ for some $C>0$. If $|t'|\le\frac{r}{2}$, then $|x'|\ge\frac{r}{2}$. Let $|\Re\psi|\le M$ in a neighborhood of $x'=0$, $t'=0$, $s'=0$. Then if $|s'|\le\frac{r}{4M}$, we have

$$|x'+s'\Re\psi(x',t',s')|\ge|x'|-M|s'|\ge\frac{r}{2}-M|s'|\ge\frac{r}{4},$$

which by (4.16) yields $\Re Q(0,0,x',t',\xi,\tau,s')\le -C|\xi|$, $C>0$ for (ξ,τ) in a conic neighborhood of $(\xi^0,0)$, while if $|s'|\ge\frac{r}{4M}$, we get a similar bound using (4.16) again. Hence there exists $C>0$, such that when $\langle\xi,b(0,0)\rangle>0$ and $s'\le 0$ or $\langle\xi,b(0,0)\rangle<0$ and $s'\ge 0$, in the region $|x'|^2+|t'|^2\ge r^2$,

$$\Re Q(0,0,x',t',\xi,\tau,s')\le -C|\xi|$$

for (ξ,τ) in a conic neighborhood of $(\xi^0,0)$. Therefore, for x near $0\in\mathbb{R}^N$ and t near 0 in $\mathbb{R}$,

$$\Re Q(x,t,x',t',\xi,\tau,s')\le -C'|\xi|,\ \ C'>0.$$

This leads to an exponential decay for the second integral. Observe that since $\Im f_\zeta^v(x,t)=t^k b(x,t)$, by (4.2), the signs of $\Im\left(\mathcal{H}^k f_\zeta\right)^v(x,0)\cdot\xi^0$ and $b(x,0)\cdot\xi^0$ are the same. Since $\Xi_0(x,t,0)=u(x,t)$, from (4.10), we get, for some $C_1, C_2>0$:

$$\left|\int_{\mathbb{R}^{N+1}} e^{\sqrt{-1}\langle(\xi,\tau),(x-x',t-t')\rangle-|(\xi,\tau)|(|x-x'|^2+|t-t'|^2)}\,\eta(x',t')u(x',t')\,dx'dt'\right|\le C_1e^{-C_2|\xi|}$$

for x and t near zero and (ξ,τ) in a conic neighborhood of $(\xi^0,0)$. Hence, $(0,0,\xi^0,0)$ is not in the analytic wave front set of $u(x,t)$.

Case (ii): Assume k is odd. We will now work in the region where $s'\ge 0$. If $t'<0$, then

$$|s'-t'|^2\ge\frac{1}{2}(|s'|^2+|t'|^2)$$

and so in this case, if $k>1$, for some $C>0$,

$$\Re Q(0,0,x',t',\xi,\tau,s')\le -C(|s'|^2+|t'|^2)|\xi|-|x'+s'\Re\psi(x',t',s')|^2|\xi|.$$

Since $p(1,1) < 0$ by (4.12) and $\langle b(0,0), \xi^0 \rangle < 0$, there exists $0 < \delta < 1$ and $C > 0$ such that if $t' \in (1-\delta, 1+\delta)s'$, for (x', t') near $(0, 0)$ and ξ in a conic neighborhood of ξ^0,

$$-s' p(t', s')\langle \xi, b(x', t')\rangle \leq -C|\xi|(|t'|^{k+1} + |s'|^{k+1}).$$

(Recall that we are assuming now that $s' \geq 0$). If $0 \leq t' \leq (1-\delta)s'$ or $t' \geq (1+\delta)s'$, then for some $C > 0$, $|s' - t'|^2 \geq C(|s'|^2 + |t'|^2)$. Hence, once again, when $k > 1$, we have

$$\Re Q(0, 0, x', t', \xi, \tau, s') \leq -C(|t'|^{k+1} + |s'|^{k+1}) - |x' + s'\Re\psi(x', t', s')|^2|\xi|.$$

We can therefore argue as before to get exponential decays for both integrals on the right in (4.10) for (x, t) near $(0, 0)$ and (ξ, τ) in a conic neighborhood of $(\xi^0, 0)$ provided that $k > 1$.

Suppose now $k = 1$. We will use an FBI transform with a parameter κ in its phase, that is, we assume that

$$\begin{aligned} Q(x, t, x', t', \xi, \tau, s') &= \sqrt{-1}\langle(\xi, \tau), (x - Z'(x', t', s'), t - Z_{N+1}(x', t', s'))\rangle \\ &\quad - \kappa|(\xi, \tau)| \left((x - Z'(x', t', s'))^2 + (t - Z_{N+1}(x', t', s'))^2\right) \end{aligned}$$

for some $\kappa > 0$ which will be determined. We may assume that $|\xi^0| = 1$. Then when $x = 0, t = 0$,

$$\begin{aligned} \Re Q(0, 0, x', t', \xi^0, 0, s') &= \langle b(0,0), \xi^0\rangle \left(t's' - \frac{s'^2}{2}\right) - \kappa(|x' + s'\Re\psi(x', t', s')|^2 \\ &\quad + |t' - s'|^2 - |s'\Im\psi(x', t', s')|^2) + O(|s'|^3 + |t'|^3). \end{aligned}$$

We will work in the region where $s' \geq 0$. Suppose first $t' < 0$. Then if $\kappa > \max\{1, -\langle b(0,0), \xi^0\rangle\}$, since $s' \geq 0$ and $t' < 0$,

$$\begin{aligned} \Re Q(0, 0, x', t', \xi^0, 0, s') &\leq \langle b(0,0), \xi^0\rangle \left(s't' - \frac{s'^2}{2} + |s' - t'|^2\right) - |x' + s'\Re\psi(x', t', s')|^2 \\ &\quad + O(|t'|^3 + |s'|^3) \\ &\leq \frac{\langle b(0,0), \xi^0\rangle}{2}(t'^2 + s'^2). \end{aligned}$$

Thus when $t' < 0$, for some $C > 0$,

$$\Re Q(0, 0, x', t', \xi^0, 0, s') \leq -C(t'^2 + s'^2) - |x' + s'\Re\psi(x', t', s')|^2.$$

Suppose now $t' \geq 0$. Let $0 < \delta < 1$ be as before so that when $t' \in (1-\delta, 1+\delta)s'$, for (x, t) near $(0, 0)$ and ξ in a conic neighborhood of ξ^0,

$$-s' p(t', s')\langle b(0,0), \xi\rangle \leq -C|\xi|(|t'|^2 + |s'|^2) \quad \text{for some } C > 0.$$

If $0 \le t' \le (1-\delta)s'$ or $(1+\delta)s' \le t'$, there is a constant $C_\delta > 0$ depending only on δ such that $|t'-s'|^2 \ge C_\delta(|t'|^2+|s'|^2)$ and hence if $\kappa > \max\{1, -\frac{2}{C_\delta}\langle b(0,0), \xi^0\rangle\}$,

$$\begin{aligned}\Re Q(0,0,x',t',\xi^0,0,s') &\le \langle b(0,0),\xi^0\rangle \left(s't' - \frac{s'^2}{2} - \kappa\frac{|s'-t'|^2}{\langle b(0,0),\xi^0\rangle}\right)\\ &\quad - |x'+s'\Re\psi(x',t',s')|^2 + O(|t'|^3+|s'|^3)\\ &\le \langle b(0,0),\xi^0\rangle \left(s't' - \frac{s'^2}{2} + 2t'^2 + 2s'^2\right)\\ &\quad - |x'+s'\Re\psi(x',t',s')|^2 + O(|t'|^3+|s'|^3)\\ &\le \langle b(0,0),\xi^0\rangle (t'^2+s'^2).\end{aligned}$$

It follows that if

$$\kappa > \max\{1, -\langle b(0,0),\xi^0\rangle, -\frac{2}{C_\delta}\langle b(0,0),\xi^0\rangle\},$$

then there exists $C > 0$ such that

$$\Re Q(0,0,x',t',\xi^0,0,s') \le -C(|t'|^2+|s'|^2) - |x'+s'\Re\psi(x',t',s')|^2$$

which is the analogue of (4.16). We can therefore argue as before to arrive at exponential decays for both integrals on the right in (4.10) for (x,t) near $(0,0)$ and (ξ,τ) in a conic neighborhood of $(\xi^0, 0)$.

Proof of Theorem 2.5 Let the V_j, $1 \le j \le 4$ be the open sets as in the proof of Theorem 2.4. Let $\Xi_0(x,t,s,\zeta_0,\zeta,\tau)$ and $Z_j(x,t,s,\zeta_0,\zeta,\tau)$ $(1 \le j \le N+1)$ be smooth functions, holomorphic in (ζ_0,ζ), such that (see [A])

$$\begin{aligned}&\mathcal{H}Z_j = O\left(s^n\right),\ n = 1,2,\ldots,\ \text{ and } Z_j(x,t,0,\zeta_0,\zeta,\tau) = x_j\ \ (1\le j\le N),\\ &\mathcal{H}\Xi_0 = O\left(s^n\right),\ n = 1,2,\ldots,\ \text{ and } \Xi_0(x,t,0,\zeta_0,\zeta,\tau) = \zeta_0.\end{aligned}$$

As before, $Z_{N+1} = t-s$. Since $\mathcal{L}_1^\upsilon f^\upsilon = (\mathcal{H}_1 f)^\upsilon$, the Z_j's and Ξ_0 (when we restrict them to $(x,t,s,u(x,t),u_x(x,t),u_t(x,t))$ are approximate solutions of $\mathcal{L}_1^\upsilon$, that is,

$$\mathcal{L}_1^\upsilon Z_j^\upsilon(x,t,s) = O(s^n),\ \ n = 1,2,\ldots$$

and this implies the analogue of (4.3):

$$\psi_i + s\partial_s\psi_i + s\partial_t\psi_i - f_{\zeta_i}^\upsilon(x,t) - s\sum_{j=1}^N f_{\zeta_j}^\upsilon(x,t)\partial_{x_j}\psi_i = O(s^n),\ \ n\in\mathbb{N}. \quad (4.18)$$

Setting $s = 0$ the latter leads to (4.4). We next apply ∂_s^m to Eq. (4.18) to get:

$$(m+1)\partial_s^m\psi_i + s\partial_s^{m+1}\psi_i + s\left(\partial_s^m\partial_t\psi_i\right) + m\left(\partial_s^{m-1}\partial_t\psi_i\right) - s\left(\sum_{j=1}^N f^{\upsilon}_{\zeta_j}\partial_s^m\partial_{x_j}\psi_i\right)$$
$$+m\sum_{j=1}^N f^{\upsilon}_{\zeta_j}\partial_s^{m-1}\partial_{x_j}\psi_i = O(s^n),\ \ n\in\mathbb{N}. \tag{4.19}$$

As before, when $s=0$, for $0\le l\le k$, we get (4.6). Therefore, using the assumptions in Theorem 2.5 and (4.3), for some functions $b_i(x,t)$,

$$\Im f^{\upsilon}_{\zeta_i}(x,t) = b_i(x,t)t^k,\ \ 1\le i\le N$$

and (4.7) holds. Assume $x_0=0$. Let M_i, $1\le i\le N+1$ be the vector fields as before. For any C^1 function $h=h(x,t,s)$,

$$dh = \sum_{i=1}^{N+1} M_i(h)\,dZ_i^{\upsilon} + \left(\mathcal{L}^{\upsilon}h - \sum_{j=1}^{N+1} M_j(h)\mathcal{L}^{\upsilon}(Z_j^{\upsilon})\right)ds$$

and hence,

$$d(h\,dZ_1^{\upsilon}\wedge\cdots\wedge dZ_{N+1}^{\upsilon}) = \left(\mathcal{L}^{\upsilon}h - \sum_{j=1}^{N+1} M_j(h)\mathcal{L}^{\upsilon}Z_j^{\upsilon}\right)ds\wedge dZ_1^{\upsilon}\wedge\cdots\wedge dZ_{N+1}^{\upsilon}. \tag{4.20}$$

If we set

$$q(x,t,x',t',\xi,\tau,s') = \eta(x)\,\Xi_0^{\upsilon}(x,t,s)e^{Q(x,t,x',t',\xi,\tau,s')},$$

using (4.20), we get

$$d(q\,dZ) = \left(\mathcal{L}^{\upsilon}(\eta\Xi_0^{\upsilon}) + (\eta\Xi_0^{\upsilon})\mathcal{L}^{\upsilon}(Q) - \sum_{j=1}^{N+1}(M_j(\eta\Xi_0^{\upsilon}) + \eta\Xi_0^{\upsilon}(M_jQ))\mathcal{L}^{\upsilon}Z_j^{\upsilon}\right)e^Q\,ds\wedge dZ. \tag{4.21}$$

By Stokes theorem we have:

$$\int_{\mathbb{R}^N} q(x,t,x',t',\xi,\tau,0)\,dx = \int_{\mathbb{R}^N} q(x,t,x',t',\xi,\tau,s_0)\,dZ(x,t,s_0) + \int_0^{s_0}\int_{\mathbb{R}^N} d(qdZ). \tag{4.22}$$

We will estimate the two integrals on the right in (4.22).

Case (i): Assume k is even.

Consider the expression for $\Re Q(x,t,x',t',\xi,\tau,s')$ in (4.11). Suppose $\langle b(0,0),\xi^0\rangle>0$. Then in the region $s'\le 0$, recall estimate (4.16):

$$\Re Q(x,t,x',t',\xi,\tau,s') \le -C(|t'|^{k+1}+|s'|^{k+1})|\xi| - |x-x'-s'\Re\psi(x',t',s')|^2|\xi|,$$

for (ξ, τ) in a conic neighborhood Γ of $(\xi^0, 0)$, and (x, t) in a neighborhood W of $(0, 0)$ in $\mathbb{R}^{N+1}$. If $\langle b(0, 0), \xi^0\rangle < 0$, we work in the region $s' \geq 0$ and arrive at (4.16). For $(x, t) \in W$ and $(\xi, \tau) \in \Gamma$, inequality (4.16) leads to an estimate

$$\left|\int_{\mathbb{R}^{N+1}} q(x, t, x', t', \xi, \tau, s_0)\, dZ(x', t', s_0)\right| \leq e^{-C|\xi|}, \quad C > 0 \tag{4.23}$$

where s_0 is a small nonzero number whose sign depends on that of $\langle b(0, 0), \xi^0\rangle$.

To estimate the second integral on the right in (4.22), we will use Eq. (4.21) which consists of two kinds of terms. The first type consists of terms involving $\mathcal{L}^{\upsilon}(h)$ where h is an approximate solution and so such terms can be bounded by constant multiples of

$$s^n e^{\Re Q(x,t,x',t',\xi,\tau,s')} \quad \text{for } n = 1, 2, \ldots$$

and so using (4.16) which in particular implies that $\Re Q(x, t, x', t', \xi, \tau, s') \leq -C|\xi||s'|^{k+1}$, the integrals of these terms decay rapidly in (ξ, τ) in Γ for $(x, t) \in W$. The second type of terms involve derivatives of $\eta(x)$ and hence since $|x'|^2+|t'|^2 \geq r^2$ in the domains of integration, we get an exponential decay for the integrals. Since $\Xi_0(x, t, 0) = u(x, t)$, it follows from (4.22) that for each $k \in \mathbb{N}$, there is $C_k > 0$ such that

$$|\mathcal{F}u(x, t, \xi, \tau)| = \left|\int e^{\sqrt{-1}\langle(\xi,\tau),(x-x',t-t')\rangle - |(\xi,\tau)|(|x-x'|^2+|t-t'|^2)}\, dx'dt'\right| \leq \frac{C_k}{|\xi|^k}, \tag{4.24}$$

for (x, t) near $(0, 0)$ and (ξ, τ) in a conic neighborhood of $(\xi^0, 0)$. By a result in [6] (see the last part of the proof of Theorem 2.1 in [6]), we conclude that $(0, 0, \xi^0, 0)$ is not in the C^∞ wave front set of $u(x, t)$.

Case (ii): Assume k is odd. We will use two regions of integration Ω_1 and Ω_2 defined as follows:

$$\begin{aligned}\Omega_1 = &\{(x', t', s') : |x'| \leq 2r^2, |t'|^2 \leq 2r^2, t'^2 + s'^2 \geq \epsilon^2, 0 \leq s' \leq t'\}\\ &\cup \{(x', t', s') : |x'|^2 \leq 2r^2, |t'|^2 \leq 2r^2, t' \leq s' \leq 0\}\\ &\cup \{(x', t', s') : |x'|^2 \leq 2r^2, t'^2 + s'^2 \leq \epsilon^2, s' \leq t', s' \leq 0\}\end{aligned}$$

where $0 < \epsilon < r$.

$$\begin{aligned}\Omega_2 = &\{(x', t', s') : |x'| \leq 2r^2, |t'|^2 \leq 2r^2, 0 \leq s' \leq t'\}\\ &\cup \{(x', t', s') : |x'|^2 \leq 2r^2, |t'|^2 \leq 2r^2, t' \leq s' \leq 0, t'^2 + s'^2 \geq \epsilon^2\}\\ &\cup \{(x', t', s') : |x'|^2 \leq 2r^2, t'^2 + s'^2 \leq \epsilon^2, t' \leq s', s' \geq 0\}.\end{aligned}$$

We will estimate the FBI transform of ηu,

$$\begin{aligned}\mathcal{F}(\eta u)(x,t,\xi,\tau) &= \int_{\mathbb{R}^{N+1}} e^{\sqrt{-1}\langle(\xi,\tau),x-x',t-t'\rangle-|(\xi,\tau)|(|x-x'|^2+|t-t'|^2)}\eta(x',t')u(x',t')\,dx'dt'\\ &= \int_{\mathbb{R}^{N+1}} q(x,t,x',t',\xi,\tau,0)\,dx'dt'\end{aligned}$$

by using the region Ω_1 when $t \leq 0$ and the region Ω_2 when $t \geq 0$. Recall that

$$q(x,t,x',t',\xi,\tau,s') = \eta(x',t')\,\Xi_0^v(x',t',s')e^{Q(x,t,x',t',\xi,\tau,s')}.$$

Suppose now $t \leq 0$. Let

$$\begin{aligned}\mathcal{C} &= \{(x',s',s') : \frac{\epsilon}{\sqrt{2}} \leq |s'| \leq 2r^2,\ |x'| \leq 2r^2\} \cup \{(x',t',s') : t'^2 + s'^2\\ &= \epsilon^2,\ s' \leq t',\ |x'|^2 \leq 2r^2\} = \mathcal{C}_1 \cup \mathcal{C}_2.\end{aligned}$$

By Stoke's theorem,

$$\mathcal{F}(\eta u)(x,t,\xi,\tau) = \int_{\mathcal{C}} q(x,t,x',t',\xi,\tau,s')\,dZ + \int_{\Omega_1} d(q(x,t,x',t',\xi,\tau,s')\,dZ) \tag{4.25}$$

where we have used the vanishing of η on part of the boundary of Ω_1. We will first show that the first integral on the right in (4.25) decays exponentially for x and t small, $t \leq 0$, and (ξ,τ) in a conic neighborhood of $(\xi^0,0)$. We begin by estimating $\Re Q(x,t,x',t',\xi,\tau,s')$ on $\mathcal{C}$ at $(\xi,\tau) = (\xi^0,0)$. Without loss of generality, we may assume that $\langle \xi^0, b(0,0)\rangle = -1$. Note that from its expression in (4.11), modulo a term of order $O(|(x',t')|(t'^{k+1}+s'^{k+1}) + |t'|^{k+2} + |s'|^{k+2})$,

$$\Re Q(x,t,x',t',\xi^0,0) = s'p(t',s') - |\xi^0|(|x-x'-s'\Re\psi(x'.t',s')|^2 + |t-(t'-s')|^2). \tag{4.26}$$

On the set $\mathcal{C}_1$, we have

$$s'p(s',s') = s'^{k+1}p(1,1) = -\frac{s'^{k+1}}{k+1} \leq -\frac{\epsilon^{k+1}}{(k+1)(\sqrt{2})^{k+1}}. \tag{4.27}$$

To estimate on the set $\mathcal{C}_2$, we define the subregions:

$$D_1 = \{(x',t',s') \in \Omega_1 : s' \geq 0\},\ \ D_2 = \{(x',t',s') \in \Omega_1 : t' \geq 0, s' \leq 0\}, \text{ and}$$

$$D_3 = \{(x',t',s') \in \Omega_1 : s' \leq t' \leq 0\}.$$

Consider now a point $(x',t',s') \in \mathcal{C}_2 \cap D_1$. We can write $s' = at'$, for some $0 \leq a \leq 1$ and we know that $t'^2 + s'^2 = \epsilon^2$, $0 \leq s' \leq t'$. From (4.12), we have

$$s'p(t', at') = \frac{s'((1-a)^{k+1}-1)}{a(k+1)} t'^k = \left(\frac{(1-a)^{k+1}-1}{k+1}\right) t'^{k+1}. \tag{4.28}$$

Since $t \leq 0$, and $t' - s' \geq 0$, we also have:

$$(t - (t' - s'))^2 \geq (t' - s')^2 = (1-a)^2 t'^2. \tag{4.29}$$

Since t' is bounded away from zero in D_1, from (4.28) and (4.29) we conclude that for some $C > 0$,

$$\Re Q(x, t, \xi, \tau, x', t', s') \leq -C|\xi| \tag{4.30}$$

for $(x', t', s') \in \mathcal{C}_2 \cap D_1$, (x, t) near $(0, 0)$, $t \leq 0$, and (ξ, τ) in a conic neighborhood of $(\xi^0, 0)$. Consider next a point $(x', t', s') \in \mathcal{C}_2 \cap D_2$. We have $s' \leq 0 \leq t'$ and since $t \leq 0$,

$$(t - (t' - s'))^2 \geq (t' - s')^2 \geq \frac{t'^2 + s'^2}{2} = \frac{\epsilon^2}{2} \tag{4.31}$$

while for some $M > 0$,

$$|s'p(s', t')| \leq M(|t'|^{k+1} + |s'|^{k+1}) \leq M\epsilon^{k+1}. \tag{4.32}$$

When $k > 1$ (then $k \geq 3$), from (4.31) and (4.32) it follows that there exists a constant $C_1 > 0$ such that

$$\Re Q(x, t, \xi, \tau, x', t', s') \leq -C_1|\xi| \tag{4.33}$$

for $(x', t', s') \in \mathcal{C}_2 \cap D_2$, (x, t) near $(0, 0)$, $t \leq 0$, and (ξ, τ) in a conic neighborhood of $(\xi^0, 0)$. Next let $(x', t', s') \in \mathcal{C}_2 \cap D_3$. Then $s' \leq t' \leq 0$ and $t' = as'$, $0 \leq a \leq 1$. From (4.12),

$$s'p(t', s') = s'p\left(t', \frac{t'}{a}\right) = as'\left(\frac{(1-a)^{k+1}-1}{k+1}\right) t'^k = \left(\frac{(1-a)^{k+1}-a^{k+1}}{k+1}\right) s'^{k+1}. \tag{4.34}$$

Since $t \leq 0$, we also have

$$(t - (t' - s'))^2 \geq (1-a)s'^2. \tag{4.35}$$

Since $|s'|$ is bounded away from zero in D_3, (4.34) and (4.35) imply that when $k > 1$, there exists a constant $C > 0$ such that

$$\Re Q(x, t, \xi, \tau, x', t', s') \leq -C|\xi| \tag{4.36}$$

for $(x', t', s') \in \mathcal{C}_2 \cap D_3$, (x, t) near $(0, 0)$, $t \leq 0$, and (ξ, τ) in a conic neighborhood of $(\xi^0, 0)$. From (4.27), (4.30), (4.33), and (4.36) we conclude that there exists a constant $C > 0$ such that

$$\Re Q(x, t, \xi, \tau, x', t', s') \leq -C|\xi| \tag{4.37}$$

for $(x', t', s') \in \mathcal{C}$, (x, t) near $(0, 0)$, $t \leq 0$, and (ξ, τ) in a conic neighborhood of $(\xi^0, 0)$. The latter implies that for some constants $C, C' > 0$,

$$\left| \int_{\mathcal{C}} q(x, t, x', t', \xi, \tau, s')\, dZ \right| \leq Ce^{-C'|\xi|} \tag{4.38}$$

for (x, t) near $(0, 0)$, $t \leq 0$, and (ξ, τ) in a conic neighborhood of $(\xi^0, 0)$.

We next estimate the second integral on the right in (4.25). Consider first a point $(x', t', s') \in D_1$. Using (4.28), setting $s' = at'$, $0 \leq a \leq 1$, we get

$$s' p(t', at') = \frac{s'((1-a)^{k+1} - 1)}{a(k+1)} t'^k = \frac{(1-a)^{k+1} - 1}{(k+1)a^{k+1}} s'^{k+1} \tag{4.39}$$

and hence since

$$\lim_{a \to 0} \frac{(1-a)^{k+1} - 1}{a^{k+1}} = -\infty,$$

there is a constant $C > 0$, such that on D_1,

$$s' p(t', s') \leq -C|s'|^{k+1}. \tag{4.40}$$

This means that there is a constant $C > 0$ such that for (x, t) near $(0, 0)$, $t \leq 0$, and (ξ, τ) in a conic neighborhood of $(\xi^0, 0)$,

$$\Re Q(x, t, x', t', \xi, \tau, s') \leq -C|s'|^{k+1}|\xi|, \quad \text{for } (x', t', s') \in D_1. \tag{4.41}$$

Let $(x', t', s') \in D_2$. We recall from (4.31) and (4.32) that

$$(t-(t'-s'))^2 \geq (t'-s')^2 \geq \frac{t'^2 + s'^2}{2}, \quad \text{and for some } M > 0 \ |s' p(s', t')| \leq M(|t'|^{k+1} + |s'|^{k+1}) \tag{4.42}$$

and hence when $k > 1$, there exists a constant $C > 0$ such that

$$\Re Q(x, t, \xi, \tau, x', t', s') \leq -C|s'|^{k+1}|\xi| \tag{4.43}$$

for $(x', t', s') \in D_2$, (x, t) near $(0, 0)$, $t \leq 0$, and (ξ, τ) in a conic neighborhood of $(\xi^0, 0)$. Consider now a point $(x', t', s') \in D_3$. We use (4.34) and (4.35) where we had

$$s' p(t', s') = \left(\frac{(1-a)^{k+1} - a^{k+1}}{k+1} \right) s'^{k+1} \text{ and } (t - (t' - s'))^2 \geq (1-a)s'^2. \tag{4.44}$$

These inequalities imply that when $k > 1$, for some $C > 0$,

$$\Re Q(x, t, \xi, \tau, x', t', s') \leq -C|s'|^{k+1}|\xi| \tag{4.45}$$

for $(x', t', s') \in D_3$, (x, t) near $(0, 0)$, $t \leq 0$, and (ξ, τ) in a conic neighborhood of $(\xi^0, 0)$. Finally, we consider a point $(x', t', s') \in D_4 = \{(x', t', s') \in \Omega_1 : t' \leq s' \leq 0\}$. Setting $s' = at'$, $0 \leq a \leq 1$ as in (4.39), we have

$$s' p(t', at') = \frac{(1-a)^{k+1} - 1}{(k+1)a^{k+1}} s'^{k+1}$$

and as in (4.40), we get

$$s' p(t', s') \leq -C|s'|^{k+1} \text{ for some } C > 0.$$

This implies that there is a constant $C > 0$ such that for (x, t) near $(0, 0)$, $t \leq 0$, and (ξ, τ) in a conic neighborhood of $(\xi^0, 0)$,

$$\Re Q(x, t, x', t', \xi, \tau, s') \leq -C|s'|^{k+1}|\xi|, \quad \text{for } (x', t', s') \in D_4. \tag{4.46}$$

From (4.41), (4.43), (4.45), and (4.46), it follows that for some constant $C > 0$,

$$\Re Q(x, t, x', t', \xi, \tau, s') \leq -C|s'|^{k+1}|\xi|, \quad \text{for } (x', t', s') \in \Omega_1, \tag{4.47}$$

(x, t) near $(0, 0)$, $t \leq 0$, and (ξ, τ) in a conic neighborhood of $(\xi^0, 0)$. This implies that the second integral on the right in (4.25), namely, $\int_{\Omega_1} d(q\, dZ)$ decays rapidly. That is, when $k > 1$, for each $n \in \mathbb{N}$, there is a constant $C_n > 0$ such that:

$$\left| \int_{\Omega_1} d(q(x, t, x', t', \xi, \tau, s')\, dZ) \right| \leq \frac{C_n}{|\xi|^n} \tag{4.48}$$

for (ξ, τ) in a conic neighborhood of $(\xi^0, 0)$, and (x, t) varying near $(0, 0)$ with $t \leq 0$.

To treat the case when $k = 1$, observe that the assumption that $k > 1$ was used in arriving at (4.33) (from (4.31) and (4.32)), (4.36) (from (4.34) and (4.35)), (4.43) (from (4.42)), and (4.45) (from (4.44)). By using an FBI transform with a sufficiently large constant κ as in the proof of Theorem 2.4, we can deduce (4.33), (4.36), (4.43), and (4.45). Thus (4.38) and (4.48) also hold when $k = 1$.

Suppose now $t \geq 0$. Let

$$\begin{aligned} \mathcal{C}' = &\{(x', s', s') : \frac{\epsilon}{\sqrt{2}} \leq |s'| \leq 2r^2,\ |x'| \leq 2r^2\} \\ &\cup \{(x', t', s') : t'^2 + s'^2 = \epsilon^2,\ s' \geq t',\ |x'|^2 \leq 2r^2\} = \mathcal{C}'_1 \cup \mathcal{C}'_2. \end{aligned}$$

By Stoke's theorem,

$$\mathcal{F}(\eta u)(x,t,\xi,\tau)=\int_{\mathcal{C}'}q(x,t,x',t',\xi,\tau,s')\,dZ+\int_{\Omega_2}d(q(x,t,x',t',\xi,\tau,s')\,dZ). \tag{4.49}$$

As before, we will first show that the first integral on the right in (4.49) decays exponentially for x and t small, $t\ge 0$, and (ξ,τ) in a conic neighborhood of $(\xi^0,0)$. We begin by estimating $\Re Q(x,t,x',t',\xi,\tau,s')$ on $\mathcal{C}'$ at $(\xi,\tau)=(\xi^0,0)$. Recall that we have the estimate (4.27) on the set $\mathcal{C}_1=\mathcal{C}_1'$. To estimate $\Re Q$ on the set $\mathcal{C}_2'$, we define the subregions:

$$D_2'=\{(x',t',s')\in\Omega_2: t'\ge 0, t'\le s'\},\ D_3'=\{(x',t',s')\in\Omega_2: t'\le 0, s'\ge 0\},\text{ and}$$

$$D_4'=\{(x',t',s')\in\Omega_2: s'\le 0\}.$$

Consider now a point $(x',t',s')\in\mathcal{C}_2'\cap D_2'$. We can write $t'=as'$, for some $0\le a\le 1$ and we know that $t'^2+s'^2=\epsilon^2$. As we saw in (4.34), we have

$$s'p(t',s')=\left(\frac{(1-a)^{k+1}-a^{k+1}}{k+1}\right)s'^{k+1}$$

and since $t\ge 0$, and $t'-s'\le 0$, we also have

$$(t-(t'-s'))^2\ge(1-a)s'^2.$$

Since $|s'|$ is bounded away from zero in D_2', it follows that when $k>1$, there exists a constant $C>0$ such that

$$\Re Q(x,t,\xi,\tau,x',t',s')\le -C|\xi| \tag{4.50}$$

for $(x',t',s')\in\mathcal{C}_2'\cap D_2'$, (x,t) near $(0,0)$, $t\ge 0$, and (ξ,τ) in a conic neighborhood of $(\xi^0,0)$. If $(x',t',s')\in\mathcal{C}_2'\cap D_3'$, since $t\ge 0$ now, we have:

$$(t-(t'-s'))^2\ge(t'-s')^2\ge\frac{1}{2}(t'^2+s'^2)=\frac{\epsilon^2}{2}$$

while $|t'p(t',s')|\le M\epsilon^{k+1}$ for some $M>0$. Hence if $k>1$, for some $C>0$,

$$\Re Q(x,t,\xi,\tau,x',t',s')\le -C|\xi| \tag{4.51}$$

for $(x',t',s')\in\mathcal{C}_2'\cap D_3'$, (x,t) near $(0,0)$, $t\ge 0$, and (ξ,τ) in a conic neighborhood of $(\xi^0,0)$. Suppose next $(x',t',s')\in\mathcal{C}_2'\cap D_4'$. Then by (4.39),

$$s'p(t',s') = \frac{(1-a)^{k+1}-1}{(k+1)a^{k+1}}s'^{k+1}$$

which we can use for s' away from zero, and when s' is near zero, we can use

$$(t-(t'-s'))^2 \geq (t'-s')^2 \geq \frac{t'^2+s'^2}{2} = \epsilon^2.$$

Thus for some $C > 0$,

$$\Re Q(x,t,\xi,\tau,x',t',s') \leq -C|\xi| \tag{4.52}$$

for $(x',t',s') \in \mathcal{C}_2' \cap D_4'$, (x,t) near $(0,0)$, $t \geq 0$, and (ξ,τ) in a conic neighborhood of $(\xi^0,0)$ whenever $k > 1$.

From (4.27), (4.50)–(4.52) we conclude that there exists a constant $C > 0$ such that

$$\Re Q(x,t,\xi,\tau,x',t',s') \leq -C|\xi| \tag{4.53}$$

for $(x',t',s') \in \mathcal{C}'$, (x,t) near $(0,0)$, $t \geq 0$, and (ξ,τ) a conic neighborhood of $(\xi^0,0)$. The latter implies that when $k > 1$, for some constants $C_1, C_2 > 0$,

$$\left|\int_{\mathcal{C}'} q(x,t,x',t',\xi,\tau,s')\,dZ\right| \leq C_1 e^{-C_2|\xi|} \tag{4.54}$$

for (x,t) near $(0,0)$, $t \geq 0$, and (ξ,τ) in a conic neighborhood of $(\xi^0,0)$.

We next estimate the second integral on the right in (4.49). Consider first a point $(x',t',s') \in D_1' = \{(x',t',s') \in \Omega_2 : 0 \leq s' \leq t'\}$. This case was treated in (4.40) and so there is a constant $C > 0$ such that for (x,t) near $(0,0)$, $t \geq 0$, and (ξ,τ) in a conic neighborhood of $(\xi^0,0)$,

$$\Re Q(x,t,x',t',\xi,\tau,s') \leq -C|s'|^{k+1}|\xi|, \quad \text{for } (x',t',s') \in D_1'. \tag{4.55}$$

Let $(x',t',s') \in D_2'$. Then $0 \leq t' \leq s'$ and if $t' = as'$,

$$s'p(t',s') = \frac{(1-a)^{k+1}-a^{k+1}}{k+1}s'^{k+1}$$

and since $t \geq 0$, $(t-(t'-s'))^2 \geq (1-a)^2 s'^2$. Therefore, when $k > 1$, for some $C > 0$,

$$\Re Q(x,t,x',t',\xi,\tau,s') \leq -C|s'|^{k+1}|\xi|, \quad \text{for } (x',t',s') \in D_2' \tag{4.56}$$

for (x,t) near $(0,0)$, $t \geq 0$, and (ξ,τ) in a conic neighborhood of $(\xi^0,0)$. If $(x',t',s') \in D_3'$,

$$(t - (t' - s'))^2 \geq (t' - s')^2 \geq \frac{1}{2}(t'^2 + s'^2)$$

while $|t' p(t', s')| \leq M\epsilon^{k+1}$ for some $M > 0$. Hence if $k > 1$, for some $C > 0$,

$$\Re Q(x, t, \xi, \tau, x', t', s') \leq -C|\xi| \tag{4.57}$$

for $(x', t', s') \in \mathcal{D}'_3$, (x, t) near $(0, 0)$, $t \geq 0$, and (ξ, τ) a conic neighborhood of $(\xi^0, 0)$. Finally, if $(x', t', s') \in D'_4$, then

$$s' p(t', s') = \left(\frac{(1-a)^{k+1} - 1}{(k+1)a^{k+1}} \right) s'^{k+1} \quad s' = at'$$

while

$$(t - (t' - s'))^2 \geq \frac{t'^2 + s'^2}{2}$$

and so for $k > 1$,

$$\Re Q(x, t, \xi, \tau, x', t', s') \leq -C|\xi| \tag{4.58}$$

for $(x', t', s') \in \mathcal{D}'_4$, (x, t) near $(0, 0)$, $t \geq 0$, and (ξ, τ) a conic neighborhood of $(\xi^0, 0)$. From (4.55)–(4.58), it follows that for some constant $C > 0$,

$$\Re Q(x, t, x', t', \xi, \tau, s') \leq -C|s'|^{k+1}|\xi|, \quad \text{for } (x', t', s') \in \Omega_2, \tag{4.59}$$

(x, t) near $(0, 0)$, $t \geq 0$, and (ξ, τ) in a conic neighborhood of $(\xi^0, 0)$. This implies that the second integral on the right in (4.49), namely, $\int_{\Omega_2} d(q\, dZ)$ decays rapidly. That is, for each $n \in \mathbb{N}$, there is a constant $C_n > 0$ such that:

$$\left| \int_{\Omega_2} d(q(x, t, x', t', \xi, \tau, s')\, dZ) \right| \leq \frac{C_n}{|\xi|^n} \tag{4.60}$$

for (ξ, τ) in a conic neighborhood of $(\xi^0, 0)$, and (x, t) varying near $(0, 0)$ with $t \geq 0$. The case $k = 1$ is treated as in the situation when $t \leq 0$.

We conclude that $(\xi^0, 0)$ is not in the C^∞ wave front set of $u(x, t)$.

5 Examples

Example 5.1 Consider the linear equation

$$\frac{\partial u}{\partial t} + \sqrt{-1} t^k \frac{\partial u}{\partial x} = 0$$

where k is an odd integer. Let $u(x,t) = \left(x - \sqrt{-1}\frac{t^{k+1}}{k+1}\right)^{\frac{2k+3}{2}}$ where we have chosen a branch of the square root. By Theorem 2.4, at the origin, the direction $(0,0,1,0)$ is not in the analytic wave front set of $u(x,t)$. Using ellipticity, we therefore know that the C^∞ wave front set of $u(x,t)$ at the origin is contained in $\{(0,0,\xi,0) : \xi < 0\}$. On the other hand, since $u(x,t)$ is not smooth, it follows that at the origin, the C^∞ and hence the analytic wave front set equals $\{(0,0,\xi,0) : \xi < 0\}$. This shows that in general, in Theorems 2.4 and 2.5, conditions (2.12) and (2.16) can not be removed.

Example 5.2 Let $u(x,t)$ be a solution of the semilinear equation

$$\frac{\partial u}{\partial t} + \sqrt{-1}t^k a(x,t)\frac{\partial u}{\partial x} = g(x,t,u),\ \ \Re a(0,0) \neq 0,\ \ -T < t < T,\ x \in (a,b)$$

where $a(x,t)$ and $g(x,t,\zeta_0)$ are C^∞ functions and g is holomorphic in ζ_0. Then if $\xi \cdot \Re a(0,0) > 0$, the point $(0,0,\xi,0)$ is not in the C^∞ wave front set of $u(x,t)$.

When k is an even integer, then u is C^∞. These assertions follow from Theorem 2.5 and [7] (see also [2]). If $g(x,t,\zeta_0)$ is real analytic, we have a similar conclusion with "real analytic" replacing "C^∞".

We remark that, in general, as was shown in the works [8, 11, 13], the vector field $L = \frac{\partial}{\partial t} + \sqrt{-1}t^k a(x,t)\frac{\partial}{\partial x}$ with $\Re a(0,0) \neq 0$ can not be transformed by a diffeomorphism to a multiple of the generalized Mizohata operator $\frac{\partial}{\partial t} + \sqrt{-1}t^k\frac{\partial}{\partial x}$. In particular, for k odd, L may not be locally integrable.

Example 5.3 Let $u(x,t)$ be a solution of

$$\frac{\partial u}{\partial t} + \sqrt{-1}t^k u\frac{\partial u}{\partial x} = g(x,t,u),\ \ -T < t < T,\ x \in (a,b)$$

where $g(x,t,\zeta_0)$ is a C^∞ function that is holomorphic in ζ_0. For any $x_0 \in (a,b)$ and ξ, if $\xi\,\Re u(x_0,0) > 0$, then the point $(x_0,0,\xi,0)$ is not in the C^∞ wave front set of $u(x,t)$. If k is even, at any point $(x_0,0)$ where $\Re u(x_0,0) \neq 0$, $u(x,t)$ is C^∞. Again when $g(x,t,\zeta_0)$ is a real analytic function, we get the corresponding analyticity results for $u(x,t)$.

Example 5.4 Let $u(x,t)$ be a solution of

$$\frac{\partial u}{\partial t} + \sqrt{-1}t\left(\frac{\partial u}{\partial x}\right)^2 = g(x,t,u),\ \ -T < t < T,\ x \in (a,b)$$

where $g(x,t,\zeta_0)$ is a C^∞ function that is holomorphic in ζ_0. For any $x_0 \in (a,b)$ and ξ, if $\xi\,\Im u_x(x_0,0) > 0$, then the point $(x_0,0,\xi,0)$ is not in the C^∞ wave front set of $u(x,t)$. Again when $g(x,t,\zeta_0)$ is a real analytic function, the corresponding analyticity result holds.

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Towards the Green-Griffiths-Lang Conjecture

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In memory of M. Salah Baouendi

Abstract The Green-Griffiths-Lang conjecture stipulates that for every projective variety X of general type over $\mathbb{C}$, there exists a proper algebraic subvariety of X containing all non constant entire curves $f : \mathbb{C} \to X$. Using the formalism of directed varieties, we prove here that this assertion holds true in case X satisfies a strong general type condition that is related to a certain jet semistability property of the tangent bundle T_X. We then give a sufficient criterion for the Kobayashi hyperbolicity of an arbitrary directed variety (X, V).

Keywords Projective algebraic variety · Variety of general type · Entire curve · Jet bundle · Semple tower · Green-griffiths-lang conjecture · Holomorphic morse inequality · Semistable vector bundle · Kobayashi hyperbolic

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1 Introduction

The goal of this paper is to study the Green-Griffiths-Lang conjecture, as stated in [7, 10]. It is useful to work in a more general context and consider the category of directed projective manifolds (or varieties). Since the basic problems we deal with are birationally invariant, the varieties under consideration can always be replaced by nonsingular models. A directed projective manifold is a pair (X, V) where X is a projective manifold equipped with an analytic linear subspace $V \subset T_X$, i.e. a closed irreducible complex analytic subset V of the total space of T_X, such that each fiber

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$V_x = V \cap T_{X,x}$ is a complex vector space [If X is not irreducible, V should rather be assumed to be irreducible merely over each component of X, but we will hereafter assume that our varieties are irreducible]. A morphism $\Phi : (X, V) \to (Y, W)$ in the category of directed manifolds is an analytic map $\Phi : X \to Y$ such that $\Phi_* V \subset W$. We refer to the case $V = T_X$ as being the *absolute case*, and to the case $V = T_{X/S} = \mathrm{Ker}\, d\pi$ for a fibration $\pi : X \to S$, as being the *relative case*; V may also be taken to be the tangent space to the leaves of a singular analytic foliation on X, or maybe even a non integrable linear subspace of T_X.

We are especially interested in *entire curves* that are tangent to V, namely non constant holomorphic morphisms $f : (\mathbb{C}, T_{\mathbb{C}}) \to (X, V)$ of directed manifolds. In the absolute case, these are just arbitrary entire curves $f : \mathbb{C} \to X$. The Green-Griffiths-Lang conjecture, in its strong form, stipulates

1.1 GGL conjecture Let X be a projective variety of general type. Then there exists a proper algebraic variety $Y \subsetneq X$ such that every entire curve $f : \mathbb{C} \to X$ satisfies $f(\mathbb{C}) \subset Y$.

[The weaker form would state that entire curves are algebraically degenerate, so that $f(\mathbb{C}) \subset Y_f \subsetneq X$ where Y_f might depend on f]. The smallest admissible algebraic set $Y \subset X$ is by definition the *entire curve locus* of X, defined as the Zariski closure

$$\mathrm{ECL}(X) = \overline{\bigcup_f f(\mathbb{C})}^{\mathrm{Zar}}. \tag{1.1}$$

If $X \subset \mathbb{P}^N_{\mathbb{C}}$ is defined over a number field $\mathbb{K}_0$ (i.e. by polynomial equations with equations with coefficients in $\mathbb{K}_0$) and $Y = \mathrm{ECL}(X)$, it is expected that for every number field $\mathbb{K} \supset \mathbb{K}_0$ the set of $\mathbb{K}$-points in $X(\mathbb{K}) \smallsetminus Y$ is finite, and that this property characterizes $\mathrm{ECL}(X)$ as the smallest algebraic subset Y of X that has the above property for all $\mathbb{K}$ [10]. This conjectural arithmetical statement would be a vast generalization of the Mordell-Faltings theorem, and is one of the strong motivations to study the geometric GGL conjecture as a first step.

1.2 Problem (*generalized GGL conjecture*) Let (X, V) be a projective directed manifold. Find geometric conditions on V ensuring that all entire curves $f : (\mathbb{C}, T_{\mathbb{C}}) \to (X, V)$ are contained in a proper algebraic subvariety $Y \subsetneq X$. Does this hold when (X, V) is of general type, in the sense that the canonical sheaf K_V is big ?

As above, we define the entire curve locus set of a pair (X, V) to be the smallest admissible algebraic set $Y \subset X$ in the above problem, i.e.

$$\mathrm{ECL}(X, V) = \overline{\bigcup_{f:(\mathbb{C},T_{\mathbb{C}})\to(X,V)} f(\mathbb{C})}^{\mathrm{Zar}}. \tag{1.2}$$

We say that (X, V) is *Brody hyperbolic* if $\mathrm{ECL}(X, V) = \emptyset$; as is well-known, this is equivalent to Kobayashi hyperbolicity whenever X is compact.

In case V has no singularities, the *canonical sheaf* K_V is defined to be $(\det \mathcal{O}(V))^*$ where $\mathcal{O}(V)$ is the sheaf of holomorphic sections of V, but in general this naive definition would not work. Take for instance a generic pencil of elliptic curves $\lambda P(z) + \mu Q(z) = 0$ of degree 3 in $\mathbb{P}^2_{\mathbb{C}}$, and the linear space V consisting of the tangents to the fibers of the rational map $\mathbb{P}^2_{\mathbb{C}} \dashrightarrow \mathbb{P}^1_{\mathbb{C}}$ defined by $z \mapsto Q(z)/P(z)$. Then V is given by

$$0 \longrightarrow \mathcal{O}(V) \longrightarrow \mathcal{O}(T_{\mathbb{P}^2_{\mathbb{C}}}) \xrightarrow{PdQ-QdP} \mathcal{O}_{\mathbb{P}^2_{\mathbb{C}}}(6) \otimes \mathcal{J}_S \longrightarrow 0$$

where $S = \mathrm{Sing}(V)$ consists of the 9 points $\{P(z) = 0\} \cap \{Q(z) = 0\}$, and $\mathcal{J}_S$ is the corresponding ideal sheaf of S. Since $\det \mathcal{O}(T_{\mathbb{P}^2}) = \mathcal{O}(3)$, we see that $(\det(\mathcal{O}(V))^* = \mathcal{O}(3)$ is ample, thus Problem 1.2 would not have a positive answer (all leaves are elliptic or singular rational curves and thus covered by entire curves). An even more "degenerate" example is obtained with a generic pencil of conics, in which case $(\det(\mathcal{O}(V))^* = \mathcal{O}(1)$ and $\#S = 4$.

If we want to get a positive answer to Problem 1.2, it is therefore indispensable to give a definition of K_V that incorporates in a suitable way the singularities of V; this will be done in Definition 2.1 (see also Proposition 2.2). The goal is then to give a positive answer to Problem 1.2 under some possibly more restrictive conditions for the pair (X, V). These conditions will be expressed in terms of the tower of Semple jet bundles

$$(X_k, V_k) \to (X_{k-1}, V_{k-1}) \to \cdots \to (X_1, V_1) \to (X_0, V_0) := (X, V) \tag{1.3}$$

which we define more precisely in Sect. 2, following [1]. It is constructed inductively by setting $X_k = P(V_{k-1})$ (projective bundle of *lines* of V_{k-1}), and all V_k have the same rank $r = \mathrm{rank}\, V$, so that $\dim X_k = n + k(r-1)$ where $n = \dim X$. Entire curve loci have their counterparts for all stages of the Semple tower, namely, one can define

$$\mathrm{ECL}_k(X, V) = \overline{\bigcup_{f:(\mathbb{C},T_{\mathbb{C}})\to(X,V)} f_{[k]}(\mathbb{C})}^{\mathrm{Zar}} \tag{1.4}$$

where $f_{[k]} : (\mathbb{C}, T_{\mathbb{C}}) \to (X_k, V_k)$ is the k-jet of f. These are by definition algebraic subvarieties of X_k, and if we denote by $\pi_{k,\ell} : X_k \to X_\ell$ the natural projection from X_k to X_ℓ, $0 \le \ell \le k$, we get immediately

$$\pi_{k,\ell}(\mathrm{ECL}_k(X, V)) = \mathrm{ECL}_\ell(X, V), \qquad \mathrm{ECL}_0(X, V) = \mathrm{ECL}(X, V). \tag{1.5}$$

Let $\mathcal{O}_{X_k}(1)$ be the tautological line bundle over X_k associated with the projective structure. We define the k-stage Green-Griffiths locus of (X, V) to be

$$\mathrm{GG}_k(X, V) = \overline{(X_k \smallsetminus \Delta_k) \cap \bigcap_{m\in\mathbb{N}} \Big(\text{base locus of } \mathcal{O}_{X_k}(m) \otimes \pi_{k,0}^* A^{-1}\Big)} \tag{1.6}$$

where A is any ample line bundle on X and $\Delta_k = \bigcup_{2\le\ell\le k} \pi_{k,\ell}^{-1}(D_\ell)$ is the union of "vertical divisors" (see Sect. 2; the vertical divisors play no role and have to be removed in this context). Clearly, $\mathrm{GG}_k(X, V)$ does not depend on the choice of A. The basic vanishing theorem for entire curves (cf. [1, 7, 16]) asserts that every entire curve $f : (\mathbb{C}, T_\mathbb{C}) \to (X, V)$ satisfies all differential equations $P(f) = 0$ arising from sections $P \in H^0(X_k, \mathcal{O}_{X_k}(m) \otimes \pi_{k,0}^* A^{-1})$, hence

$$\mathrm{ECL}_k(X, V) \subset \mathrm{GG}_k(X, V). \tag{1.7}$$

(For this, one uses the fact that $f_{[k]}(\mathbb{C})$ is not contained in any component of Δ_k, cf. [1]). It is therefore natural to define the global Green-Griffiths locus of (X, V) to be

$$\mathrm{GG}(X, V) = \bigcap_{k\in\mathbb{N}} \pi_{k,0}\left(\mathrm{GG}_k(X, V)\right). \tag{1.8}$$

By (1.5) and (1.7) we infer that

$$\mathrm{ECL}(X, V) \subset \mathrm{GG}(X, V). \tag{1.9}$$

The main result of [4] (Theorem 2.37 and Corollary 4.4) implies the following useful information:

1.3 Theorem *Assume that (X, V) is of "general type", i.e. that the canonical sheaf K_V is big on X. Then there exists an integer k_0 such that $\mathrm{GG}_k(X, V)$ is a proper algebraic subset of X_k for $k \ge k_0$ [though $\pi_{k,0}(\mathrm{GG}_k(X, V))$ might still be equal to X for all k].*

In fact, if F is an invertible sheaf on X such that $K_V \otimes F$ is big, the probabilistic estimates of [4, Corollarys 2.38 and 4.4] produce sections of

$$\mathcal{O}_{X_k}(m) \otimes \pi_{k,0}^*\mathcal{O}\Big(\frac{m}{kr}\Big(1 + \frac{1}{2} + \cdots + \frac{1}{k}\Big)F\Big) \tag{1.10}$$

for $m \gg k \gg 1$. The (long and involved) proof uses a curvature computation and singular holomorphic Morse inequalities to show that the line bundles involved in (0.11) are big on X_k for $k \gg 1$. One applies this to $F = A^{-1}$ with A ample on X to produce sections and conclude that $\mathrm{GG}_k(X, V) \subsetneq X_k$.

Thanks to (1.9), the GGL conjecture is satisfied whenever $\mathrm{GG}(X, V) \subsetneq X$. By [5], this happens for instance in the absolute case when X is a generic hypersurface of degree $d \ge 2^{n^5}$ in $\mathbb{P}^{n+1}$ (see also [13] for better bounds in low dimensions, and [14, 15]). However, as already mentioned in [10], very simple examples show that one can have $\mathrm{GG}(X, V) = X$ even when (X, V) is of general type, and this already occurs in the absolute case as soon as $\dim X \ge 2$. A typical example is a product of directed manifolds

$$(X, V) = (X', V') \times (X'', V''), \qquad V = \mathrm{pr}'^* V' \oplus \mathrm{pr}''^* V''. \tag{1.11}$$

The absolute case $V = T_X$, $V' = T_{X'}$, $V'' = T_{X''}$ on a product of curves is the simplest instance. It is then easy to check that $\mathrm{GG}(X, V) = X$, cf. (3.2). Diverio and Rousseau [6] have given many more such examples, including the case of indecomposable varieties (X, T_X), e.g. Hilbert modular surfaces, or more generally compact quotients of bounded symmetric domains of rank ≥ 2. The problem here is the failure of some sort of stability condition that is introduced in Sect. 4. This leads to a somewhat technical concept of more manageable directed pairs (X, V) that we call *strongly of general type*, see Definition 4.1. Our main result can be stated

1.4 Theorem (partial solution to the generalized GGL conjecture) *Let (X, V) be a directed pair that is strongly of general type. Then the Green-Griffiths-Lang conjecture holds true for (X, V), namely $\mathrm{ECL}(X, V)$ is a proper algebraic subvariety of X.*

The proof proceeds through a complicated induction on $n = \dim X$ and $k = \mathrm{rank}\, V$, which is the main reason why we have to introduce directed varieties, even in the absolute case. An interesting feature of this result is that the conclusion on $\mathrm{ECL}(X, V)$ is reached without having to know anything about the Green-Griffiths locus $\mathrm{GG}(X, V)$, even a posteriori. Nevetheless, this is not yet enough to confirm the GGL conjecture. Our hope is that pairs (X, V) that are of general type without being strongly of general type—and thus exhibit some sort of "jet-instability"—can be investigated by different methods, e.g. by the diophantine approximation techniques of McQuillan [11]. However, Theorem 1.4 provides a sufficient criterion for Kobayashi hyperbolicity [8, 9], thanks to the following concept of algebraic jet-hyperbolicity.

1.5 Definition A directed variety (X, V) will be said to be algebraically jet-hyperbolic if the induced directed variety structure (Z, W) on every irreducible algebraic variety Z of X such that $\mathrm{rank}\, W \geq 1$ has a desingularization that is strongly of general type [see Sects. 3 and 5 for the definition of induced directed structures and further details]. We also say that a projective manifold X is algebraically jet-hyperbolic if (X, T_X) is.

In this context, Theorem 1.4 yields the following connection between algebraic jet-hyperbolicity and the analytic concept of Kobayashi hyperbolicity.

1.6 Theorem *Let (X, V) be a directed variety structure on a projective manifold X. Assume that (X, V) is algebraically jet-hyperbolic. Then (X, V) is Kobayashi hyperbolic.*

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2 Semple Jet Bundles and Associated Canonical Sheaves

Let (X, V) be a directed projective manifold and $r = \operatorname{rank} V$, that is, the dimension of generic fibers. Then V is actually a holomorphic subbundle of T_X on the complement $X \smallsetminus \operatorname{Sing}(V)$ of a certain minimal analytic set $\operatorname{Sing}(V) \subsetneq X$ of codimension ≥ 2, called hereafter the singular set of V. If $\mu : \widehat{X} \to X$ is a proper modification (a composition of blow-ups with smooth centers, say), we get a directed manifold $(\widehat{X}, \widehat{V})$ by taking $\widehat{V}$ to be the closure of $\mu_*^{-1}(V')$, where $V' = V_{|X'}$ is the restriction of V over a Zariski open set $X' \subset X \smallsetminus \operatorname{Sing}(V)$ such that $\mu : \mu^{-1}(X') \to X'$ is a biholomorphism. We will be interested in taking modifications realized by iterated blow-ups of certain nonsingular subvarieties of the singular set $\operatorname{Sing}(V)$, so as to eventually "improve" the singularities of V ; outside of $\operatorname{Sing}(V)$ the effect of blowing-up will be irrelevant, as one can see easily. Following [4], the canonical sheaf K_V is defined as follows.

2.1 Definition For any directed pair (X, V) with X nonsingular, we define K_V to be the rank 1 analytic sheaf such that

$$K_V(U) = \text{sheaf of locally bounded sections of } \mathcal{O}_X(\Lambda^r V'^*)(U \cap X')$$

where $r = \operatorname{rank}(V)$, $X' = X \smallsetminus \operatorname{Sing}(V)$, $V' = V_{|X'}$, and "bounded" means bounded with respect to a smooth hermitian metric h on T_X.

For $r = 0$, one can set $K_V = \mathcal{O}_X$, but this case is trivial: clearly $\operatorname{ECL}(X, V) = \emptyset$. The above definition of K_V may look like an analytic one, but it can easily be turned into an equivalent algebraic definition:

2.2 Proposition *Consider the natural morphism $\mathcal{O}(\Lambda^r T_X^*) \to \mathcal{O}(\Lambda^r V^*)$ where* $r = \operatorname{rank} V$ [$\mathcal{O}(\Lambda^r V^*)$ *being defined here as the quotient of* $\mathcal{O}(\Lambda^r T_X^*)$ *by r-forms that have zero restrictions to* $\mathcal{O}(\Lambda^r V^*)$ *on* $X \smallsetminus \operatorname{Sing}(V)$]. *The bidual* $\mathcal{L}_V = \mathcal{O}_X(\Lambda^r V^*)^{**}$ *is an invertible sheaf, and our natural morphism can be written*

$$\mathcal{O}(\Lambda^r T_X^*) \to \mathcal{O}(\Lambda^r V^*) = \mathcal{L}_V \otimes \mathcal{J}_V \subset \mathcal{L}_V \tag{2.1}$$

where $\mathcal{J}_V$ is a certain ideal sheaf of $\mathcal{O}_X$ whose zero set is contained in $\operatorname{Sing}(V)$ *and the arrow on the left is surjective by definition. Then*

$$K_V = \mathcal{L}_V \otimes \overline{\mathcal{J}}_V \tag{2.2}$$

where $\overline{\mathcal{J}}_V$ is the integral closure of $\mathcal{J}_V$ in $\mathcal{O}_X$. In particular, K_V is always a coherent sheaf.

Proof Let (u_k) be a set of generators of $\mathcal{O}(\Lambda^r V^*)$ obtained (say) as the images of a basis $(dz_I)_{|I|=r}$ of $\Lambda^r T_X^*$ in some local coordinates near a point $x \in X$. Write $u_k = g_k \ell$ where ℓ is a local generator of $\mathcal{L}_V$ at x. Then $\mathcal{J}_V = (g_k)$ by definition. The boundedness condition expressed in Definition 2.1 means that we take sections of the form $f\ell$ where f is a holomorphic function on $U \cap X'$ (and U a neighborhood of x), such that

$$|f| \le C \sum |g_k| \tag{2.3}$$

for some constant $C > 0$. But then f extends holomorphically to U into a function that lies in the integral closure $\overline{\mathcal{J}}_V$, and the latter is actually characterized analytically by condition (2.3). This proves Proposition 2.2. □

By blowing-up $\mathcal{J}_V$ and taking a desingularization $\widehat{X}$, one can always find a *log-resolution* of $\mathcal{J}_V$ (or K_V), i.e. a modification $\mu : \widehat{X} \to X$ such that $\mu^*\mathcal{J}_V \subset \mathcal{O}_{\widehat{X}}$ is an invertible ideal sheaf (hence integrally closed); it follows that $\mu^*\overline{\mathcal{J}}_V = \mu^*\mathcal{J}_V$ and $\mu^* K_V = \mu^*\mathcal{L}_V \otimes \mu^*\mathcal{J}_V$ are invertible sheaves on $\widehat{X}$. Notice that for any modification $\mu' : (X', V') \to (X, V)$, there is always a well defined natural morphism

$$\mu'^* K_V \to K_{V'} \tag{2.4}$$

(though it need not be an isomorphism, and $K_{V'}$ is possibly non invertible even when μ' is taken to be a log-resolution of K_V). Indeed $(\mu')_* = d\mu' : V' \to \mu^* V$ is continuous with respect to ambient hermitian metrics on X and X', and going to the duals reverses the arrows while preserving boundedness with respect to the metrics. If $\mu'' : X'' \to X'$ provides a simultaneous log-resolution of $K_{V'}$ and $\mu'^* K_V$, we get a non trivial morphism of invertible sheaves

$$(\mu' \circ \mu'')^* K_V = \mu''^* \mu'^* K_V \longrightarrow \mu''^* K_{V'}, \tag{2.5}$$

hence the bigness of $\mu'^* K_V$ with imply that of $\mu''^* K_{V'}$. This is a general principle that we would like to refer to as the "monotonicity principle" for canonical sheaves: one always get more sections by going to a higher level through a (holomorphic) modification.

2.3 Definition We say that the rank 1 sheaf K_V is "big" if the invertible sheaf $\mu^* K_V$ is big in the usual sense for any log resolution $\mu : \widehat{X} \to X$ of K_V. Finally, we say that (X, V) is of *general type* if there exists a modification $\mu' : (X', V') \to (X, V)$ such that $K_{V'}$ is big; any higher blow-up $\mu'' : (X'', V'') \to (X', V')$ then also yields a big canonical sheaf by (2.4).

Clearly, "general type" is a birationally (or bimeromorphically) invariant concept, by the very definition. When $\dim X = n$ and $V \subset T_X$ is a subbundle of rank $r \ge 1$, one constructs a tower of "Semple k-jet bundles" $\pi_{k,k-1} : (X_k, V_k) \to (X_{k-1}, V_{k-1})$ that are $\mathbb{P}^{r-1}$-bundles, with $\dim X_k = n + k(r-1)$ and $\mathrm{rank}(V_k) = r$. For this, we take $(X_0, V_0) = (X, V)$, and for every $k \ge 1$, we set inductively $X_k := P(V_{k-1})$ and

$$V_k := (\pi_{k,k-1})_*^{-1} \mathcal{O}_{X_k}(-1) \subset T_{X_k},$$

where $\mathcal{O}_{X_k}(1)$ is the tautological line bundle on X_k, $\pi_{k,k-1} : X_k = P(V_{k-1}) \to X_{k-1}$ the natural projection and $(\pi_{k,k-1})_* = d\pi_{k,k-1} : T_{X_k} \to \pi_{k,k-1}^* T_{X_{k-1}}$ its differential (cf. [1]). In other terms, we have exact sequences

$$0 \longrightarrow T_{X_k/X_{k-1}} \longrightarrow V_k \xrightarrow{(\pi_{k,k-1})_*} \mathcal{O}_{X_k}(-1) \longrightarrow 0, \tag{2.6}$$

$$0 \longrightarrow \mathcal{O}_{X_k} \longrightarrow (\pi_{k,k-1})^* V_{k-1} \otimes \mathcal{O}_{X_k}(1) \longrightarrow T_{X_k/X_{k-1}} \longrightarrow 0, \tag{2.7}$$

where the last line is the Euler exact sequence associated with the relative tangent bundle of $P(V_{k-1}) \to X_{k-1}$. Notice that we by definition of the tautological line bundle we have

$$\mathcal{O}_{X_k}(-1) \subset \pi^*_{k,k-1} V_{k-1} \subset \pi^*_{k,k-1} T_{X_{k-1}},$$

and also $\mathrm{rank}(V_k) = r$. Let us recall also that for $k \geq 2$, there are "vertical divisors" $D_k = P(T_{X_{k-1}/X_{k-2}}) \subset P(V_{k-1}) = X_k$, and that D_k is the zero divisor of the section of $\mathcal{O}_{X_k}(1) \otimes \pi^*_{k,k-1} \mathcal{O}_{X_{k-1}}(-1)$ induced by the second arrow of the first exact sequence (2.6), when k is replaced by $k-1$. This yields in particular

$$\mathcal{O}_{X_k}(1) = \pi^*_{k,k-1} \mathcal{O}_{X_{k-1}}(1) \otimes \mathcal{O}(D_k). \tag{2.8}$$

By composing the projections we get for all pairs of indices $0 \leq j \leq k$ natural morphisms

$$\pi_{k,j} : X_k \to X_j, \quad (\pi_{k,j})_* = (d\pi_{k,j})_{|V_k} : V_k \to (\pi_{k,j})^* V_j,$$

and for every k-tuple $\mathbf{a} = (a_1, \ldots, a_k) \in \mathbb{Z}^k$ we define

$$\mathcal{O}_{X_k}(\mathbf{a}) = \bigotimes_{1 \leq j \leq k} \pi^*_{k,j} \mathcal{O}_{X_j}(a_j), \quad \pi_{k,j} : X_k \to X_j.$$

We extend this definition to all weights $\mathbf{a} \in \mathbb{Q}^k$ to get a $\mathbb{Q}$-line bundle in $\mathrm{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$. Now, Formula (2.8) yields

$$\mathcal{O}_{X_k}(\mathbf{a}) = \mathcal{O}_{X_k}(m) \otimes \mathcal{O}(-\mathbf{b} \cdot D) \quad \text{where } m = |\mathbf{a}| = \sum a_j, \mathbf{b} = (0, b_2, \ldots, b_k) \tag{2.9}$$

and $b_j = a_1 + \cdots + a_{j-1}$, $2 \leq j \leq k$.

When $\mathrm{Sing}(V) \neq \emptyset$, one can always define X_k and V_k to be the respective closures of X'_k, V'_k associated with $X' = X \smallsetminus \mathrm{Sing}(V)$ and $V' = V_{|X'}$, where the closure is taken in the nonsingular "absolute" Semple tower (X^a_k, V^a_k) obtained from $(X^a_0, V^a_0) = (X, T_X)$. We leave the reader check the following easy (but important) observation.

2.4 Fonctoriality *If $\Phi : (X, V) \to (Y, W)$ is a morphism of directed varieties such that $\Phi_* : T_X \to \Phi^* T_Y$ is injective (i.e. Φ is an immersion), then there is a corresponding natural morphism $\Phi_{[k]} : (X_k, V_k) \to (Y_k, W_k)$ at the level of Semple bundles. If one merely assumes that the differential $\Phi_* : V \to \Phi^* W$ is non zero,* *there is still a well defined meromorphic map* $\Phi_{[k]} : (X_k, V_k) \dashrightarrow (Y_k, W_k)$ *for all* $k \geq 0$.

In case V is singular, the k-th Semple bundle X_k will also be singular, but we can still replace (X_k, V_k) by a suitable modification $(\widehat{X}_k, \widehat{V}_k)$ if we want to work with a nonsingular model $\widehat{X}_k$ of X_k. The exceptional set of $\widehat{X}_k$ over X_k can be chosen to lie above $\mathrm{Sing}(V) \subset X$, and proceeding inductively with respect to k, we can also arrange the modifications in such a way that we get a tower structure $(\widehat{X}_{k+1}, \widehat{V}_{k+1}) \to (\widehat{X}_k, \widehat{V}_k)$; however, in general, it will not be possible to achieve that $\widehat{V}_k$ is a subbundle of $T_{\widehat{X}_k}$.

It is not true that $K_{\widehat{V}_k}$ is big in case (X, V) is of general type (especially since the fibers of $X_k \to X$ are towers of $\mathbb{P}^{r-1}$ bundles, and the canonical bundles of projective spaces are always negative !). However, a twisted version holds true, that can be seen as another instance of the "monotonicity principle" when going to higher stages in the Semple tower.

2.5 Lemma *If* (X, V) *is of general type, then there is a modification* $(\widehat{X}, \widehat{V})$ *such that all pairs* $(\widehat{X}_k, \widehat{V}_k)$ *of the associated Semple tower have a twisted canonical bundle* $K_{\widehat{V}_k} \otimes \mathcal{O}_{\widehat{X}_k}(p)$ *that is still big when one multiplies* $K_{\widehat{V}_k}$ *by a suitable* $\mathbb{Q}$*-line bundle* $\mathcal{O}_{\widehat{X}_k}(p)$, $p \in \mathbb{Q}_+$.

Proof First assume that V has no singularities. The exact sequences (2.6) and (2.7) provide

$$K_{V_k} := \det V_k^* = \det(T^*_{X_k/X_{k-1}}) \otimes \mathcal{O}_{X_k}(1) = \pi^*_{k,k-1} K_{V_{k-1}} \otimes \mathcal{O}_{X_k}(-(r-1))$$

where $r = \mathrm{rank}(V)$. Inductively we get

$$K_{V_k} = \pi^*_{k,0} K_V \otimes \mathcal{O}_{X_k}(-(r-1)\mathbf{1}), \qquad \mathbf{1} = (1, \ldots, 1) \in \mathbb{N}^k. \tag{2.10}$$

We know by [1] that $\mathcal{O}_{X_k}(\mathbf{c})$ is relatively ample over X when we take the special weight $\mathbf{c} = (2\,3^{k-2}, \ldots, 2\,3^{k-j-1}, \ldots, 6, 2, 1)$, hence

$$K_{V_k} \otimes \mathcal{O}_{X_k}((r-1)\mathbf{1} + \varepsilon\mathbf{c}) = \pi^*_{k,0} K_V \otimes \mathcal{O}_{X_k}(\varepsilon\mathbf{c})$$

is big over X_k for any sufficiently small positive rational number $\varepsilon \in \mathbb{Q}^*_+$. Thanks to Formula (2.9), we can in fact replace the weight $(r-1)\mathbf{1} + \varepsilon\mathbf{c}$ by its total degree $p = (r-1)k + \varepsilon|\mathbf{c}| \in \mathbb{Q}_+$. The general case of a singular linear space follows by considering suitable "sufficiently high" modifications $\widehat{X}$ of X, the related directed structure $\widehat{V}$ on $\widehat{X}$, and embedding $(\widehat{X}_k, \widehat{V}_k)$ in the absolute Semple tower $(\widehat{X}^a_k, \widehat{V}^a_k)$ of $\widehat{X}$. We still have a well defined morphism of rank 1 sheaves

$$\pi^*_{k,0} K_{\widehat{V}} \otimes \mathcal{O}_{\widehat{X}_k}(-(r-1)\mathbf{1}) \to K_{\widehat{V}_k} \tag{2.11}$$

because the multiplier ideal sheaves involved at each stage behave according to the monotonicity principle applied to the projections $\pi^a_{k,k-1} : \widehat{X}^a_k \to \widehat{X}^a_{k-1}$ and their differentials $(\pi^a_{k,k-1})_*$, which yield well-defined transposed morphisms from the $(k-1)$-st stage to the k-th stage at the level of exterior differential forms. Our contention follows. □

3 Induced Directed Structure on a Subvariety of a Jet Space

Let Z be an irreducible algebraic subset of some k-jet bundle X_k over X, $k \geq 0$. We define the linear subspace $W \subset T_Z \subset T_{X_k|Z}$ to be the closure

$$W := \overline{T_{Z'} \cap V_k} \tag{3.1}$$

taken on a suitable Zariski open set $Z' \subset Z_{\mathrm{reg}}$ where the intersection $T_{Z'} \cap V_k$ has constant rank and is a subbundle of $T_{Z'}$. Alternatively, we could also take W to be the closure of $T_{Z'} \cap V_k$ in the k-th stage (X^a_k, V^a_k) of the absolute Semple tower, which has the advantage of being nonsingular. We say that (Z, W) is the *induced* directed variety structure; this concept of induced structure already applies of course in the case $k = 0$. If $f : (\mathbb{C}, T_{\mathbb{C}}) \to (X, V)$ is such that $f_{[k]}(\mathbb{C}) \subset Z$, then

$$\text{either} \quad f_{[k]}(\mathbb{C}) \subset Z_\alpha \quad \text{or} \quad f'_{[k]}(\mathbb{C}) \subset W, \tag{3.2}$$

where Z_α is one of the connected components of $Z \smallsetminus Z'$ and Z' is chosen as in (3.1); especially, if $W = 0$, we conclude that $f_{[k]}(\mathbb{C})$ must be contained in one of the Z_α's. In the sequel, we always consider such a subvariety Z of X_k as a directed pair (Z, W) by taking the induced structure described above. By (3.2), if we proceed by induction on $\dim Z$, the study of curves tangent to V that have a k-lift $f_{[k]}(\mathbb{C}) \subset Z$ is reduced to the study of curves tangent to (Z, W). Let us first quote the following easy observation.

3.1 Observation *For $k \geq 1$, let $Z \subsetneq X_k$ be an irreducible algebraic subset that projects onto X_{k-1}, i.e. $\pi_{k,k-1}(Z) = X_{k-1}$. Then the induced directed variety $(Z, W) \subset (X_k, V_k)$, satisfies*

$$1 \leq \operatorname{rank} W < r := \operatorname{rank}(V_k).$$

Proof Take a Zariski open subset $Z' \subset Z_{\mathrm{reg}}$ such that $W' = T_{Z'} \cap V_k$ is a vector bundle over Z'. Since $X_k \to X_{k-1}$ is a $\mathbb{P}^{r-1}$-bundle, Z has codimension at most $r-1$ in X_k. Therefore $\operatorname{rank} W \geq \operatorname{rank} V_k - (r-1) \geq 1$. On the other hand, if we had $\operatorname{rank} W = \operatorname{rank} V_k$ generically, then $T_{Z'}$ would contain $V_{k|Z'}$, in particular it would contain all vertical directions $T_{X_k/X_{k-1}} \subset V_k$ that are tangent to the fibers of $X_k \to X_{k-1}$. By taking the flow along vertical vector fields, we would conclude that Z' is a union of fibers of $X_k \to X_{k-1}$ up to an algebraic set of smaller dimension, but this is excluded since Z projects onto X_{k-1} and $Z \subsetneq X_k$. □

3.2 Definition For $k \geq 1$, let $Z \subset X_k$ be an irreducible algebraic subset of X_k. We assume moreover that $Z \not\subset D_k = P(T_{X_{k-1}/X_{k-2}})$ (and put here $D_1 = \emptyset$ in what follows to avoid to have to single out the case $k = 1$). In this situation we say that (Z, W) is of general type modulo $X_k \to X$ if either $W = 0$, or $\operatorname{rank} W \geq 1$ and there exists $p \in \mathbb{Q}_+$ such that $K_W \otimes \mathcal{O}_{X_k}(p)_{|Z}$ is big over Z, possibly after replacing Z by a suitable nonsingular model $\widehat{Z}$ (and pulling-back W and $\mathcal{O}_{X_k}(p)_{|Z}$ to the nonsingular variety $\widehat{Z}$).

The main result of [4] mentioned in the introduction as Theorem 1.3 implies the following important "induction step".

3.3 Proposition *Let (X, V) be a directed pair where X is projective algebraic. Take an irreducible algebraic subset $Z \not\subset D_k$ of the associated k-jet Semple bundle X_k that projects onto X_{k-1}, $k \geq 1$, and assume that the induced directed space $(Z, W) \subset (X_k, V_k)$ is of general type modulo $X_k \to X$, $\operatorname{rank} W \geq 1$. Then there exists a divisor $\Sigma \subset Z_\ell$ in a sufficiently high stage of the Semple tower (Z_ℓ, W_ℓ) associated with (Z, W), such that every non constant holomorphic map $f : \mathbb{C} \to X$ tangent to V that satisfies $f_{[k]}(\mathbb{C}) \subset Z$ also satisfies $f_{[k+\ell]}(\mathbb{C}) \subset \Sigma$.*

Proof Let $E \subset Z$ be a divisor containing $Z_{\rm sing} \cup (Z \cap \pi_{k,0}^{-1}(\operatorname{Sing}(V)))$, chosen so that on the nonsingular Zariski open set $Z' = Z \smallsetminus E$ all linear spaces $T_{Z'}$, $V_{k|Z'}$ and $W' = T_{Z'} \cap V_k$ are subbundles of $T_{X_k|Z'}$, the first two having a transverse intersection on Z'. By taking closures over Z' in the absolute Semple tower of X, we get (singular) directed pairs $(Z_\ell, W_\ell) \subset (X_{k+\ell}, V_{k+\ell})$, which we eventually resolve into $(\widehat{Z}_\ell, \widehat{W}_\ell) \subset (\widehat{X}_{k+\ell}, \widehat{V}_{k+\ell})$ over nonsingular bases. By construction, locally bounded sections of $\mathcal{O}_{\widehat{X}_{k+\ell}}(m)$ restrict to locally bounded sections of $\mathcal{O}_{\widehat{Z}_\ell}(m)$ over $\widehat{Z}_\ell$.

Since Theorem 1.3 and the related estimate (1.10) are universal in the category of directed varieties, we can apply them by replacing X with $\widehat{Z} \subset \widehat{X}_k$, the order k by a new index ℓ, and F by

$$F_k = \mu^*\Big(\big(\mathcal{O}_{X_k}(p) \otimes \pi_{k,0}^*\mathcal{O}_X(-\varepsilon A)\big)_{|Z}\Big)$$

where $\mu : \widehat{Z} \to Z$ is the desingularization, $p \in \mathbb{Q}_+$ is chosen such that $K_W \otimes \mathcal{O}_{x_k}(p)_{|Z}$ is big, A is an ample bundle on X and $\varepsilon \in \mathbb{Q}_+^*$is small enough. The assumptions show that $K_{\widehat{W}} \otimes F_k$ is big on $\widehat{Z}$, therefore, by applying our theorem and taking $m \gg \ell \gg 1$, we get in fine a large number of (metric bounded) sections of

$$\begin{aligned}\mathcal{O}_{\widehat{Z}_\ell}(m) \otimes \widehat{\pi}_{k+\ell,k}^*\mathcal{O}\Big(\frac{m}{\ell r'}\Big(1 + \frac{1}{2} + \cdots + \frac{1}{\ell}\Big) F_k\Big) \\ = \mathcal{O}_{\widehat{X}_{k+\ell}}(m\mathbf{a}') \otimes \widehat{\pi}_{k+\ell,0}^*\mathcal{O}\Big(-\frac{m\varepsilon}{kr}\Big(1 + \frac{1}{2} + \cdots + \frac{1}{k}\Big) A\Big)_{|\widehat{Z}_\ell}\end{aligned}$$

where $\mathbf{a}' \in \mathbb{Q}_+^{k+\ell}$ is a positive weight (of the form $(0, \ldots, \lambda, \ldots, 0, 1)$ with some non zero component $\lambda \in \mathbb{Q}_+$ at index k). These sections descend to metric bounded sections of

$$\mathcal{O}_{X_{k+\ell}}((1+\lambda)m) \otimes \widehat{\pi}^*_{k+\ell,0}\mathcal{O}\Big(-\frac{m\varepsilon}{kr}\Big(1+\frac{1}{2}+\cdots+\frac{1}{k}\Big)A\Big)_{|Z_\ell}.$$

Since A is ample on X, we can apply the fundamental vanishing theorem (see e.g. [2] or [4], Statement 8.15), or rather an "embedded" version for curves satisfying $f_{[k]}(\mathbb{C}) \subset Z$, proved exactly by the same arguments. The vanishing theorem implies that the divisor Σ of any such section satisfies the conclusions of Proposition 3.3, possibly modulo exceptional divisors of $\widehat{Z} \to Z$; to take care of these, it is enough to add to Σ the inverse image of the divisor $E = Z \smallsetminus Z'$ initially selected. □

4 Strong General Type Condition for Directed Manifolds

Our main result is the following partial solution to the Green-Griffiths-Lang conjecture, providing a sufficient algebraic condition for the analytic conclusion to hold true. We first give an ad hoc definition.

4.1 Definition Let (X, V) be a directed pair where X is projective algebraic. We say that that (X, V) is "strongly of general type" if it is of general type and for every irreducible algebraic set $Z \subsetneq X_k$, $Z \not\subset D_k$, that projects onto X, the induced directed structure $(Z, W) \subset (X_k, V_k)$ is of general type modulo $X_k \to X$.

4.2 Example The situation of a product $(X, V) = (X', V') \times (X'', V'')$ described in (1.11) shows that (X, V) can be of general type without being strongly of general type. In fact, if (X', V') and (X'', V'') are of general type, then $K_V = \mathrm{pr}'^* K_{V'} \otimes \mathrm{pr}''^* K_{V''}$ is big, so (X, V) is again of general type. However

$$Z = P(\mathrm{pr}'^* V') = X'_1 \times X'' \subset X_1$$

has a directed structure $W = \mathrm{pr}'^* V'_1$ which does not possess a big canonical bundle over Z, since the restriction of K_W to any fiber $\{x'\} \times X''$ is trivial. The higher stages (Z_k, W_k) of the Semple tower of (Z, W) are given by $Z_k = X'_{k+1} \times X''$ and $W_k = \mathrm{pr}'^* V'_{k+1}$, so it is easy to see that $\mathrm{GG}_k(X, V)$ contains Z_{k-1}. Since Z_k projects onto X, we have here $\mathrm{GG}(X, V) = X$ (see [6] for more sophisticated indecomposable examples).

4.3 Remark It follows from Definition 3.2 that $(Z, W) \subset (X_k, V_k)$ is automatically of general type modulo $X_k \to X$ if $\mathcal{O}_{X_k}(1)_{|Z}$ is big. Notice further that

$$\mathcal{O}_{X_k}(1+\varepsilon)_{|Z} = \big(\mathcal{O}_{X_k}(\varepsilon) \otimes \pi^*_{k,k-1}\mathcal{O}_{X_{k-1}}(1) \otimes \mathcal{O}(D_k)\big)_{|Z}$$

where $\mathcal{O}(D_k)_{|Z}$ is effective and $\mathcal{O}_{X_k}(1)$ is relatively ample with respect to the projection $X_k \to X_{k-1}$. Therefore the bigness of $\mathcal{O}_{X_{k-1}}(1)$ on X_{k-1} also implies that every directed subvariety $(Z, W) \subset (X_k, V_k)$ is of general type modulo $X_k \to X$. If (X, V) is of general type, we know by the main result of [4] that $\mathcal{O}_{X_k}(1)$ is big for $k \geq k_0$ large enough, and actually the precise estimates obtained therein give explicit bounds for such a k_0. The above observations show that we need to check the condition of Definition 4.1 only for $Z \subset X_k$, $k \leq k_0$. Moreover, at least in the case where V, Z, and $W = T_Z \cap V_k$ are nonsingular, we have

$$K_W \simeq K_Z \otimes \det(T_Z/W) \simeq K_Z \otimes \det(T_{X_k}/V_k)_{|Z} \simeq K_{Z/X_{k-1}} \otimes \mathcal{O}_{X_k}(1)_{|Z}.$$

Thus we see that, in some sense, it is only needed to check the bigness of K_W modulo $X_k \to X$ for "rather special subvarieties" $Z \subset X_k$ over X_{k-1}, such that $K_{Z/X_{k-1}}$ is not relatively big over X_{k-1}. □

4.4 Hypersurface case Assume that $Z \neq D_k$ is an irreducible hypersurface of X_k that projects onto X_{k-1}. To simplify things further, also assume that V is nonsingular. Since the Semple jet-bundles X_k form a tower of $\mathbb{P}^{r-1}$-bundles, their Picard groups satisfy $\mathrm{Pic}(X_k) \simeq \mathrm{Pic}(X) \oplus \mathbb{Z}^k$ and we have $\mathcal{O}_{X_k}(Z) \simeq \mathcal{O}_{X_k}(\mathbf{a}) \otimes \pi^*_{k,0}B$ for some $\mathbf{a} \in \mathbb{Z}^k$ and $B \in \mathrm{Pic}(X)$, where $a_k = d > 0$ is the relative degree of the hypersurface over X_{k-1}. Let $\sigma \in H^0(X_k, \mathcal{O}_{X_k}(Z))$ be the section defining Z in X_k. The induced directed variety (Z, W) has rank $W = r - 1 = \mathrm{rank}\, V - 1$ and formula (2.11) yields $K_{V_k} = \mathcal{O}_{X_k}(-(r-1)\mathbf{1}) \otimes \pi^*_{k,0}(K_V)$. We claim that

$$K_W \supset \big(K_{V_k} \otimes \mathcal{O}_{X_k}(Z)\big)_{|Z} \otimes \mathcal{J}_S = \big(\mathcal{O}_{X_k}(\mathbf{a}-(r-1)\mathbf{1}) \otimes \pi^*_{k,0}(B \otimes K_V)\big)_{|Z} \otimes \mathcal{J}_S \quad (4.1)$$

where $S \subsetneq Z$ is the set (containing Z_{sing}) where σ and $d\sigma_{|V_k}$ both vanish, and $\mathcal{J}_S$ is the ideal locally generated by the coefficients of $d\sigma_{|V_k}$ along $Z = \sigma^{-1}(0)$. In fact, the intersection $W = T_Z \cap V_k$ is transverse on $Z \smallsetminus S$; then (4.1) can be seen by looking at the morphism

$$V_{k|Z} \xrightarrow{d\sigma_{|V_k}} \mathcal{O}_{X_k}(Z)_{|Z},$$

and observing that the contraction by $K_{V_k} = \Lambda^r V_k^*$ provides a metric bounded section of the canonical sheaf K_W. In order to investigate the positivity properties of K_W, one has to show that B cannot be too negative, and in addition to control the singularity set S. The second point is a priori very challenging, but we get useful information for the first point by observing that σ provides a morphism $\pi^*_{k,0}\mathcal{O}_X(-B) \to \mathcal{O}_{X_k}(\mathbf{a})$, hence a nontrivial morphism

$$\mathcal{O}_X(-B) \to E_{\mathbf{a}} := (\pi_{k,0})_*\mathcal{O}_{X_k}(\mathbf{a}).$$

By [1, Section 12], there exists a filtration on $E_{\mathbf{a}}$ such that the graded pieces are irreducible representations of $\mathrm{GL}(V)$ contained in $(V^*)^{\otimes \ell}$, $\ell \leq |\mathbf{a}|$. Therefore we get a nontrivial morphism

$$\mathcal{O}_X(-B) \to (V^*)^{\otimes \ell}, \qquad \ell \le |\mathbf{a}|. \tag{4.2}$$

If we know about certain (semi-)stability properties of V, this can be used to control the negativity of B. □

We further need the following useful concept that slightly generalizes entire curve loci.

4.5 Definition If Z is an algebraic set contained in some stage X_k of the Semple tower of (X, V), we define its "induced entire curve locus" $\mathrm{IEL}_{X,V}(Z) \subset Z$ to be the Zariski closure of the union $\bigcup f_{[k]}(\mathbb{C})$ of all jets of entire curves $f : (\mathbb{C}, T_{\mathbb{C}}) \to (X, V)$ such that $f_{[k]}(\mathbb{C}) \subset Z$.

We have of course $\mathrm{IEL}_{X,V}(\mathrm{IEL}_{X,V}(Z)) = \mathrm{IEL}_{X,V}(Z)$ by definition. It is not hard to check that modulo certain "vertical divisors " of X_k, the $\mathrm{IEL}_{X,V}(Z)$ locus is essentially the same as the entire curve locus $\mathrm{ECL}(Z, W)$ of the induced directed variety, but we will not use this fact here. Notice that if $Z = \bigcup Z_\alpha$ is a decomposition of Z into irreducible divisors, then

$$\mathrm{IEL}_{X,V}(Z) = \bigcup_\alpha \mathrm{IEL}_{X,V}(Z_\alpha).$$

Since $\mathrm{IEL}_{X,V}(X_k) = \mathrm{ECL}_k(X, V)$, proving the Green-Griffiths-Lang property amounts to showing that $\mathrm{IEL}_{X,V}(X) \subsetneq X$ in the stage $k = 0$ of the tower. The basic step of our approach is expressed in the following statement.

4.6 Proposition *Let (X, V) be a directed variety and $p_0 \le n = \dim X$, $p_0 \ge 1$. Assume that there is an integer $k_0 \ge 0$ such that for every $k \ge k_0$ and every irreducible algebraic set $Z \subsetneq X_k$, $Z \not\subset D_k$, such that $\dim \pi_{k,k_0}(Z) \ge p_0$, the induced directed structure $(Z, W) \subset (X_k, V_k)$ is of general type modulo $X_k \to X$. Then $\dim \mathrm{ECL}_{k_0}(X, V) < p_0$.*

Proof We argue here by contradiction, assuming that $\dim \mathrm{ECL}_{k_0}(X, V) \ge p_0$. If

$$p_0' := \dim \mathrm{ECL}_{k_0}(X, V) > p_0$$

and if we can prove the result for p_0', we will already get a contradiction, hence we can assume without loss of generality that $\dim \mathrm{ECL}_{k_0}(X, V) = p_0$. The main argument consists of producing inductively an increasing sequence of integers

$$k_0 < k_1 < \cdots < k_j < \cdots$$

and directed varieties $(Z^j, W^j) \subset (X_{k_j}, V_{k_j})$ satisfying the following properties :

(3.6.1) Z^0 is one of the irreducible components of $\mathrm{ECL}_{k_0}(X, V)$ and $\dim Z^0 = p_0$.

(3.6.2) Z^j is one of the irreducible components of $\mathrm{ECL}_{k_j}(X, V)$ and $\pi_{k_j,k_0}(Z^j) = Z^0$.

(3.6.3) For all $j \geq 0$, $\mathrm{IEL}_{X,V}(Z^j) = Z^j$ and $\mathrm{rank}\, W_j \geq 1$.

(3.6.4) For all $j \geq 0$, the directed variety (Z^{j+1}, W^{j+1}) is contained in some stage (of order $\ell_j = k_{j+1} - k_j$) of the Semple tower of (Z^j, W^j), namely

$$(Z^{j+1}, W^{j+1}) \subsetneq (Z^j_{\ell_j}, W^j_{\ell_j}) \subset (X_{k_{j+1}}, V_{k_{j+1}})$$

and

$$W^{j+1} = \overline{T_{Z^{j+1\prime}} \cap W^j_{\ell_j}} = \overline{T_{Z^{j+1\prime}} \cap V_{k_j}} \tag{4.3}$$

is the induced directed structure; moreover $\pi_{k_{j+1},k_j}(Z^{j+1}) = Z^j$.

(3.6.5) For all $j \geq 0$, we have $Z^{j+1} \subsetneq Z^j_{\ell_j}$ but $\pi_{k_{j+1},k_{j+1}-1}(Z^{j+1}) = Z^j_{\ell_j-1}$.

For $j = 0$, we simply take Z^0 to be one of the irreducible components S_α of $\mathrm{ECL}_{k_0}(X, V)$ such that $\dim S_\alpha = p_0$, which exists by our hypothesis that $\dim \mathrm{ECL}_{k_0}(X, V) = p_0$. Clearly, $\mathrm{ECL}_{k_0}(X, V)$ is the union of the $\mathrm{IEL}_{X,V}(S_\alpha)$ and we have $\mathrm{IEL}_{X,V}(S_\alpha) = S_\alpha$ for all those components, thus $\mathrm{IEL}_{X,V}(Z^0) = Z^0$ and $\dim Z^0 = p_0$. Assume that (Z^j, W^j) has been constructed. The subvariety Z^j cannot be contained in the vertical divisor D_{k_j}. In fact no irreducible algebraic set Z such that $\mathrm{IEL}_{X,V}(Z) = Z$ can be contained in a vertical divisor D_k, because $\pi_{k,k-2}(D_k)$ corresponds to stationary jets in X_{k-2} ; as every non constant curve f has non stationary points, its k-jet $f_{[k]}$ cannot be entirely contained in D_k ; also the induced directed structure (Z, W) must satisfy $\mathrm{rank}\, W \geq 1$ otherwise $\mathrm{IEL}_{X,V}(Z) \subsetneq Z$. Condition (3.6.2) implies that $\dim \pi_{k_j,k_0}(Z^j) \geq p_0$, thus (Z^j, W^j) is of general type modulo $X_{k_j} \to X$ by the assumptions of the proposition. Thanks to Proposition 3.3, we get an algebraic subset $\Sigma \subsetneq Z^j_\ell$ in some stage of the Semple tower (Z^j_ℓ) of Z^j such that every entire curve $f : (\mathbb{C}, T_\mathbb{C}) \to (X, V)$ satisfying $f_{[k_j]}(\mathbb{C}) \subset Z^j$ also satisfies $f_{[k_j+\ell]}(\mathbb{C}) \subset \Sigma$. By definition, this implies the first inclusion in the sequence

$$Z^j = \mathrm{IEL}_{X,V}(Z^j) \subset \pi_{k_j+\ell,k_j}(\mathrm{IEL}_{X,V}(\Sigma)) \subset \pi_{k_j+\ell,k_j}(\Sigma) \subset Z^j$$

(the other ones being obvious), so we have in fact an equality throughout. Let (S'_α) be the irreducible components of $\mathrm{IEL}_{X,V}(\Sigma)$. We have $\mathrm{IEL}_{X,V}(S'_\alpha) = S'_\alpha$ and one of the components S'_α must satisfy

$$\pi_{k_j+\ell,k_j}(S'_\alpha) = Z^j = Z^j_0.$$

We take $\ell_j \in [1, \ell]$ to be the smallest order such that $Z^{j+1} := \pi_{k_j+\ell,k_j+\ell_j}(S'_\alpha) \subsetneq Z^j_{\ell_j}$, and set $k_{j+1} = k_j + \ell_j > k_j$. By definition of ℓ_j, we have $\pi_{k_{j+1},k_{j+1}-1}(Z^{j+1}) = Z^j_{\ell_j-1}$, otherwise ℓ_j would not be minimal. Then $\pi_{k_{j+1},k_j}(Z^{j+1}) = Z^j$, hence $\pi_{k_{j+1},k_0}(Z^{j+1}) = Z^0$ by induction, and all properties (3.6.1–3.6.5) follow easily. Now, by Observation 3.1, we have

$$\operatorname{rank} W^j < \operatorname{rank} W^{j-1} < \cdots < \operatorname{rank} W^1 < \operatorname{rank} W^0 = \operatorname{rank} V.$$

This is a contradiction because we cannot have such an infinite sequence. Proposition 4.6 is proved. □

The special case $k_0 = 0$, $p_0 = n$ of Proposition 4.6 yields the following consequence.

4.7 Partial solution to the generalized GGL conjecture *Let (X, V) be a directed pair that is strongly of general type. Then the Green-Griffiths-Lang conjecture holds true for (X, V), namely $\mathrm{ECL}(X, V) \subsetneq X$, in other words there exists a proper algebraic variety $Y \subsetneq X$ such that every non constant holomorphic curve $f : \mathbb{C} \to X$ tangent to V satisfies $f(\mathbb{C}) \subset Y$.*

4.8 Remark The proof is not very constructive, but it is however theoretically effective. By this we mean that if (X, V) is strongly of general type and is taken in a bounded family of directed varieties, i.e. X is embedded in some projective space $\mathbb{P}^N$ with some bound δ on the degree, and $P(V)$ also has bounded degree $\le \delta'$ when viewed as a subvariety of $P(T_{\mathbb{P}^N})$, then one could theoretically derive bounds $d_Y(n, \delta, \delta')$ for the degree of the locus Y. Also, there would exist bounds $k_0(n, \delta, \delta')$ for the orders k and bounds $d_k(n, \delta, \delta')$ for the degrees of subvarieties $Z \subset X_k$ that have to be checked in the definition of a pair of strong general type. In fact, [4] produces more or less explicit bounds for the order k such that Proposition 3.3 holds true. The degree of the divisor Σ is given by a section of a certain twisted line bundle $\mathcal{O}_{X_k}(m) \otimes \pi_{k,0}^*\mathcal{O}_X(-A)$ that we know to be big by an application of holomorphic Morse inequalities – and the bounds for the degrees of (X_k, V_k) then provide bounds for m. □

4.9 Remark The condition that (X, V) is strongly of general type seems to be related to some sort of stability condition. We are unsure what is the most appropriate definition, but here is one that makes sense. Fix an ample divisor A on X. For every irreducible subvariety $Z \subset X_k$ that projects onto X_{k-1} for $k \ge 1$, and $Z = X = X_0$ for $k = 0$, we define the slope $\mu_A(Z, W)$ of the corresponding directed variety (Z, W) to be

$$\mu_A(Z, W) = \frac{\inf \lambda}{\operatorname{rank} W},$$

where λ runs over all rational numbers such that there exists $m \in \mathbb{Q}_+$ for which

$$K_W \otimes \big(\mathcal{O}_{X_k}(m) \otimes \pi_{k,0}^*\mathcal{O}(\lambda A)\big)_{|Z} \quad \text{is big on } Z$$

(again, we assume here that $Z \not\subset D_k$ for $k \ge 2$). Notice that (X, V) is of general type if and only if $\mu_A(X, V) < 0$, and that $\mu_A(Z, W) = -\infty$ if $\mathcal{O}_{X_k}(1)_{|A}$ is big. Also, the proof of Lemma 2.5 shows that

$$\mu_A(X_k, V_k) \le \mu_A(X_{k-1}, V_{k-1}) \le \ldots \le \mu_A(X, V) \quad \text{for all } k$$

(with $\mu_A(X_k, V_k) = -\infty$ for $k \geq k_0 \gg 1$ if (X, V) is of general type). We say that (X, V) is *A-jet-stable* (resp. *A-jet-semi-stable*) if $\mu_A(Z, W) < \mu_A(X, V)$ (resp. $\mu_A(Z, W) \leq \mu_A(X, V)$) for all $Z \subsetneq X_k$ as above. It is then clear that if (X, V) is of general type and A-jet-semi-stable, then it is strongly of general type in the sense of Definition 4.1. It would be useful to have a better understanding of this condition of stability (or any other one that would have better properties). □

4.10 Example (**case of surfaces**) Assume that X is a minimal complex surface of general type and $V = T_X$ (absolute case). Then K_X is nef and big and the Chern classes of X satisfy $c_1 \leq 0$ ($-c_1$ is big and nef) and $c_2 \geq 0$. The Semple jet-bundles X_k form here a tower of $\mathbb{P}^1$-bundles and $\dim X_k = k + 2$. Since $\det V^* = K_X$ is big, the strong general type assumption of 4.6 and 4.8 need only be checked for irreducible hypersurfaces $Z \subset X_k$ distinct from D_k that project onto X_{k-1}, of relative degree m. The projection $\pi_{k,k-1} : Z \to X_{k-1}$ is a ramified $m : 1$ cover. Putting $\mathcal{O}_{X_k}(Z) \simeq \mathcal{O}_{X_k}(\mathbf{a}) \otimes \pi_{k,0}(B)$, $B \in \mathrm{Pic}(X)$, we can apply (4.1) to get an inclusion

$$K_W \supset \big(\mathcal{O}_{X_k}(\mathbf{a}-\mathbf{1}) \otimes \pi_{k,0}^*(B \otimes K_X)\big)_{|Z} \otimes \mathcal{J}_S, \qquad \mathbf{a} \in \mathbb{Z}^k,\ \ a_k = m.$$

Let us assume $k = 1$ and $S = \emptyset$ to make things even simpler, and let us perform numerical calculations in the cohomology ring

$$H^\bullet(X_1, \mathbb{Z}) = H^\bullet(X)[u]/(u^2 + c_1 u + c_2), \qquad u = c_1(O_{X_1}(1))$$

(cf. [3, Section 2] for similar calculations and more details). We have

$$Z \equiv mu + b \quad \text{where} \quad b = c_1(B) \quad \text{and} \quad K_W \equiv (m-1)u + b - c_1.$$

We are allowed here to add to K_W an arbitrary multiple $\mathcal{O}_{X_1}(p)$, $p \geq 0$, which we rather write $p = mt + 1 - m$, $t \geq 1 - 1/m$. An evaluation of the Euler-Poincaré characteristic of $K_W + \mathcal{O}_{X_1}(p)_{|Z}$ requires computing the intersection number

$$\begin{aligned}\big(K_W + \mathcal{O}_{X_1}(p)_{|Z}\big)^2 \cdot Z &= \big(mt\,u + b - c_1\big)^2(mu + b)\\ &= m^2t^2\big(m(c_1^2 - c_2) - bc_1\big) + 2mt(b - mc_1)(b - c_1)\\ &\quad + m(b - c_1)^2, \qquad (4.4)\end{aligned}$$

taking into account that $u^3 \cdot X_1 = c_1^2 - c_2$. In case $S \neq \emptyset$, there is an additional (negative) contribution from the ideal $\mathcal{J}_S$ which is $O(t)$ since S is at most a curve. In any case, for $t \gg 1$, the leading term in the expansion is $m^2t^2(m(c_1^2 - c_2) - bc_1)$ and the other terms are negligible with respect to t^2, including the one coming from S. We know that T_X is semistable with respect to $c_1(K_X) = -c_1 \geq 0$. Multiplication by the section σ yields a morphism $\pi_{1,0}^*\mathcal{O}_X(-B) \to \mathcal{O}_{X_1}(m)$, hence by direct image, a morphism $\mathcal{O}_X(-B) \to S^m T_X^*$. Evaluating slopes against K_X (a big nef class), the semistability condition implies $bc_1 \leq \frac{m}{2}c_1^2$, and our leading term is bigger that

$m^3t^2(\frac{1}{2}c_1^2 - c_2)$. We get a positive answer in the well-known case where $c_1^2 > 2c_2$, corresponding to T_X being almost ample. Analyzing positivity for the full range of values (k, m, t) and of singular sets S seems an unsurmountable task at this point; in general, calculations made in [3, 12] indicate that the Chern class and semistability conditions become less demanding for higher order jets (e.g. $c_1^2 > c_2$ is enough for $Z \subset X_2$, and $c_1^2 > \frac{9}{13}c_2$ suffices for $Z \subset X_3$). When rank $V = 1$, major gains come from the use of Ahlfors currents in combination with McQuillan's tautological inequalities [11]. We therefore hope for a substantial strengthening of the above sufficient conditions, and a better understanding of the stability issues, possibly in combination with a use of Ahlfors currents and tautological inequalities. In the case of surfaces, an application of Proposition 4.6 for $k_0 = 1$ and an analysis of the behaviour of rank 1 (multi-)foliations on the surface X (with the crucial use of [11]) was the main argument used in [3] to prove the hyperbolicity of very general surfaces of degree $d \geq 21$ in $\mathbb{P}^3$. For these surfaces, one has $c_1^2 < c_2$ and $c_1^2/c_2 \to 1$ as $d \to +\infty$. Applying Proposition 4.6 for higher values $k_0 \geq 2$ might allow to enlarge the range of tractable surfaces, if the behavior of rank 1 (multi)-foliations on X_{k_0-1} can be analyzed independently.

5 Algebraic Jet-Hyperbolicity Implies Kobayashi Hyperbolicity

Let (X, V) be a directed variety, where X is an irreducible projective variety; the concept still makes sense when X is singular, by embedding (X, V) in a projective space $(\mathbb{P}^N, T_{\mathbb{P}^N})$ and taking the linear space V to be an irreducible algebraic subset of $T_{\mathbb{P}^n}$ that is contained in T_X at regular points of X.

5.1 Definition Let (X, V) be a directed variety. We say that (X, V) is algebraically jet-hyperbolic if for every $k \geq 0$ and every irreducible algebraic subvariety $Z \subset X_k$ that is not contained in the union Δ_k of vertical divisors, the induced directed structure (Z, W) either satisfies $W = 0$, or is of general type modulo $X_k \to X$, i.e. has a desingularization $(\widehat{Z}, \widehat{W})$, $\mu : \widehat{Z} \to Z$, such that some twisted canonical sheaf $K_{\widehat{W}} \otimes \mu^*(\mathcal{O}_{X_k}(\mathbf{a})_{|Z})$, $\mathbf{a} \in \mathbb{N}^k$, is big.

Proposition 4.6 then gives

5.2 Theorem *Let (X, V) be an irreducible projective directed variety that is algebraically jet-hyperbolic in the sense of the above definition. Then (X, V) is Brody (or Kobayashi) hyperbolic, i.e.* $\mathrm{ECL}(X, V) = \emptyset$.

Proof Here we apply Proposition 4.6 with $k_0 = 0$ and $p_0 = 1$. It is enough to deal with subvarieties $Z \subset X_k$ such that $\dim \pi_{k,0}(Z) \geq 1$, otherwise $W = 0$ and can reduce Z to a smaller subvariety by (3.2). Then we conclude that $\dim \mathrm{ECL}(X, V) < 1$. All entire curves tangent to V have to be constant, and we conclude in fact that $\mathrm{ECL}(X, V) = \emptyset$. □

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On Microlocal Regularity for Involutive Systems of Complex Vector Fields of Tube Type in $\mathbb{R}^{n+m}$

Makhlouf Derridj

In memory of M. Salah Baouendi

Abstract Our aim is to improve and generalize some results previously obtained in a joint work with Bernard Helffer in a series of papers, concerning microlocal subellipticity (hence microlocal hypoellipticity) and maximal estimates for the systems first studied by F. Treves in (Treves, Ann. Math. **104**(2) (1976) and **113**(2) (1981), [14]) for which he gave a necessary condition for microlocal hypoellipticity. After him, many mathematicians studied such systems in various contexts (Derridj, *Subelliptic Estimates for Some Systems of Complex Vector Fields*, (2006) [5]), (Derridj, Helffer, *Complex Analysis*, (2010), [6]), (Derridj, Helffer, Contemp. Math., **550**, 15–56 (2011), [7]), (Helffer, Nier, *Hypoelliptic estimates and spectral theory for Fokker-Planck operators and Witten Laplacians*, (2005), [8]), (Journé, Trépreau, Le cas de n champs de vecteurs complexes en $(n + 1)$ variables, Séminaire EDP, Ecole Polytechnique (2006), [11]), (Maire, Commun. Part. Diff. Equ. **5**(4) (1980), [12]), (Treves, Ann. Math. **104**(2) (1976) and **113**(2) (1981), [14]).

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1 Introduction

We recall just some facts on the systems we are studying. We refer to [7] for more details, results and references. We consider in $(\omega \subset \mathbb{R}^n_t)\times\mathbb{R}^m_x$ systems of the following type (Treves's systems)

$$L = \begin{cases} L_j = \dfrac{\partial}{\partial t_j} + i \displaystyle\sum_{k=1}^{m} \dfrac{\partial \varphi_k}{\partial t_j}(t)\dfrac{\partial}{\partial x_k}, \text{ where} \\ \Phi = (\varphi_1, \dots, \varphi_m) \colon \omega \subset \mathbb{R}^n \to \mathbb{R}^m,\ \Phi \in C^1(\omega, \mathbb{R}^m). \end{cases} \tag{1.1}$$

This system L will be sometimes denoted by L_Φ.

For the study of estimates and hypoellipticity, see for example [5–7, 11, 12]. In particular, in [5], we considered the case $m = 1$, and φ homogeneous. In [6], with B. Helffer, we studied the case $m = 1$, $n = 2$, and φ quasihomogeneous and in [7] the case $m = 2 = n$ and φ homogeneous.

Concerning microlocal subellipticity, we consider here the case where $n = 2$, $m \geq 2$ and Φ is a quasihomogeneous vector function, which means that, when denoting coordinates by (s, t)

$$\Phi(\lambda s, \lambda^\ell t) = \lambda^q \Phi(s, t),\ \forall (s, t) \in \mathbb{R}^2\,, \tag{1.2}$$

where

$$\ell \geq 1, \ell \in \mathbb{R}^+,\ \text{and } q \in \mathbb{R}^+, (q \geq 2\ell \text{ as assumption})\,.$$

The quasi-circle, defined by

$$S = \{(s^2)^\ell + t^2 = 1\} \subset \mathbb{R}^2_{(s,t)} \tag{1.3}$$

determines Φ by $\tilde{\Phi} = \Phi|_S$.

Now given a vector $\xi_0 \in \mathbb{R}^m \setminus \{0\}$, a conic neighborhood of ξ_0 in $\mathbb{R}^m$ is a set that we can define for $\delta > 0$

$$V_\delta = \{\xi = (\xi_1, \dots, \xi_m) \in \mathbb{R}^m \setminus \{0\} : \Big|\Big|\frac{\xi}{||\xi||} - \frac{\xi_0}{||\xi_0||}\Big|\Big| < \delta\},\ \delta \text{ small.} \tag{1.4}$$

Let us recall that, for $\xi \in \mathbb{R}^m \setminus \{0\}$, the following function $\Phi_\xi \colon \omega \to \mathbb{R}$

$$\Phi_\xi = \sum_{k=1}^{m} \xi_k \varphi_k \tag{1.5}$$

plays an important role.

Let us recall that since the system $L = L_\Phi$ is elliptic in the t-directions (meaning that the characteristic set of L is contained in $\{\tau = 0\}$, where (τ, ξ) is in the dual space), in order to study microlocal subellipticity for L_Φ, it is sufficient to study the validity of the following inequality, when we come back to general (t, x) variable in $\mathbb{R}^n \times \mathbb{R}^m$.

Definition 1.1 We say the the system $L = L_\Phi$ satisfies a microlocal σ-subelliptic estimate at $(0, 0; 0, \xi_0)$ if there exist neighborhoods ω of $0 \in \mathbb{R}^n_t$, Ω of $0 \in \mathbb{R}^m_x$, two conic neighborhoods V and V' of ξ_0 and a constant $C > 0$ such that

$$||u||_{\sigma,V} \leq C(||u||_{V'} + ||Lu||_{V'}),\ \forall u \in \mathcal{D}(\omega \times \Omega) \tag{1.6}$$

where

$$\begin{cases} ||u||^2_{s,V} & = \int_{\xi\in V}\int_{t\in\mathbb{R}^n}(1 + ||\xi||^2)^{s/2}|\widetilde{u}(t, \xi)|^2 dt d\xi,\ s \in \mathbb{R} \\ \widetilde{u}(t, \xi) & = (\mathcal{F}_x u)(t, \xi),\ \mathcal{F}_x \text{ is partial Fourier transform.} \end{cases} \tag{1.7}$$

Remark 1.2 To study microlocal subellipticity at $(0, 0; 0, \xi_0)$, we can assume that $\xi_0 = (1, 0, \dots, 0)$. In fact, we begin to note that we take $||\xi_0|| = 1$, in general. After just perform an orthogonal transform A in $\mathbb{R}^m$-space to reduce the situation to $\xi_0 = (1, 0, \dots, 0)$. Remark also that we have: if $\xi = A\xi_0$, $\Phi_{\xi_0} = \varphi_1$, $\Phi_\xi = \xi \cdot \Phi = A\xi_0 \cdot \Phi = \xi_0 \cdot {}^t A\Phi = ({}^t A\Phi)_1$. So in the new system of coordinates in $\mathbb{R}^m$, we have just to consider ${}^t A\Phi$ in place of Φ.

So, now, we always study microlocally $L = L_\Phi$ near the point $(0, 0; 0, (1, 0, \dots, 0))$ in $\mathbb{R}^n_t \times \mathbb{R}^m_x \times \mathbb{R}^n_\tau \times \mathbb{R}^m_\xi$. For the microlocal subellipticity, our study will be in case $n = 2$, $m \geq 2$, and considering Φ quasi-homogeneous. Concerning the question of maximal estimates, where we will work also microlocally near $\xi_0 = (1, 0, \dots, 0)$, let us also say a few words on what we mean; looking to the nature of our system (L), we look the situation for cones in $(\mathbb{R}^m)^*$. So a conic neighborhood of $(0, 0; 0, \xi_0)$ will be:

$$\omega \times (\Omega \times V),\ \text{where } 0 \in \omega \subset \mathbb{R}^n_t,\ 0 \in \Omega \subset \mathbb{R}^m_x,\ V \text{ cone around } \xi_0 \text{ in } \mathbb{R}^m. \tag{1.8}$$

When $u \in \mathcal{D}(\omega \times \Omega)$, we define:

$$||u||^2_V = \int_{\mathbb{R}^n}\int_{\xi\in V} |\widetilde{u}(t, \xi)|^2 dt d\xi, \quad \widetilde{u}(t, \xi) = (\mathcal{F}_x u)(t, \xi). \tag{1.9}$$

Definition 1.3 We say that the system (L) satisfies a maximal estimate, microlocally at ξ_0 on $\omega \times \Omega$ if there exist $V \subset V'$ cones in $\mathbb{R}^m$, with $\xi_0 \in V$ and $C > 0$ such that

$$||(\text{Re } L_j)u||_V \leq C||u||_{V'} + \sum_k ||L_k u||_{V'},\ \forall u \in \mathcal{D}(\omega \times \Omega). \tag{1.10}$$

We will give a large class of functions Φ such that the system (L)=(L_Φ) does not satisfy (1.10), for any n and any $m \geq 2$.

In the case $n = 2$, we will denote by (s, t) the variables

$$\begin{cases} s = t_1, t = t_2, \text{ so } u = u(s, t, x), \ (s, t) \in \omega, x \in \mathbb{R}^m \\ \quad \Phi \text{ given by (1.2)}, \ S \text{ by (1.3)}, \ \Phi \in C^k, \ k \geq 1. \end{cases} \tag{1.11}$$

Now the functions $\widetilde{\varphi}_j$ (recall that $\widetilde{\Phi} = \Phi|_S$) can be considered as functions of one variable near every point $(s, t) \in S$. Near the points $(0, 1)$ and $(0, -1)$, we consider $\widetilde{\varphi}_j$ as functions of s, s near 0 by

$$\begin{cases} \widetilde{\varphi}_j(s) = \varphi_j(s, (1 - s^{2\ell})^{1/2}), \text{ when } t_0 = 1 \quad s \in]-1, 1[\\ \widetilde{\varphi}_j(s) = \varphi_j(s, -(1 - s^{2\ell})^{1/2}), \text{ when } t_0 = -1 \quad s \in]-1, 1[. \end{cases} \tag{1.12}$$

In fact near every point $(s_0, t_0) \in S$ such that $t_0 \neq 0$, we can take

$$\begin{cases} \widetilde{\varphi}_j(s) = \varphi_j(s, (1 - (s^2)^\ell)^{1/2}), \text{ when } t_0 > 0 \quad s \in]s_0 - \epsilon, s_0 + \epsilon[\\ \widetilde{\varphi}_j(s) = \varphi_j(s, -(1 - (s^2)\ell)^{1/2}), \text{ when } t_0 < 0 \quad s \in]s_0 - \epsilon, s_0 + \epsilon[. \end{cases} \tag{1.13}$$

Outside the points $(1, 0)$ and $(-1, 0)$, we take as variable t, with

$$\begin{cases} \widetilde{\varphi}_j(t) = \varphi_j((1 - t^2)^{1/2\ell}, t), \text{ when } s_0 = 1 \quad t \in]-1, 1[\\ \widetilde{\varphi}_j(t) = \varphi_j(-(1 - t^2)^{1/2\ell}, t), \text{ when } s_0 = -1 \quad t \in]-1, 1[. \end{cases} \tag{1.14}$$

Definition 1.4 A point $(s_0, t_0) \in S$ is called a zero of order α, $\alpha < k$, of $\widetilde{\varphi}_j$ if

$$\begin{cases} \text{(i) if } t_0 = 1 \text{ or } t_0 = -1, \ s_0 = 0 \text{ is a zero of order } \alpha \text{ of } \widetilde{\varphi}_j(s) \text{ in (1.11)} \\ \text{(ii) if } s_0 = 1 \text{ or } s_0 = -1, \ t_0 = 0 \text{ is a zero of order } \alpha \text{ of } \widetilde{\varphi}_j(t) \text{ in (1.13)} \\ \text{(iii) if } s_0 \neq 0 \text{ (and } t_0 \neq 0), \ t_0 \text{ (or } s_0) \text{ is a zero of order } \alpha \text{ of } \widetilde{\varphi}_j(t) \text{ in (1.13)} \\ \quad \text{(resp. is a zero of order } \alpha \text{ of } \widetilde{\varphi}_j(s) \text{ in (1.12))}. \end{cases} \tag{1.15}$$

Remark that it is easy to show that in case (iii) the definition is coherent: we use the fact $\widetilde{\varphi}_j(s)$ (or $\widetilde{\varphi}_j(t)$) have s_0 (or t_0) as zeroes of the same order.

Once the parameter s or t is chosen near a point and if $(s_0, t_0) \in S$ then we write

$$\varphi_j^{(\alpha)}(s_0, t_0) = \widetilde{\varphi}_j^{(\alpha)}(s_0)$$

$$\Phi^{(\alpha)}(s_0, t_0) \text{ is the vector } (\varphi_j^{(\alpha)}(s_0, t_0))_{j=1,\dots,m}.$$

Remark 1.5 When ℓ is not an integer, $\widetilde{\Phi}$ is not smooth but of class $C^{2E(\ell),2\sigma}$ with $\ell = E(\ell) + \sigma$ near $s = 0$. But it is easy to see that, near $s = 0$, when Φ is C^k, we have either $\Phi_\xi(s,t) = s^j \cdot (\Phi_\xi)_j(s,t)$ with j strictly less than k (maybe $j = 0$), $(\Phi_\xi)_j(s,t)$ of class C^{k-j} and $(\Phi_\xi)_j$ not vanishing at (0, 1) (in that case we say that (0, 1) is a zero of order j of Φ_ξ, or, $\Phi_\xi(s,t) = s^k \cdot \Phi_k(s,t)$ for some $\Phi_k(s,t)$ continuous (recall that Φ_ξ is the scalar product of Φ and ξ).

In our second result, our goal is to give a class of vector functions Φ, generalizing considerably the simple example in $\mathbb{R}^{2+2}_{t,x}$ for which (see [7]) there is no microlocal maximal estimate at $(0, 0; 0, \xi_0)$, $\xi_0 = (1, 0)$: $\Phi = (\varphi_1(s,t) = s^3, \varphi_2(s,t) = st^2)$.

In a third result we give in Sect. 4 a subclass of the above class of Φ's, for which we have no microlocal hypoellipticity.

Finally, we recall the Baouendi-Treves condition which we always assume when studying subellipticity. It is not really an hypothesis, in the sense that as it is a necessary condition for microlocal hypoellipticity, we can not expect microlocal subellipticity without assuming it.

Definition 1.6 We say that Φ satisfies Baouendi-Treves condition on $(\omega \times \Omega, V)$ if for any $\xi \in V$, $\Phi_\xi = \xi \cdot \Phi$ has no local maximum in ω. (Here, as always, $0 \in \omega \subset \mathbb{R}^n$ and V a given cone in $\mathbb{R}^m$).

The present work follows some of my papers (alone or in joint work) on the study of estimates and regularity for systems of vector fields and related partial differential operators.

Salah Baouendi, who was my first advisor, introduced me to the study of systems of real vector fields. He obtained in 1966 (as a particular case of his work [1]) optimal sub elliptic estimates for the systems of the following kind in $\mathbb{R}^n$: $(X_1, \ldots, X_n)$ where:

$$X_j = x_n^k \partial_{x_j},\ j = 1, \ldots, n-1,\ X_n = \partial_{x_n},\ (x_1, \ldots, x_n) \in \mathbb{R}^n,\ k \in \mathbb{N}.$$

Just after, in 1967, L. Hörmander studied systems of smooth real vector fields satisfying the so-called Hörmander's bracket condition [10] for which he gave quasi-optimal sub elliptic estimates and later on, L.P. Rothschild and E. Stein proved optimal estimates [13].

2 Microlocal Subellipticity for $(L) = (L_\Phi)$

2.1 Statement of the Main Result

In this section we want to prove the following result.

Theorem 2.1 *Let Φ be as in (1.2) and of class C^k. Assume that Φ satisfies the Treves condition in $(\omega \times \Omega, V)$ and the following three hypothesis (i), (ii), (iii) which we will denote in what follows by $(H1)(\xi)$, with ξ given in $\mathbb{R}^m \setminus \{0\}$:*

(i) $\Phi(s, t) \neq 0$, $\forall (s, t) \in S$, so there exist η_1, η_2 such that $\Phi_{\eta_1}(0, 1) \neq 0$ and $\Phi_{\eta_2}(0, -1) \neq 0$.
(ii) The function $\widetilde{\Phi}_\xi$ has a finite number of zeroes of order less than k (on S). Call Z_ξ this set of zeroes on S.
(iii) Every zero $(s_0, t_0) \in Z_\xi$ satisfies one of the following conditions:
  (a) The order $o_{(s_0,t_0)}$ of the zero (s_0, t_0) of $\widetilde{\Phi}_\xi$ is ≤ 2.
  (b) The order $o_{(s_0,t_0)}$ is ≥ 3 and $\dim span\{\widetilde{\Phi}^{(\alpha)}(s_0, t_0), \alpha < o_{(s_0,t_0)}\} = 1$ if $s_0 \neq 0$. If $s_0 = 0$, the order $o_{(0,1)}$ is ≥ 3 and for all $\zeta \in \mathbb{R}^m$, Φ_ζ vanishes at $(0, 1)$, modulo Φ_{η_1}, at order $\geq o_{(0,1)}$ (same for $(0, -1)$). Moreover, if the order is even, we can relax condition i) (i.e. the vector function Φ may vanish at that zero).
  (c) The order $o_{(s_0,t_0)}$ is even, ≥ 4 and $\dim span\{\widetilde{\Phi}^{(\alpha)}(s_0, t_0), \alpha < o_{(s_0,t_0)}\} = 2$ if $s_0 \neq 0$. If $s_0 = 0$, the order $o_{(0,1)}$ is even ≥ 4 and there exist $\gamma \in \mathbb{R}^m$ and $q_0 \in \mathbb{N}$, $1 \leq q_0 < o_{(0,1)}$ such that Φ_γ vanishes at $(0, 1)$ at order q_0 and for all $\zeta \in \mathbb{R}^m$ Φ_ζ vanishes, modulo $\text{span}\{\Phi_{\eta_1}, \Phi_\gamma\}$ at $(0, 1)$ at order $\geq o_{(0,1)}$.

Then (L_Φ) is σ-microlocally subelliptic at $(0, 0; 0, \xi)$ with $\sigma = \inf(\frac{1}{q}, \frac{1}{p})$ where $p = \sup\{o_{(s_0,t_0)} : (s_0, t_0) \in Z_\xi\}$.

2.2 *Proof of Theorem 2.1*

First we recall that we can take $\xi_0 = (1, 0, \ldots, 0)$. The proof will be in several steps.

2.2.1 Step 1: More Explicit Formulation for $(H1)(\xi_0)$ and Simplification as $\Phi_{\xi_0} = \varphi_1$

So now conditions (i), (ii), (iii) can be read on φ_1 .

Let us express in particular cases (b) and (c). The order $o_{(s_0,t_0)}$ of (s_0, t_0) is its order as a zero of φ_1. Now consider the vector space defined by the vectors $\{\widetilde{\varphi}_1^{(\alpha)}, \ldots, \widetilde{\varphi}_m^{(\alpha)}\}(s_0, t_0)$ for $\alpha = 0, 1, \ldots, o_{(s_0,t_0)} - 1$. Now the condition (i) means that there exists $j \in \{2, \ldots, m\}$ such that $\widetilde{\varphi}_j(s_0, t_0) \neq 0$. We may assume without loss of generality that $j = 2$. Consider first the case $(s_0, t_0) \neq (0, 1)$ or $(0, -1)$; so $\{\widetilde{\varphi}_1^{(\alpha)}, \ldots, \widetilde{\varphi}_m^{(\alpha)}\}(s_0, t_0)$ are well defined. We may assume, working with $\widetilde{\varphi}_j(s)$, s near s_0, by small translation on s, that we work near $s = 0$, with $\widetilde{\varphi}_j$ smooth.

Moreover, in case (b), we may assume that:

$$\widetilde{\Phi}(0) = (0, \widetilde{\varphi}_2(0), 0, \ldots, 0); \;\; \widetilde{\Phi}^{(\alpha)}(0) = (0, a_\alpha, 0, \ldots, 0) \text{ for } \alpha < p. \tag{2.1}$$

Recall that a conic neighborhood of $\xi_0 \in \mathbb{R}^m$ is a cone which can be written in our case as ;

$$V_{\xi_0,\delta} = \{(\xi_1, \ldots, \xi_m) \in \mathbb{R}^m : ||\xi'|| < \delta\xi_1 \; ; \; \xi' = (\xi_2, \ldots, \xi_m), \; \delta > 0 \text{ small}\}. \tag{2.2}$$

Now, from (2.1), we can write more explicitly

$$\widetilde{\varphi}_1(s) = s^p\psi_1(s), \; \psi_1(0) \neq 0, \; \widetilde{\varphi}_2(0) \neq 0, \; \widetilde{\varphi}_j(s) = s^p\psi_j(s), \; j \geq 3, \; \text{in } I. \tag{2.3}$$

Let us now look at the much more complicated case c): we continue to assume that $\widetilde{\varphi}_2(0) \neq 0$, $\widetilde{\varphi}_j(0) = 0$, $j \neq 2$. Define now q by:

$$q \text{ is the smallest integer such that } \{\widetilde{\Phi}(0), \widetilde{\Phi}^{(q)}(0)\} \text{ is a basis of } Span\,\{\Phi^{(\alpha)}(0); \alpha < p\}. \tag{2.4}$$

We may also assume

$$\begin{cases} \quad \widetilde{\Phi}(0) = (0, \widetilde{\varphi}_2(0), 0, \ldots, 0); \; \widetilde{\Phi}^{(\alpha)}(0) = (0, a_\alpha, 0, \ldots, 0), \; \alpha < q; \\ \widetilde{\Phi}^{(q)}(0) = (0, 0, \widetilde{\varphi}_3^{(q)}(0), 0, \ldots, 0) \\ \widetilde{\Phi}^{(\alpha)}(0) = (0, a_\alpha, b_\alpha, 0, \ldots, 0); q < \alpha < p. \end{cases} \tag{2.5}$$

So, in this case, we can write:

$$\begin{cases} \widetilde{\varphi}_1(s) = s^p\psi_1(s), \; \widetilde{\varphi}_2(0) \neq 0, \; \widetilde{\varphi}_3(s) = s^q\psi_3(s), \\ \psi_1(0) \neq 0, \; \psi_3(0) \neq 0, \widetilde{\varphi}_j(s) = s^p\psi_j(s), \, j \geq 4. \end{cases} \tag{2.6}$$

As in [7], we define now $\epsilon = (\epsilon_2, \ldots, \epsilon_m)$. Here $\epsilon \in \mathbb{R}^{m-1}$, $||\epsilon|| < \epsilon_0$ and

$$\widetilde{\varphi}_\epsilon(s) = \widetilde{\varphi}_1(s) + \sum_{j=2}^{m} \epsilon_j \widetilde{\varphi}_j(s), \; s \in I = I_\sigma = (-\sigma, \sigma)\,. \tag{2.7}$$

In the case where (0, 1) or (0, −1) is a zero of φ_1, the formulas (2.3) in case (b) and (2.7) in case (c) are respectively direct consequences of the second parts of (b) and (c) respectively. We saw in [7] that we have a microlocal subellipticity in $(\omega \times \Omega, V_{\xi_0,\delta})$, if the graph of the function $\widetilde{\varphi}_\epsilon$ is of good type for every ϵ, $||\epsilon|| < \epsilon_0$, ϵ_0 small and I too.

Concerning the meaning of good type, we refer to what we did with Bernard Helffer in [7] concerning graphs in I, I small, determining sectors (here quasisectors, as Φ is quasihomogeneous) of good types $A^\pm$, $B^\pm$, $C^\pm$ ([7] pages 31–32). The graphs of $\widetilde{\varphi}_\epsilon$ are easy to see, in the case a) (pieces of parabola at worse).

2.2.2 Step 2: The Proof in the Simple Case (b)

We give the proof in the more complicated case , i.e. $(s_0, t_0) = (0, 1)$, $\ell \notin \mathbb{N}$ (case where $t(s) = (1 - |s|^{2\ell})^{1/2}$ is not smooth).

We begin by writing in a convenient way $\widetilde{\varphi}_\epsilon$. We start with (2.3)

$$\widetilde{\varphi}_\epsilon(s) = s^p \psi_1(s) + \epsilon_2 \widetilde{\varphi}_2(s) + \sum_{j=3}^{m} \epsilon_j s^p \psi_j(s), \quad \psi_1(0) \neq 0,\ \widetilde{\varphi}_2(0) \neq 0. \tag{2.8}$$

There is $\sigma > 0$ and $C_0 > 0$ such that

$$|\psi_1| \geq 2C_0,\ |\widetilde{\varphi}_2| \geq 2C_0,\ \text{in } I_\sigma = (-\sigma, \sigma)\,. \tag{2.9}$$

From (2.9), we obtain, using (2.8) and putting $\sum_{j=3}^{m} |\psi_j| \leq C_1$ in I_σ ,

$$\begin{cases} \widetilde{\varphi}_\epsilon(s) = s^p \psi_\epsilon(s) + \epsilon_2 \widetilde{\varphi}_2(s),\ |\widetilde{\varphi}_2| \geq 2C_0 \text{ in } I_\sigma,\ \text{if } ||\epsilon'|| = ||(\epsilon_3, \ldots, \epsilon_m|| < \dfrac{C_0}{C_1} \\ \qquad |\psi_\epsilon| \geq C_0,\ \text{in } I_\sigma. \text{ Put also } |\psi_\epsilon| \leq C_1,\ \text{by enlarging } C_1 \text{ if necessary.} \end{cases} \tag{2.10}$$

Now if p is odd, the situation is simple:

(A) *p odd*: as $|\psi_\epsilon| \geq C_0$, to study $\widetilde{\varphi}_\epsilon$, we first look at the zeroes of $\widetilde{\varphi}_\epsilon$ which are the zeroes of the equation on $I = I_\sigma$,

$$s^p = -\frac{\widetilde{\varphi}_2}{\psi_\epsilon}(s) \Leftrightarrow s = \left(-\frac{\widetilde{\varphi}_2}{\psi_\epsilon}(s)\right)^{1/p}. \tag{2.11}$$

As $\left|\frac{\widetilde{\varphi}_2}{\psi_\epsilon}(s)\right| \geq C_0$, then for ϵ_0 small enough, for every ϵ with $|\epsilon_2| < \epsilon_0$, Eq. (2.11) has a unique solution in I_σ. Consider now $\widetilde{\varphi}'_\epsilon(s)$. The expression of $\widetilde{\varphi}'_\epsilon(s)$ is

$$\widetilde{\varphi}'_\epsilon(s) = s^{p-1}(p\psi_\epsilon + s\psi'_\epsilon)(s) + \epsilon_2 \widetilde{\varphi}'_2(s). \tag{2.12}$$

– If $\widetilde{\varphi}'_2(s) = s^{p-1}\theta_2(s)$, then in this case

$$\widetilde{\varphi}'_\epsilon(s) = s^{p-1}(p\psi_\epsilon + \epsilon_2\theta_2 + s\psi')(s)$$

and

$$p\psi_\epsilon + \epsilon_2\theta_2 + s\psi' \neq 0, \quad \text{on} I_\sigma.$$

(σ and ϵ_0 are small enough.) So 0 is the unique zero of $\widetilde{\varphi}'_\epsilon$ on I_σ of even order, and then $\widetilde{\varphi}_\epsilon$ is monotonic on I_σ, whose graph is therefore of good type.

Otherwise, we need more precise expressions of $\widetilde{\varphi}_2(s)$ and $\widetilde{\varphi}'_2(s)$. Expressions of $\widetilde{\varphi}_2(s)$ and $\widetilde{\varphi}'_2(s)$: as $\varphi_2 \in C^k(\mathbb{R}^2)$ with $p \leq k - 1$, we may write

$$\varphi_2(s,t) = \sum_{0\le j\le p-1} \frac{s^j}{j!}\varphi_2^{(j,0)}(0,t) + s^p\theta_2(s,t)\,;\ \varphi_2(0,1)\ne 0. \tag{2.13}$$

Remark that $\varphi_2^{(j,0)}(0,t)$ is C^{k-j} and a homogeneous function of t of degree $\frac{q-j}{\ell}$, and θ_2 is $(1,\ell)$ quasi-homogeneous of degree $q-p$ and C^{k-p}. Hence, by differentiation, we deduce for $t>0$

$$\begin{cases} \varphi_2^{(1,0)}(s,t) = \sum_{0\le j\le p-2} a_j s^j t^{(q-(j+1))/\ell} + s^{p-1}\theta_3(s,t) \\ \varphi_2^{(0,1)}(s,t) = \sum_{0\le j\le p-2} b_j s^j t^{(q-j)/\ell-1} + s^{p-1}\theta_4(s,t);\ b_0\ne 0. \end{cases} \tag{2.14}$$

Now using $\widetilde{\varphi}_2(s) = \varphi_2(s,t(s))$ with $t(s) = (1-|s|^{2\ell})^{1/2}$ and $\widetilde{\varphi}_2'(s) = \varphi_2^{(1,0)}(s,t(s))$ $+\,t'(s)\varphi_2^{(0,1)}(s,t(s))$, we get

$$\begin{aligned} \widetilde{\varphi}_2'(s) = &\sum_{0\le j\le p-2} a_j s^j (t(s))^{(q-(j+1))/\ell} - \ell s|s|^{2(\ell-1)} \sum_{0\le j\le p-2} b_j s^j (t(s))^{(q-j)/\ell-1} \\ &+ s^{p-1}\theta_5(s,t(s));\ b_0\ne 0. \end{aligned} \tag{2.15}$$

Now if $a_0\ne 0$, from (2.12) and (2.15), we see that 0 is the only zero of $\widetilde{\varphi}_\epsilon'(s)$, in I_σ, σ small.

Otherwise, let α such that $\alpha>0$

$$\varphi_2^{(1,0)}(s,t(s)) = \begin{cases} s^\alpha A_\alpha(s),\ |A_\alpha|\ge c_0 \text{ on } I_\sigma \text{ if } \alpha\le p-2 \\ s^{p-1}A_{p-1}(s) \text{ otherwise.} \end{cases}$$

So,

$$\begin{cases} \varphi_2'(s) = s^\alpha A_\alpha(s) + s|s|^{2(\ell-1)}B(s),\ |B|\ge c_0 \text{ on } I_\sigma \\ \quad B = b_0 + O(s), \end{cases} \tag{2.16}$$

and hence B has the sign of $b_0\ne 0$. Now s is a zero of $\widetilde{\varphi}_2'$ hence of φ_ϵ'. To study $\widetilde{\varphi}_2'(s)$ more closely, we consider first the case

• $2\ell\notin\mathbb{N}$, hence $\alpha\ne 2\ell-1$. Let us consider the worst case where $2\ell-1<\alpha<p-1$, so that

$$\varphi_\epsilon'(s) = s|s|^{2(\ell-1)}\left(s^{p-2}|s|^{2(1-\ell)}\theta_\epsilon + \epsilon_2(B(s) + O(s^{\alpha-2\ell+1}))\right). \tag{2.17}$$

As $p-2$ is odd, it is easy to see that the expression in the factor has one zero s_1 so that $|s_1|\sim|\epsilon_2|^{1/(p-2\ell)}$ and s_1 has the sign of $(-\epsilon_2 b_0)$. If $\alpha<2\ell-1<p-1$ one has

$$\varphi_\epsilon'(s) = s^\alpha\left(s^{p-1-\alpha}\theta_\epsilon + \epsilon_2(A_\alpha(s) + O(s^{2\ell-1-\alpha}))\right). \tag{2.18}$$

So $\varphi'_\epsilon(s)$ may have 0, 1 or 2 zeroes, depending on the value of α and the sign of $\epsilon_2 \frac{A_\alpha}{\theta_\epsilon}(0)$. We remark that the value of $\widetilde{\varphi}_\epsilon$ at these zeroes is comparable to ϵ_2, obtaining a good graph.

• $2\ell \in \mathbb{N}$: in that case, we need to write $\widetilde{\varphi}'_2(s)$ more precisely, by giving an expansion of $t(s)$ or its powers, which appear in (2.15) in terms of powers of $|s|^{2\ell}$. Replacing in (2.15) we obtain, with $b_{00} = b_0 \neq 0$:

$$\widetilde{\varphi}'_2(s) = \sum_{j+2\ell\beta \leq p-2} a_{jp} s^j |s|^{2\ell\beta} - \ell s |s|^{2(\ell-1)} \sum_{2\ell-1+j+2\ell\beta \leq p-2} b_{j\beta} s^j |s|^{2\ell\beta} + s^{p-1}\theta_6(s). \tag{2.19}$$

If $a_{00} = a_0 \neq 0$, $\widetilde{\varphi}'_2(0) = a_{00} \neq 0$. If $a_0 = 0$, $s = 0$ is a zero of $\widetilde{\varphi}_2$. In order to search for other zeroes, we consider (2.19) in cases $s > 0$ and $s < 0$. So we have:

$$\begin{cases} \varphi'_2(s) = s^\alpha F_\alpha(s), \text{ with } F_\alpha(0) \neq 0 \text{ if } 0 \leq \alpha \leq p-2, s > 0 \\ \varphi'_2(s) = s^\beta G_\beta(s), \text{ with } G_\beta(0) \neq 0 \text{ if } 0 \leq \beta \leq p-2, s < 0. \end{cases} \tag{2.20}$$

Then, so, $\widetilde{\varphi}'_\epsilon$ is such that, for σ, c_0, ϵ_0 small enough

$$\begin{cases} \varphi'_2(s) = s^\alpha(s^{p-\alpha-1}\theta_\epsilon(s) + \epsilon_2 F_\alpha(s)), \ s > 0, \ \inf_{I_\sigma} (|\theta_\epsilon|, |F_\alpha|) \geq c_0 \\ \varphi'_2(s) = s^\beta(s^{p-\alpha-1}\theta_\epsilon(s) + \epsilon_2 G_\beta(s)), \ s < 0, \ \inf_{I_\sigma} (|\theta_\epsilon|, |G_\beta|) \geq c_0. \end{cases} \tag{2.21}$$

Of course, in (2.21), in case $\alpha = p-1$, $\widetilde{\varphi}'_\epsilon = s^{p-1}\tilde{\theta}_\epsilon$, $s > 0$, or in case $\beta = p-1$, $\widetilde{\varphi}'_\epsilon(s) = s^{p-1}\tilde{\theta}_\epsilon(s)$, $s < 0$, so just 0 is the zero.

So we see that, depending on the value of α, ϵ_2 and β, $\widetilde{\varphi}'(s)$ has, for $s > 0$, 0 or 1 zero, and for $s < 0$, 0 or 1 zero, and the value of $\widetilde{\varphi}_\epsilon(s)$ at these zeroes is comparable to ϵ_2. So $\widetilde{\varphi}_\epsilon$ has a good graph.

(B) *p even*: the study is quite the same as above, but $\widetilde{\varphi}_\epsilon$ may have two zeroes, giving that $\widetilde{\varphi}_\epsilon$ has a good graph.

Before going on the proof of part (c), which will be much harder to deal with, let us note that in part (b), just one parameter, ϵ_2, played a real role, corresponding to the dimension one of *Span* $\widetilde{\Phi}^{(\alpha)}(0)$, $\alpha < p$. As in (c), this dimension is two, we will face a situation where two real parameters will play a role. In part (c), we will have to work with functions $\widetilde{\varphi}_\epsilon$ depending essentially on two parameters associated to the dimension two of the span.

2.2.3 Step 3: Simplification of the Notations in Case (c), for the Functions $\widetilde{\varphi}_\epsilon$

We come back to our function $\widetilde{\varphi}_\epsilon$ given in (2.7) with the property (2.6). So we are in the case where

$$\begin{cases} \widetilde{\varphi}_\epsilon(s) = s^p\psi_1(s) + \epsilon_2\widetilde{\varphi_2}(s) + \epsilon_3 s^q\psi_3(s) + \sum_{j=4}^{m} \epsilon_j s^p \psi_j(s) \\ \qquad\qquad\qquad\qquad\qquad \widetilde{\varphi_2}(0) \neq 0,\ \psi_3(0) \neq 0. \end{cases} \tag{2.22}$$

Of course, if $m = 3$, the last term does not exist. Putting

$$\psi_\epsilon(s) = \psi_1(s) + \sum_{j=4}^{m} \epsilon_j \psi_j(s). \tag{2.23}$$

We can write more simply for $\widetilde{\varphi}_\epsilon$.

$$\widetilde{\varphi}_\epsilon(s) = s^p\psi_\epsilon(s) + \epsilon_2\widetilde{\varphi}_2(s) + \epsilon_3 s^q\psi_3(s). \tag{2.24}$$

Now we have, for C_0, σ, $(\epsilon_4, \dots, \epsilon_m)$ small enough

$$\begin{cases} |\psi_3| \geq 2C_0 \\ |\psi_\epsilon| \geq 2C_0, \quad \text{on } I_\sigma, \quad \text{because } \psi_1(0) \neq 0,\ \widetilde{\varphi}_2(0) \neq 0,\ \psi_3(0) \neq 0 \\ |\widetilde{\varphi}_2(s)| \geq 2C_0. \end{cases} \tag{2.25}$$

We see that the couple (ϵ_2, ϵ_3) will play really a role in the study of the graph of $\widetilde{\varphi}_\epsilon$ for ϵ small. For that, we put along this study, more simply

$$\epsilon_2 = \delta,\ \epsilon_3 = \gamma. \tag{2.26}$$

So we have to study

$$\begin{cases} \widetilde{\varphi}_\epsilon = s^p\psi_\epsilon(s) + \delta\widetilde{\varphi}_\epsilon + \gamma s^q\psi_3(s), \\ \text{with } q > 0,\ |\psi_\epsilon| \geq 2C_0,\ |\widetilde{\varphi}_2| \geq 2C_0,\ |\psi_3| \geq 2C_0, \text{ on } I_\sigma. \end{cases} \tag{2.27}$$

2.2.4 Step 4: Further Reductions

We know that p is even, by the condition in (c). Let us say more in q, from the Treves condition, and also on ψ_ϵ. For that we give the following very simple lemma.

Lemma 2.2 *With the hypothesis in Theorem 2.1, we have:*

$$\begin{cases} q \text{ is odd} \\ \psi_\epsilon > 0, \text{ on } I_\sigma. \end{cases} \tag{2.28}$$

Proof of Lemma 2.2 As φ_1 satisfies Treves' condition, $\widetilde{\varphi}_1$ has no local maximum at 0. As $\widetilde{\varphi} = s^p\psi_1$, p even, $\psi_1 \neq 0$, we have $\psi_1 > 0$ on I_σ, which gives $\psi_\epsilon \geq 2C_0 > 0$, on I_σ, for $||\epsilon|| < \epsilon_0$ and ϵ_0, σ, C_0 small enough. Now assume q even $0 < q < p$.

Choose $\widetilde{\epsilon}$, $||\widetilde{\epsilon}|| < \epsilon_0$, $\widetilde{\epsilon} = (\delta, \gamma, \ldots)$ such that $\gamma\psi_3(0) < 0$. Then there exists $\widetilde{\sigma} > 0$ (depending on $\widetilde{\epsilon}$) such that $\widetilde{\varphi}_{\widetilde{\epsilon}}$ vanishes at 0 and is strictly negative on $I_{\widetilde{\sigma}} \setminus \{0\}$. So $\widetilde{\varphi}_{\widetilde{\epsilon}}$ has a local maximum at 0, contradicting the Treves condition. □

So, from now on, we work under the following:

$$\begin{cases} \widetilde{\varphi}_\epsilon(s) = s^p\psi_\epsilon(s) + \delta\widetilde{\varphi}_2(s) + \gamma s^q\psi_3(s), \text{ with} \\ p \text{ even } \geq 4,\ q \text{ odd},\ |\psi_\epsilon| \geq 2C_0,\ |\widetilde{\varphi}_2| \geq 2C_0,\ |\psi_3| \geq 2C_0, \text{ on } I_\sigma. \end{cases} \tag{2.29}$$

Now we are going to make a reduction on our situation: for that, to simplify, we put:

$$\widetilde{\varphi}_2 = f,\ \psi_3 = g \text{ in (2.29)}. \tag{2.30}$$

Moreover, we give another simple lemma which will permit us to work with $\gamma g > 0$ (the case $\gamma = 0$ was studied in case (b)).

Lemma 2.3 *To study* (2.29)*, it suffices to work with* $\gamma g > 0$. □

Proof of Lemma 2.3 Let us consider the case $\gamma g < 0$, i.e. take

$$\widetilde{\varphi}_\epsilon(s) = s^p\psi_\epsilon(s) + \delta f(s) + \gamma s^q g(s);\ \gamma g < 0, \text{ on } I_\sigma$$

$$\begin{aligned} \widetilde{\varphi}_\epsilon(-s) &= s^p\psi_\epsilon(-s) + \delta f(-s) - \gamma s^q g(-s); && \gamma g < 0, \text{ as } q \text{ odd} \\ &= s^p\theta_\epsilon(s) + \delta h(s) - \gamma s^q k(s), && \text{and } -\gamma k > 0. \end{aligned}$$

So $\widetilde{\varphi}_\epsilon(-s)$ has the form (2.29) with $\gamma g > 0$. So the graph will be the symmetric, with respect to $s = 0$, of that of $\widetilde{\varphi}_\epsilon(-s)$, which will be under our study with condition $\gamma g > 0$. □

So we arrive at our step 5.

2.2.5 Step 5

The study of the graph of $\widetilde{\varphi}_\epsilon$ given by

$$\begin{cases} \widetilde{\varphi}_\epsilon(s) = s^p\psi_\epsilon(s) + \delta f(s) + \gamma s^q g(s),\ p \text{ even } \geq 4 \\ q \text{ odd};\ s \in I_\sigma,\ \psi_\epsilon(s) \geq 2C_0 > 0,\ |f(s)| \geq 2C_0 > 0, \\ |g(s)| \geq 2C_0,\ \gamma g > 0,\ ||\epsilon|| < \epsilon_0,\ q < p. \end{cases} \tag{2.31}$$

For that, we will give results on the behaviour of zeroes of $\widetilde{\varphi}_\epsilon$ and $\widetilde{\varphi}_\epsilon{}'$, with respect to δ, γ for $||\epsilon|| < \epsilon_0$, ϵ_0 small.

Lemma 2.4 *Let* $\widetilde{\varphi}_\epsilon$ *as in* (2.31)*. If* $s_\epsilon \in I_\sigma$ *is a zero of* $\widetilde{\varphi}_\epsilon$*, with* $s_\epsilon \geq 0$*, then there exists* $C > 0$ *such that* $s_\epsilon^q \leq C|\delta|$*, with* C *independent of* ϵ *for* ϵ *sufficiently small.*

Proof of Lemma 2.4 As $\gamma s_\epsilon^q g(s_\epsilon) \geq 0$, we have, for $||\epsilon|| < \epsilon_0$,

$$0 < s_\epsilon^q |\psi_\epsilon(s_\epsilon)| \leq \delta |f(s_\epsilon)| \leq \widetilde{C}\delta, \tag{2.32}$$

where $|f| \leq \widetilde{C}$ on I_σ. Hence we obtain what we wanted

$$s_\epsilon^q \leq C|\delta| = \frac{\widetilde{C}}{2C_0}|\delta|. \tag{2.33}$$

□

Our goal is to study zeroes of $\widetilde{\varphi}_\epsilon$ which are negative. For ϵ small enough (recall that $\epsilon = (\delta, \gamma, \epsilon_4, \ldots, \epsilon_m)$ i.e. $||\epsilon|| < \epsilon_0$, we have that $\widetilde{\varphi}_\epsilon$ is first decreasing in a small interval around $-\sigma$, because for ϵ small enough $\widetilde{\varphi}_\epsilon{}'$ is negative near $-\sigma$. These things are trivial due to $\psi_\epsilon \geq 2C_0$ on I_σ.

We have the following crucial lemma.

Lemma 2.5 *Let us consider the following*

$$\begin{cases} \widetilde{s}_\epsilon < s_\epsilon < 0, \quad \text{in } I_\sigma \\ \widetilde{\varphi}_\epsilon{}'(\widetilde{s}_\epsilon) = 0, \quad \widetilde{\varphi}_\epsilon(s_\epsilon) = 0. \end{cases} \tag{2.34}$$

Then there exist σ, ϵ_0 *small enough and* $C > 0$ *such that*

$$|s_\epsilon^p| \leq C|\delta|, \ |\gamma s_\epsilon^q| \leq C|\delta|, \ ||\epsilon|| < \epsilon_0. \tag{2.35}$$

Proof of Lemma 2.5 Let us first write equalities satisfied by s_ϵ and $\widetilde{s}_\epsilon$.

$$\begin{cases} \widetilde{\varphi}_\epsilon(s_\epsilon) = s_\epsilon^p \psi_\epsilon(s_\epsilon) + \delta f(s_\epsilon) + \gamma s_\epsilon^q g(s_\epsilon) = 0 \\ \widetilde{\varphi}_\epsilon{}'(\widetilde{s}_\epsilon) = \widetilde{s}_\epsilon^{p-1}(p\psi_\epsilon(\widetilde{s}_\epsilon) + \widetilde{s}_\epsilon \psi_\epsilon'(\widetilde{s}_\epsilon)) + \delta f'(\widetilde{s}_\epsilon) + \gamma \widetilde{s}_\epsilon^{q-1}(qg(\widetilde{s}_\epsilon) + s_\epsilon g'(\widetilde{s}_\epsilon)) = 0 \end{cases}$$

which we rewrite, conveniently, as

$$\begin{cases} s_\epsilon^q(s_\epsilon^{p-q}\psi_\epsilon(s_\epsilon) + \gamma g(s_\epsilon)) = -\delta f(s_\epsilon) \\ \widetilde{s}_\epsilon^{q-1}(\widetilde{s}_\epsilon^{p-q}(p\psi_\epsilon(\widetilde{s}_\epsilon) + \widetilde{s}_\epsilon\psi_\epsilon'(\widetilde{s}_\epsilon)) + \gamma(qg(\widetilde{s}_\epsilon) + s_\epsilon g'(\widetilde{s}_\epsilon)) = -\delta f'(\widetilde{s}_\epsilon) \\ p - q \text{ is odd.} \end{cases} \tag{2.36}$$

Our goal here is to prove the first inequality. The second one follows from the first. Let a constant $C_1 > 0$ such that

$$|s_\epsilon|^p > C_1|\delta|. \tag{2.37}$$

From the first relation in (2.36) and the fact that $|f| < C_2$ in I

$$|s_\epsilon^{p-q}\psi_\epsilon(s_\epsilon)+\gamma g(s_\epsilon)|<\frac{C_2}{C_1^{\frac{q}{p}}}|\delta|^{1-\frac{q}{p}}. \tag{2.38}$$

As $\widetilde{s}_\epsilon < s_\epsilon < 0$, from (2.37), we deduce

$$|\widetilde{s}_\epsilon|^p > C_1|\delta|. \tag{2.39}$$

So, from the second relation in (2.36), we get, if $|f'|<C_2$

$$|\widetilde{s}_\epsilon^{p-q}(p\psi_\epsilon(\widetilde{s}_\epsilon)+\widetilde{s}_\epsilon\psi_\epsilon'(\widetilde{s}_\epsilon))+\gamma(qg(\widetilde{s}_\epsilon)+s_\epsilon g'(\widetilde{s}_\epsilon)|<\frac{C_2}{C_1^{\frac{q-1}{p}}}|\delta|^{1-\frac{q-1}{p}}. \tag{2.40}$$

We use now the fact that $\psi_\epsilon \geq 2C_0$ on I_σ. Then from (2.38) and (2.40):

$$\left|s_\epsilon^{p-q}+\gamma\frac{g(s_\epsilon)}{\psi_\epsilon(s_\epsilon)}\right|<\frac{C_2}{C_1^{\frac{p}{q}}}\frac{1}{2C_0}|\delta|^{1-\frac{q}{p}} \tag{2.41}$$

$$\left|\widetilde{s}_\epsilon^{p-q}+\gamma\frac{qg(\widetilde{s}_\epsilon)+\widetilde{s}_\epsilon g'(\widetilde{s}_\epsilon)}{p\psi_\epsilon+\widetilde{s}_\epsilon\psi'(\widetilde{s}_\epsilon)}\right|<\frac{C_2}{C_1^{\frac{q-1}{p}}}\frac{1}{C_0}|\delta|^{1-\frac{q-1}{p}}, \tag{2.42}$$

if $\sigma \leq \sigma_0$, σ_0 small enough, in order that $p\psi_\epsilon + s\psi' > C_0$ in I_σ. Now we use the fact that $q<p$ i.e. $\frac{q}{p}<1$. So, we choose σ such that the factor of γ in (2.41) and the one of γ in (2.42) named now $A(s_\epsilon)$ and $B(\widetilde{s}_\epsilon)$ are such that with a very small to be chosen

$$\begin{cases} |A(s_\epsilon)-A(0)|<a|A(0)| \\ |B(\widetilde{s}_\epsilon)-B(0)|<a|B(0)|, \end{cases} \tag{2.43}$$

with A and B depending on ϵ. Note that the modulus of each function A and B is bounded from below by C_0, shrinking C_0 if necessary. Remark that $B(0)=\frac{q}{p}A(0)$, and $\gamma A > 0$, $p-q$ odd in our crucial lemma, we never use that p is even or q odd, we will just use that $p-q$ is odd. From (2.41), (2.42) and (2.43) we get

$$\begin{cases} -\gamma A(0)-a\gamma|A(0)|-\frac{C_2}{C_1^{\frac{p}{q}}}\frac{1}{2C_0}|\delta|^{1-\frac{q}{p}}<s_\epsilon^{p-q}<-\gamma A(0) \\ \qquad +a\gamma|A(0)|+\frac{C_2}{C_1^{\frac{p}{q}}}\frac{1}{2C_0}|\delta|^{1-\frac{q}{p}} \\ -\gamma B(0)-a\gamma|B(0)|-\frac{C_2}{C_1^{\frac{p}{q}}}\frac{1}{C_0}|\delta|^{1-\frac{q-1}{p}}<\widetilde{s}_\epsilon^{p-q}<-\gamma B(0) \\ \qquad +a\gamma|B(0)|+\frac{C_2}{C_1^{\frac{q-1}{p}}}\frac{1}{C_0}|\delta|^{1-\frac{q-1}{p}}. \end{cases} \tag{2.44}$$

Now let us use that $p-q$ is odd then, from (2.34)

$$\widetilde{s}_\epsilon^{p-q} < s_\epsilon^{p-q}. \tag{2.45}$$

Now we look at (2.44) and (2.45). Then by taking the left hand side in the second line in (2.44) and the right hand side of the first line in (2.44), we get, using (2.45):

$$-\gamma B(0) - a\gamma|B(0)| - \frac{C_2}{C_1^{\frac{p}{q}}}\frac{1}{C_0}|\delta|^{1-\frac{q-1}{p}} < -\gamma A(0) + a\gamma|A(0)| + \frac{C_2}{C_1^{\frac{p}{q}}}\frac{1}{2C_0}|\delta|^{1-\frac{q}{p}} \tag{2.46}$$

which can be rewritten as:

$$\frac{p-q}{p}\gamma A(0) < a\gamma A(0)(1+\frac{q}{p}) + \frac{C_2}{C_0 C_1^{\frac{q-1}{p}}}|\delta|^{1-\frac{q}{p}}(|\delta|^{\frac{1}{p}} + \frac{1}{2C_1}). \tag{2.47}$$

If we choose $a = \frac{p-q}{4p} > 0$, we get then, since $a\frac{1+q}{p} < 2a$

$$\begin{cases} A(0)\gamma < 2\dfrac{p}{p-q}\dfrac{C_2}{C_0 C_1^{\frac{q-1}{p}}}|\delta|^{\frac{p-q}{p}}, \quad (\text{since } |\delta|^{\frac{1}{p}} + \dfrac{1}{2C_1} < 1) \\ \qquad \text{so} \\ A(0)\gamma < C|\delta|^{\frac{p-q}{p}}, \quad \text{with } C = \dfrac{2p}{p-q}\dfrac{C_2}{C_0 C_1^{\frac{q-1}{p}}}. \end{cases} \tag{2.48}$$

Conclusion: if for some C_1 inequality (2.48) is false, then we have not (2.37), so proving our lemma. If this inequality is true, so we have to prove our lemma in this case:

$$|A(0)\gamma| < C|\delta|^{\frac{p-q}{p}}, \quad \text{with } C = \frac{\widetilde{C}_2}{C_1^{\frac{q-1}{p}}}, \ (\textit{see } (2.48)).$$

Let us come back to our equation

$$s_\epsilon^p \psi_\epsilon(s_\epsilon) + \gamma s_\epsilon^q g(s_\epsilon) = \delta f(s_\epsilon)\,. \tag{2.49}$$

As $|\gamma g| \le 2C|\delta|^{\frac{p-q}{p}}|\psi_\epsilon|$ on I_σ (from (2.48)) and $|f| \le C_2$, then

$$|s_\epsilon|^q(|s_\epsilon|^{p-q} - 2C|\delta|^{\frac{p-q}{p}}) \le \frac{C_2}{C_0}|\delta|\,. \tag{2.50}$$

Now we assume that: $|s_\epsilon|^p > C_1|\delta|$. So:

$$|s_\epsilon|^{p-q} - 2C|\delta|^{\frac{p-q}{p}} > (C_1^{\frac{p-q}{p}} - 2C)|\delta|^{\frac{p-q}{p}} . \tag{2.51}$$

As $C = \widetilde{C}_2 C_1^{\frac{1-q}{p}}$, we can take C_1 such that:

$$|s_\epsilon|^{p-q} - 2C|\delta|^{\frac{p-q}{p}} > (C_1^{\frac{p-q}{p}} - 2C)|\delta|^{\frac{p-q}{p}} > 2C|\delta|^{\frac{p-q}{p}} . \tag{2.52}$$

Hence from (2.50) and (2.52), we get

$$|s_\epsilon|^q < \frac{C_2}{C_0}|\delta|(\widetilde{C}_2^{-1} C_1^{\frac{q-1}{p}})|\delta|^{\frac{q-p}{p}}, \ ||\epsilon|| < \epsilon_0 . \tag{2.53}$$

So with a constant C_3 depending just on Φ,

$$C_1^{\frac{q}{p}} < C_3 C_1^{\frac{q-1}{p}} \text{ or } C_1^{\frac{1}{p}} < C_3 . \tag{2.54}$$

(2.54) comes from (2.53) and $|s_\epsilon|^p > C_1|\delta|$. So if we take $C_1 > C_3^p$, we have what we wanted. □

As we made the remark that all what we used is the form of the function $\widetilde{\varphi}_\epsilon$ and the fact that $p - q$ is odd, we also have the same lemma for the function $\widetilde{\varphi}_\epsilon{}'$. Hence we also have:

Lemma 2.6 *Let us consider the following*

$$\begin{cases} \widetilde{s}_\epsilon < s_\epsilon < 0, \quad \text{in } I_\sigma \\ \widetilde{\varphi}_\epsilon{}''(\widetilde{s}_\epsilon) = 0, \quad \widetilde{\varphi}_\epsilon{}'(s_\epsilon) = 0. \end{cases} \tag{2.55}$$

Then there exist σ, ϵ_0 *small enough and* $C > 0$ *such that from* (2.55) *we get*

$$|s_\epsilon^{p-1}| \le C|\delta|, \ |\gamma s_\epsilon^{q-1}| \le C|\delta|, \ ||\epsilon|| < \epsilon_0. \tag{2.56}$$

2.2.6 Consequences of the Crucial Lemmas 2.5 and 2.6

We began before to say that, for σ and ϵ_0 small enough, the function $\widetilde{\varphi}_\epsilon$, $||\epsilon|| < \epsilon_0$, is non negative in $(-\sigma, s_\epsilon)$ for some $s_\epsilon > -\sigma$, and also non negative in $(\widetilde{s}_\epsilon, \sigma)$ for some $\widetilde{s}_\epsilon < \sigma$.

The worse case is when $\widetilde{\varphi}_\epsilon$ has zeroes. Our study on the graph will work too when $\widetilde{\varphi}_\epsilon$ has no zeroes in I_σ. Recall also that we are in the situation as in (2.31).

Let us first note the following, which is trivial:

$$\begin{cases} \text{Let } s_{0,\epsilon} \text{ minimal in } I_\sigma \text{ such that } \widetilde{\varphi}_\epsilon(s_{0,\epsilon}) = 0. \\ \text{Then } \exists C > 0 \text{ such that } |s_{0,\epsilon}|^p \le C||\epsilon|| \, ; \ ||\epsilon|| < \epsilon_0. \end{cases} \tag{2.57}$$

It follows by what we said above that $\widetilde{\varphi}_\epsilon{}'$ has zeroes in I_σ, as ϵ_0 is small enough, $||\epsilon|| < \epsilon_0$. We have:

$$\begin{cases} \text{Let } \widetilde{s}_{0,\epsilon} \text{ minimal in } I_\sigma \text{ such that } \widetilde{\varphi}_\epsilon{}'(\widetilde{s}_{0,\epsilon}) = 0. \\ \text{Then } \exists C > 0 \text{ such that } |\widetilde{s}_{0,\epsilon}|^{p-1} \leq C||\epsilon|| \ ; \ ||\epsilon|| < \epsilon_0. \end{cases} \tag{2.58}$$

Call, now $s_{j,\epsilon}$, $\widetilde{s}_{j,\epsilon}$, $j \geq 1$, the other zeroes, in increasing order of $\widetilde{\varphi}_\epsilon$ and $\widetilde{\varphi}_\epsilon{}'$ respectively.

$$\begin{cases} s_{1,\epsilon} < s_{2,\epsilon} < \ldots \quad \widetilde{\varphi}_\epsilon(s_{j,\epsilon}) = 0, \ 1 \leq j \leq j_0 \\ \widetilde{s}_{1,\epsilon} < \widetilde{s}_{2,\epsilon} < \ldots \quad \widetilde{\varphi}_\epsilon{}'(\widetilde{s}_{j,\epsilon}) = 0, \ 1 \leq j \leq j_1. \end{cases} \tag{2.59}$$

It is easy to see that one has

$$\begin{cases} \text{For any } j, \ 0 \leq j \leq j_1 - 1, \text{ there exists } \widehat{s}_{j,\epsilon} \text{ such that} \\ \widetilde{s}_{j,\epsilon} < \widehat{s}_{j,\epsilon} < \widetilde{s}_{j+1,\epsilon}, \quad \widetilde{\varphi}_\epsilon{}''(\widehat{s}_{j+1,\epsilon}) = 0. \end{cases} \tag{2.60}$$

Corollary 2.7 *There exist $C > 0$, ϵ_0 such that*

$$\begin{cases} |\widetilde{s}_{j,\epsilon}|^{p-1} \leq C|\delta| \\ |\gamma \widetilde{s}_{j,\epsilon}^{q-1}| \leq C|\delta|, \quad 1 \leq j \leq j_1, \ ||\epsilon|| < \epsilon_0. \end{cases} \tag{2.61}$$

Proof of Corollary 2.7 Use crucial Lemma 2.5 and the second line in (2.60). Remark that $j_0 \leq p$ and $j_1 \leq p - 1$. □

Corollary 2.8 *Let $\widetilde{s}_{j,\epsilon}$ solutions of $\widetilde{\varphi}_\epsilon{}'(s) = 0$, $1 \leq j \leq j_1$ given in (2.59), with $\widetilde{\varphi}_\epsilon$ as given in (2.31). Given any C_0 sufficiently small, there exist $\epsilon_0 > 0$ and $\sigma > 0$ such that for $\widetilde{s}_{j,\epsilon}$ in I_σ we have*

$$\begin{cases} \widetilde{\varphi}_\epsilon(\widetilde{s}_{j,\epsilon}) = \delta f(\widetilde{s}_{j,\epsilon}) + a_{j\epsilon} \\ |a_{j\epsilon}| \leq C_0|\delta|, \quad \forall \epsilon \ ||\epsilon|| < \epsilon_0. \end{cases} \tag{2.62}$$

Proof of Corollary 2.8 We use the estimate (2.61) in $\widetilde{\varphi}_\epsilon(s)$ given by (2.31):

$$\widetilde{\varphi}_\epsilon(\widetilde{s}_{j,\epsilon}) = \widetilde{s}_{j,\epsilon}^p \psi_\epsilon(\widetilde{s}_{j,\epsilon}) + \delta f(\widetilde{s}_{j,\epsilon}) + \gamma \widetilde{s}_j^q g(\widetilde{s}_{j,\epsilon})$$

$$|a_{j,\epsilon}| = |\widetilde{s}_{j,\epsilon}^p \psi_\epsilon(\widetilde{s}_{j,\epsilon}) + \gamma \widetilde{s}_{j,\epsilon} g(\widetilde{s}_{j,\epsilon})| \leq C|\widetilde{s}_{j,\epsilon}|^p + |\widetilde{s}_{j,\epsilon}^q \gamma g(\widetilde{s}_{j,\epsilon})|.$$

So, with another constant C,

$$|a_{j\epsilon}| \leq C|\widetilde{s}_{j,\epsilon}||\delta|, \quad ||\epsilon|| < \epsilon_0.$$

If σ is small enough i.e. $|\widetilde{s}_{j,\epsilon}| < \frac{C_0}{C}$, we get $|a_{j\epsilon}| \leq C0|\delta|$. So we found σ (small enough) and ϵ_0 such that (2.62) is true. □

The following corollary is just the meaning of (2.62).

Corollary 2.9 *There exist σ and ϵ_0 such that in I_σ the extrema $\widetilde{\varphi}_\epsilon(\widetilde{s}_{j,\epsilon})$, $1 \le j \le j_1$ have all the same sign which is the sign of $\delta f(0)$ and moreover:*

$$C_0|\delta| \le |\widetilde{\varphi}_\epsilon(\widetilde{s}_{j,\epsilon})| \le C_1|\delta| \,. \tag{2.63}$$

Proof of Corollary 2.9 Use (2.62) and lower and upper bound for $|f|$ in I_σ. □

Corollary 2.10 *There exist σ and ϵ_0 such that in I_σ $\widetilde{\varphi}_\epsilon$ has at most two zeroes $s_{0,\epsilon}, s_{1,\epsilon}$, and if $\widetilde{s}_{0,\epsilon} < \widetilde{s}_{,1\epsilon} < \dots < \widetilde{s}_{j,\epsilon}$ are the zeroes of $\widetilde{\varphi}_\epsilon{}'$ in I_σ, then the extrema $\widetilde{\varphi}_\epsilon(\widetilde{s}_{\ell,\epsilon})$, $1 \le \ell \le j_1$, have all the sign of $\delta f(0)$ (recall again that $\epsilon = (\delta, \gamma, \epsilon_4, \dots, \epsilon_m)$) and satisfy* (2.63).

Proof of Corollary 2.10 If $\widetilde{\varphi}_\epsilon$ has a minimal zero $s_{0,\epsilon}$, all our preceding study shows that we have the following picture: $\widetilde{\varphi}_\epsilon$ is decreasing on $(-\sigma, \widetilde{s}_{0,\epsilon})$, $\widetilde{s}_{0,\epsilon} \ge s_{0,\epsilon}$, all the further extrema $\widetilde{\varphi}_\epsilon(s_j)$ have the same sign, with bounds (2.63). Of course $s_{0,\epsilon}$ may be the unic zero. But if $\widetilde{\varphi}_\epsilon(\widetilde{s}_{0,\epsilon}) < 0$ then we have necessarily another zero, but only one. Of course, we may have also $\widetilde{s}_{0,\epsilon} = s_{0,\epsilon}$.

If $\widetilde{\varphi}_\epsilon$, $||\epsilon|| < \epsilon_0$, have no zero, all we proved for the zeroes $\widetilde{s}_{j,\epsilon}$ of $\widetilde{\varphi}_\epsilon{}'$ works, because in the proofs on these we did not use existence of zeroes for $\widetilde{\varphi}_\epsilon$. □

So we studied the situation when $\gamma g > 0$ on I_σ. But we saw that, by taking symmetry with respect to $\{s = 0\}$, we can reduce to the case $\gamma g > 0$ on I_σ.

From Corollary 2.10, we obtain the shape of the graph of $\widetilde{\Phi}_\epsilon$ on I_σ, σ small, which is of good type, as described in [7, pages31-32].

3 Vector Functions Φ Such that the System L_Φ Does Not Satisfy a Microlocal Maximal Estimate

We want to give here a large class of Φ's for which L_Φ does not satisfy inequality (1.10). We consider the following hypothesis for $\Phi = (\varphi_1, \dots, \varphi_m)$ on $\omega \subset \mathbb{R}^n$; $n, m \ge 2$, $\Phi \in C^1(\omega)$.

There exist in ω, a piece of a hypersurface $H = \{h = 0\}$, two odd numbers $0 < q < p$, $\lambda \in \mathbb{R}^m$, such that, near H

$$(H2)\begin{cases} 1)\ \varphi_1 \text{vanishes at order p on H}(\varphi_1 = h^p\psi_1, \psi_1 \ne 0 \,\text{on H}, \psi_1 \in C^1). \\ 2)\ \Phi_\lambda = \lambda\cdot\Phi \text{ vanishes at order q on H}(\Phi_\lambda = h^q\psi_\lambda, \psi_\lambda \ne 0 \,\text{on H}, \psi_\lambda \in C^1). \end{cases}$$

Theorem 3.1 *If Φ satisfies (H2), the system L_Φ does not satisfy a microlocal maximal estimate on $(\omega \times \Omega, V)$, where V is a cone with vertex $\xi_0 = (1, 0, \dots, 0)$ and $\Omega \subset \mathbb{R}^m$, open, $0 \in \Omega$.*

Some reductions before the proof: We may first assume that $h = s_1$ in $\mathbb{R}^n, n \geq 2$, $s = (s_1, \ldots, s_n)$. Moreover, we see from (H2) that $\lambda \neq (1, 0, \ldots, 0)$ because $q < p$. Hence, we may assume also that $\lambda = (0, 1, 0, \ldots, 0)$ (by a linear transformation in $\mathbb{R}^m$). So we have the following:

$$\begin{cases} \varphi_1(s) = s_1^p \psi_1(s), \ \psi_1(0, s') \neq 0; \ s' = (s_2, \ldots, s_n), \ (0, s') \in \omega_1 \subset \omega \\ \varphi_2(s) = s_1^q \psi_2(s), \ \psi_2(0, s') \neq 0. \end{cases} \tag{3.1}$$

First we choose $\sigma > 0$, ω' a neighborhood of $(0, s'_0)$ in ω such that, for some $C_0 > 0$

$$\begin{cases} |\psi_1| \geq C_0, \ |\psi_2| \geq C_0, \ \text{in } I \times \omega' \\ \text{where } I = I_\sigma = (-\sigma, \sigma). \end{cases} \tag{3.2}$$

Before giving the proof, let us remark that the Φ's satisfying (H2) generalize considerably the example in $\mathbb{R}^{2+2}$ studied by Helffer and the author [7]: $\varphi_1 = s_1^3$, $\varphi_2 = s_1 s_2^2$.

Proof of Theorem 3.1 We use a different method than the one we used in the above mentioned example.

Now, for any δ small, we write, if C_0 is small enough, $\varphi_\delta = \varphi_1 + \delta\varphi_2$. Hence

$$\varphi_\delta(s) = s_1^q \psi_1(s)(s_1^{p-q} + \delta \frac{\psi_2}{\psi_1}(s)),$$

with

$$C_0 \leq \left|\frac{\psi_2}{\psi_1}\right| \leq C_1 \,. \tag{3.3}$$

In the study of φ_δ, mainly the zeroes of φ_δ, and $(\varphi_\delta)'_{s_1}$, we make the following more reductions, we write:

$$\begin{cases} s \text{ will denote } s_1 \\ t \text{ will denote } s'. \end{cases} \tag{3.4}$$

So we have, with these new notations:

$$\begin{cases} \varphi_\delta(s, t) = s^q \psi_1(s^{p-q} + \delta \frac{\psi_2}{\psi_1}(s, t)) \\ C_0 \leq \left|\frac{\psi_2}{\psi_1}(s, t)\right| \leq C_1 \text{ on } I \times \omega'. \end{cases} \tag{3.5}$$

We recall now that the functions $w_j(s, t, x) := x_j - i\varphi_j(s, t)$ are homogeneous solutions of L_Φ:

$$L_\Phi(w_j) = 0 \,.$$

Recall that, given $\xi = (\xi_1, \ldots, \xi_m)$ in $\mathbb{R}^m \setminus \{0\}$, the function $\Phi_\xi = \sum_1^m \xi_j \varphi_j$ plays a role in the study of L_Φ: a necessary condition for microlocal hypoellipticity at

$(0, (0, \xi))$ of the system L_Φ is given in terms of Φ_ξ. In fact, we will use the following function $w_{\lambda,\delta}$, for $\lambda > 0$ and δ small: $|\delta| < \delta_0$

$$\begin{cases} w_{\lambda,\delta} = i\lambda(w_1 + \delta w_2) = i\lambda(x_1 - i\varphi_1 + \delta(x_2 - i\varphi_2)); \text{ so} \\ \text{Re}\, w_{\lambda,\delta} = \lambda(\varphi_1 + \delta\varphi_2) = \lambda\varphi_\delta = \text{Re}\,(\lambda w_\delta), \;\; w_\delta = i(w_1 + \delta w_2). \end{cases} \tag{3.6}$$

Our aim, now, is to produce a family of functions contradicting the inequality defining microlocal maximal estimate, in $(\omega \times \Omega, V)$ V conic neighborhood of $(1, 0, \ldots, 0) \in \mathbb{R}^m$. We will call this family depending on (λ, δ) as before by $u_{\lambda,\delta} \in \mathcal{D}(\omega \times \Omega)$. We will choose suitably functions $\theta_\delta(s)$, $|\delta| < \delta_0$, $\theta_1(t) \in \mathcal{D}(\omega')$, $\theta_2(x) \in \mathcal{D}(\Omega)$ and

$$u_{\lambda,\delta} = \theta_\delta\theta_1\theta_2 \exp(w_{\lambda,\delta}), \;\; \lambda > 0, \;\; |\delta| < \delta_0,$$

in order that the inequality of microlocal maximal estimate in $(\omega \times \Omega, V)$ is not satisfied. More precisely we want to prove that the following estimate:

$$\Big|\Big|\frac{\partial u_{\lambda,\delta}}{\partial s}\Big|\Big|_{L^2_V} \leq C(\big|\big|u_{\lambda,\delta}\big|\big|_{L^2_{V'}} + \big|\big|L_\Phi u_{\lambda,\delta}\big|\big|_{L^2_{V'}}) \tag{3.7}$$

is not true for all $\lambda > 0$, $|\delta| < \delta_0$.

In order to define θ_δ, θ_1, θ_2, we need to first establish some properties of φ_δ. Put first $\mu := \frac{1}{p-q}$, $a := \big|\frac{\psi_2}{\psi_1}(0)\big|$, $b := \frac{q}{p}a$. Note that if $\delta\frac{\psi_2}{\psi_1} > 0$, then $\varphi_{\delta,t}$ has only one zero on I which is 0.So we assume $\delta\frac{\psi_2}{\psi_1} < 0$.

Lemma 3.2 *In order to study the graph of the function $\varphi_{\delta,t} \colon I \to \mathbb{R}$, $|\delta| < \delta_0$, in particular the zeroes of $\varphi_{\delta,t}$ and $\varphi'_{\delta,t}$ where $\varphi_{\delta,t}(s) = \varphi_\delta(s, t)$, we may assume*

$$\psi_1 > 0, \;\; \psi_2 < 0, \;\; \text{and so } \delta > 0. \tag{3.8}$$

Proof of Lemma 3.3 If $\psi_1 < 0$ on $I \times \omega'$. Then we take $\widetilde{\varphi}_\delta = -\varphi_\delta$. So $\widetilde{\varphi}_\delta(s) = s^q(-\psi_1(s, t))(s^{p-q} + \delta\frac{\psi_2}{\psi_1}(s, t))$. So we work with $-\psi_1 > 0$, for $\widetilde{\varphi}_\delta$. If moreover $\psi_2 > 0$, then as $\delta\psi_2 > 0$ (from the condition $\delta\frac{\psi_2}{\psi_1} < 0$), $\delta > 0$, we have just to put $\widetilde{\psi}_2 = -\psi_2$ to obtain $\widetilde{\psi}_2 < 0$

$$\widetilde{\varphi}_\delta = s^q\widetilde{\psi}_1(s, t)\left(s^{p-q} + \delta\frac{(-\psi_2)}{(-\psi_1)}\right) = s^q\widetilde{\psi}_1(s, t)\left(s^{p-q} + \delta\frac{\widetilde{\psi}_2}{\widetilde{\psi}_1}(s, t)\right)$$

with $\widetilde{\psi}_1 > 0$, $\widetilde{\psi}_2 < 0$, $\delta > 0$. □

So now will work with condition in Lemma 3.2.

Lemma 3.3 *Let us assume* (3.8). *Then for any $t \in V$, $\varphi_{\delta,t}$ has three zeroes $s_{1,t} < 0 < s_{2,t}$ in I, and $\varphi'_{\delta,t}$, its derivative, has three zeroes $\widetilde{s}_{1,t} < 0 < \widetilde{s}_{2,t}$ if $q > 1$, and two zeroes $\widetilde{s}_{1,t} < \widetilde{s}_{2,t}$ if $q = 1$, if $\delta < \delta_0$, δ_0 sufficiently small.*

Proof of Lemma 3.3 Of course 0 is a zero of $\varphi_{s,t}$. The other possible zeroes are those of:

$$s^{p-q} + \delta \frac{\psi_2}{\psi_1}(s,t) = 0. \tag{3.9}$$

As $\delta \frac{\psi_2}{\psi_1} < 0$ in $I \times \omega'$ (ω' neighborhood of t_0 in $\mathbb{R}^{n-1}$) and $p-q$ is an even number, $p-q > 0$, the solutions are those of the equations, for $t \in \omega'$ fixed

$$\begin{cases} s = \left(-\delta \frac{\psi_2}{\psi_1}(s,t)\right)^{\frac{1}{p-q}} & = \left(-\delta \frac{\psi_2}{\psi_1}\right)^{\mu} \\ s = -\left(-\delta \frac{\psi_2}{\psi_1}(s,t)\right)^{\frac{1}{p-q}} & = -\left(-\delta \frac{\psi_2}{\psi_1}\right)^{\mu}, \ \mu = \frac{1}{p-q}. \end{cases} \tag{3.10}$$

Now it is easy to see that there exists δ_0 sufficiently small such that for $0 < \delta < \delta_0$, we can apply the implicit function theorem for each of the two equations in (3.10), in order to have a unique solution in I (more elementary, just see that the derivative being without zero in I, each equation is monotone). So if $s_{1,t}$ is the solution of the second equation and $s_{2,t}$ of the first one, we have the three solutions $s_{1,t} < 0 < s_{2,t}$. Now, the expression of $\varphi'_{\delta,t}$ is

$$\varphi'_{\delta,t}(s) = s^{q-1}\left(s^{p-q}(p\psi_1 + s\psi'_1) + \delta(q\psi_2 + s\psi'_2)\right) \tag{3.11}$$

which can be rewritten, as $\psi_3 \neq 0$ in $I \times \omega'$ sufficiently small

$$\begin{cases} \varphi'_{\delta,t}(s) = s^{q-1}\psi_3(s,t)\left(s^{p-q} + \delta \frac{\psi_4}{\psi_3}(s,t)\right), \text{ where} \\ \psi_3 := p\psi_1 + s\psi'_1, \ \psi_4 := q\psi_2 + s\psi'_2. \end{cases} \tag{3.12}$$

Remark here that the prime's denote derivatives in s. First note that 0 is a zero of $\varphi'_{\delta,t}$ if and only if $q \neq 1$. The other possible zeroes of $\varphi'_{\delta,t}$ are those of

$$s^{p-q} + \delta \frac{\psi_4}{\psi_3}(s,t) = 0. \tag{3.13}$$

Now we are in the situation (as $I \times \omega'$ is shrinked in order that $|\psi_3| \geq C_0 > 0$, $|\psi_4| \geq C_0 > 0$ in $I \times \omega'$) and remark that $p\psi_1 + s\psi'_1 > 0$ in $I \times \omega'$, $q\psi_2 + s\psi'_2 < 0$ in $I \times \omega'$.

So (3.13) is similar to (3.10), then we have $\widetilde{s}_{1,t} < \widetilde{s}_{2,t}$ solutions of $\varphi'_{\delta,t}(s) = 0$. Of course, we seek that these solutions are close to the corresponding solutions when we replace $\psi_1, \psi_2, \psi_3, \psi_4$ by $\psi_1(0), \psi_2(0), \psi_3(0), \psi_4(0)$; the later solutions are

$$\begin{cases} s_1 = -(a\delta)^{\mu}, \ s_2 = (a\delta)^{\mu}, \ \widetilde{s}_1 = -(b\delta)^{\mu}, \ \widetilde{s}_2 = (b\delta)^{\mu} \\ \text{where } a = \left|\frac{\psi_2}{\psi_1}(0)\right|, \ b = \left|\frac{\psi_4}{\psi_3}(0)\right|, \text{ and } b = \frac{q}{p}a < a \\ \text{with } \mu = \frac{1}{p-q}. \end{cases} \tag{3.14}$$

□

Our aim is to give an estimate for $s_{1,t} - s_1, \dots, \widetilde{s}_{1,t} - \widetilde{s}_1, \dots$ in terms of suitable powers of δ. It is convenient to note that if we put $I = (-\sigma, \sigma)$ where σ is small enough, then the function of one variable $\varphi_{\delta,t}$, $t \in \omega'$, is increasing in $(-\sigma, \widetilde{s}_{1,t})$, with the zero $s_{1,t}$ in $(-\sigma, \widetilde{s}_{1,t})$, decreasing in $(\widetilde{s}_{1,t}, s_{2,t})$ with 0 as zero in it, and increasing in $(\widetilde{s}_{2,t}, \sigma)$ because $\psi_{1,t}$ and $\psi_{3,t}$ are positive in I, $t \in \omega'$.

Let us now give estimates related to the functions $\varphi_{\delta,t}, \varphi'_{\delta,t}$.

Proposition 3.4 *There exist $\widetilde{C}_0 > 0$, $\delta_0 > 0$, $C_1 > 0$ and $\omega' \ni t_0$ such that:*

(1) $|\varphi_\delta| \le C_1 \delta^{p\mu}$ *on* $(-2(a\delta)^\mu, (a\delta)^\mu) \times \omega'$.

(2) $\varphi_\delta \ge \widetilde{C}_0 \delta^{p\mu}$ *on* $(-\frac{2}{3}(b\delta)^\mu, -\frac{1}{3}(b\delta)^\mu) \times \omega' = -I_\delta \times \omega'$.

(3) $\varphi_\delta \le -\widetilde{C}_0 \delta^{p\mu}$ *on* $\left(\underbrace{(\frac{1}{3}(b\delta)^\mu, \frac{2}{3}(b\delta)^\mu)}_{I_\delta} \times \omega'\right) \cup \big((-\sigma, -(a\delta)\mu - \frac{1}{3}(b\delta)^\mu) \times \omega'\big)$.

(4) $|(\varphi_\delta)'_s| \ge \widetilde{C}_0 \delta^{(p-1)\mu}$ *on* $(-I_\delta \cup I_\delta) \times \omega'$.

Proof of Proposition 3.4
(1) This inequality is trivial, with C_1 suitably chosen.
(2) Note first that, as $\psi_1(s,t) > C_0$ on $I \times \omega'$, we have

$$s^q \psi_1 < 0$$

on $-I_\delta$ and

$$|s^q \psi_1| \ge (\frac{1}{3})^q (b\delta)^{q\mu} C_0 .$$

Moreover,

$$\begin{aligned} |s^{p-q} + \delta \frac{\psi_2}{\psi_1}(s,t)| &\ge \delta |\frac{\psi_2}{\psi_1}(s,t)| - s^{p-q} \\ &\ge a\delta - \delta(|\frac{\psi_2}{\psi_1}(s,t) - \frac{\psi_2}{\psi_1}(0)|) - s^{p-q} \\ &\ge \delta(a - \frac{a}{2}) - (\frac{2}{3})^{p-q}(b\delta)^{(p-q)\mu=1} = \frac{a\delta}{2} - (\frac{2}{3})^{p-q} b\delta , \end{aligned}$$

if δ_0 is small.

But $\frac{a}{2} - (\frac{2}{3})^{p-q} b \ge \frac{a}{2} - \frac{4}{9} b > 0$ $(p-q \ge 2)$. Note that we used that there exists δ_0 and ω' such that

$$\left| \frac{\psi_2}{\psi_1}(s,t) - \frac{\psi_2}{\psi_1}(0,0) \right| \le \frac{a}{2}, \text{ if } (s,t) \in (-2(a\delta)^\mu, 2(a\delta)^\mu) \times \omega',\ \delta < \delta_0.$$

Hence we obtain (as $\varphi_\delta > 0$ on $-I_\delta \times \omega'$)

$$\varphi_\delta(s,t) \geq C_0(\frac{1}{3})^q(\frac{a}{2} - \frac{4}{9}(b)\delta b^{q\mu}\delta^{q\mu} = \widetilde{C}_0\delta^{p\mu},\ \widetilde{C}_0 > 0.$$

(3) The proof is the same, after observing that $\varphi_\delta < 0$ here.

(4) Here also, the proof is basically the same than in (2), (3), because $\varphi'_{\delta,t}$ has the same form as $\varphi_{\delta,t}$ after replacing ψ_1, ψ_2 by ψ_3, ψ_4 and s^{q-1} is as a factor in place of s^q. So we will have, if ω' is shrinked more and δ_0 may be smaller, the desired estimate from below for $|(\varphi_\delta)'_s|$ in $(-I_\delta \cup I_\delta) \times \omega'$, $\delta < \delta_0$. □

Now we are ready to define our functions $u_{\lambda,\delta}$:

- Define $\theta_\delta(s)\colon I \to \mathbb{R}$ for $\delta < \delta_0$ by

(a) $\theta_\delta \in \mathcal{D}(-(a\delta)^\mu - \frac{2}{3}(b\delta)^\mu, \frac{2}{3}(b\delta)^\mu)$.

(b) $\theta_\delta = 1$ on $(-(a\delta)^\mu - \frac{1}{3}(b\delta)^\mu, \frac{1}{3}(b\delta)^\mu)$.

(c) $|\theta'_\delta| \leq C_1\delta^{-\mu}$ for some C_1 .

Note that such a family does exist.

- Define the functions $u_{\lambda,\delta}$ for $\lambda > 0$, $\delta < \delta_0$. We first choose $\theta_1 = \theta_1(t)$ with support in $\omega' \subset \mathbb{R}^{n-1}$, $\theta_2 = \theta_2(x)$ with support in the ball $B \subset \mathbb{R}^m$, $0 \leq \theta_j \leq 1$, $\theta_j \not\equiv 0$. Then, we define the family $u_{\lambda,\delta} \in \mathcal{D}(I \times \omega' \times (B)$:

$$u_{\lambda,\delta}(s,t,x) = \theta_\delta(s)\theta_1(t)\theta_2(x)\exp(w_{\lambda,\delta}(s,t,x))\,.$$

In order to contradict inequality (3.7), we first give expressions of $\frac{\partial}{\partial s}u_{\lambda,\delta}$ and $L_j u_{\lambda,\delta}$ keeping (s,t) as variables.

$$\begin{cases} \frac{\partial}{\partial s}u_{\lambda,\delta}(s,t,x) & = (\theta'_\delta(s) + \lambda\varphi'_{\delta,t}(s)\theta_\delta)\theta_1(t)\theta_2(x)\exp(w_{\lambda,\delta}(s,t,x)) \\ L_j u_{\lambda,\delta}(s,t,x) & = L_j((\theta_\delta\theta_1\theta_2)(s,t,x)\exp(w_{\lambda,\delta}(s,t,x)) \end{cases} \tag{3.15}$$

where we have, recalling that $s = s_1$ and $t = (s_2, \ldots, s_n)$

$$\begin{cases} L_1(\theta_\delta\theta_1\theta_2) = \theta'_\delta\theta_1\theta_2 + i(\sum_{k=1}^m \frac{\partial\varphi_k}{\partial s}\frac{\partial\theta_2}{\partial x_k})\theta_\delta\theta_1 \\ L_j(\theta_\delta\theta_1\theta_2) = \theta_\delta(\partial_{s_j}\theta_1)\theta_2 + i(\sum_{k=1}^m \frac{\partial\varphi_k}{\partial s_j}\frac{\partial\theta_2}{\partial x_k})\theta_\delta(\partial_{s_j}\theta_1);\ \ j = 2, \ldots, n\,. \end{cases} \tag{3.16}$$

The expression of (3.16) can now be simplified by writing with new notations

$$\begin{cases} L_1(\theta_\delta\theta_1\theta_2) = \theta'_\delta\theta_1\theta_2 + i(\sum_{k=1}^m f_k(s,t)g_k(x)\theta_1(t))\theta_\delta \\ L_j(\theta_\delta\theta_1\theta_2) = \theta_\delta\left(h_j(t)\theta_2(x) + i\sum_{k=1}^m \ell_{k,j}(s,t)\tilde{\theta}_k(x)\tilde{\theta}_j(t)\right);\ \ j = 2, \ldots, n\,. \end{cases} \tag{3.17}$$

Assume now the inequality (3.7). Then we deduce using (3.15)–(3.17), with maybe a different constant C

$$\begin{cases} \left|\left|\lambda\varphi'_{\delta,t}(s)\theta_\delta(s)\theta_1(t)\theta_2(x)\exp(w_{\lambda,\delta}(s,t,x))\right|\right|_{L^2_V} \\ \leq C\left\{\left|\left|\theta'_\delta(s)\theta_1(t)\theta_2(x)\exp(w_{\lambda,\delta}(s,t,x))\right|\right|_{L^2_{V'}} \right. \\ \qquad \left. + \sum_{finite} \left|\left|\theta_\delta(s)G_k(s,t)H_k(x)\exp(w_{\lambda,\delta}(s,t,x))\right|\right|_{L^2_{V'}}\right\},\ G_k \in \mathcal{D}(I\times\omega'). \end{cases} \tag{3.18}$$

Now we want to give a suitable bound for the second term in (3.18). For that, we have:

$$\begin{cases} \mathcal{F}_x\left\{\theta'_\delta(s)\theta_1(t)\theta_2(x)\exp(w_{\lambda,\delta}(s,t,x))\right\}(s,t,\eta) = \\ \int \theta'_\delta(s)\theta_1(t)\theta_2(x)\exp\left(i\lambda(x_1 - i\varphi_1(s,t) + \delta(x_2 - i\varphi_2(s,t))) - i\eta_1 x_1\right. \\ \qquad \left. -i\eta_2 x_2 - \ldots - i\eta_m x_m\right) \\ = \theta'_\delta(s)\theta_1(t)\widehat{\theta_2}(\eta_1 - \lambda, \eta_2 - \lambda\delta, \eta_3, \ldots, \eta_m)\exp(\lambda\varphi_\delta)(s,t). \end{cases} \tag{3.19}$$

Similarly, we have

$$\begin{cases} \mathcal{F}_x\left\{\theta_\delta(s)G_k(s,t)H_k(x)\exp(w_{\lambda,\delta}(s,t,x))\right\}(s,t,\eta) = \\ \theta_\delta(s)G_k(s,t)\widehat{H}_k(\eta_1 - \lambda, \eta_2 - \lambda\delta, \eta_3, \ldots, \eta_m)\exp(\lambda\varphi_\delta)(s,t)\,. \end{cases} \tag{3.20}$$

We want to find a suitable upper bound of the second term in (3.17) in terms of powers of δ, for $0 < \delta < \delta_0$, using what we know about θ_δ and φ_δ. First, for some $C_2 > 0$,

$$\begin{cases} \{\left|\left|\theta'_\delta\theta_1\theta_2\exp(w_{\lambda,\delta})\right|\right|_{L^2_{V'}} \leq \sup|\theta'_\delta|\left((a\delta)^\mu + \frac{4}{3}(b\delta)^\mu\right)\sup_{\mathrm{Supp}\,\theta'_\delta}|\exp(\lambda\varphi_\delta)|\ ||\widehat{\theta}_2||_{L^2} \\ \qquad\qquad \leq C_2\exp(-C_0\lambda\delta^{p\mu})||\widehat{\theta}_2||_{L^2} \\ \sum_{finite}\left|\left|\theta_\delta(s)G_k(s,t)H_k(x)\exp(w_{\lambda,\delta}(s,t,x))\right|\right|_{L^2_{V'}} \leq C_2\delta^\mu\exp(\widetilde{C}_0\lambda\delta^{p\mu})\,. \end{cases} \tag{3.21}$$

Hence from the preceding estimates, we obtain

$$\left|\left|\lambda\varphi'_{\delta,t}(s)\theta_\delta(s)\theta_1(t)\theta_2(x)\exp(w_{\lambda,\delta})\right|\right|_{L^2_V} \leq C_3(\exp(-C_0\lambda\delta^{p\mu}) + \delta^\mu\exp(\widetilde{C}_0\lambda\delta^{p\mu}))\,. \tag{3.22}$$

Let us now find a lower bound, for the first member in (3.18). For that it suffices to give a lower bound for the smaller term obtained by integrating in s just on I_δ, where we have lower bounds for φ_δ and $|(\varphi_\delta)'_s|$, given in Proposition 3.4. So we have

$$\left|\left|\lambda\varphi'_{\delta,t}\theta_\delta\theta_1\theta_2\exp(w_{\lambda,\delta})\right|\right|_{L^2_V} \geq C_0\delta^{(p-1)\mu}$$
$$\exp(C_0\lambda\delta^{p\mu})\delta^\mu||\widehat{\theta_2}(\eta_1-\lambda,\eta_2-\lambda\delta,\eta_3,\dots,\eta_m))||_{L^2(V)}\,. \quad (3.23)$$

Now we have to choose θ_2 in order that

$$||\widehat{\theta_2}(\eta_1-\lambda,\eta_2-\lambda\delta,\eta_3,\dots,\eta_m))||_{L^2(V)} \geq C > 0,\ \forall\lambda > 1,\ \delta < \delta_0\,. \quad (3.24)$$

Recall that, if $V_1 \subset\subset V$, i.e. V_1 is compactly contained in the cone V, there is an open cone V' contained in $V - (\lambda, \lambda\delta, 0, \dots, , 0)$ for any $\lambda > 1$, $0 < \delta < \delta_0$, if $\delta_0 > 0$ small, meaning $(\lambda, \lambda\delta, 0, \dots, 0) \in V_1$. Hence

$$||\widehat{\theta_2}(\eta_1-\lambda,\eta_2-\lambda\delta,\eta_3,\dots,\eta_m))||_{L^2(\eta\in V)} \geq ||\widehat{\theta_2})||_{L^2(V')},\ \forall\lambda > 1,\ \delta < \delta_0\,. \quad (3.25)$$

Now just choose $\theta_2 \in \mathcal{D}((B), 0 \leq \theta_2 \leq 1$, with $||\widehat{\theta_2})||_{L^2(V')} \geq C > 0$. Collecting all we saw, we obtain, with some positive constants C, C_0, C_1

$$\lambda\delta^{p\mu}\exp(C_0\lambda\delta^{p\mu}) \leq C(\delta^\mu\exp(C_1\lambda\delta^{p\mu}) + \exp(-C_0\lambda\delta^{p\mu})),\ \forall\lambda > 0,\ \delta < \delta_0\,. \quad (3.26)$$

Now we prove the following

Proposition 3.5 *The inequality* (3.26) *is not true.*

Proof of Proposition 3.5 Put $\gamma := \lambda\delta^{p\mu}$. So (3.26) has the form:

$$\gamma\exp(C_0\gamma) \leq C(\delta^\mu\exp(C_1\gamma) + \exp(-C_0\gamma)),\ \forall\lambda > 0,\ \delta < \delta_0\,. \quad (3.27)$$

This inequality will be false if we can choose $\gamma > 1$ and $\delta < \delta_0$ such that:

$$\begin{cases} \gamma\exp(2C_0\gamma) > C \\ \gamma\exp((C_0 - C_1)\gamma) > C\delta^\mu\,. \end{cases} \quad (3.28)$$

For that, choose $\gamma_0 > 1$ such that

$$\gamma_0\exp(2C_0\gamma_0) > C$$

and then choose $\delta_1 > 0$ such that

$$\gamma_0\exp((C_0 - C_1)\gamma_0) > C\delta_1^\mu\,.$$

(One has just to take δ_1 sufficiently small).

So we have:

$$\begin{cases} \gamma_0 \exp(2C_0\gamma_0) > C, \text{ (by choosing } \gamma_0 \text{ sufficiently large)} \\ \gamma_0 \exp((C_0 - C_1)\gamma_0) > C\delta_1^{\mu} \text{ (by choosing } \delta_1 \text{ small after the choice of } \gamma_0) . \end{cases} \tag{3.29}$$

If $\delta < \inf(\delta_0, \delta_1)$, we have $\delta < \delta_0$ and $\gamma_0 \exp((C_0 - C_1)\gamma_0) > C\delta^{\mu}$. Coming back to the notations (λ, δ), we obtained (λ_0, δ) (with $\lambda_0 = \gamma_0 \delta^{-p\mu}$) such that (3.26) is contradicted. The proof of Proposition 3.5 is complete. □

Using Proposition 3.5, we get that the system (L_Φ) does not satisfy the microlocal maximal estimate (3.7). The proof of Theorem 3.1 is therefore complete. □

4 A Class of Non Hypoelliptic Systems

We will use the notation of Sect. 3 but we consider vector functions Φ satisfying a slightly more restrictive condition than the condition (H2); more precisely, given $\xi_0 \in \mathbb{R}^m$, we assume the following on $\Phi_{\xi_0} = \xi_0 \cdot \Phi$:
(H3): There exists in ω a piece of a hypersurface $H = \{h = 0\}$, two odd numbers $0 < q < p$, and $\lambda \in \mathbb{R}^m$, such that:

(1) Φ_{ξ_0} vanishes at order p on H (i.e. $\Phi_{\xi_0} := h^p \psi_{\xi_0}$, $\psi_{\xi_0} \neq 0$ on H, and C^∞).
(2) $\Phi_\lambda = \lambda \cdot \Phi$ vanishes at order q on H (i.e. $\Phi_\lambda = h^q \psi_\lambda$, $\psi_\lambda \neq 0$ on H, and C^∞).
(3) If ∇_H defines the tangential gradient to H in ω, then $\nabla_H \Phi_{\xi_0}$ vanishes at order greater or equal to $p + 1$ on H and $\nabla_H \Phi_\lambda$ vanishes at order greater or equal to $q + 1$ on H.

Theorem 4.1 *Under Condition (H3), the system L_Φ is not microlocally hypoelliptic at* $(0, (0, \xi_0)) \in \mathbb{R}^{n+m} \times \mathbb{R}^{n+m}$.

Theorem 4.2 *Assume now $m = 2$, Condition (H3), and $0 \in H$. Then $(0, (0, \xi)) \in \mathbb{R}^{n+m} \times \mathbb{R}^{n+m}$ is singular for L_Φ if and only if $\xi = \xi_0$. Moreover L_Φ is microlocally $\frac{1}{q}$-subelliptic at $(0, (0, \xi))$, if $\xi \neq \xi_0$.*

Proof of Theorem 4.1:
We begin by the following remark.

Remark 4.3 It is easy to see that we can assume that $\xi_0 = (1, 0, 0)$ and $\lambda = (0, 1, 0, 0)$, so that $\Phi_{\xi_0} = \varphi_1$ and $\Phi_\lambda = \varphi_2$, $\psi_{\xi_0} = \psi_1$, $\psi_\lambda = \psi_2$. So, in the sequel we assume Condition (H3) on φ_1 and φ_2.

(A): Expression of Condition (H3) in suitable coordinates (s, t), $s \in \mathbb{R}$, $t \in \mathbb{R}^{n-1}$, **in** ω.

We can assume that $H = \{s = 0\}$. Then ∇_H can be expressed by ∇_t, in the coordinates (s, t). Then we have

Lemma 4.4 *Condition (H3)-(3) means:*

$$\psi_j(s,t) = f_j(s) + s\, g_j(s,t)\,,\ j = 1, 2\,, \tag{4.1}$$

where f_j and g_j are smooth respectively on the interval $I_\sigma = (-\sigma, +\sigma)$ and on the cube $(I_\sigma)^n$, σ small.

Proof The condition (3) in (H3) means that $\nabla_t \psi_j$ vanishes on H, so that

$$\nabla_t \psi_j(s,t) = s\, k_j(s,t),\ j = 1, 2\,.$$

Hence

$$\psi_j(s,t) = \psi_j(s,0) + \int_0^1 t \nabla_t \psi_j(s, \tau t) d\tau = f_j(s) + s g_j(s,t)\,,$$

where:

$$f_j(s) = \psi_j(s,0) \text{ and } g_j(s,t) = \int_0^1 k_j(s, \tau t) d\tau\,,$$

are C^∞ on I_σ and $(I_\sigma)^n$ respectively.

(B) Consequence of (4.1).

We will use expression (4.1) in order to give a precise behavior of the zeroes of the functions: $\phi_{\delta,t}(s) = \phi_\delta(s,t)$, δ small , t fixed in $(I_\sigma)^{n-1}$, and those of the derivatives $(\phi_{\delta,,t})'(s)$. We already know that, for σ small enough, there are three zeroes of $\phi_{\delta,t}$ in I_σ: 0 , $s_{(1,\delta,t)}$, $s_{(2,\delta,t)}$, with $s_{(1,\delta,t)} < 0 < s_{(2,\delta,t)}$ (see Lemma 3.2).

As in Sect. 3, the two zeroes of the derivatives $(\phi_{\delta,t})'$ different from 0 are denoted by $\tilde{s}_{(j,\delta,t)}$, $j = 1, 2$, with $\tilde{s}_{(1,\delta,t)} < 0 < \tilde{s}_{(2,\delta,t)}$.

Let us write, with this new notation, the equation satisfied by the zeroes $s_{(1,\delta,t)}$ and $s_{(2,\delta,t)}$:

$$s^{p-q} + \delta \frac{f_2(s) + s g_2(s,t)}{f_1(s) + s g_1(s,t)} = 0\,. \tag{4.2}$$

In order to work with positive δ we can assume w.l.o.g. that $\psi_1\, \psi_2 < 0$ in $(I_\sigma)^n$. From now on, we assume:

$$\delta > 0\,.$$

We remark first that if we freeze functions f_j and g_j at $(s,t) = 0$, the solutions of equation (4.2) are:

$$(-1)^j \left(-\delta \frac{f_2}{f_1}(0)\right)^\mu \text{ with } \mu = \frac{1}{p-q}\,,$$

(see (3.14)).

We will write $a = -\frac{f_2}{f_1}(0)$. So, it is natural to compare the two zeroes $s_{(1,\delta,t)}$, $s_{2j,\delta,t)}$ to the last values.

Proposition 4.5 *There exists δ_0 and C such that, for $\delta < \delta_0$, with σ small enough, we have:*

$$s_{(j,\delta,t)} = (-1)^j.(a\delta)^\mu + E_{(j,\delta,t)}\,, \tag{4.3}$$

with,

$$|E_{(j,\delta,t)}| \leq C(a\delta)^{2\mu}$$

on $(I_\sigma)^n$.

Proof: We denote here the zeroes simply by s_j. Then we have first directly from the equation (using that $0 < c \leq \frac{|\psi_2|}{|\psi_1|} \leq C$ on $(I_\sigma)^n$)

$$c\delta \leq |s_j^{p-q}| \leq C\delta, \tag{4.4}$$

where c and C may vary from line to line.

$$s_j^{p-q} - (a\delta) = s_j^{p-q} - \delta\left(\frac{f_2(s_j) + s_j g_2(s_j,t)}{f_1(s_j) + s_j g_1(s_j,t)} - \frac{f_2(0)}{f_1(0)}\right).$$

Now we need the following.

Lemma 4.6 *We have:*

$$\frac{f_2(s) + s g_2(s,t)}{f_1(s) + s g_1(s,t)} - \frac{f_2(0)}{f_1(0)} = s g_3(s,t)\,, \tag{4.5}$$

where g_3 is a smooth function.

Proof of the lemma One has just to use: $f_j(s) - f_j(0) = s v_j(s)$, replace in the expression in order to obtain (4.5), with σ small enough. So we deduce from (4.4) and Lemma 4.6:

$$|s_j^{p-q} - a\delta| = |a\delta s_j g_3(s_j,t)| \leq C\delta\delta^\mu\,. \tag{4.6}$$

So now, we obtain from (4.6), for δ_0 small enough:

$$|s_j - (-1)^j (a\delta)^\mu| \leq C\,\delta^{2\mu} \text{ for } \delta < \delta_0\,,$$

hence (4.3).

Now we want to give the same kind of behavior for the zeroes $\tilde{s}_{(j,\delta,t)}$, $j = 1, 2$. For that, we have just to show that the derivative $(\phi_{\delta,t})'$ have the same form as $\phi_{\delta,t}$. More precisely we have:

Lemma 4.7 *The $\tilde{s}_{(j,\delta,t)}$ are the zeroes of an equation, similar to Eq.* (4.2)*, namely*

$$s^{p-q} + \delta \frac{f_4(s) + sg_4(s,t)}{f_3(s) + sg_3(s,t)} = 0\,, \tag{4.7}$$

where

$$f_3(s) = pf_1(s)\,,\ f_4(s) = qf_2(s)\,.$$

Proof A simple computation gives:

$$(\phi_{\delta,t})'(s) = s^{q-1}\left(s^{p-q}(p\psi_1(s,t) + s(\psi_{1,t})'(s)) + \delta(q\psi_2(s,t) + s(\psi_{2,t})'(s))\right).$$

Now, using (4.1), we have :

$$p\psi_1(s,t) + s(\psi_{1,t})'(s) = pf_1(s) + s(pg_1(s,t) + (\psi_{1,t})'(s)) = f_3(s) + sg_3(s,t)\,,$$
$$q\psi_2(s,t) + s(\psi_{2,t})'(s) = qf_2(s) + s(qg_2(s,t) + (\psi_{2,t})'(s)) = f_4(s) + sg_4(s,t)\,.$$

This proves (4.7), so we obtain the analogue of Proposition 4.5:

Proposition 4.8 *There exist $\delta_0 > 0$ and $C > 0$ such that, for $j = 1, 2$ and $\delta < \delta_0$, and σ small enough, we have:*

$$\tilde{s}_{(j,\delta,t)} = (-1)^j\,(\frac{q}{p}a\delta)^{\mu} + F_{(j,\delta,t)}\,, \tag{4.8}$$

with,

$$|F_{(j,\delta,t)}| \le C\,\delta^{2\mu} \text{ on } (I_\sigma)^n\,.$$

Our aim is to estimate the supremum of $\phi_{\delta,t}$ on I_σ (i.e. the value of $\phi_{\delta,t}$ at the point $\tilde{s}_{(1,\delta,t)}$; see Sect. 3 for the graph of ϕ_δ), which is:

$$\phi_{\delta,t}(\tilde{s}_{(j,\delta,t)}) = \phi_{\delta,t}\left(-(\frac{q}{p}a\delta)^{\mu} + E_{(j,\delta,t)}\right). \tag{4.9}$$

It will be convenient to write in the next, with O uniform in t:

$$E_{(j,\delta,t)} = O(\delta^{2\mu})\,,\ F_{(j,\delta,t)} = O(\delta^{2\mu})\,. \tag{4.10}$$

We can assume $f_1 > 0$, $f_2 < 0$.

Proposition 4.9 *For δ_0 and σ small enough one has:*

$$\phi_{\delta,t}(\tilde{s}_{(j,\delta,t)}) = A\delta^{p\mu} + O(\delta^{(p+1)\mu}),\, (s,t) \in (I_\sigma)^n\,,\, \delta < \delta_0\,, \tag{4.11}$$

with:

$$A = \left(\frac{q}{p}a\right)^{q\mu}(1 - \frac{q}{p})f_1(0)\,, \tag{4.12}$$

the O being uniform with respect to t.

Proof Using our previous result, mainly (4.8), we obtain, for $t \in (I_\sigma)^{n-1}$,

$$\begin{aligned}\phi_{\delta,t}(\tilde{s}_{(j,\delta,t)}) &= (-\tfrac{q}{p}(a\delta)^\mu + O(\delta^{2\mu}))^p(f_1(0) + O(\delta^\mu)) \\ &\quad +\delta(-\tfrac{q}{p}(a\delta)^\mu + O(\delta^{2\mu}))^q(f_2(0) + O(\delta^\mu)) \\ &= -(\tfrac{q}{p}a\delta)^{p\mu} + O(\delta^{(p+1)\mu})(f_1(0) + O(\delta^\mu)) \\ &\quad -\delta((\tfrac{q}{p}a\delta)^{q\mu} + O(\delta^{(q+1)\mu}))(f_2(0) + O(\delta^\mu)) \\ &= -(\tfrac{q}{p}a\delta)^{p\mu} f_1(0) - \delta(\tfrac{q}{p}a\delta)\delta^{q\mu} f_2(0) + O(\delta^{(p+1)\mu})\,.\end{aligned} \tag{4.13}$$

Remember now that $a = -\frac{f_2(0)}{f_1(0)} > 0$. Then a short manipulation gives

$$\phi_{\delta,t}(\tilde{s}_{(j,\delta,t)}) = -\delta(\frac{q}{p}a\delta)^{q\mu}(1 - \frac{q}{p}) f_2(0) + O(\delta^{(p+1)\mu})\,,$$

the O being uniform for $t \in (I_\sigma)^{n-1}$.

Now, to prove Theorem 4.1 under Remark 4.3, microlocally, we will contradict an inequality satisfied (when microlocal hypoellipticity at $(0; (0, \xi_0)) \in \mathbb{R}^{n+m} \times \mathbb{R}^{n+m}$ holds), between some semi-norms of u and Lu which we now introduce.

Given (N, M, K, V), where N, M are respectively, n-uple and m-uple of integers, K a compact in $(I_\sigma)^n$, V an open cone around $\xi_0 = (1, 0, ., 0)$ in R^m, we note, for a smooth function u:

$$s_{(N,M,K,V)}(u) = \sup_{(s,t)\in K} ||\xi^M (D_{s,t})^N \tilde{u}(s, t, \xi)||_{L^2(\xi\in V)}\,. \tag{4.14}$$

We want to show that for a suitable choice of (N, M, K, V) and α, we can contradict an inequality of the form,

$$s_{(N,M,K,V)}(\alpha u) \le C\left(s_{(0,0,K',V')}(\beta u) + s_{(N',M',K',V')}(\gamma(Lu))\right)\,, \tag{4.15}$$

for any smooth function u on $(I_\sigma)^n \times R^m$, any (N', M', K', V'), β and γ being given non negative test functions in $x \in \mathbb{R}^m$ with value 1 near 0, and with $L = L_\Phi$.

Proposition 4.10 *Inequality* (4.15) *is not satisfied for the choice* (N_0, M_0, K_0, V) *,* $N_0 = 0$ *,* $M_0 = (1, 0, \dots, 0)$*,* $K_0 = [-\frac{\sigma}{2}\,,\ \frac{\sigma}{2}] \times \{t = 0\}$.

Proof We will use the family of functions $u_{(\lambda,\delta)}$ like in Sect. 3 (see (3.7)), namely here with $\alpha = \alpha(x)$,

$$\alpha u_{(\lambda,\delta)}(s, t, x) = \alpha(x)\theta_\delta(s)\exp(w_{(\lambda,\delta)}(s, t, x))\,,$$

where, having in mind (3.6), we can add the property:

$$|\theta_\delta^{(k)}| \le C_k\, \delta^{-k\mu}\,, \tag{4.16}$$

with

$$w_{(\lambda,\delta)}(s,t,x) = i\lambda(x_1 + \delta x_2 - i\phi_\delta)\,.$$

From these properties of the $\theta_\delta(s)$ and our property (4.11), we get the following facts:

$$|u_{(\lambda,\delta)}(s,t,x)| = \theta_\delta(s)\exp(\varphi_{(\delta,t)}) = A\,\delta^{p\mu} + O(\delta^{(p+1)\mu})\,, \tag{4.17}$$

$$\sup|u_{(\lambda,\delta)}(s,t,x)| = \exp(A\,\delta^{p\mu} + O(\delta^{(p+1)\mu}))\,,\ \delta < \delta_0\,, \tag{4.18}$$

$$\widehat{\alpha u_{(\lambda,\delta)}}(s,t,\xi) = \theta_\delta(s)\exp(\varphi_{(\delta,t)})\,\hat{\alpha}(\xi - \Lambda_\delta)\,, \tag{4.19}$$

where $\Lambda_\delta = \lambda\,(1,\delta,\cdots,0)$, and $\widehat{u}$ denotes the partial Fourier transform $x \to \xi$.

We can now give estimates, in terms of (λ,δ), of the two members in (4.15) with $u = u_{(\lambda,\delta)}$ and $(N, M, K, V) = (N_0, M_0, K_0, V)$.

(1) **Estimate from below of** $s_{(N_0,M_0,K_0,V)}(\alpha u_{(\lambda,\delta)})$.

We first observe that

$$s_{(N_0,M_0,K_0,V)}(\alpha u_{(\lambda,\delta)}) \geq \sup_{s\in[-\sigma,\sigma]} ||\xi_1\hat{v}_{(\lambda,\delta)}(s,0,\xi)||_{L^2(\xi\in V)}\,,$$

where $v_{(\lambda,\delta)}(s,0,x) = \alpha(x)u_{(\lambda,\delta)}(s,0,x)$.

Now, we have, from (4.18):

$$\begin{aligned}||\xi_1\,\hat{v}_{(\lambda,\delta)}(s,0,\xi)||_{L^2(\xi\in V)} &= ||\theta_\delta(s)\exp(\phi_{(\delta,0)})(s)\xi_1\,\hat{\alpha}(\xi-\Lambda_\delta)||_{L^2(\xi\in V)}\\ &\geq \theta_\delta(s)\exp(\phi_{(\delta,0)}(s))\,||(\eta_1+\lambda)\hat{\alpha}(\eta)||_{L^2(\eta\in V-\Lambda_\delta)}\,.\end{aligned} \tag{4.20}$$

But:

$$V \subset V - \Lambda_\delta\,, \tag{4.21}$$

if δ_0 is small enough.

Hence:

$$s_{(N_0,M_0,K_0,V)}(\alpha u_{(\lambda,\delta)}) \geq \exp\lambda(A\,\delta^{p\mu} + O(\delta^{(p+1)\mu})) \times ||(\eta_1+\lambda)\hat{\alpha}(\eta)||_{L^2(V)}. \tag{4.22}$$

Note from (4.12) that $A > 0$. Now we remark that $\eta_1 + \lambda \geq \lambda$, for $\eta \in V$. So, we get:

$$s_{(N_0,M_0,K_0,V)}(\alpha u_{(\lambda,\delta)}) \geq \lambda\exp\lambda(A\,\delta^{p\mu} + O(\delta^{(p+1)\mu})) \times ||\hat{\alpha}(\eta)||_{L^2(V)}. \tag{4.23}$$

At this step we can choose a test function α in $\mathbb{R}^m$, with value 1 near 0 such that, choosing first α such that $||\hat{\alpha}||_{L^2(V)} \neq 0$,

$$||\hat{\alpha}||_{L^2(V)} = 1\,. \tag{4.24}$$

So, for a given open cone V around ξ_0, taking δ_0 small enough (in order to have (4.21)), choosing α satisfying (4.24), one has

$$s_{(N_0,M_0,K_0,V)}(\alpha u_{(\lambda,\delta)}) \geq \lambda \exp \lambda(A\,\delta^{p\mu} + O(\delta^{(p+1)\mu}))\,,\ \delta < \delta_0\,,\ \lambda > 0\,. \quad (4.25)$$

(2) **An upper bound for** $s_{(0,0,K',V')}(\beta u_{(\lambda,\delta)})$**, with** $K' = [-\frac{\sigma}{2}\,,\ \frac{\sigma}{2}]\times\{||t|| \leq \sigma'\}$: This part is easier than part (1). The same computations give

$$s_{(0,0,K',V')}(\beta u_{(\lambda,\delta)}) \leq \exp \lambda(A\,\delta^{p\mu} + O(\delta^{(p+1)\mu})) \times ||\hat{\beta}(\eta)||_{L^2(\mathbb{R}^m)}. \quad (4.26)$$

(3) **An upper bound for** $s_{(N',M',K',V')}(\gamma(Lu_{(\lambda,\delta)}))$:
(a) The expression of $Lu_{(\lambda,\delta)} = (L_j u_{(\lambda,\delta)})_j$.
We have

$$\begin{aligned} L_1 u_{(\lambda,\delta)}(s,t,x) &= (\theta_\delta)'(s)\exp(w_{(\lambda,\delta)}(s,t,x))\,,\\ L_j u_{(\lambda,\delta)}(s,t,x) &= 0\,, \qquad\qquad j = 2,\cdots,m\,. \end{aligned}$$

So,we have just to consider the derivatives of the first term

$$(D_{(s,t)})^N (L_1 u_{(\lambda,\delta)})(s,t,x) = \sum_{\textit{finite}} D^{a,b}((\theta_\delta)'(s)\exp w_{(\lambda,\delta)})(s,t,x),$$

where $D^{a,b}$ are derivatives in (s,t), $b = (b_1,\cdots,b_{n-1})$.
(b) Estimates.
Then one has easily that $(D(s,t))^N(L_1 u_{(\lambda,\delta)})$ is a finite sum (with $k \leq |N|$, $v_k(s,t)$ smooth):

$$D^{a+1}(\theta_\delta(s))\lambda^k\, v_k(s,t)\exp(w_{(\lambda,\delta)}(s,t,x))\,. \quad (4.27)$$

So we search a bound to

$$\sup_{(s,t)\in K'} \lambda^k |D^{a+1}(\theta_\delta)(s)v_k(s,t)| \times ||\xi^M\,\hat{\gamma}(\xi - \Lambda_\delta)||_{L^2(\xi\in V')}\,. \quad (4.28)$$

So, we have just to give an upper bound to the second factor in (4.27) First we note that, from Proposition 3.4 -3) and (4.16), we have the following upper bounds:

$$|D^{a+1}(\theta_\delta)(s)v_k(s,t)| \leq C_k\,\delta^{-(a+1)\mu}\,. \quad (4.29)$$

$$\sup_{s\in\text{supp}(D^{a+1}(\theta_\delta))\,,\,||t||<\sigma'} |\exp(w_{(\lambda,\delta)}(s,t,x))| \leq C\,\exp\lambda(-b\delta^{p\mu})\,,\ b > 0\,. \quad (4.30)$$

$$||\xi^M\,\hat{\gamma}(\xi-\Lambda_\delta)||_{L^2(\xi\in V')} \leq |||\eta+\lambda|^{|M|}\,\hat{\gamma}(\eta)||_{L^2(V'-\Lambda_\delta)} \leq C\lambda^{|M|}\,||\gamma||_{H^{|M|}}\,, \quad (4.31)$$

(where H^ℓ is the ℓ-Sobolev space $H^\ell(\mathbb{R}^m)$).

So, we get from (4.25), (4.26), (4.29), (4.30), and (4.31):

$$\lambda \exp \lambda(A\,\delta^{p\mu} + O(\delta^{(p+1)\mu})) \le \exp \lambda(A\,\delta^{p\mu} + O(\delta^{(p+1)\mu}))||\hat{\beta}||_{L^2} + C\,\lambda^{|M|}\,C_k\,\delta^{-(a+1)\mu} \exp \lambda(-b\delta^{p\mu})||\gamma||_{H^{|M|}}\,, \tag{4.32}$$

with $\delta < \delta_0\,,\ \lambda > 0\,.$
So, we deduce the existence of $C > 0$, such that

$$\lambda \exp \lambda(A\,\delta^{p\mu} - C\delta^{(p+1)\mu}) \le C \exp \lambda(A\,\delta^{p\mu} + C\delta^{(p+1)\mu}) + C\,\lambda^{|M|}\,\delta^{-(a+1)\mu} \exp \lambda(-b\delta^{p\mu})\,, \tag{4.33}$$

with $\delta < \delta_0\,,\ \lambda > 0\,.$

Now, our theorem will be proved if the following lemma holds:

Lemma 4.11 *Inequality* (4.33) *is not satisfied for* $\delta < \delta_0\,,\ \lambda > 0\,.$

Proof
Take any pair (λ, δ), such that:

$$\lambda\delta^{(p+1)\mu} = 1\,,\ \delta < \delta_0\,,\ \lambda > 0\,.$$

Then

$$\lambda \exp(A\delta^{-\mu} - C) \le C \exp(A\delta^{-\mu} + C) + \lambda^{|M|}\,\delta^{-(|N|+1)\mu} \exp(-b\delta^{-\mu})\,,$$

or

$$(\lambda - C') \exp(A\delta^{-\mu}) \le C' + \delta^{-\mu(|N|+1+|M|(p+1))} \exp(-b\delta^{-\mu})\,, \tag{4.34}$$

with $\lambda \ge C' + 1, \delta < \delta_0\,.$

Now just observe that the right hand side in (4.34) is bounded as $\delta \to 0$, but that the left hand side (as $\lambda \ge C' + 1$) tends to ∞. So, the proof of the lemma, hence of Theorem 4.1, is finished.

Proof of Theorem 4.2
So we consider the case $m = 2$, hence with:

$$\Phi = (\varphi_1, \varphi_2), \Phi_{\xi_0} = \varphi_1, \Phi_\xi = a\varphi_1 + b\varphi_2, a, b \in \mathbb{R}, b \ne 0. \tag{4.35}$$

So

$$\Phi_\xi = b\,s^q(\psi_2 + as^{p-q}\psi_1)\,, \tag{4.36}$$

with $\psi_j \ne 0$, in a neighborhood ω of 0.

In order to prove that L_Φ is microlocally hypoelliptic at $(0, (0, \xi)) \in \mathbb{R}^{n+m} \times \mathbb{R}^{n+m}$, we use a criterion which I proved in [5], for $m = 1$, but which is true for

$m \geq 2$, in a version established in collaboration with B. Helffer in [7], proving in fact microlocal $\frac{1}{q}$-subellipticity. For that we want to construct in a neighborhood ω of 0 in $\mathbb{R}^n$, a family of curves, depending on $(s, t, \eta) \in \omega \times V$ (with V being a small cone around ξ) $\gamma_{\eta,s,t}(\tau) : [0, 1] \mapsto \mathbb{R}^n$ satisfying the properties needed in this criterion.

The choice of ω and V .

As $\xi \neq \xi_0$, we can choose V and $C > 0$ such as

$$\eta \in V \to ||\eta|| \leq C\,|\eta_2|, \text{ with } \eta = (\eta_1, \eta_2). \tag{4.37}$$

Now we have:

$$\begin{aligned}\Phi_\eta(s,t) &= s^q\,(\eta_2\,\psi_2(s,t) + s^{p-q}\,\eta_1\,\psi_1(s,t)) \qquad = s^q\,\psi_\eta(s,t) \\ &= ||\eta||\,s^q\,(\eta'_2\psi_2(s,t) + s^{p-q}\eta'_1\psi_1(s,t)) = ||\eta||\,s^q\,\psi_{\eta'}(s,t)\,,\ ||\eta'|| = 1\,.\end{aligned}$$

So, we see that there exist a neighborhood ω_0 of 0, $c_0 > 0$, $C_0 > 0$, such that (writing now η in place of η', so with $||\eta|| = 1$)

$$c_0 \leq |\psi_\eta(s,t)| \leq C_0\,,\ \eta \in V, (s,t) \in \omega_0\,. \tag{4.38}$$

Consider now :

$$F_\eta : \omega_0 \ni (s,t) \mapsto (u_\eta(s,t), t) \in \mathbb{R}^2, u_\eta(s,t) = s(\psi_\eta(s,t))^{\frac{1}{q}}\,, \tag{4.39}$$

so that

$$\Phi_\eta(s,t) = s^q\psi_\eta(s,t) = (u_\eta(s,t))^q\,. \tag{4.40}$$

Lemma 4.12 *If ω_0 is small enough, the mapping F_η is a smooth diffeomorphism from ω_0 onto $F_\eta(\omega_0)$, with $F_\eta(0) = 0$.*

Proof: Trivially: $F_\eta(0) = 0$ and from (4.39), u_η , hence F_η are smooth. The determinant of the Jacobian matrix of F_η is equal the derivative with respect to s of u_η. This derivative is equal to

$$(u_{\eta,t})'(s) = (\psi_\eta(s,t))^{\frac{1}{q}} + s(\frac{1}{q}(\psi_{\eta,t})'(s)(\psi_\eta(s,t))^{(\frac{1}{q}-1)}). \tag{4.41}$$

$$|(\psi_{\eta,t})'(s)| = |\eta_1 s^{p-q}(\psi_{1,t})'(s) + \eta_2(\psi_{2,t})'(s)| \leq C\,. \tag{4.42}$$

We deduce from (4.38), (4.41), and (4.42) that, if ω_0 is small, for some $c_1, C_1 > 0$:

$$0 < c_1 \leq |(u_{\eta,t})'(s)| \leq C_1\,,\ (s,t) \in \omega_0\,,\ ||\eta|| = 1\,. \tag{4.43}$$

This gives the lemma.

Hence F_η is a change of variable in ω_0 For every η, $\|\eta\| = 1$, we now define our family of curves by:

$$\gamma_{(\eta,s,t)}(\tau) = F_\eta^{-1}(u_\eta(s,t) + \tau, t), \tau \in [0,1], (s,t) \in \omega_0, \|\eta\| = 1. \tag{4.44}$$

In order to apply our criterion, we have now to establish the properties needed (see [5] or [7]), which are:

(i) $\gamma_{(\eta,s,t)}(0) = (s,t)$, $\gamma_{(\eta,s,t)}(1) \notin \omega_0$, if ω_0 is small enough. The first part of (i) is clear. Now take ω_0 small so that ω_0 and $F_\eta(\omega_0)$ are contained in the ball $B(\frac{1}{2})$ of radius $\frac{1}{2}$. Then:

$$(u_\eta(s,t) + 1, t) \in C(B(\frac{1}{2})).$$

This implies $\gamma_{(\eta,s,t)}(1) \in C(\omega_0)$.

(ii)

$$|(\gamma_{(\eta,s,t)})'(\tau)| \leq C\ ,\ \tau \in [0,1]\,.$$

This is clear from (4.44).

(iii) Lower bound for $\Phi_\eta(\gamma_{(\eta,s,t)}(\tau)) - \Phi_\eta(s,t)$. We have

$$\begin{aligned}\Phi_\eta(\gamma_{(\eta,s,t)}(\tau)) - \Phi_\eta(s,t) &= \Phi_\eta \circ F_\eta^{-1}(u_\eta(s,t) + \tau, t) - \Phi_\eta \circ F_\eta^{-1}(u_\eta(s,t), t)\\ &= (u_\eta(s,t) + \tau)^q - (u_\eta(s,t))^q, \text{(from 4.40)}\\ &\geq c_q \tau^q\ ,\ \tau \in [0,1]\,, (s,t) \in \omega_0\,.\end{aligned}$$

This finishes the proof of Theorem 4.2.

Remark 4.13 Looking at our Theorem 4.1, it is useful to observe that Φ_{ξ_0} satisfies the necessary condition of Treves for microlocal hypoellipticity. H. Maire gave for $n = m = 2$, an example satisfying Treves's condition without to be hypoelliptic.

Remark 4.14 There are very simple examples for which one has no microlocal maximal estimate but for which we know nothing about microlocal hypoellipticity. A simple one is the following, in case $n = m = 2$;

$$\Phi = (\varphi_1, \varphi_2) = (s^3, st^2)\ ,\ (s,t) \in \mathbb{R}^2\,.$$

We know that, for this Φ , L_Φ is not maximally microlocally hypoelliptic at $(0, (0, \xi_0)) \in \mathbb{R}^{2+2} \times \mathbb{R}^{2+2}, \xi_0 = (1, 0)$. But this example does not satisfy Hypothesis $(H3)$, so Theorem 4.1 cannot be applied.

Remark 4.15 It is useful to note, in the context of Theorem 4.2, that the operator L_Φ is analytic hypoelliptic near 0 (this is a local result, not microlocal). It is a consequence of [2, Theorem 2.1] because for any ξ different from 0, Φ_ξ has no extremum in a neighborhood of the origin.

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Submanifolds of Hermitian Symmetric Spaces

Xiaojun Huang and Yuan Yuan

In memory of M. Salah Baouendi

Abstract We study the problem of non-relativity for a complex Euclidean space and a bounded symmetric domain equipped with their canonical metrics. In particular, we answer a question raised by Di Scala. This paper is dedicated to the memory of Salah Baouendi, a great teacher and a close friend to many of us.

Keywords Hermitian symmetric space · Isometric embedding · Bergman metric · Nash algebraic function

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1 Introduction

Holomorphic isometric embeddings have been studied extensively by many authors. In the celebrated paper by Calabi [1], he obtained the global extendability and rigidity of a local holomorphic isometry into a complex space form, among many other important results. In particular, he proved that any complex space form cannot be locally isometrically embedded into another complex space form with a different curvature sign with respect to the canonical Kähler metrics. In his paper, Calabi introduced the so called diastasis function and reduced the metric tensor equation

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to the functional identity for the diastasis functions. In a later development [5], Di Scala and Loi generalized Calabi's non-embeddability result to the case of Hermitian symmetric spaces of different types.

On the other hand, Umehara [17] studied an interesting question whether two complex space forms can share a common submanifold with the induced metrics. Following Calabi's idea, Umehara proved that two complex space forms with different curvature signs cannot share a common Kähler submanifold. When two complex manifolds share a common Kähler submanifolds with induced metrics, Di Scala and Loi in [6] called them to be relatives. Furthermore, Di Scala and Loi proved that a bounded domain with its associated Bergman metric can not be a relative to a Hermitian symmetric space of compact type equipped with the canonical metric. Notice that any irreducible Hermitian symmetric space of compact type can be holomorphically isometrically embedded into a complex project space by the classical Nakagawa-Takagi embedding. Therefore in order to show that a Kähler manifold is not a relative of a projective manifold with induced metric, it suffices to show that it is not a relative to the complex projective space with the Fubini-Study metric. Meanwhile it follows from the result of Umehara [17], the complex Euclidean space and the irreducible Hermitian symmetric space of compact type cannot be relatives. After these studies, it remains to understand if a complex Euclidean space and a Hermitian symmetric space of noncompact type can be relatives.

Denote the Euclidean metric on $\mathbb{C}^n$ by $\omega_{\mathbb{C}^n}$. For each $1 \leq j \leq J$, let the bounded symmetric domain $\Omega_j \subset \mathbb{C}^{m_j}$ be the Harish-Chandra realization of an irreducible Hermitian symmetric space of noncompact type and let ω_{Ω_j} be the Bergman metric on Ω_j. Let $D \subset \mathbb{C}^\kappa$ be a connected open set and ω_D be a Kähler metric on D, not necessarily complete.

In this short paper, we show that there do not simultaneously exist holomorphic isometric immersions $F : (D, \omega_D) \to (\mathbb{C}^n, \omega_{\mathbb{C}^n})$ and $G = (G_1, \ldots, G_J) : (D, \omega_D) \to (\Omega_1, \mu_1\omega_{\Omega_1}) \times \cdots \times (\Omega_J, \mu_J\omega_{\Omega_J})$ with $\mu_1, \ldots, \mu_J$ positive real numbers. As a consequence, a complex Euclidean space and a bounded symmetric domain cannot be relatives. Indeed, we prove the following slightly stronger result:

Theorem 1.1 *Let $D \subset \mathbb{C}$ be a connected open subset. Suppose that $F : D \to \mathbb{C}^n$ and $G = (G_1, \ldots, G_J) : D \to \Omega = \Omega_1 \times \cdots \times \Omega_J$ are holomorphic mappings such that*

$$F^*\omega_{\mathbb{C}^n} = \sum_{j=1}^{J} \mu_j G_j^*\omega_{\Omega_j} \quad \text{on} \quad D \tag{1}$$

for certain real constants $\mu_1, \ldots, \mu_J$. Then F must be a constant map. Furthermore, if all μ_j's are positive, then G is also a constant map.

Corollary 1.2 *There does not exist a Kähler manifold (X, ω_X) that can be holomorphic isometrically embedded into the complex Euclidean space $(\mathbb{C}^n, \omega_{\mathbb{C}^n})$ and also into a Hermitian symmetric space of noncompact type (Ω, ω_Ω).*

2 Proof of Theorem 1.1

Our proof fundamentally uses ideas developed in our previous work [10]. Let D be a domain in $\mathbb{C}$. Let $F = (f_1, \ldots, f_n) : D \to \mathbb{C}^n$, $G = (G_1, \ldots, G_J) : D \to \Omega_1 \times \cdots \times \Omega_J$ be holomorphic maps satisfying Eq. (1). Without loss of generality, assume that $0 \in D$ and $F(0) = 0$, $G(0) = 0$. We argue by contradiction by assuming that F is not constant. By Eq. (1), we have

$$\partial\bar{\partial}\left(\sum_{i=1}^{n} |f_i(z)|^2\right) = \sum_{j=1}^{J} \mu_j \partial\bar{\partial} \log K_j(G_j(z), \overline{G_j(z)}) \quad \text{for} \quad z \in D,$$

where $K_j(\xi, \overline{\eta}) = \sum_l h_{jl}(\xi)\overline{h_{jl}(\eta)}$ is the Bergman kernel on Ω_j and $\{h_{jl}(\xi)\}$ is an orthonormal basis of L^2 integrable holomorphic functions over Ω_j. Note that Ω_j is a complete circular domain in the Harish-Chandra realization. Therefore, the Bergman kernel of Ω_j satisfies the identity $K_j(e^{\sqrt{-1}\theta}\xi, \overline{e^{\sqrt{-1}\theta}\eta}) = K_j(\xi, \overline{\eta})$ for any $\theta \in \mathbb{R}$ and any $\xi, \eta \in \Omega_j$. This implies $K_j(e^{\sqrt{-1}\theta}\xi, 0) = K_j(\xi, 0)$. Therefore $K_j(\xi, 0)$ is a positive constant. In another word, $K_j(\xi, \overline{\eta})$ does not contain any nonconstant pure holomorphic terms in ξ. Similarly, $K_j(\xi, \overline{\eta})$ does not contain any nonconstant pure anti-holomorphic terms in η. Hence $K_j(\xi, \overline{\xi})$ does not contain nonconstant pluriharmonic terms in ξ. After normalization, we can assume tha $K_j(\xi, 0) = 1$. By the standard argument in [2], one can get rid of $\partial\bar{\partial}$ to obtain the following functional identity by comparing the pure holomorphic and anti-holomorphic terms in z:

$$\sum_{i=1}^{n} |f_i(z)|^2 = \sum_{j=1}^{J} \mu_j \log K_j(G_j(z), \overline{G_j(z)}) \quad \text{for any } z \in D. \tag{2}$$

An alternative way to obtain (2) is from the explicit computation of the Calabi's diastasis function at 0 [1].

After polarization, (2) is equivalent to

$$\sum_{i=1}^{n} f_i(z)\bar{f_i}(w) = \sum_{j=1}^{J} \mu_j \log K_j(G_j(z), \bar{G_j}(w)) \quad \text{for } (z, w) \in D \times \mathrm{conj}(D), \tag{3}$$

where $\bar{f_i}(w) = \overline{f_i(\overline{w})}$ and $\mathrm{conj}(D) = \{z \in \mathbb{C} | \bar{z} \in D\}$. Notice that the Bergman kernel $K_j(\xi, \eta)$ is a rational function on ξ and η for the bounded symmetric domain Ω_j [7]. From this, we have the following algebraicity lemma. Here, we recall that a function H is called a holomorphic Nash algebraic function in $V \subset \mathbb{C}^\kappa$ if H is holomorphic over V and there is a non-zero polynomial $P(\eta, X)$ in (η, X) such that $P(\eta, H(\eta)) \equiv 0$ for $\eta \in V$.

Lemma 2.1 *For any $1 \le i \le n$, $f_i(z)$ can be written as a holomorphic Nash algebraic function in $G(z) = (G_1(z), \ldots, G_J(z))$, in the sense that, there exists a* *holomorphic Nash algebraic function* $\hat{f}_i(X_1, \ldots, X_J)$ *such that*

$$\hat{f}_i(G_1(z), \ldots, G_J(z)) = f_i(z),$$

after shrinking D *toward the origin if needed.*

Proof The proof is similar to the algebraicity lemma in Proposition 3.1 of [10]. Write $D^\delta = \frac{\partial^\delta}{\partial w^\delta}$. Applying the differentiation $\frac{\partial}{\partial w}$ to Eq. (3), we get for w near 0 the following:

$$\sum_{i=1}^{n} f_i(z) \frac{\partial}{\partial w} \bar{f}_i(w) = \sum_{j=1}^{J} \mu_j \frac{\frac{\partial}{\partial w} K_j(G_j(z), \bar{G}_j(w))}{K_j(G_j(z), \bar{G}_j(w))}. \tag{4}$$

We can rewrite (4) as follows:

$$F(z) \cdot D^1(\bar{F}(w)) = \phi_1(w, G_1(z), \ldots, G_J(z)), \tag{5}$$

where $F = (f_1, \ldots, f_n)$, and $\phi_1(w, X_1, \ldots, X_J)$ is Nash algebraic in $(X_1, \ldots, X_J)$ for each fixed w, as the Bergman kernel functions $K_j(\xi, \overline{\eta})$ are rational functions. Now, differentiating (5), we get for any δ the following equation

$$F(z) \cdot D^\delta(\bar{F}(w)) = \phi_\delta(w, G_1(z), \ldots, G_J(z)). \tag{6}$$

Here for $\delta > 0$, $\phi_\delta(w, X_1, \ldots, X_J)$ is Nash algebraic in $X_1, \ldots, X_J$ for any fixed w.

Now, let $\mathcal{L} := \mathrm{Span}_{\mathbb{C}}\{D^\delta(\bar{F}(w))|_{w=0}\}_{\delta \geq 1}$ be a vector subspace of $\mathbb{C}^n$. Let $\{D^{\delta_j}(\bar{F}(w))|_{w=0}\}_{j=1}^{\tau}$ be a basis for $\mathcal{L}$. Then for a small open disc Δ_0 centered at 0 in $\mathbb{C}$, $\bar{F}(\Delta_0) \subset \mathcal{L}$. Indeed, for any w near 0, we have from the Taylor expansion that

$$\bar{F}(w) = \bar{F}(0) + \sum_{\delta \geq 1} \frac{D^\delta(\bar{F})(0)}{\delta!} w^\delta = \sum_{\delta \geq 1} \frac{D^\delta(\bar{F})(0)}{\delta!} w^\delta \in \mathcal{L}.$$

Now, let ν_j $(j = 1, \ldots, n-\tau)$ be a basis of the Euclidean orthogonal complement of $\mathcal{L}$. Then, we have

$$F(z) \cdot \nu_j = 0, \quad \text{for each} \quad j = 1, \ldots, n - \tau. \tag{7}$$

Consider the system consisting of (6) at $w = 0$ (with $\delta = \delta_1, \ldots, \delta_\tau$) and (7). The linear coefficient matrix in the left hand side of the system at $w = 0$ with respect to $F(z)$ is

$$\begin{bmatrix} D^{\delta_1}(\bar{F}(w))|_{w=0} \\ \vdots \\ D^{\delta_\tau}(\bar{F}(w))|_{w=0} \\ \nu_1 \\ \vdots \\ \nu_{n-\tau} \end{bmatrix}$$

and is obviously invertible. Note that the right hand side of the system of equations consisting of (6) at $w = 0$ (with $\delta = \delta_1, \ldots, \delta_\tau$) and is Nash algebraic in $G_1(z), \ldots, G_J(z)$. By Cramer's rule, there exists a Nash algebraic function $\hat{F}(X_1, \ldots, X_J)$ in all variables $X_1, \cdots, X_J$ such that $F(z) = \hat{F}(G_1(z), \ldots, G_J(z))$ near $z = 0$. In fact, in our setting here, we can make $\hat{F}$ holomorphically rational in its variables. □

Let $G = (G_1, \ldots, G_J) = (g_{11}, \ldots, g_{1m_1}, \ldots, g_{J1}, \ldots, g_{Jm_J})$. Let $\mathfrak{R}$ be the field of rational functions in z over D. Consider the field extension

$$\mathfrak{F} = \mathfrak{R}(g_{11}(z), \ldots, g_{Jm_J}(z)),$$

namely, the smallest subfield of meromorphic function field over D containing rational functions and $g_{11}, \ldots, g_{Jm_J}$. Let l be the transcendence degree of the field extension $\mathfrak{F}/\mathfrak{R}$.

If $l = 0$, then each element in $\{g_{11}(z), \ldots, g_{Jm_J}(z)\}$ is a Nash algebraic function. Hence by Lemma 2.1, each $f_i(z)$ is also Nash algebraic. In this case, we arrive at a contradiction by the following lemma together with Eq. (3).

Lemma 2.2 *Let $V \subset \mathbb{C}^\kappa$ be a connected open set. Let $H_1(\xi_1, \ldots, \xi_\kappa), \ldots, H_K(\xi_1, \ldots, \xi_\kappa)$ and $H(\xi_1, \ldots, \xi_\kappa)$ be holomorphic Nash algebraic functions on V. Assume that*

$$\exp^{H(\xi_1,\ldots,\xi_\kappa)} = \prod_{k=1}^{K} (H_k(\xi_1, \ldots, \xi_\kappa))^{\mu_k}, \ \xi \in V,$$

for certain real numbers $\mu_1, \ldots, \mu_K$. Then $H(\xi_1, \ldots, \xi_\kappa)$ is constant.

Proof Suppose that H is not constant. After a linear transformation in ξ, if needed, we can assume, without loss of generality, that, $H(\xi)$ is not constant for a certain fixed $\xi_2, \ldots, \xi_\kappa$. Then H is a non-constant Nash-algebraic holomorphic function in ξ_1 for such fixed $\xi_2, \ldots, \xi_\kappa$. Hence, we can assume that $\kappa = 1$ to achieve a contradiction. Write $H = H(\xi)$ and $H_k = H_k(\xi)$ for each $1 \le k \le K$. Use $S \subset \mathbb{C}$ to denote the union of branch points, poles and zeros of $H(\xi)$ and $H_k(\xi)$ for each k. Given a $p \in \mathbb{C} \backslash S$ and a real curve in $\mathbb{C} \backslash S$ connecting p and V, by holomorphic continuation, the following equation holds on an open neighborhood of the curve:

$$\exp^{H(\xi)} = \prod_{k=1}^{K} (H_k(\xi))^{\mu_k}. \tag{8}$$

Assume that the minimal polynomial of H is given by $p(\xi, X) = A_d(\xi)X^d + \cdots + A_0(\xi)$ such that $p(\xi, H(\xi)) \equiv 0$. Denote the branches of H by $\{H^{(1)}, \ldots, H^{(d)}\}$ and these branches can be obtained through H by holomorphic continuation. Denote the corresponding branches for H_k obtained by holomorphic continuation by $\{H_k^{(1)}, \ldots, H_k^{(d)}\}$. Let ξ_0 be a zero of $\frac{A_d}{A_0}$ or $\xi_0 = \infty$ if $\frac{A_d}{A_0}$ is a constant. Then some branches of H blow up at ξ_0. Without loss of generality, assume that $\xi_0 = \infty$. Assume that (8) holds in a neighborhood of ∞ after holomorphic continuation from the original equality. By the Puiseux expansion, we can assume that

$$H(\xi) = \sum_{\beta=\beta_0,\beta_0-1,\ldots,-\infty} a_\beta \xi^{\beta/N_0} = a_{\beta_0}\xi^{\beta_0/N_0} + o(|\xi|^{\beta_0/N_0})$$

for $|\xi| >> 1$ with $a_{\beta_0} \neq 0$ and $\beta_0, N_0 > 0$. Without loss of generality, we assume that $a_{\beta_0} > 0$. Now, when $\xi \to \infty$ along the positive x-axis, for the branch $H^{(*)}$, which corresponds to ξ^{β_0/N_0} taking positive value along this ray in its Puiseux expansion, we have $|e^{H^{(*)}(x)}| \geq e^{\left(\frac{a_{\beta_0}}{2} x^{\frac{\beta_0}{N_0}}\right)}$ as $x \to +\infty$. However, the right hand side of (8) grows at most polynomially. This is a contradiction. □

Now, assume that $l > 0$. By re-ordering the lower index, let $\mathcal{G} = \{g_1(z), \ldots, g_l(z)\}$ be the maximal algebraic independent subset in $\mathfrak{F}$. It follows that the transcendental degree of $\mathfrak{F}/\mathfrak{R}(\mathcal{G})$ is 0. Then there exists a small connected open subset U with $0 \in \overline{U}$ such that for each j_α with $g_{j_\alpha} \notin \mathcal{G}$, we have a holomorphic Nash algebraic function $\hat{g}_{j_\alpha}(z, X_1, \ldots, X_l)$ in the neighborhood $\hat{U}$ of $\{(z, g_1(z), \ldots, g_l(z)) | z \in U\}$ in $\mathbb{C} \times \mathbb{C}^l$ such that it holds that $g_{j_\alpha}(z) = \hat{g}_{j\alpha}(z, g_1(z), \ldots, g_l(z))$ for any $z \in U$. Then by Lemma 2.1, for each $1 \leq i \leq n$, there exists a holomorphic Nash algebraic function $\hat{f}_i(z, X_1, \ldots, X_l)$ in $\hat{U}$ such that $f_i(z) = \hat{f}_i(z, g_1(z), \ldots, g_l(z))$ for $z \in U$. Define

$$\begin{aligned}\Psi(z, X, w) = &\sum_{i=1}^{n} \hat{f}_i(z, X)\bar{f}_i(w) \\ &- \sum_{j=1}^{J} \mu_j \log K_j(\ldots, X_\gamma, \ldots, \hat{g}_{j\alpha}(z, X), \ldots, \bar{g}_{j1}(w), \ldots, \bar{g}_{jm_j}(w))\end{aligned}$$

and

$$\Phi(z, X, w) = \frac{\partial}{\partial w}\Psi(z, X, w)$$

for $(z, X, w) \in \hat{U} \times \mathrm{conj}(U)$, where $X = (X_1, \ldots, X_l)$.

Lemma 2.3 *For any w near 0 and any $(z, X) \in \hat{U}$, $\Phi(z, X, w) \equiv 0$. As a consequence, $\Psi(z, X, w) \equiv 0$.*

Proof Assume $\Phi(z, X, w) \not\equiv 0$. Then there exists w_0 near 0, such that $\Phi(z, X, w_0) \not\equiv 0$. Since $\Phi(z, X, w_0)$ is a Nash algebraic function in (z, X), then there exists a holomorphic polynomial $P(z, X, t) = A_d(z, X)t^d + \cdots + A_0(z, X)$ of degree d in t, with $A_0(z, X) \not\equiv 0$ such that $P(z, X, \Phi(z, X, w_0)) \equiv 0$.

As $\Psi(z, g_1(z), \ldots, g_l(z), w) \equiv 0$ for $z \in U$, it follows that $\Phi(z, g_1(z), \ldots, g_l(z), w_0) \equiv 0$ and therefore $A_0(z, g_1(z), \ldots, g_l(z)) \equiv 0$. This means that $\{g_1(z), \ldots, \ldots, g_l(z)\}$ are algebraic dependent over $\mathfrak{R}$. This is a contradiction.

Since $\Psi(z, X, w)$ is holomorphic in w and $\Psi(z, X, 0) \equiv 0$, then $\Psi(z, X, w) \equiv 0$. □

Now for any $(z, X, w) \in \hat{U} \times \text{conj}(U)$, we have the following functional identity:

$$\sum_{i=1}^{n} \hat{f}_i(z, X)\bar{f}_i(w) = \sum_{j=1}^{J} \mu_j \log K_j(\ldots, X_\gamma, \ldots, \hat{g}_{j\alpha}(z, X), \ldots, \bar{g}_{j1}(w), \ldots, \bar{g}_{jm_j}(w)). \tag{9}$$

Lemma 2.4 *There exists $(z_0, w_0) \in U \times conj(U)$ such that*

$$\sum_{i=1}^{n} \hat{f}_i(z_0, X)\bar{f}_i(w_0) \not\equiv 0.$$

Proof Assume not. Letting $w = \overline{z}$ and $X = (g_1(z), \ldots, g_l(z))$, it follows that

$$\sum_{i=1}^{n} |f_i(z)|^2 = \sum_{i=1}^{n} \hat{f}_i(z, g_1(z), \ldots, g_l(z))\bar{f}_i(\overline{z}) \equiv 0, \text{ over } U.$$

This implies that $f_i(z) \equiv 0$ for all $1 \leq i \leq n$ and therefore contradicts to the assumption that $F = (f_1, \ldots, f_n)$ is a non-constant map. □

Choosing z_0, w_0 as in Lemma 2.4, $\sum_{i=1}^{n} \hat{f}_i(z, X)\bar{f}_i(w)$ is a nonconstant holomorphic Nash algebraic function in X by Lemma 2.4 and by the fact that

$$K_j(\ldots, X_\gamma, \ldots, \hat{g}_{j\alpha}(z, X), \ldots, g_{j1}(w), \ldots, g_{jm_j}(w))$$

is also Nash algebraic in X for all j as the Bergman kernel function of bounded symmetric domain is rational. Hence we arrive at a contradiction by Lemma 2.1. Thus F must be a constant map. Now if all $\mu'_j s$ are further assumed to be positive, it is obvious that G must also be constant. The proof of Theorem 1.1 is complete.

3 Further Remarks

A Hermitian symmetric space M of compact type can be holomorphically isometrically embedded into the complex projective space $\mathbb{P}^N$ by the Nakagawa-Takagi embedding. Notice that the Fubini-Study metric $\omega_{\mathbb{P}^N}$ on $\mathbb{P}^N$ in a standard holomorphic chart $\{w_1, \ldots, w_N\}$ is given by

$$\omega_{\mathbb{P}^N} = \sqrt{-1}\partial\bar{\partial}\log(1 + \sum_j |w_j|^2)$$

up to the normalizing constant, which is also of the form $\partial\bar{\partial}\log K(w, \bar{w})$, where $K(w, \bar{w})$ is an algebraic function. Therefore the same argument yields the following theorem:

Theorem 3.1 *Let $D \subset \mathbb{C}$ be a connected open subset. If there are holomorphic maps $F : D \to \mathbb{C}^n$ and $G = (G_1, \cdots, G_J) : D \to \Omega_1 \times \cdots \times \Omega_J$ and $L = (L_1, \ldots, L_K) : D \to \mathbb{P}^{N_1} \times \cdots \times \mathbb{P}^{N_K}$ such that*

$$F^*\omega_{\mathbb{C}^n} = \sum_{j=1}^{J} \mu_j G_j^* \omega_{\Omega_j} + \sum_{k=1}^{K} \lambda_k {L_k}^* \omega_{\mathbb{P}^{N_k}} \quad on \;\; D$$

for real constants $\mu_1, \ldots, \mu_J, \lambda_1, \ldots, \lambda_K$. Then F is a constant map. Moreover, if μ_j, λ_j are positive, G and L are also constant map.

Remark that the above constant $\mu_1, \ldots, \mu_J, \lambda_1, \ldots, \lambda_K$ can be positive, negative or zero. In particular, Theorem 3.1 implies that the complex Euclidean space cannot be a relative to the product space of a bounded symmetric space and a Hermitian symmetric space of compact type. Note that, in [6], Di Scala and Loi showed that any bounded domain with Bergman metric and a Hermitian symmetric space of compact type cannot be relatives. Combining their results, we actually can conclude that any Hermitian symmetric space of a particular type and the product of Hermitian symmetric spaces of two other types cannot be relatives. More precisely, we summarize the result as follows:

Theorem 3.2 *Let $D \subset \mathbb{C}$ be a connected open set. Let $\Omega, M, \mathbb{C}^n$ be a Hermitian symmetric space of noncompact, compact and Euclidean type, respectively, equipped with the canonical metrics $\omega_\Omega, \omega_M, \omega_{\mathbb{C}^n}$. If there exist non-constant holomorphic maps $F : D \to \mathbb{C}^n$, $G : D \to \Omega$ and $L : D \to M$ such that*

$$aF^*\omega_{\mathbb{C}^n} + bG^*\omega_\Omega + cL^*\omega_M = 0 \quad on \;\; D$$

for real constants a, b, c, then it must holds that $a = b = c = 0$.

Next, we let $(D_1 \subset \mathbb{C}^n, \omega_1)$ and $(D_2 \subset \mathbb{C}^m, \omega_2)$ be two Kähler manifolds with $\omega_j = \sqrt{-1}\partial\bar{\partial}\log h_j(z, \bar{z})$. Here $h_j(z, \bar{z})$ are real analytic functions in z. Assume that $0 \in D_j$ and h_j $(j = 1, 2)$ do not have any non-constant harmonic terms in its Taylor expansion at the origin with $h_j(0, 0)$ being normalized to be 1. (D_1, ω_1) and (D_2, ω_2) are relative at 0 if and only if there are non-constant holomorphic maps $\phi_1 : \Delta \to D_1$ and $\phi_2 : \Delta \to D_2$ with $\phi_j(0) = 0$ such that $\phi_1^*(\omega_1) = \phi_2^*(\omega_2)$. Here Δ is the unit disk in $\mathbb{C}^1$. As standard, this happens if and only if

$$h_1(\phi_1(\tau), \overline{\phi_1(\tau)}) = h_2(\phi_2(\tau), \overline{\phi_2(\tau)}).$$

Now, we let the real analytic set $M \subset D_1 \times D_2 \subset \mathbb{C}^{n+m}$ be defined by $h_1(z, z) = h_2(w, w)$ with $(z, w) \in D_1 \times D_2$. By the fact that h_j serve as potential functions of Kähler metrics near 0, it is not hard to show that M must be regular at the origin. Then (D_1, ω_1) and (D_2, ω_2) are relative at 0 or near a point close to 0 if and only if inside M, there is a non-trivial holomorphic curve containing the origin. Then this cannot happen if and only if M is of D'Angelo finite type at 0 [3]. Hence, by what we proved above, we have the following:

Theorem 3.3 *Let $K_j(w, \bar{w})$ $(j = 1, \ldots, \kappa)$ be positively-valued smooth Nash-algebraic functions in $(w, \bar{w})$ with $w(\in \mathbb{C}^m) \approx 0$. Assume that the complex Hessian of* $\log K_j(w, w)$ *is positive definite for each j. Then for any positive real numbers $\mu_1, \ldots, \mu_\kappa$, the following real-analytic hypersurface M defined near the origin is of finite D'Angelo type at* 0*:*

$$M := \{(z, w)(\subset \mathbb{C}^{n+m}) \approx (0, 0) : \sum_{j=1}^{n} |z_j|^2 = \sum_{l=1}^{\kappa} \mu_l \log K_l(w, \bar{w})\}.$$

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Non-closed Range Property for the Cauchy-Riemann Operator

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In memory of M. Salah Baouendi

Abstract In this paper we study the non-closed range of the Cauchy-Riemann operator for relatively compact domains in $\mathbb{C}^n$ or in a complex manifold. We give necessary and sufficient conditions for the L^2 closed range property for $\overline{\partial}$ on bounded Lipschitz domains in $\mathbb{C}^2$ with connected complement. It is proved for the Hartogs triangle that $\overline{\partial}$ does not have closed range for (0, 1)-forms smooth up to the boundary, even though it has closed range in the weak L^2 sense. An example is given to show that $\overline{\partial}$ might not have closed range in L^2 on a Stein domain in complex manifold.

Keywords Cauchy-Riemann operator · Closed range · Duality

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1 Introduction

A fundamental problem in function theory on a domain Ω in a complex manifold is the study of the Cauchy-Riemann operator, or the $\overline{\partial}$-equation. The understanding of the existence and regularity of the solutions of the system of inhomogeneous Cauchy-Riemann equations on Ω plays central role in complex analysis. On a bounded domain in $\mathbb{C}^n$ (or more generally in a Stein manifold), two theorems (see [13, 14, 16, 17]) for $\overline{\partial}$ on pseudoconvex domains are of paramount importance.

Theorem (Hörmander) *Let $\Omega \subset\subset \mathbb{C}^n$ be a pseudoconvex domain. For any $f \in L^2_{p,q}(\Omega)$, where $0 \le p \le n$ and $1 \le q < n$, such that $\overline{\partial} f = 0$ in Ω, there exists $u \in L^2_{p,q-1}(\Omega)$ satisfying $\overline{\partial} u = f$ and $\int_\Omega |u|^2 \le C \int_\Omega |f|^2$ where C depends only on the diameter of Ω and q.*

Furthermore, if the boundary $b\Omega$ is smooth, we also have the following global boundary regularity results for $\overline{\partial}$.

Theorem (Kohn) *Let $\Omega \subset\subset \mathbb{C}^n$ be a pseudoconvex domain with smooth boundary $b\Omega$. For any $f \in C^\infty_{p,q}(\overline{\Omega})$, where $0 \le p \le n$ and $1 \le q < n$, such that $\overline{\partial} f = 0$ in Ω, there exists $u \in C^\infty_{p,q-1}(\overline{\Omega})$ satisfying $\overline{\partial} u = f$.*

Let D be a domain in $\mathbb{C}^n$ or a complex manifold. A natural question to ask is when the $\overline{\partial}$ equation has closed range for forms in L^2 or smooth coefficients. The closed range property gives solvability for $\overline{\partial}$ from the point of view of functional analysis. In this paper, we survey the recent progress related to this problem. For closed-range property of the $\overline{\partial}$-equations, we refer the readers to the many books and papers and the references therein (see [2, 6, 10, 13, 14] or [28]). In this paper, we focus on the non-closed range property for the $\overline{\partial}$-equation.

In Sect. 2, we first study the non-closed range property for $\overline{\partial}$ in the L^2 setting for domains in $\mathbb{C}^n$. For any bounded non-pseudoconvex domain D in $\mathbb{C}^2$ such that its complement is connected, the $\overline{\partial}$ equation does not have closed range in L^2 for (0, 1)-forms if D. In Sect. 3 we study the Hartogs triangle H for forms smooth up to the boundary. In this case, the $\overline{\partial} : C^\infty(\overline{H}) \to C^\infty_{0,1}(\overline{H})$ does not have closed range. Hence the corresponding cohomology group is non-Hausdorff. The non-hausdorff property is new, since we only knew that the cohomology is infinite dimensional. In Sect. 4, an example of a Stein domain with smooth boundary in a compact complex manifold is given where $\overline{\partial}$ does not have closed range in L^2. In other words, the Hörmander type L^2 results do not hold on a bounded pseudoconvex domain $\Omega \subset\subset \mathcal{X}$ in a complex manifold $\mathcal{X}$ which is not Stein, even though the domain Ω is Stein and with smooth boundary.

2 Non-closed Range in L^2 for $\overline{\partial}$ on Domains in $\mathbb{C}^n$

It is well-known that in $\mathbb{C}^n$, $\overline{\partial} : L^2(\mathbb{C}^n) \to L^2_{0,1}(\mathbb{C}^n)$ does not have closed range. This follows from the fact that the Poincaré inequality does not hold for compactly supported functions in $\mathbb{C}^n$.

For bounded domains in $\mathbb{C}^n$, it is known that there exists a non-pseudoconvex domain on which the L^2 range of $\overline{\partial}$ is not closed (see the example on page 76 in Folland-Kohn [10]). One can show explicitly that $\overline{\partial}$ cannot have closed range by an explicit example of a (0, 1)-form. Using duality, one can show the following result (see [19]).

Theorem 2.1 *Let D be a bounded domain in $\mathbb{C}^2$ such that $\mathbb{C}^2 \setminus D$ is connected. Suppose D is not pseudoconvex and the boundary of D is Lipschitz. Then $\overline{\partial} : L^2(D) \to L^2_{0,1}(D)$ does not have closed range.*

The proof of Theorem 2.1 is based on the Serre duality in the L^2 sense. Let $\overline{\partial}_c$ be the strong minimal closure of the $\overline{\partial}$ operator

$$\overline{\partial}_c : \mathcal{D}_{p,q-1}(\Omega) \to \mathcal{D}_{p,q}(\Omega)$$

where $\mathcal{D}$ is the set of compactly supported functions in Ω. By this we mean that $\overline{\partial}_c$ is the minimal closed extension of the operator such that $\text{Dom}(\overline{\partial}_c)$ contains $\mathcal{D}_{p,q-1}$. The $\text{Dom}(\overline{\partial}_c)$ contains elements $f \in L^2_{p,q-1}(\Omega)$ such that there exists sequence $f_\nu \in \mathcal{D}_{p,q-1}(\Omega)$ such that $f_\nu \to f$ in $L^2_{p,q-1}(\Omega)$ and $\overline{\partial} f_\nu \to \overline{\partial} f$ in $L^2_{p,q}(\Omega)$.

Lemma 2.2 *Let Ω be a bounded domain in $\mathbb{C}^n$. The following conditions are equivalent*

(1) $\overline{\partial} : L^2_{p,q-1}(\Omega) \to L^2_{p,q}(\Omega)$ *has closed range.*
(2) $\overline{\partial}_c : L^2_{n-p,n-q}(\Omega) \to L^2_{n-p,n-q+1}(\Omega)$ *has closed range.*

Proof Let $\overline{\partial}^*$ denote the Hilbert space adjoint of $\overline{\partial}$. Following the definition, $f \in \text{Dom}(\overline{\partial}_c)$ if and only if $\star f \in \text{Dom}(\overline{\partial}^*)$. Suppose (1) holds. Then $\overline{\partial}^* : L^2_{p,q}(\Omega) \to L^2_{p,q-1}(\Omega)$ has closed range. Thus we have $\star\overline{\partial}^*\star = \overline{\partial}_c$ has closed range. Thus (1) implies (2). The other direction is proved similarly.

We may also consider $\overline{\partial}_{\tilde{c}}$, the minimal closure of $\overline{\partial}$ in the weak sense, which is related to solving $\overline{\partial}$ with prescribed support in $\overline{\Omega}$ and we refer it as the $\overline{\partial}$-Cauchy problem. When the boundary is Lipschitz, the weak and strong minimal extension are the same (see Lemma 2.4 in [19]).

Definition Let Ω be a domain in a hermitian manifold $\mathcal{X}$. We define the L^2 cohomology group for (p, q)-forms by

$$H^{p,q}_{L^2}(\Omega) = \frac{\{f \in L^2_{p,q}(\Omega) \mid \overline{\partial} f = 0 \text{ in } \Omega\}}{\{f \in L^2_{p,q}(\Omega) \mid f = \overline{\partial} u \text{ for some } u \in L^2_{p,q-1}(\Omega)\}}.$$

We also define the L^2 cohomology group with compact support by

$$H^{p,q}_{c,L^2}(\Omega) = \frac{\{f \in L^2_{p,q}(\Omega) \mid f \in \mathrm{Dom}(\overline{\partial}_c),\ \overline{\partial}_c f = 0 \text{ in } \Omega\}}{\{f \in L^2_{p,q}(\Omega) \mid f = \overline{\partial}_c u \text{ in } \Omega \text{ for some } u \in L^2_{p,q-1}(\Omega) \cap \mathrm{Dom}(\overline{\partial}_c)\}}.$$

Lemma 2.3 *Let D be a bounded Lipschitz domain in $\mathbb{C}^n$, $n \geq 2$, such that $\mathbb{C}^n \setminus D$ is connected, then $H^{0,1}_{c,L^2}(D) = 0$.*

Proof Let $f \in L^2_{0,1}(D) \cap \mathrm{Dom}(\overline{\partial}_c)$ and $\overline{\partial}_c f = 0$. Let f^0 denote the trivial extension of f to $\mathbb{C}^n$ by setting f equal to zero outside D. It follows that $\overline{\partial} f^0 = 0$ in $\mathbb{C}^n$ in the distribution sense. This follows from the assumption that the boundary of D is Lipschtiz (see Lemma 2.4 in [19]). The form f^0 is a compactly supported (0, 1)-form in $\mathbb{C}^n$.

Let B be a large ball in $\mathbb{C}^n$ containing $\overline{D}$. By Hörmander's theorem (see [13]), we can solve $\overline{\partial} u = f^0$ in B and the solution u is in $W^1(D)$ from the interior regularity for $\overline{\partial}$. The function u is holomorphic on $B \setminus \overline{D}$. From our assumption that $\mathbb{C}^n \setminus D$ is connected and $n \geq 2$, the holomorphic function $u \mid_{B \setminus \overline{D}}$ can be extended as a holomorphic function h in B. Let $U = u - h$ in B. Then U is a compactly supported solution in $W^1(D)$ such that $\overline{\partial} U = f^0$ in $\mathbb{C}^n$ in the distribution sense. The solution $U \in \mathrm{Dom}(\overline{\partial}_c)$ and $\overline{\partial}_c U = f$. This proves that $H^{0,1}_{c,L^2}(D) = 0$.

Lemma 2.4 *Let D be a bounded domain in $\mathbb{C}^n$ with Lipschitz boundary. Then the following conditions are equivalent:*

(1) The domain D is pseudoconvex.
(2) $H^{0,q}_{L^2}(D) = 0, \qquad 1 \leq q \leq n - 1.$

Proof If $D \subset\subset \mathbb{C}^n$ is bounded pseudoconvex, then $H^{0,q}_{L^2}(D) = 0$ for all $1 \leq q \leq n - 1$ by Hörmander L^2-theory. The converse is true provided D has Lipschitz boundary or more generally, D satisfies interior$(\overline{D}) = D$ (see e.g. the remark at the end of the paper in [11]).

Proof of Theorem 2.1. Suppose that

$$\bar{\partial} \ : \ L^2(D) \to L^2_{0,1}(D) \tag{2.1}$$

has closed range. Using Lemma 2.2, we have that

$$\bar{\partial}_c \ : \ L^2_{2,1}(D) \to L^2_{2,2}(D) \tag{2.2}$$

has closed range.

On the other hand, for top degree (0, 2)-forms, we always have that

$$\bar{\partial} \ : \ L^2_{0,1}(D) \to L^2_{0,2}(D) \tag{2.3}$$

has closed range since D is a bounded domain in $\mathbb{C}^2$. From the L^2 Serre duality (see [4]), we have

$$H^{0,1}_{L^2}(D) \simeq H^{2,1}_{c,L^2}(D). \tag{2.4}$$

Since the domain D is in $\mathbb{C}^n$, the exponent p plays no role, the same proof for Lemma 2.3 also holds for $(n, 1)$-forms or any $(p, 1)$-forms. Thus we have $H^{2,1}_{c,L^2}(D) = 0$. From (2.4), this will gives that $H^{0,1}_{L^2}(D) = 0$ and from Lemma 2.4, D is pseudoconvex, a contradiction. The theorem is proved. □

Corollary 2.5 *Let D be a bounded Lipschitz domain in $\mathbb{C}^2$ such that $\mathbb{C}^2 \setminus D$ is connected. Then*

(1) If the domain D is pseudoconvex, then $H^{0,1}_{L^2}(D) = 0$.

(2) If the domain D is non-pseudoconvex, then $H^{0,1}_{L^2}(D)$ is not Hausdorff.

In other words, for bounded Lipschitz domains in $\mathbb{C}^2$ with connected complement we have only two kinds of L^2 cohomology groups $H^{0,1}_{L^2}(D)$: either it is trivial or it is non-Hausdorff. There is nothing in between. We remark that some related results for the Fréchet space cohomology were proved by Trapani ([29], Theorem 2), where a characterization of Stein domains in a Stein manifold of complex dimension 2 is given. In particular he proves that Corollary 2.5 also holds for $H^{0,1}(D)$ the cohomology of C^∞-smooth (0, 1)-forms in D.

Let Ω be a relatively compact pseudoconvex domain with smooth boundary in a Stein manifold with a hermitian metric. We use $H^{p,q}(\Omega)$ or $H^{p,q}(\overline{\Omega})$ to denote the cohomology group of (p, q)-forms with $C^\infty(\Omega)$ coefficients or $C^\infty(\overline{\Omega})$ respectively and since $\mathcal{X}$ is hermitian, we use $H^{p,q}_{L^2}(\Omega)$ to denote the cohomology group of (p, q)-forms with L^2 coefficients. Then

$$H^{p,q}(\Omega) = H^{p,q}_{L^2}(\Omega) = H^{p,q}(\overline{\Omega}) = 0, \qquad q > 0.$$

Next we will compare the L^2-cohomology groups $H^{p,q}_{L^2}(\Omega)$ and $H^{p,q}(\overline{\Omega})$ for a domain Ω when the boundary is not smooth.

3 The Hartogs Triangle

Let us consider the Hartogs triangle H in $\mathbb{C}^2$

$$H = \{(z, w) \in \mathbb{C}^2 \mid |z| < |w| < 1\}.$$

It is a bounded pseudoconvex domain in $\mathbb{C}^2$ with Lipschitz boundary outside the origin. Near the origin, it is not a Lipschitz domain since its boundary is not the graph of a Lipschitz function. The Hartogs triangle and it smooth cousins, the Diederich-Fornaess (see [8]) worm domains, provide many counter examples for function theory on pseudoconvex domains in $\mathbb{C}^n$.

Let $C^{k,\alpha}(H)$ denote the Hölder space of functions in H whose kth derivatives are C^α in H, where $k \in \mathbb{N}$ and $0 < \alpha < 1$. On H, we can consider the space $C^\infty(\overline{H})$ of smooth functions on the closure of H. Several natural definitions could be used.

Definition We define the smooth functions on $\overline{H}$ as follows:

(1) The space of the restrictions to $\overline{H}$ of C^∞-smooth functions on $\mathbb{C}^2$, which can be identified with the quotient of the space of C^∞-smooth functions on $\mathbb{C}^2$ by the ideal of the functions vanishing with all their derivatives on $\overline{H}$.
(2) The intersection for some $0 < \alpha < 1$ of all spaces $C^{k,\alpha}(H)$, $k \in \mathbb{N}$.
(3) The space of C^∞-smooth functions on $\overline{H}$ in the sense of Whitney's jets.

Lemma 3.1 *The three definitions are equivalent.*

Proof The space defined by (1) is clearly contained in the other two. The Whitney extension theorem implies that the spaces defined by (1) and (3) coincide. Using the extension theorem for uniformly continuous functions, it is easy to see that the space defined by (2) is included in the space defined by (3), which implies finally that all the three definitions are equivalent.

For a bounded domain D in $\mathbb{C}^n$ with Lipschitz boundary, it is well-known that the dual of the topological vector space $C^\infty(\overline{D})$ is the space $\mathcal{E}'_{\overline{D}}(\mathbb{C}^n)$ of distributions with compact support in $\overline{D}$ ([19], Lemma 2.3). We will determine the dual of the topological vector space $C^\infty(\overline{H})$ endowed with the Fréchet topology of uniform convergence on $\overline{H}$ of functions and all derivatives.

Theorem 3.2 *The spaces $C^\infty(\overline{H})$ and $\mathcal{E}'_{\overline{H}}(\mathbb{C}^2)$ are dual to each other.*

Proof By the definition of $C^\infty(\overline{H})$, the restriction map $R : \mathcal{E}(\mathbb{C}^2) \to C^\infty(\overline{H})$ is continuous and surjective. Taking the transpose map ${}^tR$ we get an injection from $(C^\infty(\overline{H}))'$ into $\mathcal{E}'(\mathbb{C}^2)$ the space of distributions with compact support in $\mathbb{C}^2$. More precisely the image of $(C^\infty(\overline{H}))'$ by ${}^tR$ is clearly included in $\mathcal{E}'_{\overline{H}}(\mathbb{C}^2)$, the space of distributions on $\mathbb{C}^2$ with support contained in $\overline{H}$.

For any current $T \in \mathcal{E}'_{\overline{H}}(\mathbb{C}^2)$, we set, for $f \in C^\infty(\overline{H})$, $T(f) =< T, \widetilde{f} >$, where $\widetilde{f}$ is a C^∞-smooth extension of f to $\mathbb{C}^2$. We have to prove that $T(f)$ is independent of the choice of the extension $\widetilde{f}$ of f.

Since the difference of two extensions of f is an infinite order flat function on H, we will prove that for any C^∞-smooth function φ flat to infinite order on H and with compact support in $\mathbb{C}^2$, we have $< T, \varphi >= 0$. Since T has compact support, it is a distribution of finite order k_0.

Let r be a positive real number and χ a C^∞-smooth function in $\mathbb{C}^2$ with compact support in the ball $B(0, r)$ of radius r, centered at the origin and equal to 1 on the closed ball $\overline{B(0, r/2)}$. Then $\chi\varphi$ is flat to infinite order on H. Notice that though the Hartogs triangle is not Lipschitz near the origin, it satisfies the exterior cone property at the origin. Thus for any $z \in \mathbb{C}^2 \setminus H \cap B(0, r)$, $d(z, bH) \leq |z|$. This implies that for any $k \in \mathbb{N}$ and any multi-index α with $|\alpha| \leq k$, there exists a positive real constant C_k such that

$$\|D^\alpha(\chi\varphi)\|_\infty \leq C_k r. \tag{3.1}$$

Fix $\varepsilon > 0$ and choose r such that

$$\sup_{k\leq k_0}(C_k r) < \frac{\varepsilon}{2\|T\|}. \tag{3.2}$$

Since $\mathbb{C}^2 \setminus (H \cup B(0,r))$ has Lipschitz boundary, there exists a C^∞-smooth function θ with compact support in $\mathbb{C}^2 \setminus (\overline{H \cup B(0,r)})$ such that

$$\sum_{|\alpha|\leq k_0} \|D^\alpha(1-\chi)\varphi - D^\alpha\theta\|_\infty \leq \frac{\varepsilon}{2\|T\|}. \tag{3.3}$$

Then, since T has support in $\overline{H}$ and θ in $\mathbb{C}^2 \setminus (\overline{H \cup B(0,r)})$, $< T, \theta >= 0$ and we have from (3.2) and (3.3) that

$$\begin{aligned} | < T,\varphi > | &\leq | < T,\chi\varphi > | + | < T,(1-\chi)\varphi > | \\ &\leq | < T,\chi\varphi > | + | < T,(1-\chi)\varphi - \theta > | + | < T,\theta > | \\ &\leq \|T\|\frac{\varepsilon}{2\|T\|} + \|T\|\frac{\varepsilon}{2\|T\|} \leq \varepsilon. \end{aligned} \tag{3.4}$$

This gives that $< T,\varphi >= 0$. Consequently T defines a linear form on $C^\infty(\overline{H})$, which is continuous by the open mapping theorem. This proves that R^t is one-to-one with range equal to $\mathcal{E}'_{\overline{H}}(\mathbb{C}^2)$. □

Since H is a pseudoconvex domain, we have $H^{0,1}(H) = 0$. Moreover it follows from results by Sibony [26, 27] (see also the paper by Chaumat and Chollet [5]) that for any ζ in the bidisc $P = \Delta \times \Delta$ and $\zeta \in P \setminus \overline{H}$, their exists a C^∞-smooth, $\overline{\partial}$-closed $(0,1)$-form α_ζ defined in $\mathbb{C}^2 \setminus \{\zeta\}$ such that there does not exist any C^∞-smooth function β on $\overline{H}$ such that $\overline{\partial}\beta = \alpha_\zeta$, which means that $H^{0,1}(\overline{H}) \neq 0$.

Theorem 3.3 *The cohomology group $H^{0,1}(\overline{H})$ is not Hausdorff.*

Proof By Theorem 3.2, for $0 \leq p \leq 2$, the complexes $(C^\infty_{p,\bullet}(\overline{H}), \overline{\partial})$ and $(\mathcal{E}'^{2-p,\bullet}_{\overline{H}}(\mathbb{C}^2), \overline{\partial})$ are dual from each other. So it follows from Serre duality (see [22] or Corollary 2.6 in [19]) that it is sufficient to prove that we can solve the $\overline{\partial}$ with prescribe support in $\overline{H}$ in the current category.

Lemma 3.4 *For each current $T \in \mathcal{E}'_{2,1}(\mathbb{C}^2)$ with support contained in $\overline{H}$ there exists a $(2,0)$-current S with compact support in $\mathbb{C}^2$, whose support is contained in $\overline{H}$, such that $\overline{\partial}S = T$.*

Proof Let $T \in \mathcal{E}'_{2,1}(\mathbb{C}^2)$ be a current with support contained in $\overline{H}$. Since one can solve the $\overline{\partial}$ equation with compact support for bidegree $(2,1)$ in $\mathbb{C}^2$, there exists a $(2,0)$-current S with compact support in $\mathbb{C}^2$ such that $\overline{\partial}S = T$. The support of T is contained in $\overline{H}$, so the current S is an holomorphic (2, 0)-form on $\mathbb{C}^2 \setminus \overline{H}$. Moreover S has compact support in $\mathbb{C}^2$ and hence vanishes on an open subset of $\mathbb{C}^2 \setminus \overline{H}$. By the analytic continuation theorem, the connectedness of $\mathbb{C}^2 \setminus H$ implies that the support of S is contained in $\overline{H}$. □

Remark 1 One does have *almost* smooth solutions to the $\overline{\partial}$ problem on the Hartogs triangle. Let $H^{p,q}_{C^{k,\alpha}}(H)$ denote the Dolbeault cohomology of (p, q)-forms with $C^{k,\alpha}(H)$ coefficients. Using integral kernel method, Chaumat and Chollet [5] prove that $H^{0,1}_{C^{k,\alpha}}(H) = 0$.

Notice that the intersection $\cap_k C^{k,\alpha}(H) = C^\infty(\overline{H})$. This shows the delicate nature of such problem on non-Lipschitz domains.

Remark 2 We also mention that if Ω is a bounded pseudoconvex domain in $\mathbb{C}^n$ with a good Stein neighborhood basis, then one has that $H^{0,1}(\overline{\Omega}) = 0$ (see [9]). The Hartogs triangle is a prototype of domains without Stein neighborhood basis.

Let $\overline{\partial}_s$ denote the strong maximal extension of $\overline{\partial}$. By this we mean that $\overline{\partial}_s$ is the maximal closed extension of the operator such that $\mathrm{Dom}(\overline{\partial}_s)$ contains $C^\infty_{p,q-1}(\overline{H})$. The $\mathrm{Dom}(\overline{\partial}_s)$ contains elements $f \in L^2_{p,q-1}(H)$ such that there exists sequence $f_\nu \in C^\infty_{p,q-1}(\overline{H})$ such that $f_\nu \to f$ in $L^2_{p,q-1}(H)$ and $\overline{\partial} f_\nu \to \overline{\partial} f$ in $L^2_{p,q}(H)$. Since the boundary of H is rectifiable, $\overline{\partial}_{\tilde{c}}$, the weak minimal closure of $\overline{\partial}$, is dual to $\overline{\partial}_s$.

We do know $H^{0,1}_{L^2}(H) = 0$ from Hörmander's result. It is not known if the weak maximal extension $\overline{\partial} \ : L^2(H) \to L^2_{0,1}(H)$ is the same as the strong maximal extension $\overline{\partial}_s$. So we only get from Lemma 2.3 (the Lipschitz hypothesis is only used to get $\overline{\partial}_c$ at the place of $\overline{\partial}_{\tilde{c}}$) and L^2 Serre duality (see [4]) that

Proposition 3.5 *The cohomology group $H^{0,1}_{\overline{\partial}_s, L^2}(H)$ is either* 0 *or not Hausdorff.*

Again, if the boundary is Lipschitz, then the weak $\overline{\partial}$ and strong $\overline{\partial}_s$ are the same following the Friedrichs' lemma (see [13] or [6]).

Consider the annulus between a pseudoconvex domain and the Hartogs triangle. We have the following result (see Corollary 4.6 in [19]).

Theorem 3.6 *Let Ω be a pseudoconvex domain in $\mathbb{C}^2$ such that $\overline{H} \subset \Omega$. Then $H^{0,1}(\Omega \setminus \overline{H})$ is not Hausdorff.*

Let D be the annulus between two bounded domains $\Omega_1 \subset\subset \Omega \subset\subset \mathbb{C}^2$. Suppose that the Dolbeault cohomology $H^{0,1}(D)$ is Hausdorff. It follows from a result of Trapani (see Theorem 3 in [30]) that both Ω and Ω_1 have to be pseudoconvex. Theorem 3.6 shows that the converse is not true.

If we replace H by the bidisc $\triangle^2$, then $H^{0,1}(\Omega \setminus \overline{\triangle^2})$ is Hausdorff since $\triangle^2$ has a Stein neighborhood basis (see [18] or Corollary 4.3 in [19]).

In fact, Trapani (see Theorem 4 in [30]) proves that, if D has smooth boundary, a sufficient condition for $H^{0,1}(D)$ to be Hausdorff is that Ω is pseudoconvex and Ω_1 is strictly pseudoconvex. It is no longer true if Ω_1 is only pseudoconvex, taking for example Ω_1 to be the Diederich-Fornaes domain [7].

On the other hand, if Ω and Ω_1 are pseudoconvex and we assume that the boundary of Ω_1 is C^2-smooth, then the L^2 cohomology $H^{0,1}_{L^2}(D)$ is Hausdorff (Hörmander [15] or Shaw [23–25]). The following problem remains unsolved.

Question *Let B be a ball of radius two in $\mathbb{C}^2$ and $\triangle^2$ be the bidisc. Determine if the L^2 cohomology $H^{0,1}_{L^2}(B \setminus \overline{\triangle^2})$ is Hausdorff.*

4 Non-closed Range Property for $\overline{\partial}$ on Pseudoconvex Domains in Complex Manifolds

When $\mathcal{X}$ is a Hermitian complex manifold and $\Omega \subset\subset \mathcal{X}$ is a pseudoconvex domain with smooth boundary, the $\overline{\partial}$ problem could be very different if $\mathcal{X}$ is not Stein. We first note that if Ω is strongly pseudoconvex, Grauert proved that the $\overline{\partial}$ has closed range in the Fréchet space of C^∞-smooth forms. If the domain Ω is relatively compact strongly pseudoconvex, (or more generally of finite type) with smooth boundary, the closed range property for the $\overline{\partial}$ equation in the L^2 setting has been established by Kohn [17] via the $\overline{\partial}$-Neumann problem.

However, function theory on general weakly pseudocnvex domains in a complex manifold can be quite different. Grauert (see [12]) first gives an example of a pseudoconvex domain Ω in a complex torus which is not holomorphically convex. He shows that the only holomorphic functions on Ω are constants. The domain in the Grauert's example actually has Levi-flat boundary. The boundary splits the complex two torus into two symmetric parts. Based on the example of Grauert, Malgrange proves the following theorem.

Theorem (Malgrange [20]) *There exists a pseudoconvex domain Ω with Levi-flat boundary in a complex torus of dimension two whose Dolbeault cohomology group $H^{p,1}(\Omega)$ is non-Hausdorff in the Fréchet topology, for every* $0 \le p \le 2$.

Malgrange shows that for the Grauert's example, the $\overline{\partial}$ equation does not necessarily have closed range in the Fréchet space of C^∞-smooth forms and the corresponding Dolbeault cohomology $H^{p,1}(\Omega)$ is non-Hausdorff. The domain Ω is not holomorphically convex. Recently the following result is proved in [3].

Theorem 4.1 *There exists a compact complex manifold $\mathcal{X}$ of complex dimension two and a relatively compact, smoothly bounded, Stein domain Ω with smooth boundary in X, such that the range of $\overline{\partial} : L^2_{2,0}(\Omega) \to L^2_{2,1}(\Omega)$ is not closed. Consequently, the L^2-cohomology space $H^{2,1}_{L^2}(\Omega)$ is not Hausdorff.*

The domain Ω is defined as follows. Let $\alpha > 1$ be a real number and let Γ be the subgroup of $\mathbb{C}^*$ generated by α. We will standardize $\alpha = e^{2\pi}$. Let $T = \mathbb{C}^*/\Gamma$ be the torus.

Let $\mathcal{X} = \mathbb{C}P^1 \times T$ be equipped with the product metric ω from the Fubini-Study metric for $\mathbb{C}P^1$ and the flat metric for T. Let Ω be the domain in $\mathcal{X}$ defined by

$$\Omega = \{(z, [w]) \in \mathbb{C}P^1 \times T \mid \Re zw > 0\} \tag{4.1}$$

where z is the inhomogeneous coordinate on $\mathbb{C}P^1$. The domain Ω was first introduced by Ohsawa [21] and used in Barrett [1]. It was proved in [21] that Ω is biholomorphic to the product domain of an annulus and a pictured disc in $\mathbb{C}^2$. In particular, Ω is Stein.

Thus we have

$$H^{p,q}(\Omega) = 0, \qquad q > 0.$$

The range of $\overline{\partial} : L^2_{(2,0)}(\Omega) \to L^2_{(2,1)}(\Omega)$ is not closed (see [3]). Theorem 4.1 shows that on a pseudconvex domain in a complex manifold X, there is no connection between the Dolbeault cohomology groups in the classical Fréchet space of smooth forms and the L^2 space. This is in sharp contrast with the case when the manifold X is Stein. We note that Ω is Stein, but the ambient space $\mathcal{X}$ is not Stein. The idea of the proof is to use the L^2 Serre duality and the extension of holomorphic functions. For details of the proof of the theorem, we refer the reader to [3]. We end the paper by raising the following question.

Question *Let Ω be defined by (4.1). Determine if the range of $\overline{\partial} : L^2_{p,0}(\Omega) \to L^2_{p,1}(\Omega)$ is closed, where $p = 0$ or $p = 1$.*

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Quasicrystals and Control Theory

Yves Meyer

In memory of M. Salah Baouendi

Abstract Cross-fertilization between control theory and irregular sampling is illustrated by two examples. Salah Baouendi applauded new ideas and welcomed new cultures. Interdisciplinary research is crossing frontiers, as Baouendi did all along his life. Two examples of cross-fertilization between harmonic analysis and control theory will be discussed in this homage. In 1983 Jacques-Louis Lions raised a problem in control theory. The solution I gave was grounded on a theorem on trigonometric sums proved by Arne Beurling. This will be our first example. The second example goes the other way around. A problem on trigonometric sums is solved using tools from control theory. Frontiers are erased as Baouendi wished.

Keywords Control theory · Quasicrystals · Irregular sampling

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1 Memories

Baouendi was born in Tunis in 1937. I was born in Paris in 1939. My family moved to North Africa when I was five. We could have met in Tunis. Instead we met in Paris in the late sixties. Baouendi was working with Charles Goulaouic at that time. We had many discussions about Tunisia. Baouendi was a native and I was a foreigner. Baouendi used to criticize what I was telling. He was shocked by my ignorance of the Arabic culture. For example in my high school years I never heard of Ibn Khaldoun, which is a shame since Ibn Khaldoun, born in Tunis in 1332, was one of the major intellectual figure in medieval Islam. The Tunis I liked so much was a melting pot where people from all over the Mediterranean sea found a peaceful exile. Baouendi

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respected every way of living, every culture and every religion. He aimed to construct peace and to abolish frontiers. In mathematics, too, frontiers shall be abolished. The two examples which will be discussed now are backing this claim.

2 The First Example

It happened the 28th of February, 1984. I was running the Goulaouic-Schwartz seminar after Charles Goulaouic's death. Baouendi had already moved to Purdue. Jacques-Louis Lions was then the head of the French Space Agency (CNES). Lions accepted to give a talk at the seminar. The main issue raised by Lions was the control and stabilization of some large oscillations which might occur on the Space Laboratory and be dangerous for this flexible structure. The problem had to be fixed before launching the satellite and beginning the construction of the International Space Station. Lions was suggesting that one could attenuate and eventually cancel the vibrations by commanding a tiny rocket fixed on the structure. Lions built a mathematical model for addressing this problem and asked us about a solution. To my greatest surprise I succeeded in solving the problem raised by Lions. My proof [18] relied on some properties of nonharmonic Fourier series. Soon after Louis Nirenberg found a simpler proof. Finally Lions discovered a third proof which is detailed in the appendix of [9].

3 The Second Example

Thirty years later I had to review Enrique Zuazua's mathematical production and, almost the same day, I was asked to evaluate a proposal by Nir Lev, from Bar-Ilan University, Israel. I accepted these duties. Zuazua is an "applied mathematician" working on control theory and fluid dynamics. Zuazua wrote his Ph.D. under the guidance of Jacques-Louis Lions. Lev is a "pure mathematician" working in harmonic analysis. His Ph.D. was supervised by Alexander Olevskii. This will be our second example of an unexpected interaction between nonharmonic Fourier series, irregular sampling and control theory. Sergei A. Avdonin made the link between Lev and Zuazua. Indeed Avdonin's work [1, 2] is playing a seminal role in [4] and is present in most of Zuazua's achievements. I was overwhelmed by surprise and happiness. I was back in the streets of Tunis hearing Arabic, French or Italian. A peaceful melting pot.

The spectrum $\sigma(f)$ of $f \in L^2(\mathbb{R}^n)$ is the support of its Fourier transform $\hat{f}$. This Fourier transform is normalized by

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} \exp(-2\pi i\, x \cdot \xi) f(x)\, dx. \tag{1}$$

Let $K \subset \mathbb{R}^n$ be a Borel set. The Paley Wiener space PW_K is defined by

$$PW_K = \{f \in L^2(\mathbb{R}^n);\ \ \sigma(f) \subset K\}. \tag{2}$$

If K is a compact set, every function $f \in PW_K$ is bandlimited and can be *sampled efficiently*. Let us define this sampling. A set $\Lambda \subset \mathbb{R}^n$ is uniformly discrete (u.d.) if there exists a positive number γ such that

$$\lambda, \lambda' \in \Lambda,\ \ \lambda \neq \lambda' \Rightarrow |\lambda - \lambda'| \geq \gamma > 0. \tag{3}$$

If $\Lambda \subset \mathbb{R}^n$ is uniformly discrete and if K is compact, every $f \in PW_K$ is a smooth function which can be sampled on Λ. The operator $S\ :PW_K \mapsto l^2(\Lambda)$ defined by $S(f) = (f(\lambda))_{\lambda\in\Lambda}$ is continuous. Is this mapping left-invertible ? A left-inverse is a continuous operator $T\ : l^2(\Lambda) \mapsto PW_K$ fulfilling $TS = \mathrm{I}$ on PW_K. If it is the case, every $f \in PW_K$ can be retrieved from its sample $(f(\lambda))_{\lambda\in\Lambda}$. The Nyquist-Shannon theorem answers this question when $n = 1$, $K = [-\omega, \omega]$, and $\Lambda = h\mathbb{Z}$. Then S is left-invertible if and only if $0 < h \leq 1/2\omega$: the density of Λ shall be larger than or equal to the measure of K.

Following the paradigm of "compressed sensing" a signal (or an image) f is *sparse* if the Lebesgue measure $|\sigma(f)|$ of its spectrum $\sigma(f)$ is *small*. Similarly if the Lebesgue measure of a compact set K is small every function $f \in PW_K$ is sparse. What is the most efficient way to sample sparse functions ? If $|K|$ is small and if the functions $f \in PW_K$ are sampled on a u.d. set Λ, then an efficient sampling algorithm minimizes the *density* of Λ under the constraint that $S\ : PW_K \mapsto l^2(\Lambda)$ is left-invertible. This density is the *sampling rate*. It is defined below and denoted by $\operatorname{dens}\Lambda$. Landau proved that S cannot be left-invertible when $\operatorname{dens}\Lambda < |K|$ ([7] and Theorem 3.1). What happens when $\operatorname{dens}\Lambda > |K|$? A. Olevskii and A. Ulanovskii addressed this issue in [19] and proposed the following definition:

Definition 1 A u.d. set Λ is a universal sampling set if the following three conditions are satisfied:

(i) Λ has a density denoted by $\operatorname{dens}\Lambda$.
(ii) For every compact set K such that $|K| < \operatorname{dens}\Lambda$, every $f \in PW_K$ can be recovered from its sample $f(\lambda)$, $\lambda \in \Lambda$.
(iii) More precisely, for every compact set K such that $|K| < \operatorname{dens}\Lambda$, there exists a constant $C = C(K, \Lambda)$ such that for every $f \in PW_K$ we have $\|f\|_2 \leq C(\sum_{\lambda\in\Lambda} |f(\lambda)|^2)^{1/2}$.

The lattice $\mathbb{Z}^n$ is not a universal sampling set. A counterexample is $f(x) = (\exp(2\pi i x_1) - 1)g(x)$ where the Fourier transform of g is supported in the ball B centered at 0 with radius ϵ. Let $e_1 = (1, 0, \ldots, 0) \in \mathbb{R}^n$ and $K = B \cup (B + e_1)$. Then $f \in PW_K$. Moreover $|K| = C_n\epsilon^n < 1 = \operatorname{dens}\mathbb{Z}^n$ if ϵ is small enough. Finally $f = 0$ on $\mathbb{Z}^n$, but f is not identically 0. This is called *aliasing* in signal processing.

The problem of recovering f from its sample has two versions, denoted by (a) and (b). In (a) Λ and K are given and we want to retrieve an unknown function

$f \in PW_K$ from its sample on Λ. Theorem 3.3 will settle this issue. In (b) Λ and a "small" constant β are given, both K and $f \in PW_K$ are unknown and the only information which is at our disposal is $|K| \leq \beta$ and $f \in PW_K$. We then want to retrieve the pair (K, f) with $f \in PW_K$ from the sample of f on Λ. Theorem 3.5 will take care of this problem.

We now consider the first issue (K is given) and follow Landau [7]. Problem (b) will be addressed later on.

Definition 2 Let K be a compact subset of $\mathbb{R}^n$. A collection of points $\Lambda\subset\mathbb{R}^n$ is a set of stable sampling for PW_K if there exists a constant C such that

$$f \in PW_K \Rightarrow \|f\|_2^2 \leq C\sum_{\lambda\in\Lambda}|f(\lambda)|^2. \tag{4}$$

Stable sampling is identical to (iii) in Definition 1. The property of stable sampling is equivalent to the following one: every $F \in L^2(K)$ is the sum of a generalized Fourier expansion

$$F(x) = \sum_{\lambda\in\Lambda} c(\lambda)\exp(2\pi i\,\lambda\cdot x) \tag{5}$$

where

$$\sum_{\lambda\in\Lambda}|c(\lambda)|^2 \leq C\int_K |F(x)|^2\,dx \tag{6}$$

and where the series (5) converges to F in $L^2(K)$. If Λ is a set of stable sampling for the space PW_K, any "band-limited" $f \in PW_K$ can be reconstructed from its sample $f(\lambda),\ \lambda \in \Lambda$, and this reconstruction is given by a linear algorithm. This remark follows from some general properties of frames. Therefore the sampling operator $S : PW_K \mapsto l^2(\Lambda)$ has a left-inverse T as it was stated above.

We also consider the property of stable interpolation [7].

Definition 3 Let K be a compact subset of $\mathbb{R}^n$. Then $\Lambda \subset \mathbb{R}^n$ is a set of stable interpolation for PW_K if there exists a constant C such that for every square-summable sequence $c(\lambda),\ \lambda \in \Lambda$, we have

$$\sum_{\lambda\in\Lambda}|c(\lambda)|^2 \leq C\int_K \Big|\sum_{\lambda\in\Lambda} c(\lambda)\exp(2\pi i\,x\cdot\lambda)\Big|^2\,dx. \tag{7}$$

A set of stable interpolation is uniformly discrete. An equivalent definition of a set of stable interpolation for PW_K is the following property. Every square summable sequence $a(\lambda),\ \lambda \in \Lambda$, can be interpolated by a function f in $PW_K : f(\lambda) = a(\lambda),\ \lambda \in \Lambda$. The sampling operator $S : PW_K \mapsto l^2(\Lambda)$ is onto.

If Λ is both a set of stable sampling and of stable interpolation for PW_K, then the family of functions $\exp(2\pi i\lambda\cdot x),\ \lambda \in \Lambda$, is a Riesz basis of $L^2(K)$.

Landau [7] discovered necessary conditions for sampling and interpolation. These conditions are relating the upper or lower density of Λ to the measure of K.

If $\Lambda \subset \mathbb{R}^n$ is a collection of points, the upper density of Λ is defined as

$$\overline{\text{dens}}\,\Lambda = \overline{\lim}_{R\to\infty} c_n R^{-n} \sup_{x\in\mathbb{R}^n} \#\{\Lambda \cap B_{x,R}\} \tag{8}$$

where $B_{x,R}$ is the ball centered at x with radius R, c_n is the inverse of the volume of the unit ball and $\#E$ denotes the cardinality of E. In this definition balls could be replaced by cubes as well. The lower density is defined similarly:

$$\underline{\text{dens}}\,\Lambda = \underline{\lim}_{R\to\infty} c_n R^{-n} \inf_{x\in\mathbb{R}^n} \#\{\Lambda \cap B_{x,R}\}. \tag{9}$$

A collection Λ of points has a uniform density (denoted by dens Λ) if the upper density and the lower density of Λ coincide. Here is Landau's theorem:

Theorem 3.1 *If Λ be a set of stable sampling for PW_K, then*

$$\underline{\text{dens}}\,\Lambda \geq |K|. \tag{10}$$

Similarly if Λ be a set of stable interpolation for PW_K, then

$$\overline{\text{dens}}\,\Lambda \leq |K|. \tag{11}$$

These necessary conditions are not sufficient. A counterexample has already been given. But aliasing does not occur in the one-dimensional case when K is an interval. Indeed Arne Beurling proved the following [3]:

Theorem 3.2 *Let Λ be a uniformly discrete set of real numbers. For any interval J the condition*

$$\underline{\text{dens}}\,\Lambda > |J| \tag{12}$$

implies that Λ is a set of stable sampling for PW_J and similarly

$$\overline{\text{dens}}\,\Lambda < |J| \tag{13}$$

implies that Λ is a set of stable interpolation for PW_J.

Another example where $\underline{\text{dens}}\,\Lambda > |K|$ implies stable sampling and $\overline{\text{dens}}\,\Lambda < |K|$ implies stable interpolation is given now.

3.1 Simple Quasicrystals

Let us begin with "model sets". Model sets $\Lambda \subset \mathbb{R}^n$ are defined by the "cut and projection" scheme [14]. Let $m \in \mathbb{N}$, $N = n + m$, $\mathbb{R}^N = \mathbb{R}^n \times \mathbb{R}^m$ and consider a lattice $\Gamma \subset \mathbb{R}^N$. If $(x, y) = X \in \mathbb{R}^n \times \mathbb{R}^m$, we write $x = p_1(X)$ and $y = p_2(X)$. The dual lattice $\Gamma^* \subset \mathbb{R}^N$ is defined by $\exp(2\pi i\, y \cdot x) = 1$ for every $x \in \Gamma$ and every $y \in \Gamma^*$. Then $p_1^* : \Gamma^* \mapsto \mathbb{R}^n$ and $p_2^* : \Gamma^* \mapsto \mathbb{R}^m$ are defined as p_1, p_2. Let us assume that $p_1 : \Gamma \to p_1(\Gamma)$ is a one-to-one mapping and that $p_1(\Gamma)$ is a dense subgroup of $\mathbb{R}^n$. We impose the same properties on p_2. Then $p_1^* : \Gamma^* \to p_1^*(\Gamma^*)$ is a one-to-one mapping and $p_1^*(\Gamma^*)$ is a dense subgroup of $\mathbb{R}^n$ and the same holds for p_2^*. A set $W \subset \mathbb{R}^m$ is Riemann integrable if its boundary has a zero Lebesgue measure.

Definition 4 Let W be a Riemann integrable Borel subset of $\mathbb{R}^m$ with a positive measure. Then the model set $\Lambda \subset \mathbb{R}^n$ defined by Γ and W is

$$\Lambda = \{\lambda = p_1(\gamma);\ \gamma \in \Gamma,\ p_2(\gamma) \in W\}. \tag{14}$$

The compact set W is named the *window* of the model set Λ.

Definition 5 A model set is *simple* if $m = 1$ and $W = [a, b)$ is a semi-closed interval.

A *simple quasicrystal* is defined as a *simple model set* in this essay. B. Matei and the author proved the following theorem [10]:

Theorem 3.3 *Let Λ be a simple quasicrystal and $K \subset \mathbb{R}^n$ be a compact set. Then $|K| <$ dens Λ implies that Λ is a set of stable sampling for the Hilbert space PW_K. If K is Riemann integrable, then $|K| >$ dens Λ implies that Λ is a set of stable interpolation.*

If $\Lambda = \mathbb{Z}^n$ is an ordinary lattice these implications are no longer valid and the obstacle is aliasing. In order to suppress aliasing one shall use simple quasicrystals instead of lattices. Theorem 3.3 is no longer true if K is not compact as it is stated in the following theorem:

Theorem 3.4 *Let Λ be a simple quasicrystal. For every positive ϵ there exists a closed set F with $|F| \le \epsilon$ such that Λ is not a set of stable sampling for W_F.*

The proof can be found in [16].

Theorem 3.3 paves the way to the solution of Problem (b).

Theorem 3.5 *For any $\beta > 0$, let Λ be a simple quasicrystal with* dens $\Lambda > 2\beta$ *Let $f \in L^2(\mathbb{R}^n)$ be a function satisfying the following property: the spectrum $\sigma(f)$ of f is contained in a compact set K such that $|K| \le \beta$. Then f can be recovered from its sample $f(\lambda),\ \lambda \in \Lambda$. This property is not valid if* dens $\Lambda < 2\beta$.

The proof is straightforward. If f_1 and f_2 are two competitors we denote by K_1 and K_2 their spectrum and set $K = K_1 \cup K_2$. Then the spectrum of $f = f_1 - f_2$ lies in K and we have $f(\lambda) = 0,\ \lambda \in \Lambda$. Moreover $|K_1| < \beta$ and $|K_2| < \beta$ imply $|K| <$ dens Λ. Therefore Λ is a set of stable sampling for PW_K and $f = 0$ on Λ

implies $f = 0$. This proof does not give any algorithm for retrieving a sparse signal from its sample. The optimality is proved by the argument used in [12]. Let us assume $\beta > d/2$ where $d = \text{dens}\,\Lambda$. Let K and L be two Riemann integrable compact sets such that $|K| = |L| \in (d/2, \beta)$ and $K \cap L = \emptyset$. There exist two functions g and h such that $g \in W_K$, $h \in W_L$, $g(\lambda) = h(\lambda)$, $\forall \lambda \in \Lambda$, and $g \neq h$. Here is the construction. Consider the union $K' = K \cup L$. Then $|K'| > d$ which implies that Λ is a set of stable interpolation for $W_{K'}$. Redundancy in Theorem 3.3 implies that there exists a function $f \in W_{K'}$ which is not identically 0 and which vanishes on Λ. Then $f = g + h$ where $g \in W_K$ and $h \in W_L$. Finally g and $-h$ are the functions we are looking for.

Theorem 3.3 does not cover the limiting case $|K| = \text{dens}\,\Lambda$. This was achieved by Sigrid Grepstad and Nir Lev [4]. Their results are unveiled now.

3.2 Avdonin's Theorem

Grepstad and Lev used two ingredients to address the limiting case $|K| = \text{dens}\,\Lambda$. The first one is a duality between stable sampling and stable interpolation. Lemma 3.1 is an improved version of an observation which was already used in [10].

Let $\Gamma \subset \mathbb{R}^n \times \mathbb{R}$ be a lattice as in Definition 4, $W = [a, b)$ be a semi-closed interval, and let $\Lambda(\Gamma, I) \subset \mathbb{R}^n$ be the simple quasicrystal defined by (14). The "dual" quasicrystal $\Lambda^*(\Gamma^*, K) \subset \mathbb{R}$ is defined by

$$\Lambda^*(\Gamma^*, K) = \{\lambda^* = p_2^*(\gamma^*);\ \gamma^* \in \Gamma^*,\ p_1^*(\gamma^*) \in K\}. \tag{15}$$

This duality is not the one which was studied in [13] and which was consistent with the definition of the dual lattice.

Lemma 3.1 *Let us assume that $\partial K \cap p_1(\Gamma^*) = \emptyset$. Then $\Lambda(\Gamma, I)$ is a set of stable sampling for PW_K if and only if $\Lambda^*(\Gamma^*, K)$ is a set of stable interpolation for PW_I. Similarly $\Lambda(\Gamma, I)$ is a set of stable interpolation for PW_K if and only if $\Lambda^*(\Gamma^*, K)$ is a set of stable sampling for PW_I.*

This lemma will be used to reduce the proof of Theorem 3.8 to the one-dimensional case. Then Grepstad and Lev rely on Avdonin's theorem which is an outstanding improvement on Kadec's 1/4 theorem. Here is the result:

Theorem 3.6 *Let α be a positive constant and λ_j, $j \in \mathbb{Z}$, a sequence of real numbers fulfilling (a), (b) and (c)*

(a) $|\lambda_j - \lambda_k| \geq \alpha > 0,\ j \neq k.$
(b) $\sup_{j \in \mathbb{Z}} |\lambda_j - j| \leq C.$
(c) There exists a constant γ and a positive integer N such that

$$\sup_{k\in\mathbb{Z}} \Big|\frac{1}{N}\sum_{j=k}^{k+N-1}\big(\lambda_j - j - \gamma\big)\Big| = \theta < \frac{1}{4}. \tag{16}$$

Then the system $\exp(2\pi i\, x\lambda_j)$, $j \in \mathbb{Z}$, *is a Riesz basis for* $L^2[0, 1]$.

An example is given by $\lambda_j = j + \frac{1}{3}\sin(2\pi\omega j)$ where ω is irrational. Beurling's theorem suffices for proving Theorem 3.3 while the spectacular results by Grepstad and Lev depend on Avdonin's theorem. It is interesting that the precise value $1/4$ in Avdonin's theorem does not play any role in the proof and could be replaced by any smaller constant. Theorem 3.6 is a special case of the result proved in [1].

3.3 Bounded Remainder Sets

The definition of bounded remainder sets is the second seminal ingredient in Theorem 3.8. After a linear change of variables it can be assumed that the simple model set Γ is given by the following variant of Definition 4. We have $\gamma = (\gamma_1, \gamma_2) \in \Gamma$ if and only if $\gamma_1 = m + \beta(\alpha\cdot m - l)$, $\gamma_2 = l - \alpha\cdot m$ where $\alpha, \beta \in \mathbb{R}^n$, $m \in \mathbb{Z}^n$, $l \in \mathbb{Z}$. Here $x \cdot y$ denotes the inner product between x and y. Moreover the definition of model sets implies that the numbers $1, \alpha_1, \alpha_2, \ldots, \alpha_n$ are linearly independent over $\mathbb{Q}$ and that $\beta_1, \beta_2, \ldots, \beta_n, 1 + \alpha\cdot\beta$ are also linearly independent over $\mathbb{Q}$. To every compact set $K \subset \mathbb{R}^n$ we associate the $\mathbb{Z}^n$ periodic version of its indicator function defined as $\chi_K(x) = \sum_{k\in\mathbb{Z}^n} \mathbf{1}_K(x + k)$.

Definition 6 Let $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n$ and assume that $1, \alpha_1, \alpha_2, \ldots, \alpha_n$ are linearly independent over $\mathbb{Q}$. Then a compact set K is a bounded remainder set with respect to α if there exists a constant $C = C(\alpha, K)$ such that

$$\Big|\sum_{l=0}^{m-1}\chi_K(x + l\alpha) - m|K|\Big| \le C \tag{17}$$

holds for every $m \in \mathbb{N}$ and for almost every $x \in \mathbb{T}^n$.

3.4 Main Facts

Here are the two main theorems by Grepstad and Lev [4]:

Theorem 3.7 *Let* Λ *be a simple quasicrystal defined by a window* $I = [a, b)$ *such that* $b - a = |I| \notin p_2(\Gamma)$. *Then there does not exists a Riemann integrable compact set* K *such that* Λ *is simultaneously a set of stable sampling and stable interpolation for* PW_K.

This negative result is completed by the following positive result:

Theorem 3.8 *We now assume $|I| \in p_2(\Gamma)$ and that α and β satisfy the requirements listed above. Then the simple quasicrystal Λ is simultaneously a set of stable sampling and of stable interpolation for PW_K for every Riemann integrable compact set K which is a bounded remainder set with respect to α with $|K| = |I|$.*

In a remarkable paper [5] Grepstad and Lev characterized these bounded remainder set.

In one dimension Gady Kozma and Nir Lev [6] proved the following special case of Theorem 3.8. We begin with some notations. Let $\theta > 1$ be a real number. The integral part $[x]$ of a real number x is the largest integer $k \in \mathbb{Z}$ such that $k \leq x$. Define $\Lambda_\theta \subset \mathbb{Z}$ by $\Lambda_\theta = \{[k\theta],\ k \in \mathbb{Z}\}$. With these notations we have

Theorem 3.9 *Let us assume that $\theta > 1$ is irrational. Let $S \subset \mathbb{T}$ be a finite union of intervals J_m, $1 \leq m \leq M$. We assume that the sum of the lengths of these intervals equals the density θ^{-1} of Λ_θ and that the length of each J_m belongs to $\theta^{-1}\mathbb{Z} + \mathbb{Z}$.*

Then any square summable function f defined on S can be uniquely written as a generalized Fourier series

$$f(x) = \sum_{\lambda \in \Lambda_\theta} c_\lambda \, exp(2\pi i \lambda x) \tag{18}$$

where the frequencies λ belong to Λ_θ and the coefficients c_λ belong to $l^2(\Lambda_\theta)$. This series converges to f in $L^2(S)$.

4 Open Problems

The first problem is to extend Theorems 3.3, 3.5, 3.7 and 3.8 to general quasicrystals. Our second problem is an issue raised by Theorem 3.5. A sparse function f is uniquely defined by its sample on Λ but the proof of Theorem 3.5 is not constructive. Is it possible to retrieve f by an algorithm ? A third problem is to extend Theorem 3.8 to L^p, $1 < p \leq \infty$. There are two options. In the first one PW_K is replaced by $PW_K \cap L^p(\mathbb{R}^n)$. Then a definition of stable sampling is the following:

$$f \in PW_K \cap L^p(\mathbb{R}^n) \Rightarrow \|f\|_p \leq C\Big(\sum_{\lambda \in \Lambda} |f(\lambda)|^p\Big)^{1/p}. \tag{19}$$

Is Theorem 3.8 still valid if this new definition is adopted ? Unfortunately Lemma 3.1 which is seminal in the proof of Theorem 3.8 does not generalize to L^p. In the second option one considers the L^p norm of the Fourier transforms $\hat{f}$ of $f \in PW_K$. Before giving more details let us make a detour which is seminal in the proof of Theorem 4.1 [17].

In the one-dimensional case, if K is an interval, and if $p = \infty$, Beurling found a necessary and sufficient condition for (19):

Lemma 4.1 *Let I be an interval and Λ a u.d. sequence of real numbers. Then the following two properties are equivalent:*

(a) $f \in PW_I \Rightarrow \|f\|_\infty \leq C \sup_{\lambda\in\Lambda} |f(\lambda)|$.
(b) $\underline{\mathrm{dens}}\,\Lambda > |I|$.

Similarly Beurling proved the following lemma:

Lemma 4.2 *Let I be an interval and Λ a u.d. sequence of real numbers. Then the following two properties are equivalent:*

(a) For every sequence $c(\lambda) \in l^\infty(\Lambda)$ there exists a $f \in PW_I \cap L^\infty(\mathbb{R})$ such that $f(\lambda) = c(\lambda),\ \lambda \in \Lambda$.
(b) $\overline{\mathrm{dens}}\,\Lambda < |I|$.

There exists a second option for defining stable sampling and stable interpolation in the L^p setting. Instead of using (19) one can try to generalize (5) to L^p. In this generalization the Stepanov space S_p of almost periodic functions will be needed. The Stepanov space S_p is the completion of trigonometric polynomials for the norm $\sup_{x\in\mathbb{R}^n}(\int_{x+B} |f(y)|^p\, dy)^{1/p}$ where B is the unit ball. A compact set $K \subset \mathbb{R}^n$ and a u.d. set $\Lambda \subset \mathbb{R}^n$ are given.

Definition 7 A u.d. set Λ is a set of stable sampling with respect to $L^p(K)$ and the Stepanov space S_p if every function $f \in L^p(K)$ is the restriction to K of an almost periodic function $F \in S_p$ whose frequencies belong to Λ.

When $p = 2$ this definition is identical to Definition 2 or to (19). When $p = \infty$ we have:

Theorem 4.1 *Let $\Lambda \subset \mathbb{R}^n$ be a simple quasicrystal and let $K \subset \mathbb{R}^n$ be a compact set. Then the following properties are equivalent*

(a) Each continuous function f on K is the restriction to K of an almost periodic function $F(x) = \sum_{\lambda\in\Lambda} c(\lambda) \exp(2\pi i \lambda \cdot x)$ whose frequencies belong to Λ.
(b) $|K| < \mathrm{dens}\,\Lambda$.

Similarly the following properties of a Riemann integrable compact set K are equivalent

(c) There exists a constant C such that $\|F\|_\infty \leq C \sup_{x\in K} |F(x)|$ for every finite trigonometric sum $F(x) = \sum_{\lambda\in\Lambda} c(\lambda) \exp(2\pi i \lambda \cdot x)$ whose frequencies belong to Λ.
(d) $|K| > \mathrm{dens}\,\Lambda$.

Property (a) is similar to stable sampling and is implied by $|K| < \mathrm{dens}\,\Lambda$ as in the L^2 setting. Property (c) is similar to stable interpolation and is implied by $|K| > \mathrm{dens}\,\Lambda$ as in the L^2 case. In contrast when $|K| = \mathrm{dens}\,\Lambda,\ K$ cannot be a set of stable sampling or a set of stable interpolation in the sense given by (a) and (c). The proof of Theorem 4.1 which is given in [17] is based on a variant of Lemma 3.1 and on Beurling's theorem.

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Holomorphic Approximation in Banach Spaces: A Survey

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In memory of M. Salah Baouendi

Abstract We give a survey about the Runge approximation problem for a holomorphic function defined on the unit ball of a complex Banach space. More precisely, we ask whether such a holomorphic function can be uniformly approximated on smaller balls by functions that are holomorphic on the entire space. This turns out to be a subtle (open) question, whose (partial) resolution in the past 15 years played a central role in deeper investigations in complex analysis in Banach spaces.

Keywords Infinite-dimensional holomorphy · Entire functions · Finite-dimensional Schauder decomposition

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1 Introduction

Given a complex Banach space X of *infinite dimension*, we recall the following Runge approximation problems:

(1) *Given* $r \in (0, 1)$, $\epsilon > 0$, *and* f *a holomorphic function on the open unit ball of* X, *is there an entire function* h *satisfying* $|f - h| < \epsilon$ *on the open ball of radius* r *centered at the origin? See for instance* [8].

(2) *Is there* $r \in (0, 1)$ *such that for any* $\epsilon > 0$, *and any holomorphic function* f *on the open unit ball of* X, *there exists a entire function* h *satisfying* $|f - h| < \epsilon$ *on the open ball of radius* r *centered at the origin? See for instance* [16].

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We note that problem (2) is clearly independent of the choice of an equivalent norm defining the topology of X, while problem (1) is the analogous version of what is known to be true when X is finite dimensional. The main difficulty here is due to a theorem of Riesz asserting that the unit ball is not relatively compact when $\dim X = \infty$. In other words, we may possibly deal with holomorphic functions defined on the unit ball that are not bounded on smaller balls. The following example shows that this is indeed the case.

Example 1.1 Consider on the Banach space $l^1(\mathbb{N}) = \{z : \mathbb{N} \longrightarrow \mathbb{C},\ ||z|| = \sum_{j=1}^{\infty} |z(j)| < \infty, \}$ the function given by

$$f(z) = \sum_{j=1}^{\infty} 2^j z(j)^j,$$

and let e_k be defined by $e_k(j) = \delta_{jk}$. One can check that f is holomorphic on the unit ball, and since $f(\frac{2}{3}e_k) = 2^k(\frac{2}{3})^k$, f is unbounded on $\overline{B(0, \frac{2}{3})}$.

On the other hand, since unbounded holomorphic functions on the unit ball do exist, we can not replace, as in the finite dimensional case, *h entire holomorphic function* by *h holomorphic polynomial* in the setting of the Runge approximation problems. Indeed, holomorphic polynomials are bounded on the unit ball.

The first result in the direction of problem (1) is due to L. Lempert. He proves the following theorem for the Banach space $l^1(\Gamma) = \{z : \Gamma \longrightarrow \mathbb{C},\ ||z|| = \sum_{\gamma \in \Gamma} |z(\gamma)| < \infty, \}$ where Γ is any set.

Theorem 1.2 ([8]) *Let* $(X, \{\phi_\alpha\})$ *be a locally convex space and* M *be a Stein manifold. Let* $K \subset M$ *be a compact set that is convex with respect to the set of holomorphic functions on* M*, and* $V \subset M$ *be an open neighborhood of* K*. Define*

$$\Omega := \{(m, z) \in V \times l^1(\Gamma) : ||z|| < R(m)\}, \quad \omega := \{(m, z) \in \Omega : ||z|| < r(m)\},$$

where r *and* R *are positive continuous functions on* V*, satisfying* $r < R$*.*

Then for every seminorm ϕ_α*, for every* $\epsilon > 0$*, for every* X*-valued holomorphic map* f *defined on* Ω*, there exists an* X*-valued holomorphic map* g *defined on* $M \times l^1(\Gamma)$ *such that* $\phi_\alpha(f - g) < \epsilon$ *on* $\omega|_K$*.*

One of Lempert's motivations to get interested into the Runge approximation problems has been for instance the solvability of the $\bar{\partial}$ equation in Banach spaces. See [9, 11, 17]. In particular, he obtains the following theorem:

Theorem 1.3 ([11]) *If* $\Omega \subset l^1(\mathbb{N})$ *is pseudoconvex and* $f \in C_{0,1}(\Omega)$ *is a closed locally Lipschitz continuous* $(0, 1)$ *form, then the equation* $\bar{\partial}u = f$ *has a solution* $u \in C^1(\Omega)$*.*

When solving the problem when Ω is the ball centered at 0 of radius R in [9], he shows first that $\bar{\partial}u = f$ is solvable on balls centered at 0 of radius $r < R$, and then constructs a global solution on the ball centered at 0 of radius R using the Runge approximation problem (1).

2 Preliminaries

Let X be a complex Banach space. For $r > 0$, we use $B(0, r)$ to denote the ball of radius r. Recall that a function $f : U \subset X \longrightarrow \mathbb{C}$, where U is an open subset of X, is *holomorphic* if f is continuous on U, and $f_{|U \cap X_1}$ is holomorphic, in the classical sense, as a function of several complex variables, for each finite dimensional subspace X_1 of X. (See [2].) We note that unlike the finite dimensional case, the continuity of f is not automatic as it is shown in the following example:

Example 2.1 Let $\{e_i\}$ be an algebraic basis of X with $||e_i|| = 1$, and let $f : X \longrightarrow \mathbb{C}$ be the linear form satisfying $f(e_i) = i$. Since f is not bounded, f is not holomorphic.

We now recall the definition of a holomorphic polynomial.

Definition 2.2 Let $\Delta_n : X \longrightarrow X^n$ be the mapping defined by $\Delta_n(x) = (x, \ldots, x)$. An $n-$homogeneous holomorphic polynomial $P : X \longrightarrow \mathbb{C}$ is the composition of Δ_n with any continuous $n-$linear map $L : X^n \rightarrow \mathbb{C}$, that is, $P = L \circ \Delta_n$. A holomorphic polynomial is a finite sum of homogeneous holomorphic polynomials.

Definition 2.3 Let H be a holomorphic function on $B(0, r),\ r > 0$. We say that H is $n-$homogeneous if $H(\lambda x) = \lambda^n H(x)$ for $\lambda \in \mathbb{C},\ |\lambda| \leq 1$.

For the convenience of the reader, we give the proof of the next propositions.

Proposition 2.4 *Let $f_n : B(0, r) \longrightarrow \mathbb{C}$ be holomorphic functions such that $\lim_{n\to\infty} f_n = f$, uniformly on compact subsets of $B(0, r)$. Then f is holomorphic on $B(0, r)$.*

Proof Since f is clearly holomorphic when restricted to finite dimensional subspaces of X, we only have to show the continuity of f. Let $\{z_n\}$ be a sequence in X with $\lim_{n\to\infty} z_n = z \in X$. By assumption, given $\epsilon > 0$, there exists on the compact subset $\{\{z_n\}, z\}$, M depending only on ϵ such that

$$|f(z) - f_M(z)| + |f_M(z_n) - f(z_n)| < \frac{2}{3}\epsilon. \tag{2.1}$$

Hence, there exists $N > 0$ such that for $n \geq N$

$$|f(z)-f(z_n)| \leq |f(z)-f_M(z)|+|f_M(z)-f_M(z_n)|+|f_M(z_n)-f(z_n)| < \epsilon. \tag{2.2}$$

□

Proposition 2.5 *Let $H : B(0, r) \longrightarrow \mathbb{C}$ be an $n-$homogeneous holomorphic function. Then H is an entire function, that is a holomorphic function defined on X.*

Proof For $z \in X$, we may write $z = \lambda_1 z_1,\ z_1 \in B(0, r)$. We define $\tilde{H}(z) := {\lambda_1}^n H(z_1)$ and claim that it is a well defined entire function. Indeed if z has another representation $z = \lambda_2 z_2,\ z_2 \in B(0, r)$, then, without loss of generality, we will have $z_1 = \frac{\lambda_2}{\lambda_1} z_2,\ |\frac{\lambda_2}{\lambda_1}| \leq 1$, which implies $H(z_1) = H(\frac{\lambda_2}{\lambda_1} z_2)$. □

Proposition 2.6 *Let $H : X \longrightarrow \mathbb{C}$ be an $n-$homogeneous holomorphic function. Then H is an $n-$homogeneous holomorphic polynomial. More precisely, there is a unique symetric continuous $n-$linear map L such that $H = L \circ \Delta_n$.*

Proof If L exists, it has to satisfy

$$H(z_1 x_1 + \cdots + z_n x_n) = L(\sum z_j x_j, \ldots, \sum z_j x_j) = \tag{2.3}$$

$${z_1}^n L(x_1, \ldots, x_1) + \cdots + n! z_1 z_2 \ldots z_n L(x_1, \ldots, x_n) + \cdots + {z_n}^n L(x_n, \ldots, x_n).$$

Hence, we have the integral formula

$$L(x_1, \ldots, x_n) = \frac{1}{n!}(2\pi i)^{-n} \int_{|z_1| = \epsilon_1} \cdots \int_{|z_n| = \epsilon_n} \frac{H(z_1 x_1 + \cdots + z_n x_n)}{{z_1}^2 \ldots {z_n}^2} dz_1 \ldots dz_n. \tag{2.4}$$

Using (2.4), we obtain the uniqueness. For the existence, we use (2.4) as a definition and show that L is a (symetric) $n-$linear map using a linear change of variables, the continuity of L beeing obtained by using ϵ_j small enough. □

3 Bounded Holomorphic Functions

We start this section by recalling the following definition:

Definition 3.1 Let $f : B(0, 1) \longrightarrow \mathbb{C}$ be a holomorphic function. We define $f_n(z)$ as

$$f_n(z) := \int_0^{2\pi} f(e^{2\pi i t} z) e^{-2\pi i n t} dt. \tag{3.1}$$

Lemma 3.2 *Let $f : B(0, 1) \longrightarrow \mathbb{C}$ be a holomorphic function. Then f_n defined by (3.1) is an $n-$homogeneous holomorphic polynomial.*

Proof We first notice that f_n is a holomorphic fonction on the unit ball that is $n-$homogeneous along any finite dimensional subspace. We conclude then, using Proposition 2.6, that f_n is an $n-$homogeneous holomorphic polynomial. □

Lemma 3.3 *Let $f : B(0, 1) \longrightarrow \mathbb{C}$ be a holomorphic function. Then*

$$f(z) = \sum_{n=1}^{\infty} f_n(z). \tag{3.2}$$

Proof The proof is achieved by restricting to finite dimensional subspaces of X. □

The following theorem shows that uniform approximation by holomorphic polynomials is possible for bounded holomorphic functions. For the convenience of the reader, we give the proof.

Theorem 3.4 *Let $f : B(0,1) \longrightarrow \mathbb{C}$ be a holomorphic function that is bounded on smaller balls $B(0,r)$, $0 < r < 1$. Then given $\epsilon > 0$, there exists a holomorphic polynomial h satisfying $|f - h| < \epsilon$ on $B(0,r)$.*

Proof We claim that, for $0 < \sigma < 1$, we have

$$\overline{\lim_{n\to\infty}}\, \sigma(|f_n|_{B(0,1)})^{\frac{1}{n}} < 1. \tag{3.3}$$

Indeed, assuming without loss of generality that f is bounded on $B(0,1)$, we have $M \geq |f_n(\sigma z)| = \sigma^n |f_n(z)|$. Hence $\sigma |f_n(z)|^{\frac{1}{n}} < M^{\frac{1}{n}}$, which implies $\overline{\lim}_{n\to\infty}\, \sigma(|f_n|_{B(0,1)})^{\frac{1}{n}} \leq 1$. Choosing $\tilde{\sigma} < 1$ satisfying $\tilde{\sigma} = \lambda\sigma$, $\lambda > 1$, we obtain (3.3). We then consider, for $r < 1$, the ball $B(0,r) = rB(0,1)$. Using (3.3), we obtain that there exists $\mu < 1$ such that $r^n |f_n|_{B(0,1)} < \lambda^n$, for n large enough. Using Lemma 3.3, we then conclude that $f(z) = \sum_{n=1}^{\infty} f_n(z)$ uniformly on $B(0,r)$. The proof of the proposition is achieved using Lemma 3.2. □

4 A Counterexample

In this section, we discuss the counterexample given by Lempert in [13]. We start with the following definition:

Definition 4.1 A set S is called a bounding set if $||f||_S < \infty$ for every entire function.

In [1], Dineen shows that l^∞ admits non compact closed bounding subsets. More precisely, he shows that any entire function on l^∞ is bounded on the set $S = \{e_n\}$, where $e_n(j) = \delta_{jn}$.

Lempert considers a sequence of norms on l^∞, all equivalent to the sup norm. More precisely, he defines

$$||z||_k = \frac{2}{k} \sup_{j_1 < j_2 < \cdots < j_k} |z(j_1)| + |z(j_2)| + \cdots + |z(j_k)|, \quad z : \mathbb{N} :\longrightarrow \mathbb{C} \in l^\infty. \tag{4.1}$$

Then he considers on $(l^\infty, ||z||_k)$, the following holomorphic function on the unit ball

$$f(z) = \sum_{j=1}^{\infty} j z(j)^j, \tag{4.2}$$

that is unbounded on the set $S = \{e_n\}$. This shows, using Dineen's result, that the answer to Problem (1) is negative on $(l^\infty, ||z||_k)$. We refer the interested reader to [13] for the details and for the negative answer to Problem (2).

5 The General Case

In the light of the counterexample given by Lempert, we may ask the following question:

- Do Runge approximations (1) and (2) hold in any separable Banach space?

Of particular interest is the space $C[0, 1]$ since every separable Banach space is isometric to a subspace of $C[0, 1]$. The Runge approximations are still open for this space as well as for the Banach space $L^1(0, 1)$. As said in the introduction, the obstruction to Runge approximations is the theorem of Riesz. Therefore, one needs to look for "good" relatively compact subsets that "replace" the unit ball. The following lemma gives some understanding of what kind of sufficient conditions are needed to obtain a positive answer to Problem (1) and Problem (2).

Lemma 5.1 ([3]) *Let T_n be a uniformly bounded sequence of linear operators in X. If $\lim_n T_n x = x$ for every $x \in X$, then this limit exists uniformly on any compact set. Conversely, if $\lim_n T_n x = x$ uniformly for x in a bounded set K, and if, in addition, $T_n K$ is relatively compact for each n, then K relatively is compact.*

We recall the following definitions.

Definition 5.2 A series $\sum_{n=1}^{\infty} x_n$ is said to converge unconditionally if $\sum_{n=1}^{\infty} x_{\pi(n)}$ converges for every permutation π of the integers.

Definition 5.3 Let X be a complex Banach space. A sequence $\{X_n\}$ of closed subspaces of X is called a Schauder decomposition of X if every $x \in X$ has a unique representation of the form $x = \sum_{n=1}^{\infty} x_n$, with $x_n \in X$. The Schauder decomposition is unconditional if for every $x \in X$, the series $\sum_{n=1}^{\infty} x_n$ which represents x converges unconditionally.

Remark 5.4 ([18]) A decomposition $\{X_n\}$ of a Banach space X is a Schauder decomposition of X if and only if the projections $\{P_j\}$ defined by

$$P_j \sum_{n=1}^{\infty} x_n = \sum_{n=1}^{j} x_n \tag{5.1}$$

are continuous. Moreover $\{P_j\}$ is a uniformly bounded sequence of linear operators in X.

Example 5.5 $l^p(\mathbb{N}) = \{z : \mathbb{N} \longrightarrow \mathbb{C},\ ||z|| = (\sum_{j=1}^{\infty} |z(j)|^p)^{\frac{1}{p}} < \infty\}$ admits an unconditional Schauder basis, i.e. $\dim X_n = 1$.

Example 5.6 Let X be the space of compact operators on $l^2(\mathbb{N})$ which have a triangular representing matrix with respect to $\{e_n\}$, where $e_n(j) = \delta_{jn}$. Let X_n be the subspace of X defined as

$$X_n = \{T \in X \mid Te_j = 0,\ j \neq n\}.$$

Then $\{X_n\}$ is an unconditional Schauder decomposition of X with $\dim X_n < \infty$. In this case, we say that X has a UFDD (unconditional finite dimensional decomposition). Moreover, by a result of Gordon and Lewis, X does not have an unconditional Schauder basis. See [4, 15].

Example 5.7 It is well known that the space of compact operators on $l^2(\mathbb{N})$ has an unconditional Schauder decomposition into UFDD, but has no UFDD itself. See [7].

Assume now that X admits an unconditional Schauder decomposition $\{X_n\}_{n=1}^{\infty}$. Let $x = \sum_{n=1}^{\infty} x_n$, $x_n \in X_n$, be the unique representation of x. It is known that for every sequence of complex numbers $\theta = \{\theta_n\}$, $|\theta_n| \leq 1, n \in \mathbb{N}$, the operator M_θ defined by

$$M_\theta \sum_{n=1}^{\infty} x_n = \sum_{n=1}^{\infty} \theta_n x_n$$

is a bounded linear operator. The (finite) constant $\sup_\theta \|M_\theta\|$ is called the *unconditional constant* of the decomposition.(See [15, 18] for details). It is clear that one can always define on X an equivalent norm $\|\,\|_1$ so that the unconditional constant becomes 1. (Take $\|x\|_1 = \sup_\theta \|M_\theta x\|$). In other words, we have

$$\|\sum_{n=1}^{\infty} \theta_n x_n\|_1 \leq \|\sum_{n=1}^{\infty} x_n\|_1, \quad |\theta_n| \leq 1,\ n \in \mathbb{N}. \tag{5.2}$$

Using Lemma 5.1 and (5.2), one proves

Proposition 5.8 *Let X be a complex Banach space admitting an unconditional Schauder decomposition $\{X_n\}$ with unconditional constant one. Then the following holds.*

(1) $M_\theta(B(0, R))$ is relatively compact in $B(0, R)$ for any sequence of complex numbers $\theta = \{\theta_n\}$, $|\theta_n| < 1$, which converges to 0 if and only if $\dim X_n < \infty$, for any n.

(2) For any compact $K \subset B(0, R)$, there exists a sequence of complex numbers $\theta = \{\theta_n\}$, $|\theta_n| < 1$ which converges to 0, and a compact $L \subset B(0, R)$ so that $M_\theta L = K$.

Proof We note that $P_j(M_\theta(B(0, R)))$ is relatively compact if and only if $\dim X_n < \infty$, for any n. The rest of the proof is similar to the one of Proposition 1.2 in [10]. □

Remark 5.9 The sets $M_\theta(B(0, R))$ in Proposition 5.8 will play the role of "good" relatively compact sets that replace the unit ball in the case of a space admitting a UFDD.

6 The Case of Banach Spaces Admitting a UFDD

In the light of the proof of Theorem 3.4, Remark 5.9 leads to the following definitions that appear in [5, 8, 10, 16].

Let X be a complex Banach space admitting an unconditional Schauder decomposition $\{X_n\}$ with unconditional constant one. Let $z \in X$ be given by its unique representation $z = \sum_{n=1}^{\infty} z_n$, with $z_n \in X_n$ for every n. For $m \in \mathbb{N}$, one defines

$$T(m) := \{t : \mathbb{N} \to S^1, n \to e^{2\pi i t_n},\ t_n \in \mathbb{R},\ t_n = 0 \text{ for } n > [\sqrt{m}\,]\}, \tag{6.1}$$

endowed with the product topology, and

$$K(m) := \{k = \{k_n\},\ k_n \in \mathbb{N} \cup \{0\},\ k_n = 0 \text{ for } n > [\sqrt{m}\,],\ \sum_{n=1}^{[\sqrt{m}]} k_n \leq m\}. \tag{6.2}$$

By abuse of notation, for $t \in T(m)$, we write $e^{2\pi i t} := \{e^{2\pi i t_n}\}$.

Definition 6.1 Let f be a holomorphic function on $B(0, R)$. For $k \in K(m)$ and $s \in T(m)$, we define $f^k(z)$ as

$$f^k(z) := \int_{T(m)} f(M_{e^{2\pi i s}} z) e^{-2\pi i k.s} ds, \tag{6.3}$$

where ds is the normalized Haar measure on $T(m)$.

Note that f^k is homogeneous of degree k_n in z_n.

Definition 6.2 The formal series associated to f given by

$$\sum_{m=0}^{\infty} \sum_{k \in K(m)} (f_m)^k(z) \tag{6.4}$$

is called the Josefson series.

We have the following proposition:

Proposition 6.3 ([16]) *Let X be a complex Banach space admitting an unconditional Schauder decomposition $\{X_n\}$ with unconditional constant one. Then the Josefson series converges to f, uniformly on compact sets of $B(0, R)$ if* $\dim X_n < \infty$ *for any n.*

Remark 6.4 In Proposition 6.3, the proof of the uniform convergence on compact sets relies on the fact that the sets $M_\theta(B(0, R))$ are compact. It would be interesting to know if it holds without the assumption on the dimension of each X_n.

Making use of the Josefson series, one can show that problem (2) holds in spaces admitting an unconditional Schauder decomposition $\{X_n\}$ with $\dim X_n < \infty$ for any n. More precisely, we have

Theorem 6.5 ([16]) *Let X be a complex Banach space admitting an unconditional Schauder decomposition* $\{X_n\}$ *with* $\dim X_n < \infty$ *for any n. Then there exists an equivalent norm* $\|\|$ *on X for which approximation as in problem (1) is possible.*

Corollary 6.6 ([10]) *Let X be a separable Hilbert space. Then approximation as in problem (1) is possible.*

Remark 6.7 More generally, L. Lempert shows in [10] that approximation as in problem (1) is possible in any Hilbert space.

We may ask the following question

- Does Runge approximation (2) hold in complex Banach spaces admitting an unconditional Schauder decomposition into UFDD?

Note that the "model" case of such a space is given by Example 5.7.

Remark 6.8 It is known that the spaces $C[0, 1]$ and $L^1[0, 1]$ do not admit any unconditional Schauder decomposition into UFDD. Indeed, Lindenstrauss and Pelczynski in [14] showed that if (X_n) is an unconditional Schauder decomposition of $C[0, 1]$, then at least one $X_n \approx C[0, 1]$; but $C[0, 1]$ has no UFDD [18]. N.J. Kalton in [6] showed that the same holds for $L^1[0, 1]$.

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Pseudoconvex Mizohata Structures on Compact Manifolds

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In memory of M. Salah Baouendi

Abstract The main result of this paper establishes the rigidity of pseudoconvex Mizohata structures on compact manifolds with abelian fundamental groups. Any simply connected $(n+1)$-dimensional compact manifold with a pseudoconvex Mizohata structure is equivalent the standard Mizohata structure on the sphere $\mathbb{S}^{n+1}$. If an $(n+1)$-dimensional connected compact manifold with a nontrivial abelian fundamental group carries a pseudoconvex Mizohata structure, then it is equivalent to a structure on $\mathbb{S}^1 \times \mathbb{S}^n$.

Keywords Compact manifold · Fundamental group · Mizohata structure · Pseudoconvex

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1 Introduction

In this paper we study pseudoconvex Mizohata structures on smooth compact manifolds M of dimension $n+1$. Such a structure is given by a rank 1, formally integrable smooth subbundle $\mathcal{V}$ of the complexified cotangent bundle $\mathbb{C}T^*M$, for which the Levi form is positive definite at each point of the characteristic set Σ (set of points where $\mathcal{V}$ is nonelliptic).

Mizohata structures are strongly linked to CR structures with nondegenerate Levi forms and their study (local and global aspects) has attracted the attention of many authors [1–12, 15, 16].

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Global aspects of Mizohata structures in the two-dimensional case ($n = 1$) have been investigated in [5, 7, 10, 11]. Here, we consider the higher dimensional case ($n > 1$) when the structure is pseudoconvex. In this case, $\mathcal{V}$ generates a codimension 2 singular foliation $\mathcal{F}(\mathcal{V})$ whose leaves are all diffeomorphic to the sphere $\mathbb{S}^{n-1}$ and whose singularity is carried by the characteristic set Σ (Theorems 2.1 and 2.2). In Sect. 4, we prove (Theorem 4.1) that the leaf space R is a Riemann surface with boundary and that R has the same fundamental group as the manifold M. In Sect. 5, we prove that the structure is trivial in a neighborhood of Σ (Theorem 5.1). Using this trivialization, we construct (Theorem 5.2) a desingularization of the foliation $\mathcal{F}(\mathcal{V})$. That is a map $\Pi : (X, \widetilde{\mathcal{F}}) \longrightarrow (M, \mathcal{F})$, where X is an $(n + 1)$-dimensional (noncompact) manifold, $\widetilde{\mathcal{F}}$ is a nonsingular foliation on X, such that $\Pi : X\backslash\Pi^{-1}(\Sigma) \longrightarrow M\backslash\Sigma$ is a local diffeomorphism and the leaves of $\widetilde{\mathcal{F}}$ are sent onto those of $\mathcal{F}$. In Sect. 6, we consider pseudoconvex Mizohata structures $(M, \mathcal{V})$ over compact manifolds with abelian fundamental groups $\pi_1(M)$. We prove (Theorem 6.1) that if M is simply connected and equipped with such a structure, then it is diffeomorphic to the sphere $\mathbb{S}^{n+1}$ and the Mizohata structure is equivalent to the standard structure on the sphere. If $\pi_1(M)$ is nontrivial, we prove (Theorem 6.2) that M is diffeomorphic to $\mathbb{S}^n \times \mathbb{S}^1$ and the structure is again equivalent to a particular model (see Sect. 3).

2 Definitions and Local Trivialization

We give the basic definitions, recall the local integrability of Mizohata structures, and prove that the associated foliation is locally trivial.

Let $\widetilde{\omega}$ be a smooth (C^∞), $\mathbb{C}$-valued 1-form defined on an open set $U \subset \mathbb{R}^{n+1}$ with $0 \in U$. Suppose that

$$\widetilde{\omega} \neq 0\,, \quad \text{and} \quad \widetilde{\omega} \wedge d\widetilde{\omega} = 0. \tag{2.1}$$

Then $\widetilde{\omega}$ generates a smooth rank 1 subbundle $\mathcal{V}$ of the complexified cotangent bundle $\mathbb{C}T^*U$ and that $\mathcal{V}$ satisfies the formal integrability condition. Let $\widetilde{\omega}_1 = \mathfrak{Re}\widetilde{\omega}$ and $\widetilde{\omega}_2 = \mathfrak{Im}\widetilde{\omega}$ be the real and imaginary parts of $\widetilde{\omega}$. It follows at once, from (2.1), that

$$\widetilde{\omega}_1 \wedge \widetilde{\omega}_2 \wedge d\widetilde{\omega}_i = 0 \qquad i = 1, 2. \tag{2.2}$$

Hence, $\widetilde{\omega}$ generates (via $\{\widetilde{\omega}_1, \widetilde{\omega}_2\}$) a codimension 2 foliation $\mathcal{F}(\mathcal{V})$ on $U\backslash\Sigma$, where

$$\Sigma = \{p \in U : \widetilde{\omega}_1 \wedge \widetilde{\omega}_2 = 0\} = \{p \in U : \widetilde{\omega} \wedge \overline{\widetilde{\omega}} = 0\}\,. \tag{2.3}$$

It is proved (see [2] or [16] for example) that there exist coordinates $(x, t_1, \ldots, t_n)$ centered at 0 and C^∞ functions $\lambda_1, \ldots, \lambda_n$ defined near 0 such that $\mathcal{V}$ is generated near 0 by the form

$$\omega = dx + \sum_{j=1}^{n} \lambda_j(x,t)dt_j \, . \tag{2.4}$$

The bundle $\mathcal{V}$ is said to be elliptic at $p \in U$ if ω and $\overline{\omega}$ are independent at p. In this case coordinates $(x, y, s_1, \ldots, s_{n-1})$, defined near p, can be found so that $\mathcal{V}$ is generated near p by the exact form $d(x+iy)$. Note that if $\mathcal{V}$ is elliptic in open sets O_1 and O_2 and if f_1, f_2 are C^∞ functions defined in O_1 and O_2, respectively, such that

$$\mathcal{V}\,|_{O_1} = \text{span}\{df_1\} \quad \text{and} \quad \mathcal{V}\,|_{O_2} = \text{span}\{df_2\}\,,$$

then $f_2 = h \circ f_1$ on $O_1 \cap O_2$, where h is a holomorphic function defined on $f_1(O_1 \cap O_2)$.

The bundle $\mathcal{V}$ is said to be of Mizohata type at $p \in U$, if it is not elliptic at p and the Levi form of $\mathcal{V}$ is nondegenerate at p. This means that local coordinates $(x, t_1, \ldots, t_n)$ can be found so that $\mathcal{V}$ is generated near p by the form

$$d(x + iQ(t)) + \sum_{j=1}^{n} a_j(x,t)dt_j \, , \tag{2.5}$$

where

$$Q(t) = t_1^2 + \cdots + t_k^2 - t_{k+1}^2 - \cdots - t_n^2 \tag{2.6}$$

and the a_j's are C^∞ functions that vanish to higher order at p. The signature at p of the Mizohata structure is $|n-2k|$. The structure is pseudoconvex if its signature is n ($Q(t)$ is positive definite). With respect to these coordinates, the characteristic set is defined by $t = 0$. The following local integrability result can be found in [8].

Theorem 2.1 *If ω defines a Mizohata structure near $0 \in \mathbb{R}^{n+1}$ with signature different from $n-2$, then there exist coordinates $(x, t_1, \ldots, t_n)$ centered at 0 such that*

$$\omega = \mu(x,t)\, d(x + iQ(t)) \tag{2.7}$$

where $Q(t) = t_1^2 + \cdots + t_k^2 - t_{k+1}^2 - \cdots - t_n^2$ and $\mu(x,t)$ is a C^∞ function.

Let M be a C^∞ manifold of dimension $n+1$ and let $\mathcal{V}$ be a smooth (C^∞), rank 1 subbundle of the complexified cotangent bundle $\mathbb{C}T^*M$. Suppose that $\mathcal{V}$ satisfies the formal integrability condition ($\mathcal{V} \wedge d\mathcal{V} = 0$). Denote by $\overline{\mathcal{V}}$ the complex conjugate of $\mathcal{V}$. The characteristic set Σ of $\mathcal{V}$ is defined as

$$\Sigma = \left\{ p \in M : \ \mathcal{V} \cap \overline{\mathcal{V}} \neq \{0\} \right\}. \tag{2.8}$$

Note that $\mathcal{V}$ is elliptic in $M \backslash \Sigma$. The bundle $\mathcal{V}$ defines a Mizohata structure on M if $\Sigma \neq \emptyset$ and if for every $p \in \Sigma$, the bundle $\mathcal{V}$ has a generator of the form (2.5) near p. It follows at once from the local description of $\mathcal{V}$ that Σ is an embedded one-dimensional submanifold of M. It also follows that $\mathcal{V}$ generates a codimension two foliation on $M\backslash\Sigma$. We denote this foliation by $\mathcal{F}(\mathcal{V})$. The next theorem describes $\mathcal{F}(\mathcal{V})$ when the structure is pseudoconvex.

Theorem 2.2 *Let $(M, \mathcal{V})$ be a pseudoconvex Mizohata structure with characteristic set Σ. Assume that the $(n+1)$-dimensional manifold M is compact and connected. Let $p \in M\backslash\Sigma$. Then there exist an open set $U \ni p$ and a diffeomorphism*

$$\Phi : U \longrightarrow \mathbb{S}^{n-1} \times \Delta\,, \tag{2.9}$$

where Δ is the unit disc in $\mathbb{C}$, such that $\mathcal{V}$ is generated over U by $\Phi^*(d(x+iy))$, where $x + iy$ is the coordinate in $\mathbb{C}$.

Proof It follows at once from Theorem 2.1 that $\mathcal{F}(\mathcal{V})$ is trivial near each point $q \in \Sigma$. We need then to prove that this local triviality propagates to all points of M. For this we first prove that all leaves of $\mathcal{F}(\mathcal{V})$ are diffeomorphic to $\mathbb{S}^{n-1}$.

Let $p \in M\backslash\Sigma$. Denote by L_p the leaf of $\mathcal{F}(\mathcal{V})$ through p. Let $p_0 \in M\backslash\Sigma$ be close enough to Σ so that $L_{p_0} \cong \mathbb{S}^{n-1}$ and $\mathcal{F}(\mathcal{V})$ is trivial near p_0 (note that when $n > 2$, L_{p_0} is simply connected and the stability theorem of foliations (see [14]) implies that all leaves of $\mathcal{F}(\mathcal{V})$ are diffeomorphic to $\mathbb{S}^{n-1}$). Let

$$\Gamma : [0, 1] \longrightarrow M \tag{2.10}$$

be a simple smooth curve in M joining p_0 to p and such that Γ is transversal to $\mathcal{F}(\mathcal{V})$. The interval $[0, 1]$ can be divided into subintervals $[0, t_1], [t_1, t_2], \ldots, [t_N, 1]$ in such a way that each arc $\Gamma([t_{i-1}, t_i))$ is contained in a surface S_i transversal to $\mathcal{F}(\mathcal{V})$, and S_1 is contained in an open set U_0, where $\mathcal{V}$ is trivial (Theorem 2.1). Hence, $L_{\Gamma(t)} \cong \mathbb{S}^{n-1}$ for $0 \le t < t_1$ and has trivial holonomy.

Now we prove that $L_{\Gamma(t_1)}$ is diffeomorphic to $\mathbb{S}^{n-1}$ and has trivial holonomy. As noted earlier, only the case $n = 2$ needs to be considered. Let τ be a surface transversal to $\mathcal{F}(\mathcal{V})$ and containing $\Gamma(t_1) = p_1$. Let $\Psi : \tau \longrightarrow \tau$ be the first return map. Thus, Ψ is a local diffeomorphism and

$$\Psi(p) = p \quad \text{for} \quad p \in \tau' = \tau \cap \left(\bigcup_{q \in S_1} L_q \right). \tag{2.11}$$

Note that τ' has a nonempty interior in τ. Since the bundle $\mathcal{V}$ is elliptic at p_1, there exist a neighborhood V of p_1 and a C^∞ function $z : V \longrightarrow \mathbb{C}$ such that dz generates $\mathcal{V}$ over V. Let z_0 be the restriction of z to $\tau \cap V$. It can be shown (see [8]) that

$$z_0 \circ \Psi = h \circ z_0\,, \tag{2.12}$$

where h is a holomorphic function defined near $z_0(p_1) \in \mathbb{C}$. Since Ψ is the identity function on τ', then h is the identity on $z_0(\tau' \cap V)$ (which has a nonempty interior in $\mathbb{C}$). Thus, h is the identity and consequently Ψ is the identity. This proves the triviality of the holonomy of L_{p_1}. This argument can be repeated to show that each leaf $L_{\Gamma(t_i)}$ is diffeomorphic to $\mathbb{S}^{n-1}$ and has trivial holonomy.

We complete the proof as follows. Since $\mathcal{V}$ is elliptic at p, it has a local first integral defined near p, i.e., there are an open set $V \ni p$ and a function $z : V \longrightarrow \mathbb{C}$ (with $z(p) = 0$) such that

$$\mathcal{V}|_V = \text{span}\{z^* d(x + iy)\}\,. \tag{2.13}$$

Let $U = \bigcup_{q \in V} L_q$ and define

$$f : U \longrightarrow \mathbb{C}; \quad f(m) = z(L_m \cap V)\,. \tag{2.14}$$

Thus $f^*(d(x + iy))$ generates $\mathcal{V}$ over U. Finally, since f is a submerssion, and $f^{-1}(0)$ is compact ($\cong \mathbb{S}^{n-1}$), there is a tubular neighborhood $\widehat{U}$ of L_p and a diffeomorphism

$$\Phi : \widehat{U} \longrightarrow \mathbb{S}^{n-1} \times \Delta\,, \tag{2.15}$$

so that $\Phi^* d(x + iy)$ generates $\mathcal{V}$ over $\widehat{U}$. □

To summarize, we have proved that if $\mathcal{V}$ defines a pseudoconvex Mizohata structure on a connected and compact $(n + 1)$-dimensional manifold M ($n \geq 2$), then:

- For $p \in \Sigma$, there exist an open set $U_p^0 \subset M$ ($p \in U_p^0$) and a diffeomorphism

$$\Phi_p^0 : U_p^0 \longrightarrow B^{n+1}, \mathcal{V}|_{U_p^0} = \text{span}\{(\Phi_p^0)^* d(x + i(t_1^2 + \cdots + t_n^2))\}\,, \tag{2.16}$$

  where B^{n+1} is the unit ball in $\mathbb{R}^{n+1}$ and $(x, t_1, \ldots, t_n)$ the coordinates in $\mathbb{R}^{n+1}$.
- For $p \in M \backslash \Sigma$, there exists an open set $U_p \subset M$ with $L_p \subset U_p$ and a diffeomorphism

$$\Phi_p : U_p \longrightarrow \mathbb{S}^{n-1} \times \Delta, \mathcal{V}|_{U_p} = \text{span}\{(\Phi_p)^* d(x + iy)\}\,, \tag{2.17}$$

  where Δ is the unit disc in $\mathbb{C}$ and (x, y) the coordinates in $\mathbb{R}^2 \cong \mathbb{C}$.

We call such charts (U_p^0, Φ_p^0), and (U_p, Φ_p) the local trivialization charts of the Mizohata structure.

3 Models of Mizohata Structures

In this section, we give examples of Mizohata structures on compact manifolds. The first two are the standards structures on the sphere $\mathbb{S}^{n+1}$ and on $\mathbb{S}^n \times \mathbb{S}^1$. These two models give a complete description of compact manifolds with abelian fundamental groups that carry pseudoconvex Mizohata structures (see Sect. 6). The third is an example of a Mizohata structure (with a global first integral) on a manifold with a nonabelian fundamental. The fourth is an example of a structure without global first integral. The fifth is an example of a non-pseudoconvex Mizohata structure.

3.1 Standard Structure on $\mathbb{S}^{n+1}$

The idea for this type of construction goes back to Jacobowitz [7]. Let

$$\mathbb{S}^{n+1} = \{x = (x_1, \ldots, x_{n+2}) \in \mathbb{R}^{n+2} :\ x_1^2 + \cdots + x_{n+2}^2 = 1\}\,.$$

Denote by $\overline{\Delta}$ the closure of the unit disc $\Delta \subset \mathbb{C}$ and let

$$z : \mathbb{S}^{n+1} \longrightarrow \overline{\Delta}; \quad z(x) = x_1 + i x_2\,.$$

Note that if $\zeta = u + iv \in \Delta$, then $z^{-1}(\zeta)$ is an $(n-1)$-sphere:

$$z^{-1}(\zeta) = \{(u, v, x_3, \ldots, x_{n+2}) \in \mathbb{R}^{n+2} :\ x_3^2 + \cdots + x_{n+2}^2 = 1 - |\zeta|^2\}\,,$$

and if $\zeta = \mathrm{e}^{i\sigma} \in \partial\Delta$, then $z^{-1}(\zeta)$ is the single point

$$z^{-1}(\zeta) = \{(\cos\sigma, \sin\sigma, 0, \ldots, 0)\}\,.$$

Hence z is injective on the equatorial circle

$$\Sigma = \{(x_1, x_2, 0, \ldots, 0) \in \mathbb{R}^{n+2} :\ x_1^2 + x_2^2 = 1\}\,,$$

and z is a fibration with fibre $\mathbb{S}^{n-1}$ on $\mathbb{S}^{n+1}\backslash\Sigma$. Let

$$\mathcal{V}_0 = \mathrm{span}(dz)\,.$$

Clearly, $\mathcal{V}_0$ is elliptic on $\mathbb{S}^{n+1}\backslash\Sigma$. To determine the nature of $\mathcal{V}_0$ near a point $m \in \Sigma$, we use spherical coordinates. Without loss of generality we can assume that $m = (1, 0, \ldots, 0)$. Let $\theta_1, \ldots, \theta_{n+1} \in (-1, 1)$ and write

$$\begin{aligned}
x_1 &= \cos\frac{\pi\theta_1}{2}\cos\frac{\pi\theta_2}{2}\cdots\cos\frac{\pi\theta_{n+1}}{2}\\
x_2 &= \sin\frac{\pi\theta_1}{2}\cos\frac{\pi\theta_2}{2}\cdots\cos\frac{\pi\theta_{n+1}}{2}\\
x_3 &= \sin\frac{\pi\theta_2}{2}\cos\frac{\pi\theta_3}{2}\cdots\cos\frac{\pi\theta_{n+1}}{2}\\
&\vdots
\end{aligned}$$

$$x_{n+1} = \sin\frac{\pi\theta_n}{2}\,\cos\frac{\pi\theta_{n+1}}{2}$$

$$x_{n+2} = \sin\frac{\pi\theta_{n+1}}{2}\,.$$

In this chart, $\mathcal{V}_0$ is determined by the form

$$\omega = d\left(\cos\frac{\pi\theta_2}{2}\,\cdots\,\cos\frac{\pi\theta_{n+1}}{2}\,\mathrm{e}^{i\pi\theta_1/2}\right)$$

$$= i\mathrm{e}^{i\pi\theta_1/2}\,\cos\frac{\pi\theta_2}{2}\,\cdots\,\cos\frac{\pi\theta_{n+1}}{2}\,d\left(\theta_1 - i\sum_{j=2}^{n+1}\log\cos\frac{\pi\theta_j}{2}\right).$$

Since the point m has coordinates $\theta_1 = 0, \ldots \theta_{n+1} = 0$, then, for θ_j near 0, we have

$$\log\cos\frac{\pi\theta_j}{2} = -\frac{\pi\theta_j^2}{4} + \cdots,$$

where $\cdots$ denote higher order terms, and thus ω has the form (2.5). Therefore $\mathcal{V}_0$ generates a (standard) pseudoconvex Mizohata structure on $\mathbb{S}^{n+1}$.

3.2 Standard Structure on $\mathbb{S}^n \times \mathbb{S}^1$

Let

$$\mathbb{S}^n = \{x = (x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} : \; x_1^2 + \cdots + x_{n+1}^2 = 1\}$$

and

$$l : \; \mathbb{S}^n \longrightarrow [-1, 1]; \; l(x) = x_{n+1}\,.$$

Note that if $-1 < a < 1$, then $l^{-1}(a)$ is an $(n-1)$-sphere of radius $\sqrt{1-a^2}$ contained in $\mathbb{S}^n$, and if $a = \pm 1$, then $l^{-1}(\pm 1) = \{p_\pm\} = \{(0, \ldots, 0, \pm 1)\}$.

For a given $\rho \in (0, 1)$, let

$$R_\rho = \{\zeta \in \mathbb{C} : \; \rho < |\zeta| < 1\}$$

and let $\overline{R_\rho}$ be the closure of R_ρ. Define

$$z_\rho : \; \mathbb{S}^n \times \mathbb{S}^1 \longrightarrow \overline{R_\rho}; \quad z_\rho(x, \mathrm{e}^{i\sigma}) = \left(\frac{1-\rho}{2}l(x) + \frac{1+\rho}{2}\right)\mathrm{e}^{i\sigma}\,.$$

Note that if $\zeta \in R_\rho$, then $z_\rho^{-1}(\zeta)$ is diffeomorphic to $\mathbb{S}^{n-1}$ and that

$$z_\rho^{-1}(\mathrm{e}^{i\sigma}) = \{(p_+, \mathrm{e}^{i\sigma})\}, \quad \text{and} \quad z_\rho^{-1}(\rho\mathrm{e}^{i\sigma}) = \{(p_-, \mathrm{e}^{i\sigma})\}.$$

Let $\Sigma^+ = \{p_+\} \times \mathbb{S}^1$, $\Sigma^- = \{p_-\} \times \mathbb{S}^1$, and $\Sigma = \Sigma^+ \cup \Sigma^-$. Let

$$\mathcal{V}_\rho = \text{span}\{dz_\rho\}.$$

It is clear that $\mathcal{V}_\rho$ is elliptic on $\mathbb{S}^n \times \mathbb{S}^1 \setminus \Sigma$. To determine $\mathcal{V}_\rho$ near a point $(p_+, \mathrm{e}^{i\sigma}) \in \Sigma^+$, we use coordinates $x_1, \ldots, x_n$ near $p_+ = (0, \ldots, 0, 1) \in \mathbb{R}^{n+1}$. With respect to these coordinates, we have

$$l(x) = \sqrt{1 - (x_1^2 + \cdots + x_n^2)} = 1 - \frac{1}{2}(x_1^2 + \cdots + x_n^2) + \cdots$$

Thus

$$z_\rho(x, \mathrm{e}^{i\sigma}) = \left(1 - \frac{1-\rho}{4}(x_1^2 + \cdots + x_n^2) + \cdots\right) \mathrm{e}^{i\sigma}$$

and dz_ρ defines a pseudoconvex Mizohata structure near Σ^+. Similarly, we can prove that dz_ρ defines a pseudoconvex Mizohata structure near Σ^-. Therefore $\mathcal{V}_\rho$ is a pseudoconvex Mizohata structure on $\mathbb{S}^n \times \mathbb{S}^1$ with characteristic set $\Sigma = \Sigma^+ \cup \Sigma^-$ and with global first integral z_ρ.

In the next proposition, we prove that if $\rho \neq \rho'$, the structures $\mathcal{V}_\rho$ and $\mathcal{V}_{\rho'}$ are not equivalent.

Proposition 1 *Let $\rho, \rho' \in (0, 1)$ and let $\mathcal{V}_\rho$ and $\mathcal{V}_{\rho'}$ be the corresponding Mizohata structures on $\mathbb{S}^n \times \mathbb{S}^1$ as constructed above. Then $\mathcal{V}_\rho$ and $\mathcal{V}_{\rho'}$ are equivalent if and only if $\rho = \rho'$.*

Proof Suppose $\mathcal{V}_\rho$ and $\mathcal{V}_{\rho'}$ are equivalent. Then there exists a diffeomorphism Ψ on $\mathbb{S}^n \times \mathbb{S}^1$ such that $\Psi^*\mathcal{V}_\rho = \mathcal{V}_{\rho'}$. It follows that

$$d(z_\rho \circ \Psi) \wedge dz_{\rho'} = 0$$

and so $z_\rho \circ \Psi = H \circ z_{\rho'}$ where $H : R_{\rho'} \longrightarrow R_\rho$ is a conformal map. We know from elementary complex analysis that two such annuli are conformally equivalent if and only $\rho = \rho'$. □

3.3 Structures with Global First Integrals

Consider again the manifold $M = \mathbb{S}^n \times \mathbb{S}^1$ equipped with the Mizohata structure $\mathcal{V}_\rho$ constructed above. Let $a_1, \ldots, a_m \in R_\rho$ (where R_ρ is the annulus defined above), and let $\epsilon_1, \ldots, \epsilon_m > 0$ be such that

$$\overline{\Delta}(a_j, \epsilon_j) \subset R_\rho\,, \quad \overline{\Delta}(a_j, \epsilon_j) \cap \overline{\Delta}(a_k, \epsilon_k) = \emptyset\,, \quad j, k \le m\,,\ j \ne k\,,$$

where $\Delta(a_j, \epsilon_j)$ denotes the disc with center a_j and radius ϵ_j. For each j, let

$$O_j = z_\rho^{-1}(\Delta(a_j, \epsilon_j)) \quad \text{and} \quad N = M \backslash \bigcup_{j=1}^{m} O_j\,.$$

N is an $(n+1)$-dimensional compact manifold with boundary

$$\partial N = \bigcup_{j=1}^{m} \partial O_j = \bigcup_{j=1}^{m} z_\rho^{-1}(\partial \Delta(a_j, \epsilon_j))\,.$$

In N consider the equivalence relation that identifies each fiber of z_ρ that lies on ∂N with a single point. More precisely, if $\overline{x}$ denotes the equivalence class of x, then

$$\overline{x} = \{x\} \quad \text{if } x \notin \partial N \quad \text{and} \quad \overline{x} = z_\rho^{-1}(z_\rho(x)) \quad \text{if } x \in \partial N\,.$$

Denote by $\widetilde{N}$ the quotient space and by q the quotient map. We will equip $\widetilde{N}$ with a differentiable structure in such a way that $\widetilde{N}$ becomes a compact manifold (without boundary) and the function $z_\rho \circ q^{-1}$ generates a Mizohata structure.

First note that $\widetilde{N}$, equipped with the quotient topology, is compact. Since

$$q:\ N\backslash \partial N \longrightarrow \widetilde{N}\backslash q(\partial N)$$

is a homeomorphism, then it induces a C^∞ structure on $\widetilde{N}\backslash q(\partial N)$. Now we show that this structure extends to $q(\partial N)$. For this consider the map

$$W:\ \mathbb{S}^{n-1} \times R_\rho \longrightarrow \mathbb{S}^n \times \mathbb{S}^1$$

$$W(\theta, r\mathrm{e}^{i\sigma}) = \left(\left(\theta \cos \frac{\pi(2r-1-\rho)}{2(1-\rho)},\ \sin \frac{\pi(2r-1-\rho)}{2(1-\rho)}\right),\ \mathrm{e}^{i\sigma}\right)\,.$$

W is a diffeomorphism of $\mathbb{S}^{n-1} \times R_\rho$ onto $(\mathbb{S}^n \backslash \{p_+, p_-\}) \times \mathbb{S}^1$ and we have

$$z_\rho \circ W(\theta, r\mathrm{e}^{i\sigma}) = \left(\frac{1-\rho}{2} \sin \frac{\pi(2r-1-\rho)}{2(1-\rho)} + \frac{1+\rho}{2}\right) \mathrm{e}^{i\sigma}\,.$$

Note that for $r = 1$ or $r = \rho$, we have $z_\rho \circ W(\theta, r\mathrm{e}^{i\sigma}) = r\mathrm{e}^{i\sigma}$. Let $\delta > 0$ be such that

$$\Delta(a_j, \epsilon_j + \delta) \cap \Delta(a_k, \epsilon_k + \delta) = \emptyset \quad \text{if } j \ne k\,,$$

and let

$$A_j = \Delta(a_j, \epsilon_j + \delta)\backslash\Delta(a_j, \epsilon_j) = \{\zeta \in \mathbb{C} : \epsilon_j \le |\zeta| < \epsilon_j + \delta\}.$$

Define the blowing down map g_j as follows:

$$g_j : \mathbb{S}^{n-1} \times A_j \longrightarrow B^n(\delta) \times \mathbb{S}^1 ; \quad g_j(\theta, a_j + r\mathrm{e}^{i\sigma}) = (\theta\sqrt{r - \epsilon_j}, \mathrm{e}^{i\sigma}),$$

where $B^n(\delta) = \{x \in \mathbb{R}^n : |x|^2 < \delta\}$. Note that,

$$g_j : \mathbb{S}^{n-1} \times \big(A_j\backslash\{\zeta : |\zeta - a_j| = \epsilon_j\}\big) \longrightarrow \big(B^n(\delta)\backslash\{0\}\big) \times \mathbb{S}^1$$

is a C^∞ diffeomorphism and that

$$g_j^{-1}(0, \mathrm{e}^{i\sigma}) = \mathbb{S}^{n-1} \times \{(a_j + \epsilon_j)\mathrm{e}^{i\sigma}\}.$$

Let

$$\widetilde{U}_j = q(W(\mathbb{S}^{n-1} \times A_j)).$$

Then $\widetilde{U}_j$ is an open neighborhood of the set $\Sigma_j = q(z_\rho^{-1}(C_j))$ in $\widetilde{N}$, where C_j is the circle $|\zeta - a_j| = \epsilon_j$. Consider the homeomorphism

$$\Psi_j : \widetilde{U}_j \longrightarrow B^n(\delta) \times \mathbb{S}^1; \quad \Psi_j(\overline{m}) = g_j \circ W^{-1} \circ q^{-1}(\overline{m}).$$

It can be verified that Ψ_j is compatible on $\widetilde{U}_j\backslash q(z_\rho^{-1}(C_j))$ with the C^∞ structure defined by q on $\widetilde{N}\backslash q(\partial N)$. Thus, $\widetilde{N}$ is a C^∞ compact manifold without boundary.

Let

$$\widetilde{z}_\rho : \widetilde{N} \longrightarrow R_\rho\backslash \bigcup_{j=1}^{m} \Delta(a_j, \epsilon_j); \quad \widetilde{z}_\rho(\overline{m}) = z_\rho(q^{-1}(\overline{m})).$$

Clearly $\widetilde{z}_\rho$ is well defined, surjective, and

$$d\widetilde{z}_\rho \wedge d\overline{\widetilde{z}_\rho} \ne 0 \quad \text{on} \quad \widetilde{N}\backslash\widetilde{\Sigma}^+ \cup \widetilde{\Sigma}^- \cup \widetilde{\Sigma}_1 \cup \cdots \cup \widetilde{\Sigma}_m,$$

where

$$\widetilde{\Sigma}^\pm = q(\Sigma^\pm) \quad \text{and} \quad \widetilde{\Sigma}_j = q(z_\rho^{-1}(\partial\Delta(a_j, \epsilon_j))),$$

and $d\widetilde{z}_\rho$ defines a Mizohata structure in a neighborhood of $\Sigma^\pm$. It remains to show that $d\widetilde{z}_\rho$ is also of Mizohata type on $\widetilde{\Sigma}_j$. For this, we use the expression of $\widetilde{z}_\rho$ in the chart $(\widetilde{U}_j, \Psi_j)$:

$$\begin{aligned}\widetilde{z}_\rho \circ \Psi_j^{-1}(x, \mathrm{e}^{i\sigma}) &= \widetilde{z}_\rho \circ q \circ W \circ g_j^{-1}(x, \mathrm{e}^{i\sigma}) \\ &= z_\rho \circ W\left(\frac{x}{|x|}\,,\ a_j + (\epsilon_j + |x|^2)\mathrm{e}^{i\sigma}\right) \\ &= a_j + (\epsilon_j + |x|^2)\mathrm{e}^{i\sigma}\,.\end{aligned}$$

Thus $\widetilde{\mathcal{V}}_\rho = \operatorname{span}\{d\widetilde{z}_\rho\}$ is a pseudoconvex Mizohata structure on $\widetilde{N}$ with characteristic set $\widetilde{\Sigma}^+ \cup \widetilde{\Sigma}^- \cup \widetilde{\Sigma}_1 \cup \cdots \cup \widetilde{\Sigma}_m$.

3.4 Structures Without Global First Integrals

Let S be a compact Riemann surface without boundary and let $D_1, \ldots, D_m$ be m open sets in S such that each D_j is diffeomorphic to the unit disc and such that $D_j \cap D_k = \emptyset$ if $j \neq k$. Let

$$S' = S \setminus \bigcup_{j=1}^{m} D_j$$

and let $M = \mathbb{S}^{n-1} \times S'$. Then M is an $(n+1)$-dimensional manifold with boundary

$$\partial M = \mathbb{S}^{n-1} \times \partial S' = \mathbb{S}^{n-1} \times \bigcup_{j=1}^{m} \partial D_j\,.$$

Let $z :\ M \longrightarrow S'$ be defined by $z(\theta, p) = p$. Consider the equivalence relation in M that identifies each fiber of z that lies on ∂M with a single point. That is, if $\overline{(\theta, p)}$ denotes the equivalence class of (θ, p), then

$$\overline{(\theta, p)} = \{(\theta, p)\} \text{ if } p \notin \partial S' \quad \text{and} \quad \overline{(\theta, p)} = \mathbb{S}^{n-1} \times \{p\} \text{ if } p \in \partial S'\,.$$

Let $\widetilde{M}$ be the quotient space and $q : M \longrightarrow \widetilde{M}$ be the quotient map. As in the previous example, $\widetilde{M}$ can be equipped with a differentiable structure so that the map $\widetilde{z} :\ \widetilde{M} \longrightarrow S'$ defined by $\widetilde{z}(\overline{m}) = z \circ q^{-1}(\overline{m})$ generates a pseudoconvex Mizohata structure on $\widetilde{M}$ with characteristic set $\widetilde{\Sigma} = q(\partial M)$.

3.5 Non Pseudoconvex Structures

Let M be compact manifold and let $f :\ M \longrightarrow \mathbb{R}$ be a Morse function. The function

$$z :\ M \times \mathbb{S}^1 \longrightarrow \mathbb{C},\ z(m, \mathrm{e}^{i\sigma}) = f(m)\mathrm{e}^{i\sigma}$$

generates a (not necessarily pseudoconvex) Mizohata structure on $M \times \mathbb{S}^1$ with characteristic set

$$\Sigma = \bigcup_{j=1}^{l} \{m_j\} \times \mathbb{S}^1 ,$$

where $\{m_1, \ldots, m_l\}$ is the set of critical points of f. The signature of the structure along the component $\{m_j\} \times \mathbb{S}^1$ is the signature of the Hessian of f at m_j.

4 The Leaf Space

Let $(M, \mathcal{V})$ be a pseudoconvex Mizohata structure on the $(n+1)$-dimensional compact and connected manifold M with $n \geq 2$. Let

$$\Sigma = \Sigma_1 \cup \cdots \cup \Sigma_k \tag{4.1}$$

be the characteristic set (the Σ_j's are the connected components of Σ). For $m \in M \backslash \Sigma$, denote by L_m the leaf through m of the foliation $\mathcal{F}(\mathcal{V})$ (note that L_m is diffeomorphic to $\mathbb{S}^{n-1}$). For $m \in \Sigma$, we set $L_m = \{m\}$. Consider the equivalence relation $\sim$ on M defined by $m \sim m'$ if and only if $L_m = L_{m'}$. Let $R = M/\sim$ be the quotient space, and let

$$p : M \longrightarrow R \tag{4.2}$$

be the quotient map. The remainder of this section is devoted to the proof of the following theorem.

Theorem 4.1 *The Mizohata structure $\mathcal{V}$ on M induces a structure of Riemann surface with boundary on R. Moreover,*

$$\partial R = p(\Sigma) \quad \textit{and} \quad \pi_1(R) \cong \pi_1(M) ,$$

where $\pi_1(\cdot)$ denotes the fundamental group.

Since the manifold M is compact, then we can find a collection of charts $\{(U_j^0, \phi_j^0\}_{j=1}^K$ and $\{(U_j, \phi_j\}_{j=1}^N$ such that

$$\Sigma \subset \bigcup_{j=1}^{K} U_j^0 , \qquad M = \left(\bigcup_{j=1}^{K} U_j^0 \right) \cup \left(\bigcup_{j=1}^{N} U_j \right) , \tag{4.3}$$

and $\mathcal{V}$ is trivial over each chart. That is, for each j the following holds

$$\phi_j^0 : U_j^0 \longrightarrow (-1, 1) \times B^n , \quad \phi_j : U_j \longrightarrow \Delta \times \mathbb{S}^{n-1} \text{ where}$$

$$\mathcal{V}|_{U_j^0} = \text{span}\{(\phi_j^0)^* d(u + i(t_1^2 + \cdots + t_n^2))\} \text{ and } \mathcal{V}|_{U_j} = \text{span}\{(\phi_j)^* d(x + iy))\} \tag{4.4}$$

and where $(u, t_1, \ldots, t_n)$ are coordinates in $(-1, 1) \times B^n$ and $(x + iy)$ is coordinate in the disc Δ. Hence, it follows at once that we have homeomorphisms

$$p(U_j^0) \cong (-1, 1) \times [0, 1) \quad \text{and} \quad p(U_j) \cong \Delta \, . \tag{4.5}$$

These homeomorphisms are achieved via the maps

$$\psi_j^0(\overline{m}) = z^0(\phi_j^0(p^{-1}(\overline{m}))) \quad \text{and} \quad \psi_j(\overline{m}) = z(\phi_j(p^{-1}(\overline{m}))) \, , \tag{4.6}$$

where

$$z^0(u, t_1, \ldots, t_n) = u + i(t_1^2 + \ldots + t_n^2) \, , \quad \text{and} \quad z(x + iy, \theta) = x + iy \, .$$

It follows that the family of charts $\{p(U_j^0), \psi_j^0\}_{j=1}^K$ and $\{p(U_j), \psi_j\}_{j=1}^N$ defines a complex structure on R. The boundary of R is $p(\Sigma)$. Note that, since p is injective on Σ, then ∂R has k connected components.

Now we proceed to prove that $\pi_1(R) \cong \pi_1(M)$. First we prove some lemmas. Choose points $x_0 \in \Sigma$ and $\overline{x}_0 = p(x_0) \in \partial R$ as base points for loops in M and R, respectively. Consider the space $\mathcal{C}_{\mathcal{F}}(\mathcal{V})$ of loops in M that are transversal to the foliation $\mathcal{F}(\mathcal{V})$, i.e. $\mathcal{C}_{\mathcal{F}}(\mathcal{V})$ consists of smooth loops

$$\gamma : [0, 1] \longrightarrow M, \quad \gamma(0) = \gamma(1) = x_0 \text{ and } \frac{d\gamma}{dt} \notin T_{\gamma(t)} L_{\gamma(t)} \tag{4.7}$$

We have the following lemmas.

Lemma 1 *Let $\sigma : [0, 1] \longrightarrow M$ be a loop based at x_0. Then γ is homotopic to a loop $\widehat{\gamma} \in \mathcal{C}_{\mathcal{F}}(\mathcal{V})$.*

Proof Let γ be a loop in M based at x_0. Since M is smooth with dimension ≥ 3 and Σ is a 1-dimensional submanifold, then we can assume (after replacing γ by a homotopic loop) that γ is smooth, that it is an immersion, and that $\gamma([0, 1]) \cap \Sigma = \{x_0\}$. Divide $[0, 1]$ into subintervals by points

$$t_0 = 0 < t_1 < t_2 < \cdots < t_{p-1} < t_p = 1$$

such that for $i = 2, \ldots, p-1$, the arc $\gamma([t_{i-1}, t_i])$ is contained in a trivial chart of the form U_i (considered above) and the arcs $\gamma([t_0, t_1])$ and $\gamma([t_{p-1}, t_p])$ are contained in a trivial chart $U_1^0 = U_p^0 \ni x_0$. Thus

$$U_1^0 = U_p^0 \cong (-1, 1) \times B^n \quad \text{and} \quad U_i \cong \Delta \times S^{n-1} .$$

Since U_1^0 is simply connected, we can assume that

$$\frac{d\gamma}{dt}(t) \notin T_{\gamma(t)} L_{\gamma(t)} \quad \text{for} \quad 0 \le t \le t_1 \quad \text{and} \quad t_{p-1} \le t \le 1 . \tag{4.8}$$

To continue, we need to consider two cases

Case 1 $n > 2$. In this case each U_i is simply connected and $\gamma : [t_{i-1}, t_i] \longrightarrow U_i$ is homotopic (with fixed endpoints) to an $\mathcal{F}(\mathcal{V})$-transversal path. Therefore, $\gamma : [0, 1] \longrightarrow M$ is homotopic to a loop in $\mathcal{C}_{\mathcal{F}}(\mathcal{V})$.
Case 2 $n = 2$. Consider the arc

$$\gamma : [t_{p-2}, t_{p-1}] \longrightarrow U_{p-1} \cong \Delta \times \mathbb{S}^1 .$$

Note that $\pi_1(U_{p-1}) \cong \mathbb{Z}$. Let $m_i = \gamma(t_i)$ and let

$$c_{p-1} : [t_{p-1}, 1] \longrightarrow U_{p-1}, \text{with } c_{p-1}(t_{p-1}) = m_{p-1}, \; c_{p-1}(1) = m_{p-2}, \tag{4.9}$$

be an arc transversal to the foliation $\mathcal{F}(\mathcal{V})$. Consider the loop

$$\sigma_{p-1} : [t_{p-2}, 1] \longrightarrow U_{p-1} , \quad \sigma_{p-1}(t) = \begin{cases} \gamma(t) & \text{if } t_{p-2} \le t \le t_{p-1} \\ c_{p-1}(t) & \text{if } t_{p-1} < t \le 1 . \end{cases} \tag{4.10}$$

Let $k \in \mathbb{Z}$ be the homotopy class of σ_{p-1}. Since $L_{m_{p-2}}$ ($\cong \mathbb{S}^1$) is a deformation retract of U_{p-1}, then we can find a loop

$$l : [t_{p-2}, r] \longrightarrow L_{m_{p-2}} , \quad l(t_{p-2}) = l(r) = m_{p-2}$$

with homotopy class k (here we are taking $r < t_{p-1}$), and an $\mathcal{F}(\mathcal{V})$-transversal arc

$$c : [r, t_{p-1}] \longrightarrow U_{p-1} , \quad c(r) = m_{p-2}, \quad c(t_{p-1}) = m_{p-1}$$

such that σ_{p-1} is homotopic to the loop $\widehat{\sigma}_{p-1}$ defined by

$$\widehat{\sigma}_{p-1}(t) = \begin{cases} l(t) & \text{if } t_{p-2} \le t \le r \\ c(t) & \text{if } r \le t \le t_{p-1} \\ c_{p-1}(t) & \text{if } t_{p-1} < t \le 1 . \end{cases} \tag{4.11}$$

This means that the loop γ is homotopic to the loop γ_{p-1} defined by

$$\gamma_{p-1}(t) = \begin{cases} \gamma(t) & \text{if } t \in [0, t_{p-2}] \cup [t_{p-1}, 1] \\ \widehat{\sigma}_{p-1}(t) & \text{if } t \in [t_{p-2}, t_{p-1}] . \end{cases} \tag{4.12}$$

Note that the nontransversality (if any) of the path $\gamma_{p-1} : [t_{p-2}, 1] \longrightarrow M$ is carried by the leaf $L_{m_{p-2}}$ and that $L_{m_{p-2}} \subset U_{p-2}$.

We repeat the argument to construct a loop γ_{p-2}:

$$\gamma_{p-2}(t) = \begin{cases} \gamma_{p-1}(t) & \text{if } t \in [0, t_{p-3}] \cup [t_{p-2}, 1] \\ \widehat{\sigma}_{p-2}(t) & \text{if } t \in [t_{p-3}, t_{p-2}] \end{cases} \tag{4.13}$$

such that γ_{p-2} is homotopic to γ_{p-1} and the nontransversality (if any) of the arc $\gamma_{p-2} : [t_{p-3}, 1] \longrightarrow M$ is carried by the leaf $L_{m_{p-3}} \subset U_{p-3}$. By repeating the argument $(p-1)$ times, we can construct a loop γ_1 homotopic to γ and such that the nontransversality of γ_1 is carried by the chart U_1^0. Since U_1^0 is simply connected, then γ_1 is homotopic to a transversal loop.

Lemma 2 *Let $\sigma : [0, 1] \longrightarrow R$ be a smooth loop in R based at $\overline{x}_0$. Then there exists a loop $\widehat{\sigma} \in \mathcal{C}_{\mathcal{F}}(\mathcal{V})$ such that*

$$\widehat{\sigma}(t) = p \circ \sigma(t) \quad t \in [0, 1] . \tag{4.14}$$

Proof We can assume that $\sigma([0, 1]) \cap \partial R = \{\overline{x}_0\}$. Let

$$t_0 = 0 < t_1 < \cdots < t_m < t_{m+1} = 1$$

be such that there exist open sets $D_0, D_1, \ldots D_m = D_0$ in R satisfying $\sigma([t_i, t_{i+1}]) \subset D_i$ and each $U_i = p^{-1}(D_i)$ is a trivial chart for $\mathcal{V}$. Let

$$\begin{gathered} \phi_0 : U_0 \longrightarrow (-1, 1) \times B^n \\ \phi_0(x_0) = (0, 0), \quad \phi_0(\Sigma \cap U_0) = (-1, 1) \times \{0\} \end{gathered} \tag{4.15}$$

be a local trivialization of $\mathcal{V}$ and let

$$S_0 = \phi_0^{-1}\left((-1, 1) \times \{(x_1, 0, \ldots, 0) : 0 \le x_1 < 1\}\right) . \tag{4.16}$$

Then $p : S_0 \longrightarrow D_0$ is a diffeomorphism and S_0 is transversal to $\mathcal{F}(\mathcal{V})$. Let

$$\widehat{\sigma}(t) = p^{-1}(\sigma(t)) \cap S_0 \quad \text{for } 0 \le t \le t_1 . \tag{4.17}$$

The arc $\widehat{\sigma}$ is then transversal to $\mathcal{F}(\mathcal{V})$. Let

$$\phi_1 : U_1 \longrightarrow \Delta \times S^{n-1}$$

be a local trivialization of $\mathcal{V}$. Set

$$(z_1, \theta_1) = \phi_1(\widehat{\sigma}(t_1)) \quad \text{and} \quad S_1 = \phi_1^{-1}(\Delta \times \{\theta_1\}) . \tag{4.18}$$

The surface S_1 is transversal to $\mathcal{F}(\mathcal{V})$ and $p : S_1 \longrightarrow D_1$ is a diffeomorphism. Define

$$\widehat{\sigma}(t) = p^{-1}(\sigma(t)) \cap S_1 \quad \text{for } t_1 \le t \le t_2 . \tag{4.19}$$

The repetition of this construction leads to a path

$$\widehat{\sigma} : [0, 1] \longrightarrow M , \quad p \circ \widehat{\sigma} = \sigma \tag{4.20}$$

such that $\widehat{\sigma}$ is piecewise smooth and transversal to $\mathcal{F}(\mathcal{V})$. Since p is injective on Σ, then $\widehat{\sigma}$ is a loop based at x_0. □

Lemma 3 *Let σ be a loop in R based at $\overline{x}_0$. If $\widehat{\sigma}_1$ and $\widehat{\sigma}_2$ are lifts to M of σ and are transversal to $\mathcal{F}(\mathcal{V})$, then $\widehat{\sigma}_1$ and $\widehat{\sigma}_2$ are homotopic.*

Proof We can find a subdivision of [0, 1]: $t_0 = 0 < t_1 < \cdots < t_m = 1$ and surfaces $S_0^i, \ldots, S_m^i$ $(i = 1, 2)$ in M transversal to $\mathcal{F}(\mathcal{V})$ such that

$$p : S_j^i \longrightarrow D_j = p(S_j^i) , \quad j = 1, \ldots, m, i = 1, 2$$

are diffeomorphisms,

$$\widehat{\sigma}_i(t) = p^{-1}(\sigma(t)) \cap S_j^i \;\; \text{for } t_{j-1} \le t \le t_j$$

and that $\mathcal{V}$ is trivial over $U_j = p^{-1}(D_j)$. Consider a curve of class C^1:

$$l_1 : [0, 1] \longrightarrow p^{-1}(\sigma(t_1)) \quad l_1(0) = \widehat{\sigma}_1(t_1), l_1(1) = \widehat{\sigma}_2(t_1) . \tag{4.21}$$

Define a continuous map

$$\begin{gathered} \widehat{F} : [0, 1] \times [0, t_1] \longrightarrow U_0 \subset M , \\ \widehat{F}(0, t) = \widehat{\sigma}_1(t) , \quad \widehat{F}(1, t) = \widehat{\sigma}_2(t) \quad \text{for } t \le t_1 \\ \widehat{F}(s, 0) = x_0 \;\; \text{and} \;\; \widehat{F}(s, 1) = l_1(s) \quad \text{for } 0 \le s \le 1 . \end{gathered} \tag{4.22}$$

It is clear that this construction can be repeated a number of times to define $\widehat{F}$ on $[0, 1] \times [0, t_{m-1}]$. Finally, we define $\widehat{F}$ on $[0, 1] \times [t_{m-1}, 1]$ in such a way that

$$\begin{gathered} \widehat{F}(0, t) = \widehat{\sigma}_1(t), \quad \widehat{F}(1, t) = \widehat{\sigma}_2(t) \quad \text{for } t_{m-1} \le t \le 1 , \\ \widehat{F}(s, t_{m-1}) = l_{m-1}(s) \;\; \text{and} \;\; \widehat{F}(s, 1) = x_0 \quad \text{for } 0 \le s \le 1 , \end{gathered} \tag{4.23}$$

where $l_{m-1} : [0, 1] \longrightarrow p^{-1}(\sigma(t_{m-1}))$ is a smooth curve. Then, $\widehat{F}$ is a homotopy between $\widehat{\sigma}_1$ and $\widehat{\sigma}_2$.

Using analogous arguments, we can prove the following lemma.

Lemma 4 *If σ_1 and σ_2 are homotopic loops in R (based at $\overline{x}_0$), then their transversal lifts $\widehat{\sigma}_1$ and $\widehat{\sigma}_2$ are homotopic in M.*

The above lemmas show that the projection map $p : M \longrightarrow R$ induces an isomorphism p_* between the homotopy groups $\pi_1(M)$ and $\pi_1(R)$. This completes the proof of Theorem 4.1.

5 Trivialization in a Neighborhood of Σ and Desingularization of $\mathcal{F}(\mathcal{V})$

Let $(M, \mathcal{V})$ be a pseudoconvex Mizohata structure with characteristic set Σ on the $(n+1)$-dimensional compact and connected manifold M. Let Σ_j be a connected component of Σ. In Theorem 5.1, we prove that $\mathcal{V}$ is trivial in a tubular neighborhood of Σ_j. Using this trivialization, we construct an $(n+1)$-dimensional (noncompact) manifold X, a codimension two (nonsingular) foliation $\widehat{\mathcal{F}(\mathcal{V})}$ on X whose leaves are diffeomorphic to $\mathbb{S}^{n-1}$, and a smooth map $\Pi : X \longrightarrow M$ such that Π is a local diffeomorphism outside the preimage of Σ and such that Π sends the leaves of $\widehat{\mathcal{F}(\mathcal{V})}$ to those of $\mathcal{F}(\mathcal{V})$.

Theorem 5.1 *Let $(M, \mathcal{V})$ and Σ_j be as above. Then there exist an open set $V \subset M$ with $\Sigma_j \subset V$ and a diffeomorphism*

$$\Phi : V \longrightarrow \mathbb{S}^1 \times B^n_\epsilon \,,$$

where $\mathbb{S}^1 = \{e^{i\sigma} \in \mathbb{C}, \ \ \sigma \in \mathbb{R}\}$, $B^n_\epsilon = \{t = (t_1, \dots, t_n) \in \mathbb{R}^n, |t| < \epsilon\}$, such that $\mathcal{V}$ is generated, over V, by the exact form

$$\Phi^* d((1 - |t|^2)e^{i\sigma}) \,. \tag{5.1}$$

To prove this theorem, we need a Morse lemma with parameter on $\mathbb{S}^1$.

Lemma 5 *Let $g : \mathbb{R}^n \times \mathbb{S}^1 \longrightarrow \mathbb{R}$ be a C^∞ function satisfying the following*

$(\imath)$ $g(0, e^{i\sigma}) = 0, \forall e^{i\sigma} \in \mathbb{S}^1$;

$(\imath\imath)$ $\dfrac{\partial g}{\partial x_j}(0, e^{i\sigma}) = 0, \forall e^{i\sigma} \in \mathbb{S}^1, \forall j \in \{1, \dots, n\}$; *and*

$(\imath\imath\imath)$ *$Hess(g(0, e^{i\sigma}))$ is positive definite for all $e^{i\sigma} \in \mathbb{S}^1$*

where $Hess(g)$ denotes the Hessian of the function g with respect to the variable $x \in \mathbb{R}^n$. Then, there exists a change of coordinates

$$(y, e^{i\sigma}) = \Psi(x, e^{i\sigma}) = (\psi(x, e^{i\sigma}), e^{i\sigma})$$

defined in a neighborhood of $\{0\} \times \mathbb{S}^1$ *such that*

$$g(\Psi^{-1}(y, e^{i\sigma})) = |y|^2 = y_1^2 + \cdots + y_n^2 \,. \tag{5.2}$$

Proof We use Taylor's formula to write

$$g(x, \mathrm{e}^{i\sigma}) = \sum_{j,k=1}^{n} \int_0^1 \int_0^t \frac{\partial^2 g}{\partial x_j \partial x_k}(sx, \mathrm{e}^{i\sigma}) x_j x_k \, ds \, dt \; = \; x^T H(x, \mathrm{e}^{i\sigma}) x \,, \tag{5.3}$$

where x^T denotes the transpose of the vector $x \in \mathbb{R}^n$ and where $H(x, \mathrm{e}^{i\sigma})$ is the $n \times n$ symmetric matrix

$$\left(\int_0^1 \int_0^t \frac{\partial^2 g}{\partial x_j \partial x_k}(sx, \mathrm{e}^{i\sigma}) \, ds \, dt \right)_{j,k} \,. \tag{5.4}$$

Note that $H(0, \mathrm{e}^{i\sigma})$ has positive eigenvalue for all $\mathrm{e}^{i\sigma}$. Let $R(\mathrm{e}^{i\sigma})$ be a smooth $n \times n$ nonsingular matrix such that

$$R(\mathrm{e}^{i\sigma})^T \; H(0, \mathrm{e}^{i\sigma}) \; R(\mathrm{e}^{i\sigma}) \tag{5.5}$$

has positive and distinct eigenvalues. Such matrix R do exist since the set defined by $\det(R^T HR - \lambda I) = 0$ is an algebraic variety of codimension one in $GL(n, \mathbb{R}) \times \mathbb{R}$ while the set given by the two equations, $\det(R^T HR - \lambda I) = 0$ and $\dfrac{\partial}{\partial \lambda}\det(R^T HR - \lambda I) = 0$ has codimension two. Consider the change of variables

$$z = R(\mathrm{e}^{i\sigma})^{-1} x \,. \tag{5.6}$$

With respect to the variables $(z, \mathrm{e}^{i\sigma})$, the function g has the form

$$g(z, \mathrm{e}^{i\sigma}) = z^T \widehat{H}(z, \mathrm{e}^{i\sigma}) z \,, \tag{5.7}$$

where

$$\widehat{H}(z, \mathrm{e}^{i\sigma}) = R(\mathrm{e}^{i\sigma})^T H(z, \mathrm{e}^{i\sigma}) R(\mathrm{e}^{i\sigma}) \,.$$

Since $\widehat{H}(0, \mathrm{e}^{i\sigma})$ has positive and distinct eigenvalues, then $\widehat{H}(z, \mathrm{e}^{i\sigma})$ has positive and distinct eigenvalues for each $\mathrm{e}^{i\sigma} \in \mathbb{S}^1$ and for $|z|$ small enough. Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of $\widehat{H}(z, \mathrm{e}^{i\sigma})$. Note that since the λ_j's are simple roots of the characteristic polynomial of $\widehat{H}(z, \mathrm{e}^{i\sigma})$, then they depend smoothly on the variable $(z, \mathrm{e}^{i\sigma})$. For each j the eigenspace S_j associated with the eigenvalue λ_j is one-dimensional. Thus the linear equation

$$(\widehat{H} - \lambda_j I) u = 0 \tag{5.8}$$

for $u = (u_1, \ldots, u_n)$ is equivalent (Gaussian elimination) to a system of $(n-1)$ equations of the form

$$\begin{aligned} E_{11}(z, \mathrm{e}^{i\sigma})u_1 + \cdots + E_{1n}(z, \mathrm{e}^{i\sigma})u_n &= 0 \\ &\vdots \\ E_{(n-1)1}(z, \mathrm{e}^{i\sigma})u_1 + \cdots + E_{(n-1)n}(z, \mathrm{e}^{i\sigma})u_n &= 0 \end{aligned} \tag{5.9}$$

where the $(n-1) \times n$ matrix $E = (E_{ij})$ has rank $n-1$. We can therefore take for a nonzero solution of (5.8) the vector u^j whose k-th component is

$$u_k^j = (-1)^{j+1} \det \begin{pmatrix} E_{11} & \cdots & E_{1(k-1)} & E_{1(k+1)} & \cdots & E_{1n} \\ \vdots & & \vdots & \vdots & & \vdots \\ E_{(n-1)1} & \cdots & E_{(n-1)(k-1)} & E_{(n-1)(k+1)} & \cdots & E_{(n-1)n} \end{pmatrix}. \tag{5.10}$$

For each j let $v^j = \dfrac{u^j}{|u^j|}$ and consider the orthogonal matrix

$$P(z, \mathrm{e}^{i\sigma}) = \left(v^1(z, \mathrm{e}^{i\sigma}), \ \ldots \ v^n(z, \mathrm{e}^{i\sigma}) \right) .$$

Clearly for $|z|$ small we have

$$P(z, \mathrm{e}^{i\sigma})^T \widehat{H}(z, \mathrm{e}^{i\sigma}) P(z, \mathrm{e}^{i\sigma}) = D(z, \mathrm{e}^{i\sigma}) , \tag{5.11}$$

where D is the diagonal matrix with the λ_j's on the diagonal. Hence

$$g(z, \mathrm{e}^{i\sigma}) = (P(z, \mathrm{e}^{i\sigma})^T z)^T \, D(z, \mathrm{e}^{i\sigma}) \, P(z, \mathrm{e}^{i\sigma})^T z \, . \tag{5.12}$$

If we let $w = P^T z$, then

$$g(w, \mathrm{e}^{i\sigma}) = \sum_{j=1}^{n} \lambda_j(z, \mathrm{e}^{i\sigma}) w_j^2 \, . \tag{5.13}$$

Finally, with respect to the coordinates $y_j = \sqrt{\lambda_j(z, \mathrm{e}^{i\sigma})}\, w_j$ ($j = 1, \ldots, n$), the function g has the desired form of the lemma. □

Proof of Theorem 5.1 Let $p : M \longrightarrow R$ be the quotient map considered in Sect. 4, $C_j = p(\Sigma_j)$, and let O be a tubular neighborhood of C_j in R such that O is diffeomorphic to $\mathbb{S}^1 \times [0, 1)$. We use the uniformization theorem of the planar domain O (see [13]) to find a conformal map

$$f : O \longrightarrow \mathbb{C} \tag{5.14}$$

such that $f(O) \subset \overline{\Delta}$ and $f : C_j \longrightarrow \partial\Delta$ is a diffeomorphism. Let $\widehat{O} = p^{-1}(O)$. Then $\widehat{O}$ is an open neighborhood of Σ_j in M and

$$\mathcal{V}|_{\widehat{O}} = \text{span}\{\, d(f \circ p)\,\}\,. \tag{5.15}$$

Now let U be a tubular neighborhood of Σ_j contained in $\widehat{O}$ and such that there exists a diffeomorphism

$$\Lambda : \mathbb{S}^1 \times B^n \longrightarrow U\,, \quad \Lambda(\mathbb{S}^1 \times \{0\}) = \Sigma_j\,. \tag{5.16}$$

Consider the function F defined on $\mathbb{S}^1 \times B^n$ by

$$F(\mathrm{e}^{i\sigma}, x) = f \circ p \circ \Lambda(\mathrm{e}^{i\sigma}, x)\,. \tag{5.17}$$

The differential form dF generates a pseudoconvex Mizohata structure on $\mathbb{S}^1 \times B^n$ whose characteristic set is $\mathbb{S}^1 \times \{0\}$. Note that the restriction F_0 of F to $\mathbb{S}^2 \times \{0\}$ is a diffeomorphism onto $\partial\Delta$. Consider the diffeomorphisms

$$\begin{gathered} G : \mathbb{S}^1 \times B^n \longrightarrow \mathbb{S}^1 \times B^n, G(\mathrm{e}^{i\sigma}, x) = (F_0^{-1}(\mathrm{e}^{i\sigma}), x) \quad \text{and} \\ \widehat{\Lambda} : \mathbb{S}^1 \times B^n \longrightarrow U, \widehat{\Lambda} = \Lambda \circ G\,. \end{gathered} \tag{5.18}$$

The differential of the function $\widehat{F} = F \circ G$ generates a pseudoconvex Mizohata structure on $\mathbb{S}^1 \times B^n$ with characteristic set $\mathbb{S}^1 \times \{0\}$ and such that $\widehat{F}(\mathrm{e}^{i\sigma}, 0) = \mathrm{e}^{i\sigma}$. In fact $\widehat{\Lambda}^*\mathcal{V} = \text{span}\{d\widehat{F}\}$. We write

$$\widehat{F}(\mathrm{e}^{i\sigma}, x) = (1 - N(\mathrm{e}^{i\sigma}, x))\mathrm{e}^{i\Theta(\mathrm{e}^{i\sigma}, x)}\,, \tag{5.19}$$

where $N = 1 - |\widehat{F}|$ and Θ is the argument of $\widehat{F}$. It follows from the construction of $\widehat{F}$ that

$$\Theta(\mathrm{e}^{i\sigma}, 0) = \sigma\,, \quad N(\mathrm{e}^{i\sigma}, 0) = 0 \quad \frac{\partial N}{\partial x_j}(\mathrm{e}^{i\sigma}, 0) = 0 \quad (j = 1, \ldots, n) \tag{5.20}$$

and that $\text{Hess}(N)$ is positive definite on $\mathbb{S}^1 \times \{0\}$. We start by taking coordinates $x' = x$, $\sigma' = \Theta(\mathrm{e}^{i\sigma}, x)$ near $\mathbb{S}^1 \times \{0\}$. With respect to these coordinates the function $\widehat{F}$ has the form

$$\widehat{F}(x', \sigma') = (1 - N(x', \sigma'))\mathrm{e}^{i\sigma'}\,.$$

Since the function N satisfies the hypotheses of Lemma 5, then we can find new coordinates x'' and $\sigma'' = \sigma'$ in which $\widehat{F}$ has the form

$$\widehat{F}(x'', \sigma'') = (1 - |x''|)\mathrm{e}^{i\sigma''}\,.$$

This completes the proof of the theorem. □

Theorem 5.2 *Let M, $\mathcal{V}$, and Σ be as above. Then there exist a manifold X, a (nonsingular) codimension two foliation $\widehat{\mathcal{F}}(\mathcal{V})$ on X, and a map $\Pi : X \longrightarrow M$ with the following properties*

(1) $\widehat{\Sigma} = \Pi^{-1}(\Sigma)$ is a compact n-dimensional submanifold of X;
(2) $\Pi : X\backslash\widehat{\Sigma} \longrightarrow M\backslash\Sigma$ is a local diffeomorphism;
(3) $\widehat{\mathcal{F}}(\mathcal{V}) = \Pi^*(\mathcal{F}(\mathcal{V}))$ on $X\backslash\widehat{\Sigma}$; and
(4) there exists a component $\widehat{M}$ of $X\backslash\widehat{\Sigma}$ such that $\Pi : \widehat{M} \longrightarrow M\backslash\Sigma$ is a diffeomorphism.

Proof Suppose that Σ has k components $\Sigma_1, \ldots, \Sigma_k$. It follows from Theorem 5.1 that we can find an open neighborhood U of Σ_1 in M and an embedding

$$\Phi : M \longrightarrow \mathbb{R}^2 \times \mathbb{R}^n \times \mathbb{R}^d$$

such that

$$\begin{gathered} \Phi(U) = \mathbb{S}^1 \times B^n \times \{0\}\,, \quad \Phi(\Sigma_1) = \mathbb{S}^1 \times \{0\} \times \{0\}\,, \\ \mathcal{V}|_U = \operatorname{span}\{\, \Phi^* d((1-|x|^2)\mathrm{e}^{i\sigma})\,\} \quad \text{where } x \in \mathbb{R}^n \text{ and } \mathrm{e}^{i\sigma} \in \mathbb{S}^1 \subset \mathbb{R}^2\,. \end{gathered} \tag{5.21}$$

Furthermore, we can assume (by increasing the dimension d and deforming the mapping Φ if necessary) that

$$\begin{aligned} \Phi(M) \cap (\mathbb{R}^2 \times \{0\} \times \mathbb{R}^d) &= \Phi(U) \cap (\mathbb{R}^2 \times \{0\} \times \mathbb{R}^d) \\ &= \mathbb{S}^1 \times \{0\} \times \{0\}\,. \end{aligned} \tag{5.22}$$

Now consider the map (spherical coordinates in $\mathbb{R}^n$)

$$\begin{gathered} B_1 : \mathbb{R}^2 \times \mathbb{S}^{n-1} \times (-1,\infty) \times \mathbb{R}^d \longrightarrow \mathbb{R}^2 \times \mathbb{R}^n \times \mathbb{R}^d\,, \\ B_1(p,\theta,r,v) = (p, r\theta, v) \end{gathered} \tag{5.23}$$

It is clear that the following restrictions of B_1 are diffeomorphisms

$$\begin{gathered} B_1 : \mathbb{R}^2 \times \mathbb{S}^{n-1} \times (0,\infty) \times \mathbb{R}^d \longrightarrow \mathbb{R}^2 \times (\mathbb{R}^n\backslash\{0\}) \times \mathbb{R}^d\,, \\ B_1 : \mathbb{R}^2 \times \mathbb{S}^{n-1} \times (-1,0) \times \mathbb{R}^d \longrightarrow \mathbb{R}^2 \times (B^n\backslash\{0\}) \times \mathbb{R}^d\,. \end{gathered}$$

Moreover,

$$\begin{gathered} B_1^{-1}(\mathbb{R}^2 \times \{0\} \times \mathbb{R}^d) = \mathbb{R}^2 \times \mathbb{S}^{n-1} \times \{0\} \times \mathbb{R}^d \text{ and} \\ B_1^*((1-|x|^2)\mathrm{e}^{i\sigma}) = (1-|x|^2)\mathrm{e}^{i\sigma}\,. \end{gathered} \tag{5.24}$$

Let

$$\Pi_1 : X_1 = B_1^{-1}(M) \longrightarrow M\,, \quad \Pi_1(p) = \Phi^{-1} \circ B_1(p)\,. \tag{5.25}$$

Then clearly $\Pi_1^{-1}(\Sigma_1)$ is a compact n-dimensional submanifold of X_1 that divides X_1 into two components

$$\begin{aligned} M_1 &= X_1 \cap (\mathbb{R}^2 \times \mathbb{S}^{n-1} \times (0,\infty) \times \mathbb{R}^d) \quad \text{and} \\ U_1 &= X_1 \cap (\mathbb{R}^2 \times \mathbb{S}^{n-1} \times (-1,0) \times \mathbb{R}^d)\,. \end{aligned} \tag{5.26}$$

It follows from the properties of B_1 and Φ that

$$\Pi_1 : M_1 \longrightarrow M \backslash \Sigma_1 \quad \text{and} \quad \Pi_1 : U_1 \longrightarrow U \backslash \Sigma_1 \tag{5.27}$$

are diffeomorphisms and that the lift via Π_1 of $\mathcal{F}(\mathcal{V})$ is defined near $\Pi_1^{-1}(\Sigma_1)$ by the nonsingular differential forms $\{d\sigma, dr\}$ (see (5.21) and (5.24)). Thus the map Π_1^{-1} desingularizes the foliation along Σ_1 and $\mathcal{V}^* = \Pi_1^*(\mathcal{V})$ defines a pseudoconvex Mizohata structure on M_1 whose characteristic set is

$$\Pi_1^{-1}(\Sigma_2) \cup \cdots \cup \Pi_1^{-1}(\Sigma_k)\,. \tag{5.28}$$

We repeat this construction $(k-1)$ times to produce a manifold X and foliation $\widehat{\mathcal{F}}(\mathcal{V})$ satisfying the conditions of the theorem. □

Remark 5.1 Let $\widehat{R}$ be the leaf space of $\widehat{\mathcal{F}}(\mathcal{V})$ and $\widehat{p} : X \longrightarrow \widehat{R}$ be the quotient map. Then $\widehat{R}$ is a Riemann surface (without boundary) and $\widehat{p} : X \longrightarrow \widehat{R}$ is an $\mathbb{S}^{n-1}$-bundle. It follows from Theorem 5.2 that R (leaf space of $\mathcal{F}(\mathcal{V})$) can be viewed as a submanifold of $\widehat{R}$ and $\pi_1(\widehat{R}) = \pi_1(R)$. Another way of viewing the relationship between $\widehat{R}$ and R is by considering the compact Riemann surface (without) boundary $\widetilde{R}$ obtained by gluing two copies of R along their boundaries. The surface $\widehat{R}$ is then an ϵ-neighborhood of R in $\widetilde{R}$.

6 Pseudoconvex Mizohata Structures on Manifolds with Abelian Fundamental Groups

We say that two Mizohata structures $(M, \mathcal{V})$ and $(M', \mathcal{V}')$ are equivalent if there exists a diffeomorphism $\Phi : M \longrightarrow M'$ such that $\Phi^*\mathcal{V}' = \mathcal{V}$. In this section, we prove that the only simply connected compact manifold capable of carrying a pseudoconvex Mizohata structure is the sphere $\mathbb{S}^{n+1}$ and that the structure is equivalent to the standard structure given in Sect. 3.4 We also prove that if M has a nontrivial abelian fundamental group and carries a pseudoconvex Mizohata structure, then it is equivalent to $(\mathbb{S}^1 \times \mathbb{S}^n, \mathcal{V}_\rho)$ where $\mathcal{V}_\rho$ (with $0 < \rho < 1$) is the Mizohata structure given in Sect. 3.2.

Theorem 6.1 *Let* $(M, \mathcal{V})$ *be a pseudoconvex Mizohata structure on the* $(n+1)$-*dimensional compact manifold* M. *Then* $(M, \mathcal{V})$ *is equivalent to* $(\mathbb{S}^{n+1}, \mathcal{V}_0)$, *where* $\mathcal{V}_0$ *is the standard structure on the sphere* $\mathbb{S}^{n+1}$ *given in Sect. 3.1.*

Theorem 6.2 *Let $(M, \mathcal{V})$ be a pseudoconvex Mizohata structure on the $(n+1)$-dimensional compact manifold M. If $\pi_1(M)$ is nontrivial and abelian, then $(M, \mathcal{V})$ is equivalent to the structure $(\mathbb{S}^1 \times \mathbb{S}^n, \mathcal{V}_\rho)$ for some $0 < \rho < 1$, where $\mathcal{V}_\rho$ is the structure defined in Sect. 3.2.*

Proof of Theorem 6.1 Let $\Pi : X \longrightarrow M$ be the desingularization map of $\mathcal{F}(\mathcal{V})$ (see Theorem 5.2) and let $p : M \longrightarrow R$, $\widehat{p} : X \longrightarrow \widehat{R}$ be the quotient maps onto the leaf spaces. Since M is simply connected, then

$$\pi_1(R) \cong \pi_1(\widehat{R}) \cong \pi_1(M) \cong 0\,, \tag{6.1}$$

(see Theorem 4.1 and Remark 5.1). Thus $\widehat{R}$ and R are simply connected Riemann surfaces (R with boundary). Since $(X, \widehat{R}, \widehat{p})$ is a fiber bundle with fiber $\mathbb{S}^{n-1}$ and since $\widehat{R}$ is contractible, then there exists a diffeomorphism

$$\widehat{\Phi} : X \longrightarrow \widehat{R} \times \mathbb{S}^{n-1} \tag{6.2}$$

such that $\widehat{p}_1 \circ \widehat{\Phi} = \widehat{p}$, where $\widehat{p}_1 : \widehat{R} \times \mathbb{S}^{n-1} \longrightarrow \widehat{R}$ is the projection map. Let $\widehat{\Sigma} = \Pi^{-1}(\Sigma)$ and $\widehat{M} \subset X$ be such that $\Pi : \widehat{M} \longrightarrow M \backslash \Sigma$ is a diffeomorphism (Theorem 5.2). When R is considered a submanifold of $\widehat{R}$, the map

$$\widehat{\Phi} : \widehat{M} \cup \widehat{\Sigma} \longrightarrow R \times \mathbb{S}^{n-1}$$

satisfies $\widehat{p}_1 \circ \widehat{\Phi} = \widehat{p}$. Let

$$f : R \longrightarrow \overline{\Delta} = \{\, \zeta \in \mathbb{C},\ |\zeta| \leq 1 \,\} \tag{6.3}$$

be a conformal mapping and let

$$F : R \times \mathbb{S}^{n-1} \longrightarrow \overline{\Delta} \times \mathbb{S}^{n-1}\,, \quad F(p, \theta) = (f(p), \theta). \tag{6.4}$$

Consider the maps

$$\begin{aligned} \widehat{\Psi} : \widehat{M} \cup \widehat{\Sigma} &\longrightarrow \overline{\Delta} \times \mathbb{S}^{n-1}\,, \quad \widehat{\Psi} = F \circ \widehat{\Phi} \\ \widetilde{p} : \widehat{M} \cup \widehat{\Sigma} &\longrightarrow \overline{\Delta}\,, \quad \widetilde{p} = f \circ \widehat{p}\,. \end{aligned} \tag{6.5}$$

It follows that $\widetilde{p} = p_1 \circ \widehat{\Psi}$, where $p_1 : \overline{\Delta} \times \mathbb{S}^{n-1} \longrightarrow \overline{\Delta}$ is the projection map. Now consider the (blowing down) map

$$B_0 : \overline{\Delta} \times \mathbb{S}^{n-1} \longrightarrow \mathbb{S}^{n+1}\,, \quad B_0(r\mathrm{e}^{i\sigma}, \theta) = \left(\mathrm{e}^{i\sigma} \sin \frac{\pi r}{2}\,, \, \theta \cos \frac{\pi r}{2}\right). \tag{6.6}$$

It can be verified that the map Ψ in the following commutative diagram is a diffeomorphism

$$\begin{array}{ccc} \widehat{M} \cup \widehat{\Sigma} & \xrightarrow{\widehat{\Psi}} & \overline{\Delta} \times \mathbb{S}^{n-1} \\ \Pi \big\downarrow & & \big\downarrow B_0 \\ M & \xrightarrow{\Psi} & \mathbb{S}^{n+1} \end{array} \tag{6.7}$$

The bundle $\mathcal{V}' = (\Psi^{-1})^*\mathcal{V}$ defines a pseudoconvex Mizohata structure on $\mathbb{S}^{n+1}$ whose characteristic set is the equatorial circle

$$\Sigma_0 = \{ (x_1, x_2, 0, \ldots, 0) \in \mathbb{S}^{n+1} \}.$$

To complete the proof of the theorem, we need to show that $(\mathbb{S}^{n+1}, \mathcal{V}')$ is equivalent to the standard structure $(\mathbb{S}^{n+1}, \mathcal{V}_0)$. Note that the codimension two foliations $\mathcal{F}(\mathcal{V}_0)$ and $\mathcal{F}(\mathcal{V}')$ defined by $\mathcal{V}_0$ and $\mathcal{V}'$ have the same leaves. Let $\mathcal{F}^T$ be the transversal foliation defined on $\mathbb{S}^{n+1}$ as follows. For $\zeta_0 \in \overline{\Delta}$ and $\theta_0 \in \mathbb{S}^{n-1}$ define the leaf of $\mathcal{F}^T$ through the point $B_0(\zeta_0, \theta_0)$ to be

$$L_{B_0(\zeta_0,\theta_0)} = B_0(\overline{\Delta} \times \{\theta_0\}). \tag{6.8}$$

Finally, the diffeomorphism

$$G : \mathbb{S}^{n+1} \longrightarrow \mathbb{S}^{n+1}, \quad G(m) = L_m \cap z_0^{-1}(z(m)), \tag{6.9}$$

where

$$z_0 : \mathbb{S}^{n+1} \longrightarrow \overline{\Delta} \quad \text{and} \quad z : \mathbb{S}^{n+1} \longrightarrow \overline{\Delta}$$

are first integrals of $\mathcal{V}_0$ and $\mathcal{V}'$, respectively, satisfies $G^*\mathcal{V}_0 = \mathcal{V}'$. □

Proof of Theorem 6.2 Let $\Pi : X \longrightarrow M$ be the desingularization map of $\mathcal{F}(\mathcal{V})$ and $p : M \longrightarrow R$, $\widehat{p} : X \longrightarrow \widehat{R}$ be the quotient maps onto the leaf spaces. Since any Riemann surface with a nontrivial fundamental group is a topological cylinder, then it follows from Theorem 4.1 that

$$\pi_1(M) \cong \pi_1(R) \cong \pi_1(\widehat{R}) \cong \mathbb{Z}. \tag{6.10}$$

The characteristic set of $\mathcal{V}$ has two connected components Σ^+ and Σ^-. We use the uniformization theorem (see [13]) to find a conformal mapping

$$f : R \longrightarrow R_\rho = \{ \zeta \in \mathbb{C} : \ \rho \le |\zeta| \le 1 \} \tag{6.11}$$

for some unique $\rho \in (0, 1)$. When the surface R is viewed as a submanifold (with boundary) of $\widehat{R}$ (see Remark 5.1), the map f can be extended to a diffeomorphism

$$\widehat{f} : \widehat{R} \longrightarrow R_{\rho,\epsilon} = \{\, \zeta \in \mathbb{C} : \ \rho - \epsilon < |\zeta| < 1 + \epsilon \,\}, \tag{6.12}$$

for some $\epsilon > 0$, such that $\widehat{f} = f$ on R. Note that we are not imposing on $\widehat{f}$ to be conformal on $\widehat{R} \backslash R$. Let

$$T : X \longrightarrow (\rho - \epsilon \ 1 + \epsilon), \quad T(x) = |\widehat{f}(\widehat{p}(x))| . \tag{6.13}$$

Then T is a submersion and

$$T^{-1}(r) \cong \mathbb{S}^1 \times \mathbb{S}^{n-1} \quad \forall r \in (\rho - \epsilon, 1 + \epsilon) . \tag{6.14}$$

Thus T generates a fiber bundle on the (contractible) interval $(\rho - \epsilon \ 1 + \epsilon)$. Therefore there exists a diffeomorphism

$$\widehat{\Psi} : X \longrightarrow (\rho - \epsilon, 1 + \epsilon) \times \mathbb{S}^1 \times \mathbb{S}^{n-1} \tag{6.15}$$

such that $T = p_1 \circ \widehat{\Psi}$, where

$$p_1 : (\rho - \epsilon, 1 + \epsilon) \times \mathbb{S}^1 \times \mathbb{S}^{n-1} \longrightarrow (\rho - \epsilon, 1 + \epsilon)$$

is the projection map. Let $\widehat{M} \subset X$ be such that $\Pi : \widehat{M} \longrightarrow M \backslash \Sigma$ is a diffeomorphism (Theorem 5.2) and let $\widehat{\Sigma}^{\pm} = \Pi^{-1}(\Sigma^{\pm})$. It follows from the above discussion that the following diagram commutes

$$\begin{array}{ccc} \widehat{M} \cup \widehat{\Sigma}^{+} \cup \widehat{\Sigma}^{-} & \xrightarrow{\ \widehat{\Psi}\ } & [\rho, 1] \times \mathbb{S}^1 \times \mathbb{S}^{n-1} \\ {\scriptstyle p \circ \Pi} \big\downarrow & & \big\downarrow {\scriptstyle p_1} \\ R & \xrightarrow{\ f\ } & R_{\rho} \end{array} \tag{6.16}$$

and that

$$\widehat{\Psi}(\widehat{\Sigma}^{+}) = \{1\} \times \mathbb{S}^1 \times \mathbb{S}^{n-1}, \quad \widehat{\Psi}(\widehat{\Sigma}^{-}) = \{\rho\} \times \mathbb{S}^1 \times \mathbb{S}^{n-1},$$

where the map p_1 is defined by $p_1(r, \mathrm{e}^{i\sigma}, \theta) = r\mathrm{e}^{i\sigma}$. Let

$$\begin{gathered} B_0 : [\rho, 1] \times \mathbb{S}^1 \times \mathbb{S}^{n-1} \longrightarrow \mathbb{S}^1 \times \mathbb{S}^n , \\ B_0(r, \mathrm{e}^{i\sigma}, \theta) = \left(\mathrm{e}^{i\sigma} , \left(\theta \cos \frac{(2r - 1 - \rho)\pi}{2 - 2\rho} , \sin \frac{(2r - 1 - \rho)\pi}{2 - 2\rho} \right) \right) . \end{gathered} \tag{6.17}$$

It can be verified that the map

$$\Psi : M \longrightarrow \mathbb{S}^1 \times \mathbb{S}^n , \quad \Psi(m) = B_0 \circ \widehat{\Psi} \circ \Pi^{-1}(m) \tag{6.18}$$

is a diffeomorphism and that $\mathcal{V}' = (\Psi^{-1})^*\mathcal{V}$ is a pseudoconvex Mizohata structure on $\mathbb{S}^1 \times \mathbb{S}^n$ with characteristic set

$$\Sigma = \Sigma_0^+ \cup \Sigma_0^-, \text{with } \Sigma_0^\pm = \mathbb{S}^1 \times \{A^\pm\}, \text{ and } A^\pm = (0, \ldots, 0, \pm 1) \in \mathbb{S}^n .$$

To prove that $(\mathbb{S}^1 \times \mathbb{S}^n, \mathcal{V}')$ is equivalent to the standard structure $(\mathbb{S}^1 \times \mathbb{S}^n, \mathcal{V}_\rho)$ (of Sect. 3.2), we can use an argument similar to that used at the end of the proof of Theorem 6.1. □

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