# Constructing Abstract Mathematical Knowledge in Context

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**Abstract** Understanding how students construct abstract mathematical knowledge is a central aim of research in mathematics education. Abstraction in Context (AiC) is a theoretical-methodological framework for studying students' processes of constructing abstract mathematical knowledge as they occur in a mathematical, social, curricular and learning-environmental context. AiC builds on ideas by Freudenthal, Davydov, and others. According to AiC, processes of abstraction have three stages: need, emergence and consolidation. The emergence of new (to the student) constructs is treated by means of a model of three observable epistemic actions: Recognizing, Building-with and Constructing—the RBC-model. This paper presents a theoretical and methodological introduction to AiC including to the RBC-model, and an overview of pertinent research studies.

**Keywords** Abstraction · Knowledge construction · Context · RBC-model

## Introduction

The approach described in this paper took shape in the course of research that accompanied innovative curriculum development, when questions arose such as "What did students learn? What new deep mathematical knowledge, what concepts and strategies have been consolidated? And how did the processes of learning and consolidation happen?" The salient characteristics of mathematical curricula and classroom learning environments in which these questions have been investigated are that curricula are organized as successions of activities proposed to the students, and that mathematical themes arising along these activities very often are transformations of previous mathematical themes. Hence curricula express an intention of continuous transformation. There is an underlying expectation of students'

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responsibility for their own learning in an environment that encourages inquiry. Students have the responsibility to report and justify their work and their conclusions to their peers and their teacher, for example during whole class discussions. The reader is referred to the literature for a more detailed treatment of the curriculum standards and design principles (Hadas et al. 2008; Hershkowitz et al. 2002).

The research problem that arose was how to understand students' construction of knowledge, especially of deep, abstract mathematical knowledge such as concepts and strategies in learning situations, in particular in classrooms, using these curricula. A main aim was to describe processes of constructing knowledge in order to get insight into such processes and the conditions under which they happen or fail to happen. Additional aims were to use the understanding of students' learning processes in order to improve the design of activities and inform teacher behaviour.

For this purpose, a theoretical framework for describing processes of abstraction was required. Rina Hershkowitz, Baruch Schwarz and the author developed such a framework over the past 15 years (Dreyfus et al. 2015; Hershkowitz et al. 2001; Schwarz et al. 2009). The framework takes into account the particularities of the context of learning. This context includes the students' prior history of learning, the learning environment, including possibly available technological and other tools, as well as mathematical, curricular and social components. In particular, the social and interactional context may vary considerably from one class to another according to the teacher's decisions. In view of the above, we called our theoretical framework Abstraction in Context, or briefly, AiC.

## Theoretical Background and the AiC Framework

The attention to a special kind of curriculum and to learning processes within various contexts required a rather hybrid reference to theoretical forefathers that belong to different traditions, Freudenthal and Davydov. Freudenthal (1991) provided what mathematicians have in mind when they think of abstraction. Freudenthal has brought forward some of the most important insights to mathematics education in general, and to mathematical abstraction in particular. These insights constitute a cultural legacy that led his collaborators to the idea of vertical mathematisation (Treffers and Goffree 1985). Vertical mathematisation points to a process that typically consists of the reorganization of previous mathematical constructs within mathematics and by mathematical means, by which students construct a new abstract construct. In vertical reorganisation, previous constructs serve as building blocks in the process of constructing. Often these building blocks are not only reorganised but also integrated and interwoven, thus adding a layer of depth to the learner's knowledge, and giving expression to the composite nature of the mathematics. Sequences of problem situations provide opportunities to capitalise on the new constructs repeatedly, and to turn them into building blocks for further constructions where each construct includes 'pockets' of past constructs on one hand, and is itself a potential component for new constructs.

Davydov was one of the most prominent followers of the historical cultural theory of human development initiated by Vygotsky. For Davydov (1990), scientific knowledge is not a simple expansion of people's everyday experience. It requires the cultivation of particular ways of thinking, which permit the internal connections of ideas and their essence to emerge; it also requires enriching rather than impoverishing reality. According to Davydov's "method of ascent to the concrete", abstraction starts from an initial, simple, undeveloped and vague first form, which often lacks consistency. The development of abstraction proceeds from analysis, at the initial stage of the abstraction, to synthesis. It ends with a more consistent and elaborated form. It does not proceed from concrete to abstract but from an undeveloped to a developed form.

The reference to both Freudenthal and Davydov's theories of abstraction implies that the curriculum affords certain kinds of abstraction, but at the same time, students and teachers are free to capitalize or not on those affordances. AiC adopts the views of vertical mathematisation and ascent to the concrete and builds on them to define abstraction as a process of vertically reorganizing some of the learner's previous mathematical constructs within mathematics and by mathematical means so as to lead to a construct that is new to the learner. Activity theory proposes an adequate framework to consider processes that are fundamentally cognitive while taking into account the mathematical, historical, social and learning contexts in which these processes occur. In this, AiC follows Bauersfeld (1992) and Giest (2005), who considers activity theory as a theoretical basis, which has an underlying constructivist philosophy but allows avoiding a number of problems presented by constructivism.

According to activity theory, outcomes of previous activities naturally turn to artefacts in further ones, a feature which is crucial to trace the genesis and the development of abstraction through a succession of activities. The kinds of actions that are relevant to abstraction are epistemic actions—actions that pertain to the knowing of the participants and that are observable by participants and researchers. As researchers with loyalty to Freudenthal, we were a priori attentive to certain constructs afforded by the activities we designed. In tune with Davydov and a cultural-historical theory of development, we also looked at other constructs that emerged from classroom activities.

This is well expressed by Kidron and Monaghan (2009) when dealing with the need that brings students to engage in abstraction, a need that emerges from a suitable design and from an initial vagueness of the learner's notions:

> The learners' need for new knowledge is inherent to the task design but this need is an important stage of the process of abstraction and must precede the constructing process, the vertical reorganization of prior existing constructs. This need for a new construct permits the link between the past knowledge and the future construction. Without the Davydovian analysis, this need, which must precede the constructing process, could be viewed naively and mechanically, but with Davydov's dialectic analysis the abstraction proceeds from an initial unrefined first form to a final coherent construct in a two-way relationship between the concrete and the abstract – the learner needs the knowledge to make sense of a situation. At the moment when a learner realizes the need for a new construct, the learner already has

> an initial vague form of the future construct as a result of prior knowledge. Realizing the need for the new construct, the learner enters a second stage in which s/he is ready to build with her/his prior knowledge in order to develop the initial form to a consistent and elaborate higher form, the new construct, which provides a scientific explanation of the reality. (pp. 86–87)

Hence we postulate that the genesis of an abstraction passes through a three-stage process, which includes the need for a new construct, the emergence of the new construct, and the consolidation of that construct.

A central component of AiC is a theoretical—methodological model, according to which the emergence of a new construct is described and analysed by means of three observable epistemic actions: recognizing (R), building-with (B) and constructing (C). Recognizing refers to the learner realizing that a specific previous knowledge construct is relevant in the situation at hand. Building-with comprises the combination of recognized constructs, in order to achieve a localized goal such as the actualization of a strategy, a justification or the solution of a problem. The model suggests constructing as the central epistemic action of mathematical abstraction. Constructing consists of assembling and integrating previous constructs by vertical mathematisation to produce a new construct. It refers to the first time the new construct is expressed or used by the learner. This definition of constructing does not imply that the learner has acquired the new construct once and forever; the learner may not even be fully aware of the new construct, and the learner's construct is often fragile and context dependent. Constructing does not refer to the construct becoming freely and flexibly available to the learner. Becoming freely and flexibly available pertains to consolidation.

Consolidation is a never-ending process through which students become aware of their constructs, the use of the constructs becomes more immediate and self-evident, the students' confidence in using the construct increases, and the students demonstrate more and more flexibility in using the construct (Dreyfus and Tsamir 2004). Consolidation of a construct is likely to occur whenever a construct that emerged in one activity is built-with in further activities. These further activities may lead to new constructs. Hence consolidation connects successive constructing processes and is closely related to the design of sequences of activities.

In processes of abstraction, the epistemic actions are nested. C-actions depend on R- and B-actions; the R- and B-actions are the building blocks of the C-action; at the same time, the C-action is more than the collection of all R- and B-actions that make up the C-action, in the same sense as the whole is more than the sum of its parts. The C-action draws its power from the mathematical connections, which link these building blocks and make them into a single whole unity. It is in this sense that we say that R- and B-actions are constitutive of and nested in the C-action. Similarly, R-actions are nested within B-actions since building-with a previous construct necessitates recognising this construct, at least implicitly. Moreover, a lower level C-action may be nested in a more global one, if the former is made for the sake of the latter. This nested character was observed in classrooms and in interviews in which we studied abstraction and it substantiated our theoretical tenets according to which the curriculum was intended to afford a continuous transformation of constructs.

Given these characteristics, we named the model the dynamically nested epistemic actions model of abstraction in context, more simply the RBC model, or RBC+C model using the second C in order to point to the important role of consolidation. The RBC-model is the theoretical and micro-analytic lens, through which we observe and analyse the dynamics of abstraction in context.

We will below come back to the RBC-model in order to show how the model as a part of the theory interacts with the same model as methodological tool, and hence theory and methodology mutually depend on and influence each other in AiC. The successive analyses by several researchers who used the RBC-model to identify abstraction processes through the unveiling of its epistemic actions not only helped understanding these abstraction processes: The theory as well as the methodology underwent successive refinements as they served as lenses to understand mathematical learning activities.

## An Example

While it is not possible to illustrate all aspects of AiC by means of a single example, the example used in this section fairly well illustrates many of the main aspects. It stems from a 7th grade class whose beginning algebra curriculum consisted of seventeen activities, thirteen with a spreadsheet, and four without computer. In these activities students learned to use algebra to express generality. For example, by generalizing from a few numerical examples or from a "story", they generated an algebraic relation, which they could insert into the spreadsheet. By dragging, they could then obtain sequences of numbers to describe and investigate a phenomenon. Hence, in terms of Kieran's (2004) framework, the activities were generational.

We know from weekly observations and teacher reports that the students increasingly used algebra for expressing generality throughout the year. For example, the following fact will be crucial below: The students had experience with the use of the simple distributive laws $a(b+c) = ab + ac$ and $(a+b)c = ac + bc$ but had never yet used the extended distributive law $(a+b)(c+d) = ac + ad + bc + bd$. Even more importantly, students had never yet been asked to justify general properties by using algebraic manipulation.

The following activity was carried out by the class toward the end of the school year. The students were asked to consider two-by-two number arrays, called *seals*, of the form shown in the diagram; two numerical seals were given to the students, but not the general form. The students were asked to find as many properties of the seals as possible, and to establish whether some of these properties were true for all seals. The research focus was on the diagonal product property (DPP): The difference of the products of the diagonals in the array equals 12. More detail on the curriculum and the activity may be found elsewhere (Dreyfus et al. 2001).

| | |
|---|---|
| $X$ | $X$+2 |
| $X$+6 | $X$+8 |

According to the a priori analysis, the activity was designed toward two new (to the students) constructs: algebra as a tool for justifying a general statement, and the extended distributive law. We will call the corresponding knowledge elements E and E1, respectively.

We focus on how a pair of students, Ha and Ne, collectively called HaNe, dealt with justifying the DPP. After about 25 min of discussing other properties of the seals, HaNe reached the stage in the worksheet where they are explicitly asked whether the DPP holds for all seals.

| | | |
|---|---|---|
| Ha | 111 | We didn't want to think about this before. This is really somehow like that. Because this is the first diagonal, and that is the second, right? |
| Ne | 112 | No. Yes |
| Ha | 113 | So, well, see: *X* plus 8, *X*, well, then it is 8*X* plus *XX* |
| Ne | 114 | Again? |
| Ha | 115 | Wait. *X* times *X* plus 8, right? This is *XX*, like, *X*, *X* twice, so it's *XX* |
| Ne | 116 | Yes |
| Ha | 117 | And X times 8 is 8*X* |
| Ne | 118 | Why times 8? |
| Ha | 119 | Why not? Because, one multiplies this by this, right? |
| Ne | 120 | Yes |
| Ha | 121 | So, like, one does the distributive law |

Our interpretation of 111–121 according to the RBC-model is that the students recognize the simple distributive law as relevant in this new context, and build-with it a component of what they apparently need for justifying the DPP. Hence they consolidate the simple distributive law by making use of it in a new context. They then soon turn their attention to the other diagonal.

| | | |
|---|---|---|
| Ha | 133 | And this… |
| Ne | 134 | It's impossible to do the distributive law here. Wait, one can do |
| Ha | 135 | This is 6*X* |
| Ne | 136 | This is 6*X* times *X* and 6*X* times 2 |
| Ha | 137 | Wait, first, no… |
| Ne | 138 | Yes |
| Ha | 139 | No because this is *X* plus 6, this is not 6*X*, it's different. Wait. First one does… *X*; then it's *XX* plus 2*X*, and here 6*X* plus 24. Then… |

In 133–139 HaNe deal with the applicability and application of the distributive law to the more complex case of the second diagonal. The strategy of comparing the two diagonals caused the students' need (134, see also 152) for a technique that

would be applicable in this context, and this need led to some progress in constructing such a technique, namely the extended distributive law (135–139), to be completed later (157–163).

| | | |
|---|---|---|
| Ha | 141 | Ah, 12. Like, for this one needs to know *X*… never mind |
| N | 145 | One simply needs to know what *X* is |

In 141, 145 he students wonder whether they need to know the value of *X* to proceed. The interviewer reminds them that they were able to deal with the first diagonal without knowing *X*. This leads to progress.

| | | |
|---|---|---|
| Ha | 152 | Ah, it's *XX* plus 8*X*, but I don't know, like, how this will also be *XX* plus 8*X*. Like, it HAS to be |
| Ne | 153 | XX is a square root? |
| Ha | 154 | I have the first part. This is *XX*, so this is OK |
| Ne | 156 | XX is a square root? |
| Ha | 157 | … plus 8*X*. Here I have 6*X*… |
| Ha | 159 | Ah, and 2*X*, can I do this? Because 6*X*… |
| Ne | 160 | Is XX a square root? |
| Ha | 161 | You can write this. Ah, yes, *XX* is *X* to the power 2, because it is *X* times *X*. Wait. *XX* is *X* to the power 2 plus 8*X*, wait… |
| Ha | 163 | Wait, it's X to the power 2 plus 6*X*, plus 8*X*, but there is also, like, plus 12. Ah, so, like, plus 12 because this is bigger by 12. Understand? |
| Ne | 164 | Like, yes, it's the same thing but this is bigger by 12 |

In 152–164, the students construct the extended distributive law in the context of the comparison of the two diagonals. We denote this constructing process by C1 since it leads to a student's construct that corresponds to knowledge element E1. Utterance 163 has been interpreted as the completion of C1: Even though the students may not be aware that they have used a new law, they have, in fact, for the first time applied the extended distributive law. Moreover, the difference of 12 between the two diagonals becomes significant in 163–164. The students use this soon thereafter for justifying the DPP—they construct C, the construct corresponding to knowledge element E, a justification by means of algebraic manipulation.

A number of comments on these constructing processes are in order.

*Nesting* During C1, the students have recognized as relevant and built-with the simple distributive law, in 135–136 as well as in 157–163; elaborating the inappropriate $6X(X+2) = 6XX + 6X2$ (in 135–136) appears to have helped them bridge from the simple to the extended distributive law. Not only are R- and B-actions with previous constructs, such as the simple distributive law, nested in C1 but C1 itself is nested in C.

*Need* The need for C1 (134, 152) derives from the mathematical situation as discussed above. A need for C can be identified from the beginning (111), but this appears to be an external need imposed by the worksheet. In fact, the students had earlier stated the DPP and referred to it without exhibiting any need for justifying it. However, as they progress, this need becomes an internal personal one as expressed by the “it HAS to be” in 152, interpreted as “must be equal in order for an algebraic comparison to become possible”.

*Vertical mathematisation* When approaching C1—constructing a new algebraic law—the students enter an “adventure” without any knowledge about the needed mathematical structures; they have to discover as well as to construct these structures. The C1-action thus makes the C-action, within which it is nested into a deep holistic construction, which goes beyond the specific construction of the DPP justification, and in which the construction of an unfamiliar algebraic structure is nested. In this sense C is an activity of vertically reorganising previously constructed mathematical knowledge into a new mathematical structure, which is the AiC definition of abstraction.

*Building-with versus constructing* We claim that HaNe built-with the simple distributive law (113–121, 135–136), but constructed the extended one (154–163). The difference between building-with and constructing lies in the students’ personal learning history: They had previously constructed the simple law and here applied it —built-with it—in a somewhat different context, whereas they had never yet met the extended law and hence needed to construct it as a new (to them) mathematical law. Other students, for whom the extended law was a previous construct, would presumably have built-with it to justify the DPP.

*Co-constructing* The above analysis did not attempt to separate the roles of Ha and Ne in the constructing process. While Ha is leading in 152–163, Ne has made two possibly crucial contributions: the inappropriate but helpful application of the simple distributive law in 136, and a three times repeated question whether *XX* is a square root, in 153, 156, and 160 that led to Ha’s “*XX* is *X* to the power 2” in 161. Social interaction during constructing processes will be further discussed below.

*Unexpected constructs* A careful re-analysis of the data (Kidron and Dreyfus 2009) revealed an additional C-action, to be called C2, that had not been predicted by the a priori analysis: the transition from a procedural mode, in which students ‘do’ expressions (115, 121, 134, 139) to an object mode, which allows to ‘trade’ expressions against each other because they ‘are’ (152) something that one ‘has’ (157).

*Combining constructing actions* This re-analysis showed that C1 and C2 are two strands that come closer to each other and start combining in 152. The justification of the DPP was the motivation, which enticed the students to construct C1 and C2, and the combining of C1 and C2 enabled the justification. C1 and C2 had to combine in order to enable to students to reach the goal C, the justification of the DPP. This could not have happened in the process mode afforded by C1. The transition C2 to the object mode was necessary for the justification to be completed. Hence, C1 and C2 are interacting parallel constructions, which complete and reinforce each other, and the combining between C1 and C2 constitutes C.

The aspects discussed here, in particular nesting, need, unexpected constructs, social interaction during constructing, and combining constructions in justification have been observed in other research studies; in the next section, some of these aspects will be discussed more generally and connections will be established.

## Other AiC Based Studies

In this section, a selection of other research studies that used AiC as theoretical framework are reviewed without being described in detail. The reader is referred to the research literature for more information on these studies. Studies have been selected for inclusion in order to point out some of the main achievements of the AiC research program.

Constructing processes may have characteristics linked to the kind of construct (a concept, a strategy, a justification), which students are constructing. A sequence of studies has established such a result for constructing justifications; this is discussed in the first subsection. In the second subsection, a sequence of studies is presented, that deals with the central issue of partial correctness of students' constructs. Both, the studies on justification and the studies on partially correct constructs show the analytic power of the AiC framework and the RBC-model associated with it. In the third subsection, studies relating to consolidation and the mechanisms that support it are discussed. In the fourth and fifth subsections, two components of the context and their influence on constructing processes are reviewed: the role of the social context and the role of technological and other tools in the construction of knowledge.

### *Constructing Justifications*

The nesting of two combining constructing actions C1 and C2 within a more encompassing constructing action C in the case of HaNe constitutes a rather elaborate interaction between constructing actions going on in parallel. Kidron and Dreyfus (2010a) have pointed out the importance of such interacting parallel constructing actions. They identified this specific pattern of interaction as being typical for constructing a justification. They established this in the case of justifying the second bifurcation point in a logistic dynamical system by a solitary learner L, an experienced mathematician. L's motivation for finding a justification drives her learning process. The researchers inferred her epistemic actions from her detailed notes during the learning experience and from her interaction with the computer. They found an overarching constructing action C, within which four secondary constructing actions were nested. These secondary constructing actions relate to different modes of thinking: numerical (C1), algebraic (C2), analytic (C3), and visual (C4). They are not linearly ordered but took place in parallel and interacted. Interactions included branching of a new constructing action from an ongoing one, combining or recombining of constructing actions, and interruption and resumption of constructing actions. Here, the combining of constructing actions is of particular interest.

L aimed to justify results obtained empirically from her interaction with a computer. Her aim was not to convince herself or others, nor was she looking for conviction in the logical sense of the term; rather, she wanted to gain more insight into the phenomena causing the second bifurcation point. The term enlightenment, introduced by Rota (1997) seems appropriate to express her interpretation of the word justification. Rota also pointed out that contrary to mathematical proof, enlightenment is a phenomenon, which admits degrees. Kidron and Dreyfus (2010a) show how, at each of three successive stages during L's learning experience, combining constructing actions indicate steps in the justification process that lead to enlightenment.

The relationship between combining constructions and justification has been confirmed in other contexts with students of different age groups dealing with different mathematical topics. One of these is the case of HaNe discussed above. Still another one has been briefly discussed by Kidron and Dreyfus (2009) elsewhere. Combining of constructing actions leads to enlightenment, not in the sense of a formal proof of the statement the learner wants to justify but as an insight into the understanding of the statement. This observation gives an analytic dimension to the RBC model and to its parallel constructions aspect: It allows researchers to use RBC analysis in order to identify a learner's enlightening justification.

Moreover, the analysis of the relationships between justification and parallel constructions led to the realization that often a weak and a strong branch are involved in the combining constructions, and that reinforcement of the weak branch plays a crucial role in the construction of a justification. The realization that a weak and a strong branch combine considerably strengthens the theoretical root of the RBC-model in Davydov's ideas as exposed above. Indeed, reinforcing the weak branch towards combination of constructions closely matches the description of the genesis of abstraction as expressed by Davydov's (1990) method of ascent, according to which abstraction starts from an initial, simple, undeveloped first form, which need not be internally and externally consistent, and ends with a consistent and elaborate final form.

Just like the case of HaNe, the case of L demonstrates vertical mathematisation representing a process of constructing new mathematical knowledge within mathematics and by mathematical means. These processes often include a kind of insight or 'AHA'. This expresses that the reorganisation processes of the already constructed pieces of knowledge into a new construct are driven and strengthened by the genuine and creative mathematical thinking of the learner.

## *Partially Correct Constructs*

It is an open secret that students' correct answers sometimes hide knowledge gaps. On the other hand, incorrect answers often overshadow substantial knowledge students have constructed. These two phenomena, which can reflect aspects of partially correct knowledge, raise questions about the essence of partially correct knowledge and about its emergence. It is important to understand situations in which partially correct knowledge emerges because these situations are very common, and because of the role of existing knowledge in the constructing of further knowledge.

Ron (2009) proposed the term *Partially Correct Construct* (PaCC), as a general term for constructs that only partially match the corresponding mathematical knowledge elements that underlie the learning context. Obviously one cannot expect that a student will construct every aspect and meaning of a knowledge element. In this sense, knowledge is always partial. Thus, discussion of PaCCs requires clarifying with respect to which whole entity a construct is partially correct. For that reason the research on PaCCs was restricted to intentional learning situations, like school learning, which is directed by teachers and designers to the constructing of specific knowledge elements that can be identified by an a priori analysis. Hence, the identification of a construct as a PaCC is always related to a specific learning context and requires a detailed a priori analysis of the content and of the knowledge elements that the student is intended to construct.

In a study in the content domain of elementary probability, Ron et al. (2010) found that situations in which articulations or actions of a student seem inconsistent with other articulations or actions of the same student, can be explained, at least in some cases, by the identification of some of the student's constructs as PaCCs. The researchers used micro-analysis of students' knowledge constructing processes by means of the epistemic actions of the RBC-model to identify PaCCs. They found PaCCs that arose in the early stages of a learning activity, when all the student's articulations and actions were correct; in other words, the PaCCs arose long before the student's actions raised any clues for the existence of the PaCC or for an expected difficulty.

Knowledge constructing processes, in which PaCCs emerge and are possibly consolidated, take place in parallel and simultaneously with the construction and consolidation of knowledge that does fit the learning aims. These processes are not different in their essence from other processes of knowledge constructing as described in the AiC framework and they are based on the same epistemic actions. Characteristics that are specific to the knowledge construction processes that lead to PaCCs find their expression in different types of PaCCs. The partiality of the fit between the student's construct and the corresponding mathematical knowledge element can be related to the building blocks of the knowledge elements and/or to the context in which the student recognizes the construct as relevant and makes use of it. This distinction is a basis for defining two categories of PaCCs: structural PaCCs and contextual PaCCs (Ron et al. 2009).

Three types of structural PaCCs were identified: a missing-element PaCC in which at least one of the constituent elements of the knowledge element is missing from the student's construct, an incompatible-element PaCC in which the student's construct includes an element that contradicts the mathematical knowledge element, and a disconnected-element PaCC which is characterized by disconnected constituent elements in the student's construct. The two types of contextual PaCCs that were identified are a narrow-context PaCC, where the student recognizes the construct as relevant in a too narrow a context, and a wide-context PaCC, where the student implements his constructs in a context that is wider than warranted.

In summary, PaCCs are useful as explanatory tools for correct answers based on (partially) faulty knowledge and for wrong answers based on largely correct knowledge. Ron's research shows that AiC is a suitable framework for defining the notion of PaCC and that the RBC-model is an efficient tool for identifying PaCCs and their nature.

## *Consolidation*

Consolidation of students' constructs has been conceptualized and studied within the AiC framework by Tsamir and Dreyfus (Dreyfus and Tsamir 2004; Tsamir and Dreyfus 2002), as well as by Monaghan and Ozmantar (2006). Dreyfus and Tsamir (2004) proposed the criteria of immediacy, self-evidence, confidence, flexibility, and awareness, as indicating that a construct has been consolidated. The study of consolidation usually requires data taken over a longer period than the study of constructing; typically studying consolidation requires data from several subsequent activities.

Using a study of a 10th grade student learning about the comparison of infinite sets, Dreyfus and Tsamir (2004) have identified mechanisms of consolidation. In particular, consolidating a recent construct during building-with this construct is the most frequent and most easily observed one. Another mechanism, consolidating a recent construct when recognising it as an object of reflection, often stems from opportunities for reflection provided to students (e.g., requests for written reports). Reflection tends to lead to the use of more elaborate language about the construct, expressing a more acute and fine-tuned awareness; and as mentioned above, awareness is an important characteristic of consolidation. Dreyfus et al. (2006) have identified a third mechanism of consolidation: Consolidating a previous construct as it is used as a building block in the course of a new constructing process. The example of HaNe above provides an excellent example of this: The students used the simple distributive law as building block when constructing the extended one, and in the process became progressively quicker, more flexible and more self-confident in applying the simple distributive law. An independent instance of the same mechanism of consolidation is a student's consolidation of her construct of derivative as limit during the process of constructing Euler's numerical method of solving differential equations (Kidron 2008).

Notwithstanding these and several other examples of apparently clear consolidation, the question how well a construct has been consolidated is a delicate one. This has been shown by Tsamir and Dreyfus (2005), who showed that under slight variations of context, knowledge structures that have apparently been well consolidated may become inactive and subordinate to more primitive ones. In other words, even when a construct has apparently been consolidated, it is a delicate issue to determine the extent of such consolidation.

## Social Interaction in Constructing Processes

In the above example, Ha and Ne were mostly treated as a single student. While this was a conscious and acceptable methodological decision by the researchers, it provides only a limited view of the learning processes that occur. Dreyfus et al. (2001) have considered processes of abstraction in pairs of collaborating peers and investigated the distribution of the process of abstraction in the context of peer interaction. This was done by carrying out two parallel analyses of the protocols of the work of the student pairs, an analysis of the epistemic actions of abstraction as well as an analysis of the peer interaction. The parallel analyses led to the identification of types of social interaction that support processes of abstraction. In classrooms, however, the situation is often even much more complex. Abstraction often takes place in interacting groups of students. Hence, the focus ideally should be on groups as composed of individuals and two dual issues become central: On the one hand, constructing by individual students, and on the other, the knowledge shared by the group. Hershkowitz et al. (2007) dealt with the relationship and interaction between these two dual issues. Their data emphasize the interactive flow of knowledge from one student to the others in the group, until they reach some shared knowledge—a common basis of knowledge, which allows them to continue the construction of further knowledge in the same topic together.

The issues involved in knowledge construction by groups of students are too complex to allow detailed discussion here. However, some of these issues will be discussed from a wider perspective in the concluding section.

## Computer Tools and Other Artefacts

Kidron and Dreyfus (2010b) have re-examined the study of L's justification of bifurcations in a dynamical system described above with a view to how instrumentation led to constructing actions and how the roles of the learner and a computer algebra system (CAS) intertwine during the process of constructing the justification. The main contribution of this research lies in showing that certain patterns of epistemic actions, specifically those of branching and combining, have been facilitated by certain contextual factors, specifically the CAS context. They found that the branching and combining patterns have been enabled by the work with the CAS. This is due to the fact that the computer provides a context that is very rich in resources. The richness of these resources activates the branching of constructions even if the learner is unable to immediately make sense of the input provided by the CAS. Constructions are interrupted by lack of knowledge. Nevertheless, the seeds for the future combinations may already be present. The fact that the CAS can perform the computations even if the learner does not really understand its mechanism encourages the learner to make sense of the rich resources offered by the CAS. Therefore the branching, interruptions after branching and resumptions of the interrupted constructions were necessary stages preceding the integration of the knowledge structures. The combining process, which ends in the integration of knowledge structures, was facilitated by the potential offered by the CAS and the learner's ability to make sense of the resources offered by the computer. The relations between the learner and the computer as a dynamic partner were different in the branching and in the combining phases: the researchers suggested that the computer had the upper hand during branching and the learner took command during combining.

A recent study by Weiss (2010) also focuses on the role of tools in the construction of knowledge; more specifically, Weiss considers teaching and learning situations that utilize the potential of an analogical model for creating meaningful abstraction processes. Such situations combine challenging tasks that have a high potential for abstraction with an analogical model, which supports students in handling the task. In order to create a meaningful combination between the challenging task and the analogical model, Weiss has developed model-based tasks—tasks that lead naturally to using the analogical model. He then described and analysed the role of the analogical model in knowledge constructing during model-based tasks with a high potential for abstraction.

The subject matter that was chosen for designing the model-based construction tasks is taken from the domain of fractions, and more specifically the complete-to-whole rule, which says that if the completion of fraction A to a whole unit is smaller than the completion of fraction B to a whole unit, then fraction A is bigger than fraction B. Weiss developed a unit for 4th graders based on an analogical model. In the last decade researchers have been calling for more emphasis on linear models in the learning of fractions in general and of fraction comparison in particular. In light of the above, the tower-of-bars model, an analogical model for fractions, was chosen as model.

For the purpose of analysing the role of the model in knowledge construction, AiC turned out to be insufficient. Hence, Weiss also used RME (Realistic Mathematics Education)—a theoretical framework developed at the Freudenthal Institute—which is also dealing with the role of mathematical models in the learning of mathematics based on the emergent models approach of Gravemeijer (1999). Transcripts were subjected to a dual-lens analysis by using the two theoretical frameworks: RBC analysis focusing on constructing actions and RME analysis of the role of the model with the emphasis on the transition between Model-Of and Model-For modes according to the emergent models approach.

Weiss found that among the 21 students (out of 24) who had constructed the complete-to-whole rule there was a linkage between the transition from Model-Of to Model-For and knowledge construction. In addition to that, three empirical linkages between the functionality of the model (taking the RME approach) and knowledge constructing (taking the RBC approach) were discovered: The linkage between the model as a tool for visual reasoning and comparative building-with; the linkage between the model as cognitive desktop and connective building-with; and the linkage between the model as a tool for mental justification and justifying during knowledge construction and consolidation. These three empirical linkages reveal an important aspect of the analogical model as a tool for mathematical reasoning, refine and enrich the descriptive language of the RBC methodology, and link RBC to RME empirically as well as theoretically.

## Concluding Remarks

AiC has been successfully used to analyse the constructing of abstract mathematical knowledge by students aged nine to adult, and in many mathematical content domains including fractions, beginning algebra, probability, geometrical proofs, rate of change, function transformations, integration, bifurcations in dynamical systems. The longitudinal dimension of the studies varied from a single session to sequences of ten and more lessons. A large variety of learning environments have been studied, and social settings ranged from activities of individuals via tutoring situations, and small-group work to (teacher-led) classroom discussions. Researchers' aims have included the relationships between affect, creativity and constructing (Williams 2002, 2011), issues of social interaction when co-constructing, the role of technological tools in knowledge construction, questions of cognition relating to justifications and conceptual change, and others. This has rendered the use of the RBC-model a well-validated methodological and analytical tool for research. It has also led to instances where the micro-design of activities was improved on the basis of the RBC-microanalysis of student's learning processes (e.g., Kouropatov and Dreyfus 2011).

In the course of these research studies, it became apparent that the RBC-model in fact carries a dual role as a methodological tool and as a theory in development. This was first pointed out by Hershkowitz et al. (2001): "Our definition is thus a product of our oscillating between our theoretical perspective on abstraction and experimental observations of actions (experimental data)" (p. 202). Hershkowitz (2009) discussed this phenomenon in some detail and named it the contour lines (boundaries) between the theoretical framework and the methods and methodological tools within the same research. She argued that these boundaries may be flexible and even a bit vague in the sense that the same scheme or model, in this case the RBC-model, may serve as a theoretical framework in one piece of research, as a methodological tool in a second one, and as both of them in a third piece of research.

This apparently came about as follows: The researchers approached the problem of investigating the construction of abstract mathematical knowledge. They began with a first hypothesis for a scheme or a model, using both theoretical considerations and the analysis of considerable amounts of data. In this undertaking, they were led by the need to give theoretical expression to the specific characteristics of their data, which pointed to constructing of knowledge by means of mathematical thinking. In the process, they took into account and incorporated elements of existing theories. Abstracting, for example, was taken as human activity of mathematization, specifically vertical mathematization. They realized the importance of contextual factors and described illustrative examples in different contexts.

At that stage, a circular situation arose where theory stemmed from the analysis of data, and the analysed data served as evidence for validating the theory. The researchers were quite aware of this situation and explained: “This definition [of abstraction] is a result of the dialectical bottom-up approach described above… a product of our oscillating between our theoretical perspective on abstraction and experimental observations of students’ actions, actions we judged to be evidence of abstracting” (Hershkowitz et al. 2001, p. 202). It is clear that for analysing the above actions, we had to use some basic methodologies, which fit transcript analyses of an individual and the more complicated analysis of cognitive and interactive work within pairs and groups. The three epistemic actions, recognizing, building-with and constructing, and the dynamically nested relationships between them were hypothesized as the main building blocks of the model, and at the same time used as the lens and compass to describe and interpret the data analyses themselves. Such a situation held for the first steps towards the validation of the model as a theoretical framework.

Further research made it clear that the RBC+C model for AiC is an appropriate theoretical tool and methodology to describe and provide insight into processes of abstraction and consolidation in a wide range of situations. About ten years of research and more than 50 research publications, contributed by more than a few people, separate the ‘birth’ of the AiC framework and the RBC+C model, as an empirically based theoretical framework, from recent publications that use this model as one of two or more ‘conceptual frameworks’ (Bikner-Ahsbahs et al. 2010; Weiss 2010; Wood et al. 2006). These studies show some maturation of the model as a theoretical framework and as a methodology. In each case, the researchers needed two or more conceptual frameworks in their study and AiC was one of them. The RBC+C model is then not any more the focus of the study but exemplifies the flexible contour lines between a model as theory and a model as methodological tool: the model aims to serve as a framework for describing, analysing and interpreting a human mental activity and at the same time is appropriate for exploring individual student mental activity as well as for exploring collective mental activity that is distributed in a group or a classroom among different individuals.

The model with its three epistemic actions, has a very general nature, general in the sense that it can be used in many and varied contexts. The nested relationships among the epistemic actions of the RBC+C model are global and the three actions of the model are observable and can be identified. Therefore the model lends itself easily to be adapted and to contribute to research in many different contexts of constructing abstract knowledge.

However, the notion of collective abstracting raises many questions, such as: What can we learn from this kind of research about abstracting, or more generally about learning processes and knowledge constructing in classrooms? What can we say about the individual students in the classroom and the classroom community as a community that consists of all the individuals who belong to this community? Do we have a methodology/methodological tool by which we will be able to conduct the kind of research that gives some answers to such questions?

Looking back on the development of AiC over the years, one may discern a trend from investigating an individual learner or dyad with an interviewer in a laboratory setting via investigating focus groups in a working classroom, to investigating students' knowledge construction, and shifts of the constructed knowledge in a working mathematical classroom. The first phase served to develop the AiC framework and the RBC+C model, whereby the RBC+C model was used in two parallel roles: as a methodological tool for analysing the data and for validating the theory. In the second phase, the RBC+C model was applied for analysing students' processes of abstraction as they worked in a group in a working classroom. This is a big challenge because of the many variables that play a role in a whole class situation and the potentially messy data. Hence we had to deal with the issue of the shared knowledge of a group of individual students as they construct and consolidate it in the mathematical classroom. In the third phase, which is an even bigger challenge, we aim to develop a methodology, based on the RBC model and other methodological tools, in order to coordinate analyses of the individual, the group in the classroom and the classroom collective in a working mathematical classroom. Tabach et al. (2014) and Hershkowitz et al. (2014) have recently presented first results from this line of research.

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