Developing Free Computer-Based Learning Objects for High School Mathematics: Examples, Issues and Directions

Humberto José Bortolossi

Abstract In late 2007, the Brazilian government launched a grant program offering 42 million dollars to support the production of digital contents to high school level in the following areas: Portuguese, biology, chemistry, physics and mathematics. Of this amount, the CDME Project (http://www.cdme.im-uff.mat.br/) of the Fluminense Federal University won 124 thousand dollars to develop educational software, manipulative materials and audio clips to the area of mathematics. In this article, we report our experience (and what we learned from it) within this project, regarding the development of educational software as learning objects. We hope that the examples, issues and directions shown here are useful for other teams concerned about cost, time and didactic quality in the development of their applications and online teaching systems. Learning objects in mathematics, software development technologies, visualization in the teaching and learning of mathematics.

Introduction

The use of computers in teaching mathematics is a topic that has been studied for some time (for instance, the theme already appears in the ICME-1 conference held in France in 1969). With the prices falling faster, the processing power increasing and the advent of the Internet, computers became more affordable and popular, gaining a place in our homes and our schools. On the other hand, recent studies show that the simple fact of having access to a computer or Internet technology has little impact on student achievement in basic education (Papanastasiou et al. 2003; OECD 2006; Lei and Zhao 2007; Witter and Senkbeil 2008; Roschelle et al. 2010). In the case of mathematics, the impact may be even negative! These same studies indicate that the problem is not the computer itself, but rather how it is used as a

H.J. Bortolossi (✉)
Fluminense Federal University, Niterói, Brazil
e-mail: hjbortol@vm.uff.br

S.J. Cho (ed.), *Selected Regular Lectures from the 12th International Congress on Mathematical Education*, DOI 10.1007/978-3-319-17187-6_3

tool for teaching and learning (see, for instance, Fig. 1 in Witter and Senkbeil 2008). The availability of quality digital contents accompanied by methodological guidelines for their effective use in teaching practice is fundamental! In this context, various materials have been proposed: the Atractor Project[1] (funded by the Ministry of Education of Portugal), the Descartes Project[2] (funded by the Ministry of Education of Spain), the WisWeb Project[3] (of the Freudenthal Institute for Science and Mathematics Education in the The Netherlands), the Illuminations Project[4] (of the National Council of Teachers of Mathematics in the United States) and the Shodor Project[5] (funded by National Science Foundation in the United States). In Brazil, one of the last major initiatives occurred in 2007 with the release of 42 million dollars from the Ministry of Science and Technology and the Ministry of Education to fund the production of digital multimedia educational contents to high school level in the following areas: Portuguese, biology, chemistry, physics and mathematics. For the area of mathematics, five projects were awarded, one of them proposed by the Fluminense Federal University: the CDME Project.[6] With a budget of 124 thousand dollars and a staff of five professors and twenty students, this project produced 66 educational software, 12 practical experiments with manipulative materials and 15 audio clips. Of the 124 thousand dollars, 107 thousand dollars were spent on payment of scholarships for students. The remaining balance was used to purchase computer equipment. In this article, we report our experience (and what we learned from it), within this project, regarding the development of educational software as learning objects. We hope that the examples, issues and directions shown here are useful for other teams concerned about cost, time and didactic quality in the development of their applications and online teaching systems.

Our Instructional and Technological Choices

In the development of educational software, we have adopted the following instructional principles which, in our experience, proved to be very robust:

- Each software seeks to explore a specific topic in a self-contained way (following a style that some scholars call *learning objects*).
- To persuade and maintain students' attention, each software has been written in such a way as to lead them as quickly as possible to the main object of interaction under study.

[1] http://www.atractor.pt/.

[2] http://recursostic.educacion.es/descartes/web/ingles/.

[3] http://www.fi.uu.nl/wisweb/en/.

[4] http://illuminations.nctm.org/.

[5] http://www.shodor.org/.

[6] http://www.cdme.im-uff.mat.br/ or http://www.uff.br/cdme/.

- Our suggestions of exercises, reflections and questions to be worked out with the software are not embedded in the software itself: we prefer to state them in a separate text file (the Student Handout). This scheme has several advantages: (a) the content becomes more flexible, allowing the teacher to make adjustments according to the profile of his or her class (after all, an easy exercise for a class may be difficult for other); (b) it promotes the so important *writing practice* by the students; (c) one obtains a written record that can support discussions, reflections and assessments.

These principles have allowed that the structure of each software was designed to provide different logistical uses. The teacher can, for example, (1) to conduct the activity with all students in the school's computer laboratory (if the physical structure of the laboratory and the planning of lessons allow), (2) to conduct the activity in the classroom with the aid of a computer and a multimedia projector (something simpler and easier to organize) or, even, (3) to propose the activity as an extra exercise to be carried out during a certain period of time (a week, for example). In the latter case, once the period of time is over, the teacher can use the initial time of a class to discuss, along with their students, the experience they have had with the content (even in situations where the interaction of students is not so intense, we believe, even so, that carrying it will bring benefits).

While instructional aspects are very important in the development of educational software, technical aspects also deserve equal attention. After all, the range of technologies employed in the development has a direct influence on the available building blocks that will constitute the software. In fact, depending on the technologies chosen, certain didactic components can be more difficult to implement (spatial geometry and the two-dimensional mathematical notation are classic examples of this situation). Even more, the range of technologies has a direct impact on the cost and development time of the project. For the CDME Project, all contents were built as interactive and dynamic web pages (which can run on all major browsers) with the integrated use of the following technologies: the mathematical components were implemented with the help of the Java language, the software GeoGebra (Hohenwarter 2012) and the software JavaView (Polthier et al. 2012), structure and organization are designed using the HTML and MathML languages, elements of style with CSS language, interactivity and dynamism with the JavaScript language. All software can be stored on a server for online use through the Internet or, alternatively, they may be downloaded or made available on CDs and flash drives for offline use (that is, in a browser, but without the need of an Internet connection).

In the following sections, we will present, briefly, some of the digital educational contents produced by the CDME Project.

Learning Objects on Spatial Geometry

Typically, the teaching of spatial geometry is done using the textbook and the chalkboard as the only tools. Thus, the student of mathematics has to face the arduous task of studying three-dimensional objects from two-dimensional representations presented statically in a book page or on a chalkboard. This transition from a drawing on a sheet of paper to the actual object in space is configured as a significant problem in the study of geometry in three dimensions. In fact, it appears that both students and teachers face difficulties in the construction and interpretation of two-dimensional representations of three-dimensional objects. Moreover, even books and periodicals often display wrong drawings of three-dimensional objects: (Grünbaum 1985) and (Casselman 2000). Therefore, any tool that can assist the teacher in the teaching of spatial geometry is welcome.

Certainly, the use of concrete manipulative materials is an indispensable educational resource, especially in the early grades. On the other hand, due to technical limitations, there are certain configurations and geometrical properties that are difficult to be worked out using such materials. In this context, the computer puts itself as a promising tool for the teaching of spatial geometry. However, only recently 3D dynamic geometry software began to appear: Archimedes Geo3D,[7] Cabri 3D,[8] GEUP 3D[9] and GeoGebra 3D.[10] Of these four applications, the first three are not free and the last one is in beta test. Thus, there is still an enormous lack of free educational software geared towards the teaching and learning of spatial geometry. To try to fill this gap, it seemed to us that it would be much appropriated to prioritize and to put efforts on the production of learning objects for this theme.

Trip-Lets

Inspired by the cover of the book (Hofstadter 1999), in this activity, we propose an interactive game to exercise three-dimensional thinking in an interdisciplinary work with Portuguese, Spanish and English languages: the student must manipulate a virtual special solid and identify three letters of the alphabet. With these three letters, he or she must compose a word or an acronym in the chosen language (Fig. 1). To assist him or her in the viewing process, coordinate axes and coordinate planes can be displayed in the software. In addition to the vocabulary exercise and the cognitive exercise of perception of three-dimensional forms, the software can also be used to address questions on symmetry (of the shape of letters) and counting (of the possible words for a given set of letters).

[7] http://raumgeometrie.de.

[8] http://www.cabri.com/cabri-3d.html.

[9] http://www.geup.net/en/geup3d/.

[10] http://www.geogebra.org/forum/viewtopic.php?f=52&t=19846.

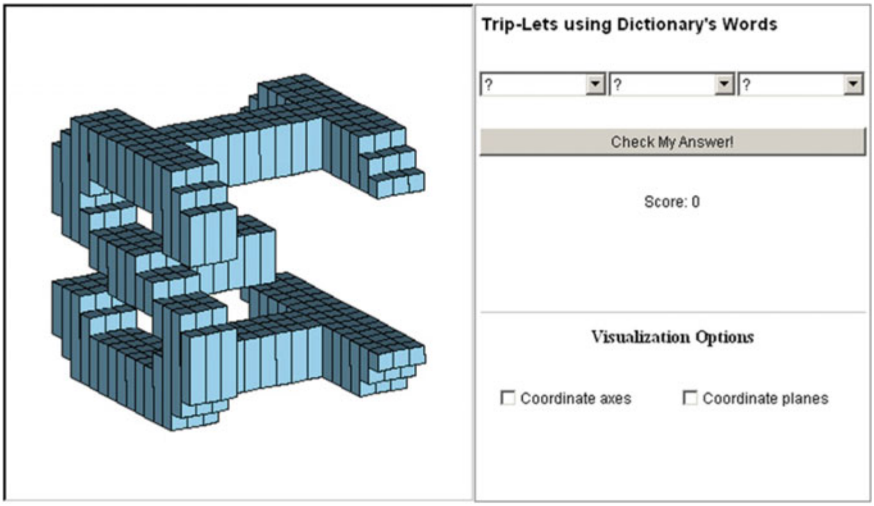


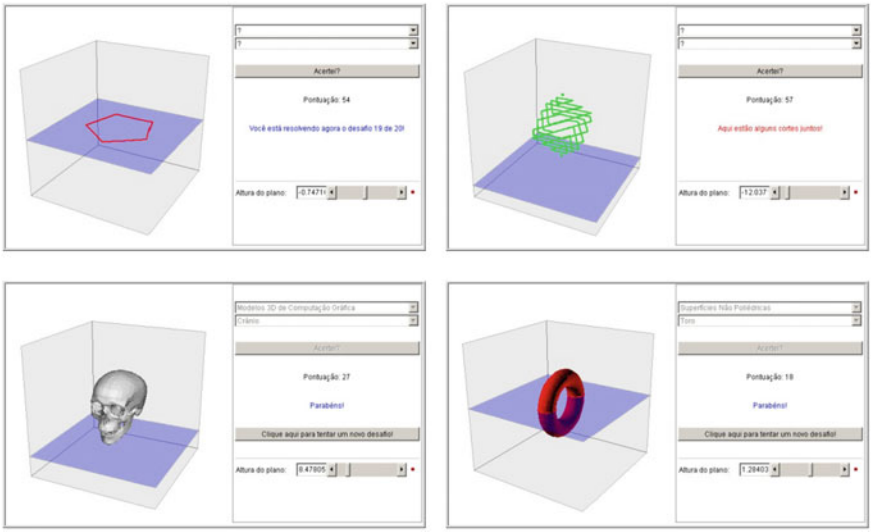
Fig. 1 Trip-Lets game with the English word SHE

The Tomography Game

The objective of this game is to identify an object that is partially invisible: only the intersection of the object with a plane is displayed. The height of this plane can be changed and, thus, the student has access to the various cross sections of the object, one at a time (Fig. 2). There are three categories of objects: polyhedra (with eight objects: a regular tetrahedron, a cube, a regular octahedron, a regular dodecahedron, a regular icosahedron, a pyramid of quadrangular base, a pyramid of pentagonal base and a regular pentagonal prism), non-polyhedral surfaces (with four objects: a right circular cone, a right circular cylinder, a sphere and a torus) and 3D models in computer graphics (with four objects: a skull, a dromedary, an octopus and a cow). In the game, a same object can appear in different positions. If the student fails to identify the object in two attempts, the program provides a hint: 10 cross sections of the object are displayed simultaneously. Along with the software, we present the use of cross sections in computed tomography and rapid prototyping technology (3D printers).

Projections in Perspective

Through various interactive modules, this content discusses some important aspects of the perspective projections, including questions of ambiguity (different objects that have the same perspective projection) (Fig. 3) and questions of deformation (as, for example, circles, which are projected in ellipses or straight line segments).


?
?
Acertei?
Acertei?
Acertei?
Acertei?
Clique aqui para tentar um novo desafio!
Clique aqui para tentar um novo desafio!

Pontuação: 54

Você está resolvendo agora o desafio 19 de 20!

Altura do plano: -0.7471

Pontuação: 57

Aqui estão alguns cortes juntos!

Altura do plano: -12.037

Modelos 3D de Computação Gráfica

Crânio

Pontuação: 27

Parabéns!

Altura do plano: 8.47805

Superfícies Não Poliédricas

Toro

Pontuação: 18

Parabéns!

Altura do plano: 1.28403

Fig. 2 The tomography game

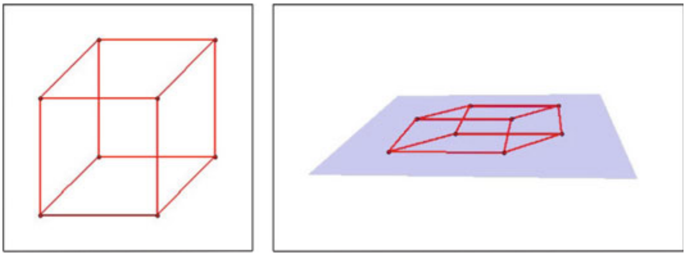


Fig. 3 Perspective projections: a cube?

The content also explores the use of perspective projections in photos and works of art (vanish points), including the analysis of some "impossible objects" of the Dutch artist Maurits Cornelis Escher (Fig. 4).

A Plethora of Polyhedra

The main element of this content is an application that allows the student to view and manipulate various types of polyhedra, such as the Platonic solids, the Archimedean solids, the Johnson solids, the Catalan solids, the isohedra, prisms, pyramids, etc. Several operations are on hand: polar reciprocation (to compute dual

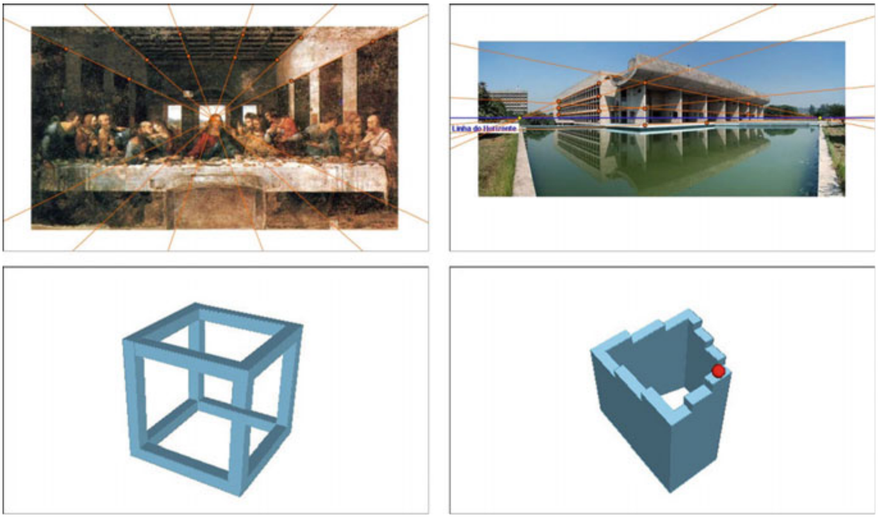
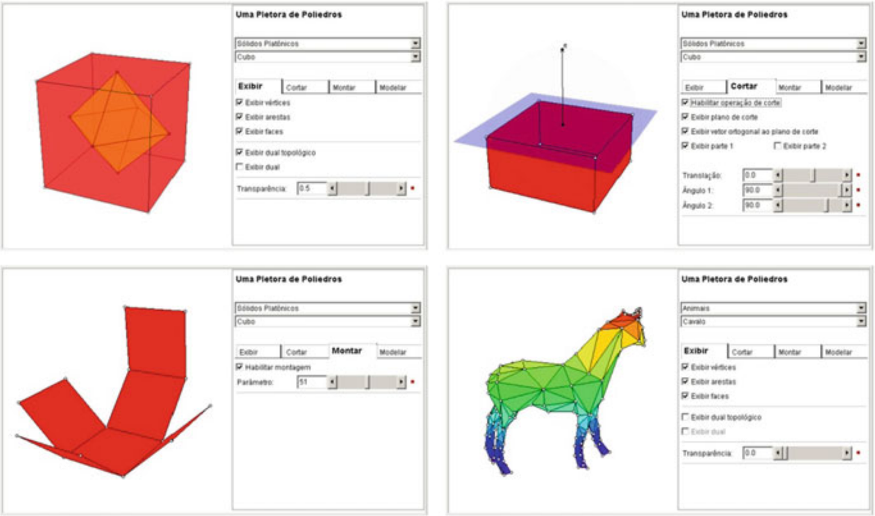


Fig. 4 Perspective projections in photos and works of art

polyhedra), Euler's characteristic computation, cuts by planes (cross sections), unfolding, truncation and stellation (Fig. 5). There are over 300 convex and non-convex polyhedra available.



Uma Pletora de Poliedros

Sólidos Platônicos
Cubo

Exibir | Cortar | Montar | Modelar

☑ Exibir vértices
☑ Exibir arestas
☑ Exibir faces
☑ Exibir dual topológico
□ Exibir dual

Transparência: 0.5

Uma Pletora de Poliedros

Sólidos Platônicos
Cubo

Exibir | **Cortar** | Montar | Modelar

☑ Habilitar operação de corte
☑ Exibir plano de corte
☑ Exibir vetor ortogonal ao plano de corte
☑ Exibir parte 1 □ Exibir parte 2

Translação: 0.0
Ângulo 1: 90.0
Ângulo 2: 90.0

Uma Pletora de Poliedros

Sólidos Platônicos
Cubo

Exibir | Cortar | **Montar** | Modelar

☑ Habilitar montagem

Parâmetro: 51

Uma Pletora de Poliedros

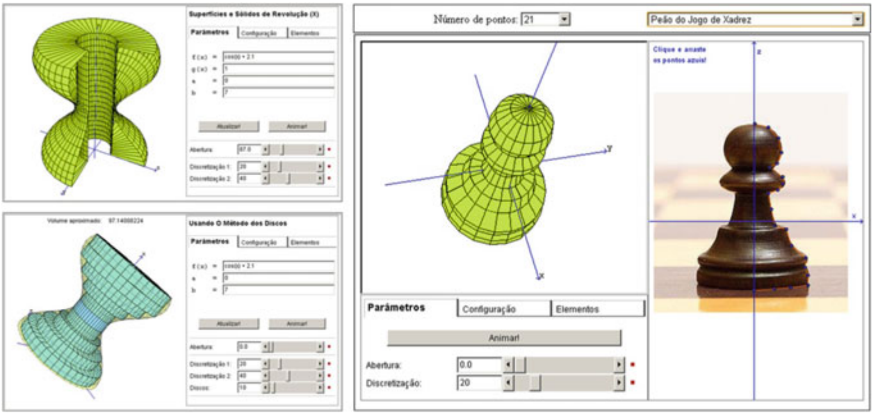
Animais
Cavalo

Exibir | Cortar | Montar | Modelar

☑ Exibir vértices
☑ Exibir arestas
☑ Exibir faces
□ Exibir dual topológico
□ Exibir dual

Transparência: 0.0

Fig. 5 A plethora of polyhedra



f(x) = cos(x) + 2.1
g(x) = 1
a = 0
b = 7
Abertura: 87.0
Discretização 1: 20
Discretização 2: 40

Volume aproximado: 97.14088224
f(x) = cos(x) + 2.1
a = 0
b = 7
Abertura: 0.0
Discretização 1: 20
Discretização 2: 40
Discos: 10

Número de pontos: 21
Peão do Jogo de Xadrez
Abertura: 0.0
Discretização: 20

Fig. 6 Surfaces and solids of revolution

Surfaces and Solids of Revolution

This software offers a three-dimensional interactive environment where it is possible to view and to study a solid of revolution whose generatrix is described by real functions of one variable. With it, students can investigate how to use different types of functions (including functions defined by parts) to obtain solids of revolution with different shapes (cylinders, spheres, cones, tori, bottles, etc.). Moreover, through this description of solids of revolution via functions, students can see how approximations of these functions may be used to compute approximations of the volume of the original solid (using the disc and the cylindrical shell methods). With the objective to promote the perception of the presence and the use of surfaces of revolution in our lives, we have also prepared a special module where the student can interactively model a 3D object that is a surface of revolution using a photo of its profile (Fig. 6).

Matrices and Digital Images

This software (a joint work with Dirce Uesu Pesco) has two main objectives: (1) to present an application of matrices and their operations in digital images and (2) to explore the use of functions in the context of image processing, that is, to investigate how various effects that may be applied to an image (such as grayscale, brightness, quantization and transition) can be obtained through a convenient real function.

The content of this activity is divided into five main modules. The first module shows how binary digital images can be represented by matrices and how certain

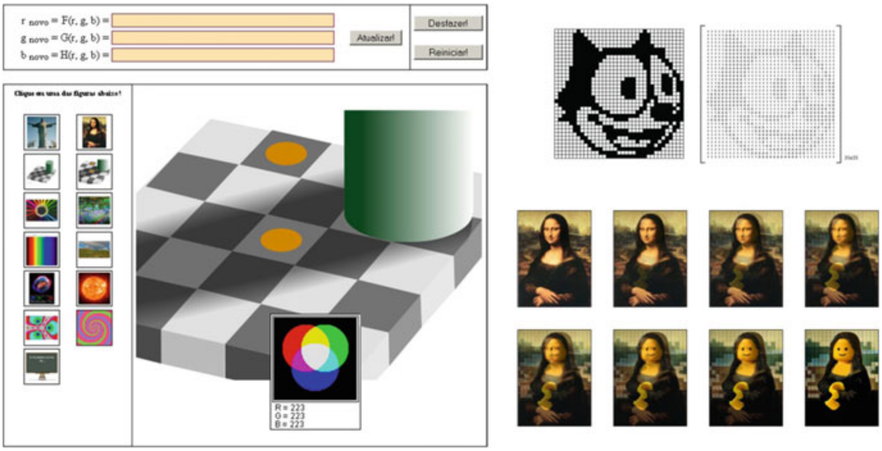

$r_{novo} = F(r, g, b) =$
$g_{novo} = G(r, g, b) =$
$b_{novo} = H(r, g, b) =$
Atualizar!
Desfazer!
Reiniciar!
Clique em uma das figuras abaixo!
R = 223
G = 223
B = 223

Fig. 7 Matrices and digital images

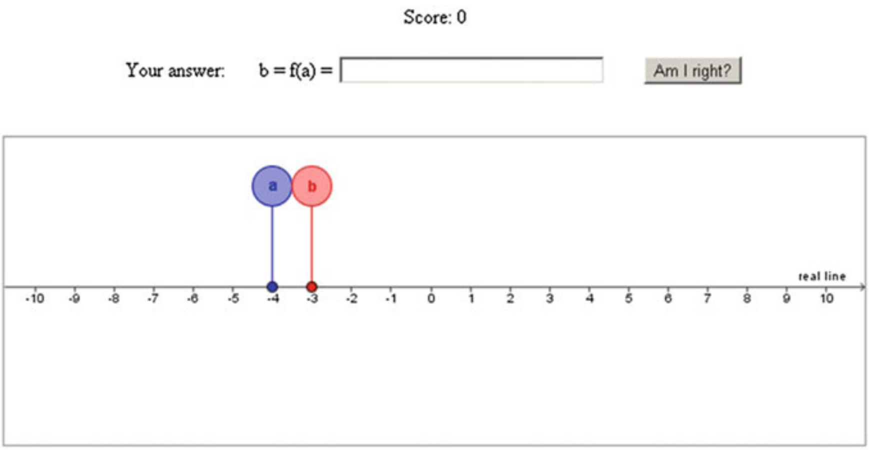
image manipulations are performed through operations on rows and columns of the corresponding matrix. The second module examines grayscale images through a program that allows the student to modify the intensity of a pixel through a user-defined function. The third module studies color images using the RGB system. In this system, three matrices are used to specify the amount of red, green and blue that makes up the image. Among the images studied in this module, there are the optical illusions created by Edward H. Adelson and Akiyoshi Kitaoka. The fourth module shows how a color in the RGB system can be identified with a point of a cube (the Color Cube). The fifth module shows how to use the operations of multiplication by scalar and sum of matrices to create a transition effect between images of the same size (Fig. 7). The material produced for this activity was the source of a submitted vignette for the Klein Project: (Pesco and Bortolossi 2012).

Learning Objects on Functions

How b *Depends on* a?

This software (a joint work with Dirce Uesu Pesco) is presented as a game with 16 challenges. For each one, two points, a and b, are marked on a number line. The student can then click and drag the point a. In doing so, the point b will change its position in accordance with the position of the point a, that is, b is a function of a: $b = f(a)$. To win the challenge, the student has just to find out which is the expression that defines the function f (Fig. 8).

If the student types an expression as an answer to the challenge and if he or she presses the “Am I right?” button, then the application will tell if he or she succeeded



Score: 0

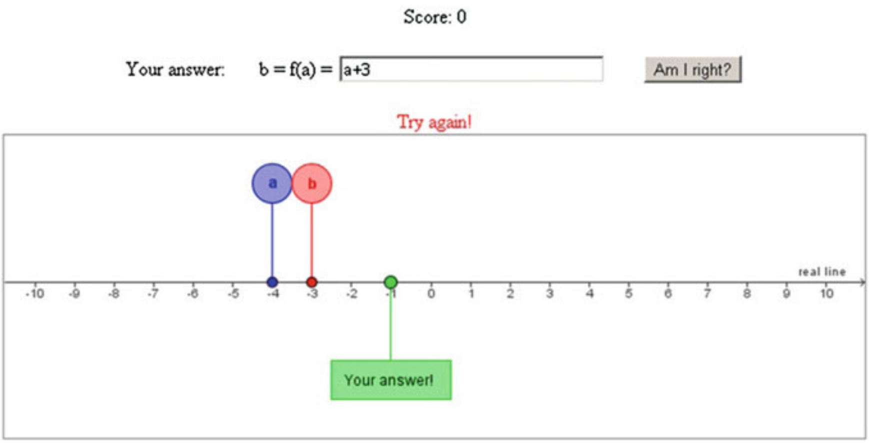
Your answer: b = f(a) =

Fig. 8 How *b* depends on *a*?

or not, and at the same time, the application will create a third point (in green) on the number line corresponding to the value of the function of the student evaluated at the point *a* (Fig. 9). If the point *a* is dragged, the point *b* and the green point will move dynamically, thus allowing the student to reassess him or her answer in case of error.

If the student misses two times, the application will give a hint (the function is linear, the function is quadratic, etc.). If he or she misses again, the application will reveal the correct answer.

Traditionally, real functions are studied using algebraic techniques and graphs. In this activity, the subject is explored in a different and unusual way, namely, when



Score: 0

Your answer: b = f(a) = a+3

Try again!

Fig. 9 Software's feedback for a wrong answer of the student

a point of the domain and its image are represented in the same number line. Thus, when interacting with the proposed game, students can assess their knowledge of properties of elementary functions.

The idea of the game is a limiting case of the *dynagraphs*, originally proposed by Goldenberg et al. (1992). In our opinion, working with a single number line (instead of the two number lines of the *dynagraphs*) has an advantage: it is easier to make comparisons between the values of a and b and thus try to establish some functional dependence. With different axes, there is an extra cognitive effort of transferring values from one axis to another.

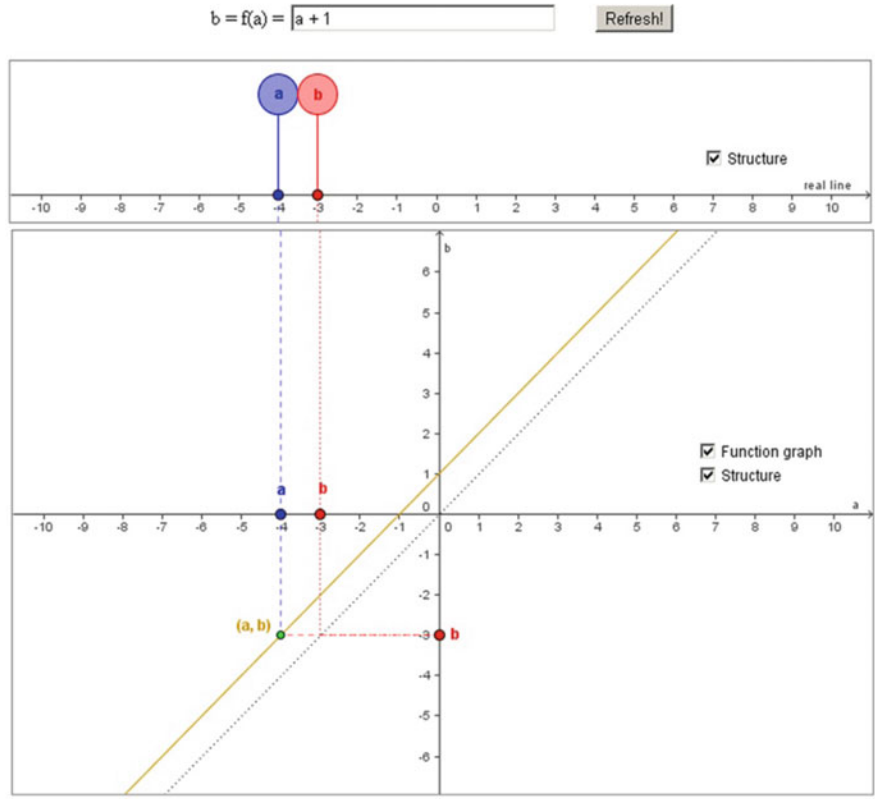
Our experience with this activity has shown that students use the following typical strategy: change the position of point a, observe the position of point b, generate a table of functional values and, from this table, try to deduce an interpolating function. Other strategies may be induced: what is the variation Δb of b in terms of a variation Δa of a? (This strategy is particularly useful for linear functions.) Are there symmetries? Is the function even? Is the function increasing? Is the function injective? Trying to identify such properties in this new environment is a great exercise to practice these concepts.

At the end of the game, the software displays a page where students can visualize the relationship between the representation used in the game with graphs of functions (Fig. 10).

The Optimal Project

This activity (a joint work with Gilda de la Rocque Palis and Silvana Marini Rodrigues Lopes) is divided into 21 modules. In each module, the statement of an optimization problem is presented. The student should read it, interpret it and then perform a sequence of tasks with the use of two computational resources available in the module. In the first computational resource, the student can enter values for the independent variable x of the problem. If the value of x is not admissible, the program will acknowledge the fact. If the value of x is admissible, then the program will (1) calculate the value of the dependent variable V for that value of x, (2) record the values of x and V in a table, (3) draw the point (x, V) in a Cartesian coordinate system and (4) update the geometric configurations of the statement of the problem for these values. If the student clicks on one of the lines of the table, the program will update the geometric configurations of the statement in accordance with the values recorded on this line. Figure 11 shows the graphical interface of the first computational resource for the classic box problem: “Equal squares are to be cut off from the four corners of a flat rectangular cardboard sheet measuring 30 cm by 50 cm. The remaining flaps are then folded up to form an open box. What should be the value of x, the measure of the sides of the squares that were removed, to make the volume V of the corresponding box as large as possible?”.

As indicated by Palis (2011): The tasks are designed to advance the issue of reading and interpretation of word problems by students, the discrimination



b = f(a) = a + 1

☑ Structure

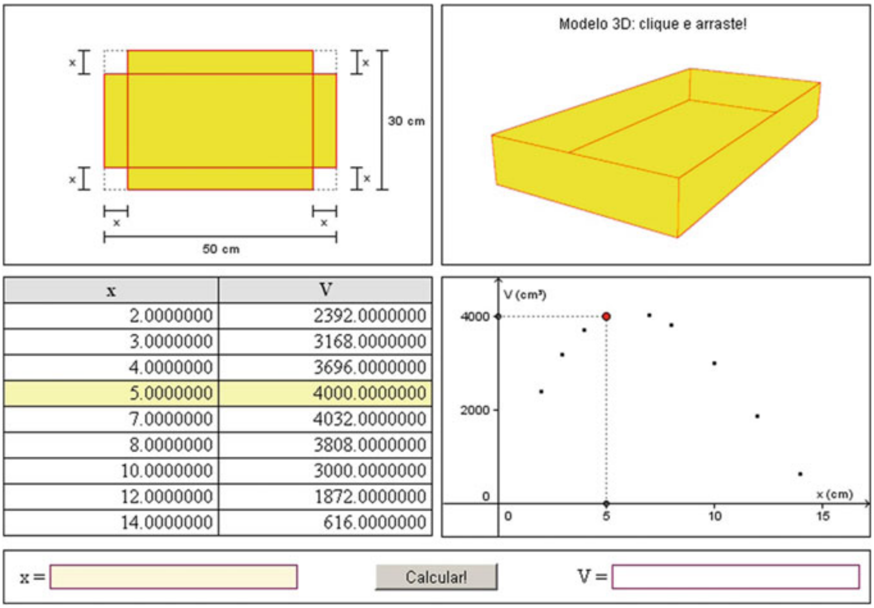
☑ Function graph
☑ Structure

Fig. 10 Relationship between the representation used in the game with graphs of functions

between constants and variables, the development of the concept of approximation of a real number, the justification process in problem solving, the use of multiple representations of functions. The proposed tasks culminate in two points: (1) the student must determine (with pencil and paper) the objective function and the admissible set of the optimization problem (and, here, the second computational resource can be used to check the answer), and (2) the student must determine the optimal solution explicitly for the case of quadratic objective functions or an approximation of the optimal solution with error less than 0.01 for all other cases.

Epicycles and Trigonometric Interpolation

In this activity (a joint work with Carlos Tomei), we propose an interactive application that explores, through the theory of epicycles, the connections between trigonometric functions and the circle (with special emphasis on amplitudes and



x	V
2.0000000	2392.0000000
3.0000000	3168.0000000
4.0000000	3696.0000000
5.0000000	4000.0000000
7.0000000	4032.0000000
8.0000000	3808.0000000
10.0000000	3000.0000000
12.0000000	1872.0000000
14.0000000	616.0000000

Fig. 11 The optimal project: the classic box problem

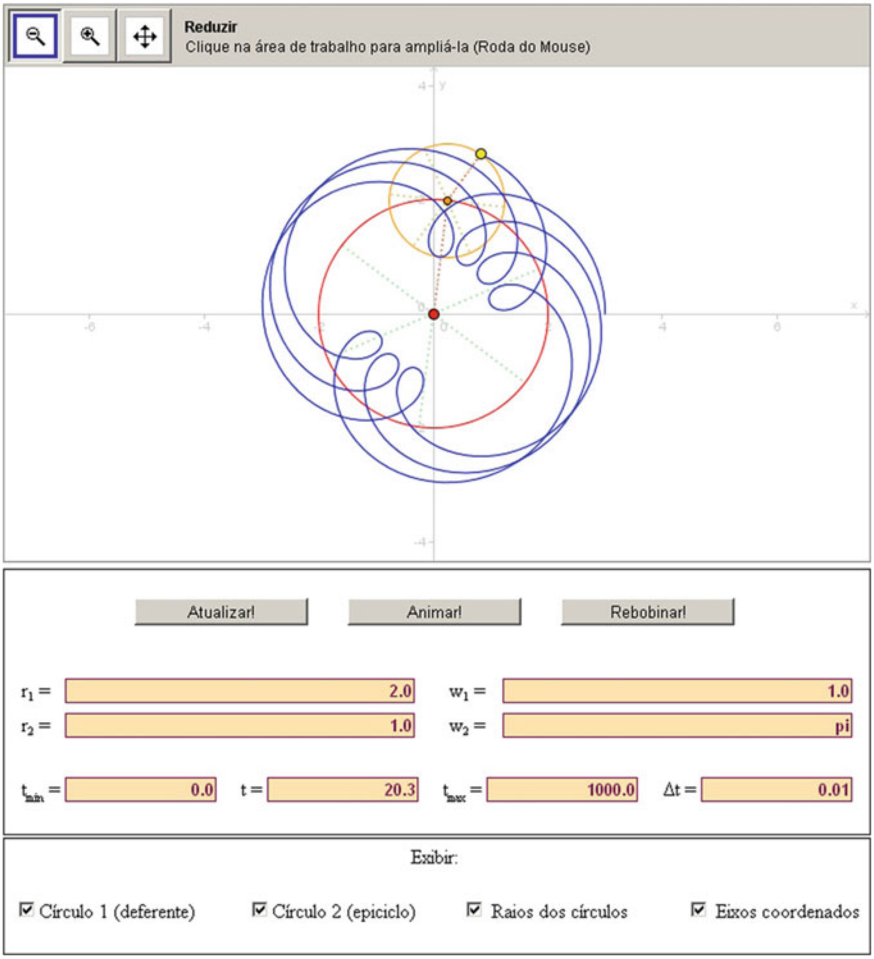
frequencies), parametrized curves, rational and irrational numbers in a geometric context (periodic orbits).

The activity is divided into three parts. In the first part, we present the basic model of the theory of epicycles, with only two coupled circles. The student can change the values of the radii of the circles (input fields r_1 and r_2 in Fig. 12) the values of the angular velocities with which the points move (input fields w_1 and w_2 in Fig. 12). If the student presses the “Animate!” button, then the application will animate, in function of the time t, the point of coordinates (r_1 cos(w_1 t), r_1 sin(w_1 t)), the circle with center in this point and radius r_2, and the yellow point with coordinates

$$x = r_1\cos(w_1t) + r_2\cos(w_2t) \quad \text{and} \quad y = r_1\sin(w_1t) + r_2\sin(w_2t).$$

Several questions can be explored: is it possible to make the yellow point draw a circle or an ellipse or a straight line segment? Roses (epicycles with $r_1 = r_2$) can be used to study how the ratio $|w_1/w_2|$ affects the geometry of the curve (a rose “never closes” if w_1/w_2 is an irrational number, and if w_1/w_2 is a rational number, investigate the relationship between the number of petals and the value of w_1/w_2).

In the second part, we describe how the basic model of the theory of epicycles with two coupled circles can be used to explain the retrograde motion of the planets (the Ptolemaic system).



Reduzir
Clique na área de trabalho para ampliá-la (Roda do Mouse)

$r_1 =$ 2.0 $w_1 =$ 1.0
$r_2 =$ 1.0 $w_2 =$ pi

$t_{min} =$ 0.0 $t =$ 20.3 $t_{max} =$ 1000.0 $\Delta t =$ 0.01

Exibir:

☑ Círculo 1 (deferente) ☑ Círculo 2 (epiciclo) ☑ Raios dos círculos ☑ Eixos coordenados

Fig. 12 The basic model of the theory of epicycles

In the third and final part, we show how to use epicycles with several coupled circles to approximate curves (a prelude to the theory of trigonometric interpolation and Fourier series). Figure 13 shows an approximation of the Batman symbol obtained with 32 circles coupled.

The Anatomy of a Quadratic Function

Quadratic functions are usually studied in connection with quadratic equations. Often the approach is restricted to algebraic manipulation, leaving aside important

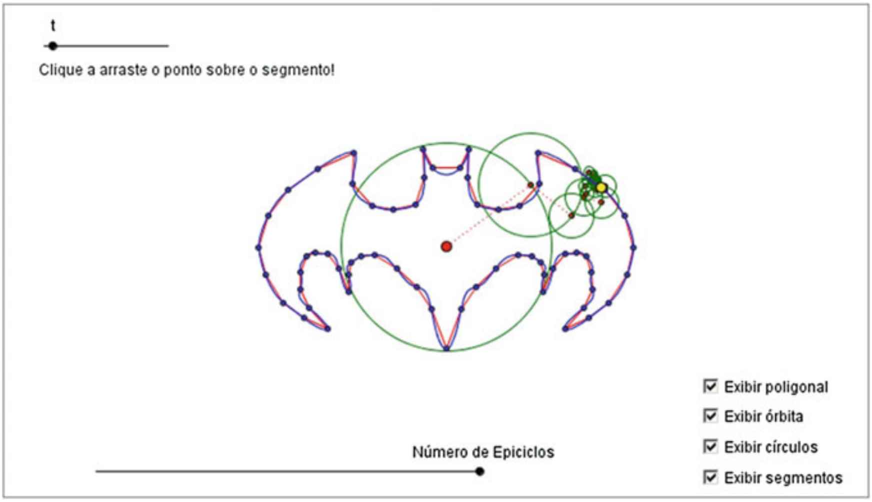
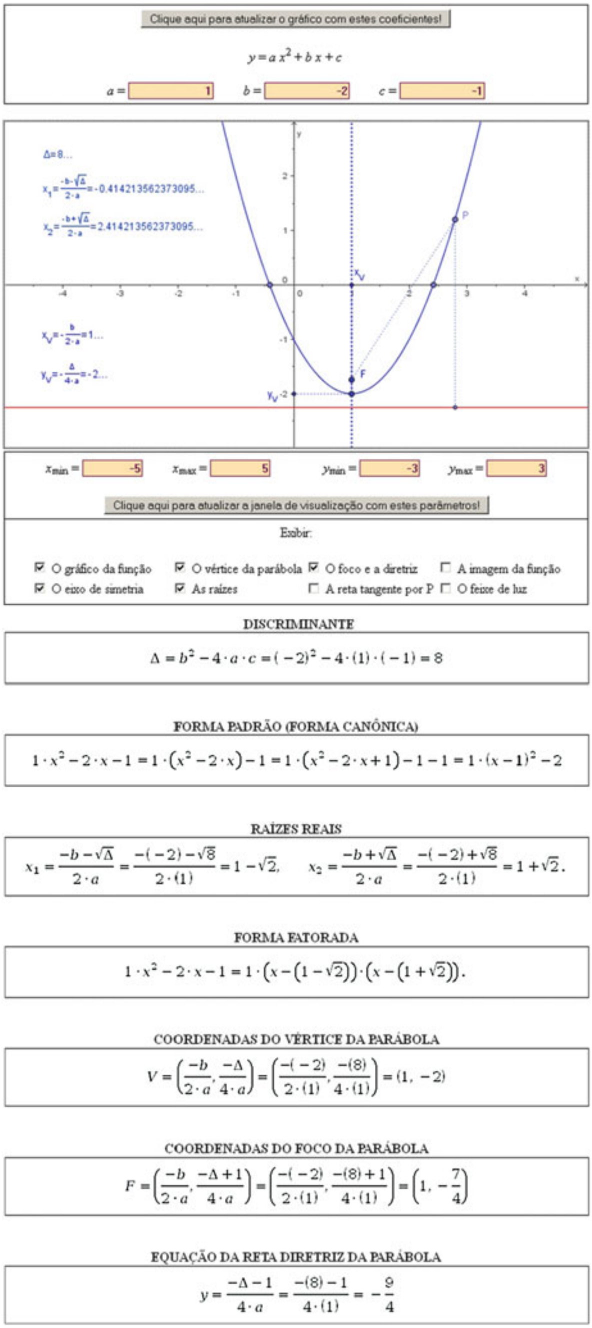


Fig. 13 Epicycles drawing the batman symbol

geometrical aspects. However, even some algebraic aspects are not developed with the proper intensity, as, for example, the fundamental technique of completing the square. Another remarkable characteristic of the teaching of this subject is the almost exclusive use of integer coefficients, which inhibits certain investigations on the dependence of geometric elements of the graph of a quadratic function with respect to their coefficients. Divided into four interfaces, this software offers to math teachers graphical, symbolical and numerical resources to subsidize the presentation and development of the theory of quadratic functions.

The Interface 1 (Fig. 14) allows the student to define a quadratic function with integer coefficients. This restriction is deliberate, because we believe that the first contact with the software should occur through an interface that is closer to that usually presented in textbooks. In this interface, there are two main areas: the graphic area and the algebraic area. In the graphic area of the Interface 1, students can view the following elements: the graph of the function; the vertex, focus and directrix of the parabola; the image of the function; the symmetry axis; the real roots (if any); a draggable tangent line; and a light beam (a set of parallel lines to the y-axis that hits the graph of the function and, then, converges at the focus). The viewport can be configured and there is an input field where the student can mark points on the graph of the function by specifying their abscissas. The coordinates of the vertex and the roots (if any) are also presented numerically, that is, with their values approximated by decimal numbers. In the algebraic area of the Interface 1, the discriminant, the real roots (if any), the standard form, the factored form, the coordinates of the vertex, the coordinates of the foci and the directrix equation are presented symbolically, that is, without approximations. By changing the coefficients of the quadratic function at the top of the interface, all graphical, symbolical and numerical elements are updated automatically.

Fig. 14 The anatomy of a quadratic function



Clique aqui para atualizar o gráfico com estes coeficientes!

$$y = a\,x^2 + b\,x + c$$

$a =$ 1 $b =$ -2 $c =$ -1

Δ=8...
$x_1 = \frac{-b-\sqrt{\Delta}}{2\cdot a} = -0.414213562373095...$
$x_2 = \frac{-b+\sqrt{\Delta}}{2\cdot a} = 2.414213562373095...$
$x_V = -\frac{b}{2\cdot a} = 1...$
$y_V = -\frac{\Delta}{4\cdot a} = -2...$
y
x
-4
-3
-2
-1
0
1
2
3
4
-1
-2
P
F
x_V
y_V

$x_{\min} =$ -5 $x_{\max} =$ 5 $y_{\min} =$ -3 $y_{\max} =$ 3

Clique aqui para atualizar a janela de visualização com estes parâmetros!

Exibir:

☑ O gráfico da função ☑ O vértice da parábola ☑ O foco e a diretriz ☐ A imagem da função

☑ O eixo de simetria ☑ As raízes ☐ A reta tangente por P ☐ O feixe de luz

DISCRIMINANTE

$$\Delta = b^2 - 4\cdot a\cdot c = (-2)^2 - 4\cdot(1)\cdot(-1) = 8$$

FORMA PADRÃO (FORMA CANÔNICA)

$$1\cdot x^2 - 2\cdot x - 1 = 1\cdot\left(x^2 - 2\cdot x\right) - 1 = 1\cdot\left(x^2 - 2\cdot x + 1\right) - 1 - 1 = 1\cdot(x-1)^2 - 2$$

RAÍZES REAIS

$$x_1 = \frac{-b-\sqrt{\Delta}}{2\cdot a} = \frac{-(-2)-\sqrt{8}}{2\cdot(1)} = 1-\sqrt{2}, \qquad x_2 = \frac{-b+\sqrt{\Delta}}{2\cdot a} = \frac{-(-2)+\sqrt{8}}{2\cdot(1)} = 1+\sqrt{2}.$$

FORMA FATORADA

$$1\cdot x^2 - 2\cdot x - 1 = 1\cdot\left(x-\left(1-\sqrt{2}\right)\right)\cdot\left(x-\left(1+\sqrt{2}\right)\right).$$

COORDENADAS DO VÉRTICE DA PARÁBOLA

$$V = \left(\frac{-b}{2\cdot a}, \frac{-\Delta}{4\cdot a}\right) = \left(\frac{-(-2)}{2\cdot(1)}, \frac{-(8)}{4\cdot(1)}\right) = (1, -2)$$

COORDENADAS DO FOCO DA PARÁBOLA

$$F = \left(\frac{-b}{2\cdot a}, \frac{-\Delta+1}{4\cdot a}\right) = \left(\frac{-(-2)}{2\cdot(1)}, \frac{-(8)+1}{4\cdot(1)}\right) = \left(1, -\frac{7}{4}\right)$$

EQUAÇÃO DA RETA DIRETRIZ DA PARÁBOLA

$$y = \frac{-\Delta-1}{4\cdot a} = \frac{-(8)-1}{4\cdot(1)} = -\frac{9}{4}$$

For convenience, the Interface 2 presents only the algebraic area of the Interface 1, in order to prevent the student only interested in algebraic calculations to have to scroll the HTML page in the browser. The Interface 3 allows the student to provide decimal numbers with a finite number of figures as coefficients of the quadratic function. The Interface 4 replaces the input fields by sliders in the specification of the coefficients and, so, the student may visualize, in a dynamic way, how the various graphical elements change as the values of the sliders are modified.

Among the proposed exercises, two classes are noteworthy (because they are particularly difficult to explore using only the chalkboard): (1) exercises that explore graphs drawn in a coordinate system with different scales and (2) the exercises that explore the dependence of the several geometric elements of the graph of a quadratic function with respect to their coefficients.

The Triangle Classification Game

This is one of our most accessed software (in Portuguese and English). In it, we present an interactive game to train the classification of triangles according to the sides and angles. The idea is quite simple: in the game, the student must move the vertices of the triangle on a Cartesian grid in the plane (so, the vertices of the triangle can be only placed on points with integer coordinates), in order to form the triangle that is asked in each challenge (Fig. 15).

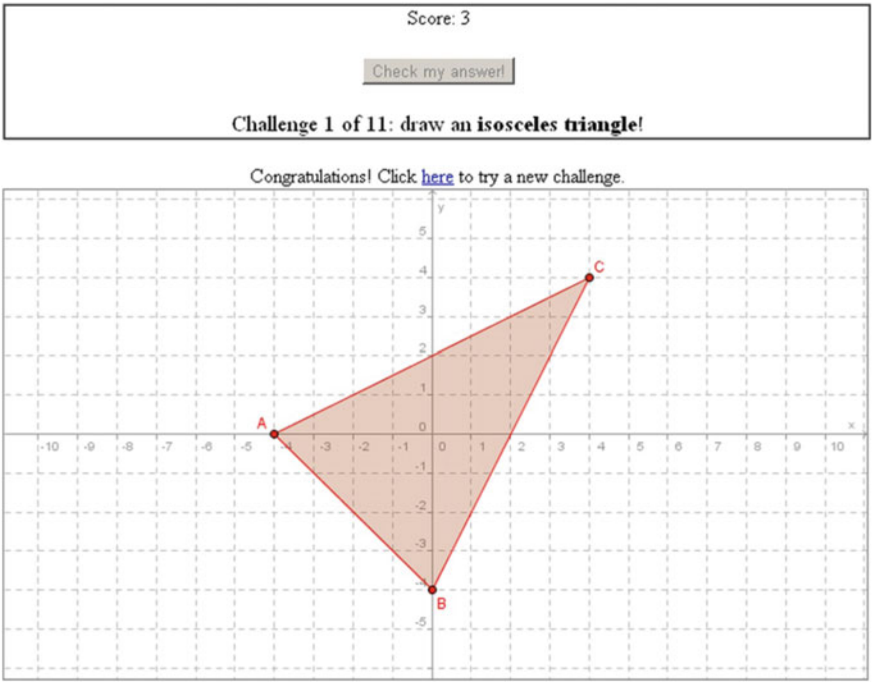
The environment integrates algebra and geometry, and it is a place where some interesting questions may be posed: (1) Is it is possible to form a triangle rectangle with a horizontal hypotenuse? (2) Is it possible to form an isosceles triangle with a non-horizontal and a non-vertical base? (3) Is it possible to form an isosceles triangle with all sides not horizontal and not vertical? (4) Is it possible to form an equilateral triangle whose vertices have integer coordinates?

We also created a game similar to this, but that explores the classification of quadrilaterals: parallelograms, rhombi, kites, trapeziums, orthodiagonal quadrilaterals, cyclic quadrilaterals and inscriptible quadrilaterals.

Learning Objects on Probability and Statistics

Wheels of Fortune

This software offers various modules for the simulation of random experiments on the computer using pseudo random number generation algorithms, with emphasis on geometric probability. The Module 1 simulates a wheel of fortune where the sample space and probabilities can be configured, allowing the simulation of classical experiments like the toss of a coin and the roll of a dice (Fig. 16). There are



Score: 3

Challenge 1 of 11: draw an **isosceles triangle**!

Congratulations! Click here to try a new challenge.

Fig. 15 The triangle classification game

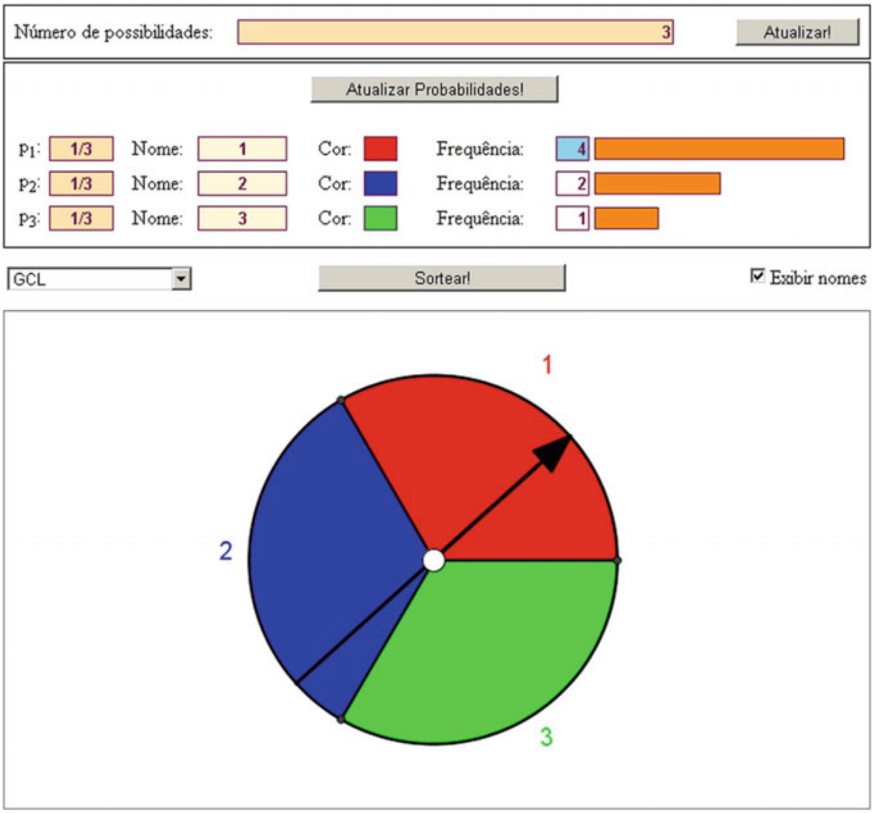
three pseudo random number generators: the linear congruential generator (LCG), the Mersenne Twister and the Mersenne Twister 53.

The Module 2 explains the mathematics behind an algorithm widely used in the generation of pseudo random numbers on the computer: the linear congruential generator. The other modules are simulations where it is possible to test the linear congruential generator. The last three simulations are experiments in geometric probability: estimation of the area of a circle, estimation of the volume of a sphere (Fig. 17) and the classical Buffon's needle problem.

The material produced for this activity was the source of a submitted vignette for the Klein Project in Portuguese: (Bortolossi 2011a).

Statistics of Letters, Words and Periods

This software performs statistical analysis of strings. More precisely, the application counts the number of letters, digits, accents, punctuation marks, words and periods of a text. It also calculates the number of letters per word and words per period (showing position and dispersion measures of these quantitative variables), the longest periods, the shortest periods, the longest words and the shortest words.



Número de possibilidades: 3 Atualizar!

Atualizar Probabilidades!

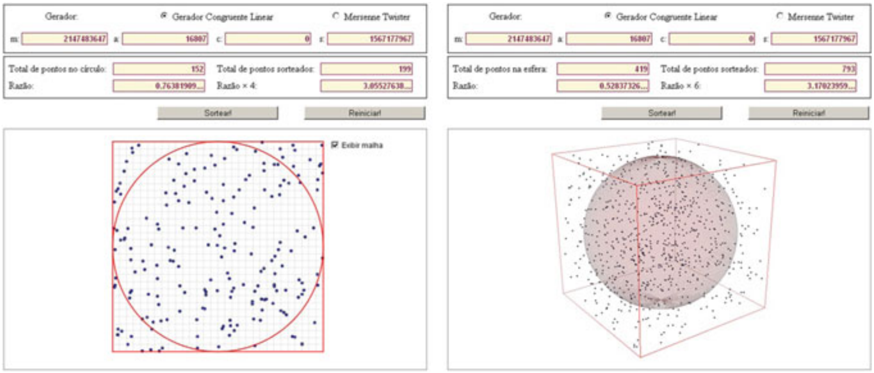
p_1: 1/3 Nome: 1 Cor: 🟥 Frequência: 4

p_2: 1/3 Nome: 2 Cor: 🟦 Frequência: 2

p_3: 1/3 Nome: 3 Cor: 🟩 Frequência: 1

GCL ▾ Sortear! ☑ Exibir nomes

Fig. 16 Wheels of fortune



Gerador: ● Gerador Congruente Linear ○ Mersenne Twister

m: 2147483647 a: 16807 c: 0 s: 1567177967

Total de pontos no círculo: 152 Total de pontos sorteados: 199

Razão: 0.76381909... Razão × 4: 3.05527638...

Sortear! Reiniciar!

☑ Exibir malha

Gerador: ● Gerador Congruente Linear ○ Mersenne Twister

m: 2147483647 a: 16807 c: 0 s: 1567177967

Total de pontos na esfera: 419 Total de pontos sorteados: 793

Razão: 0.52837326... Razão × 6: 3.17023959...

Sortear! Reiniciar!

Fig. 17 Using Monte Carlo methods to estimate areas and volumes

There are two special modules: the first module lets the student use the frequency distribution of letters to decode messages encrypted with the Caesar cipher, and the second one allows him or her to investigate power laws in frequency distributions of words in a text (Zipf's Law).

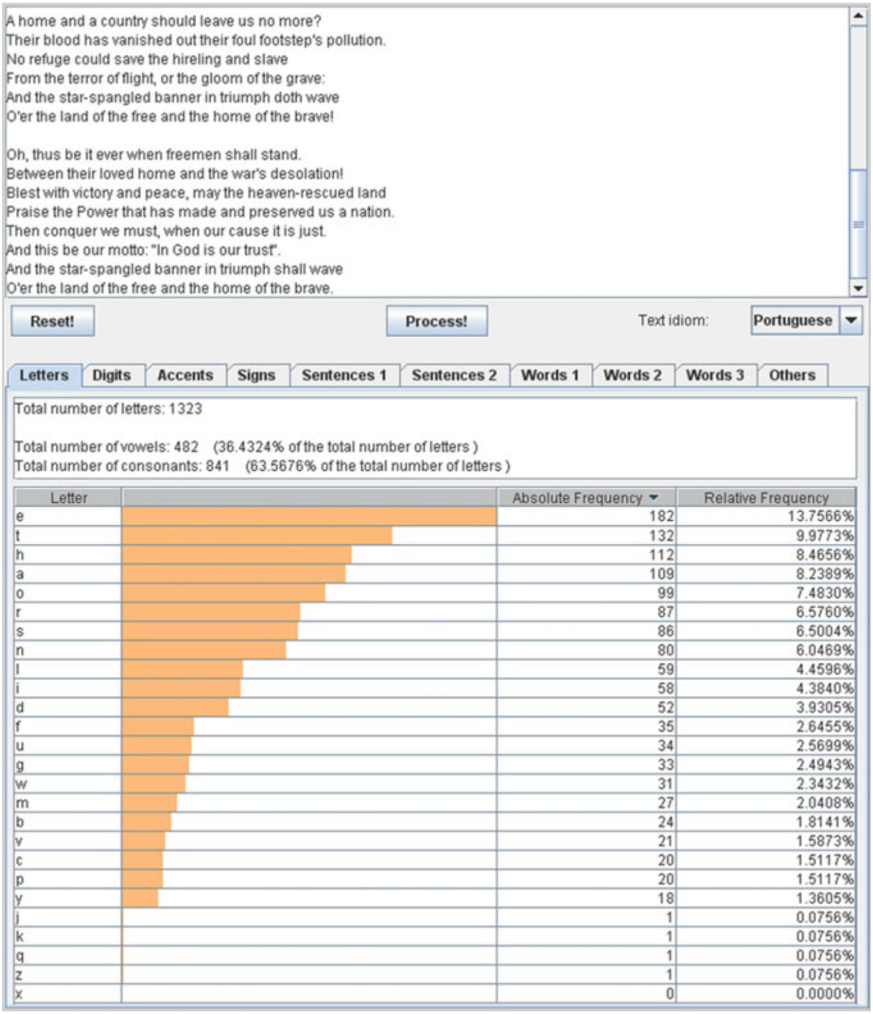
The main objective of our software is to provide an interactive environment in which students and teachers can experiment, explore and enjoy the use of statistics in a real-world application (namely, text mining), and through this exercise in the linguistic context, promote the learning of statistical concepts. In addition, this proposal has a practical feature: it is really very easy to find out data for analysis on the Internet (free books, poems, speeches, song lyrics, etc.).

Figure 18 displays the graphical user interface of the Module 2 (the main module of the software). Data may be entered using the text area at the top of the interface (this area is filled with the United States National Anthem in Fig. 18). The student can type some text directly, or use the "copy and paste" technique (ctrl + c/ctrl + v). If he or she presses the "Process!" button, then the text entered will be processed, and the results of the analysis will be informed in the various tabs of the program. The "Letters" tab, for instance, reports the total number of letters, the total number of vowels (that is, the letters *a*, *e*, *i*, *o* and *u*), the total number of consonants and, in the format of a table, the (absolute and relative) distribution frequency of the letters of the text. The informations in the table can be sorted by clicking repeatedly on the corresponding column header.

As an exercise, we encourage the student to investigate the frequency distribution of letters in texts with different sizes, languages and narratives. The objective is to make him or her realize that this distribution can be used to identify (automatically) the language of the text and, in this case, the sample size is important. This question is taken a step further: since the frequency distribution of the letters is a characteristic of the language, it follows that it is invariant under permutations of the names of the letters. Therefore, this property can be used to decode an encrypted text with the Caesar cipher (whose encoding principle is just the permutation of the names of the letters).

The "Digits" tab reports the total number of digits (0, 1, 2, 3, 4, 5, 6, 7, 8 and 9) in the text and its (absolute and relative) frequency distribution. Although the Module 2 has been designed for text analysis, it is also possible to use it to study frequency distributions of digits in decimal numbers and, thus, investigate questions on the normality of real numbers.

Another example of investigative work that may be carried out by students refers to the surprising Zipf's Law. This empirical power law, proposed by the linguist George Kingsley Zipf (1902–1950) of Harvard University, suggests that in a text with a large number of words, the frequency f of occurrence of a word as a function of its position r in a list sorted by frequency of occurrence has the following form: $f = C/r^a$, where C and a are constants, with the value of a close to 1. Note that, in the variables $y = \log(f)$ and $x = \log(r)$, the Zipf's Law is expressed as an affine function:



A home and a country should leave us no more?
Their blood has vanished out their foul footstep's pollution.
No refuge could save the hireling and slave
From the terror of flight, or the gloom of the grave:
And the star-spangled banner in triumph doth wave
O'er the land of the free and the home of the brave!

Oh, thus be it ever when freemen shall stand.
Between their loved home and the war's desolation!
Blest with victory and peace, may the heaven-rescued land
Praise the Power that has made and preserved us a nation.
Then conquer we must, when our cause it is just.
And this be our motto: "In God is our trust".
And the star-spangled banner in triumph shall wave
O'er the land of the free and the home of the brave.

Text idiom: Portuguese

Total number of letters: 1323

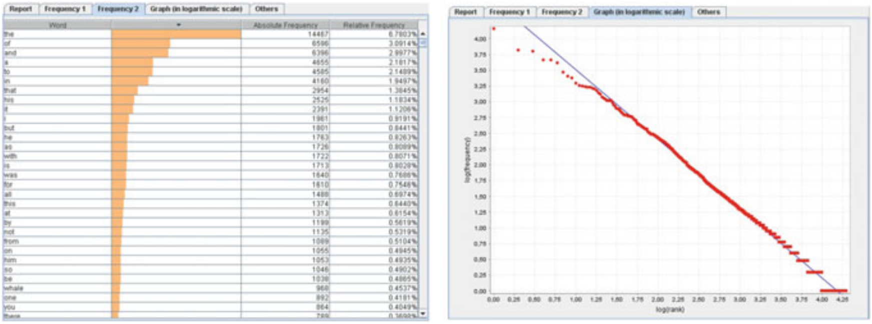
Total number of vowels: 482 (36.4324% of the total number of letters)
Total number of consonants: 841 (63.5676% of the total number of letters)

Letter		Absolute Frequency ▼	Relative Frequency
e		182	13.7566%
t		132	9.9773%
h		112	8.4656%
a		109	8.2389%
o		99	7.4830%
r		87	6.5760%
s		86	6.5004%
n		80	6.0469%
l		59	4.4596%
i		58	4.3840%
d		52	3.9305%
f		35	2.6455%
u		34	2.5699%
g		33	2.4943%
w		31	2.3432%
m		27	2.0408%
b		24	1.8141%
v		21	1.5873%
c		20	1.5117%
p		20	1.5117%
y		18	1.3605%
j		1	0.0756%
k		1	0.0756%
q		1	0.0756%
z		1	0.0756%
x		0	0.0000%

Fig. 18 Statistcs of the United States National Anthem

$y = b + a\,x$, with $b = \log(C)$. Thus, the coefficients a and b can be estimated, for example, using the method of least squares. This whole process is automated in the Module 3 of our software, an adaptation of the Module 2 oriented to the specific study of the Zipf's Law. Figure 19 illustrates the Zipf's Law for the novel "Moby Dick" of Herman Melville ($C = 40536.4574$ and $a = 1.1025$).

The material produced for this activity was the source of a submitted vignette for the Klein Project in Portuguese: (Bortolossi 2011b).



Word		Absolute Frequency	Relative Frequency
the		14467	6.7803%
of		6596	3.0914%
and		6396	2.9977%
a		4655	2.1817%
to		4585	2.1489%
in		4160	1.9497%
that		2954	1.3845%
his		2525	1.1834%
it		2391	1.1206%
i		1961	0.9191%
but		1801	0.8441%
he		1763	0.8263%
as		1726	0.8089%
with		1722	0.8071%
is		1713	0.8028%
was		1640	0.7686%
for		1610	0.7546%
all		1488	0.6974%
this		1374	0.6440%
at		1313	0.6154%
by		1199	0.5619%
not		1135	0.5319%
from		1089	0.5104%
on		1055	0.4945%
him		1053	0.4935%
so		1046	0.4902%
be		1038	0.4865%
whale		968	0.4537%
one		892	0.4181%
you		864	0.4049%
there		789	0.3698%

Fig. 19 Zipf's Law for the novel "Moby Dick"

Acknowledgments We would like to thank the Brazilian Ministry of Education, the Brazilian Ministry of Science and Technology, the National Fund for Educational Development (FNDE), the Project Klein in Portuguese, the Brazilian Mathematical Society (SBM) and the Institute of Mathematics and Statistics of the Fluminense Federal University for the financial support that subsidized this work. We also thank Ana Maria Martensen Roland Kaleff, Bernardo Kulnig Pagnoncelli, Dirce Uesu Pesco and Wanderley Moura Rezende for their suggestions to improve the manuscript.

References

Bortolossi, H. J. (2011a). Números (Pseudo) Aleatórios, Probabilidade Geométrica, Métodos de Monte Carlo e Estereologia [(Pseudo) Random Numbers, Geometric Probability, Monte Carlo Methods and Stereology]. Projeto Klein em Língua Portuguesa. Rio de Janeiro: Sociedade Brasileira de Matemática.

Bortolossi, H. J. (2011b). A Lei de Zipf e Outras Leis de Potência em Dados Empíricos [Zipf's Law and Other Power Laws in Empirical Data]. Projeto Klein em Língua Portuguesa. Rio de Janeiro: Sociedade Brasileira de Matemática.

Casselman, B. (2000). Pictures and proofs. *Notices of the American Mathematical Society, 47*(10), 1257–1266.

Goldenberg, P., Lewis, P., & O'Keefe, J. (1992). Dynamic representation and the development of a process understanding of functions. In G. Harel & E. Dubinsky (Eds.), *The concept of functions: Aspects of epistemology and pedagogy: MAA notes* (Vol. 25, pp. 235–260). Washington, DC: Mathematical Association of America.

Grünbaum, B. (1985). Geometry strikes again. *Mathematics Magazine, 58*(1), 12–17.

Hofstadter, D. R. (1999). *Gödel, Escher, Bach: An eternal golden braid*. New York, USA: Basic Books.

Hohenwarter, M. (2012). GeoGebra: Dynamics mathematics for everyone (Version 4.2) [Software]. Linz, Austria: The Johannes Kepler University. Retrieved from http://www.geogebra.org.

Lei, J., & Zhao, Y. (2007). Technology uses and student achievement: A longitudinal study. *Computers & Education, 49*(2), 284–296.

OECD. (2006). *Are students ready for a technology-rich world? What PISA studies tell us*. Paris, France: OECD.

Palis, G. (2011). O Conceito de Função: Da Concepção Ação à Concepção Processo. Desenvolvimento de Tarefas Instrucionais [The Concept of Function: From Action Conception to Process Conception. Development of Instructional Tasks]. *Boletim do LABEM*, *2*(2), 1–5.

Papanastasiou, E. C., Zembylas, M., & Vrasidas, C. (2003). Can computer use hurt science achievement? The USA results from PISA. *Journal of Science Education and Technology, 12* (3), 325–332.

Pesco, D. U., & Bortolossi, H. J. (2012). Matrices and digital images. Retrieved from http://wikis.zum.de/dmuw/Klein_Vignettes.

Polthier, K., Hildebrandt, K., Preuss, E., & Reitebuch, R. (2012). JavaView (Version 3.95) [Software]. Berlin, Germany: Freie Universität Berlin. Retrieved from http://www.javaview.de.

Roschelle, J., Shechtman, N., Tatar, D., Hegedus, S., Hopkins, B., Empson, S., et al. (2010). Integration of technology, curriculum, and professional development for advancing middle school mathematics: Three large-scale studies. *American Educational Research Journal, 47*(4), 833–878.

Witter, J., & Senkbeil, M. (2008). Is students' computer use at home related to their mathematical performance at school? *Computers & Education, 50*(4), 1558–1571.