Hands that See, Hands that Speak: Investigating Relationships Between Sensory Activity, Forms of Communicating and Mathematical Cognition

Lulu Healy

Abstract This contribution explores the role of the body's senses in the constitution of mathematical practices. It examines the mathematics activities of learners with disabilities, with the idea being that by identifying the differences and similarities in the practices of those whose knowledge of the world is mediated through different sensory channels, we might not only become better able to respond to their particular needs, but also to build more robust understandings of the relationships between experience and cognition more generally. To focus on connections between perceptual activities, material and semiotic resources and mathematical meanings, the discussion concentrates on the mathematical practices of learners who see with their hands or who speak with their hands. This discussion centres around two examples from our research with blind learners and deaf learners and, in particular, analyses the multiple roles played by their hands in mathematical activities.

Keywords Blind mathematics learners · Deaf mathematics learners · Embodied cognition · Gestures

Introduction

Can listening to learners' hands help us understand mathematical cognition? The search for answers to this question, and perhaps even its very posing, requires some reflections about how our bodies are involved in thinking mathematically and about the relationships between thinking and doing. In relation to these questions, recent years have seen a growing interest in the embodied nature of human cognition, with increasing evidence and support for idea that the ways in which we think should not (or cannot) be separated from the ways in which we act and that both doing and

L. Healy (✉)
Universidade Anhanguera de São Paulo, São Paulo, Brazil
e-mail: lulu@pq.cnpq.br

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imagining have their basis in our bodies, its physical capacities and its location in space and time (Gallese and Lakoff 2005; Barsalou 2008, 2009). Embodied approaches emphasising how the meanings assigned to abstract symbols are shaped by aspects of the body's sensory and motor systems and its interactions in its environments have also begun to permeate research in mathematics education (Nemirovsky and Borba 2004; Lakoff and Núñez 2000; Radford 2009; Roth 2010). Indeed the legitimacy of the opening question and the current attention to gestures in mathematical activity represent indications of the arrival of embodiment on the scene (see for example, the special issue of *Educational Studies in Mathematics*, Radford et al. 2009).

Yet, while the argument that our mathematical understandings are structured by our bodies' encounters and interactions with actual and virtual worlds might be gaining force, it is also important not to forget either that mathematics as discipline of knowledge is also a cultural affair, defined by the set of the artefacts and practices created in the historical trajectory of its construction, or that learning mathematics occurs in settings which too are associated with particular forms of social practices. In this context, it can hardly be controversial to claim that we develop and that we learn by interacting within the various biological, social and cultural systems that make up the world as we experience it. Individuals construct their own meanings for the mathematics they encounter which depend upon the ways and means through which they come into contact with the knowledge culturally labelled as mathematics, as well as upon their individual resources. Attention to how mathematics is encountered emphasises its social, cultural and political nature—as it cannot be assumed that all learners have the same opportunities to participate in the same learning activities.

In our work, which involves students with disabilities,[1] this is especially the case. Rather little attention has yet been given either to the particular ways in which students with different kinds of disabilities make sense of the mathematical artefacts which compose school mathematics or to the material and semiotic tools which best support their participation in mathematical activity. Our work with these students is in part motivated by our belief that if we understand the differences and similarities in the mathematical practices of those with and without disabilities, and particularly those with different physical (sensory or motor) means of experiencing the world, then we may build a more robust understanding of the relationships between experience and cognition more generally. The approach that we adopt attempts to combine the premise of embodied cognition, that mathematical (like all) thinking has its basis in our bodily capacities, with Vygotsky's ideas about how the inclusion of tools, be they material or semiotic in nature, in the process of activity alters its entire structure and flow (Vygotsky 1981).

The two approaches appear to us as complementary, as the Vygotskian notion of tool mediation actually has its roots in his work with differently-abled individuals

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[1]The research reported in this paper is being carried out together with other members of the research programme *Rumo à Educação Matemática Inclusiva*, PROESP-CAPES, No. 23038.019444/2009–33).

(Vygotsky 1997), and his view that bodily organs can be thought of as tools, "*the eye, like the ear, is an instrument that can be substituted by another*" (p. 83). In his perspective, such a substitution is expected to cause a profound restructuration of the intellect. Here, it is important to stress that Vygotsky believed that the adequate substitution of a missing or disabled tool would enable those with sensory impairments to attain as highly as their hearing and sighted counterparts, but that the paths by which they would do so might be quite different. That is, in adopting this position, rather than seeing difference as equated to a state of deficiency, difference can be treated as just that, difference.

The rest of this paper returns to a modified version of the question posed at its beginning. It aims to examine what might be learnt from listening to the hands of blind learners and deaf learners as they engage in mathematics learning activities. Before presenting two examples from our ongoing work, it begins with an outline, in the context of the theoretical approach briefly introduced above, of why gestures have come to assume such a central role in our research.

Hands, Gestures and Embodiment

Rotman (2009) argues that gestures have tended to be seen rather as the poor relations of the spoken word and as inferior too to visual images: "*one encounters everywhere a dismissive attitude if not hostility to the notion that gesture be taken seriously*". However, he also suggests that current attention to the embodied nature of being is calling such a view into question. Defining a gesture as "*any body-movement that can be identified, repeated, and assigned significance or affect as a sign, a function, or an experience*", he divides gestures along three modes of embodiment. In the semiotic mode, the gesture represents a form of conveying meaning, significance or affect. In an instrumental gesture, the body becomes akin to a machine, performing a goal-related action to produce a particular effect. In the third, immersive mode, the gesture indicates the body's experience of, or participation in, forms of cultural activity. Rotman's view of gesture is more inclusive than the definition more usually found in the mathematics education literature, in which McNeill defines gestures as "*movements of the arms and hands … closely synchronized with the flow of speech*" (McNeill 1992; p. 11). In contrast to Rotman, McNeill appears to privilege mainly gesture as sign, stressing the speech-gesture relationships and, perhaps because of this, many of those interested in gesture research limit themselves to "*the study of hand and arm movements that are interpreted by others as part of what a person says*" (Roth 2001; p. 368). In McNeill's definition, then, gestures are to be treated as part of the same semiotic system as speech or, more precisely, as "*elements of a single integrated process of utterance formation in which there is a synthesis of opposite modes of thought—global-synthetic and instantaneous imagery with linear segmented temporally extended verbalization*" (ibid. p. 35). In the context of our research with deaf students and blind students, we have some difficulties with this view. The first question that it raises for us is if we are to see gesture as the visual manifestation of imagistic aspects of cognition, then can we expect gestures to be used in the same ways by the blind and the sighted? And the second is what this means in the case of those who speak with their hands rather than with their mouths.

In the first case, we might ask if global-synthetic, instantaneous imagery comes about at all for those who are blind, since when hands are used as substitute eyes the process of seeing can no longer be described as synthetic and global: touch permits a gradual analysis from parts to the whole. Does "seeing" in this case become more sequential?

In the second case, the separation is perhaps even more problematic, since Sign[2] languages are in themselves visual-gestual languages and of a rather different nature to sequential-auditory spoken languages. Rotman (2009) goes as far as to offer the suppression of the use of Sign languages in the education of deaf learners as the strongest evidence on offer of how gestures have been traditionally placed as lower-level or more primitive in relation to spoken language. It is true that there have been times in the historical trajectory associated with the education of deaf learners during which oral methods have been championed at the expense of Sign languages. Without going into all the details, two important moments of this history will perhaps suffice to synthesise the split between the two language forms. The first occurred in 1880, at the Congress of Milan, where hearing participants "voted to proclaim that the German oral method should be the official method used in the schools of many nations" (Lang 2003, p. 15). Deaf people were excluded from this vote. It took until the 1960s and the scientific recognition of American Sign Language (and consequently the Sign languages of other countries throughout the world) as a true and natural language (Stokoe 1960/2005) before this dominance began to be challenged. It is now generally accepted both that Sign languages exhibit the fundamental properties that exist in all natural languages and are just as rich and complex as any oral one (Kilma and Bellugi 1979), with lexicons, grammar, syntax and morphological rules, and that the manual modality is as good a medium for language as the oral modality (see, for example, Goldin-Meadow 2003). But where does this recognition leave gestures?

Certainly, gestures accompany Sign languages just as they accompany spoken languages, but McNeill's sequential/imagistic split is not easy to apply, since phrases in Sign languages are not linearly segmented in the same way as spoken languages. One way out would be to label gestures those hand and arm movements which accompany speech but are not officially recognised parts of any language. Once again, this would limit gestures to complements of speech acts.

If we return to Rotman's alternative view however, instead of treating gestures as some kind of semiotic supplement to speech, speech itself is treated as a species of gesture, sometimes perceived by auditory rather than visual means, but a gesture

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[2]To distinguish between the use of the world "sign" as the generalized manifestation of a signifier-signified relationship and the Signs that compose the manually-expressed languages of the Deaf, a capital "S" will be used in the second case.

nevertheless. In subordinating speech, he underlines the embodied nature of language, but he is by no means the first to do so. In the 18th century, the sensualist Condillac was describing signs as "transformed sensations" and suggesting that the transformation of embodied experiences into shared material signs holds the key to human knowledge:

> Condillac called signs "sensations transformées", transformed sensations, by which he meant the entire complex of affect, desire, sensory perception, and motor action that make up what nowadays we might call "embodied experience" … In his view … [t]he formation of a symbol is a defining moment in the fabrication of shared knowledge because it allows the participants to focus upon and re-invoke previously shared experiences and to plan and conduct shared activities in their wake (LeBaron and Streeck 2000 pp. 118–119).

For the phenomenologist Merleau-Ponty, like Rotman, speech is gesture: "*the spoken word is a gesture, and its meaning, a world*" (Merleau-Ponty 1945/1962, p. 215). In common with Condillac, for him the importance is not so much the form in which a sign representing experience is expressed, it is what this sign incites, invokes, that is, how it is felt by the interlocutors.

> I do not see anger or a threatening attitude as a psychic fact hidden behind the gesture, I read anger in it. The gesture does not make me think of anger, it is the anger itself […] The sense of gestures is not given, it is understood, that is, recaptured in an act on the spectator's part […] The communication or comprehension of gestures comes about through the reciprocity of my intentions and the gestures of others, of my gestures and intentions discernible in the conduct of other people. It is as if the other person's intention inhabited my body and mine his. (pp. 214–215)

Hence, Condillac, Merleau-Ponty and Rotman all appear to agree that the signs through which we communicate experiences have their bases in the body. These positions can be connected to more contemporary writing about embodied cognition as well as to tenets of socio-cultural approaches.

In relation to embodied cognition, Merleau-Ponty's view that the sign *is* the experience can be associated with recent research into the multimodality of human cognition and the idea that both doing something and imagining doing the same something involve a shared neural substrate. Gallese and Lakoff (2005) offer neurological evidence to support this position, arguing that circuitry across brain regions link different sensorial modalities "*infusing each with the properties of others*" (p. 456). In the case of mathematics learning, for some at least, accepting that doing and imagining involve the same cognitive resources, might go somewhat against the grain. Mathematical imaginings are frequently revered as the means of engaging with the abstract, while doing is seen as a less sophisticated interaction with concrete situations. It also raises questions about what we understand as mathematical concepts. If cognition is multimodal and if imagining involves reliving—and re-feeling—previous doings, then concepts cannot be seen as mental representations in which the abstract, logical universal properties of an object are stored in a somehow transcendental form stripped of the particularities of the settings in which it was encountered.

Instead, as Rosch (1999) has it, *"[C]oncepts and categories do not represent the world in the mind; they are a participating part of the mind-world whole*" (p. 70). Concepts in this perspective resemble what Barsalou (2009) defines as *simulators,* the distributed neural systems of the multimodal content associated with a particular category (p. 1282). According to him, these simulators are self-extending systems that enable us to interpret new instances as representatives of a type (Barsalou 2003). In this viewpoint, understanding anything—something immediately perceivable by our sensory apparatus or something that only manifests itself only through semiotic sign systems—involves activating and reliving, that is, simulating, any part of the previous experiences that have come to be allied with it. In this sense, when we come across any object, or a representation of an object, the plurality of perceptions associated with it are reactivated, leading Roth and Thom (2009) to suggest that we experience a single instance associated with any concept as its whole.

Thus far, this description may seem to privilege only the web of relations which exist between an individual subject (body and mind) and manifestations of a class of objects in actual or virtual worlds. However, as the above quotes indicate, Condillac and Merleau-Ponty believed that we are essentially social, not individual beings. They stress our attempts to communicate our feelings and interpretations so that others can understand (experience) them and they signal the role of semiotic tools for achieving this. To a certain degree, this takes us into the realm of Vygotsky's socio-cultural perspective, with its emphasis on our need and ability as human beings to both create and use artefacts and to encourage their appropriation by subsequent generations (Cole and Wertsch 1996). Gestures, be they idiosyncratic body movements or representatives of organised language systems, can hence be understood as bodily-based semiotic means of mediation. Through them, we both feel our own interpretations and communicate them to others.

Given that the students with whom we work lack access to one or other sensory field, it seems reasonable to ask what the gestures they produce tell us about their experiences of mathematical objects. It is to this that we now turn. The main focus in the examples that follow is on the production of gestures intended to some extent as shared signs.

Mathematical Gestures of Blind Students and Deaf Students

The two examples are both drawn from our ongoing programme of research which aims to (1) investigate forms of accessing and expressing mathematics which respect the diverse needs of all our students; (2) contribute to the development of teaching strategies which recognise this diversity; and (3) explore the relationships between sensory experience and mathematical knowledge. The research strategy used in this project is based on establishing collaborative partnerships between school- and university-based participants. The methodology used is a kind of action research, co-generative inquiry, in which all participants co-generate knowledge through a process of collaborative communication (Greenwood and Levin 2000).

Hence, the project involves researchers and practicing mathematics teachers working together to design learning activities that might be used to contribute, in the long run, to the development of a more inclusive school mathematics and which, in the short run, the teachers can implement with their own students.

Example 1 Exploring symmetry and reflection with deaf and hearing students.

The first example[3] involves a group of eight 7th grade students, composed of five deaf and three hearing students. It is drawn from a sequence of activities involving the investigation of the transformation reflection. The example has been selected to illustrate our attempts to understand the challenges associated with learning mathematics in Libras, the Sign language used by the deaf community in Brazil. In particular, the example explores how deaf mathematics learners express mathematical objects and their properties and relations in their visual-gestual language. The students were all from the same class of a mainstream school in the municipal of Barueri, São Paulo. This school is known as one in which teaching is conducted in both Portuguese and Libras (although only Portuguese and not Libras, is actually studied as a school discipline). Not all the teachers in the school are bilingual, but interpreters are present to assist in the classes of the monolingual teachers. Reflection and symmetry were chosen as objects of study in accordance with the mathematics curriculum of the school.

The first step in designing the activities had been to search the literature for previous studies involving the same mathematical object. The literature concerning mathematics learning of deaf students tends to emphasise their performance in relation to arithmetic tasks and to focus on students working individually or with other deaf students (see, for example, Bull 2008; Kelly et al. 2003; Nunes and Moreno 1998; Nunes 2004; Pagliaro 2006). Indeed, in general, we were not able to find studies specifically investigating the relationships between the Bilingual approach adopted in the school we worked and the learning of geometry. As far as the bilingual approach is concerned, most research attention has been given to literacy and to the complexity of applying the principles of linguistic interdependence derived from studies of second (spoken) language learning to deaf learners. In this respect, it has been argued that to understand deaf literacy learners, it is necessary to take into account the set of sensory modalities available to them, to ensure they have the opportunity to appropriate and manipulate all possible meditational means at their disposal (Mayer and Akamatsu 2003). This was hence a principle design concern and one of the reasons that we decided to structure the learning activities in our study around a digital microworld, *Transtaruga,*[4] which offered different modes of interacting with geometrical ideas (visual, dynamic and

[3]This research was carried out in collaboration with the researchers Heliel Ferreira dos Santos (also the students' mathematics teacher), Solange Hassan Ahmad Ali Fernandes, Fabiane Guimarães Vieira Marcondes and Kauan Espósito da Conceição.

[4]Borrowing "*trans*" from the Portuguese word for transformation and "*taruga*" from tartaruga or turtle in Portuguese.

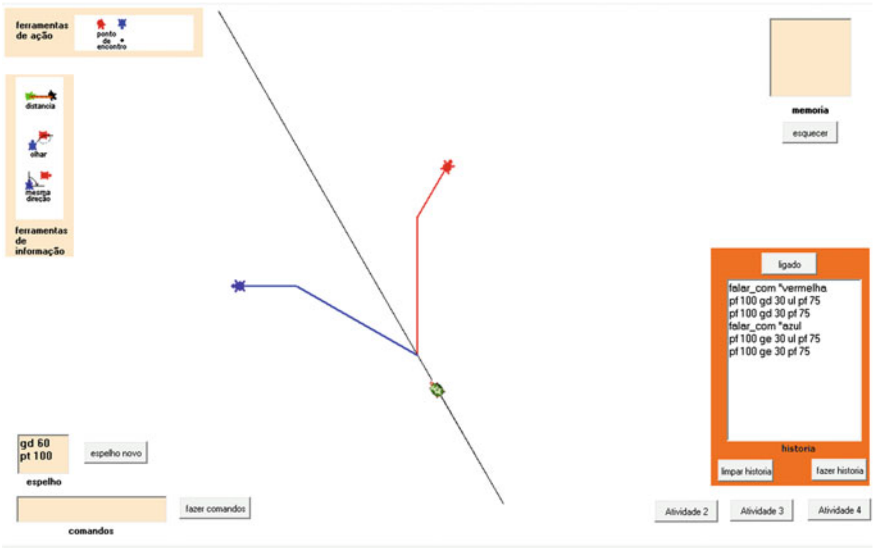

Fig. 1 Symmetrical trajectories in the Transtaruga microworld

symbolic). *Transtaruga* is a multiple-turtle geometry microworld written in the Imagine version of the programming language Logo. It inherited many features from two earlier *Turtle Mirrors* microworlds used in previous research with hearing students (Hoyles and Healy 1997; Healy 2002). Figure 1 presents a page from this microworld.

None of the eight students who participated in this study had had any previous experience with the Logo programming language and, in fact, digital resources had never featured in their mathematics lessons before. They were divided into pairs to work on the activities, with each pair positioned in front of one of four laptops, arranged in a semicircle so that, during whole group discussions, the Signs of those using Sign language could be easily seen by all the participants. Two pairs were comprised of only deaf students, one pair was made up of two hearing students and the fourth pair had one deaf and one hearing member. The hearing student in this pair spoke some Libras and the deaf student was partially oralised. Each laptop had a webcam and it was thus possible to capture not only the four screens, but also the faces of the students and the discussions between them while they worked on the activity. In addition, three other video cameras were positioned to capture all the interactions of the pairs using Libras as a communication form. Clearly the presence of all this equipment along with the researchers operating it, made this learning setting rather different from their usual classroom, however, it was necessary in order not to lose the discussions between the deaf students. A total of five researchers participated in the research session from which the episodes that follow were drawn, one of whom was the students' mathematics teacher. He spoke some Libras, but was not completely fluent and hence an interpreter was also present. All the researchers participated actively in the sessions, with two, the class teacher Heliel and myself assuming the main teaching roles.

In the first session, the students were introduced to a limited subset of turtle geometry commands: **pf** (forward, *para frente*), **pt** (backwards, *para trás*), **gd** (right, *girar direita*), **ge** (left, *girar esquerda*). **ul** (pendown, *usar lápis*) and **un** (penup or *usar nada*) and shown how to activate different turtles by clicking on them. Four additional microworld tools were also presented: **ponto de encontro** (meeting-point), which constructed a point (turtle) where two turtles intersect; **distância** (distance), which measured the distance between two turtles, displaying the value in the memory box (*memória*); **olhar** (towards), which measured the angle one turtle should turn to point towards another, again with the result displayed in the memory box; and **mesma direção** (same heading), which measured the angle one turtle should turn so as to have the same heading as another. These four tools were represented as icons which displayed the tools' name but also by means of a visual image which aimed to summarise function of the tool. This design decision was motivated by the fact that the deaf students had some difficulties with written Portuguese and tended to avoid reading any written text. A history box was also available on screen so that the students could build up a symbolic record of the commands that they used in the negotiation of the different tasks.

The first task given to the students involved constructing a symmetrical trajectory—that is, given the set of commands used to construct a trajectory of the red turtle, the students' task was to construct a trajectory for the blue turtle which would leave a trace symmetrical to that of the red turtle in relation to the axis of symmetry. Both the red and blue turtles started from the same location. Previous research with hearing students suggested that this task is very accessible to the majority of students (Healy and Hoyles 1997), who rapidly perceive that symmetrical traces can be obtained by using the same commands to move both turtles forward and back, while swopping the directions in the turning command.

When we attempted to explain this task to our student group, an unpredicted problem emerged almost immediately. Neither the class-teacher nor the interpreter knew the Sign for "symmetry" in Libras. The interpreter's first instinct was to substitute the word "symmetric" with a Sign representing "equal", but this substitution was rejected on the grounds that it would emphasise only the congruency relationship (common to all the isometric transformations). One option at this point would have been to finger spell the word, that is, to use the Libras signs for the letters that compose the word in written Portuguese. In itself, this does not represent a very satisfactory solution as simply spelling the word could not be expected to help the students understand what was required of them, since it would communicate nothing of the meaning of the word in question. A third option was chosen: that of re-introducing the task, suggesting that the students imagine that the line representing the axis of symmetry is like a mirror, and their task is to produce the image of the trajectory of the red turtle in this mirror. Our belief was that this explanation would serve equally for the hearing and deaf students—as the reflection transformation is frequently introduced to learners in relation to activities with mirrors. Perhaps it is worth pausing to comment on the embodied nature of this explanation: its intention is to invite the students to simulate their previous experiences of looking into mirrors, and to use associated feelings to imagine what they would see. Doing and the imagining of the doing are treated as alike in this invitation.

To a certain extent, the need for this association between mirrors and symmetry had already been anticipated in the design of the microworld, even though the absence of the Sign for symmetry had not been predicted, with the turtle responsible for producing the axis of symmetry named *espelho* (mirror). In the event, this association did indeed appear to make the task requirements clear to all four student-pairs. The trajectory to be reflected is shown in Fig. 2.

All the students seemed confident as to where the blue trace should appear on the screen, but we noticed two differences in the strategies used by the pairs composed only of deaf students in comparison to the other two pairs. First, the deaf students were more likely to confuse the functions of the Logo drawing commands. This difficulty is relatively easy to understand. The choice of the two letters that compose the Portuguese version of the forward command (**pf**) is not arbitrary, but an abbreviation of ***para frente***.

The hearing students sense the non-arbitrariness of these commands, the sounds associated with the letters representing an indication of the action they serve. This was not the case for the deaf students, which made it harder for them to remember which command to use when. The result being that the process of appropriating the functions of the tools, or we might say, the meaning of the written "pf" gesture, took longer for the deaf students.

The second difference is to a certain extent related to the first and concerns where the students directed their attention. The students in the two deaf-only groups initially paid only limited attention to the commands displayed in the history box (*história*) and focussed their attention mostly on the part of the screen displaying the visual turtle traces. When they did look at the information displayed in this box, they seemed to be drawn more to the numbers than to the letters representing commands. In contrast, the groups with at least one hearing student tended to move their attention equally between the history box and the visual traces. One interpretation of this difference in attention would be to suggest that the deaf students

Fig. 2 The first image to reflect

were less aware of the dependence of the blue turtles trace on the commands used by the red. However, on a closer look, this would not seem to be entirely the case, the numbers used as inputs to the commands used by the deaf-only groups were all numbers associated with commands given to the red turtle. Because the deaf students took longer to understand which command was associated with which movements, at times this made it look as if they were using rather random strategies of assigning any of the numbers with any of the commands. Perhaps too their initial difficulties in assigning letters to movements also discouraged them to look for regularities in the symbolic records of the two turtles' trajectories.

If we look only at their interaction with the symbolic code, without looking at how they attempted to visually decode the necessary movements, however, we gain rather a biased picture of their solution strategies. A closer look at the work of Pedro and Daniel shows that, they knew exactly what they wanted to produce. The symmetrical gesture produced by Pedro to indicate the desired relationship between the two trajectories provides evidence of this (Fig. 3).

Figure 3 shows Pedro's hands only in their final configuration. In fact, the gesture as a whole was a dynamic one in which his hands traced out a symmetrical movement. His gesture in this case resembles the specific image that they are trying to construct, but the way Pedro produced it, indicates how he sensed symmetry, perhaps even the gesture was being offered to his partner as a Sign for symmetry—or, following Roth and Thom (2009), might indicate that he was experiencing this single instance as representative of symmetry as a whole.

It is also the case that the relationship between turns in symmetrical tasks was signalled in the gesture; Pedro bodily enacts a process of rotating his hands in opposite directions. He may not yet be conscious of doing so, the gesture might be felt as the whole movements and not broken down into constituent parts. One thing that we can be sure of is at this point the movement is by no means connected to the Logo language which potentially provides another means of expressing it.

Fig. 3 A gesture for symmetry

Despite knowing the trajectory that the blue turtle should produce, it was their unfamiliarity with these commands, and in particular a mixing up the commands **`pt`** and **`pf`** that eventually led Pedro and Daniel to abandon the given task and attempt to construct a different pair of symmetrical paths. This decision serves as further evidence that they were thinking beyond the specific case. A major concern of theirs continued to be the appropriation of the functions of the Logo commands, and this time, as they discussed the commands to be used, Daniel uses his fingers to represent the two turtles they were communicating with (Fig. 4). The pair decided to move the turtles forward 100 and they agree to try **`pt 100`** (Fig. 5). During this conversation Daniel was using his right hand simultaneously to both Sign and to represent the turtle, but this did not seem to be a problem for him.

When the command **`pt 100`** was typed, they both saw the associated turtle movement and Daniel also acted out its backward movement. Realizing that they had used the wrong command, Daniel made a kind of rubbing-out gesture (Fig. 6).

Fig. 4 Daniel's fingers become turtles

Fig. 5 **`pt`** is suggested to move forward

It was as if he was somehow guarding a visual image of the conversation and then rubbing out this image when a mistake was made. We saw Daniel use this same gesture on other occasions, such as when he was performing calculations. Whatever this gesture implies about the mental processing of Sign language, the gesture was clearly understood by both Pedro and Daniel to mean they should start again. Accompanying the movement of the turtles with their fingers seemed to have been useful in helping the pair to decide which command should be used when and, in Fig. 7, Daniel is able to feel and to articulate to Pedro that the turns should be made in opposite directions. This time the relationship between these turns is the explicit focus of attention Daniel's gestures, rather than part of what might be described as the more composite gesture originally made by Pedro.

While it took Pedro and Daniel much longer to appropriate the function of the written Logo commands and to explicitly articulate the relationships between the turns made in symmetrical paths, physically in gestures as well as in the Logo

Fig. 6 Daniel "rubs out" the exchange

Fig. 7 Feeling opposite turns

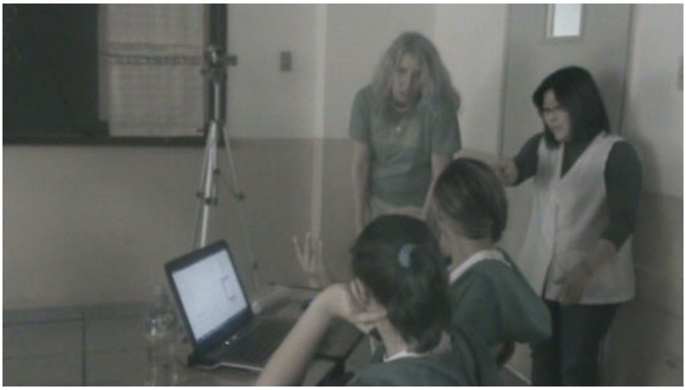

Fig. 8 Lara's suggestion

languages of right and left turns, their efforts to make sense of the activity were eventually so successful that they took responsibility for explaining this relationship to Aline and Lara. At the end of this explanation, not only did all the students seem to be confident about producing symmetrical trajectories, the deaf students wanted to negotiate a Sign to represent the process of reflecting a turtle path. Figure 8 shows Lara's sign which involved the use of a flipping movement of one hand in a way reminiscent of the flipping of space over an axis of symmetry.

In this process of negotiating a shared Sign, which is a common activity for these students and for members of the deaf community in Brazil in general, both suggestions incorporate bodily movements which represent what reflection means to the students in the context of their work with Turtle Geometry. Embedded in the gestures are aspects of the reflection process that the students want particularly to stress. While it might not be clear in the case of Pedro's symmetrical gesture whether he was thinking only of a particular image, or whether the instance represented the whole, in the negotiation of the Sign for reflection there can be no doubt. The students were not representing specific cases, they were offering gestures in which aspects of the process of reflecting were abstracted in the production of a dynamic Sign intended to bring this process to mind.

Example 2 Blind students sharing their feelings of area.

The second example involves an episode from a sequence of learning situations undertaken with a group of four blind learners who attended a mainstream school, which was part of the public education system of the State of São Paulo. The learners, whose ages were between 14 and 18 years, were first-year high school students. The data reported here come from a sequence of activities associated with the study of volume, area and perimeter. The activities were designed to include tools intended to favour tactile exploration of the mathematical objects in question and were implemented over four research sessions, each approximately 90 min in duration. Activities which took places during the first two session contributed to

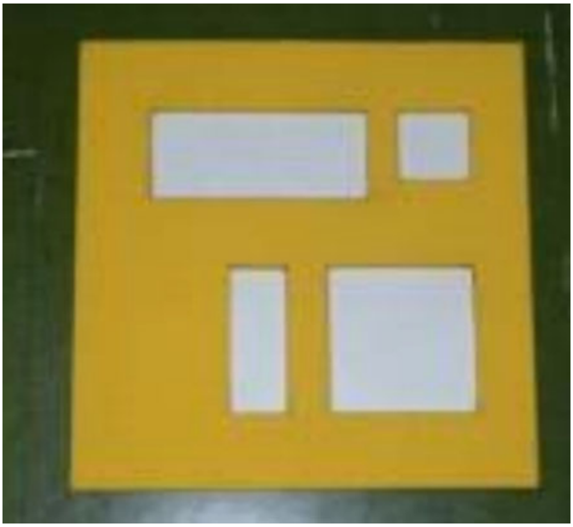

Fig. 9 Impressions of rectangles

understanding the interactions in the episode presented here (more details on the students' interactions on the tasks in this learning sequence are available in Fernandes and Healy (2010) and Healy and Fernandes (2014)). All the activities took place in a room which houses the resources specifically designed for the blind students and substituted their regular mathematics lessons.

Before they began to work on the activities, the students had explained to us that it was rare for them to interact with representations of geometrical shapes, and the tendency was for measures of shapes to be given to them to operate with, rather than being expected to perform the measurements for themselves.

For this reason, we[5] wanted to make sure that they had access to tactile materials that would make this possible. During the first session, they were given a board with the impressions of four rectangles, which could be filled either with wooden unit cubes or with rectangular and triangular shapes in foam rubber (Fig. 9). In this session, the students explored the areas of rectangles and triangles. In particular, they experienced how the area of a triangle could be perceived as half the area of a rectangle with the same height and base.

In the second session, the students worked on determining the areas of the plane figures represented in foldable cardboard. During this session, adapted rulers, in which the number marks were raised so they could be read tactilely, were available, as were the wooden-unit cubes.

The episode which follows occurred during the second session, when the students were working on finding the areas of plane figures and in particular relates to

[5]This work was carried out together with Solange Hassan Ahmad Ali Fernandes.

their attempts to determine the area a parallelogram with a length of 12 cm and a height of 7 cm. Each student received his own cardboard representation of this parallelogram.

The four previous shapes that they had worked with were triangles. The area of the third shape was easy for them to work-out, as it was a right-angled triangle, the area for which they already knew how to obtain from their activities in the first session. They were shown how other triangles could be divided into two right-angled triangles, either physically by folding along a line perpendicular to one of the sides and which passed through the vertex, or virtually, using the ruler to mark this same line segment and then measuring the height and base of both the resulting triangles. Using these methods they successfully calculated the areas of three non-right-angled triangles.

After their initial explorations of the new parallelogram shape, it looked as if the students were about to agree that its area was 42 cm^2. This measure had been obtained by measuring the length of the parallelogram (which they called the "base"), its height (correctly located, by measuring the perpendicular distance from the base to the vertex on the opposite side, as they had done with the triangles), and then dividing by 2, also using the strategy that had worked for the triangles. Just as they were about to confirm to the researchers that this was the area on which they were all agreed, first Fabio, then, Caio began to have doubts about whether or not they ought to be dividing by 2.

Fabio seemed to be thinking about straightening the two parallel sides, measuring 8 cm, joined to the base, so that they were perpendicular to it, forming a rectangle, with what he called "*a better-behaved area*", which he believed would have an area of 8 times 12. His mentioning of a rectangle, led Caio also to question the appropriateness of dividing by 2 in the case of the figure they were now exploring, although he didn't agree that the measure of 8 cm should be used. His view was that the area of the as yet unnamed shape could be determined by multiplying 12 by 7.

As Leandro and Marcos were attempting to defend the method which included dividing by 2, Caio and Fabio had become convinced that this was no longer an appropriate option. To persuade the other two, Caio did not discount the idea that there was something about this shape that resembled the triangles, but rather used this as the basis to argue that they were now investigating a different kind of shape "*It seems like a triangle, but it isn't a triangle, it's something close to a rhombus or something like that*". Taking up Caio's position, and perhaps also the inclusion in the conversation of another kind of quadrilateral, Fabio went on to state "*We are not working with a triangle anymore; we are working with a whole figure.*" This comment is particularly interesting. Again, his words suggest the simulation of a previous experience. This time, the experience in question is one in which all four of students had participated in the first research session—that is, the treating of a triangle as half an impression of a rectangle in the material presented in Fig. 9. It was as if, by choosing to describe the four-sided shape as a "whole-figure", Fabio was bringing to the present aspects of a previous activity, in which, he knew, all the others had shared and offering this verbal gesture as a sign which might help the others to appreciate his point of view. He does seem to have managed to communicate the idea, as, when Caio attempted to defend his value for the area of the parallelogram, he used exactly the same combination of words once more "*12 times 7, which is base times height, and since it's a whole figure you don't divide by 2*". Moreover, from this point on, none of the students adopted the method they had used for a three-sided shape when working with quadrilaterals.

Shared Signs as Embodied Abstractions

The two examples presented in this paper are offered with the aim of exploring the mathematical sense-making activities that occur when students speak and see with their hands. In both examples, there are indications of how the particularities associated with the sensory channels through which these learners experience the world, contribute to the ways in which they appropriate knowledge. For example, for the blind students, using hands as seeing tools in the absence of eyes appears to lead to particular views of geometrical figures, emphasised by the dynamic gestures that emerge during this kind of seeing process. While in the case of the deaf students, the gesture in which Daniel seems to rub out an ongoing argument in order that it might be reconstructed from scratch, offers some evidence that reasoning is in some sense a visual process for those who communicate in Libras.

Yet despite the obvious differences between the groups of students who participated in the different learning scenarios, their sense-making activities are also characterised by a number of similarities and, in particular, by the efforts of the students to create sharable signs intended to re-invoke, both by themselves and by the others present, previously shared experiences. That is, their aim was to permit that all those involved in the interactions might feel what they were feeling. In both cases, the signs offered resulted from what we have elsewhere termed *embodied abstractions* (Healy and Fernandes 2011). We call them abstractions because the gestures offered as shared signs not only expressed the learner's own sense of an experience of a specific mathematical situation by relating it to some other, they also imply some conscious appreciation by learners of the generalised relationships implied in their expressions. The shared signs might be expressed in manual or verbal forms—in this paper both are considered as gestures—and it is important to stress that one form cannot be considered as more or less embodied, nor more or less abstract than the other. Regardless of their form, the gestures expressed in both example episodes were not only material manifestations, traces, of aspects of physical activities, they also had a semiotic role, carrying a sense of the mathematical understanding that had come to be associated with the activities of those who made them.

Another commonality that can be identified in both examples is that although the different gestures were expressed by particular individuals, the abstractions that they expressed should be considered as social rather than individual acts—the imagined, or actual, actions they conveyed, involved the reactivation of perceptive, motor and introspective states associated with such objects in previous activities. In this sense, gestures were outward signs of *imagined re-enactments* of previous doings with the things and the other people involved in the activity in question (Healy and Fernandes 2014). In the case of both the gesture offered by the deaf students to express the word "reflection" in Libras and the use of the expression "whole-figure" by the blind students to indicate a significant difference in determining areas of triangles as compared to quadrilaterals, there was evidence of a conscious attempt to choose a gesture so that it might re-invoke the multimodal content associated with this concept by the others—or, putting it another way, so it could be felt by the others. What the gestures of these students suggest is that in order to best communicate the meanings that we hold for particular mathematical objects and activities, we need to look for ways of permitting that others sense how we feel these objects. Perhaps, this is what Merleau-Ponty meant when he talked of how when we interpret the gestures of others, their intentions come to inhabit our bodies, and reciprocally, our intentions inhabit theirs. Perhaps even we might suggest that as teachers of mathematics we should seek to do precisely that: permit that others feel mathematics as we do, whilst simultaneously attempting to appropriate their feelings.

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