Minimal Realizations of Syndrome Formers of a Special Class of 2D Codes

Ettore Fornasini, Telma Pinho, Raquel Pinto, and Paula Rocha

Abstract In this paper we consider a special class of 2D convolutional codes (composition codes) with encoders $G(d_1, d_2)$ that can be decomposed as the product of two 1D encoders, i.e., $G(d_1, d_2) = G_2(d_2)G_1(d_1)$. In case that $G_1(d_1)$ and $G_2(d_2)$ are prime we provide constructions of syndrome formers of the code, directly from $G_1(d_1)$ and $G_2(d_2)$. Moreover we investigate the minimality of 2D state-space realization by means of a separable Roesser model of syndrome formers of composition codes, where $G_2(d_2)$ is a quasi-systematic encoder.

Keywords Encoders and syndrome forms • 2D composition codes • 2D state-space models

1 Introduction and Preliminary Concepts

Minimal state-space realization of convolutional codes play an important role in efficient code generation and verification. This question has been widely investigated in the literature for 1D codes [3, 6], however it is still open for the 2D case. Preliminary results concerning 2D encoder and code realizations have been presented in [10]. In this paper we study the syndrome former realization problem for a special class of 2D codes.

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We consider 2D convolutional codes constituted by sequences indexed by $\mathbb{Z}^2$ and taking values in $\mathbb{F}^n$, where $\mathbb{F}$ is a field. Such sequences $\{w(i,j)\}_{(i,j)\in\mathbb{Z}^2}$ can be represented by bilateral formal power series

$$\hat{w}(d_1,d_2) = \sum_{(i,j)\in\mathbb{Z}^2} w(i,j)d_1^i d_2^j.$$

For $n \in \mathbb{N}$, the set of 2D bilateral formal power series over $\mathbb{F}^n$ is denoted by $\mathscr{F}_{2D}^n$. This set is a module over the ring $\mathbb{F}[d_1,d_2]$ of 2D polynomials over $\mathbb{F}$. The set of matrices of size $n \times k$ with elements in $\mathbb{F}[d_1,d_2]$ will be denoted by $\mathbb{F}^{n\times k}[d_1,d_2]$.

Given a subset $\mathscr{C}$ of sequences indexed by $\mathbb{Z}^2$, taking values in $\mathbb{F}^n$, we denote by $\hat{\mathscr{C}}$ the subset of $\mathscr{F}_{2D}^n$ defined by $\hat{\mathscr{C}} = \{\hat{w} \mid w \in \mathscr{C}\}$.

Definition 1 A 2D convolutional code is a subset $\mathscr{C}$ of sequences indexed by $\mathbb{Z}^2$ such that $\hat{\mathscr{C}}$ is a submodule of $\mathscr{F}_{2D}^n$ which coincides with the image of $\mathscr{F}_{2D}^k$ (for some $k \in \mathbb{N}$) by a polynomial matrix $G(d_1,d_2)$, i.e.,

$$\hat{\mathscr{C}} = \operatorname{im} G(d_1,d_2) = \{\hat{w}(d_1,d_2) \mid \hat{w}(d_1,d_2) = G(d_1,d_2)\hat{u}(d_1,d_2),\ \hat{u}(d_1,d_2) \in \mathscr{F}_{2D}^k\}.$$

It follows, as a consequence of [Theorem 2.2, [7]], that a 2D convolutional code can always be given as the image of a full column rank polynomial matrix $G(d_1,d_2) \in \mathbb{F}^{n\times k}[d_1,d_2]$. Such polynomial matrix is called an *encoder* of $\mathscr{C}$. A code with encoders of size $n \times k$ is said to have rate k/n.

A 2D convolutional code $\mathscr{C}$ of rate k/n can also be represented as the kernel of a $(n-k)\times n$ left-factor prime polynomial matrix (i.e. a matrix without left nonunimodular factors), as follows from [Theorem 1, [12]].

Definition 2 Let $\mathscr{C}$ be a 2D convolutional code of rate k/n. A left-factor prime matrix $H(d_1,d_2) \in \mathbb{F}^{(n-k)\times n}[d_1,d_2]$ such that

$$\hat{\mathscr{C}} = \ker H(d_1,d_2),$$

is called a syndrome former of $\mathscr{C}$.

Note that w is in $\mathscr{C}$ if and only if $H(d_1,d_2)\hat{w} = 0$.

Remark 3 This means that whereas codewords are output sequences of an encoder, they constitute the *output-nulling inputs* of a syndrome former of the code.

Given an encoder $G(d_1,d_2)$ of $\mathscr{C}$, a syndrome former of $\mathscr{C}$ can be obtained by constructing a $(n-k)\times n$ left-factor prime matrix $H(d_1,d_2)$ such that $H(d_1,d_2)G(d_1,d_2) = 0$. Moreover all syndrome formers of $\mathscr{C}$ are of the form $U(d_1,d_2)H(d_1,d_2)$, where $U(d_1,d_2) \in \mathbb{F}^{(n-k)\times(n-k)}[d_1,d_2]$ is unimodular.

2 Composition Codes and Their Syndrome Formers

In this section we consider a particular class of 2D convolutional codes generated by 2D polynomial encoders that are obtained from the composition of two 1D polynomial encoders. Such encoders/codes will be called *composition encoders/codes*. Our goal is to characterize the syndrome formers of such codes. The formal definition of composition encoders is as follows.

Definition 4 An encoder $G(d_1, d_2) \in \mathbb{F}^{n\times k}[d_1, d_2]$ such that

$$G(d_1, d_2) = G_2(d_2)G_1(d_1), \tag{1}$$

where $G_1(d_1) \in \mathbb{F}^{p\times k}[d_1]$ and $G_2(d_2) \in \mathbb{F}^{n\times p}[d_2]$ are 1D encoders, is said to be a composition encoder.

Note that the requirement that $G_i(d_i)$, for $i = 1, 2$, is a 1D encoder implies the condition that $G_i(d_i)$ is a full column rank matrix. Moreover this requirement clearly implies that $G_2(d_2)G_1(d_1)$ has full column rank, hence the composition $G_2(d_2)G_1(d_2)$ of two 1D encoders is indeed a 2D encoder.

The 2D composition code $\mathscr{C}$ associated with $G(d_1, d_2)$ is such that

$$\begin{aligned}\hat{\mathscr{C}} &= \operatorname{im} G(d_1, d_2) = G_2(d_2)(\operatorname{im} G_1(d_1))\\ &= \{\hat{w}(d_1, d_2) \mid \exists\, \hat{z}(d_1, d_2) \in \operatorname{im}(G_1(d_1)) \text{ such that } \hat{w}(d_1, d_2) = G_2(d_2)\hat{z}(d_1, d_2)\}.\end{aligned}$$

We shall concentrate on a particular class of composition codes, namely on those that admit a composition encoder $G(d_1, d_2)$ as in (1) with $G_2(d_2)$ and $G_1(d_1)$ both right-prime encoders (i.e., they admit a left polynomial inverse), and derive a procedure for constructing the corresponding syndrome formers based on 1D polynomial methods. This procedure will be useful later on for the study of state-space realizations.

It is important to observe that as $G_2(d_2)$ and $G_1(d_1)$ are both assumed to have polynomial inverses, then $G(d_1, d_2)$ also has a 2D polynomial left inverse (given by the product of the left inverses of $G_1(d_1)$ and $G_2(d_2)$) and therefore $G(d_1, d_2)$ is right-zero prime[1](*rZP*). Recall that if a 2D convolutional code admits a right-zero prime encoder then all its *rFP* encoders are *rZP*. Moreover, the corresponding syndrome formers are also *lZP* (see Prop. A.4 of [4]).

[1]A polynomial matrix $G(d_1, d_2)$ is right/left-zero prime (*rZP/lZP*) if the ideal generated by the maximal order minors of $G(d_1, d_2)$ is the ring $\mathbb{F}[d_1, d_2]$ itself, or equivalently if and only if admits a polynomial left/right inverse. Moreover right/left-zero primeness implies right/left-factor primeness(*rFP/lFP*).

Since $G_2(d_2) \in \mathbb{F}^{n\times p}[d_2]$ is right-prime there exists a unimodular matrix $U(d_2) \in \mathbb{F}^{n\times n}[d_2]$ such that

$$U(d_2)G_2(d_2) = \begin{bmatrix} I_p \\ 0 \end{bmatrix}.$$

We shall partition $U(d_2)$ as

$$U(d_2) = \begin{bmatrix} L_2(d_2) \\ H_2(d_2) \end{bmatrix}, \tag{2}$$

where $L_2(d_2)$ has p rows.

It is easy to check that, if $H_1(d_1) \in \mathbb{F}^{(p-k)\times p}[d_1]$ is a syndrome former of the 1D convolutional code $\operatorname{im} G_1(d_1)$ (i.e., $H_1(d_1)$ is left-prime and is such that $H_1(d_1)G_1(d_1) = 0$), then

$$\begin{bmatrix} H_1(d_1)L_2(d_2) \\ H_2(d_2) \end{bmatrix} G_2(d_2)G_1(d_1) = 0. \tag{3}$$

This reasoning leads to the following proposition.

Proposition 5 *Let $\mathscr{C}$, with $\hat{\mathscr{C}} = \operatorname{im} G(d_1,d_2)$, be a composition code with $G(d_1,d_2) \in \mathbb{F}^{n\times k}[d_1,d_2]$ such that $G(d_1,d_2) = G_2(d_2)G_1(d_1)$, where $G_2(d_2) \in \mathbb{F}^{n\times p}[d_2]$ and $G_1(d_1) \in \mathbb{F}^{p\times k}[d_1]$ are both right-prime 1D encoders. Let further $H_1(d_1)$ be a $(p-k)\times p$ 1D syndrome former of $\operatorname{im} G_1(d_1)$ and define $\begin{bmatrix} L_2(d_2) \\ H_2(d_2) \end{bmatrix}$ as in (2). Then*

$$H(d_1,d_2) = \begin{bmatrix} H_1(d_1)L_2(d_2) \\ H_2(d_2) \end{bmatrix}$$

is a syndrome former of $\mathscr{C}$.

Proof Since (3) is obviously satisfied and $H(d_1,d_2)$ has size $(n-k)\times n$, we only have to prove that $H(d_1,d_2)$ is left-factor prime. Note that as $H_1(d_1)$ is left-prime, there exists $R_1(d_1) \in \mathbb{F}^{p\times(p-k)}[d_1]$ such that $H_1(d_1)R_1(d_1) = I_{p-k}$. Now it is easy to see that

$$R(d_1,d_2) = U(d_2)^{-1} \begin{bmatrix} R_1(d_1) & 0 \\ 0 & I_{n-p} \end{bmatrix}.$$

constitutes a polynomial right inverse of $H(d_1,d_2)$. Consequently $H(d_1,d_2)$ is left-zero prime which implies that it is left-factor prime as we wish to prove. □

3 State-Space Realizations of Encoders and Syndrome Formers

In this section we recall some fundamental concepts concerning 1D and 2D state-space realizations of transfer functions, having in mind the realizations of encoders and syndrome formers.

A 1D state-space model

$$\begin{cases} x(t+1) = Ax(t) + Bu(t) \\ w(t) = Cx(t) + Du(t) \end{cases}$$

denoted by $\Sigma^{1D}(A, B, C, D)$ is a realization of dimension m of $M(d) \in \mathbb{F}^{s\times r}[d]$ if $M(d) = C(I_m - Ad)^{-1}Bd + D$. Moreover, it is a minimal realization if the size of the state x is minimal among all the realizations of $M(d)$. The dimension of a minimal realization of $M(d)$ is called the *McMillan degree* of $M(d)$ and is given by $\mu(M) = \operatorname{int deg} \begin{bmatrix} M(d) \\ I_r \end{bmatrix}$, where $\operatorname{int deg} M(d)$ is the maximum degree of its r-order minors [11].

As for the 2D case, there exist several types of state-space models [1, 2]. In our study we shall consider *separable Roesser models* [13]. These models have the following form:

$$\begin{cases} x_1(i+1, j) = A_{11}x_1(i, j) + A_{12}x_2(i, j) + B_1u(i, j) \\ x_2(i, j+1) = A_{21}x_1(i, j) + A_{22}x_2(i, j) + B_2u(i, j) \\ y(i, j) = C_1x_1(i, j) + C_2x_2(i, j) + Du(i, j) \end{cases} \tag{4}$$

where A_{11}, A_{12}, A_{21}, A_{22}, B_1, B_2, C_1, C_2 and D are matrices over $\mathbb{F}$, with suitable dimensions, u is the input-variable, y is the output-variable, and $x = (x_1, x_2)$ is the state variable where x_1 and x_2 are the horizontal and the vertical state-variables, respectively. The dimension of the system described by (4) is given by the size of x. Moreover either $A_{12} = 0$ or $A_{21} = 0$. The separable Roesser model corresponding to Eqs. (4) with $A_{12} = 0$ is denoted by $\Sigma_{12}^{2D}(A_{11}, A_{21}, A_{22}, B_1, B_2, C_1, C_2, D)$, whereas the one with $A_{21} = 0$ is denoted by $\Sigma_{21}^{2D}(A_{11}, A_{12}, A_{22}, B_1, B_2, C_1, C_2, D)$.

The remaining considerations of this section can be stated for both cases when $A_{12} = 0$ or $A_{21} = 0$, however we just consider $A_{12} = 0$; the case $A_{21} = 0$ is completely analogous, with the obvious adaptations.

Definition 6 $\Sigma_{12}^{2D}(A_{11}, A_{21}, A_{22}, B_1, B_2, C_1, C_2, D)$ is said to be a realization of the 2D polynomial matrix $M(d_1, d_2) \in \mathbb{F}^{s\times r}[d_1, d_2]$ if

$$M(d_1, d_2) = \begin{bmatrix} C_1 & C_2 \end{bmatrix} \begin{bmatrix} I - A_{11}d_1 & 0 \\ -A_{21}d_2 & I - A_{22}d_2 \end{bmatrix}^{-1} \left(\begin{bmatrix} B_1 \\ 0 \end{bmatrix} d_1 + \begin{bmatrix} 0 \\ B_2 \end{bmatrix} d_2 \right) + D.$$

As it is well known different realizations of $M(d_1, d_2)$ may not have the same dimension. For the sake of efficient implementation, we are interested in studying the realizations of $M(d_1, d_2)$ with minimal dimension. Such realizations are called *minimal*. The *Roesser McMillan degree* of $M(d_1, d_2)$, $\mu_R(M)$, is defined as the dimension of a minimal realization of $M(d_1, d_2)$.

Note that every polynomial matrix $M(d_1, d_2) \in \mathbb{F}^{s\times r}[d_1, d_2]$ can be factorized as follows:

$$M(d_1, d_2) = M_2(d_2)M_1(d_1), \tag{5}$$

where $M_2(d_2) = \left[I_n \mid \cdots \mid I_n d_2^{\ell_2}\right] N_2 \in \mathbb{F}^{s\times p}[d_2]$ and $M_1(d_1) = N_1 \left[I_k \ \ldots \ I_k d_1^{\ell_1}\right]^T \in \mathbb{F}^{p\times r}[d_1]$, with N_2 and N_1 constant matrices.

If N_2 has full column rank and N_1 has full row rank we say that (5) is an *optimal decomposition* of $M(d_1, d_2)$. As shown in [8, 9], if (5) is an optimal decomposition, given a minimal realization $\Sigma^{1D}(A_{11}, B_1, \bar{C}_1, \bar{D}_1)$ of $M_1(d_1)$ (of dimension $\mu(M_1)$) and a minimal realization $\Sigma^{1D}(A_{22}, \bar{B}_2, C_2, \bar{D}_2)$ of $M_2(d_2)$ (of dimension $\mu(M_2)$) then the 2D system $\Sigma_{12}^{2D}(A_{11}, A_{21}, A_{22}, B_1, B_2, C_1, C_2, D)$, where $A_{21} = \bar{B}_2\bar{C}_1$, $B_2 = \bar{B}_2\bar{D}_1$, $C_1 = \bar{D}_2\bar{C}_1$ and $D = \bar{D}_2\bar{D}_1$, is a minimal realization of $M(d_1, d_2)$ of dimension $\mu_R(M) = \mu(M_1) + \mu(M_2)$. A similar reasoning can be made if we factorize $M(d_1, d_2) = \bar{M}_1(d_1)\bar{M}_2(d_2)$, where $\bar{M}_1(d_1) \in \mathbb{F}^{s\times \bar{p}}[d_1]$ and $\bar{M}_2(d_2) \in \mathbb{F}^{\bar{p}\times r}[d_2]$, for some $p \in \mathbb{N}$, to obtain a minimal realization $\Sigma_{21}^{2D}(A_{11}, A_{12}, A_{22}, B_1, B_2, C_1, C_2, D)$ of $M(d_1, d_2)$.

Note that, since both encoders and syndrome formers are (2D) polynomial matrices, they both can be realized by means of (4). However, when considering realizations of an encoder $G(d_1, d_2) = G_2(d_2)G_1(d_1)$ we shall take $A_{12} = 0$ and $y = w$; on the other hand when considering realizations of a syndrome former $H(d_1, d_2) = H_1(d_1)H_2(d_2)$, we shall take $A_{21} = 0$, $u = w$ and $y = 0$, (cf. Remark 3).

4 Minimal Syndrome Former Realizations of a Special Class of Composition Codes

In the sequel the composition codes $\mathscr{C}$ to be considered are such that $\hat{\mathscr{C}} = \operatorname{im} G(d_1, d_2)$, where the encoder $G(d_1, d_2)$ is as in (1) and satisfies the following properties:

(P1) $G_1(d_1)$ is a minimal 1D polynomial encoder[2] (for instance, prime and column reduced[3]), with full row rank over $\mathbb{F}$;

(P2) $G_2(d_2)$ is a quasi-systematic 1D polynomial encoder, i.e., there exists an invertible matrix $T \in \mathbb{F}^{n\times n}$ such that $TG_2(d_2) = \begin{bmatrix} I_p \\ \bar{G}_2(d_2) \end{bmatrix}$, $\bar{G}_2(d_2) \in \mathbb{F}^{(n-p)\times p}[d_2]$.

Note that both $G_1(d_1)$ and $G_2(d_2)$ are minimal encoders of the corresponding 1D convolutional codes. Moreover, $G(d_1, d_2)$ is a *minimal encoder* of $\mathscr{C}$, i.e., it has minimal Roesser McMillan degree among all encoders of $\mathscr{C}$, [9, 10], in the sequel we denote this minimal degree by $\mu(\mathscr{C})$.

In what follows, we shall derive a syndrome former construction for the code $\mathscr{C}$, based on Proposition 5. Define

$$H_1(d_1) = \begin{bmatrix} L_1(d_1) & 0 \\ 0 & I \end{bmatrix} \in \mathbb{F}^{(n-k)\times n}[d_1] \text{ and } H_2(d_2) = \begin{bmatrix} I & 0 \\ -\bar{G}_2(d_2) & I \end{bmatrix} T \in \mathbb{F}^{n\times n}[d_2],$$

where $L_1(d_1) \in \mathbb{F}^{(p-k)\times p}[d_1]$ and $\begin{bmatrix} -\bar{G}_2(d_2) & I \end{bmatrix} \in \mathbb{F}^{(n-p)\times n}[d_2]$ are 1D syndrome formers of the 1D convolutional codes $\operatorname{im} G_1(d_1)$ and $\operatorname{im} G_2(d_2)$, respectively. Let

$$H(d_1, d_2) = H_1(d_1)H_2(d_2) \tag{6}$$

$$= \begin{bmatrix} L_1(d_1) & 0 \\ -\bar{G}_2(d_2) & I \end{bmatrix} T. \tag{7}$$

It is easy to see that $H(d_1, d_2)$ is a syndrome former of $\mathscr{C}$. It can be shown that it is possible to assume, without loss of generality, that (6) is an optimal decomposition of $H(d_1, d_2)$. Then

$$\mu_R(H) = \mu(H_1) + \mu(H_2) = \mu(L_1) + \mu(-\bar{G}_2) = \mu(L_1) + \mu(G_2).$$

Note that since $L_1(d_1)$ is a syndrome former of the 1D convolutional code $\operatorname{im} G_1(d_1)$ and $G_1(d_1)$ is a minimal encoder of $\operatorname{im} G_1(d_1)$, it follows that $\mu(L_1) \geq \mu(G_1)$, [5, 6], and hence $\mu_R(H) \geq \mu_R(G)$. Moreover, $\mu(L_1) = \mu(G_1)$ if $L_1(d_1)$ has minimal McMillan degree among all syndrome formers of $\operatorname{im} G_1(d_1)$, for instance, if $L_1(d_1)$ is row reduced, [5, 6], (which can always be assumed without loss of generality, since otherwise pre-multiplication of $H(d_1, d_2)$ by a suitable unimodular matrix $U(d_1)$ yields another syndrome former for $\mathscr{C}$, with $L_1(d_1)$ row reduced); in this case $\mu_R(H) = \mu_R(G)$.

[2] A minimal 1D encoder is an encoder with minimal McMillan degree among all the encoders of the same code.

[3] A full row (column) rank matrix $M(d) \in \mathbb{F}^{n\times k}[d]$ is said to be row (column) reduced if $\operatorname{int\,deg} M(d)$ is equal to the sum of the row (column) degrees of $M(d)$; in that case $\mu(M) = \operatorname{int\,deg} M(d)$.

Thus given the encoder $G(d_1, d_2)$ we have constructed a syndrome former $H(d_1, d_2)$, as in Proposition 5. Moreover, based on the special properties of $G(d_1, d_2)$, we have shown that the minimal realizations of $H(d_1, d_2)$ have dimension $\mu_R(H) = \mu_R(G) = \mu(\mathscr{C})$ (recall that $G(d_1, d_2)$ is a minimal encoder).

We next show that $\mu_R(H)$ is minimal among the McMillan degree of all syndrome formers of $\mathscr{C}$ with similar structure as $H(d_1, d_2)$.

Theorem 7 *Let $\mathscr{C}$, with $\hat{\mathscr{C}} = \operatorname{im} G(d_1, d_2)$, be a 2D composition code, and assume that $G(d_1, d_2) = G_2(d_2)G_1(d_1)$, where $G_1(d_1)$ and $G_2(d_2)$ satisfy properties (P1) and (P2), respectively. Let further $\tilde{H}(d_1, d_2) = \begin{bmatrix} X_1(d_1) & 0 \\ X_{21}(d_2) & X_{22}(d_2) \end{bmatrix} T$ be a syndrome former of $\mathscr{C}$, where $X_1(d_1) \in \mathbb{F}^{(p-k)\times p}[d_1]$, $X_{21}(d_2) \in \mathbb{F}^{(n-p)\times p}[d_2]$, $X_{22}(d_2) \in \mathbb{F}^{(n-p)\times(n-p)}[d_2]$ and $T \in \mathbb{F}^{n\times n}$ as in (P2). Then $\mu_R(\tilde{H}) \geq \mu(\mathscr{C})$.*

Proof Note that $\tilde{H}(d_1, d_2)G(d_1, d_2) = 0$ if and only if

$$\begin{cases} X_1(d_1)G_1(d_1) = 0 \\ \left(X_{21}(d_2) + X_{22}(d_2)\bar{G}_2(d_2)\right) G_1(d_1) = 0. \end{cases} \tag{8}$$

Then $X_1(d_1)$ must be a syndrome former of the 1D convolutional code $\operatorname{im} G_1(d_1)$ and consequently $\mu(X_1) \geq \mu(G_1)$ [6]. On the other hand we have that $X_{21}(d_2) + X_{22}(d_2)\bar{G}_2(d_2) = 0$, that is equivalent to $\begin{bmatrix} X_{21}(d_2) & X_{22}(d_2) \end{bmatrix} \begin{bmatrix} I \\ \bar{G}_2(d_2) \end{bmatrix} = 0$, and therefore $\begin{bmatrix} X_{21}(d_2) & X_{22}(d_2) \end{bmatrix}$ is a syndrome former of the 1D convolutional code $\begin{bmatrix} I \\ \bar{G}_2(d_2) \end{bmatrix}$. Hence $\mu\left(\begin{bmatrix} X_{21} & X_{22} \end{bmatrix}\right) \geq \mu\left(\begin{bmatrix} I \\ \bar{G}_2 \end{bmatrix}\right)$, since $\begin{bmatrix} I \\ \bar{G}_2(d_2) \end{bmatrix}$ is a minimal encoder of $\operatorname{im} \begin{bmatrix} I \\ \bar{G}_2(d_2) \end{bmatrix}$. Now, since $\tilde{H}(d_1, d_2) = \begin{bmatrix} X_1(d_1) & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} I & 0 \\ X_{21}(d_2) & X_{22}(d_2) \end{bmatrix} T$, it is not difficult to see that

$$\begin{aligned} \mu_R(\tilde{H}) &= \mu(X_1) + \mu\left(\begin{bmatrix} X_{21} & X_{22} \end{bmatrix}\right) \geq \mu(G_1) + \mu\left(\begin{bmatrix} I \\ \bar{G}_2 \end{bmatrix}\right) \\ &= \mu(G_1) + \mu\left(T^{-1} \begin{bmatrix} I \\ \bar{G}_2 \end{bmatrix}\right) = \mu_R(G) = \mu(\mathscr{C}). \end{aligned}$$

□

Corollary 8 *Using the notation and conditions of Theorem 7, the syndrome former of $\mathscr{C}$ given by (7) has minimal Roesser McMillan degree among all syndrome formers of the same structure.*

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