Sung Je Cho
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Selected Regular Lectures from the 12th International Congress on Mathematical Education

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Preface

This book is a result of the 12th International Congress on Mathematical Education (ICME-12), which was held at Seoul, Korea from July 8, 2012 to July 15, 2012.

The International Program Committee (IPC-12) of ICME-12 took on the task of acting as the editorial board to publish the Proceedings of the ICME-12 and Selected Regular Lectures of the ICME-12 in two separate volumes. The Proceedings of the ICME-12 is accessible through the Open Access Program by Springer, the ICME-12 publisher.

The second volume, The Selected Regular Lectures from the ICME-12 contains 51 Regular Lectures. The Regular Lecturers were invited from different parts of world by the IPC-12. The Regular Lectures covered a wide spectrum of topics, themes, and issues. Originally, 78 Regular Lectures were planned but some of the lecturers could not make trips to ICME-12. Among the lectures given at ICME-12, some of them are not contained in this volume for various reasons.

ICME-12 would not have been possible without the contribution from its members and strategic partners. For the first time, all of the Korean mathematical societies united to bid and host ICME-12. The success of ICME-12 is closely tied to the tireless efforts of all.

A considerable amount of the ICME-12 budget was funded through private donations by mathematically minded individuals and businesses. ChunJae Education Inc. was one of the largest contributors of funds and services. Printing of the ICME-12 Program Booklets and Abstracts were paid for by ChunJae Education Inc.

The Korean Ministry of Education helped to secure the balance of the budget and assisted in the operation of ICME-12. The City of Seoul, Korea Foundation for the Advancement of Science & Creativity, and Korea Tourism Organization were significant funding bodies as well.

The dedicated members of the Local Organizing Committee, skilled professional conference organizers at MCI, and staff at the COEX (Convention and Exhibition) were integral in the successful planning and execution of ICME-12. The dedication shown by the Local Organizing Committee for the conference was second to none and well beyond expectations.

Finally, the Chief Editor would like to express his sincere thanks to all Regular Lecturers. Without their efforts and dedications this volume was not possible at all. Also, the Chief Editor would like to thank all the members of IPC-12, who read all or part of the Regular Lecture papers for improvements. It is needless to say that without Professor Hee-chan Lew's work and devotion, this volume could not have been completed. The Chief Editor would like to express his heartfelt thanks to him. The Chief Editor believes that the world mathematical education society is closer than before and leading toward more productive and friendly mathematics classrooms around the world.

Sung Je Cho

Contents

Understanding the Nature of the Geometric Work Through Its Development and Its Transformations

Alain Kuzniak

Abstract The question of the teaching and learning of geometry has been profoundly renewed by the appearance of Dynamic Geometry Software (DGS). These new artefacts and tools have modified the nature of geometry by changing the methods of construction and validation. They also have profoundly altered the cognitive nature of student work, giving new meaning to visualisation and experimentation. In our presentation, we show how the study of some geneses (figural, instrumental and discursive) could clarify the transformation of geometric knowledge in school context. The argumentation is supported on the framework of Geometrical paradigms and Spaces for Geometric Work that articulates two basic views on a geometer's work: cognitive and epistemological.

Keywords Geometric work · Visualisation · Geometrical paradigm

Introduction

The influence of tools, especially drawing tools, on Geometry development at school has recently improved greatly due to the appearance of DGS. The traditional opposition between practical and theoretical aspects of geometry has to be rethought. It's well known that we can approach Geometry through two main routes:

1. A concrete approach which tends to reduce geometry to a set of spatial and practical knowledge based on material world.
2. An abstract approach oriented towards well organized discursive reasoning and logical thinking.

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With the social cynicism of the Bourgeoisie in the mid-nineteenth century, the first approach was for a long time reserved to children coming from the lower class and the second was introduced to train the elite who needed to think and manage society.

Today, in France, this conflict between both approaches stays more hidden in Mathematics Education but such discussions have reappeared with the social expectation supported by the Organisation for Economic Co-operation and Development (OECD) and its “bras armé” Programme for International Student Assessment (PISA) with the opposition between “Mathematical literacy” and “Advanced Mathematics”.

In the present paper, I will leave aside sociological and ideological aspects and focus on what could be a didactic approach, keeping in mind a possible scientific approach to a more practical geometry referring to approximation and measure, in the sense Klein used when he suggested a kind of approximated Pascal’s theorem on conics:

> Let six points be roughly located on a conic: if we draw the lines roughly joining points and they intersect at a, b and c, then these points are roughly aligned. (Klein 1903).

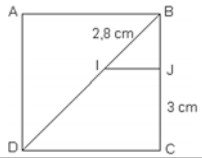
The present presentation will be supported by a first example showing what kind of contradiction exists in French Education where no specific work on approximation exists during compulsory school. This contradiction appears as a source of confusion and misunderstandings between teachers and students. We were lead to introduce some theoretical perspectives aiming at understanding and solving this trouble. In the following, our theoretical framework for studies in Geometry will be introduced and used to launch some perspectives.

Complexity of the Geometric Work

Mathematical domains are constituted by the aggregation and organization of knowledge. A mathematical domain is the object of various interpretations when it is transformed to be taught. These interpretations will also depend on school institutions. The case of geometry is especially complex at the end of compulsory school, as we will show in the following.

The following problem was given for the French examination at Grade 9 in 1991 (Table 1).

Table 1 A geometric problem

Construct a square ABCD with side 5 cm 1. Compute BD 2. Draw the point I on [BD] such that BI = 2.8 cm, and then the point J on [BC] such that JC = 3 cm Is the line (IJ) parallel to the line (DC)?	

The intuitive evidence (the lines are parallel) contradicts the conclusion expected from a reasoning based on properties (the lines are not parallel). Students are faced with a variety of tasks referring to different, somewhat contradictory conceptions and the whole forms a fuzzy landscape:

1. In the first question, a real drawing is requested. Students need to use some drawing and measure tools to build the square and control and validate the construction.
2. Students then have to compute a length BD using the Pythagorean theorem and not measure it with drawing tools. But which is the nature of the numbers students have to use to give the result: An exact value with square roots, or an approximate one with decimal numbers which is well adapted to using constructions and that allows students to check the result on the drawing?
3. In the third question—are the lines parallel?—students work again with constructions and have to place two points (I and J) by measuring lengths. Moreover, giving the value 2.8 can suggest that the length is known up to one digit and could encourage students to use approximated numbers rounded to one digit. In that case is equal to 1.4 and both ratios are equal, which implies the parallelism by the Thales' Theorem related to similarity. If students keep exact values and know that is irrational, the same Theorem implies that the lines are not parallel.

With Grade 9 Students

The problem was given in a Grade 9 class (22 students), one week after a lecture on exact value with square roots and its relationships to length measurement. After they had spent 30 min working on the problem, half of the students answered that the lines were parallel and the other half answered that they were not. On the teacher's request, they used the problem of approximated values to explain the differences among them. At the teacher's invitation, they started again to think about their solutions. At the end, 12 concluded the lines were not parallel, 8 that they were and 2 hesitated.

Indeed, after studying their solutions and their comments on the problem, we can conclude that students' difficulties did not generally relate to a lack of knowledge on geometric properties, but to their interpretations of the results. They had trouble with the conclusions to be drawn from Thales' Theorem. Even after discussion, students expressed their perplexity about the result and its fluctuation. One student said "I don't know if they are parallel for when I round off, the ratios are equal and so the lines are parallel, but they are not parallel when I take the exact values". For students, one answer is not more adequate than another. This gives birth to a geometric conception where some properties could be sometimes true or false. How to make students overcome the contradiction? A first possibility is to force the entrance in the didactical contract expected by the class's teacher, who explained that at this moment in Grade 9, it must be clear that "a figure is not a proof".

Working on approximation and thinking about the nature of geometry taught during compulsory school open a second way we will explore with geometrical paradigms in the following.

Geometrical Paradigms and Three Elementary Geometries

The previous example and numerous others of the same kind show that a single viewpoint on geometry would miss the complexity of the geometric work, due to different meanings that depend both on the evolution of mathematics and school institutions. At the same time, we saw that students are strongly disturbed by this diversity of approaches. Geometrical paradigms were introduced into the field of didactics of geometry to take into account the diversity of points of view (Houdement and Kuzniak 1999, 2003).

The idea of geometrical paradigms was inspired by the notion of paradigm introduced by Kuhn (1966) in his work on the structure of the scientific revolutions. In a global view, one paradigm consists off all the beliefs, techniques and values shared by a scientific group. It indicates the correct way for putting and starting the resolution of a problem. Within the restricted frame of the teaching and learning of geometry, our study is limited to elementary geometry, and the notion of paradigm is used to pinpoint the relationships between geometry and belief or mathematical theories.

With the notion of paradigms, Kuhn has enlarged the idea of a theory to include the members of a community who share a common theory.

> A paradigm is what the members of a scientific community share, and, a scientific community consists of men who share a paradigm (Kuhn 1966, p. 180).

When people share the same paradigm, they can communicate very easily and in an unambiguous way. By contrast, when they stay in different paradigms, misunderstandings are frequent and can lead, in certain cases, to a total lack of comprehension. For instance, the use and meaning of figures in geometry depend on the paradigm. Sometimes it's forbidden to use the drawing to prove a property by measuring and only heuristic uses of figures are allowed.

To bring out geometrical paradigms, we used three viewpoints: epistemological, historical and didactical. That led us to consider the three following paradigms described below.

Geometry I: Natural Geometry

Natural Geometry has the real and sensible world as a source of validation. In this Geometry, an assertion is supported using arguments based upon experiment and deduction. Little distinction is made between model and reality and all arguments are allowed to justify an assertion and convince others of its correctness. Assertions are proven by moving back and forth between the model and the real: The most important thing is to develop convincing arguments. Proofs could lean on drawings or observations made with common measurement and drawing tools such as rulers, compasses and protractors. Folding or cutting the drawing to obtain visual proofs are also allowed. The development of this geometry was historically motivated by practical problems.

The perspective of Geometry I is of a technological nature.

Geometry II: Natural Axiomatic Geometry

Geometry II, whose archetype is classic Euclidean Geometry, is built on a model that approaches reality. Once the axioms are set up, proofs have to be developed within the system of axioms to be valid. The system of axioms could be incomplete and partial: The axiomatic process is a work in progress with modelling as its perspective. In this geometry, objects such as figures exist only by their definition even if this definition is often based on some characteristics of real and existing objects.

Both Geometries have a close link to the real world even if it is in different ways.

Geometry III: Formal Axiomatic Geometry

To these two approaches, it is necessary to add a third Geometry (Formal Axiomatic Geometry) which is little present in compulsory schooling but which is the implicit reference of teachers' trainers when they have studied mathematics in university, which is very influenced by this formal and logical approach.

In Geometry III, the system of axioms itself, disconnected from reality, is central. The system of axioms is complete and unconcerned with any possible applications in the world. It is more concerned with logical problems and tends to complete "intuitive" axioms without any "call in" to perceptive evidence such as convexity or betweenness. Moreover, axioms are organized in families which structure geometrical properties: affine, euclidean, projective, etc.

These three approaches (and this is one original aspect of our viewpoint) are not ranked: Their perspectives are different and so the nature and the handling of problems change from one to the next. More than the name, what is important here is the idea of three different approaches of geometry: Geometry I, II and III.

Back to the Example

If we look again at our example, students—and teachers—are not explicitly aware of the existence of two geometrical approaches to the problem, each coherent and possible. And students generally think within the paradigm which seems natural to them and close to perception and instrumentation—Geometry I. But in this geometry, measurement is approximated and known only over an interval. Parallelism of lines depends on the degree of approximation. Teachers insist on a logical approach—Geometry II—which leads the students to conclude blindly that the lines are not parallel, against what they see.

It could be interesting to follow Klein's ideas and introduce a kind of "approximated" theorems, more specifically here an "approximated" Thales' Theorem: If the ratios are "approximately" equal then the lines are "almost" parallel. In that case, it would be possible to reconcile what is seen on the drawing and what is deduced based on properties.

Developing thinking on approximation in Geometry can be supported by DGS which favour a geometric work into Geometry I but with a better control of the degree of approximation. It is the case, for instance, with the CABRI version we used during the session with students. In this version, an "oracle" is available which can confirm or not the validity of a property seen on the drawing. Here, the parallelism of the two lines was confirmed by the "oracle" according to the approach with approximation of the problem.

Many problems allow discussion of the validity of a theorem or property in relationship to numerical fields. For instance, CABRI oracle asserts that (EF) and (BC) are parallel lines in a triangle ABC when E and F are respectively defined as the midpoints of [AB] and [AC]. But, if E is defined as the midpoint of [AB], when we drag a point F on [AC] it is possible that CABRI oracle never concludes that (EF) and (BC) are parallel for any position of F. These variations in the conclusion need an explanation and provoke a discussion among students which can be enriched by the different perspectives on Geometry introduced by geometrical paradigms.

To discuss the question in-depth and think about new routes in the teaching and learning of geometry, we will introduce some details about the notion of Space for Geometric Work.

The Notion of Space for Geometric Work Within the Framework of Didactics of Geometry

At school, Geometry is not a disembodied set of properties and objects reduced to signs manipulated by formal systems: It is at first and mainly a human activity. Considering mathematics as a social activity that depends on the human brain leads to understanding how a community of people and individuals use geometrical paradigms in everyday practice of the discipline. When specialists are trying to solve geometric problems, they go back and forth between the paradigms and they use figures in various ways, sometimes as a source of knowledge and, at least for a while, as a source of validation of some properties. However, they always know the exact status of their hypotheses and the confidence they can give to each one of these conclusions.

When students do the same task, we are not sure about their ability to use knowledge and techniques related to Geometry. That requires an observation of geometric practices set up in a school frame, and, more generally, in professional and everyday contexts, if we aim to know common uses of mathematics tools. The whole work will be summarized under the notion of *Space for Geometric Work* (SGW), a place organized to enable the work of people solving geometric problems. Individuals can be experts (the mathematician) or students or senior students in mathematics. Problems are not a part of the Work Space but they justify and motivate it.

Architects define Work Spaces as places built to ensure the best practice of a specific work (Lautier 1999). To conceive a Work Space, Lautier suggests thinking of it according to three main issues: a material device, an organization left at the designers' responsibility and finally a representation which takes into account the way the users integrate this space. We do not intend to take up this structure oriented to the productive work without any modifications, but it seems to us necessary to keep in mind these various dimensions, some more material and the others intellectual.

The Epistemological Level

To define the Space for Geometric Work, we introduced three characteristic components of the geometrical activity into its purely mathematical dimension. These three interacting components are the following:

A real and local space as material support with a set of concrete and tangible objects.
A set of artefacts such as drawing instruments or software.
A theoretical frame of reference based on definitions and properties.

These components are not simply juxtaposed but must be organized with a precise goal depending on the mathematical domain in its epistemological dimension. This justifies the name *epistemological plane* given to this first level. In our theoretical frame, the notion of paradigms brings together the components of this epistemological plane. The components are interpreted through the reference paradigm and in return, through their different functions, the components specify each paradigm. When a community can agree on one paradigm, they can then formulate problems and organize their solutions by favouring tools or thought styles described in what we name the reference SGW. To know this SGW, it will be necessary to bring these styles out by describing the geometrical work with rhetoric rules of discourse, treatment and presentation.

The Cognitive Level

We introduced a second level, centred on the cognitive articulation of the SGW components, to understand how groups, and also particular individuals, use and appropriate the geometrical knowledge in their practice of the domain. From Duval (2005), we adapted the idea of three cognitive processes involved in geometrical activity.

A visualization process connected to the representation of space and material support;
A construction process determined by instruments (rules, compass, etc.) and geometrical configurations;
A discursive process which conveys argumentation and proofs.

From Gonseth (1945–1952), we retained the idea of conceiving geometry as the synthesis between different modes of knowledge: intuition, experiment and deduction (Houdement and Kuzniak 1999).

The real space will be connected to visualization by intuition, artefacts to construction by experiment and the reference model to the notion of proof by deduction. This can be summarized in the following diagram (Fig. 1).

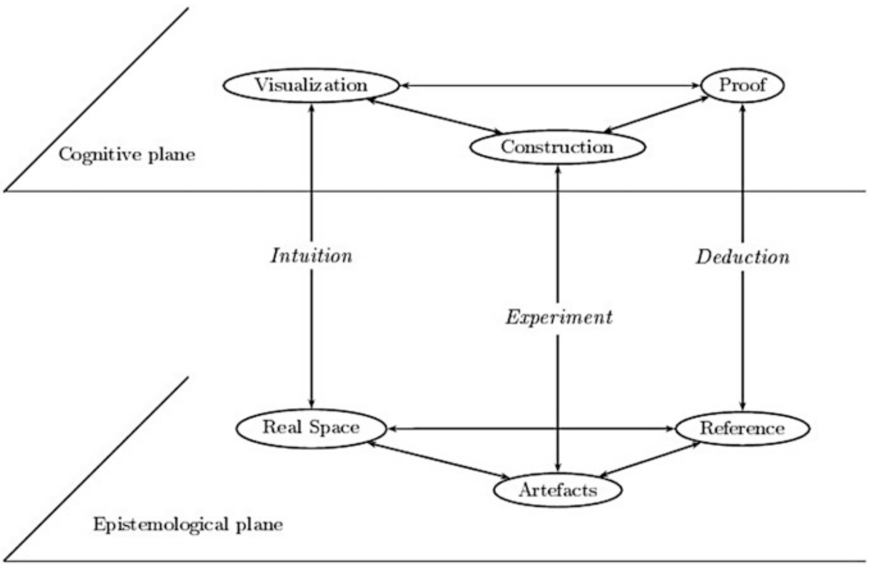


Fig. 1 The space for geometric work

Building a Space for Geometric Work: A Transformation Process

On the Meaning of Genesis

In the following, we will consider the formation of SGW by teachers and students. Our approach intends to better understand the creation and development of all components and levels existing in the diagram above. The geometric work will be considered as a process involving creation, development and transformation. The whole process will be studied through the notion of genesis, used in a general meaning which is not only focused on origin but also on development and transformation of interactions. The transformation process takes place and, finally, forms a structured space, the Space for Geometric Work.

Various SGW Levels

In a particular school institution, the resolution of geometric tasks implies that one specific SGW has been developed and well organized to allow students to enter into the problem solving process. This SGW has been named appropriately and the appropriate SGW needs to meet two conditions: it enables the user to solve the problem within the right geometrical paradigm, and it is well built, in the sense in which its various components are organized in a valid way. The designers play a role similar to architects conceiving a working place for prospective users. When the problem is put to an actual individual (young student, student or teacher), the problem will be treated in what we have named a *personal SGW*. The geometric work at school can be described thanks to three SGW levels: Geometry intended by the institution is described in the reference SGW, which must be fitted out in an appropriate SGW, enabling an actual implementation in a classroom where every student works within his or her personal SGW.

Various Geneses of the Space for Geometric Work

As we have seen, geometrical work is framed through the progressive implementation of various SGW. Each SGW, and specifically the personal SGW, requires a general genesis which will lean on particular geneses connecting the components and cognitive processes essential to the functioning of the whole Geometric Working Space. The SGW epistemological plane needs to be structured and organized through a process oriented by geometrical paradigms and mathematical considerations. This process has been named *epistemological genesis*. In the same way, the cognitive plane needs a cognitive genesis when it is used by a generic or

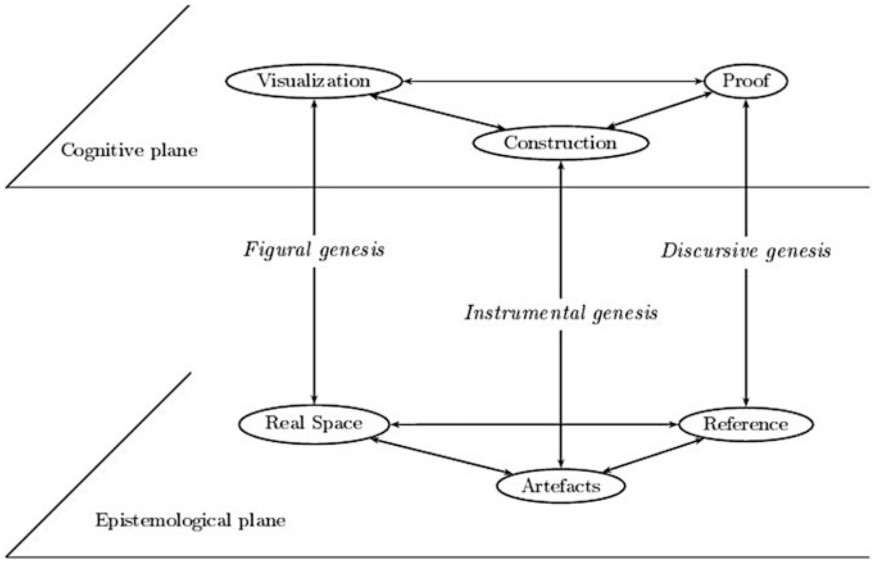


Fig. 2 Geneses into the geometrical work space

particular individual. Specific attention is due for some cognitive processes such as visualization, construction and discursive reasoning.

Both levels, cognitive and epistemological, need to be articulated in order to ensure a coherent and complete geometric work. This process supposes some transformations that can be pinpointed through three fundamental geneses strictly related to our first diagram (Fig. 2):

An *instrumental* genesis which transforms artefacts in tools within the construction process.

A *figural* and *semiotic* genesis which provides the tangible objects their status of operating mathematical objects.

A *discursive* genesis of proof which gives a meaning to properties used within mathematical reasoning.

We will examine how it comes into geometrical work by clarifying each genesis involved into the process.

On Figural Genesis

The visualization question came back recently to the foreground of concerns in mathematics and didactics after a long period of ostracism and exclusion for suspicion.

In geometry, figures are the visual supports favoured by geometrical work. This led us, in a slightly restrictive way, to introduce a figural genesis within the SGW framework to describe the semiotic process associated with visual thinking and involved in geometry. This process has been especially studied by Duval (2005) and Richard (2004). Duval has given some perspectives to describe the transition from a drawing seen as a tangible object to the figure conceived as a generic and abstract object. For instance, he spoke of a biologist viewpoint when it is enough to recognize and classify geometric objects such as triangle or Thales' configurations often drawn in a prototypical way. He also introduced the idea of dimensional deconstruction to explain the visual work required on a figure to guide the perceptive process. In that case, a figure needs to be seen as a 2D-object (a square as an area), a set of 1D-objects (sides) or 0D-objects (vertices). Conversely, Richard insists on the coming down process from the abstract and general object to a particular drawing.

On Instrumental Genesis

A viewpoint on traditional drawing and measuring instruments depends on geometrical paradigms. These instruments are usually used for verifying or illustrating some properties of the studied objects. The appearance of computers has completely renewed the question of the role of instruments in mathematics by facilitating their use and offering the possibility of dynamic proofs. This aspect is related to the question of proof mentioned in the preceding paragraph, but the ability to drag elements adds a procedural dimension which further increases the strength of proof in contrast to static perception engaged in paper and pencil environments. But the ability with the use of artefacts is not easy to reach by the students. At the same time, teachers need to develop specific knowledge for implementing software in a classroom. Based on Rabardel's works on ergonomic, Artigue (2002) stressed the necessity of an instrumental genesis with two main phases that we can insert in our frame. The coming up transition, from the artefacts to the construction of geometric configurations, is called instrumentation and gives information on how users manipulate and master the drawing tools. The coming down process, from the configuration to the adequate choice and the correct use of one instrument, related to geometric construction procedures, is called instrumentalisation. In this second process, geometric knowledge are engaged and developed.

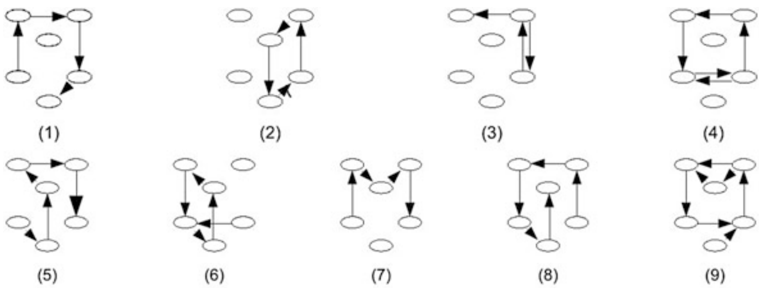
On Discursive Genesis of Reasoning

The geometrization process, which combines geometric shapes and mathematical concepts, is central to mathematical understanding. We saw the strength of images or experiments in developing or reinforcing certainty in the validity of an announced result. However, how can we make sure that students understand the logic of proof when they do not express their argumentation in words, but instead base it on visual reconstructions that can create illusions? A discursive explanation with words is necessary to argue and to convince others.

The nature and importance of written formulations differ from one paradigm to another. In most axiomatic approaches, it is possible to say that mathematical objects exist only in and by their definition. This is obviously not the case in the empiricist approach, where mathematical objects are formed from a direct access to more or less prototypical concrete objects.

Towards a Coherent Geometric Work at the End of Compulsory School

Using the theoretical framework introduced above, we will insist here on some contradictory ways we encountered in French geometry education and highlight what could be a coherent approach using both geometric paradigms. For that, we draw some conclusions from a work of Lebot (2011) who has studied different ways of teaching the introduction for the notion of angles at Grades 6 to 8. Using the SGW diagram, it is possible to describe possible routes students may take when they use software or drawing tools to construct figures and solve problems. Lebot has observed interesting differences visible on the following diagrams and we will discuss some among them.



A Coherent GI Work Space

Generally, a geometric task begins with a construction performed using either traditional drawing tools or digital geometric software. Each time, the construction is adjusted and controlled by the gesture and vision.

In this approach to geometry, the trail into the SGW diagram is like the one of Diagram 5 and done in a first sense (Instrumental—Figural and then Discursive) which characterizes an empirical view on geometric concepts.

A coherent way to work theoretically in Geometry I would be to use "approximated" theorems in the sense we introduced (Section "A Coherent GII Work Space") where the numerical domain is based on decimal numbers rather than real numbers. Theoretical discourse must justify what we see and not contradict it. This approach has been developed by Hjelmsev (1939) among others.

A Coherent GII Work Space

In the Geometry II conception, the focus is first on the discourse that structures the figure and controls its construction. This time, the route is trailed (Diagram 8) in an opposite sense (Discursive—Figural—Instrumental) and the figure rests on its definition: All properties could be derived from the definition without surprises.

In the traditional teaching and learning of geometry, students are frequently asked to start geometric problems with the construction of real objects. This leads them to work in the sense (I-F-D) of the Diagram 5. But for the teacher, the actual construction of an object is not really important. The discursive approach is preferred and expected, as in the Diagram 8 covered in sense (D-F-I): what I know is stronger than what I see and measure. In this pedagogical approach, elements coming from Geometry I support students' intuition for working in Geometry II, leading the formation of a (GII/GI) Work Space. But at the same time, students may believe that they work in a (GI/GII) Work Space where the objective is to think about real objects using some properties coming from Geometry II (Thales and Pythagorean Theorems) to avoid direct measurement on the drawing. The geometric work made by students could be incomplete as in Diagram 6 where students stay in an experimental approach without any discursive conclusion. They have paid attention to the construction task which requires time and care, but this work is neglected in the proof process expected by the teacher, where figures play only a heuristic supporting role. That can lead to another form of incomplete work but this time favoured by teachers as in Diagram 4 where there exists only interaction between proof and figure.

The inverse circulation of the geometric work in Geometry I and Geometry II can lead to a break in the geometric work that forms, when only one approach is explicitly privileged. We support the idea that both geometric paradigms must be included in geometry learning to develop a coherent (G|GII) Work Space where both paradigms have the same importance. Only when this condition is met, can an approximation have both a numerical and geometrical meaning, and can a work space be created suitable for introducing "almost parallel" lines in relationship to decimal numbers and where "strictly parallel" relate to real numbers. That would help resolve problems of mathematical coherency such as those experienced by students who asserted that they did not know if the lines were parallel because "the lines (IJ) and (DC) are parallel if we round off, but they are not if we take the exact value".

Beyond the Space for Geometric Work

How can the notion of SGW be extended beyond the Geometry? First, we can take into account the context within which the geometric work is developed. This context can be of social nature or could deal with the cognitive dimension in the teaching and learning processes as Arzarello and Robutti (2008) did by introducing the "Space of Action, Production and Communication" viewed as metaphorical space where the student's cognitive processes mature through a variety of social interactions. Within these frameworks, it is clear that the notion of SGW can operate and pinpoint on what, at the end, is the goal of an educational approach in mathematics: to make an adequate mathematical work. This assertion leads us to another kind of generalization related to what is mathematical work. In this direction, we have started some investigations with researchers interested in Calculus, Probability or Algebra. A third symposium on this topic has been held in Montreal in 2012 and some elements on this approach are given in Kuzniak (2011). The generalization supposes an epistemological study in-depth of the specific mathematical domain and of its relationships to other domains. Indeed, each domain relates to a particular class of problems and the crucial question is to find an equivalent to the role that space has in geometry. Variations and functions for calculus, chance and data for probability and statistics, can play the same role as space and figures in geometry. If it seems that the two planes, epistemological and cognitive, keep the same importance as in geometric work, figural genesis and vizualisation should be changed and reinterpreted through semiotic and representation processes in relationship to the mathematical domain concerned. But it is another story and work in progress.

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Integration of Technology into Mathematics Teaching: Past, Present and Future

Adnan Baki

Abstract This paper deals with my endeavor as a researcher and lecturer within the world of educational computing to integrate technology into mathematics teaching. I started with the book titled "New Horizons in Educational Computing". In this book Saymor Papert enthusiastically says that computers as powerful learning tools will change tomorrow's classrooms. It is difficult to use this potential of computers for changing teacher's role and practice within an educational setting based on telling and showing. It was not easy for me to shift from traditional notions of teacher to constructivist teacher using Logo, Cabri and GeoGebra as primary tools for doing and exploring mathematics in classrooms.

Introduction

This paper represents a more than 20 year effort made relentlessly since I have been started doing postgraduate studies at the UNB in Canada. As a learner I started with Logo in 80s and continues up till today with Cabri and GeoGebra. When I was an undergraduate student in 70s I just heard the name of the computer, but I have never seen it. I touched the computer for the first time in my life, year 1988. I came up with a book titled "Mathematical Applications of Electronic Spreadsheets" by Deane Arganbright, It was the first book of mine about educational computer. Activities and problems in this book were all what I already knew in school mathematics. They were not really interesting for me in terms of learning and teaching mathematics from a constructivist paradigm. Nothing was new for me in this book in terms of constructing and exploring new mathematical ideas.

My second book on educational education was "Computers in the Mathematics Curriculum" published by The Mathematics Association and edited by David Tall. This book included many open ended activities for constructing mathematical ides.

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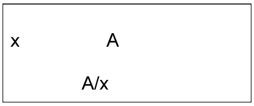
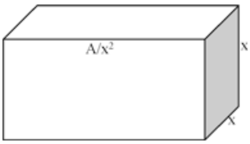
One of them was an example of iteration for finding the square root of a number A. When A is a square number like 49, we can use square to represent the number geometrically and one side of the square (7) will be square root of the number A. When A is not a square number like 55, we can use a rectangle to represent the number geometrically and calculate the square root by using the iteration method. In the classroom this method may be introduced by starting with a rectangle of area A.



The problem is to find the length of one side of the square which has the same area as the rectangle. If one side of the rectangle is of length x, the other side will be of length $\frac{A}{x}$. It is clear that the length of the square of area A will lie between x and $\frac{A}{x}$. In this case, the best approximation is likely to be given by replacing x by $\frac{1}{2}(x+\frac{A}{x})$ and continue with this iteration until the difference between x^2 and A is less than a prescribed amount, say 0.0001.

	A	B	C	D	E
1	A =	55	x^2	ROOT	DIFFERENCE
2	X=	28			
3	28	14,98214	419,5	DEVAM	-364,5
4	14,982143	9,32659	139,7323	DEVAM	-84,7323023
5	9,3265899	7,611854	70,99264	DEVAM	-15,9926396
6	7,6118539	7,418713	56,47016	DEVAM	-1,4701598
7	7,4187131	7,416199	55,018652	DEVAM	-0,01865169
8	7,4161989	7,416198	55,000003	DEVAM	-3,1605E-06
9	7,4161985	7,416198	55	DEVAM	-9,2371E-14
10	7,4161985	7,416198	55	7,41619849	0
11	7,4161985	7,416198	55	7,41619849	0
12	7,4161985	7,416198	55	7,41619849	0
13	7,4161985	7,416198	55	7,41619849	0
14	7,4161985	7,416198	55	7,41619849	0

In a similar way, an iteration for cube roots can be obtained by starting with a cuboid of square cross section whose volume is A and finding the length of side of the equivalent cube.



Later, I came up with the book titled "New Horizons in Educational Computing". 1984 This was the real turning point for me in my endeavor of the educational computing. In this book, Saymor Papert enthusiastically says that "*computers as powerful learning tools will change tomorrow's classrooms*".

At the beginning, I actually had difficulty to see this potential of computers in changing teacher's role and classroom practice. I tried to compromise my teaching approach based on telling and showing with the approach based on Papert's constructivist ideas about using Logo. It was not easy for me to shift from traditional notions of teacher to constructivist teacher using Logo as primary tools for doing and exploring mathematics in classrooms.

When I was a postgraduate student at the Institute of Education in University of London I found opportunity to work in Microworld Project with Celia Hoyles and Richard Noss. Their perspectives and approaches to educational technology helped me gradually to see what Seymor Papert points out about the potential of computers in changing teacher's role and classroom practice. I saw Logo as a paradigm for thinking about the use of mathematical software. My experiences at the Institute of Education had led me to believe that Logo is a powerful medium for confronting teachers with their preconceptions about teaching and learning mathematics. Many of the mathematical ideas which are used within the Logo environment (e.g., turtle geometry and recursion) were knew for me and excited me to learn more about Logo.

Programming as a Problem Solving

After finishing my doctoral program I returned to Karadeniz Technical University as a lecturer with many books on Logo such as "Approaching Precalculus Mathematics Discretely" edited by Philip Lewis, "Learning Mathematics and Logo" edited by Celia Hoyles and Richard Noss, "Turtle Geometry" edited by Harold Abelson and Andrea diSessa. With my expecting to explore mathematical concepts in a Logo-based environment I used these books in mathematics courses at undergraduate level and to investigate a model of developing the concepts of calculus and algebra using Logo. We (I and my students in these courses) learned some mathematics by experimenting with the ideas and developed our own structures.

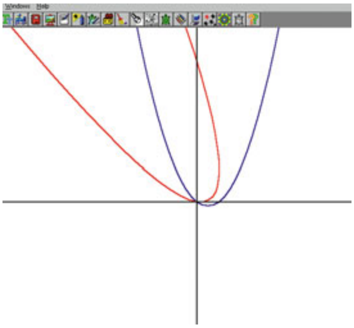
Through my first experience as learner and as teacher within a Logo based environment, I have realized that programming as a mathematical activity. Although I have sought to use Logo as a mathematical language and a good deal of the problem-solving activity is mathematical, it is also clear that many of the problems are problems of programming.

Problem solving	Programming
Understanding the problem	Understanding the problem
Planning for solution	Coding the program
Carrying out the plan	Running the program
Evaluation	Debugging the program

Representing function graphically and rotate the graphs of functions in Logo environment, it is possible both representing the function graphically and rotate the graphs. I worked with my students on the following task:

First, we define f(x) = x(x − 2) function in Logo. If we want to rotate the graph $\theta°$ counter-clockwise about the origin. We need to use the matrix.

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

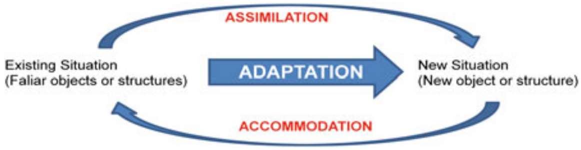


For 30°

```
to f :x
op :x*(:x-2)
end
to dönx :x :y :k
op ((:x*cos :k)+(:y*(-sin :k)))
end
to döny :x :y :k
op ((:x*sin :k)+(:y*cos :k))
end
to dönme :fonk :x :k
setpenwidth 2
if :x=15 [pu home pd penerase setpos se -75 (koş :fonk -15) pu home
Setheading 0 setpc 1 pd eksen stop]
make “y koş :fonk :x pd setpc 5 setpos se (5*:x) :y
make “t koş :fonk (:x+1) setpos se 5*(:x+1) :t
make “m dönx (5*:x) :y :k
make “n döny (5*:x) :y :k pu setpc 10 setpos se :m :n
make “r dönx (5*(:x+1)) :t :k
make “l döny (5*(:x+1)) :t :k
pd setpos se :r :l pu setpc 5 setpos se (5*(:x+1)) :t pd
dönme :fonk :x+1 :k
end
```

```
to koş :fonk :x
op run se :fonk :x
end
to eksen
fd 400 bk 800 fd 400 rt 90 fd 400 bk 800 fd 400 lt 90
end
```

Piaget and Logo

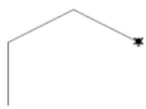


In Piaget's terms, when the individual is confronted with conflicts during mathematical activities, there are two possibilities for him/her: either she/he ignores the problem or accommodation process takes place with some modifications. This experience, therefore, enables the individual to conceptualize new situation from previous existing knowledge. Let us see how this occurs within a Logo-based environment:

Suppose that student's previous knowledge consists of writing small procedures in Logo. And also she/he knows the basic properties of square and equilateral-triangle (all sides are equal, all angles are equal and 60°. When we ask him to write a procedure for a square, he can write the following procedure and check it on the screen. Everything is going well.

```
to square
repeat 4 [fd 40 rt 90]
end
```

These are all his existing previous knowledge about drawing geometric figures in 2-D. After this experience, when we ask him to write a procedure for equilateral-triangle, this task is a new situation for him. By using his previous knowledge about Logo and triangle, probably he may just change only **repeat** line in the **square** procedure, and then write the following procedure:

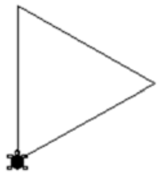


```
to equı-trıangle
repeat 3 [fd 40 rt 60]
end
```

When he run the procedure, he will see this figure.

This figure is entirely different what he expected to see.

This new situation is disequilibrium for him. In order to conceptualize this new situation, accommodation process needs to work. This process can work like this way; he can turn to the procedure and try to modify it, or he can put himself into the position of the turtle and traces the path of the turtle on the figure. When he realizes that the turtle turns according to the exterior angle rather than interior angle, it means that accommodation process is completed. Now student get new knowledge that drawing geometric shapes with Logo in 2-D space, we should use the sum of exterior angle of the shape which is 360°. After this adaptation, the procedure for an equilateral-triangle will be:



```
to equı-trıangle
repeat 3 [fd 40 rt 60]
end
```

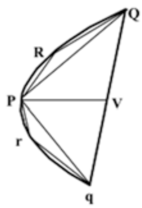
Dynamic Geometry Software

Doing mathematics in a dynamic geometry software environment is a process consisting of:

Making experimentation
Making conjecture
Proving the conjecture.

Let me give an example from the course of computer-based mathematics teaching which I have taught since 1996. I and my undergraduate students worked on the proposition of Archimedes by using CABRI:

Every segment bounded by a parabola and a chord Qq is equal to four-thirds of the triangle which has the same base as the segment and equal height.

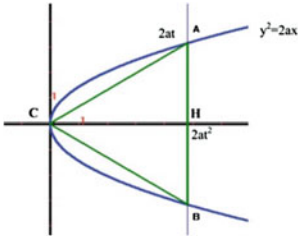


Making experimentation consists of three steps:

1. Specifying the proposition (Special Case)
2. Constructing the general form of the proposition (General Case)
3. Explaining the empirical findings

First, we constructed a **special case** as in the figure. In this case, we drew the largest triangle in the parabolic segment with Cabri.

In this case, the vertex is in the origin and the segment AB is perpendicular to the X-axis. Let A_T be the area of the triangle ABC, then $A(ABC) = 4a^2t^3$. The area of the half of the parabolic segment above X-axis will be:



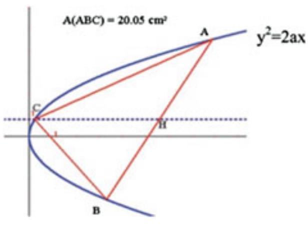
$$A_P = \int_0^{2at^2} \sqrt{2ax}\,dx = \frac{2}{3}\sqrt{2a}x^{\frac{3}{2}}\Big]_0^{2at^2} = \frac{8}{3}a^2t^3 \Rightarrow 2A_P = \frac{16}{3}a^2t^3$$

$$\frac{A_P}{A_T} = \frac{\frac{16}{3}a^2t^3}{4a^2t^3} = \frac{4}{3}$$

2. Constructing the general form of the proposition

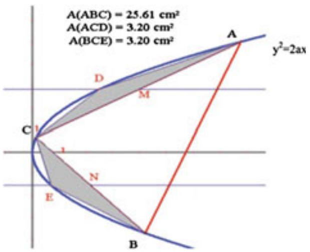
In order to expand our special case to the general case, we drew an arbitrary parabola with an arbitrary segment AB as in the figure in Cabri.

We construct triangle ABC with movable point C in order to search for the largest triangle ABC. Having located point C that maximized the area of triangle ABC, we marked the point C as the vertex. In the midst of this investigation we tried to find an answer to this question: Does the locus of the point C as a vertex have any geometrical property? The answer to this question would be the heart of the investigation. We observed that the point H is the midpoint of the segment AB and the line CH is parallel to X-axis.



(a) Making conjecture

After this observation, we conjectured that the locus of the highest point of the largest triangle in the parabolic segment is on the line passing through the midpoint of the base of the triangle and parallel to the X-axis. We continued to construct two second-tier triangles on the rest of the parabolic segment by using the same conjecture. We observed that the areas of two second-tier triangles are equal.



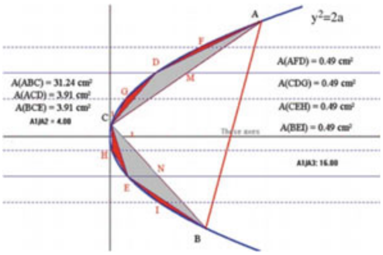
(b) Proving the conjecture

Similarly we continued to construct new triangles on the parabolic segment.

By stretching and shrinking the parabolic segment we got a series from the comparison of the areas of the triangles.

Let the area of the original triangle ABC be $A_T = a$ and the area of the parabolic segment be A_P. Then we got a series for the area of the parabolic segment as:

$$A_P = a[1 + \frac{1}{4} + \frac{1}{4^2} + \frac{1}{4^3} + \cdots + \frac{1}{4^{n-1}}] \Rightarrow A_P = \sum_{k=0}^{\infty} \frac{a}{4^k} = \frac{4}{3}a$$



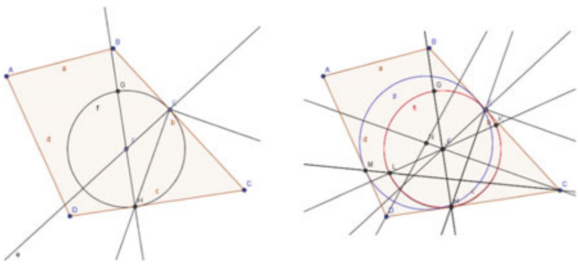
This concluded our proof.

Continuing with GeoGebra

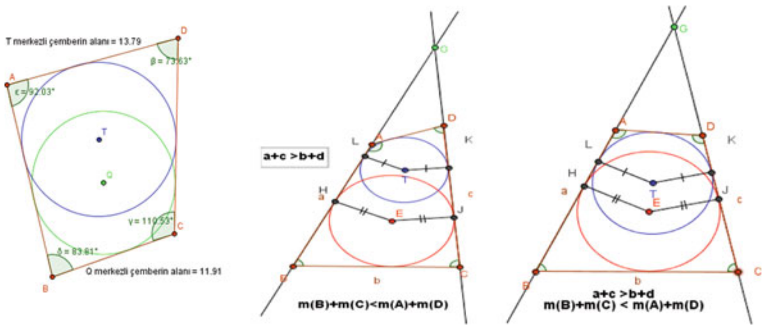
As a final example, I would like to share my individual exploration with GeoGebra as a final example of my presentation. The power of GeoGebra enables mathematicians to manipulate algebraic expressions and to construct geometric figures and drag them dynamically on the computer screen. Through using this powerful software work on the the problem dealing with the inscribing of the biggest circle in polygons. Although Euclidean geometry exists since two thousand years many interesting and challenging problems and theorems still remain to be explored by mathematicians. As a first step of the study I started with the problem stated that "*how to inscribe the biggest circle in a given regular polygon?*"

I easily solved this problem with GeoGebra. Successively, I checked whether if the solution of the initial problem is valid for all convex polygons.

Upon realizing that it is not valid for all convex polygons, then I re-stated the problem as "*how to inscribe the biggest circle in a given convex quadrilateral?*"

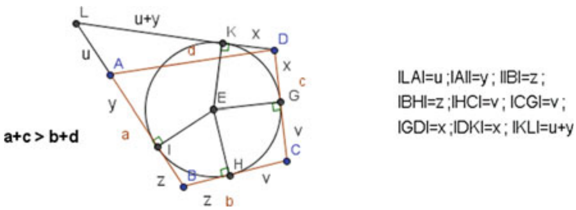


As a result of my investigation, we found that there is an original relationship between a non-regular quadrilateral and the biggest circle inscribed within it.



As a final step of the study I construct a formal proof of this relationship:

Proposition: Let ABCD be a convex quadrilateral, respectively a, b, c and d are length of IABI, IBCI, ICDI, IADI and a + c > b + d. In this case, circles are inside which are tangent to sides of AB, BC and CD or ABAD and CD.



Assert the contrary, let under the condition of $a + c > b + d \Rightarrow (y + z) + (x + v) > (z + v) + d \Rightarrow x + y > d \Rightarrow x + y + u > d + u$. This is a contradiction (according to triangle inequality).

In short, our journey does not end here; we continue to run towards new educational horizons.

Developing Free Computer-Based Learning Objects for High School Mathematics: Examples, Issues and Directions

Humberto José Bortolossi

Abstract In late 2007, the Brazilian government launched a grant program offering 42 million dollars to support the production of digital contents to high school level in the following areas: Portuguese, biology, chemistry, physics and mathematics. Of this amount, the CDME Project (http://www.cdme.im-uff.mat.br/) of the Fluminense Federal University won 124 thousand dollars to develop educational software, manipulative materials and audio clips to the area of mathematics. In this article, we report our experience (and what we learned from it) within this project, regarding the development of educational software as learning objects. We hope that the examples, issues and directions shown here are useful for other teams concerned about cost, time and didactic quality in the development of their applications and online teaching systems. Learning objects in mathematics, software development technologies, visualization in the teaching and learning of mathematics.

Introduction

The use of computers in teaching mathematics is a topic that has been studied for some time (for instance, the theme already appears in the ICME-1 conference held in France in 1969). With the prices falling faster, the processing power increasing and the advent of the Internet, computers became more affordable and popular, gaining a place in our homes and our schools. On the other hand, recent studies show that the simple fact of having access to a computer or Internet technology has little impact on student achievement in basic education (Papanastasiou et al. 2003; OECD 2006; Lei and Zhao 2007; Witter and Senkbeil 2008; Roschelle et al. 2010). In the case of mathematics, the impact may be even negative! These same studies indicate that the problem is not the computer itself, but rather how it is used as a

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tool for teaching and learning (see, for instance, Fig. 1 in Witter and Senkbeil 2008). The availability of quality digital contents accompanied by methodological guidelines for their effective use in teaching practice is fundamental! In this context, various materials have been proposed: the Atractor Project[1] (funded by the Ministry of Education of Portugal), the Descartes Project[2] (funded by the Ministry of Education of Spain), the WisWeb Project[3] (of the Freudenthal Institute for Science and Mathematics Education in the The Netherlands), the Illuminations Project[4] (of the National Council of Teachers of Mathematics in the United States) and the Shodor Project[5] (funded by National Science Foundation in the United States). In Brazil, one of the last major initiatives occurred in 2007 with the release of 42 million dollars from the Ministry of Science and Technology and the Ministry of Education to fund the production of digital multimedia educational contents to high school level in the following areas: Portuguese, biology, chemistry, physics and mathematics. For the area of mathematics, five projects were awarded, one of them proposed by the Fluminense Federal University: the CDME Project.[6] With a budget of 124 thousand dollars and a staff of five professors and twenty students, this project produced 66 educational software, 12 practical experiments with manipulative materials and 15 audio clips. Of the 124 thousand dollars, 107 thousand dollars were spent on payment of scholarships for students. The remaining balance was used to purchase computer equipment. In this article, we report our experience (and what we learned from it), within this project, regarding the development of educational software as learning objects. We hope that the examples, issues and directions shown here are useful for other teams concerned about cost, time and didactic quality in the development of their applications and online teaching systems.

Our Instructional and Technological Choices

In the development of educational software, we have adopted the following instructional principles which, in our experience, proved to be very robust:

- Each software seeks to explore a specific topic in a self-contained way (following a style that some scholars call *learning objects*).
- To persuade and maintain students' attention, each software has been written in such a way as to lead them as quickly as possible to the main object of interaction under study.

[1] http://www.atractor.pt/.

[2] http://recursostic.educacion.es/descartes/web/ingles/.

[3] http://www.fi.uu.nl/wisweb/en/.

[4] http://illuminations.nctm.org/.

[5] http://www.shodor.org/.

[6] http://www.cdme.im-uff.mat.br/ or http://www.uff.br/cdme/.

- Our suggestions of exercises, reflections and questions to be worked out with the software are not embedded in the software itself: we prefer to state them in a separate text file (the Student Handout). This scheme has several advantages: (a) the content becomes more flexible, allowing the teacher to make adjustments according to the profile of his or her class (after all, an easy exercise for a class may be difficult for other); (b) it promotes the so important *writing practice* by the students; (c) one obtains a written record that can support discussions, reflections and assessments.

These principles have allowed that the structure of each software was designed to provide different logistical uses. The teacher can, for example, (1) to conduct the activity with all students in the school's computer laboratory (if the physical structure of the laboratory and the planning of lessons allow), (2) to conduct the activity in the classroom with the aid of a computer and a multimedia projector (something simpler and easier to organize) or, even, (3) to propose the activity as an extra exercise to be carried out during a certain period of time (a week, for example). In the latter case, once the period of time is over, the teacher can use the initial time of a class to discuss, along with their students, the experience they have had with the content (even in situations where the interaction of students is not so intense, we believe, even so, that carrying it will bring benefits).

While instructional aspects are very important in the development of educational software, technical aspects also deserve equal attention. After all, the range of technologies employed in the development has a direct influence on the available building blocks that will constitute the software. In fact, depending on the technologies chosen, certain didactic components can be more difficult to implement (spatial geometry and the two-dimensional mathematical notation are classic examples of this situation). Even more, the range of technologies has a direct impact on the cost and development time of the project. For the CDME Project, all contents were built as interactive and dynamic web pages (which can run on all major browsers) with the integrated use of the following technologies: the mathematical components were implemented with the help of the Java language, the software GeoGebra (Hohenwarter 2012) and the software JavaView (Polthier et al. 2012), structure and organization are designed using the HTML and MathML languages, elements of style with CSS language, interactivity and dynamism with the JavaScript language. All software can be stored on a server for online use through the Internet or, alternatively, they may be downloaded or made available on CDs and flash drives for offline use (that is, in a browser, but without the need of an Internet connection).

In the following sections, we will present, briefly, some of the digital educational contents produced by the CDME Project.

Learning Objects on Spatial Geometry

Typically, the teaching of spatial geometry is done using the textbook and the chalkboard as the only tools. Thus, the student of mathematics has to face the arduous task of studying three-dimensional objects from two-dimensional representations presented statically in a book page or on a chalkboard. This transition from a drawing on a sheet of paper to the actual object in space is configured as a significant problem in the study of geometry in three dimensions. In fact, it appears that both students and teachers face difficulties in the construction and interpretation of two-dimensional representations of three-dimensional objects. Moreover, even books and periodicals often display wrong drawings of three-dimensional objects: (Grünbaum 1985) and (Casselman 2000). Therefore, any tool that can assist the teacher in the teaching of spatial geometry is welcome.

Certainly, the use of concrete manipulative materials is an indispensable educational resource, especially in the early grades. On the other hand, due to technical limitations, there are certain configurations and geometrical properties that are difficult to be worked out using such materials. In this context, the computer puts itself as a promising tool for the teaching of spatial geometry. However, only recently 3D dynamic geometry software began to appear: Archimedes Geo3D,[7] Cabri 3D,[8] GEUP 3D[9] and GeoGebra 3D.[10] Of these four applications, the first three are not free and the last one is in beta test. Thus, there is still an enormous lack of free educational software geared towards the teaching and learning of spatial geometry. To try to fill this gap, it seemed to us that it would be much appropriated to prioritize and to put efforts on the production of learning objects for this theme.

Trip-Lets

Inspired by the cover of the book (Hofstadter 1999), in this activity, we propose an interactive game to exercise three-dimensional thinking in an interdisciplinary work with Portuguese, Spanish and English languages: the student must manipulate a virtual special solid and identify three letters of the alphabet. With these three letters, he or she must compose a word or an acronym in the chosen language (Fig. 1). To assist him or her in the viewing process, coordinate axes and coordinate planes can be displayed in the software. In addition to the vocabulary exercise and the cognitive exercise of perception of three-dimensional forms, the software can also be used to address questions on symmetry (of the shape of letters) and counting (of the possible words for a given set of letters).

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[7] http://raumgeometrie.de.

[8] http://www.cabri.com/cabri-3d.html.

[9] http://www.geup.net/en/geup3d/.

[10] http://www.geogebra.org/forum/viewtopic.php?f=52&t=19846.

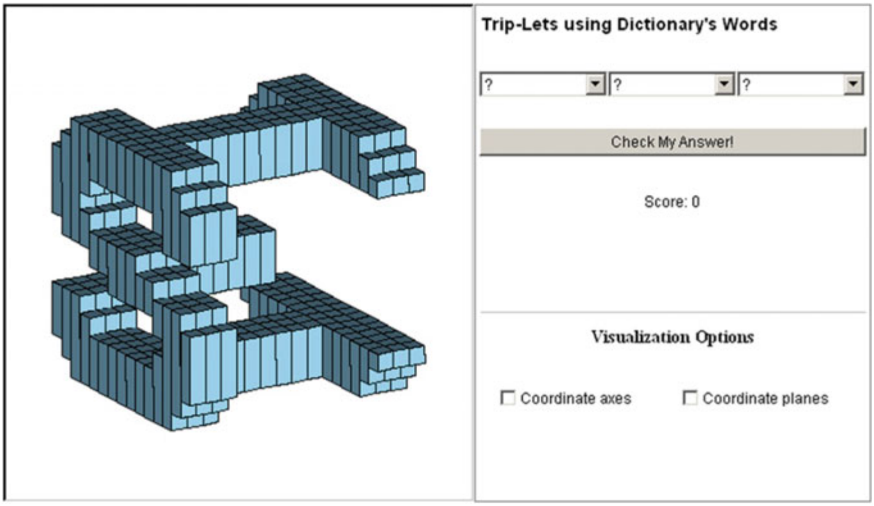

Fig. 1 Trip-Lets game with the English word SHE

The Tomography Game

The objective of this game is to identify an object that is partially invisible: only the intersection of the object with a plane is displayed. The height of this plane can be changed and, thus, the student has access to the various cross sections of the object, one at a time (Fig. 2). There are three categories of objects: polyhedra (with eight objects: a regular tetrahedron, a cube, a regular octahedron, a regular dodecahedron, a regular icosahedron, a pyramid of quadrangular base, a pyramid of pentagonal base and a regular pentagonal prism), non-polyhedral surfaces (with four objects: a right circular cone, a right circular cylinder, a sphere and a torus) and 3D models in computer graphics (with four objects: a skull, a dromedary, an octopus and a cow). In the game, a same object can appear in different positions. If the student fails to identify the object in two attempts, the program provides a hint: 10 cross sections of the object are displayed simultaneously. Along with the software, we present the use of cross sections in computed tomography and rapid prototyping technology (3D printers).

Projections in Perspective

Through various interactive modules, this content discusses some important aspects of the perspective projections, including questions of ambiguity (different objects that have the same perspective projection) (Fig. 3) and questions of deformation (as, for example, circles, which are projected in ellipses or straight line segments).

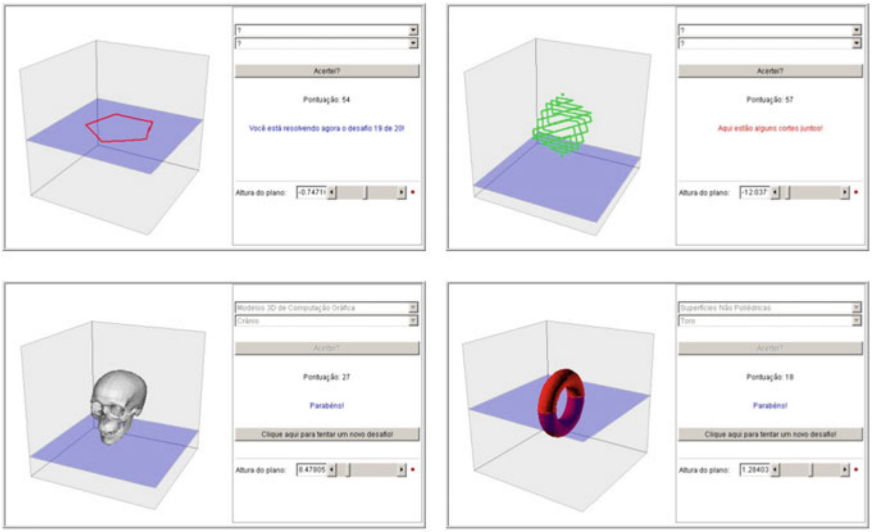

Fig. 2 The tomography game

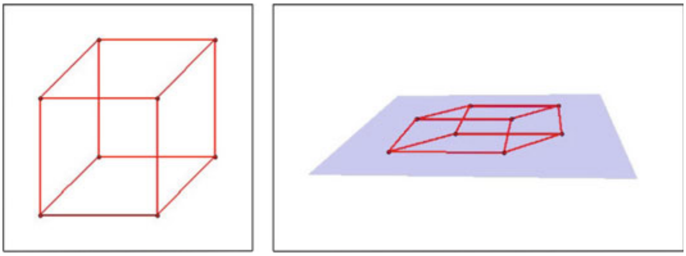

Fig. 3 Perspective projections: a cube?

The content also explores the use of perspective projections in photos and works of art (vanish points), including the analysis of some "impossible objects" of the Dutch artist Maurits Cornelis Escher (Fig. 4).

A Plethora of Polyhedra

The main element of this content is an application that allows the student to view and manipulate various types of polyhedra, such as the Platonic solids, the Archimedean solids, the Johnson solids, the Catalan solids, the isohedra, prisms, pyramids, etc. Several operations are on hand: polar reciprocation (to compute dual

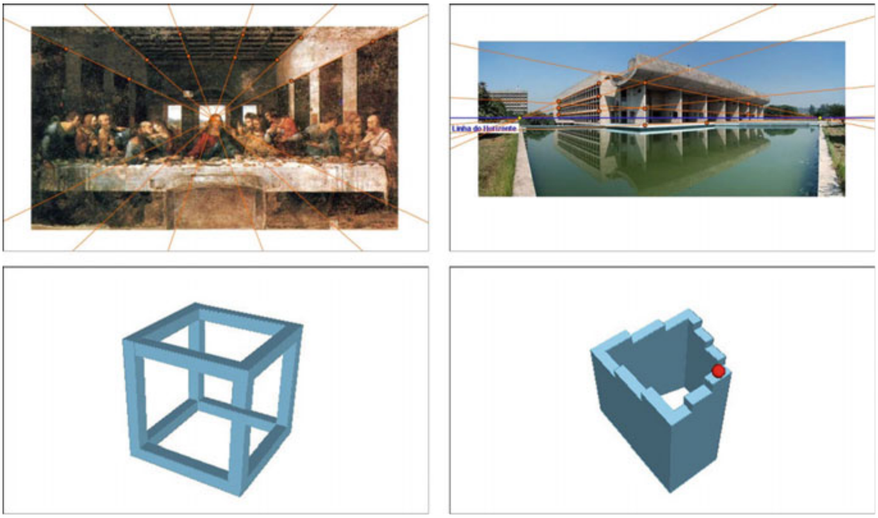

Fig. 4 Perspective projections in photos and works of art

polyhedra), Euler's characteristic computation, cuts by planes (cross sections), unfolding, truncation and stellation (Fig. 5). There are over 300 convex and non-convex polyhedra available.

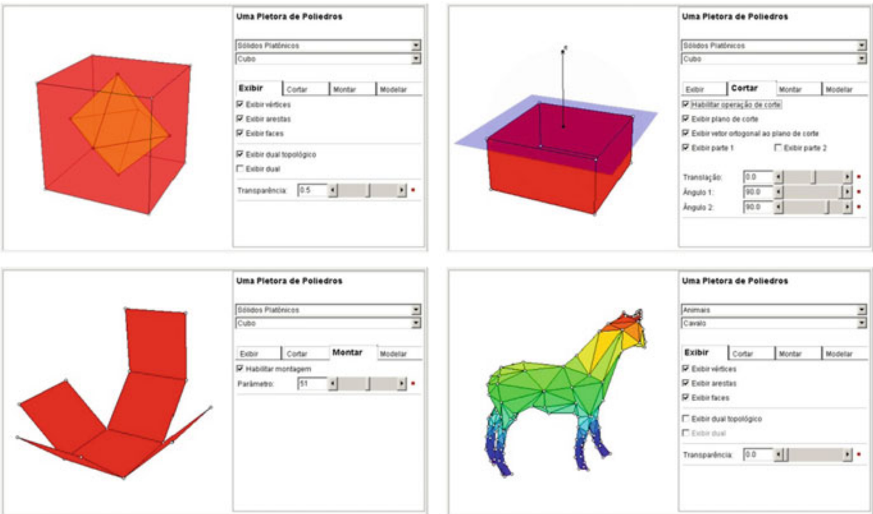

Fig. 5 A plethora of polyhedra

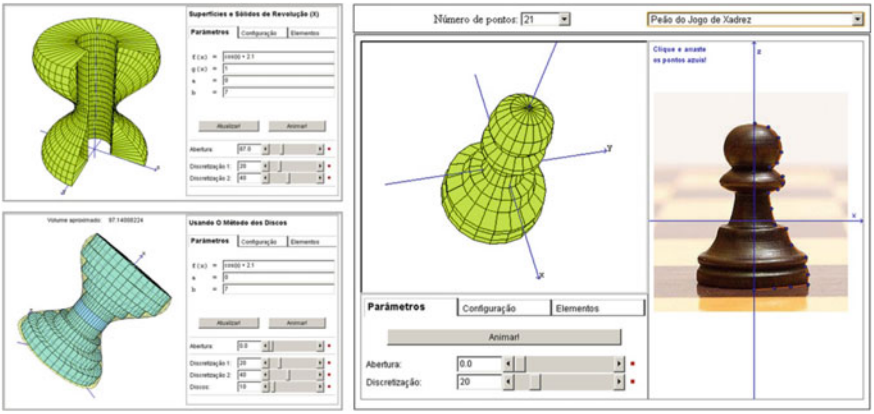

Fig. 6 Surfaces and solids of revolution

Surfaces and Solids of Revolution

This software offers a three-dimensional interactive environment where it is possible to view and to study a solid of revolution whose generatrix is described by real functions of one variable. With it, students can investigate how to use different types of functions (including functions defined by parts) to obtain solids of revolution with different shapes (cylinders, spheres, cones, tori, bottles, etc.). Moreover, through this description of solids of revolution via functions, students can see how approximations of these functions may be used to compute approximations of the volume of the original solid (using the disc and the cylindrical shell methods). With the objective to promote the perception of the presence and the use of surfaces of revolution in our lives, we have also prepared a special module where the student can interactively model a 3D object that is a surface of revolution using a photo of its profile (Fig. 6).

Matrices and Digital Images

This software (a joint work with Dirce Uesu Pesco) has two main objectives: (1) to present an application of matrices and their operations in digital images and (2) to explore the use of functions in the context of image processing, that is, to investigate how various effects that may be applied to an image (such as grayscale, brightness, quantization and transition) can be obtained through a convenient real function.

The content of this activity is divided into five main modules. The first module shows how binary digital images can be represented by matrices and how certain

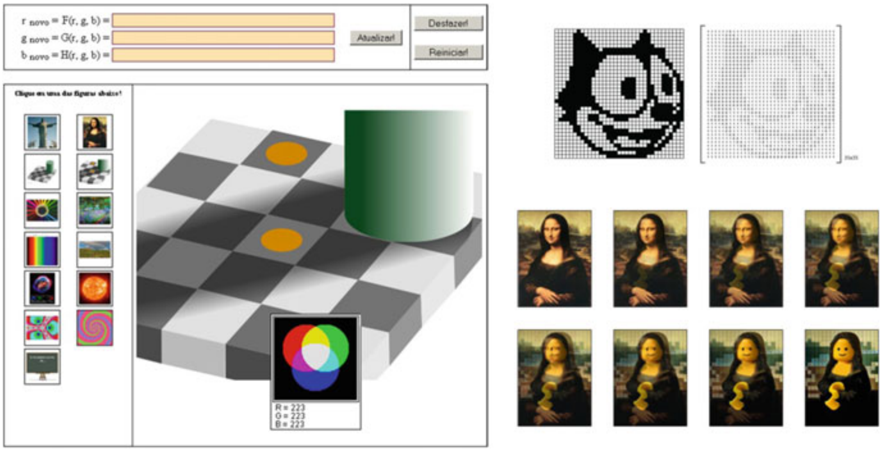

Fig. 7 Matrices and digital images

image manipulations are performed through operations on rows and columns of the corresponding matrix. The second module examines grayscale images through a program that allows the student to modify the intensity of a pixel through a user-defined function. The third module studies color images using the RGB system. In this system, three matrices are used to specify the amount of red, green and blue that makes up the image. Among the images studied in this module, there are the optical illusions created by Edward H. Adelson and Akiyoshi Kitaoka. The fourth module shows how a color in the RGB system can be identified with a point of a cube (the Color Cube). The fifth module shows how to use the operations of multiplication by scalar and sum of matrices to create a transition effect between images of the same size (Fig. 7). The material produced for this activity was the source of a submitted vignette for the Klein Project: (Pesco and Bortolossi 2012).

Learning Objects on Functions

How b *Depends on* a*?*

This software (a joint work with Dirce Uesu Pesco) is presented as a game with 16 challenges. For each one, two points, *a* and *b*, are marked on a number line. The student can then click and drag the point *a*. In doing so, the point *b* will change its position in accordance with the position of the point *a*, that is, *b* is a function of *a*: $b = f(a)$. To win the challenge, the student has just to find out which is the expression that defines the function *f* (Fig. 8).

If the student types an expression as an answer to the challenge and if he or she presses the "Am I right?" button, then the application will tell if he or she succeeded

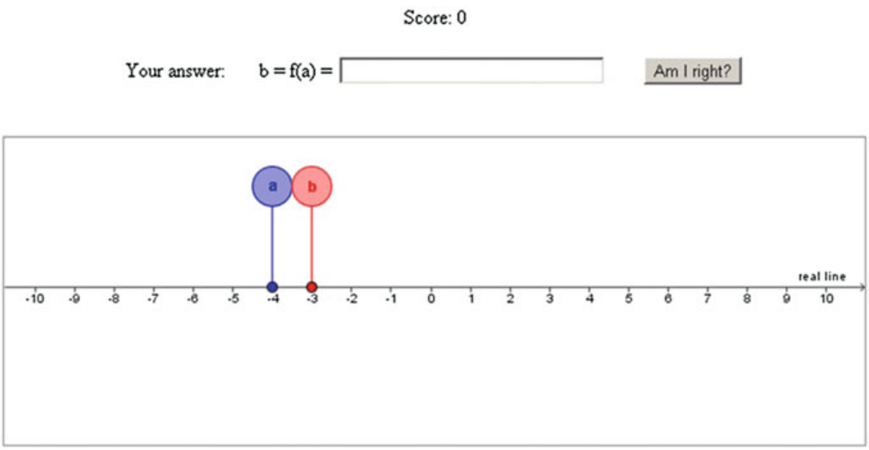

Fig. 8 How *b* depends on *a*?

or not, and at the same time, the application will create a third point (in green) on the number line corresponding to the value of the function of the student evaluated at the point *a* (Fig. 9). If the point *a* is dragged, the point *b* and the green point will move dynamically, thus allowing the student to reassess him or her answer in case of error.

If the student misses two times, the application will give a hint (the function is linear, the function is quadratic, etc.). If he or she misses again, the application will reveal the correct answer.

Traditionally, real functions are studied using algebraic techniques and graphs. In this activity, the subject is explored in a different and unusual way, namely, when

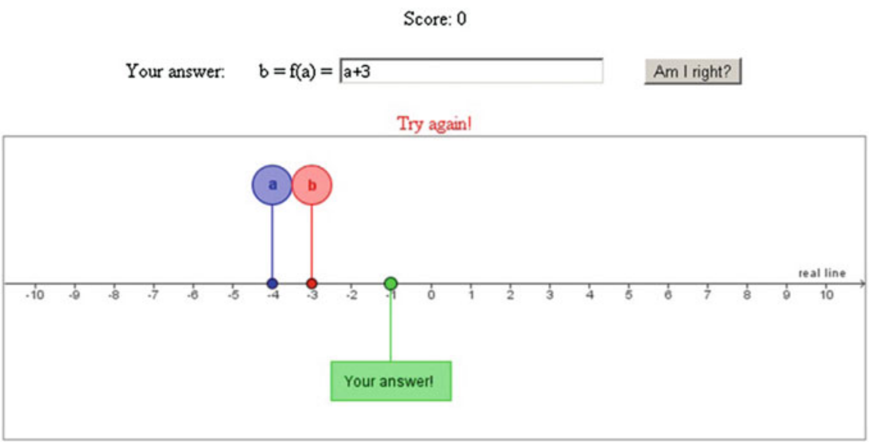

Fig. 9 Software's feedback for a wrong answer of the student

a point of the domain and its image are represented in the same number line. Thus, when interacting with the proposed game, students can assess their knowledge of properties of elementary functions.

The idea of the game is a limiting case of the *dynagraphs*, originally proposed by Goldenberg et al. (1992). In our opinion, working with a single number line (instead of the two number lines of the *dynagraphs*) has an advantage: it is easier to make comparisons between the values of a and b and thus try to establish some functional dependence. With different axes, there is an extra cognitive effort of transferring values from one axis to another.

Our experience with this activity has shown that students use the following typical strategy: change the position of point a, observe the position of point b, generate a table of functional values and, from this table, try to deduce an interpolating function. Other strategies may be induced: what is the variation Δb of b in terms of a variation Δa of a? (This strategy is particularly useful for linear functions.) Are there symmetries? Is the function even? Is the function increasing? Is the function injective? Trying to identify such properties in this new environment is a great exercise to practice these concepts.

At the end of the game, the software displays a page where students can visualize the relationship between the representation used in the game with graphs of functions (Fig. 10).

The Optimal Project

This activity (a joint work with Gilda de la Rocque Palis and Silvana Marini Rodrigues Lopes) is divided into 21 modules. In each module, the statement of an optimization problem is presented. The student should read it, interpret it and then perform a sequence of tasks with the use of two computational resources available in the module. In the first computational resource, the student can enter values for the independent variable x of the problem. If the value of x is not admissible, the program will acknowledge the fact. If the value of x is admissible, then the program will (1) calculate the value of the dependent variable V for that value of x, (2) record the values of x and V in a table, (3) draw the point (x, V) in a Cartesian coordinate system and (4) update the geometric configurations of the statement of the problem for these values. If the student clicks on one of the lines of the table, the program will update the geometric configurations of the statement in accordance with the values recorded on this line. Figure 11 shows the graphical interface of the first computational resource for the classic box problem: "Equal squares are to be cut off from the four corners of a flat rectangular cardboard sheet measuring 30 cm by 50 cm. The remaining flaps are then folded up to form an open box. What should be the value of x, the measure of the sides of the squares that were removed, to make the volume V of the corresponding box as large as possible?".

As indicated by Palis (2011): The tasks are designed to advance the issue of reading and interpretation of word problems by students, the discrimination

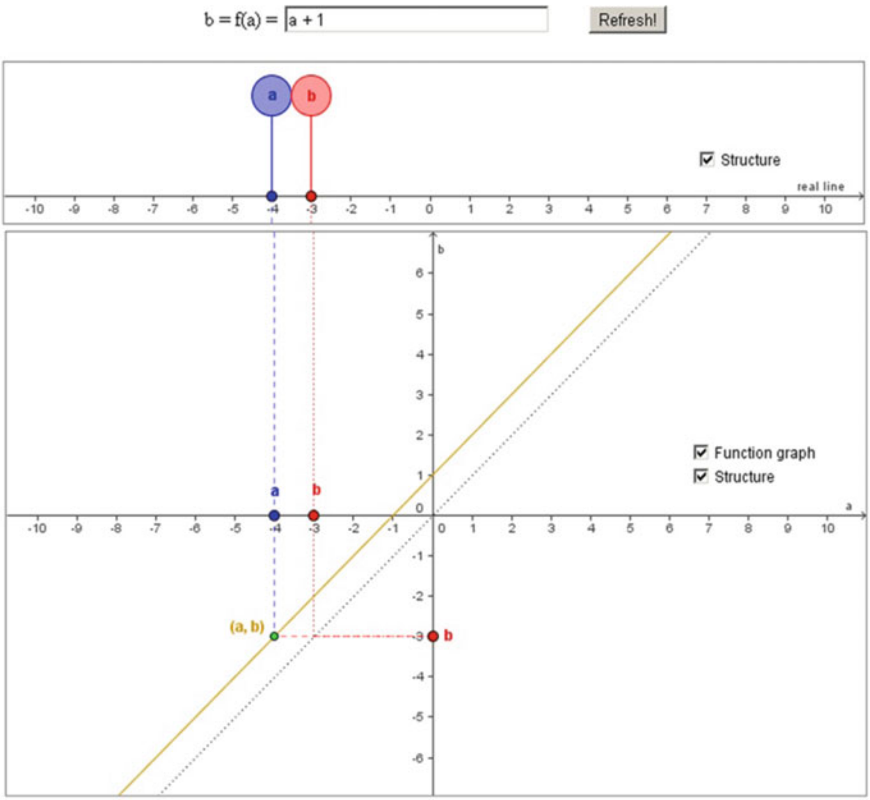

Fig. 10 Relationship between the representation used in the game with graphs of functions

between constants and variables, the development of the concept of approximation of a real number, the justification process in problem solving, the use of multiple representations of functions. The proposed tasks culminate in two points: (1) the student must determine (with pencil and paper) the objective function and the admissible set of the optimization problem (and, here, the second computational resource can be used to check the answer), and (2) the student must determine the optimal solution explicitly for the case of quadratic objective functions or an approximation of the optimal solution with error less than 0.01 for all other cases.

Epicycles and Trigonometric Interpolation

In this activity (a joint work with Carlos Tomei), we propose an interactive application that explores, through the theory of epicycles, the connections between trigonometric functions and the circle (with special emphasis on amplitudes and

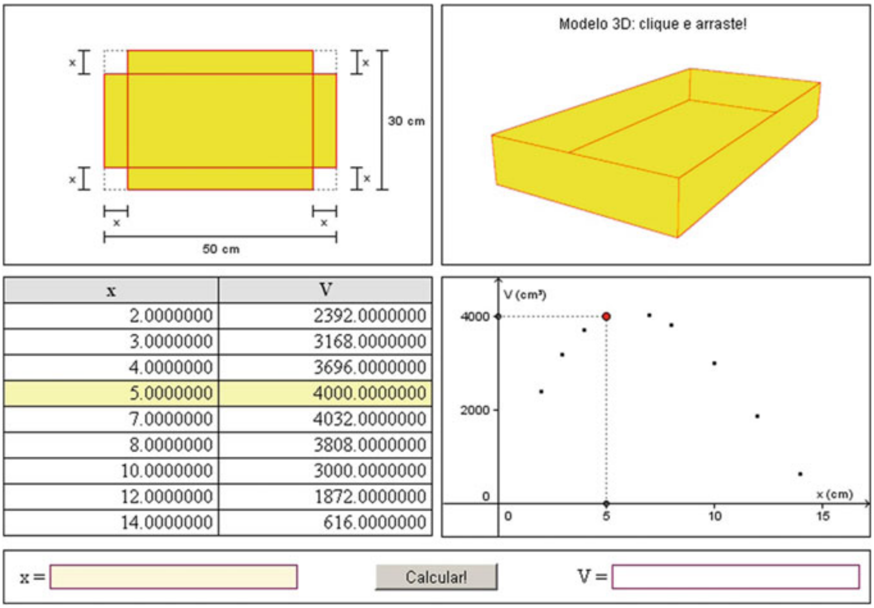

Fig. 11 The optimal project: the classic box problem

frequencies), parametrized curves, rational and irrational numbers in a geometric context (periodic orbits).

The activity is divided into three parts. In the first part, we present the basic model of the theory of epicycles, with only two coupled circles. The student can change the values of the radii of the circles (input fields r_1 and r_2 in Fig. 12) the values of the angular velocities with which the points move (input fields w_1 and w_2 in Fig. 12). If the student presses the "Animate!" button, then the application will animate, in function of the time t, the point of coordinates $(r_1 \cos(w_1 t), r_1 \sin(w_1 t))$, the circle with center in this point and radius r_2, and the yellow point with coordinates

$$x = r_1 \cos(w_1 t) + r_2 \cos(w_2 t) \quad \text{and} \quad y = r_1 \sin(w_1 t) + r_2 \sin(w_2 t).$$

Several questions can be explored: is it possible to make the yellow point draw a circle or an ellipse or a straight line segment? Roses (epicycles with $r_1 = r_2$) can be used to study how the ratio $|w_1/w_2|$ affects the geometry of the curve (a rose "never closes" if w_1/w_2 is an irrational number, and if w_1/w_2 is a rational number, investigate the relationship between the number of petals and the value of w_1/w_2).

In the second part, we describe how the basic model of the theory of epicycles with two coupled circles can be used to explain the retrograde motion of the planets (the Ptolemaic system).

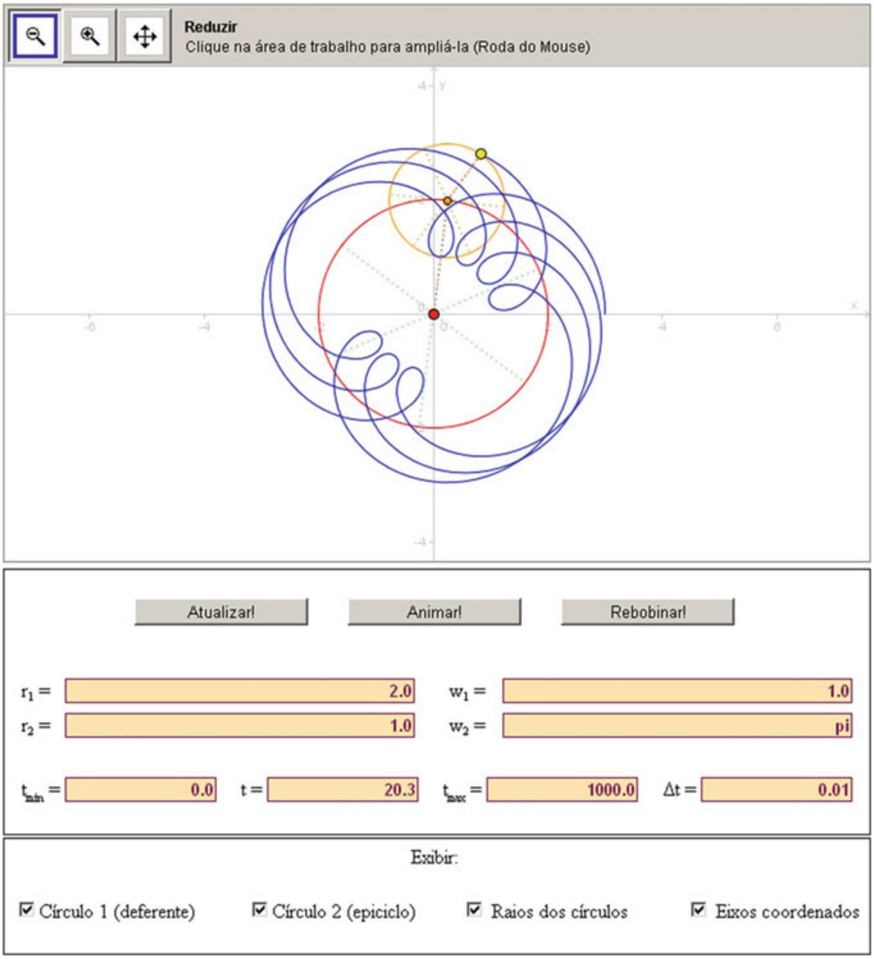

Fig. 12 The basic model of the theory of epicycles

In the third and final part, we show how to use epicycles with several coupled circles to approximate curves (a prelude to the theory of trigonometric interpolation and Fourier series). Figure 13 shows an approximation of the Batman symbol obtained with 32 circles coupled.

The Anatomy of a Quadratic Function

Quadratic functions are usually studied in connection with quadratic equations. Often the approach is restricted to algebraic manipulation, leaving aside important

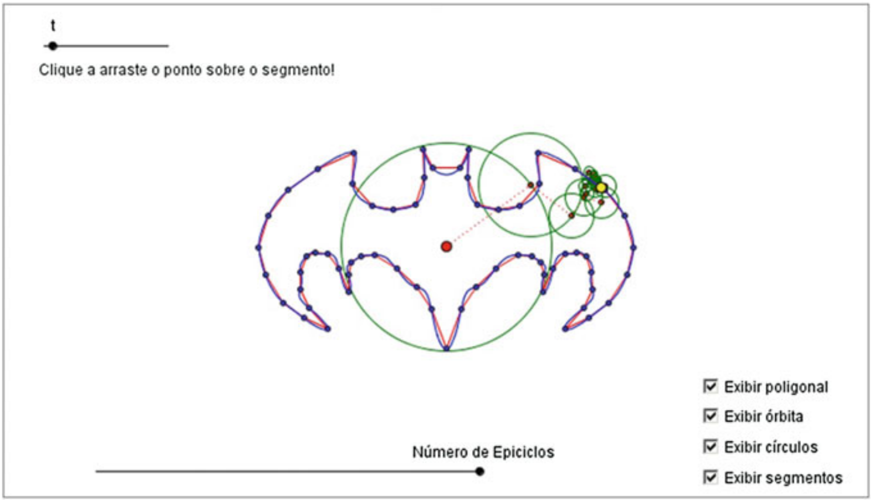

Fig. 13 Epicycles drawing the batman symbol

geometrical aspects. However, even some algebraic aspects are not developed with the proper intensity, as, for example, the fundamental technique of completing the square. Another remarkable characteristic of the teaching of this subject is the almost exclusive use of integer coefficients, which inhibits certain investigations on the dependence of geometric elements of the graph of a quadratic function with respect to their coefficients. Divided into four interfaces, this software offers to math teachers graphical, symbolical and numerical resources to subsidize the presentation and development of the theory of quadratic functions.

The Interface 1 (Fig. 14) allows the student to define a quadratic function with integer coefficients. This restriction is deliberate, because we believe that the first contact with the software should occur through an interface that is closer to that usually presented in textbooks. In this interface, there are two main areas: the graphic area and the algebraic area. In the graphic area of the Interface 1, students can view the following elements: the graph of the function; the vertex, focus and directrix of the parabola; the image of the function; the symmetry axis; the real roots (if any); a draggable tangent line; and a light beam (a set of parallel lines to the y-axis that hits the graph of the function and, then, converges at the focus). The viewport can be configured and there is an input field where the student can mark points on the graph of the function by specifying their abscissas. The coordinates of the vertex and the roots (if any) are also presented numerically, that is, with their values approximated by decimal numbers. In the algebraic area of the Interface 1, the discriminant, the real roots (if any), the standard form, the factored form, the coordinates of the vertex, the coordinates of the foci and the directrix equation are presented symbolically, that is, without approximations. By changing the coefficients of the quadratic function at the top of the interface, all graphical, symbolical and numerical elements are updated automatically.

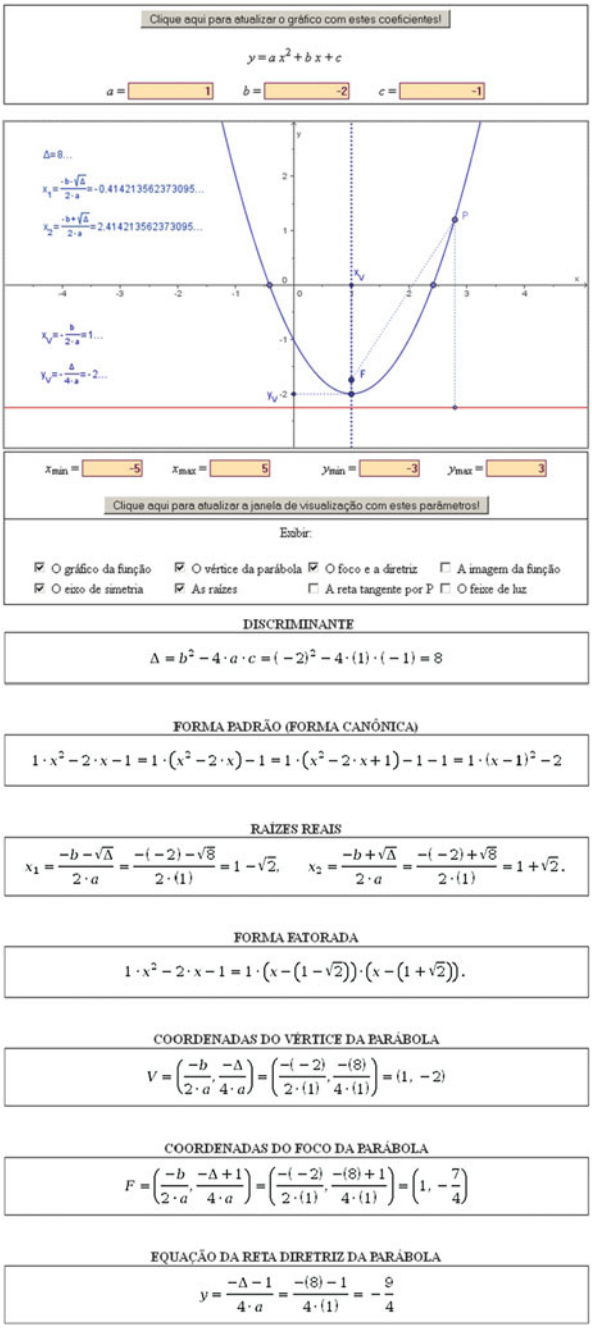

Fig. 14 The anatomy of a quadratic function

For convenience, the Interface 2 presents only the algebraic area of the Interface 1, in order to prevent the student only interested in algebraic calculations to have to scroll the HTML page in the browser. The Interface 3 allows the student to provide decimal numbers with a finite number of figures as coefficients of the quadratic function. The Interface 4 replaces the input fields by sliders in the specification of the coefficients and, so, the student may visualize, in a dynamic way, how the various graphical elements change as the values of the sliders are modified.

Among the proposed exercises, two classes are noteworthy (because they are particularly difficult to explore using only the chalkboard): (1) exercises that explore graphs drawn in a coordinate system with different scales and (2) the exercises that explore the dependence of the several geometric elements of the graph of a quadratic function with respect to their coefficients.

The Triangle Classification Game

This is one of our most accessed software (in Portuguese and English). In it, we present an interactive game to train the classification of triangles according to the sides and angles. The idea is quite simple: in the game, the student must move the vertices of the triangle on a Cartesian grid in the plane (so, the vertices of the triangle can be only placed on points with integer coordinates), in order to form the triangle that is asked in each challenge (Fig. 15).

The environment integrates algebra and geometry, and it is a place where some interesting questions may be posed: (1) Is it is possible to form a triangle rectangle with a horizontal hypotenuse? (2) Is it possible to form an isosceles triangle with a non-horizontal and a non-vertical base? (3) Is it possible to form an isosceles triangle with all sides not horizontal and not vertical? (4) Is it possible to form an equilateral triangle whose vertices have integer coordinates?

We also created a game similar to this, but that explores the classification of quadrilaterals: parallelograms, rhombi, kites, trapeziums, orthodiagonal quadrilaterals, cyclic quadrilaterals and inscriptible quadrilaterals.

Learning Objects on Probability and Statistics

Wheels of Fortune

This software offers various modules for the simulation of random experiments on the computer using pseudo random number generation algorithms, with emphasis on geometric probability. The Module 1 simulates a wheel of fortune where the sample space and probabilities can be configured, allowing the simulation of classical experiments like the toss of a coin and the roll of a dice (Fig. 16). There are

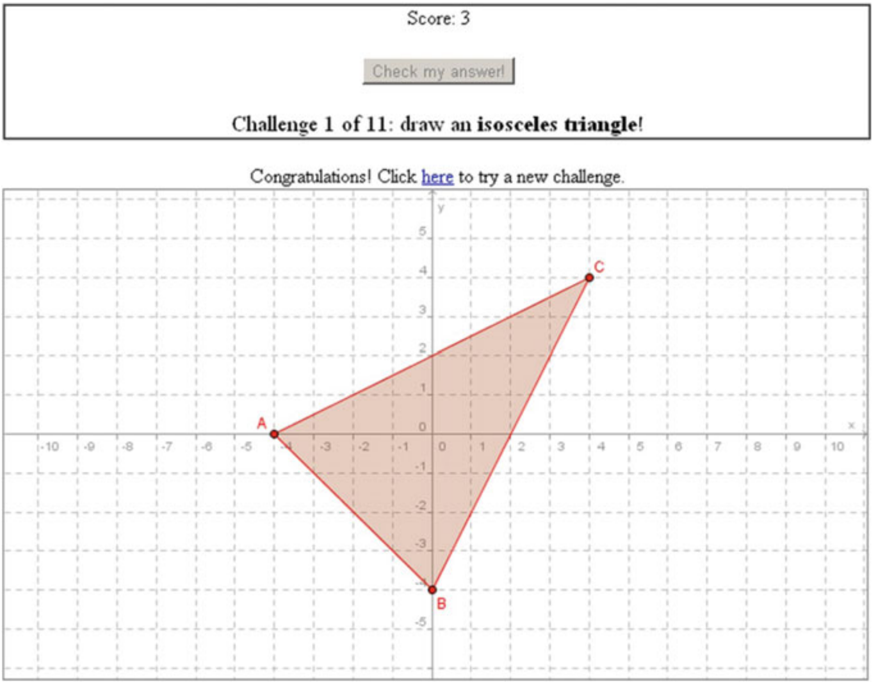

Fig. 15 The triangle classification game

three pseudo random number generators: the linear congruential generator (LCG), the Mersenne Twister and the Mersenne Twister 53.

The Module 2 explains the mathematics behind an algorithm widely used in the generation of pseudo random numbers on the computer: the linear congruential generator. The other modules are simulations where it is possible to test the linear congruential generator. The last three simulations are experiments in geometric probability: estimation of the area of a circle, estimation of the volume of a sphere (Fig. 17) and the classical Buffon's needle problem.

The material produced for this activity was the source of a submitted vignette for the Klein Project in Portuguese: (Bortolossi 2011a).

Statistics of Letters, Words and Periods

This software performs statistical analysis of strings. More precisely, the application counts the number of letters, digits, accents, punctuation marks, words and periods of a text. It also calculates the number of letters per word and words per period (showing position and dispersion measures of these quantitative variables), the longest periods, the shortest periods, the longest words and the shortest words.

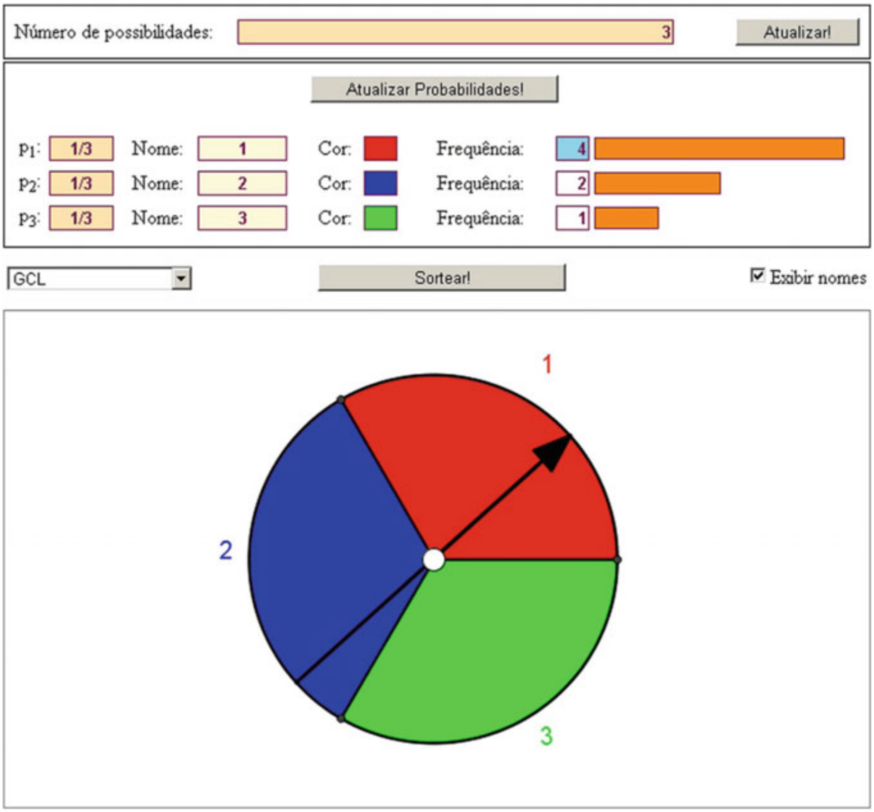

Fig. 16 Wheels of fortune

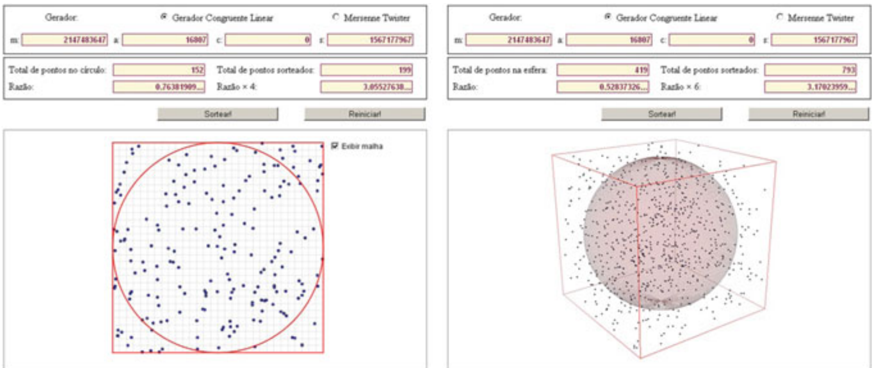

Fig. 17 Using Monte Carlo methods to estimate areas and volumes

There are two special modules: the first module lets the student use the frequency distribution of letters to decode messages encrypted with the Caesar cipher, and the second one allows him or her to investigate power laws in frequency distributions of words in a text (Zipf's Law).

The main objective of our software is to provide an interactive environment in which students and teachers can experiment, explore and enjoy the use of statistics in a real-world application (namely, text mining), and through this exercise in the linguistic context, promote the learning of statistical concepts. In addition, this proposal has a practical feature: it is really very easy to find out data for analysis on the Internet (free books, poems, speeches, song lyrics, etc.).

Figure 18 displays the graphical user interface of the Module 2 (the main module of the software). Data may be entered using the text area at the top of the interface (this area is filled with the United States National Anthem in Fig. 18). The student can type some text directly, or use the "copy and paste" technique (ctrl + c/ctrl + v). If he or she presses the "Process!" button, then the text entered will be processed, and the results of the analysis will be informed in the various tabs of the program. The "Letters" tab, for instance, reports the total number of letters, the total number of vowels (that is, the letters *a*, *e*, *i*, *o* and *u*), the total number of consonants and, in the format of a table, the (absolute and relative) distribution frequency of the letters of the text. The informations in the table can be sorted by clicking repeatedly on the corresponding column header.

As an exercise, we encourage the student to investigate the frequency distribution of letters in texts with different sizes, languages and narratives. The objective is to make him or her realize that this distribution can be used to identify (automatically) the language of the text and, in this case, the sample size is important. This question is taken a step further: since the frequency distribution of the letters is a characteristic of the language, it follows that it is invariant under permutations of the names of the letters. Therefore, this property can be used to decode an encrypted text with the Caesar cipher (whose encoding principle is just the permutation of the names of the letters).

The "Digits" tab reports the total number of digits (0, 1, 2, 3, 4, 5, 6, 7, 8 and 9) in the text and its (absolute and relative) frequency distribution. Although the Module 2 has been designed for text analysis, it is also possible to use it to study frequency distributions of digits in decimal numbers and, thus, investigate questions on the normality of real numbers.

Another example of investigative work that may be carried out by students refers to the surprising Zipf's Law. This empirical power law, proposed by the linguist George Kingsley Zipf (1902–1950) of Harvard University, suggests that in a text with a large number of words, the frequency f of occurrence of a word as a function of its position r in a list sorted by frequency of occurrence has the following form: $f = C/r^a$, where C and a are constants, with the value of a close to 1. Note that, in the variables $y = \log(f)$ and $x = \log(r)$, the Zipf's Law is expressed as an affine function:

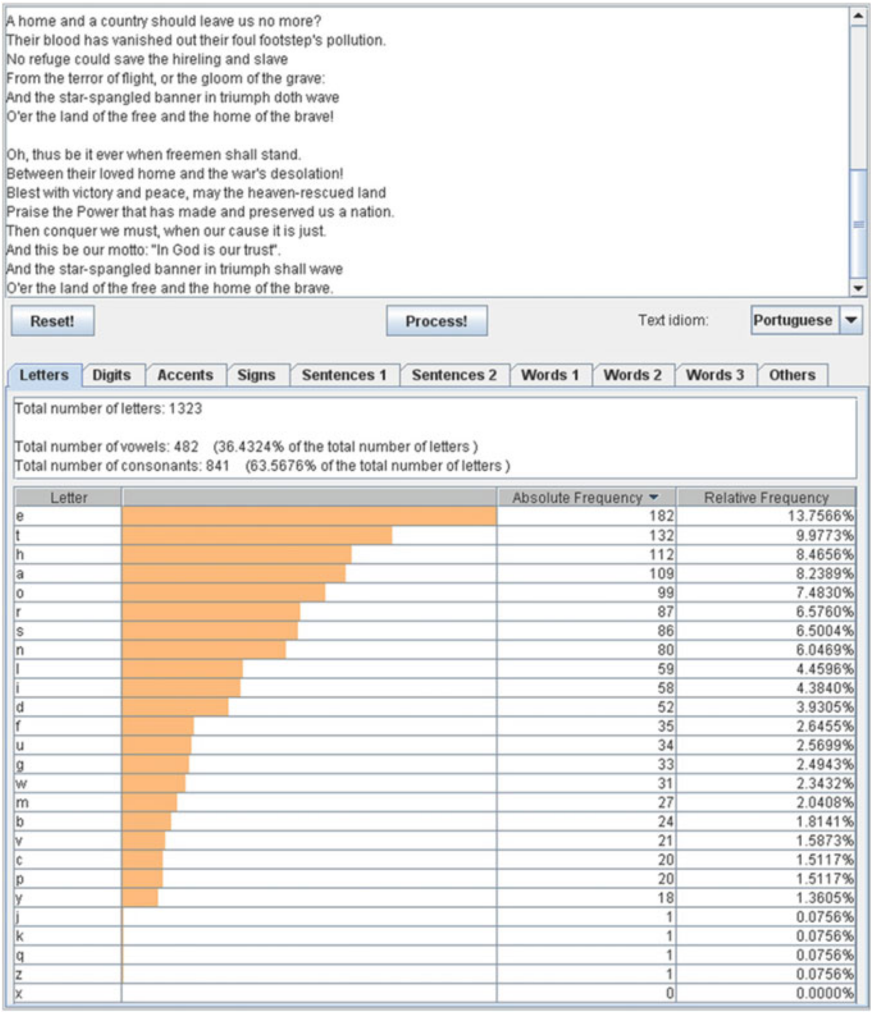

Fig. 18 Statistcs of the United States National Anthem

$y = b + a\ x$, with $b = \log(C)$. Thus, the coefficients a and b can be estimated, for example, using the method of least squares. This whole process is automated in the Module 3 of our software, an adaptation of the Module 2 oriented to the specific study of the Zipf's Law. Figure 19 illustrates the Zipf's Law for the novel "Moby Dick" of Herman Melville ($C = 40536.4574$ and $a = 1.1025$).

The material produced for this activity was the source of a submitted vignette for the Klein Project in Portuguese: (Bortolossi 2011b).

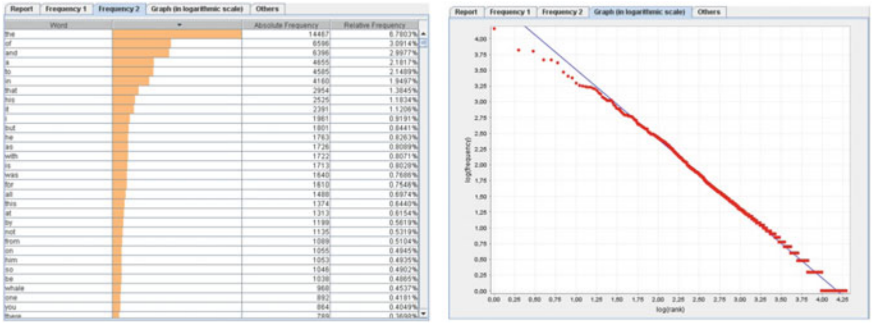

Fig. 19 Zipf's Law for the novel "Moby Dick"

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Doing Research Within the Anthropological Theory of the Didactic: The Case of School Algebra

Marianna Bosch

Abstract Since its emergence in the early 80s with the study of didactic transposition processes, the Anthropological Theory of the Didactic maintains a privileged relationship with school algebra and its diffusion, both in school and outside school. I have chosen this case study to introduce the main "gestures of research" promoted by this framework and the methodological tools used to help researchers detach from the dominant viewpoints of the institutions where teaching and learning processes take place or which affect these processes in the distance. The construction of alternative reference models concerning school algebra and teaching and learning processes leads to some recent teaching experiences that break down the established didactic contracts, raising new research questions that need more in-depth analysis in the way opened by the "procognitive paradigm".

Keywords School algebra · Anthropologic theory of the didactic · Didactic transposition · Arithmetic calculation programme · Algebraisation process

Research, Theory and the "Detachment Principle"

This paper is an introduction, or better an invitation, to the Anthropological Theory of the Didactic (ATD), a research framework where I have been working for more than twenty years now, growing in it as a researcher and having the chance of participating in its development. At the beginning we were a small team of French and Spanish people collaborating with Yves Chevallard in Marseilles, a group that

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has now become a community of about one hundred researchers mainly from Europe, Canada and Latin American countries.[1]

In spite of the word "theory", the ATD is, as the Hans Freudenthal Award recognises it, "a major cumulative programme of research" in mathematics education. As in many other cases, theory is used here as a synecdoche to refer to a whole research activity naming only one of its elements: the organisation of concepts, assumptions, relationships and other notional tools used to problematise reality in order to get more insight and modify it in a given wished direction. Many centuries of diffusion strategies seem to have overvalued theories as the main entrance to knowledge organisations. However, and even if we are used to it, the theory is not always the best way to access a research approach. Another entry is chosen here: the one of the research problems raised and of the main methodological "gestures" used to approach them, including the kind of empirical evidence considered. To be more concrete, we will restrict the entry and focus on a single case study, the problem of school algebra, which has been at the core of the ATD development since its very beginning and can provide a rich illustration of the different treatments this research framework proposes.

Even if the ATD is much more than a theory, it is also true that the role played by its theoretical constructions is essential in a very specific sense, which can be subsumed in a basic principle that permeates all its methods of study. I will call it the "detachment principle", after the work of the sociologist Norbert Elias (1987). Because researchers in didactics deal with a reality that takes place in social institutions, and because they often participate at these institutions (as researchers, teachers, students, or in several positions at the same time), we need to protect ourselves—to emancipate—from the institutional viewpoints on this reality and, more particularly, from the common-sense models used to understand it. This effort of detachment is a basic gesture in sociological and anthropological research (see, for instance, Bourdieu et al. 1968; Berger and Luckmann 1966; Elias 1978; among many others). It is also coherent with the double assumption made by the ATD that persons are the *subject* of the set of institutions they enter during their lives and that what they think or do (their knowledge and know-how) derives in a personalised way from *institutional* knowledge and know-how.

The word "institution" is taken here in a non-bureaucratic sense, as it is used by the anthropologist Mary Douglas in her work *How institutions think* (Douglas 1986). As Y. Chevallard presents it (Chevallard 2005, our translation):

> An institution lives through its actors, that is, the persons that are subjected to it—its subjects—and serve it, consciously or unconsciously. […] *Freedom* of people results from the power conferred by their institutional subjections, together with the capacity of choosing to play such or such subjection against a given institutional yoke.

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[1]An outline of the problems approached and the results obtained by this community can be found in the proceedings of the three International ATD Conferences held since 2005 in Spain and France (Estepa et al. 2007; Bronner et al. 2010; Bosch et al. 2011).

In this context, human practices and human knowledge are entities arising in institutional settings. A person acquires knowledge and practice by entering the institutions where this knowledge and practice exist. At the same time, it is through the changes introduced by their subjects that institutions evolve. This dialectic between the personal and the institutional perspective is at the core of the ATD. It is important to say that the personal subjection to institutions must be understood as a productive subjection instead of as a loss of freedom. We do not act nor know as individuals, but as part of some collective constructions we participate in, assuming their rules and contributing to making them evolve. The idea of being empowered (both cognitively and practically) through the subjection to institutions can be illustrated by the metaphor of the bicycle: when the wheels are free, the bicycle does not move; movement is possible through the subjection of the wheels. The principle of "detachment" has to be understood in this context, since researchers' institutional subjections affect the way of conceiving and understanding reality: one has to get off the bicycle to understand the mechanism of the subjection of the chain.

When trying to adopt an external perspective of the reality we want to study, we often need to question the institutional dominant viewpoints, which initially appear as "transparent" or natural to the subjects of the institution. It is here where theoretical constructions acquire their functionality, by providing alternative conceptions about this reality. Furthermore, except if we adopt a hyper-empiricist perspective, which we will not, the way of delimiting and even defining this reality also depends on the perspective adopted. As will be shown later, the detachment required by the ATD methodology also implies an important enlargement of the empirical unit of analysis considered.

But let me first introduce one of the main theoretical notions of ATD which is also part of the effort of detachment we are considering here. In didactics research, almost all problems deal with teaching and learning processes where "something" is learnt or taught. This something is usually a particular "piece of knowledge" that can be of a different size: the whole of "mathematics", the practice of "mathematical modelling", a whole domain as when we talk about "algebra" or "geometry", a sector of this domain like "first degree equations" or "similar triangles", or even a smaller piece like "the concept of variable" or "transposing and cancelling". The ATD proposes that we talk about praxeologies to refer to any kind of knowledge and, more generally, to any human practice, including mathematical and teaching and learning activities (Chevallard 1999, 2006; see also Barbé et al. 2005). The term "praxeology", made of the Greek words *praxis* and *logos*, enables us to consider two terms that are often opposed within the same entity: the "practical block" or know-how and the "theoretical block" or knowledge (in its narrow sense) made of the discursive elements (*logos*) used to describe and justify the practice. A praxeology is made of four components: types of tasks, techniques, technologies and theories (sometimes called the "four Ts"). The *praxis* or "practical block" contains a set of *types of tasks* to be carried out and a set of *techniques* to do so, technique being considered here in a very general sense of ways of doing. The *logos,* or theoretical block, is made of a double-levelled discourse. A *technology* or "discourse on the technique" to explain what is done, to let others interpret it and to provide a first justification or control of it. The general models, notions and basic assumptions that validate the technological discourse and organise the praxeological elements as a whole, form what we call the theory. Types of tasks, techniques, technologies, and theoretical discourses can be elaborated, made explicit and well-grounded, but they can also be just incipient, as growing entities or, on the contrary, long-established, routine-based and naturalised.

Scientific praxeologies try to make their technologies and theories explicit, so as to control the assumptions made, to formulate the problems and phenomena approached and, as Allan Schoenfeld outlines it in his *Reflections of an Accidental Theorist*, "to elaborate clearly for yourself 'what counts' and how things supposedly fit together" […] as well as to "hold yourself accountable to data" (Schoenfeld 2011, p. 220). The synecdoche I mentioned before about referring to a whole research praxeology by naming only its theoretical component is a classic one when dealing with scholar knowledge. In contrast, praxeologies culturally considered of a lower level are usually designated through an opposite synecdoche, naming only the practical component as if there were no theoretical block associated to it, that is, as if there was nothing to say about the practice or, at least, as if there was not a strong enough institutional theoretical construction around it. The use of the term "praxeology" enables us to escape from these institutional evaluations and consider the different mathematical, teaching and learning praxeologies through the same prism. It is meaningful, for instance, that we can easily talk about educational theories, but tend to refer to teaching practices much more than to teaching theories.

What Is "School Algebra"? Didactic Transposition Processes

The first "detachment gesture" proposed by the anthropological approach has to be found in the initial formulation of the didactic transposition process (Chevallard 1985). It consists in questioning the nature and origin of the mathematical knowledge that is taught at school, looking at the work done by different institutions during different periods of time to select, reorganise, adapt and develop the mathematical praxeologies from their first appearances in the scholar institution (the main site responsible for the production of knowledge) to their designation as "knowledge to be taught" and their implementation at school as taught knowledge. A lot of decisions are made during this transposition process that should be taken into consideration to better understand what conditions (in terms of praxeologies) are made available to teachers and students and what constraints hinder or even impede the development of many others.

The notion of didactic transposition appeared as a powerful theoretical tool to break with the dominant viewpoints with regard to the "disciplinary knowledge" didactics research has to deal with. Before focusing on how children learn and how we can teach them—the viewpoint of the teachers' institution—the attention is first put on what is learnt and taught, its nature (what it is made of), origin (where it comes from) and function (what it is for). In spite of the dominant viewpoint on mathematics brought about by the scholar and the school institution, leading to the impression that there is only *one* school algebra and that the problem is how to teach or learn *it* (as if these decisions were always beyond the epistemological dimension of teaching and learning processes), the ATD starts questioning "what is being taught" and showing its undefined nature. What is this thing called school algebra? What kind of praxeologies is it made of? What could it be made of under other institutional constraints? How does it vary from one school institution to another, both in time (from one historical period to another) and in the institutional space (from one country or educational system to another)? Where does it come from? What legitimates its teaching?

To answer those questions, the kind of empirical evidence necessary may not be reduced to the teaching and learning processes as they are currently taking place in the classroom. It becomes necessary to look into the different institutions (present and also past ones) that influence transposition processes, amongst them the institution responsible for producing mathematical praxeologies, that we will call "scholar mathematics", and the one responsible for selecting it and introducing it at school, called the "noosphere", that is, the sphere of people who think and make decisions about educational processes, such as curriculum developers, policy makers, associations of teachers, educational researchers, etc.

Research about the teaching of elementary algebra in France (Chevallard 1984, 1989a, b, c, 1990; Assude 1993; Grugeon 1995; Coulange 2001a, b; Artigue et al. 2001) and their contrast with the Spanish case (Gascón 1993, 1999, 2011; Bolea 2003; García et al. 2006; Ruiz-Munzón 2010) have all shown a similar evolution of the didactic transposition processes that has led to a dispersion of the content traditionally assigned to "elementary algebra" in secondary school curricula, splitting up the classic triad of arithmetic-algebra-geometry that used to structure school mathematics curricula before the New Mathematics reform. With slight variations depending on the historical periods and regions, we can observe that the existence of algebra as a school mathematical domain (or "block of content") is, at the most, fluctuating. For instance, it disappeared from the French and Spanish official curricula, and has only recently been reintroduced in some Spanish regions. It is not the case of Catalonia, where the present curriculum (2007) proposes five blocks of content: *Numeration and calculation; Relations and change; Space and shape; Measure; Statistics and randomness*. These blocks appear to be very similar to the overarching ideas proposed by the OECD/PISA commission: *Quantity; Space and shape; Change and relationship; Uncertainty* (OECD 2009). In this new organisation of mathematics proposed by the PISA evaluators, the correspondence with what is called the "traditional topics classification" confines algebra to the "Change and relationship" strand (OECD 2009, p. 28), as if there were no need for algebraic techniques in the other domains. It could be interesting to study how transpositive processes are currently influenced by this type of international evaluation, a phenomenon that is certainly affecting the different societies that take part in these processes in a similar way, although the effects appearing in each educational system may be fairly different.

Apart from the loss of visibility of the mathematical organisation of school algebra as a mathematical domain, which may vary significantly from one country to another, what is much more common and has been commented in numerous research projects is the establishment of a *formal* approach to the algebraic tool and the difficulty to move it into a *functional* approximation, in which algebra would appear as a way of modelling other kinds of systems or mathematical realities (Chevallard 1989a). In the traditional teaching preceding Modern Mathematics, the introduction of algebra and the use of letters to name both known and unknown quantities allowed students to systematically solve the corpus of problems of elementary and mercantile arithmetic that represented most of the mathematical work done at primary school. At that time, arithmetic calculations and the structured corpus of arithmetic problems acted as the reference and starting point of the new algebraic construction, which in turn marked the entrance to a higher level of education. For algebra to appear as a generalised arithmetic in this sense, the interplay between parameters and unknowns is essential to cope with the richness of the discursive models that support arithmetical techniques (Chevallard 1989a, b, c; Bosch 1994).

Nowadays, however, the reference to traditional arithmetic and its important corpus of problems that used to give the teaching of algebra its rationale has disappeared. The opposition—which was also a connection—between arithmetic and algebraic problem-solving techniques, which for a long time marked the entrance to algebraic work, no longer makes sense. In contemporary secondary schools, elementary algebra is largely identified with solving equations, mainly of first and second degrees, with some subsequent applications to a set of word problems coming out of nowhere. This limited domain is often preceded by a short introduction to the language of algebra used to introduce the specific terminology required (algebraic expression; evaluation; terms, members and coefficients; similar terms; equations, equalities and identities; etc.), a formal frame where students learn how to develop, factorise and simplify expressions as a goal in itself. Without the possibility to refer to the arithmetic world, algebraic expressions and equalities between expressions cannot be presented by what they designate, but only by their formal structure and their mathematical objects. This formal learning is unable to recreate the large variety of manipulations that are needed to use algebra in a functional way, and which will be required when students arrive at higher secondary education and suddenly find "completely algebraised" mathematics.

The analysis of the didactic transposition processes and the way they may hinder teaching and learning processes should not be taken as simple criticism of the praxeological entities that comprise school algebra. Knowing how algebra is understood in mathematics classrooms, at school and even in our societies, as well as the kind of praxeological elements that are not (but could be) conceived as part of it, is, however, an essential questioning to investigate the conditions of possibility for educational changes not being reduced to mere local innovations. It is important to understand the transpositive constraints that have shaped school algebra, especially when some of the detected traits (that have only been briefly described here) seem robust and stable enough to remain in most current educational systems.

The Didactic Ecology of School Algebra

The analysis of didactic transposition processes points out the existence of different constraints influencing the teaching of algebra at secondary school. Its study concerns the ecology of the praxeologies (Chevallard 2002), that is, the set of conditions necessary for a specific praxeological organisation to exist in a given institutional setting and the constraints hindering its possible evolution. For instance, the possibility (or impossibility) of referring to long-established arithmetic techniques when introducing algebra is a condition that shapes the kind of praxeologies that can be taught and, at the same time, restricts their development to other forms of activity. These conditions and constraints can be of a very specific nature, related to the way different domains, sectors, themes and questions are organised in a given mathematical curriculum. They can also be more generic, not directly related to mathematics and affecting the teaching and learning of any discipline at school, or at any educational institution, or even affecting the dissemination of any kind of knowledge in the society at large. Chevallard (2002, 2007) introduced a hierarchy of "levels of didactic codetermination" to clarify the scope of the considered constraints and also to uphold the view that the study of phenomena arising at very general levels of determination should be taken into account by research in didactics, since they can strongly affect the conditions of possibility and the evolution of teaching and learning processes. The scale consists in the following sequence (Fig. 1).

The case of school algebra leads us to identify important constraints in almost all levels of codetermination, especially phenomena arising at the level of our Western civilisation. It thus provides a good illustration of how the most generic levels can influence mathematical praxeologies at the lowest levels of specification.

The Western Relationship to Orality and Literacy

According to the work of the classical and humanistic scholars, Eric Havelock (1963) and Walter Ong (1982), in traditional Western cultures, oral formulations are regarded as the direct expression of thought, and writings are viewed as the mere

Civilization ↔ Society ↔ School ↔ Pedagogy ↔ Discipline ↔ Domain ↔ Sector ↔ Theme ↔ Question

Fig. 1 Scale of levels of didactic codetermination written transposition of oral discourse. The French philosopher Jacques Derrida (1967) describes this metaphysical position as *logocentrism.* It is assumed that thought is something residing in our head that first comes out through the discourse before being transcribed to writings. Thus, (verbal) reasoning is often opposed to (written) calculations, as illustrated by the current recommendation "First say it with words, then write it down". This assumption permeates our teaching practices and can explicitly be found in several teaching documents about the danger of introducing writing manipulation too early, before the meaning is constructed. See, for instance, the following suggestion about the construction of number sense in early arithmetic by Julia Anghileri (2006, p. 45) quoting the British Department of Education and Employment:

> Current recommendations propose that "oral and mental competence" is established "before written calculation methods are introduced" […]. This does not mean that there will be no written recording but that children will learn to record their thinking with progressive formalization, learning first to use words to record results they can already talk about.

A comment that is preceded by a synthetic indication about how "Progression in learning may be summarized" (*Ibid.*, p. 44):

> DOING … TALKING ABOUT … WRITING ABOUT … SYMBOLIZING

It is important to underline that in the algebraic manipulations, this relationship between oral and written work is reversed: algebraic objects are written by nature, they are not the written transcription of oral objects. Thus orality becomes a secondary accompaniment of written algebraic formulations, which are furthermore not always easy to "oralise" (*ecs squared minus three equals*…). Contrary to our mental habits, written algebraic symbolism is not a derivation of oral language: it is the source, the manifestation and the touchstone of algebraic thinking.

The school ecology of algebra has always been hindered by what we can call a cultural incomprehension of its written nature. In fact, the relationship to symbolism is still an important barrier to the acceptance of scientific work in the realm of highly valued cultural practices. A small sample of this situation can be the number of books in different languages pretending to popularise scientific fields using no or very little symbolism: 'Spaceflight without formulae', 'Special relativity without formulae', 'Quantum mechanics without formulae', 'Statistics without formulae', and even the Russian 'Mathematics without formulae' in two volumes! (Pujnachov and Popov 2008). In the introduction to the book, we can read the following statement that the authors attribute to the famous mathematician Sofia Kolvalésvkaya that reintroduces the common idea that formulae are something secondary in the production of knowledge (our translation):

> In mathematical works, the most important is the content, ideas, concepts, and only afterwards, to express all this, mathematicians have their language: formulae.

The lack of meaning assigned to written formulae by our Western culture has its effects in the school introduction of algebra. As we have shown in our research on the *ostensive* or *semiotic tools* used in mathematics (Bosch and Chevallard 1999), the “rupture” between arithmetic and algebra is also a cultural break from an essentially discursive world, based on oral techniques scanned by simple operations—the “reasoning” realm—to a mostly written world, where techniques are difficult to “oralise” and where a specific descriptive discourse (a *technology* of the written calculations) has to be explicitly constructed. Algebra thus appears as a kit of tools that enables one to do things more quickly to the detriment of the meaning or reasoning, as written mechanics against verbal thought. A quotation of an old French textbook of elementary algebra would give an idea of this dominant viewpoint that has still not completely disappeared (Blanc and Soler 1933, p. 12, our translation):

> If the algebraic solution is quicker than the arithmetic solution, we do not have to forget than it is the latter which mainly contributes to develop reasoning. Thus with problems the solution of which includes reasoning, it is necessary to find both solutions: the arithmetic and the algebraic one.

The Cultural Pejoration of Algebra

The first investigations on school algebra carried out within the framework of the ATD (Chevallard 1985, 1994) immediately highlighted a fact of society closely related to the primarily written nature of algebra and that can be designed as *the cultural pejoration of algebra*. As we showed (Chevallard and Bosch 2012), research carried out at the beginning of the 1980s using a *semantic differentiator* technique displayed what seems to be an almost invariable trait in secondary school students: while to them geometry would be pretty, warm, deep and feminine, algebra turned out to be ugly, cold, superficial and masculine. Again, we can find several pieces of evidence that our society maintains such a relationship with algebra. A quite surprising one comes from a voluntarily provocative comment from the great mathematician Sir Michael Atiyah, clearly expressing this cultural pejoration of algebra (Atiyah 2001, p. 659):

> Algebra is the offer made by the devil to the mathematician. The devil says: ‘I will give you this powerful machine, it will answer any question you like. All you need to do is give me your soul: give up geometry and you will have this marvellous machine. […] the danger to our soul is there, because when you pass over into algebraic calculation, essentially you stop thinking; you stop thinking geometrically, you stop thinking about the meaning. I am a bit hard on the algebraists here, but fundamentally the purpose of algebra always was to produce a formula that one could put into a machine, turn a handle and get the answer. You took something that had a meaning; you converted it into a formula; and you got out the answer.

It is not strange that, in this state of mind, the mathematical domain *par excellence* to introduce students to “proof”, “demonstration” or “deductive reasoning” is usually geometry, and rarely algebra. It will be difficult to accept algebra as a domain of proof when algebraic work seems to consist mainly in calculations, supposedly implying little “reasoning”.

Let me finish this illustration of constraints coming from the generic levels of the scale of didactic codetermination with a final example of a fact that can be located at the society level, even if in this case the society is one that I, as a citizen, am not so familiar with. Some years ago, to introduce the proposal of “algebrafying” into an elementary mathematics experience, James Kaput depicted the result of an evolution of the American didactic transposition process that led to what the author names “Algebra the Institution” (Kaput 1998, p. 25):

> ‘Algebra the Institution’ is a peculiarly American enterprise embodying the standard courses, textbooks, tests, remediation industry, and their associated economic arrangements, as well as the supporting intellectual and social infrastructure of course and workplace prerequisites, cultural expectations relating success in algebra to intellectual ability and academic promise, special interests, relations between levels of schooling, and so on. Exhortation for and legislation of Algebra For All tacitly assume the viability and legitimacy of this Institution. But this algebra is the disease for which it purports to be the cure! It alienates even nominally successful students from genuine mathematical experience, prevents real reform, and acts as an engine of inequity for egregiously many students, especially those who are the least advantaged of our society.

Again algebra is detached from cultural and useful practices. And the process of didactic transposition seems to have imposed specific restrictions that seem as serious as unforeseen.

The scale of levels of didactic codetermination is a productive methodological tool for the detachment principle I mentioned at the beginning of this paper: to be aware of the factors that influence what can or could be done at school related to the teaching and learning of algebra, and avoid taking for granted the current assumptions, evaluations and judgements about the nature of algebra and its functions in knowledge practices. However, as we said before, the best way to free research from all these implicit institutional assumptions that always impregnate teaching and learning processes, is to build an alternative reference model from which to look at the phenomena from another point of view and, of course, with other assumptions that research theory should try to make as explicit as possible. This is especially important when dealing with the specific levels of codetermination, when we are considering what school algebra is made of.

What Could Algebra Be? a Reference Epistemological Model

When analysing any teaching or learning process of mathematical content, questions arise related to the interpretation of the mathematics involved in it. The different institutions interfering in the didactic processes propose more or less explicit answers to said questions. If researchers assume those answers uncritically, they run the risk of not dealing with the empirical facts observed in a sufficiently unbiased way. Therefore the ATD proposes to elaborate what are called *reference epistemological models* for the different mathematical sectors or domains involved in teaching and learning processes (Bosch and Gascón 2005). This explanation of the specific epistemological viewpoint adopted—which is always an a priori assumption constantly evolving and continuously questioned—determines, amongst other things, the amplitude of the mathematical field where research problems are set out; the didactic phenomena which will be "visible" to researchers; and the attempted explanations and actions that are considered "acceptable" in a given field of research.

In the ATD, those reference epistemological models are formulated in terms of local and regional praxeologies and of sequences of linked praxeologies. With respect to school algebra, our proposal is to interpret it as a *process of algebraisation* of already existing mathematical praxeologies, considering it as a tool to carry out a modelling activity that ends up affecting all sectors of mathematics. Therefore, algebra does not appear as one more piece of content of compulsory mathematics, at the same level as the other mathematical praxeologies learnt as school (like arithmetic, statistics or geometry) but as a general modelling tool of *any* school mathematical praxeology, that is, as a tool to model previously mathematised systems (Bolea et al. 2001a, b, 2004; Ruiz-Munzón 2010; Ruiz-Munzón et al. 2007, 2011). In this interpretation, algebra appears as a practical and theoretical tool, enhancing our power to solve problems, but also of questioning, explaining and rearranging already existing bodies of knowledge.

This vision of algebra can provide an answer to the problem of the status and rationale of school algebra in current secondary education. On the one hand, algebra appears as a privileged tool to approach theoretical questions arising in different domains of school mathematics (especially arithmetic and geometry) that cannot be solved within these domains. A well-known example is the work with patterns or sequences where a building principle is given and one needs to make a prediction and then find the rule or general law that characterises it. This feature highlights another differential feature of algebra that is usually referred to as *universal arithmetic*: the possibility of using it to study relationships independently of the nature of the related objects, leading to generalised solutions of a whole type of problems, instead of a single answer to isolated problems, as is the case in arithmetic. Another essential aspect of the rationale of algebra is the need to organise mathematical tasks in types of problems and to introduce the idea of generalisation in the resolution process, a process making full use of letters as parameters.

In this perspective, the introduction of the algebraic tool at school needs to previously have a system to model, that is, a well-known praxeology that could act as a *milieu* (in the sense given to this term in the Theory of Didactic Situations) and that is rich enough to generate, through its modelling, the different entities (algebraic expressions, equations, inequalities, formulae, etc.) essential to the subsequent functioning of the algebraic tool. In the model proposed, this initial system is the set of *calculation programmes* (CP). A CP is a sequence of arithmetic operations applied to an initial set of numbers or quantities that can be effectuated "step by step"—mostly orally and writing the partial results—and provides a final number of quantity as a result. The corpus of problems of classic elementary arithmetic (and also some geometrical ones) can all be solved through the verbal description of a CP and its execution: what was called a "rule" in the old arithmetic books. The starting point of the reference epistemological model for elementary algebra is therefore a compound of elementary arithmetical praxeologies with techniques based on the verbal description of CP and their step by step effectuation.

Working with CP soon presents some technical limitations and also raises theoretical questions about, on the one hand, the reasons for obtaining a given result, justifying and interpreting it and, on the other hand, the possible connections between different kinds of problems and techniques. All these questions lead to an enlargement of the initial system through successive modelling processes giving rise to different stages of the "algebraisation" process that we will briefly summarise hereafter. A more detailed description can be found in Ruiz-Munzón (2010), Ruiz-Munzón et al. (2012).

The first stage of the algebraisation process starts when it is necessary to consider a CP not only as a process but as a whole, representing it in a sufficiently material way—for instance written or graphically—to manipulate it. This does not necessarily mean the use of letters to indicate the different numbers or quantities intervening in a CP (the variables or arguments of a CP). However, it requires making the global structure of the CP explicit and taking into account the hierarchy of arithmetic operations (the "bracket rules"). This new practice generates the need of new techniques to create and simplify algebraic expressions and a new theoretical environment to justify these techniques. It is here where the notions of algebraic expression—as the symbolic model of a CP—and of equivalence between two CP can be defined. Following the classic terminology about equations, we can say that this stage requires the operation of *simplifying* and *transposing* equivalent terms but not the operation of *cancelling*.

The passage to the second stage of algebraisation occurs when the identity between CP needs to be manipulated. In this stage, algebraic techniques include considering equations (of different degrees) as new mathematical objects, as well as the technical transformations needed to solve them. This case includes the resolution of equations with one unknown and one parameter, that is, the case where problems are modelled with CP with two arguments and the solutions are given as a relationship between the arguments involved. In the specific case where one of the numeric arguments takes a concrete value, the problem is reduced to solving a one-variable equation. Nowadays, school algebra mainly remains in this last case (without necessarily having passed through the first one): solving one-variable equations of first and second degree and the word problems that can be modelled with these equations, without achieving the second stage of the algebraisation process.

The third stage of the algebraisation process appears when the number of arguments of the CP is not limited and the distinction between unknowns and parameters is eliminated. The new praxeology obtained contains the work of production, transformation and interpretation of formulae. It is not often present in contemporary secondary schools even if it appears under a weak form in other disciplines (like physics or chemistry). At least in Spain, the use of algebraic techniques to deal with formulae is hardly disseminated outside the study of the general linear and quadratic cases. However, they play an essential role in the transition from elementary algebra to functions and differential calculus, a transition that is nowadays quite weakened in school mathematics. Furthermore, secondary school mathematics does not usually include the systematic manipulation of the global structure of the problems approached, which can be reflected in the fact that letters used in algebraic expressions only play the role of unknowns (in equations) or variables (in functions), while parameters are rarely present. However, it can be argued (Chevallard and Bosch 2012) in which sense the omission of parameters—that is, the use of letters to designate known as well as unknown quantities—can limit the development of efficient modelling algebraic tools and constitutes a clear denaturalisation of the algebraic activity carried out at school.

The effort to explicitly state an epistemological reference model for elementary algebra has different purposes. It can first be used as a descriptive tool to analyse what kind of algebra is taught and learnt in the different educational systems, what elements are left out of the teaching process and what other elements could be integrated under specific conditions (to be established). It is also a productive tool when trying to connect investigations about school algebra carried out from different theoretical perspectives as it helps specify the reference epistemological model of algebra more or less explicitly assumed by each research, and compare the results provided by each one. An example would be looking at the similarities and differences between the structural approach of the research strand on *Early algebra* (Carraher et al. 2000, 2006; Malara 2003; Subramaniam and Banerjee 2004; Warren 2004) or the “algebrafying” paradigm promoted by Kaput (2000) and the first stage of the algebraisation process with its possible implementation in the classroom. Another interesting exploitation consists in considering what aspects of elementary algebra are not taught at school and inquire about the possible reasons of their absence, as well as the nature and origin of these reasons. This kind of study, which in the ATD is called the “possibilistic problem” (Chevallard and Bosch 2012), would help us progress in our knowledge of the conditions needed to modify a given institutional ecology in a given way. As we will see in the next section, the epistemological reference model also provides a way to experiment new teaching processes that are supposed to bring a new insight on this institutional ecology from the response obtained to the changes operated in it. A clinical analysis of the teaching interventions can really reinforce the approach of the possibilistic problem, as it usually highlights restrictions that are normally hidden or silent. Finally, we will just mention a last important use of reference epistemological models in the research cooperative work with teachers or directly in teachers’ training programmes (Sierra et al. 2011).

How to Teach Algebra at School? Study and Research Paths

Given the results obtained by the ecological analysis of school systems related to the teaching and learning of algebra, it could seem that the only possible way to integrate algebra as a modelling tool in compulsory education is to operate effective changes in both the pedagogical and epistemological models prevailing in these institutions. However, the final aim of the ecological analysis cannot be reduced to the description of how things are and why they seem to be as they are, but to enquire into the possible ways of making them evolve. Of course not much can be changed without understanding the constraints or barriers of any kind (material as well as ideological or conceptual) that hinder the set of praxeologies that can be brought into play in the classroom, at school as well as outside school. The phenomenon of *logocentrism* and the written symbolism pejoration, the cultural supremacy of discourse and of geometrical work in front of algebraic calculations, or the disappearing of formulae from school mathematical work are part of these constraints and are affecting any local proposal of modification. As a consequence, it could seem that any attempt to renew the teaching of school algebra requires significant changes going far away from the classroom.

The way chosen by the anthropological approach to face this situation is to carry out clinical analyses of teaching and learning processes (Chevallard 2010), proposing strong local modifications, studying the conditions of possibility of such modifications and exploring the answers or reactions to them. To progress in this way, and in the frame of investigations focused on the new problematic opened by the *paradigm of questioning the world* recently introduced by Yves Chevallard, most of our investigations since 2005 have been centred on the implementation of new teaching proposals based on *research and study paths* (RSP), working in close collaboration with secondary school teachers from the metropolitan area of Barcelona. In the case of school algebra, these RSP have been designed so that the initial questions that are at the starting point of the process would promote the transition through the different stages of the process of algebraisation.

The first type of RSP are built around the well-known “Think-of-a-number” games, which are used as a milieu to informally introduce the students to simple arithmetical calculation programmes. Carrying out these games can soon highlight the limitations of arithmetical techniques (based on step-by-step calculations) and raises new theoretical questions about how to justify the magic of the games, for instance that the result of a given CP is always 75 or that, independently of the initial number taken, the final result of two different CP is always the same, etc. The work carried out during this study generates the need to progress through the first and second stages of the algebraisation process.

In close relation to this RSP, and once students can work at the first level of algebraisation with the writing and simplification of algebraic expressions (without solving equations yet), a second kind of didactic process is introduced, more tightly led by the teacher, with the aim of introducing negative numbers in the context of the algebraic work (Cid and Bolea 2010; Cid and Ruiz Munzón 2011). In this proposal, instead of putting the conceptual construction of negative numbers as vector quantities before their formal manipulation, the chosen option is to propose situations where negative numbers appear as natural needs of the algebraic work (for instance to simplify expressions obtained by a modelling process, such as $(3x + 2) - (x + 8)$ or $(a - 3)(b - 4)$) and afterwards deduce the kind of theoretical construction that can give coherence to the manipulations carried out.

The second type of RSP has been carried out with school students in the transition from lower to upper secondary level. They are based on initial questions of different natures, related to economics and financial issues ("Selling T-shirts", "Saving plans") so that their study and resolution need the transition from the second to the third stage of the algebraisation process and the connection with functional modelling, which is usually absent from Spanish secondary school curricula (Ruiz-Munzón 2010; Ruiz-Munzón et al. 2012).

These investigations have shown different gaps to make the ecology of algebraic teaching practices evolve. We can mention, for instance, the possibility to introduce algebraic techniques of the different stages of the algebraisation process, motivated by the study of questions related to the technical and theoretical limitations of the previous stage. These questions can also be taken from situations where algebra appears as a tool to progress in the modelling of both mathematical and extra-mathematical issues. Furthermore, we have confirmed the possibility for the students to work, from the first stages of algebraisation, with expressions involving several variables, exchanging the role of letters as unknowns and as parameters. However, a lot of constraints have appeared, some of which can be located at the levels of didactic codetermination linked to the curricular organisation of contents (sublevels of the discipline) and to the discipline and pedagogy levels, especially related to the change of the didactic and pedagogical contract that hinder the passage from the paradigm of "visiting the works" to the one of "questioning the world" (Bosch 2010).

We are currently studying the new needs in mathematical and didactic infrastructures required by the implementation of SRP at secondary and tertiary level, and beginning to analyse the possible use of SRP, together with the reference epistemological and pedagogical models that support them, in pre- and in-service teachers' training programmes. This work is part of the latest developments of the ATD which focus its research efforts on the study and development of a new school ecology based on the "questioning the world" paradigm. This opens new and complex problems the scope of which seem to go beyond the research work done in classroom laboratories and even beyond the collaborative research work with pre- and in-service teachers. However, the small progress already made in these contexts seems to open a fruitful line of research. It also shows that the "detachment gestures" I mentioned at the beginning of this paper are completely useless if we are not able to get efficiently involved in the social problems that we should face as mathematics educators.

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Curriculum Reform and Mathematics Learning: Evidence from Two Longitudinal Studies

Jinfa Cai

Abstract Drawing on longitudinal evidence from the LieCal Project, issues related to mathematics curriculum reform and student learning are discussed. The LieCal Project was designed to longitudinally investigate the impact of a reform mathematics curriculum called the Connected Mathematics Project (CMP) in the United States on teachers' teaching and students' learning. Using a three-level conceptualization of curriculum (intended, implemented and attained), a variety of evidence from the LieCal Project is presented to show the impact of mathematics curriculum reform on teachers' teaching and students' learning. The findings from the two longitudinal studies in the LieCal Project serve both to show the kind of impact curriculum has on teachers' teaching and students' learning and to suggest powerful ways researchers can investigate curriculum effect on both teaching and learning.

Keywords Curriculum, Education reform · Mathematics learning · Longitudinal studies · LieCal project, Problem solving, Algebra, Standards

Mathematics Education Reform

Education is commonly seen as the key to a nation's economic growth and prosperity and to its ability to compete in the global economy. Like many other countries, the United States of America has, for years, adopted national strategies for development and reform in education with a focus on improving the quality of individual life and the competitiveness of the nation (National Commission on Excellence in Education

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1983; National Council of Teachers of Mathematics 2000; National Governors Association Center for Best Practices and Council of Chief State School Officers 2010; National Science Board 2010; Cai 2010; Ravitch 2000). Historically, across the nations changing the curriculum has been viewed and used as an effective way to change classroom practice and to influence student learning to meet the needs of an ever-changing world (Cai et al. 2011a, b, 2013; Howson et al. 1981; Senk and Thompson 2003). In fact, curriculum has been called a change agent for educational reform (Ball and Cohen 1996; Darling-Hammond 1993). Thus, the school mathematics curriculum remains a central issue in efforts to improve students' learning.

The curriculum plays a significant role in mathematics education because it effectively determines what students learn, when they learn it, and how well they learn it. In recent years, some reform materials have been accepted into the curriculum and some have been rejected, leading towards more commonly accepted learning goals in school mathematics (Cai and Howson 2013). In addition to developing traditionally accepted mathematical knowledge and skills through mathematics instruction, increasing emphasis has been placed on developing students' higher-order thinking skills. Although there are no commonly accepted definitions of such skills, the frequently cited list found in Resnick (1987) provides a helpful perspective. According to Resnick, higher-order thinking:

- Is *non algorithmic*. That is, the path of action is not fully specified in advance.
- Tends to be *complex*. The total path is not "visible" (mentally speaking) from any single vantage point.
- Often yields *multiple solutions*, each with costs and benefits, rather than unique solutions.
- Involves *nuanced judgment* and interpretation.
- Involves the application of *multiple criteria*, which sometimes conflict with one another.
- Often involves *uncertainty*; not everything that bears on the task at hand is known.
- Involves *self-regulation* of the thinking process.
- Involves *imposing meaning*, finding structure in apparent disorder.
- Is *effortful*; considerable mental work is involved in the kinds of elaborations and judgments required.

This list clearly shows that higher-order thinking skills involve the ability to think flexibly so as to make sound decisions in complex and uncertain problem situations. In addition, such skills involve monitoring one's own thinking—metacognitive skills. In particular, mathematics instruction should ideally provide students with opportunities to: (1) think about things from different points of view, (2) step back to look at things again, and (3) consciously think about what they are doing and why they are doing it. Resnick's list does not include the ability to collaborate with others, but being able to work together with others is also an essential higher-order thinking skill. Collaborative work encourages students to think together about ideas and problems as well as to challenge each other's ideas.

The desirable aim of developing such skills is related to the view that mathematics education should be seen as contributing to the intellectual development of individual students: preparing them to live as informed and functioning citizens in contemporary society, and providing them with the potential to take their places in the fields of commerce, industry, technology, and science (Robitaille and Garden 1989). In addition, mathematics education should seek to teach students about the nature of mathematics. In this view, mathematics is no longer simply a prerequisite subject but rather a fundamental aspect of *literacy* for a citizen in contemporary society (Mathematics Sciences Education Board (MSEB) 1993; NCTM 1989). Education in general and mathematics education in particular have the responsibility for nurturing students' creativity and critical thinking skills not only for their lifelong learning but also for their general benefit and pleasure.

In the United States, NCTM specified five goals for students in its monumental *Standards* document published in 1989: (1) learn to value mathematics, (2) learn to reason mathematically, (3) learn to communicate mathematically, (4) become confident of their mathematical abilities, and (5) become mathematical problem solvers. NCTM also specified major shifts to achieve these goals in teaching mathematics, including movement toward: (1) Classrooms as mathematical communities–away from classrooms as simply collections of individuals; (2) Logic and mathematical evidence as verification–away from the teacher as the sole authority for right answers; (3) Mathematical reasoning–away from merely memorizing procedures; (4) Conjecturing, inventing, and problem solving–away from an emphasis on mechanistic answer-finding; and (5) Connecting mathematics, its ideas, and its applications–away from treating it as a body of isolated concepts and procedures.

The LieCal Project

With extensive support from the National Science Foundation (NSF), a number of school mathematics curricula were developed and implemented to align with the recommendations of the NCTM *Standards*. The Connected Mathematics Project (CMP) is one of the *Standards*-based middle school curricula developed with NSF funding. CMP is a complete middle-school mathematics program designed to build students' understanding of important mathematics through explorations of real-world situations and problems. Students using the CMP curriculum are guided to investigate important mathematical ideas and develop robust ways of thinking as they try to make sense of and resolve problems based on real-world situations.

The research reported here is part of a large project designed to longitudinally compare the effects of a *Standards*-based curriculum (CMP) to the effects of more traditional middle school curricula on students' learning of algebra (hereafter called non-CMP curricula). In this project, *Longitudinal Investigation of the Effect of Curriculum on Algebra Learning* (LieCal), we investigated not only the ways and circumstances under which these curricula could or could not enhance student learning in algebra, but also the characteristics of the curricula that led to student achievement gains (Cai et al. 2011a, b).

In the LieCal Project, we used a quasi-experimental design with statistical controls to examine longitudinally the relationship between students' learning and their curricular experiences. The LieCal Project was first conducted in 14 middle schools in an urban school district serving a diverse student population in the United States. Approximately 85 % of the participants were minority students: 64 % African American, 16 % Hispanic, 4 % Asian, and 1 % Native American. Male and female students were about evenly distributed.

By longitudinally comparing the effects of the CMP curriculum on students' learning of algebra to the effects of more traditional middle-school mathematics curricula, the LieCal Project was designed to provide: (a) a profile of the intended treatment of algebra in the CMP curriculum and a contrasting profile of the intended treatment of algebra in non-CMP curricula; (b) a profile of classroom experiences that CMP students and teachers had, with a contrasting profile of experiences in non-CMP classrooms; and (c) a profile of student performance resulting from the use of the CMP curriculum, with a contrasting profile of student performance resulting from the use of non-CMP curricula. Accordingly, the project was designed to answer three research questions:

- What are the similarities and differences between the intended treatment of algebra in the CMP curriculum and in the non-CMP curricula?
- What are key features of the CMP and non-CMP experience for students and teachers, and how might these features explain performance differences between CMP and non-CMP students?
- What are the similarities and differences in performance between CMP students and a comparable group of non-CMP students on tasks measuring a broad spectrum of mathematical thinking and reasoning skills, with a focus on algebra?

In the LieCal Project, we have subsequently followed the same cohort of middle school students through their four high school years in 10 high schools in the same urban school district. The CMP and non-CMP students were mixed in classes in each of the 10 high schools and thus used the same curriculum and were taught by the same teachers in their classrooms. Our goal in following this cohort through high school is to investigate how the use of different types of middle school mathematics curricula affects their learning of high school mathematics in the same urban school district. More specifically, we examined how students' curricular experiences in the middle grades effect their algebra learning in high school by providing empirical evidence about the relationships between the development of conceptual understanding, symbol manipulation skills, and problem-solving skills in middle school and the learning of mathematics in high school.

Three Levels of Curriculum

In the LieCal Project, we made use of a three-level conceptualization of curriculum (intended, implemented, and attained) which has been widely accepted in mathematics education (Cai 2010). The intended curriculum refers to the formal documents that set system-level expectations for the learning of mathematics. These usually include goals and expectations set for the educational system along with textbooks, official syllabi, and/or curriculum standards. The implemented curriculum refers to school and classroom processes for teaching and learning mathematics as interpreted and implemented by the teachers, according to their experience and beliefs for particular classes. Thus, the implemented curriculum deals with the classroom level. The classroom is central to students' learning since students acquire most of their knowledge and form their attitudes from classroom instruction (Robitaille and Garden 1989). Regardless of how well a curriculum is designed, it has little value outside of its implementation in classrooms. Finally, the attained curriculum refers to what is learned by students and is manifested in their achievements and attitudes. It exists at the level of the student, and deals with the aspects of the intended curriculum that are taught by teachers and actually learned by students.

As shown in Fig. 1 below, conceptualization of the three levels of curriculum is quite useful for comparative studies of mathematics curriculum. It highlights the differences between what a society would like to have taught, what is actually taught, and what students have actually learned. At the same time, all three levels are related to each other, and each one supports the others in the evaluation process. In the following sections, I will specifically discuss the issues and methods of studying mathematics curricula using this conceptualization. I will draw examples from the LieCal Project to discuss the theoretical and methodological issues that arise in each of these three levels.

Intended Curriculum

The intended curriculum specifies goals, topics, sequences, instructional activities, and assessment methods and instruments. The most common method of evaluating

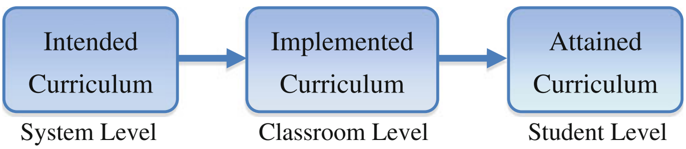

Fig. 1 The conceptualization of the three levels of curriculum

Table 1 Factors to consider in content analysis of mathematics materials (Adapted from NRC (2004), p. 42.)

Listing of topics
Sequence of topics
Clarity, accuracy, and appropriateness of topic presentation
Frequency, duration, pace, depth, and emphasis of topics
Grade level of introduction
Overall structure: integrated, interdisciplinary, or sequential
Types of tasks and activities, purposes, and level of engagement
Use of prior knowledge, attention to (mis)conceptions, and student strategies
Reading level
Focus on conceptual ideas and algorithmic fluency
Emphasis on analytic/symbolic, visual, or numeric approaches
Types and levels of reasoning, communication, and reflection
Type and use of explanation
Form of practice
Approach to formalization
Use of contextual problems and/or elements of quantitative literacy
Use of technology or manipulatives
Ways to respond to individual differences and grouping practices
Formats of materials
Types of assessment and relation to classroom practice

an intended curriculum is content analysis, which involves judging the quality of the content of a curriculum and the quality of its presentation. The National Research Council (2004) has proposed a list of factors to consider when conducting content analysis to evaluate the intended curriculum (see Table 1). When conducting comparative studies of curricula, we may focus on one or more factors, depending on the specific purpose of the study.

In the LieCal Project, we first searched for evidence of the impact of reform by conducting a detailed analysis of the intended curriculum. If a curriculum is to be considered a reform curriculum, it must have conceptualizations and features which distinguish it from traditional curricula. I highlight two sets of findings from the LieCal Project that identify such distinguishing characteristics at the level of the intended curriculum: (1) the introduction of mathematical concepts and (2) the analysis of mathematical problems.

Introduction of Mathematical Concepts. A common approach in curricular comparisons is to examine how a mathematical concept is introduced in various curricula (Cai et al. 2002). In the LieCal Project, we conducted detailed analyses of the introduction of key mathematical concepts in the CMP and non-CMP curricula and found significant differences between them (Cai et al. 2010; Moyer et al. 2012; Nie et al. 2009). Overall, our research revealed that the CMP curriculum takes a *functional approach* to the introduction of algebraic concepts in the teaching of algebra, whereas the non-CMP curricula take a *structural approach*. The functional approach emphasizes the important ideas of change and variation in situations. It also emphasizes the representation of relationships between variables. In contrast, the structural approach avoids contextual problems in order to concentrate on developing the abilities to generalize, work abstractly with symbols, and follow procedures in a systematic way (Cai et al. 2010). In this section, we highlight specific differences in the ways that the CMP and the non-CMP curricula define and introduce variables, equations, equation solving, and functions.

Defining and introducing the concept of variables. Because of the importance of variables in algebra, and in order to appreciate the differences between the CMP and non-CMP curricula, it is necessary to understand how the CMP and non-CMP curricula introduce variable ideas (Nie et al. 2009). The learning goals of the CMP curriculum characterize variables as quantities used to represent relationships. Though the CMP curriculum does not formally define variable until 7th grade, CMP's informal characterization of a variable as a quantity that changes or varies makes it convenient to use variables informally to describe relationships long before formally introducing the concept of variables in 7th grade. The choice to define variables in terms of quantities and relationships reflects the functional approach that the CMP curriculum takes.

In contrast, the learning goals in the non-CMP curricula characterize variables as placeholders or unknowns. The non-CMP curricula formally define a variable in 6th grade as a symbol (or letter) used to represent a number. Variables are treated predominantly as placeholders and are used to represent unknowns in expressions and equations. By introducing the concept of variables in this fashion, the non-CMP curricula support a structural approach to algebra.

Defining and introducing the concept of equations. Given the functional approach to variables in the CMP curriculum and the structural approach in the non-CMP curricula, it is not surprising that the concept of equation is similarly defined functionally in CMP, but structurally in the non-CMP curricula. In CMP, equations are a natural extension of the development of the concept of variable as a changeable quantity used to represent relationships. At first, CMP expresses relationships between variables with graphs and tables of real-world quantities rather than with algebraic equations. Later, when CMP introduces equations, the emphasis is on using them to describe real-world situations. Rather than seeing equations simply as objects to manipulate, students learn that equations often describe relationships between varying quantities (variables) that arise from meaningful, contextualized situations (Bednarz et al. 1996). In the non-CMP curricula, the definition of a variable as a symbol develops naturally into the use of context-free equations with the emphasis on procedures for solving equations. These are all hallmarks of a structural focus. For example, one non-CMP curriculum defines an equation as "…a sentence that contains an equals sign, =" illustrated by examples such as $2 + x = 9$, $4 = k - 6$, and $5 - m = 4$. Students are told that the way to solve an equation is to replace the variable with a value that results in a true sentence.

Defining and introducing equation solving. The CMP and non-CMP curricula use functional and structural approaches, respectively, to introduce equation solving, consistent with their approaches to defining equations. In the CMP curriculum, equation solving is introduced within the context of discussing linear relationships. The initial treatment of equation solving does not involve symbolic manipulation, as found in most traditional curricula. Instead, the CMP curriculum introduces students to linear equation solving by using a graph to make visual sense of what it means to find a solution. Its premise is that a linear equation in one variable is, in essence, a specific instance of a corresponding linear relationship in two variables. It relies heavily on the context in which the equation itself is situated and on the use of a graphing calculator.

After CMP introduces equation solving graphically, the symbolic method of solving linear equations is finally broached. It is introduced within a single contextualized example, where each of the steps in the equation-solving process is accompanied by a narrative that demonstrates the connection between what is happening in the procedure and in the real-life situation. In this way, CMP justifies the equation-solving manipulations through contextual sense-making of the symbolic method. That is, CMP uses real-life contexts to help students understand the meaning of each step of the symbolic method of equation solving, including why inverse operations are used. As with the introduction of variables and equations, CMP's functional approach to equation solving maintains a focus on contextualized relationships among quantities.

In the non-CMP curricula, contextual sense-making is not used to justify the equation-solving steps as it is in the CMP curriculum. Rather, the non-CMP curricula first introduce equation solving as the process of finding a number to make an equation a true statement. Specifically, *solving* an equation is described as replacing a variable with a value (called the *solution*) that makes the sentence true. Equation solving is introduced in the non-CMP curricula symbolically by using the additive property of equality (equality is maintained if the same quantity is added to or subtracted from both sides of an equation) and the multiplicative property of equality (equality is maintained if the same non-zero quantity is multiplied by or divided into both sides of an equation). This approach to equation solving is aligned with the non-CMP curricula's structural focus on working abstractly with symbols and procedures.

Defining and introducing functions. Consistent with their approaches to variables and equations, the CMP and non-CMP curricula once again use functional and structural approaches, respectively, to introduce the concept of functions. Their respective approaches can be seen quite clearly in the differences between their stated learning goals for the concept. CMP's learning goals for students are (1) that they be able to understand and predict patterns of change in variables, and (2) that they be able to represent relationships between real-world quantities using word descriptions, tables, graphs, and equations. In contrast, the stated learning goals from a non-CMP curriculum are (1) that students explore the use of algebraic equations to represent functions, and (2) that they be able to identify and graph functions, calculate slope, and distinguish linear from nonlinear functions.

The CMP curriculum informally introduces the concepts of function and variable at the same time in 6th grade, identifying a function as a relationship between real-world quantities (variables). At the beginning of 7th grade, when the concept of variable is formally introduced, coordinate graphs are used as a way to tell a story of how changes in one variable are related to changes in another. In an introductory investigation, students graph how many jumping jacks they can do in successive 10-second intervals for 2 min. Then they analyze the graph to determine whether a relationship exists between time and the number of jumping jacks. At the same time, students are exposed to the concepts of independent variable and dependent variable. This occurs well before the concept of function is formally introduced during the second half of 7th grade. Although the concept of function is introduced in this unit, the term "relationship" is almost always used instead of the word "function." Furthermore, in the teacher's guide, the term "function" is explicitly identified as nonessential. In fact, the term "function" is not given any importance in the CMP curriculum until the introduction of quadratic functions in the 8th grade.

In a representative non-CMP curriculum, the concept of function is informally introduced in the preview to Lesson 9–6 in 6th grade by having students make a function machine out of paper. The function machine has three key elements: input, output, and operation. The operation, or rule, lies at the core of the function machine, while input and output are external to it. Immediately after the introduction of the function machine, the non-CMP curriculum formally introduces the concepts of function, function table, and function rule in Lesson 9–6. This formal introduction begins with the following situation: "A brown bat can eat 600 mosquitoes an hour." The student is then asked to write expressions to represent the number of mosquitoes a brown bat can eat in 2 h, 5 h, and t hours. Finally, the terms function and function table are illustrated, and the term function rule is defined. The function rule is characterized as a rule giving the operation(s) that will transform an input into an output. The non-CMP curriculum defines a function as a relationship where one thing depends on another. However, it treats a function as a process of starting with an input number, performing one or more operations on it, and getting an output number. The main purpose of the function machine and the function table seems to be for students to experience the process of computing the output values from given input values and vice versa. That is, the development of the concept of function in the non-CMP curriculum emphasizes operations on input variables rather than the relationship between two variables.

Analysis of Mathematical Problems. Comparative studies of intended curricula must also take into account the quality of activities, their use in instruction, and their frequency of use. Indeed, a number of researchers have analyzed problems and worked examples in mathematics curricula (e.g., Cai et al. 2002, 2010; Fan and Zhu 2007). In the LieCal Project, we compared both the types and the level of cognitive demand of mathematical problems involving linear equations in the CMP curriculum and a representative non-CMP curriculum.

Types of Problems Involving Linear Equations. In both the CMP and non-CMP curricula, the vast majority of the equation problems involved linear equations.

Table 2 Percentage distribution of problems involving linear equations in the CMP and non-CMP curricula

	1Equation 1va	1Equation 2va	2Equation 2va
CMP (n = 402)	5.72	93.03	1.24
Non-CMP (n = 2339)	86.19	11.67	2.14

Thus we further classified problems involving linear equations in the CMP and non-CMP curricula into three categories:

- One equation with one variable (1 eq 1va)–e.g., 2x + 3 = 5;
- One equation with two variables (1 eq 2va)–e.g., y = 6x + 7;
- Two equations with two variables (2 eq 2va)–e.g., the system of equations y = 2x + 1 and y = 8x + 9.

Table 2 shows the percentage distribution of these categories of problems involving linear equations in each of the two curricula. The two distributions are significantly different (χ^2 (2) = 1262.0, $p < 0.0001$). The CMP curriculum includes a significantly greater percentage of one equation with two variables problems than the non-CMP curriculum (z = 35.49, p < 0.0001). However, the non-CMP curriculum includes a significantly greater percentage of one equation with one variable problems than the CMP curriculum (z = 34.15, p < 0.0001). These results resonate with the findings reported above. Namely, the CMP curriculum emphasizes an understanding of the relationships between the variables of equations, rather than an acquisition of the skills needed to solve them. In fact, of the 402 equation-related problems in the CMP curriculum, only 33 of them (about 8 % of the linear equation-solving problems) involve decontexualized symbolic manipulations of equations. However, the non-CMP curriculum includes 1550 problems involving decontexualized symbolic manipulations of equations (nearly 70 % of the linear equation solving problems in the non-CMP curriculum).

The non-CMP curriculum not only incorporates many more linear equation-solving problems into the curriculum, but it also carefully sequences them based on the number of steps required to solve them. Of the 2339 problems involving linear equations, over 50 % are one-step problems like, $x + b = c$, $ax = c$ or $x = a * b$. About 30 % of the problems are two-step problems, like $ax + b = c$ or $x/a = b/c$. Only a small fraction of the linear equations involve three steps or more, like $ax + bx + c = d$ or $ax + b = cx + d$. Each grade level of the non-CMP curriculum includes one-step, two-step, and three-plus-step problems involving linear equations. As the grade level increases, however, the curriculum provides increasingly more comprehensive procedures, suitable for solving all forms of linear equations.

Cognitive Demand of Mathematical Problems. If an intended curriculum claims to be problem-based, we should expect to see it contain a large proportion of cognitively demanding tasks. In the LieCal Project, we analyzed the cognitive demand of mathematical problems in both the CMP curriculum and a representative non-CMP curriculum (Cai et al. 2010). We classified the problems into four increasingly demanding categories of cognition: memorization, procedures without

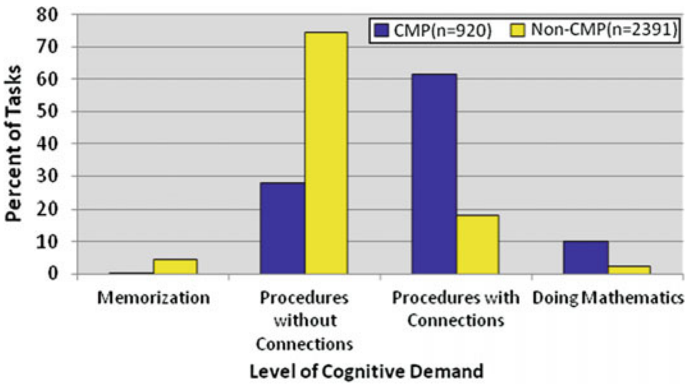

Fig. 2 Percentages of various types of tasks in CMP and non-CMP curricula

connections, procedures with connections, and doing mathematics (Stein and Lane 1996). As Fig. 2 illustrates, proportionally the CMP curriculum had significantly more high-level tasks (procedures with connections or doing mathematics) ($\chi^2(3, N = 3311) = 759.52$, $p < 0.0001$) than the non-CMP curricula. This kind of analysis of the intended curriculum provides insight into the degree to which different curricula expect students to engage in higher-level thinking and problem solving.

Implemented Curriculum

The implemented curriculum is concerned with *what* mathematics is actually taught in the classroom and *how* that mathematics is taught. Therefore, a key issue for the implemented curriculum is the recognition that what teachers teach may or may not be consistent with the intended curriculum. When the implemented curriculum, as seen in teachers' instruction, is congruent with the goals of the intended curriculum, we may say that there is fidelity of implementation. Teachers may vary widely in their commitment to the intended curriculum. Therefore when evaluating the implemented curriculum, it is important to determine whether, how, and to what extent teachers' instruction is influenced by the intended curriculum.

In the LieCal Project, we collected data on multiple aspects of implementation. We conducted 620 detailed lesson observations of CMP and non-CMP lessons over a three-year period. Approximately half of the observations were of teachers using the CMP curriculum, while the other half were observations of teachers using non-CMP curricula. Two retired mathematics teachers conducted and coded all the observations. The observers received extensive training that included frequent checks for reliability and validity throughout the 3 years (Moyer et al. 2011).

Each class of LieCal students was observed four times, during two consecutive lessons in the fall and two in the spring. The observers recorded extensive information about each lesson using a 28-page project-developed observation instrument. During each observation, the observer made a minute-by-minute record of the lesson on a specially designed form. This record was used later to code the lesson. The coding system had three main components: (1) the structure of the lesson and use of materials, (2) the nature of the instruction, and (3) the analysis of the mathematical tasks used in the lesson.

The analyses of the data we obtained from the classroom observations revealed striking differences between classroom instruction using the CMP and non-CMP curricula. In this paper, we briefly discuss the differences related to three important instructional variables: (1) the level of conceptual and procedural emphases in the lessons, (2) the cognitive level of the instructional tasks implemented, and (3) the cognitive level of the homework problems.

Conceptual and Procedural Emphases. The second component of the coding section included twenty-one 5-point Likert scale questions that the observers used to rate the nature of instruction in a lesson. Of the 21 questions, four were designed to assess the extent to which a teacher's lesson had a conceptual emphasis. Another four questions were designed to determine the extent to which the lesson had a procedural emphasis. Factor analysis of the LieCal observation data confirmed that the four procedural-emphasis questions loaded on a single factor, as did the four conceptual-emphasis questions.

There was a significant difference across grade levels between the levels of conceptual emphasis in CMP and non-CMP instruction ($F = 53.43$, $p < 0.001$). The overall (grades 6–8) mean of the summated ratings of conceptual emphasis in CMP classrooms was 13.41, whereas the overall mean of the summated ratings of conceptual emphasis in non-CMP classrooms was 10.06. The summated ratings of conceptual emphasis were obtained by adding the ratings on the four items of the conceptual-emphasis factor in the classroom observation instrument, which implies that the mean rating on the conceptual-emphasis items was 3.35 (13.41/4) for CMP instruction and 2.52 (10.06/4) for non-CMP instruction. That is, CMP instruction was rated 0.40 points above the midpoint, whereas non-CMP instruction was rated 0.5 points below the midpoint. Thus, on average, CMP instruction was rated about 4/5 of a point higher (out of 5) on each conceptual emphasis item than non-CMP instruction, which was a significant difference ($t = 11.44$, $p < 0.001$).

In contrast, non-CMP lessons had significantly more emphasis on the procedural aspects of learning than the CMP lessons. The procedural-emphasis ratings for the non-CMP lessons were significantly higher than the procedural-emphasis ratings for the CMP lessons ($F = 37.77$, $p < 0.001$). Also, the overall (grades 6–8) mean of summated ratings of procedural emphasis in non-CMP classrooms (14.49) was significantly greater than the overall mean of the summated ratings of procedural emphasis in CMP classrooms, which was 11.61 ($t = -9.43$, $p < 0.001$). The summated ratings of procedural emphasis were obtained by adding the ratings on the four items of the procedural-emphasis factor, which implies that the mean rating on the procedural emphasis items was 3.62 (14.49/4) for non-CMP instruction and 2.91 (11.61/4) for non-CMP instruction. On average, non-CMP instruction was

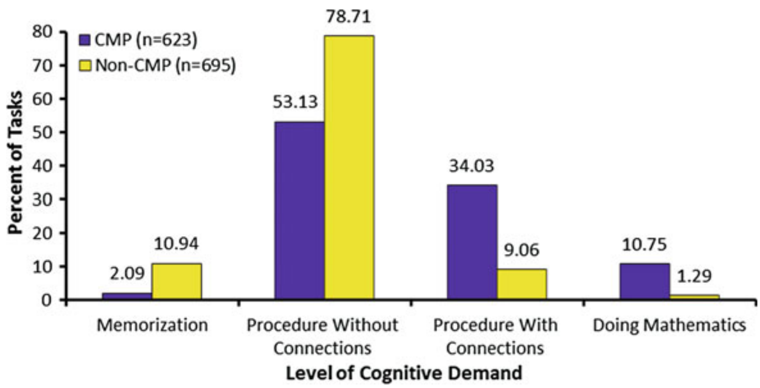

Fig. 3 The percentage distributions of the cognitive demand of the instructional tasks implemented in CMP and non-CMP classrooms

rated about 7/10 of a point higher (out of 5) on each procedural emphasis item than CMP instruction, which was a significant difference.

Instructional tasks. As we did with the mathematical problems in the intended curricula, we again used the scheme developed by Stein et al. (1996) to classify the instructional tasks actually implemented in the CMP and non-CMP classrooms into four increasingly demanding categories of cognition: memorization, procedures without connections, procedures with connections, and doing mathematics. Figure 3 shows the percentage distributions of the cognitive demand of the instructional tasks implemented in CMP and non-CMP classrooms (note that Fig. 2 referred to problems from the intended, not the implemented curricula).

The percentage distributions in CMP and non-CMP classrooms were significantly different ($X^2(3, N = 1318) = 219.45$, $p < 0.0001$). The difference confirms that a larger percentage of high cognitive demand tasks (procedures with connection or doing mathematics) were implemented in CMP classrooms than were implemented in non-CMP classrooms ($z = 14.12$, $p < 0.001$). On the other hand, a larger percentage of low cognitive demand tasks (procedures without connection or memorization) were implemented in non-CMP classrooms than were implemented in CMP classrooms. In addition, not only did CMP teachers implement a significantly higher percentage of cognitively demanding tasks than non-CMP teachers across the three grades, but also within each grade (z values range from 6.06 to 11.28 across the three grade levels, $p < 0.001$). Over 45 % of the CMP lessons implemented at least one high level task (involving either procedures with connections or doing mathematics), but only 10 % of the non-CMP lessons did so ($z = 14.12$, $p < 0.0001$). Nearly 90 % of the non-CMP lessons implemented low-level tasks involving procedures without connections, whereas only 55 % of the CMP lessons did so ($z = 14.12$, $p < 0.0001$).

Homework Problems. Each of the participating CMP and non-CMP teachers was asked to keep logs and submit all of their assigned homework problems as part of the logs. Each homework problem was coded in terms of its source, contexts, representations, and cognitive demand.

We randomly sampled half of the homework problems in each grade. A total of 10,310 of the homework problems assigned by middle school teachers during the 3 years were included in the analysis. Most of the homework problems (about 90 %) came from the respective textbooks for each curriculum; only a small proportion of the assigned homework problems (about 10 %) was supplemented by teachers. Overall, the profile of representations used in CMP homework problems was significantly different from the profile of representations used in non-CMP homework problems ($\chi^2(1, N = 10{,}310) = 34.95$, $p < 0.0001$). Of note, a larger percentage of non-CMP homework problems (39 %) than CMP homework problems (20 %) involved symbolic representations ($z = 19.90$, $p < 0.0001$). In contrast, a larger percentage of CMP problems (45 %) than non-CMP problems (22 %) involved a table, picture or graph ($z = 24.49$, $p < 0.0001$). However, nearly all homework problems, CMP or non-CMP, involved written words (97.7 % of the non-CMP problems and 99.8 % of the CMP problems).

We examined the contexts of the homework problems using the following categories: no context, context without tables or pictures, context with tables and pictures, and context with manipulatives. Overall, the distributions of homework problem contexts for CMP and non-CMP students were significantly different ($\chi^2(3, N = 10{,}310) = 431.43$, $p < 0.0001$). Non-CMP teachers assigned a larger percentage of homework problems without contexts than CMP teachers (56 and 37 %, respectively) ($z = 18.30$, $p < 0.0001$). CMP students were assigned a larger percentage of homework problems involving contexts with tables or pictures than non-CMP students (39 and 22 %, respectively) ($z = 18.92$, $p < 0.0001$). In both the CMP and non-CMP groups, about one quarter of the homework problems involved contexts without tables or pictures. There were very few homework problems in either group with contexts involving manipulatives.

Our analysis of the cognitive demand of the homework problems produced similar results to the instructional tasks. The levels of cognitive demand in the CMP and non-CMP homework problems were significantly different ($\chi^2(3, N = 10{,}310) = 793.08$, $p < 0.0001$). A larger percentage of CMP homework problems (29 %) than non-CMP homework problems (9 %) were high cognitive demand problems (procedures with connections or doing mathematics) ($z = 26.08$, $p < 0.0001$). However, a larger percentage of non-CMP homework problems (91 %) than CMP homework problems (71 %) were low cognitive demand problems (memorization or procedures without connections) ($z = 26.08$, $p < 0.0001$).

Attained Curriculum

The ultimate goal of educational research, curriculum development, and instructional improvement is to enhance student learning. Thus the evaluation of a mathematics curriculum at the student level—evaluation of the attained curriculum

—is of critical importance. In studies of the attained curriculum, we must address multiple facets of mathematical thinking (Cai 1995; Sternberg and Ben-Zeev 1996). Therefore, mixed methods such as observing students doing mathematics, performing tasks, and taking tests, should be used to collect information to evaluate the attained curriculum. Special attention must be paid to the selection of assessment tasks and methods of analysis when conducting comparative studies of attained curricula.

Assessment Tasks. Even though various methods can be used to measure students' learning, the heart of measuring mathematical performance is the set of tasks on which students' learning is to be evaluated (National Research Council 2001). It is desirable to use various types of assessment tasks, thereby measuring different facets of mathematical thinking. For example, different formats of assessment tasks (such as multiple-choice and open-ended tasks) may be used to measure students' learning. Multiple-choice tasks have many advantages. For example, more items can be administered within a given time period, and scoring can be done quickly and reliably. However, it can be difficult to infer students' cognitive processes based on their responses to such items. To that end, open-ended tasks may be used to supplement multiple-choice tasks. In open-ended tasks, students are asked to produce answers, but also to show their solution processes and provide justifications for their answers. In this way, open-ended tasks provide a better window into the thinking and reasoning processes involved in students' mathematics learning. Of course, a disadvantage of open-ended tasks is that only a small number of these tasks can be administered within a given period of time. Also, grading students' responses is labor-intensive. To help overcome the disadvantages of using open-ended tasks, we recommend using a matrix design with samples of students' responses to the administered open-ended tasks. This reduces both testing time and grading time while maintaining a good overall estimate of students' learning of mathematics.

In the LieCal Project, we used both multiple-choice tasks and open-ended problems to assess student learning. On the open-ended tasks, which assessed conceptual understanding and problem solving, the growth rate for CMP students over the 3 years was significantly greater than that for non-CMP students (Cai et al. 2011a, b). Figure 4 shows the mean scores for CMP and non-CMP students on the open-ended tasks. In particular, our analysis using Growth Curve Modeling showed that over the three middle school years the CMP students' scores on the open-ended tasks increased significantly more than the non-CMP students' scores ($t = 2.79$, $p < 0.01$). CMP students had an average annual gain of 25.09 scale points whereas non-CMP students had an average annual gain of 19.39. An additional analysis using Growth Curve Modeling showed that the CMP students' growth rate remained significantly higher than non-CMP students on open-ended tasks even when students' ethnicity was controlled ($t = 3.61$, $p < 0.01$). Moreover, CMP and non-CMP students showed similar growth over the three middle school years on the multiple-choice tasks assessing computation and equation solving skills.

These findings suggest that, regardless of ethnicity, the use of the CMP curriculum was associated with a significantly greater gain in conceptual understanding

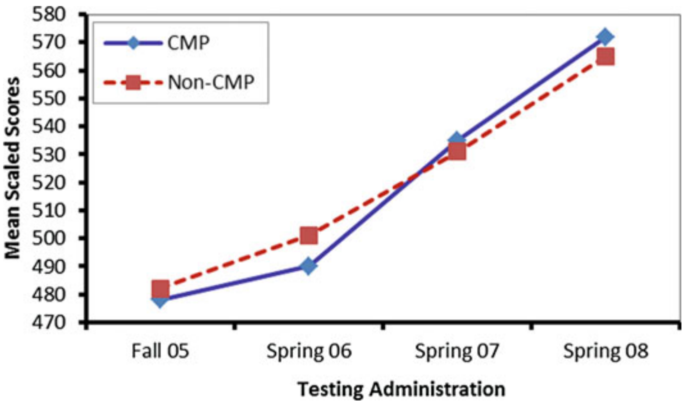

Fig. 4 Mean for CMP and non-CMP middle school students on the open-ended tasks

and problem solving than was associated with the use of the non-CMP curricula. However, those relatively greater conceptual gains did not come at the cost of basic skills, as evidenced by the comparable results attained by CMP and non-CMP students on the computation and equation solving tasks. Thus, by using both multiple-choice and open-ended assessment tasks in the LieCal Project, we were able to obtain a more comprehensive comparison of the attained CMP and non-CMP curricula.

Performance Beyond Middle School. In the 2008–2009 academic year, the CMP and non-CMP LieCal middle school students entered high school as 9th graders. We followed those CMP and non-CMP students who were enrolled in 10 high schools in the same urban school district. In these high schools, the CMP and non-CMP students were mixed together in the same mathematics classrooms and used the same curriculum.

The findings from the LieCal Project (Cai et al. 2011a, b) exhibit parallels to the findings from research on the effectiveness of Problem-Based Learning (PBL) on the performance of medical students (Barrows 2000; Hmelo-Silver 2004; Norman and Schmidt 1992; Vernon and Blake 1993). Researchers found that medical students trained using a PBL approach performed better than non-PBL students (trained, for example, using a lecture approach) on clinical components in which conceptual understanding and problem-solving ability were assessed. However, PBL and non-PBL students performed similarly on measures of factual knowledge. When these same medical students were assessed again at a later time, the PBL students not only performed better than the non-PBL students on clinical components, but also on measures of factual knowledge (Norman and Schmidt 1992; Vernon and Blake 1993). This result may imply that the conceptual understanding and problem-solving abilities learned in the context of PBL facilitated the retention and acquisition of factual knowledge over longer time intervals. As we described above, the CMP curriculum can be characterized as a problem-based curriculum. Analogous to the results of research on PBL in medical education, in the LieCal

Project, CMP students outperformed non-CMP students on measures of conceptual understanding and problem solving during middle school. In addition, CMP and non-CMP students performed similarly on measures of computation and equation solving. Continuing the analogy, it is reasonable to hypothesize that the superior conceptual understanding and problem-solving abilities gained by CMP students in middle school could result in better performance on a delayed assessment of manipulation skills such as equation solving, in addition to better performance on tasks assessing conceptual understanding and problem solving in high school.

We used various student learning outcome measures to examine the impact of middle school curriculum on students' learning in high school. For example, we developed open-ended problem solving and problem posing tasks to assess student conceptual understanding and problem solving. We developed multiple-choice tasks to assess students' basic skills in algebra. We also collected state assessment data, mathematics grades, enrollments in advanced mathematics courses, and SAT/ACT registrations and scores to assess student learning. In general, on all of the student learning outcome measures, CMP students performed better than or as well as non-CMP students in high school (Cai et al. 2013). Here, we present evidence from three outcome measures .

Ninth grade achievement. In the school district, Classroom Assessments Based on Standards (CABS) were administered to the 9th graders every 6 weeks. Each CABS task typically consists of a single open-ended mathematics problem that students are asked to solve and explain. In the 2008–2009 school year, the LieCal Researchers provided the school district with field-tested CABS open-ended problems that aligned with the adopted high school curriculum. Every 6 weeks, the participating teachers administered one of the LieCal-provided CABS assessments to the 9th grade students in the 10 LieCal high schools. An analysis of covariance (with middle school achievement as the covariate) showed that the 9th graders who used CMP in middle school performed significantly better than those 9th graders who used non-CMP curricula in middle school on four of the nine tasks ($F = 4.69$, $p < 0.05$) or performed equally well as those 9th graders who used non-CMP in middle school on the remaining five tasks.

Tenth Grade State Math Test. All 10th grade students in the school district are required to take a state standardized test. This test is specifically designed for the state. The purpose of the state test is to provide information about student attainment of mathematical proficiency to students, parents, and teachers; information to support curriculum and instructional planning; and a measure of accountability for schools and districts. Because of the importance of this state test, we examined the data to see how CMP and non-CMP students performed.

As mentioned above, in the LieCal Project we used both open-ended tasks to measure student conceptual understanding and problem solving and multiple-choice tasks to measure students' basic mathematical skills. We conducted analyses of covariance using the students' 6th grade baseline data on both the open-ended tasks and the multiple-choice tasks as covariates with the 10th grade state math test scaled score as the dependent variable. As shown in Table 3 below, CMP students had significantly higher 10th grade scaled scores than the non-CMP students

Table 3 Analysis of co-variance on 10th grade state math scaled score

Covariate	F-value	Significant level
PI-Developed 6th Grade Multiple-Choice (MC) Tasks	5.13	<0.05
PI-Developed 6th Grade Open-ended (OE) Tasks	3.90	<0.05
Both PI Developed 6th Grade MC and OE tasks	7.76	<0.01
6th grade State math scaled score	9.58	<0.01
7th grade State math scaled score	9.57	<0.01
8th grade State math scaled score	11.79	<0.001

(F(1, n = 492) = 7.76, p < 01). In particular, the adjusted mean for CMP students on the 10th grade state math test was 533.5, but only 525.9 for non-CMP students. When we used the students' 6th grade baseline data on open-ended tasks and multiple-choice tasks separately in the ANCOVA, the findings were similar. That is, CMP students had significantly higher 10th grade scaled scores than the non-CMP students when using open-ended tasks as the covariate (F(1, n = 500) = 3.90, p < 05) and using multiple-choice tasks (F(1, n = 502) = 5.13, p < 05) as the covariate.

We conducted further analyses of covariance using the 6th grade state math test scaled score, the 7th grade state math test scaled score, and the 8th grade state math test scaled score as the covariate. For each of these analyses, we again found that the CMP students had significantly higher 10th grade scaled scores than the non-CMP students, as shown in Table 3. For example, the adjusted mean for the CMP students on the 10th grade state math test is 531, but 523 for non-CMP students using the 8th grade state math scaled score as covariate.

Eleventh Grade Problem Posing. In the 11th grade, we administered a number of open-ended tasks to assess the impact of middle school curriculum on students' high school learning. Two of the tasks were problem-posing tasks (Cai et al. 2013). Our prior research has shown that problem posing can be a feasible, viable, and valid measure of the effect of middle-school curriculum on students' learning in high school. A total of 390 11th graders were included in this study (243 former CMP and 147 former non-CMP students). In order to compare the high school performance of those students who had used the CMP curriculum in middle school to that of students who had used more traditional curricula, we divided their scores from components of the baseline examination (such as equation solving) taken in the 6th grade into thirds and compared students whose baseline scores fell in the same third. We found that the CMP students performed as well or better on the problem posing tasks than the non-CMP students in the same third. For example, when grouped into thirds using the baseline equation-solving scores, the CMP students in the top third were more likely than the non-CMP students in the top third ($z = 2.01$, $p < 0.05$) to generate a problem situation that matched at least one of the given graph conditions (slope and intercept). Similarly, the CMP students in the top third were more likely than the non-CMP students in the top third to generate a problem situation that reflected the linearity of the given graph ($z = 2.40$, $p < 0.05$).

Conclusion

Curriculum reform is often seen as holding great promise for the improvement of mathematics teaching and learning. However, the realization of that promise requires careful attention to the different levels on which curriculum exists and functions. In the LieCal Project, we have analyzed the nature and impact of the intended, implemented, and attained levels of the Connected Mathematics Project curriculum as compared to more traditional middle-school mathematics curricula. Our goals have been correspondingly threefold: to characterize the intended treatment of algebra in the CMP curriculum and identify how it is different from the intended treatment of algebra in non-CMP curricula; to understand how the intentions of the CMP and non-CMP curricula are implemented and embodied in the classroom experiences of students and teachers; and to understand how these distinct experiences may translate into different levels of student attainment.

With respect to the intended curriculum, CMP paints a distinctly different picture from traditional curricula of what middle school mathematics, and particularly the learning of algebraic concepts, should be. The stated goals and their embodiment in texts and mathematical problems indicate that the CMP curriculum intends for students to take a functional approach to algebra, focusing on understanding relationships between quantities in contextualized, real-life problems. This stands in contrast to a more traditional, structural approach to algebra that puts the focus on decontextualized operations and procedures with symbols and mathematical objects. These two approaches to the learning of algebra are evident in the ways that the two types of curricula introduce concepts such as variables and equations and in the kinds of problems the curricula provide.

The implementations we have observed of the CMP and non-CMP curricula strongly reflect the intentions embedded in the curriculum materials. As teachers take each curriculum and shape it into actual instruction in their classrooms, the underlying functional and structural approaches continue to be evident in the choices that teachers make in balancing the conceptual and procedural aspects of the mathematics. In addition, the types of instructional tasks that teachers choose to use and the homework problems they assign to their students further illustrate that the implementation of the CMP curriculum looks very different from the implementation of the non-CMP curricula.

Finally, when we look to the results of the implementation of the CMP and non-CMP curricula in terms of student attainment, we see that CMP students experience greater growth in their conceptual understanding and problem-solving abilities than their non-CMP counterparts without having to sacrifice procedural skills. It would appear that the intentions that guided the development of the CMP curriculum materials, combined with classroom implementations that reflect those intentions, are associated with student learning along the intended lines. In other words, students using the CMP curriculum experience instruction that emphasizes a conceptual understanding of algebra as a way to represent and solve problems involving relationships among quantities, and they learn accordingly. In addition, they continue to develop procedural skill on par with other students, even when those students use more traditional curricula that have very different intentions and implementations. The advantages of CMP students continue as they enter high school. In fact, in various learning outcome measures, the CMP students performed better than or as well as the non-CMP students.

The persistence of the CMP students' advantages into high school is quite significant from two perspectives. First, these findings not only show the necessity of examining curriculum effect beyond the grade levels in which the curriculum is implemented, but they also suggest an effective way to investigate curriculum effect beyond the grade levels of implementation. In the past there has been no study that has systematically examined curriculum effect beyond the implemented grade levels. Thus, this work breaks new ground in curriculum studies. Second, the findings of this study provide evidence for a potential long-term effect of problem-based mathematics instruction on student learning. In mathematics education, there is a growing consensus among researchers, educators, and teachers that problem-based mathematics instruction offers considerable promise. Theoretically, this approach makes sense. In teaching through problem solving, learning takes place during the process of problem solving. As students solve problems, they can use any approach they can think of, draw on any piece of knowledge they have learned, and justify their ideas in ways they feel are convincing. The learning environment of teaching through problem solving provides a natural setting for students to present various solutions to their group or class and learn mathematics through social interactions, meaning negotiation, and reaching shared understanding. Empirically, there remains a need for more data to confirm the promise of problem-based mathematics instruction. Given that CMP is a problem-based curriculum, our research provides such data. The use of the CMP curriculum in middle school not only has a positive effect on students' high school performance on open-ended problem solving (9th grade results) and problem posing (11th grade results), but also on basic mathematical skills as assessed by the state test (10th grade results).

Thus, the three-level construct of curriculum we have used to examine the CMP curriculum affords us a powerful mechanism for understanding how curriculum reform can have an impact. In addition, we suggest that this conceptualization of curriculum provides a fruitful structure for curricular comparisons, both within and across nations. Indeed, as cross-national comparisons of curricula continue to be conducted, it is important to recognize and remember the relationships between the levels. Understanding how curriculum can be used to improve student learning requires an understanding of the goals of the curriculum and their embodiment in instruction.

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Mathematical Problem Solving Beyond School: Digital Tools and Students' Mathematical Representations

Susana Carreira

Abstract By looking at the global context of two inclusive mathematical problem solving competitions, the Problem@Web Project intends to study young students' beyond-school problem solving activity. The theoretical framework is aiming to integrate a perspective on problem solving that emphasises understanding and expressing thinking with a view on the representational practices connected to students' digital mathematical performance. Two contextual problems involving motion are the basis for the analysis of students' digital answers and an opportunity to look at the ways in which their conceptualisations emerge from a blend of pictorial and schematic digital representations.

Keywords Problem solving · Expressing thinking · Digital mathematical performance · Competitions

Introduction

For many years, mathematical problem solving has been positioned as a central research theme in mathematics education even though with fluctuations in intensity level and obvious nuances in trends across countries and research groups (Törner et al. 2007). Alongside mathematics curricula and educational orientations tend to renew the attention devoted to problem solving skills among the range of mathematical abilities that students are expected to develop in general and vocational studies throughout their school trajectories.

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Not less significant than mathematical problem solving is the mobilization of efforts in the research on technology use in mathematics teaching and learning. So far most of the research has put its gravitational centre in particular resources (software packages, calculators, spreadsheets, applets, or interactive whiteboards) that continue to be developed and implemented in direct connection with the teaching and learning of particular mathematical topics and mathematical methods. Moreover the studies on the use of digital technology focusing on classroom tasks are in obvious dominance. This body of research has already shown that a crucial factor in the success of ICT integration lies in the teacher and the students' roles and in the ways in which the content material is approached, i.e., in the type of learning environment that is generated through the use of technology.

It is now expected that research into problem solving aims for new advancements by taking into account the solver's use of technological devices. Although not many studies consider these two aspects in a clear and neat way it may be argued that research involving the integration of technology in the classroom generally reflects a problem solving approach or an investigative perspective into mathematical topics. In general there is a growing awareness of the ways in which the use of digital tools, namely computer technology, largely due to their multi-representational and dynamical nature, changes and reshapes mathematical problems.

Furthermore we are now educating a generation of students who more often than not use digital technologies out of school, regardless the serious efforts that many countries are endorsing to commit schools and educators to the technological uptake. Surely many of those students are regularly surfing the Internet or communicating on Facebook and doing a lot of things far from mathematics and mathematical problem solving. Nevertheless an increasing number of youngsters engage in web-based mathematical competitions, taking advantage of their own personal computers, mostly at home, to work on mathematical challenges.

One of such web-based environments, which will be presented in detail in the following section, combines explicit mathematical activity in problem solving with the use of home digital tools, thus extending beyond school. It is therefore a propitious context to study specific aspects of students' mathematical problem solving in view of their use of digital tools. It renders the opportunity to discover how students engage in problem solving and what kind of technology use they reveal in their solutions to mathematical problems beyond their school learning.

As part of the ongoing research work within the Problem@Web project, I will present some examples of the *forms of expressing mathematical thinking* displayed in two motion problems from a digital mathematical competition. Such problems may naturally evoke kinesthetic images on the problem solver and our purpose is to identify aspects of their use of digital representations that reflect their understanding of time-variation situations and reveal powerful conceptual models beyond typical school-like solutions.

The Context of Web-Based Mathematical Competitions

One of the conclusions drawn by large scale international studies on students' mathematical performance is the fact that students really learn mathematics outside school. On the other hand learning mathematics beyond the classroom is particularly supported today (sometimes encouraged by peers, teachers and parents) considering the availability of versatile technological environments (Freiman et al. 2009; Haapasalo 2007).

> The fact that students seem to learn as well mathematical as technical skills effectively outside the classroom, forces us to ask if there is something wrong inside school as far as the question "how to learn" is concerned (Haapasalo 2007, p. 9).

The numerous and diverse mathematical competitions and enrichment activities taking place regionally, nationally or internationally are a way of extending mathematics learning beyond the classroom. As the recent 16th ICMI Study has pointed out learning mathematics beyond the classroom may be based upon multi-day mathematical competitions for students of a wide range of mathematical abilities (Kenderov et al. 2009). Also, as remarked by authors from countries with a long tradition in mathematical competitions and highly competitive-driven systems, the uprising of a large number of new competitions "reflects a possible shift in the focus and purpose of competitions away from a strictly talent-search model to a more inclusive 'enrichment' approach" (Stockton 2012, p. 37).

Despite the many variants in content, duration and participants in today's widespread competitions, it has been shown that they all can have high motivational effects, especially for younger students (Kenderov et al. 2009; Freiman et al. 2005; Freiman and Vézina 2006). For example, in what concerns the benefits perceived by the students, the CAMI project highlights very positive and largely distributed advantages for different ability-level students (Freiman and Vézina 2006).

The overall aims of mathematical competitions are therefore reaching much further than the identification and selection of mathematically gifted students. They are becoming one of the many places of mathematics education where mathematics is presented as challenging, exciting, accessible to average students, socially and emotionally engaging—since many of them set up team work or inter-schools tournaments or mathematics dissemination to the public—and closer to the daily aspects of students' lives. As Freiman and Vézina (2006) state, these new forms of mathematical competitions, including virtual and on-line contests and the surrounding attractive materials created, become good examples of partnerships between schools, universities and families.

While these new formats and purposes of mathematical competitions are being established, the type of mathematical knowledge targeted and the questions aiming for higher problem solving skills start to give way to the idea of challenges and challenging situations to the average people. "While challenges have always been part of mathematical exposition in some small way, they have now come to the forefront in our conception of classroom practice and public exposition" (Barbeau 2009, p. 9).

Thus, according to Kenderov et al. (2009), mathematics education "beyond the classroom" has attained an important and irreplaceable role: to challenge the minds, skills and talent of youngsters. This explains why mathematical competitions are being conceived as challenging environments where young people can expose and develop their skills in the field of mathematics. As a result, there is now a "world of mathematics competitions" involving millions of young people, teachers, mathematicians, educators and schools, sponsors and parents.

Two different kinds of competitions are coexisting today around the world: inclusive, for students with a wide variety of capacities, and exclusive for especially talented students generally requiring filtering and selection of the participants. Web-based mathematical competitions that run online and involve substantial electronic communication are typically among those of the inclusive genre, characterised by being open to a large number of participants, closer to schools and teachers, and surrounded by a certain sense of community development.

Two of such competitions are promoted by the Mathematics Department of the Faculty of Sciences and Technology of the University of Algarve, in the southern region of Portugal—SUB12® and SUB14®.

A Description of SUB12 and SUB14

The two competitions have been running since 2005 and are aimed at two school levels: the SUB12 addresses students in 5th and 6th grades (10–12 year-olds) and the SUB14 addresses students in 7th and 8th grades (12–14 year-olds). The two competitions are web-based, located in the same web-site (http://fctec.ualg.pt/matematica/5estrelas/), have similar rules and operate in parallel. They involve two distinct phases: the *Qualifying* and the *Final*. The Qualifying phase develops entirely at distance through the web-site and consists of a set of ten problems each posted every two weeks. Students are invited to engage with the competition by means of informative flyers distributed to all the schools of the region covered by the competition. There is no formal enrolment process since the way to step in is just answering to the first problem posted. Failure in one of the problems will be signalled with a yellow card, and those participants who exceed two yellow cards get a red card and are eliminated from the competition. The failure cases fall into three modalities: no answer given to the problem, a wrong answer, or critical omissions in the solution, especially the lack of an explanation of the problem solving process.

During the Qualifying participants may participate individually or in small teams of two or three elements. They send their answers to the problems by e-mail or through the electronic message editor available at the web-site, having the choice of writing their solutions completely on the e-mail window or attaching files to their messages. The problem statement is displayed on the web-page and it can be downloaded. Students' answers are received in e-mail accounts specifically devoted to that purpose and a team of senior mathematics teachers reply to every participant, by giving a formative and encouraging feedback, suggesting reformulations when needed and offering clues to help overcoming obstacles or just praising good answers and cheering the progresses made. Students are allowed to submit revised solutions as many times as needed within the respective deadline.

The web-page also includes a specific area for news where the organising team places relevant information not only to the participants but also to teachers and parents, concerning the rules of the competition or announcements, or even an incentive for teachers and parents to encourage their youngsters. A table with the participants' results is periodically posted and 10–20 selected answers, seeking to illustrate the diversity of solutions produced and showing good clear explanations of the mathematical processes, are also made available.

The number of students participating in the two competitions has been growing over the years and it now reaches about 2000 participants in SUB12 and 800 in SUB14. The development of the Qualifying is characterised by an asymptotic decrease in the number of participants, mostly caused by attrition, which is more pronounced around the middle of the competition. Usually 10–20 % of the total initial participants reach the Final phase.

The Final is a half-day on-site contest held at the university campus with the presence of the finalists, their families and also teachers. At the Final students are given a set of five problems to be solved in one hour and a half. Everyone is competing individually and there is no technology available. The students' written answers to the problems are corrected anonymously by a jury. In the meantime parents, teachers and other accompanying guests have a program devoted to them, including a workshop, an exhibition, a seminar or other forms of interactive activities related to mathematics and mathematical problem solving. The Final includes a coffee-break, sometimes music and folk dance, and culminates with the awarding ceremony of the three winners who receive prizes and honour diplomas.

Throughout the history of this competition a number of distinctive characteristics have been standing out: (i) it is based on mathematical challenges that can be labelled as *contextual word problems* usually allowing several ways to be solved; (ii) it values communication competences and explicitly requires exposing the strategy and the procedures followed to find a solution; (iii) it ensures that students have the mathematical knowledge required do deal with the problems proposed; (iv) it is curriculum-detached, meaning that problems are not chosen to fit any particular school curricular topic (yet they may involve solid and plane geometry, algebra, counting, logic, numbers, variables and change); (v) it is close to teachers and families in the sense that it encourages their support to the young participants; (vi) it is formative and friendly by offering opportunities for problem solving improvement and giving hints if necessary; (vii) it welcomes all types of media to attain and to deliver solutions (either the use of digital tools or image-scanned paper and pencil work); (viii) it favours persistence and commitment often related to the involvement of parents (although e-mail communication always addresses the participants themselves); (ix) it gives public recognition to the more precise, creative, aesthetic, interesting solutions by publishing them on the website; x) it concentrates the competitive component in the Final, promoting collaboration and sharing during the Qualifying (for instance some teachers discuss the problems with their students in mathematics classes or in monitored study and help them on technological matters).

The Problem@Web Project: Researching Web-Based Mathematical Problem Solving

In Portugal, despite curricular orientations, problem solving based practices have been proved deficient over the years in school mathematics (Matos 2008). Official reports have highlighted important weaknesses regarding students' mathematical competence at the end of compulsory school: low in problem solving across curricular topics; low in communication, namely in the interpretation and use of diverse mathematical representations; fair in reasoning about simple situations but very low in deductive thinking. One of the presumed reasons to explain these facts is that students are not frequently exposed, in their regular schooling, to problem solving where analysis and interpretation are required.

In 2006, the Ministry of Education has launched a Plan of Action for Mathematics, aimed at raising students' attainment in mathematics (Ministério da Educação 2006). As a result, the development of three transversal capacities is now considered mandatory in K-9 mathematics: (a) problem solving, (b) mathematical reasoning and (c) communication (Ministério da Educação 2007). Therefore, in conjunction with the growing interest of teachers and mathematics educators on mathematical competitions, mathematical problem solving has been repositioned on the agenda of mathematics education in Portugal.

On the other hand the massive research produced for more than five decades on developing students' abilities in problem solving is judged to having little to offer to school practice because it misses to explain average students' difficulties related to devising a model of the situation presented in a problem (Lester and Kehle 2003; Lesh and Zawojewski 2007).

Also Francisco and Maher (2005) reflect on the excessive focusing of research on generating and describing taxonomies of students' problem-solving heuristics which results in an inability to perceive what is actually the mathematical reasoning of students when they solve problems. In their study, they view "mathematical learning and reasoning as integral parts of the process of problem solving" (p. 362). They have concluded that engaging students in strands of complex tasks promotes meaningful and thoughtful mathematical activity and showed that students were able to overcome cognitive obstacles sometimes by reference to some prototypical problems. Moreover, their research sparked the idea of the ownership of students' activity in problem solving, accounting for the fact that they were able to come up with different and interesting ways of thinking about the problems.

It is now becoming consensual that research on problem solving needs to find new directions and new empirical fields to understand the nature of humans' approaches to *mathematizable* situations. English et al. (2008) have identified some of the important drawbacks in the existing knowledge about mathematical problem solving, including the following: limited research on concept development and problem solving, and limited knowledge of students' problem solving beyond the classroom.

Simultaneously it is necessary to consider a new generation of students—youngsters who are developing, mostly out of school, a large number of competences, which grant them the skills and sophistication required to learn beyond the school barriers. Often described as "digital natives" (Prensky 2001, 2006), they access information very fast, are able to process several tasks simultaneously, prefer working when connected to the Web and their achievement increases by frequent and immediate rewards. In the specific context of digital mathematical competitions participants can communicate their reasoning on the problems in an inventive way and can resort to any type of technological tools. Home digital technologies play a role in tackling the problems and in communicating about them, thus adding competences that sometimes school neglects or forgets (Jacinto et al. 2009).

In light of the above directions and trends, the Problem@Web project was launched to seize the opportunity of studying students' mathematical problem solving beyond the mathematics classroom. By looking at the global context of the competitions SUB12 and SUB14 as a rich multi-faceted environment, the project intends to explore, in an integrated way, issues that combine cognitive, affective and social aspects of the problem solving activity of 21st century young students. The research field is clearly based on inclusive mathematical competitions, mainly taking place through the Internet, inducing strong digital communicative activity and having resonance with students' homes and lives.

The research focuses of the project concern:

(a) Ways of thinking and strategies in mathematical problem solving, forms of representation and expression of mathematical thinking and technology-supported problem solving approaches;
(b) Creativity as manifested in the expression of mathematical solutions to problems and its relation to the use of digital home technologies;
(c) Attitudes and affect in mathematics and mathematical problem solving, considering students, parents and teachers.

The empirical work integrates two main modes of data collection (extensive and detailed) and the data analysis will combine quantitative and qualitative methods.

The extensive data include:

- Records of the exchanged e-mail messages in each "15 days stage of the competition";
- Records of selected participants' solutions to all the problems in the course of the competition;
- Online questionnaires to the participants;
- Collection of all the finalists' papers in the Final.

The detailed data include:

- Interviews with a small number of participants who get to the Final;
- Interviews with a small number of participants who drop out of the competition;
- Interviews with parents (or family members) of participants;
- Interviews with teachers who have students participating;
- Video recordings of school classes or other school sites where teachers work on the problems with their students;
- Video recordings of the on-site Final.

The project also aims to develop a coherent theoretical framework to investigate problem solving within the context of participation in virtual mathematical competitions. The theoretical developments are being undertaken in three directions: (i) Problem solving and the use of digital technologies; (ii) Affect and beyond school mathematics; and (iii) Problem solving and creativity.

Theoretical Framework

Concerning the overarching theoretical perspectives from the ongoing research project, the emphasis is being placed on problem solving as part of understanding mathematics and being able to engage in mathematizable situations. At the same time, theoretical concepts regarding the use of digital technologies in students' mathematical activity are focusing on mathematical representation and on the role of imagery for explaining thinking.

Problem Solving from the Point of View of Expressing Thinking

One of the characteristics of the data gathered from students' answers to the problems proposed in the competitions SUB12 and SUB14 is the fact that they have a digital format, totally framed by electronic communication. Participants are required to give a clear explanation of the problem solving process to people on the other side of the e-mail connection. This well marked aspect of our data is leading us to pursue the notion of *explaining*, *exposing*, or *expressing* mathematical problem solving as a fundamental aspect of the problem solving process as a whole. In this sense, we share the kind of questioning formulated by Lesh and English (2005, p. 193):

> In what ways is "mathematical thinking" becoming more multi-media—and more contextualized (in the sense that knowledge and abilities are organized around experience as much as around abstractions, and in the sense that relevant ways of thinking usually need to draw on ways for thinking that seldom fall within the scope of a single discipline or textbook topic area)?

One of the fundamental ideas we endorse is that mathematical problem solving means *ways of thinking about challenging situations where a mathematical approach is appropriate*, even if the problem solver may not recognise such thinking as being a typical mathematical activity or may not draw on specific school mathematics knowledge, as it is often the case of our participants.

In this sense, particular concepts and notions derived from the models-and-modelling perspective (MMP) (Lesh and Doerr 2003a) reveal promising directions to understand mathematical problem solving as an activity mainly organized around experience. One of such notions refers to productive ways of thinking. Lesh and Zawojewski (2007) put it clearly by stating that a problem may be any situation or task where the problem solver feels the need to find a productive way of thinking about it. Productive ways of thinking do not mean direct paths between the givens and the goals of the situation; on the contrary they are the result of *seeing* the situation in effective ways that may involve several iterations of *interpreting*, *describing* and *explaining*. The proponents of MMP have provided evidences that students are able to create conceptual tools while looking for ways of thinking about a situation. A result of productive thinking is a *conceptual model* of the situation.

> Students produce conceptual tools that include explicit descriptive or explanatory systems that function as models which reveal important aspects about how students are interpreting the problem-solving situations (Lesh and Doerr 2003b, p. 9).

The authors further explain that such conceptual models are expressed in several different ways that students resort to, like images, diagrams, symbols, and representational materials, all explicit elements that give visibility to their understandings, as for example, the quantities they think about, the rules they consider, the relationships established, etc.

For our present research it is also important to replace the notion of "getting an answer to the problem" with the idea of "creating an explanation"—a more useful construct that encapsulates the answer and the process.

> …descriptions, explanations, and constructions are not simply processes students use on the way to "producing the answer", and, they are not simply postscripts that students give after the "answer" has been produced. They ARE the most important components of the responses that are needed (Lesh and Doerr 2003b, p. 3).

As argued by Reeuwijk and Wijers (2003), getting students to show their reasoning, thinking or strategy may be a question of introducing mathematical norms or including prompts to incite students to give justifications and to show their work. Part of such mathematical norms should be the practice of equally valuing all strategies although it may also depend on the way the tasks are defined.

In SUB12 and SUB14, all problems include a prompt of the kind: "Do not forget to explain your problem-solving process". This is one of the rules of the competition that is announced from the beginning in the news board of the competition web-page. Additionally, as students get regular feedback on the solutions they send by e-mail, they are impelled to present their ways of thinking about the problem, in their own words and with whatever means they decide to use, or else to elaborate more on their responses. Omissions in problem-solving explanation become a penalty to the participant who fails to offer a sufficiently understandable and convincing picture of the reasoning developed. Considering problem solving performance in those terms is also a consequence of the underlying assumption that it necessarily involves mathematization and mathematical communication—or rather, achieving a model and its presentation (Reeuwijk and Wijers 2003). Thus a mathematical representation, such as for instance an equation or a tree diagram, should not be taken as "the reasoning" even if it is a key part of the solution process. Instead it has to be placed within a descriptive story that contains both the original context of the problem and the mathematical representation, in a way that echoes the following conception of mathematical understanding:

> …a blurring of task, person, mathematical activity, nonmathematical activity, learning, applying what has been learned, and other features of problem solving (Lester and Kehle 2003, p. 516).

Rather than having problem solving subsuming mathematical understanding, it is proposed that mathematical understanding subsumes problem solving and posing. Thus when looking at problem solving we should be looking primarily at mathematical understandings or, more precisely, mathematical ways of understanding situations.

Problem Solving as Expository Narrative

The number of research studies addressing students' problem solving in virtual and beyond-school empirical fields is still very small especially when compared with the considerable corpus of research carried out in classroom settings. Nonetheless the study developed by Stahl and his collaborators on virtual math teams (Stahl 2009a) is particularly helpful in offering clues to our present research.

The Virtual Math Teams project (VMT) consists of one of the many online services offered by the quite well known Math Forum website, currently accessed by millions of visitors a month. The VMT service has grown out of another service in the Forum, the Problem of the Week (PoW), where challenging mathematical problems are posted and students can send their solutions and receive feedback for improvement. The VMT is another way of working on more open-ended problems, in a collaborative mode, with students interacting in groups of peers in mathematical discussion chat rooms. Specific software tools available in the VMT environment allow for maintaining group coordination and mathematical problem solving, such as the case of the whiteboard for graphical representations or the tools to edit mathematical symbols.

Given the fact that the data stored in the Problem@Web project consist mainly of digital solutions that can not provide any information regarding immediate or face-to-face interactions, our logs are clearly different from the ones captured during chats running in group problem solving sessions. Nevertheless, following the research of Stahl and his team, the concept of *expository discourse* (which they distinguish from exploratory discourse and see as complementary in their data analysis) is an important tool for an analysis of problem solving as *expressing thinking*. In fact, the aforementioned view on problem solving highlights representation, communication and explanation of thinking. From this perspective a large number of signs, considerably propelled by the use of digital tools, become significant as part of an expository discourse: the use of colour, natural language, mathematical language, drawings, pictures, photos, icons, diagrams, arrows, labels, notations, pre-symbols, symbols, outputs of specific software (spreadsheets, dynamic geometry systems, graphing tools), tables, letters, numbers, characters, and so on. As Stahl (2009b) describes it, expository discourse is the telling of a story about how the problem was solved, usually providing a sequential account of the essential elements that constitute the problem solving process. Medina et al. (2009) also reporting on the study of VMT, move into the question of *representational practices* and describe how inscriptions become representations in students' problem solving attempts. In describing a group of students engaged in finding a formula to translate a geometrical pattern, the authors highlight how students' inscriptions in the whiteboard guided the group's activity and turned into representational resources functioning as indexical signs to the problem solvers and the solution readers. This is also the case of many of the stories told by participants in SUB12 and SUB14 in their answers to the problems. Their expository narrative is often rich in inscriptions with strong indexical value: "this is how the water level rose in the tank", or "the arrows indicate opposite directions of walkers". Most of these pieces of information are meaningless without the original context of the problem and outside of the complete story of the problem solving. But they actually have a profound role not just as a post-script of the problem solving but as part of the representational practices students engage in.

Another idea that may deepen the concept of expository narratives, as thought-revealing activities or as ways of communicating math experiences, is the notion of *performance*. Gadanidis et al. (2010) used the performance lens to analyse students who wrote scripts about their mathematical experiences, and afterwards performed, recorded and shared the videos online to a wider community. For the authors those are also "digital stories" that reveal a relationship between the performance and the audience. And the new digital media are obviously offering new possibilities for storytelling and for communication to be displayed. Thus technology itself becomes an actor in the *digital mathematical performance* of the students. Referring to the work of Hughes (2008, cited by Gadanidis et al.), the authors also mention the idea of authorship as being akin to the new digital power of publishing your own stories, contents, emotional states. Students' digital mathematical performance seems to embody this sense of authorship, sometimes described as being the intersection of personalization, participation and productivity representative of the networked society (McLoughlin and Lee 2008). It is also convergent with the students' sense of ownership reported by Francisco and Maher (2005) that emerges when mathematical reasoning becomes the fundamental aspect of problem solving.

Humans-with-Media *Solving Mathematical Problems*

At the same time we are emphasising digital tools as powerful media to *express* mathematical thinking in problem solving processes, the educational community is calling attention to this new power of digital technologies already impacting on mathematical practices outside school, as stated in the report of the Joint Mathematical Council of the United Kingdom:

> School and college mathematics should acknowledge the significant use of digital technologies for *expressive and analytic purposes* both in mathematical practice outside the school and college and in the everyday lives of young people (Clark-Wilson et al. 2011, p. 18) (my emphasis).

"Because technology has the potential for broadening the representational horizon" (Zbiek et al. 2007), representational fluency is acquiring obvious importance, including knowing how to use particular representations to describe, illustrate or justify assertions and ideas. Representational fluency can also be considered as a lens to examine mathematical activity on technology-based representational media (Zbiek et al. 2007; Johnson and Lesh 2003).

Results from our research have already pointed out that representational fluency flows from the expository narratives of the participants in the competition and such fluency is strongly interlinked with their use of digital media. For example, Jacinto et al. (2009) looked at how participants in SUB14 perceived the role of the technological tools they used during the competition. The participants valued the opportunity of *communicating their reasoning in an inventive way*, since they could resort to any type of attachments, in particular those they felt more comfortable with or found adequate to the problem itself. They resorted mainly to the text editor MSWord, but also to MSPaint and MSExcel, all examples of home digital technology. The use of images is often a result of their efforts on expressing their reasoning in the best possible way. Moreover, we noticed their awareness of the different representations that could materialize their reasoning and even some decision ability when selecting among the options they had at hand. In another study, Nobre et al. (2012) reported on how students dealt with one of the competition problems with the use of a spreadsheet. It was clear that students interpreted the problem in light of their mathematical knowledge and of their knowledge of the digital tool. When the problem was later explored in the classroom with their mathematics teacher, the relationship between the symbolic language of the spreadsheet and the algebraic language was clear to the students.

These results converge with conclusions drawn by Johnson and Lesh (2003), according to whom "important functions of technology-based representational media (eMedia) are: (a) to describe or explain complex systems, and (b) to express complex constructs by providing new ways for people to communicate with both others and with themselves" (p. 273).

Finally, we endorse the theoretical stand that rejects a separation between the user and the mediational means, as it is elaborated by Borba and Villareal (2004) through the notion of humans-with-media, and also by Moreno-Armella and

Hegedus (2009) through the idea of co-action. In the former perspective, we want to highlight one particular point that resonates with our data and knowledge on students' online participation in the competitions: visualization and humans-with-media. Computers change the status of visualization in mathematical activity and bring in new tools to express ideas through visual forms. It has been a constant trace in many of the students' answers that the media used to develop visual representations goes much further than just embellishment. The ways in which students *see* the problem solution and *express* it with digital media supports the statement: "what we see is always shaped by the technologies of intelligence that form part of a given collective of humans-with-media, and what is seen shapes our cognition" (Borba and Villareal 2004, p. 99).

Expression of mathematical problem solving mediated by home computer technologies has been pushing our research into the question: how do young problem solvers expose their problem solving processes and how do visual aspects of home digital technologies emerge as part of their digital-mathematical-performance?

Problem Solving Involving Motion

To get some insight into the question above, I will be focusing on two of the problems proposed in the competitions, one from SUB12 and the other from SUB14, both involving imaginary situations that concern motion and time-variation.

The purpose is to look at how students deal with time-variation in their ways of representing the situation and to identify features of their mathematical representations within the media used to express their thinking.

A Problem from the SUB12 Competition

The following is Problem #8 proposed in the 2010/11 of SUB12 (Fig. 1). This was a problem that came out near the end of the Qualifying and therefore participants were quite familiarised with the rules and with the operational aspects of the competition. In particular, they had already solved seven problems and got a fairly good experience on sending their answers by e-mail. At this stage, many of the

On some of the week days Paulo gets a ride from a schoolmate to go from home to school but at the end of classes he walks back home. On such days he takes a total of 40 minutes to go and return. On the other week days he gets a ride to go to school and also to come back home. On such days he takes half of the total time to go and return.
Unfortunately last week, as his mate's mum was ill and could not drive, he had to walk to school and to walk back home after classes. How long did it take him to walk to school and back home?

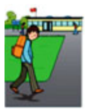

Fig. 1 Problem #8 of SUB12, edition 2010/11

students were choosing to send attached files rather than just typing their answers in the e-mail window. The majority were Word documents, but a few answers came also with Excel, PowerPoint, and Publisher files.

Very few (only two in total) of the 5th and 6th graders exposed their mathematical thinking through symbolic equations. It is worth mentioning that students in this grade levels have a limited knowledge on equations and variables although they may have learnt introductory algebraic language in pattern description and generalization.

Most of the participants took different approaches to the problem. It became evident from their answers that their understanding of the problem stood on realising that it involved the identification of three distinct possibilities of the journey from home to school and back.

Those students who managed to understand this aspect were able to solve the problem. Therefore, it becomes relevant to consider in what ways the students manifested this kind of thinking and how this was expressed in their digital-mathematical-performance.

Three excerpts of students' answers featuring their approaches to the problem are given in Fig. 2 (snapshots a, b). Both of them depict the situations described in the problem and a common trace is the use of iconic signs to represent them.

In the first solution (a), the student starts to present a picture of a car, with an arrow beneath pointing to the right, and an iconic version of two feet, with an arrow beneath pointing the left. One colour (green) and a large letter size are used to write '40 Minutes' on the side. It indicates a fundamental piece of information extracted from the problem. Both the two arrows pointing to opposite directions and the two images used reveal an understanding of the information: going to school by car and returning home on foot takes 40 min. The next piece of the answer (second paragraph) only uses the image of the car. Natural language and mathematical symbolism

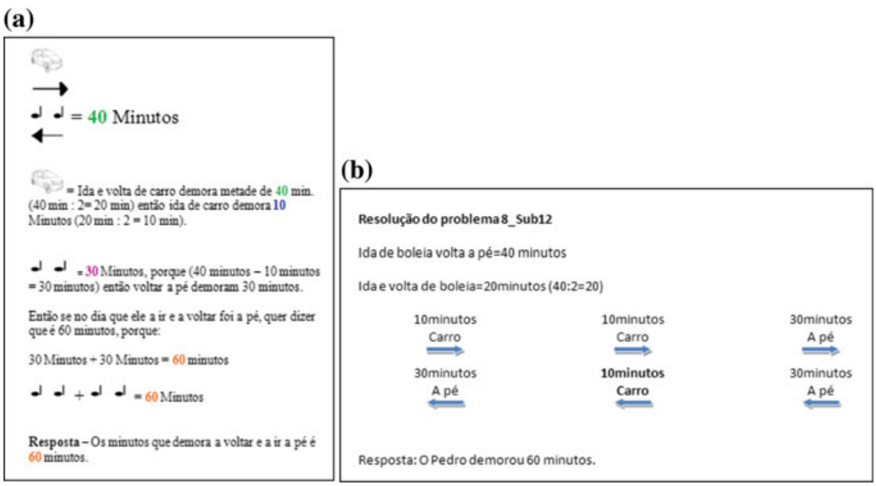

Fig. 2 Print-screens of excerpts of students' answers: the use of iconic signs

(indicating elementary computations) are introduced to state that going and returning by car takes half of 40 min, and thus dividing 20 by 2 gives the time of '10 Minutes' for a one-way trip by car. In the sentence, the number 10 is written in blue, signalling a new relevant piece of information and one that refers to a different aspect of the problem. The third paragraph starts with the icon for the walking trip immediately followed by the words '30 Minutes', with the number 30 coloured in pink. It goes on explaining how this result is obtained from the difference between 40 and 10. Afterwards, the answer states that walking to school and back takes 30 plus 30 min, in a total of '60 Minutes'. The number 60 is written in a new colour (orange), again showing that it refers to another of the cases described in the problem. Another line is finally included, where the student rephrases the previous conclusion using the iconic language and showing the sum of two walking icons being equal to 60 min (maintaining the orange colour).

In general, both the iconic elements and the use of four different colours to display different numbers were relevant inscriptions embodying the reasoning: 40 (the time for going by car and returning on foot); 10 (the time of a one-way trip by car); 30 (the time for a one-way walking trip); and 60 (the time for going and returning on foot). These pictorial-visual meanings keep the thinking directed to the several cases of the trip and allow having them differentiated while integrating the data and intermediate results to get the answer.

The second solution (b) also describes the three situations with the use of arrows and labels. The labels indicate 'by car' and 'on foot'. And in each part, the two arrows beneath each case point to opposite directions. The reasoning developed is not so much detailed as in solution (a). The answer starts with the calculation of 40 divided by 2 for knowing the time of the round trip by car. The central iconic display indicates the two one-way trips by car and the respective time, suggesting that the 20 min were divided in 10 min for either way. The label concerning the car trip from school to home is highlighted in bold, also suggesting the relevance of this data. It suggests that it was the key to find the time for the same trip on foot (on the left side), which lead to the time spent in each of the one-way walking trips (on the right) and thus to the total time.

Further examples of students' graphical arrangements of information, explanations and representational forms (Fig. 3) also highlight the pictorial use of arrows and colours to dissect the situation in the two cases. The reasoning flows out of the well distinguished cases, digitally described through the use of diagrammatic elements.

A Problem from the SUB14 Competition

The Problem #1 from the 2011/12 edition of SUB14 (for 7th and 8th graders) also involved motion (Fig. 4). This problem describes two individuals walking towards each other with different velocities and different departure times. Although this was the first problem of the season, many of the participants were already used to send attached files.

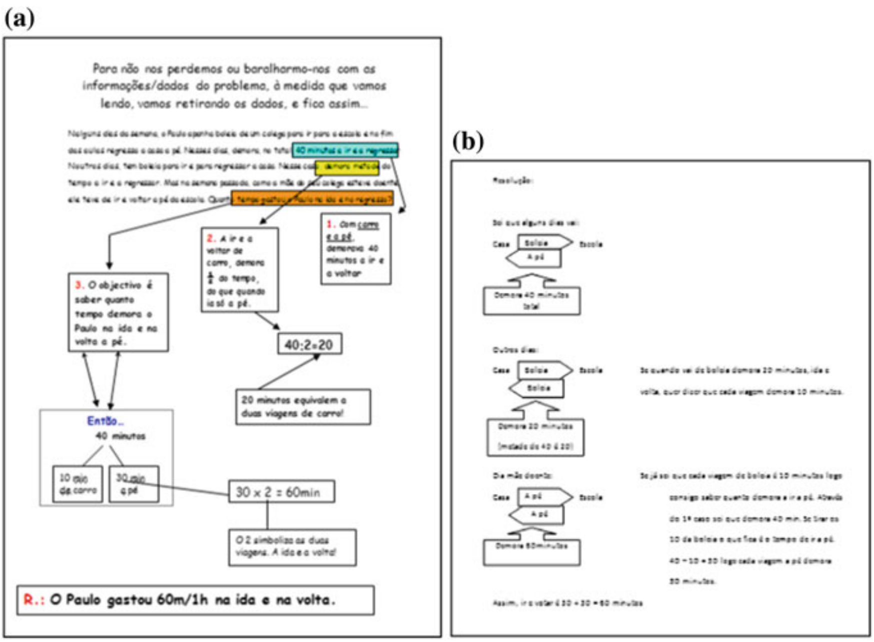

Fig. 3 Print-screens of excerpts of students' answers: the use of graphical arrangements

Alexander and Bernard live at a distance of 22 km from one another and they want to meet but the only way to do it is… walking! On a holiday morning they decided to walk towards one another to get together. Alexander left his home at 8 a.m. and went walking at a speed of 4 km per hour. Bernard left his home an hour later and walked at a speed of 5 km per hour. Neither of the two friends took his watch but we can know at what time they met. What time was it?

Fig. 4 Problem #1 of SUB14, edition 2011/12

Most of the participants who got the problem wrong on their first attempts solved it as if the two friends were walking in the same direction; there were also a number of seventh graders who invoked the notion of the least common multiple of 4 and 5 to address the problem. Such answers showed a weak understanding of the problem conditions and apparently a tendency to apply school knowledge to a problem that may have looked like a standard situation for using the least common multiple.

On the contrary, students who got correct answers payed attention to the fact that the two friends walked towards each other. Many of the solutions were presented with a table describing the positions of each of the friends at every hour, from 8 a.m. until 11 a.m., the time of the meeting.

There were also many others that involved graphical and pictorial representations, consistently highlighting the distance walked by each of the two friends and their opposite directions (Fig. 5). The use of colours and indicators of constant steps (curved lines and/or a line scale marked) were quite frequent. Usually, the drawings

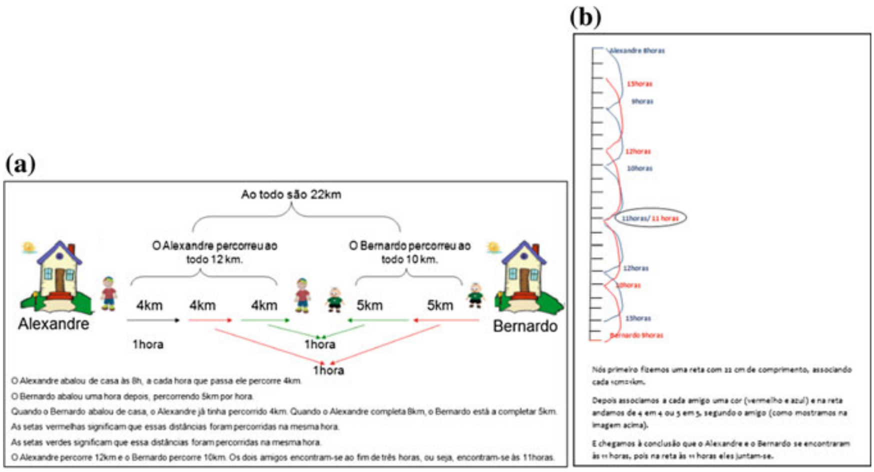

Fig. 5 Print-screens of excerpts of students' answers: the use of pictures and diagrams

are followed by natural language explanations, as in the case of Fig. 5b, where the students state: "We first draw a line with 22 cm and considered 1 cm to be equivalent to 1 km. Then we assigned a colour to each of the friends (blue and red) and on the line we followed in spaces of 4 marks or 5 marks, according to the friend (as shown in the picture). And we concluded that Alexander and Bernard met at 11 a.m. because it's where they joined on the line". Also it is very clear, in both the answers of Fig. 5, the use of labels that seek to explain the movement developing: the time changing and the position of both friends getting closer to each other.

Two additional examples (Fig. 6) illustrate the movements of the two friends and highlight the fact that the distances travelled by each of them must add up to the distance of 22 km between the houses. This explanation was given emphasis with colour (b) or it was included in the text to underline the match between what the picture showed and what the problem meant (a).

The above set of answers is rich in details, showing that different students presented several important aspects of the situation: the time-variation, the changing position relative to the origin, the distance travelled by each individual and the position relative to the other individual. Thus good models of the situation or productive ways of thinking about it are clearly expressed in students' story telling mediated by their effective use of digital representations.

Conclusions and Further Developments

The solutions to the problems concerning time-variation provide some important insights into the nature of representational practices in students' problem solving processes when digital tools become natural tools to express thinking.

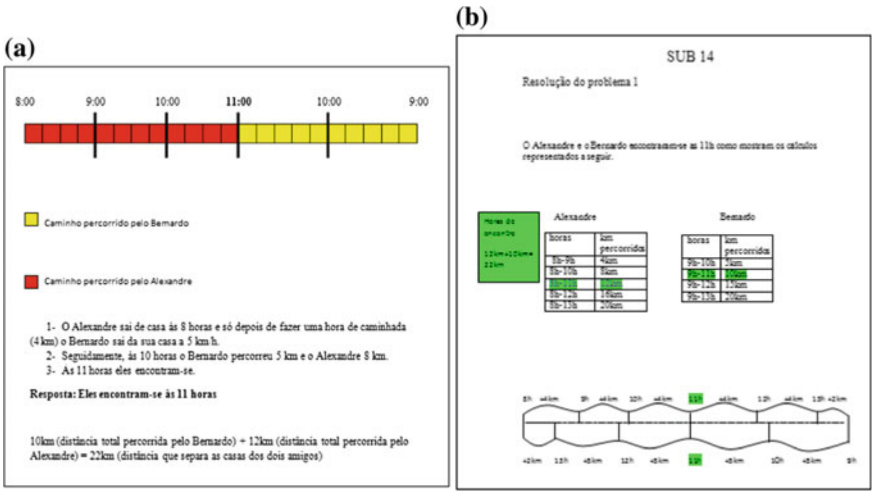

Fig. 6 Print-screens of excerpts of students' answers: the use of colours, graphical arrangements and tables

Online mathematical competitions, where communicating and expressing reasoning through electronic media is a central feature of the mathematical activity, can reveal the forms of expository narratives that youngsters produce (Stahl 2009b). Such expression of mathematical thinking becomes an integral part of the problem solving process and seems to be sustained and reinforced by the use of digital tools beyond the direct prompts that may be offered by the competition itself. It may be described as a digital-mathematical-performance, in the sense suggested by Gadanidis et al. (2010), where graphical, iconic, pictorial, indexical, and schematic means are smoothly intertwined with mathematical thinking and become inherent to the thinking.

Most of students' answers as the ones considered in the previous section are not sophisticated solutions inasmuch as they primarily intend to create and present a clear picture of the problem. Neither their use of digital tools can actually be seen as sophisticated. Yet both the solution approach and the representations afforded by the use of the tools look as friendly and clear-cut ways of creating mathematical models and performing mathematically.

Creativity is also an important aspect of many of students' visual ways of expressing thinking. It indicates how representational fluency is clearly tied to the problem solving environment and it suggests that co-action between the human agent and the digital medium (as described by Moreno-Armella and Hegedus 2009) is actually a source of creative activity. As the proponents of the concept argue, mathematical objects are refracted in the digital medium and as a result new ways of justifying and presenting mathematical ideas come to the surface. This stands out from the solutions produced by the young participants—mathematical objects, ideas, and models are being refracted in the digital media they use to think and express their thinking. Models are therefore more than mathematical expressions, algorithms or symbols. Models are essentially forms of understanding and they lead much of the successful problem solving processes of the participants in the two competitions. They reveal how situations are conceptualised and how such conceptualisations develop from inscriptions: pictures, schematic representations, language, letters, and iconic elements easily available and displayable through digital tools.

Research has provided evidence of the differences between visualisers and verbalisers in problem solving. In particular, Kozhevnikov et al. (2002) described two types of visualisers, the iconic type and the spatial type: those whose imagery is primarily pictorial and those whose imagery is primarily spatial, abstract and schematic. They also found that the first group had more difficulties in kinematics problems, especially in understanding graphs of motions. Although the problems here discussed were not real kinematics problems they involved motion and time-variation. The data presented reflect a relevant type of answers given in the competitions that may relate to *expressing thinking* when developing a solution. On the other hand, many of the digitally mediated solutions not only exhibit pictorial representations but a combination of those with spatial schematic representations, thus suggesting a blend of iconic and spatial characteristics. This brings the question of whether digital mathematical representations with which young students are fluently expressing mathematical activity influence their representational preferences.

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Constructing Abstract Mathematical Knowledge in Context

Tommy Dreyfus

Abstract Understanding how students construct abstract mathematical knowledge is a central aim of research in mathematics education. Abstraction in Context (AiC) is a theoretical-methodological framework for studying students' processes of constructing abstract mathematical knowledge as they occur in a mathematical, social, curricular and learning-environmental context. AiC builds on ideas by Freudenthal, Davydov, and others. According to AiC, processes of abstraction have three stages: need, emergence and consolidation. The emergence of new (to the student) constructs is treated by means of a model of three observable epistemic actions: Recognizing, Building-with and Constructing—the RBC-model. This paper presents a theoretical and methodological introduction to AiC including to the RBC-model, and an overview of pertinent research studies.

Keywords Abstraction · Knowledge construction · Context · RBC-model

Introduction

The approach described in this paper took shape in the course of research that accompanied innovative curriculum development, when questions arose such as "What did students learn? What new deep mathematical knowledge, what concepts and strategies have been consolidated? And how did the processes of learning and consolidation happen?" The salient characteristics of mathematical curricula and classroom learning environments in which these questions have been investigated are that curricula are organized as successions of activities proposed to the students, and that mathematical themes arising along these activities very often are transformations of previous mathematical themes. Hence curricula express an intention of continuous transformation. There is an underlying expectation of students'

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responsibility for their own learning in an environment that encourages inquiry. Students have the responsibility to report and justify their work and their conclusions to their peers and their teacher, for example during whole class discussions. The reader is referred to the literature for a more detailed treatment of the curriculum standards and design principles (Hadas et al. 2008; Hershkowitz et al. 2002).

The research problem that arose was how to understand students' construction of knowledge, especially of deep, abstract mathematical knowledge such as concepts and strategies in learning situations, in particular in classrooms, using these curricula. A main aim was to describe processes of constructing knowledge in order to get insight into such processes and the conditions under which they happen or fail to happen. Additional aims were to use the understanding of students' learning processes in order to improve the design of activities and inform teacher behaviour.

For this purpose, a theoretical framework for describing processes of abstraction was required. Rina Hershkowitz, Baruch Schwarz and the author developed such a framework over the past 15 years (Dreyfus et al. 2015; Hershkowitz et al. 2001; Schwarz et al. 2009). The framework takes into account the particularities of the context of learning. This context includes the students' prior history of learning, the learning environment, including possibly available technological and other tools, as well as mathematical, curricular and social components. In particular, the social and interactional context may vary considerably from one class to another according to the teacher's decisions. In view of the above, we called our theoretical framework Abstraction in Context, or briefly, AiC.

Theoretical Background and the AiC Framework

The attention to a special kind of curriculum and to learning processes within various contexts required a rather hybrid reference to theoretical forefathers that belong to different traditions, Freudenthal and Davydov. Freudenthal (1991) provided what mathematicians have in mind when they think of abstraction. Freudenthal has brought forward some of the most important insights to mathematics education in general, and to mathematical abstraction in particular. These insights constitute a cultural legacy that led his collaborators to the idea of vertical mathematisation (Treffers and Goffree 1985). Vertical mathematisation points to a process that typically consists of the reorganization of previous mathematical constructs within mathematics and by mathematical means, by which students construct a new abstract construct. In vertical reorganisation, previous constructs serve as building blocks in the process of constructing. Often these building blocks are not only reorganised but also integrated and interwoven, thus adding a layer of depth to the learner's knowledge, and giving expression to the composite nature of the mathematics. Sequences of problem situations provide opportunities to capitalise on the new constructs repeatedly, and to turn them into building blocks for further constructions where each construct includes 'pockets' of past constructs on one hand, and is itself a potential component for new constructs.

Davydov was one of the most prominent followers of the historical cultural theory of human development initiated by Vygotsky. For Davydov (1990), scientific knowledge is not a simple expansion of people's everyday experience. It requires the cultivation of particular ways of thinking, which permit the internal connections of ideas and their essence to emerge; it also requires enriching rather than impoverishing reality. According to Davydov's "method of ascent to the concrete", abstraction starts from an initial, simple, undeveloped and vague first form, which often lacks consistency. The development of abstraction proceeds from analysis, at the initial stage of the abstraction, to synthesis. It ends with a more consistent and elaborated form. It does not proceed from concrete to abstract but from an undeveloped to a developed form.

The reference to both Freudenthal and Davydov's theories of abstraction implies that the curriculum affords certain kinds of abstraction, but at the same time, students and teachers are free to capitalize or not on those affordances. AiC adopts the views of vertical mathematisation and ascent to the concrete and builds on them to define abstraction as a process of vertically reorganizing some of the learner's previous mathematical constructs within mathematics and by mathematical means so as to lead to a construct that is new to the learner. Activity theory proposes an adequate framework to consider processes that are fundamentally cognitive while taking into account the mathematical, historical, social and learning contexts in which these processes occur. In this, AiC follows Bauersfeld (1992) and Giest (2005), who considers activity theory as a theoretical basis, which has an underlying constructivist philosophy but allows avoiding a number of problems presented by constructivism.

According to activity theory, outcomes of previous activities naturally turn to artefacts in further ones, a feature which is crucial to trace the genesis and the development of abstraction through a succession of activities. The kinds of actions that are relevant to abstraction are epistemic actions—actions that pertain to the knowing of the participants and that are observable by participants and researchers. As researchers with loyalty to Freudenthal, we were a priori attentive to certain constructs afforded by the activities we designed. In tune with Davydov and a cultural-historical theory of development, we also looked at other constructs that emerged from classroom activities.

This is well expressed by Kidron and Monaghan (2009) when dealing with the need that brings students to engage in abstraction, a need that emerges from a suitable design and from an initial vagueness of the learner's notions:

> The learners' need for new knowledge is inherent to the task design but this need is an important stage of the process of abstraction and must precede the constructing process, the vertical reorganization of prior existing constructs. This need for a new construct permits the link between the past knowledge and the future construction. Without the Davydovian analysis, this need, which must precede the constructing process, could be viewed naively and mechanically, but with Davydov's dialectic analysis the abstraction proceeds from an initial unrefined first form to a final coherent construct in a two-way relationship between the concrete and the abstract – the learner needs the knowledge to make sense of a situation. At the moment when a learner realizes the need for a new construct, the learner already has

> an initial vague form of the future construct as a result of prior knowledge. Realizing the need for the new construct, the learner enters a second stage in which s/he is ready to build with her/his prior knowledge in order to develop the initial form to a consistent and elaborate higher form, the new construct, which provides a scientific explanation of the reality. (pp. 86–87)

Hence we postulate that the genesis of an abstraction passes through a three-stage process, which includes the need for a new construct, the emergence of the new construct, and the consolidation of that construct.

A central component of AiC is a theoretical—methodological model, according to which the emergence of a new construct is described and analysed by means of three observable epistemic actions: recognizing (R), building-with (B) and constructing (C). Recognizing refers to the learner realizing that a specific previous knowledge construct is relevant in the situation at hand. Building-with comprises the combination of recognized constructs, in order to achieve a localized goal such as the actualization of a strategy, a justification or the solution of a problem. The model suggests constructing as the central epistemic action of mathematical abstraction. Constructing consists of assembling and integrating previous constructs by vertical mathematisation to produce a new construct. It refers to the first time the new construct is expressed or used by the learner. This definition of constructing does not imply that the learner has acquired the new construct once and forever; the learner may not even be fully aware of the new construct, and the learner's construct is often fragile and context dependent. Constructing does not refer to the construct becoming freely and flexibly available to the learner. Becoming freely and flexibly available pertains to consolidation.

Consolidation is a never-ending process through which students become aware of their constructs, the use of the constructs becomes more immediate and self-evident, the students' confidence in using the construct increases, and the students demonstrate more and more flexibility in using the construct (Dreyfus and Tsamir 2004). Consolidation of a construct is likely to occur whenever a construct that emerged in one activity is built-with in further activities. These further activities may lead to new constructs. Hence consolidation connects successive constructing processes and is closely related to the design of sequences of activities.

In processes of abstraction, the epistemic actions are nested. C-actions depend on R- and B-actions; the R- and B-actions are the building blocks of the C-action; at the same time, the C-action is more than the collection of all R- and B-actions that make up the C-action, in the same sense as the whole is more than the sum of its parts. The C-action draws its power from the mathematical connections, which link these building blocks and make them into a single whole unity. It is in this sense that we say that R- and B-actions are constitutive of and nested in the C-action. Similarly, R-actions are nested within B-actions since building-with a previous construct necessitates recognising this construct, at least implicitly. Moreover, a lower level C-action may be nested in a more global one, if the former is made for the sake of the latter. This nested character was observed in classrooms and in interviews in which we studied abstraction and it substantiated our theoretical tenets according to which the curriculum was intended to afford a continuous transformation of constructs.

Given these characteristics, we named the model the dynamically nested epistemic actions model of abstraction in context, more simply the RBC model, or RBC+C model using the second C in order to point to the important role of consolidation. The RBC-model is the theoretical and micro-analytic lens, through which we observe and analyse the dynamics of abstraction in context.

We will below come back to the RBC-model in order to show how the model as a part of the theory interacts with the same model as methodological tool, and hence theory and methodology mutually depend on and influence each other in AiC. The successive analyses by several researchers who used the RBC-model to identify abstraction processes through the unveiling of its epistemic actions not only helped understanding these abstraction processes: The theory as well as the methodology underwent successive refinements as they served as lenses to understand mathematical learning activities.

An Example

While it is not possible to illustrate all aspects of AiC by means of a single example, the example used in this section fairly well illustrates many of the main aspects. It stems from a 7th grade class whose beginning algebra curriculum consisted of seventeen activities, thirteen with a spreadsheet, and four without computer. In these activities students learned to use algebra to express generality. For example, by generalizing from a few numerical examples or from a "story", they generated an algebraic relation, which they could insert into the spreadsheet. By dragging, they could then obtain sequences of numbers to describe and investigate a phenomenon. Hence, in terms of Kieran's (2004) framework, the activities were generational.

We know from weekly observations and teacher reports that the students increasingly used algebra for expressing generality throughout the year. For example, the following fact will be crucial below: The students had experience with the use of the simple distributive laws $a(b + c) = ab + ac$ and $(a + b)c = ac + bc$ but had never yet used the extended distributive law $(a + b)(c + d) = ac + ad + bc + bd$. Even more importantly, students had never yet been asked to justify general properties by using algebraic manipulation.

The following activity was carried out by the class toward the end of the school year. The students were asked to consider two-by-two number arrays, called *seals*, of the form shown in the diagram; two numerical seals were given to the students, but not the general form. The students were asked to find as many properties of the seals as possible, and to establish whether some of these properties were true for all seals. The research focus was on the diagonal product property (DPP): The difference of the products of the diagonals in the array equals 12. More detail on the curriculum and the activity may be found elsewhere (Dreyfus et al. 2001).

X	$X+2$
$X+6$	$X+8$

According to the a priori analysis, the activity was designed toward two new (to the students) constructs: algebra as a tool for justifying a general statement, and the extended distributive law. We will call the corresponding knowledge elements E and E1, respectively.

We focus on how a pair of students, Ha and Ne, collectively called HaNe, dealt with justifying the DPP. After about 25 min of discussing other properties of the seals, HaNe reached the stage in the worksheet where they are explicitly asked whether the DPP holds for all seals.

Ha	111	We didn't want to think about this before. This is really somehow like that. Because this is the first diagonal, and that is the second, right?
Ne	112	No. Yes
Ha	113	So, well, see: *X* plus 8, *X*, well, then it is 8*X* plus *XX*
Ne	114	Again?
Ha	115	Wait. *X* times *X* plus 8, right? This is *XX*, like, *X*, *X* twice, so it's *XX*
Ne	116	Yes
Ha	117	And X times 8 is 8*X*
Ne	118	Why times 8?
Ha	119	Why not? Because, one multiplies this by this, right?
Ne	120	Yes
Ha	121	So, like, one does the distributive law

Our interpretation of 111–121 according to the RBC-model is that the students recognize the simple distributive law as relevant in this new context, and build-with it a component of what they apparently need for justifying the DPP. Hence they consolidate the simple distributive law by making use of it in a new context. They then soon turn their attention to the other diagonal.

Ha	133	And this…
Ne	134	It's impossible to do the distributive law here. Wait, one can do
Ha	135	This is 6*X*
Ne	136	This is 6*X* times *X* and 6*X* times 2
Ha	137	Wait, first, no…
Ne	138	Yes
Ha	139	No because this is *X* plus 6, this is not 6*X*, it's different. Wait. First one does… *X*; then it's *XX* plus 2*X*, and here 6*X* plus 24. Then…

In 133–139 HaNe deal with the applicability and application of the distributive law to the more complex case of the second diagonal. The strategy of comparing the two diagonals caused the students' need (134, see also 152) for a technique that

would be applicable in this context, and this need led to some progress in constructing such a technique, namely the extended distributive law (135–139), to be completed later (157–163).

Ha	141	Ah, 12. Like, for this one needs to know *X*… never mind
N	145	One simply needs to know what *X* is

In 141, 145 he students wonder whether they need to know the value of *X* to proceed. The interviewer reminds them that they were able to deal with the first diagonal without knowing *X*. This leads to progress.

Ha	152	Ah, it's *XX* plus 8*X*, but I don't know, like, how this will also be *XX* plus 8*X*. Like, it HAS to be
Ne	153	XX is a square root?
Ha	154	I have the first part. This is *XX*, so this is OK
Ne	156	XX is a square root?
Ha	157	… plus 8*X*. Here I have 6*X*…
Ha	159	Ah, and 2*X*, can I do this? Because 6*X*…
Ne	160	Is XX a square root?
Ha	161	You can write this. Ah, yes, *XX* is *X* to the power 2, because it is *X* times *X*. Wait. *XX* is *X* to the power 2 plus 8*X*, wait…
Ha	163	Wait, it's X to the power 2 plus 6*X*, plus 8*X*, but there is also, like, plus 12. Ah, so, like, plus 12 because this is bigger by 12. Understand?
Ne	164	Like, yes, it's the same thing but this is bigger by 12

In 152–164, the students construct the extended distributive law in the context of the comparison of the two diagonals. We denote this constructing process by C1 since it leads to a student's construct that corresponds to knowledge element E1. Utterance 163 has been interpreted as the completion of C1: Even though the students may not be aware that they have used a new law, they have, in fact, for the first time applied the extended distributive law. Moreover, the difference of 12 between the two diagonals becomes significant in 163–164. The students use this soon thereafter for justifying the DPP—they construct C, the construct corresponding to knowledge element E, a justification by means of algebraic manipulation.

A number of comments on these constructing processes are in order.

Nesting During C1, the students have recognized as relevant and built-with the simple distributive law, in 135–136 as well as in 157–163; elaborating the inappropriate $6X(X+2) = 6XX + 6X2$ (in 135–136) appears to have helped them bridge from the simple to the extended distributive law. Not only are R- and B-actions with previous constructs, such as the simple distributive law, nested in C1 but C1 itself is nested in C.

Need The need for C1 (134, 152) derives from the mathematical situation as discussed above. A need for C can be identified from the beginning (111), but this appears to be an external need imposed by the worksheet. In fact, the students had earlier stated the DPP and referred to it without exhibiting any need for justifying it. However, as they progress, this need becomes an internal personal one as expressed by the “it HAS to be” in 152, interpreted as “must be equal in order for an algebraic comparison to become possible”.

Vertical mathematisation When approaching C1—constructing a new algebraic law—the students enter an “adventure” without any knowledge about the needed mathematical structures; they have to discover as well as to construct these structures. The C1-action thus makes the C-action, within which it is nested into a deep holistic construction, which goes beyond the specific construction of the DPP justification, and in which the construction of an unfamiliar algebraic structure is nested. In this sense C is an activity of vertically reorganising previously constructed mathematical knowledge into a new mathematical structure, which is the AiC definition of abstraction.

Building-with versus constructing We claim that HaNe built-with the simple distributive law (113–121, 135–136), but constructed the extended one (154–163). The difference between building-with and constructing lies in the students’ personal learning history: They had previously constructed the simple law and here applied it —built-with it—in a somewhat different context, whereas they had never yet met the extended law and hence needed to construct it as a new (to them) mathematical law. Other students, for whom the extended law was a previous construct, would presumably have built-with it to justify the DPP.

Co-constructing The above analysis did not attempt to separate the roles of Ha and Ne in the constructing process. While Ha is leading in 152–163, Ne has made two possibly crucial contributions: the inappropriate but helpful application of the simple distributive law in 136, and a three times repeated question whether *XX* is a square root, in 153, 156, and 160 that led to Ha’s “*XX* is *X* to the power 2” in 161. Social interaction during constructing processes will be further discussed below.

Unexpected constructs A careful re-analysis of the data (Kidron and Dreyfus 2009) revealed an additional C-action, to be called C2, that had not been predicted by the a priori analysis: the transition from a procedural mode, in which students ‘do’ expressions (115, 121, 134, 139) to an object mode, which allows to ‘trade’ expressions against each other because they ‘are’ (152) something that one ‘has’ (157).

Combining constructing actions This re-analysis showed that C1 and C2 are two strands that come closer to each other and start combining in 152. The justification of the DPP was the motivation, which enticed the students to construct C1 and C2, and the combining of C1 and C2 enabled the justification. C1 and C2 had to combine in order to enable to students to reach the goal C, the justification of the DPP. This could not have happened in the process mode afforded by C1. The transition C2 to the object mode was necessary for the justification to be completed. Hence, C1 and C2 are interacting parallel constructions, which complete and reinforce each other, and the combining between C1 and C2 constitutes C.

The aspects discussed here, in particular nesting, need, unexpected constructs, social interaction during constructing, and combining constructions in justification have been observed in other research studies; in the next section, some of these aspects will be discussed more generally and connections will be established.

Other AiC Based Studies

In this section, a selection of other research studies that used AiC as theoretical framework are reviewed without being described in detail. The reader is referred to the research literature for more information on these studies. Studies have been selected for inclusion in order to point out some of the main achievements of the AiC research program.

Constructing processes may have characteristics linked to the kind of construct (a concept, a strategy, a justification), which students are constructing. A sequence of studies has established such a result for constructing justifications; this is discussed in the first subsection. In the second subsection, a sequence of studies is presented, that deals with the central issue of partial correctness of students' constructs. Both, the studies on justification and the studies on partially correct constructs show the analytic power of the AiC framework and the RBC-model associated with it. In the third subsection, studies relating to consolidation and the mechanisms that support it are discussed. In the fourth and fifth subsections, two components of the context and their influence on constructing processes are reviewed: the role of the social context and the role of technological and other tools in the construction of knowledge.

Constructing Justifications

The nesting of two combining constructing actions C1 and C2 within a more encompassing constructing action C in the case of HaNe constitutes a rather elaborate interaction between constructing actions going on in parallel. Kidron and Dreyfus (2010a) have pointed out the importance of such interacting parallel constructing actions. They identified this specific pattern of interaction as being typical for constructing a justification. They established this in the case of justifying the second bifurcation point in a logistic dynamical system by a solitary learner L, an experienced mathematician. L's motivation for finding a justification drives her learning process. The researchers inferred her epistemic actions from her detailed notes during the learning experience and from her interaction with the computer. They found an overarching constructing action C, within which four secondary constructing actions were nested. These secondary constructing actions relate to different modes of thinking: numerical (C1), algebraic (C2), analytic (C3), and visual (C4). They are not linearly ordered but took place in parallel and interacted. Interactions included branching of a new constructing action from an ongoing one, combining or recombining of constructing actions, and interruption and resumption of constructing actions. Here, the combining of constructing actions is of particular interest.

L aimed to justify results obtained empirically from her interaction with a computer. Her aim was not to convince herself or others, nor was she looking for conviction in the logical sense of the term; rather, she wanted to gain more insight into the phenomena causing the second bifurcation point. The term enlightenment, introduced by Rota (1997) seems appropriate to express her interpretation of the word justification. Rota also pointed out that contrary to mathematical proof, enlightenment is a phenomenon, which admits degrees. Kidron and Dreyfus (2010a) show how, at each of three successive stages during L's learning experience, combining constructing actions indicate steps in the justification process that lead to enlightenment.

The relationship between combining constructions and justification has been confirmed in other contexts with students of different age groups dealing with different mathematical topics. One of these is the case of HaNe discussed above. Still another one has been briefly discussed by Kidron and Dreyfus (2009) elsewhere. Combining of constructing actions leads to enlightenment, not in the sense of a formal proof of the statement the learner wants to justify but as an insight into the understanding of the statement. This observation gives an analytic dimension to the RBC model and to its parallel constructions aspect: It allows researchers to use RBC analysis in order to identify a learner's enlightening justification.

Moreover, the analysis of the relationships between justification and parallel constructions led to the realization that often a weak and a strong branch are involved in the combining constructions, and that reinforcement of the weak branch plays a crucial role in the construction of a justification. The realization that a weak and a strong branch combine considerably strengthens the theoretical root of the RBC-model in Davydov's ideas as exposed above. Indeed, reinforcing the weak branch towards combination of constructions closely matches the description of the genesis of abstraction as expressed by Davydov's (1990) method of ascent, according to which abstraction starts from an initial, simple, undeveloped first form, which need not be internally and externally consistent, and ends with a consistent and elaborate final form.

Just like the case of HaNe, the case of L demonstrates vertical mathematisation representing a process of constructing new mathematical knowledge within mathematics and by mathematical means. These processes often include a kind of insight or 'AHA'. This expresses that the reorganisation processes of the already constructed pieces of knowledge into a new construct are driven and strengthened by the genuine and creative mathematical thinking of the learner.

Partially Correct Constructs

It is an open secret that students' correct answers sometimes hide knowledge gaps. On the other hand, incorrect answers often overshadow substantial knowledge students have constructed. These two phenomena, which can reflect aspects of partially correct knowledge, raise questions about the essence of partially correct knowledge and about its emergence. It is important to understand situations in which partially correct knowledge emerges because these situations are very common, and because of the role of existing knowledge in the constructing of further knowledge.

Ron (2009) proposed the term *Partially Correct Construct* (PaCC), as a general term for constructs that only partially match the corresponding mathematical knowledge elements that underlie the learning context. Obviously one cannot expect that a student will construct every aspect and meaning of a knowledge element. In this sense, knowledge is always partial. Thus, discussion of PaCCs requires clarifying with respect to which whole entity a construct is partially correct. For that reason the research on PaCCs was restricted to intentional learning situations, like school learning, which is directed by teachers and designers to the constructing of specific knowledge elements that can be identified by an a priori analysis. Hence, the identification of a construct as a PaCC is always related to a specific learning context and requires a detailed a priori analysis of the content and of the knowledge elements that the student is intended to construct.

In a study in the content domain of elementary probability, Ron et al. (2010) found that situations in which articulations or actions of a student seem inconsistent with other articulations or actions of the same student, can be explained, at least in some cases, by the identification of some of the student's constructs as PaCCs. The researchers used micro-analysis of students' knowledge constructing processes by means of the epistemic actions of the RBC-model to identify PaCCs. They found PaCCs that arose in the early stages of a learning activity, when all the student's articulations and actions were correct; in other words, the PaCCs arose long before the student's actions raised any clues for the existence of the PaCC or for an expected difficulty.

Knowledge constructing processes, in which PaCCs emerge and are possibly consolidated, take place in parallel and simultaneously with the construction and consolidation of knowledge that does fit the learning aims. These processes are not different in their essence from other processes of knowledge constructing as described in the AiC framework and they are based on the same epistemic actions. Characteristics that are specific to the knowledge construction processes that lead to PaCCs find their expression in different types of PaCCs. The partiality of the fit between the student's construct and the corresponding mathematical knowledge element can be related to the building blocks of the knowledge elements and/or to the context in which the student recognizes the construct as relevant and makes use of it. This distinction is a basis for defining two categories of PaCCs: structural PaCCs and contextual PaCCs (Ron et al. 2009).

Three types of structural PaCCs were identified: a missing-element PaCC in which at least one of the constituent elements of the knowledge element is missing from the student's construct, an incompatible-element PaCC in which the student's construct includes an element that contradicts the mathematical knowledge element, and a disconnected-element PaCC which is characterized by disconnected constituent elements in the student's construct. The two types of contextual PaCCs that were identified are a narrow-context PaCC, where the student recognizes the construct as relevant in a too narrow a context, and a wide-context PaCC, where the student implements his constructs in a context that is wider than warranted.

In summary, PaCCs are useful as explanatory tools for correct answers based on (partially) faulty knowledge and for wrong answers based on largely correct knowledge. Ron's research shows that AiC is a suitable framework for defining the notion of PaCC and that the RBC-model is an efficient tool for identifying PaCCs and their nature.

Consolidation

Consolidation of students' constructs has been conceptualized and studied within the AiC framework by Tsamir and Dreyfus (Dreyfus and Tsamir 2004; Tsamir and Dreyfus 2002), as well as by Monaghan and Ozmantar (2006). Dreyfus and Tsamir (2004) proposed the criteria of immediacy, self-evidence, confidence, flexibility, and awareness, as indicating that a construct has been consolidated. The study of consolidation usually requires data taken over a longer period than the study of constructing; typically studying consolidation requires data from several subsequent activities.

Using a study of a 10th grade student learning about the comparison of infinite sets, Dreyfus and Tsamir (2004) have identified mechanisms of consolidation. In particular, consolidating a recent construct during building-with this construct is the most frequent and most easily observed one. Another mechanism, consolidating a recent construct when recognising it as an object of reflection, often stems from opportunities for reflection provided to students (e.g., requests for written reports). Reflection tends to lead to the use of more elaborate language about the construct, expressing a more acute and fine-tuned awareness; and as mentioned above, awareness is an important characteristic of consolidation. Dreyfus et al. (2006) have identified a third mechanism of consolidation: Consolidating a previous construct as it is used as a building block in the course of a new constructing process. The example of HaNe above provides an excellent example of this: The students used the simple distributive law as building block when constructing the extended one, and in the process became progressively quicker, more flexible and more self-confident in applying the simple distributive law. An independent instance of the same mechanism of consolidation is a student's consolidation of her construct of derivative as limit during the process of constructing Euler's numerical method of solving differential equations (Kidron 2008).

Notwithstanding these and several other examples of apparently clear consolidation, the question how well a construct has been consolidated is a delicate one. This has been shown by Tsamir and Dreyfus (2005), who showed that under slight variations of context, knowledge structures that have apparently been well consolidated may become inactive and subordinate to more primitive ones. In other words, even when a construct has apparently been consolidated, it is a delicate issue to determine the extent of such consolidation.

Social Interaction in Constructing Processes

In the above example, Ha and Ne were mostly treated as a single student. While this was a conscious and acceptable methodological decision by the researchers, it provides only a limited view of the learning processes that occur. Dreyfus et al. (2001) have considered processes of abstraction in pairs of collaborating peers and investigated the distribution of the process of abstraction in the context of peer interaction. This was done by carrying out two parallel analyses of the protocols of the work of the student pairs, an analysis of the epistemic actions of abstraction as well as an analysis of the peer interaction. The parallel analyses led to the identification of types of social interaction that support processes of abstraction. In classrooms, however, the situation is often even much more complex. Abstraction often takes place in interacting groups of students. Hence, the focus ideally should be on groups as composed of individuals and two dual issues become central: On the one hand, constructing by individual students, and on the other, the knowledge shared by the group. Hershkowitz et al. (2007) dealt with the relationship and interaction between these two dual issues. Their data emphasize the interactive flow of knowledge from one student to the others in the group, until they reach some shared knowledge—a common basis of knowledge, which allows them to continue the construction of further knowledge in the same topic together.

The issues involved in knowledge construction by groups of students are too complex to allow detailed discussion here. However, some of these issues will be discussed from a wider perspective in the concluding section.

Computer Tools and Other Artefacts

Kidron and Dreyfus (2010b) have re-examined the study of L's justification of bifurcations in a dynamical system described above with a view to how instrumentation led to constructing actions and how the roles of the learner and a computer algebra system (CAS) intertwine during the process of constructing the justification. The main contribution of this research lies in showing that certain patterns of epistemic actions, specifically those of branching and combining, have been facilitated by certain contextual factors, specifically the CAS context. They found that the branching and combining patterns have been enabled by the work with the CAS. This is due to the fact that the computer provides a context that is very rich in resources. The richness of these resources activates the branching of constructions even if the learner is unable to immediately make sense of the input provided by the CAS. Constructions are interrupted by lack of knowledge. Nevertheless, the seeds for the future combinations may already be present. The fact that the CAS can perform the computations even if the learner does not really understand its mechanism encourages the learner to make sense of the rich resources offered by the CAS. Therefore the branching, interruptions after branching and resumptions of the interrupted constructions were necessary stages preceding the integration of the knowledge structures. The combining process, which ends in the integration of knowledge structures, was facilitated by the potential offered by the CAS and the learner's ability to make sense of the resources offered by the computer. The relations between the learner and the computer as a dynamic partner were different in the branching and in the combining phases: the researchers suggested that the computer had the upper hand during branching and the learner took command during combining.

A recent study by Weiss (2010) also focuses on the role of tools in the construction of knowledge; more specifically, Weiss considers teaching and learning situations that utilize the potential of an analogical model for creating meaningful abstraction processes. Such situations combine challenging tasks that have a high potential for abstraction with an analogical model, which supports students in handling the task. In order to create a meaningful combination between the challenging task and the analogical model, Weiss has developed model-based tasks—tasks that lead naturally to using the analogical model. He then described and analysed the role of the analogical model in knowledge constructing during model-based tasks with a high potential for abstraction.

The subject matter that was chosen for designing the model-based construction tasks is taken from the domain of fractions, and more specifically the complete-to-whole rule, which says that if the completion of fraction A to a whole unit is smaller than the completion of fraction B to a whole unit, then fraction A is bigger than fraction B. Weiss developed a unit for 4th graders based on an analogical model. In the last decade researchers have been calling for more emphasis on linear models in the learning of fractions in general and of fraction comparison in particular. In light of the above, the tower-of-bars model, an analogical model for fractions, was chosen as model.

For the purpose of analysing the role of the model in knowledge construction, AiC turned out to be insufficient. Hence, Weiss also used RME (Realistic Mathematics Education)—a theoretical framework developed at the Freudenthal Institute—which is also dealing with the role of mathematical models in the learning of mathematics based on the emergent models approach of Gravemeijer (1999). Transcripts were subjected to a dual-lens analysis by using the two theoretical frameworks: RBC analysis focusing on constructing actions and RME analysis of the role of the model with the emphasis on the transition between Model-Of and Model-For modes according to the emergent models approach.

Weiss found that among the 21 students (out of 24) who had constructed the complete-to-whole rule there was a linkage between the transition from Model-Of to Model-For and knowledge construction. In addition to that, three empirical linkages between the functionality of the model (taking the RME approach) and knowledge constructing (taking the RBC approach) were discovered: The linkage between the model as a tool for visual reasoning and comparative building-with; the linkage between the model as cognitive desktop and connective building-with; and the linkage between the model as a tool for mental justification and justifying during knowledge construction and consolidation. These three empirical linkages reveal an important aspect of the analogical model as a tool for mathematical reasoning, refine and enrich the descriptive language of the RBC methodology, and link RBC to RME empirically as well as theoretically.

Concluding Remarks

AiC has been successfully used to analyse the constructing of abstract mathematical knowledge by students aged nine to adult, and in many mathematical content domains including fractions, beginning algebra, probability, geometrical proofs, rate of change, function transformations, integration, bifurcations in dynamical systems. The longitudinal dimension of the studies varied from a single session to sequences of ten and more lessons. A large variety of learning environments have been studied, and social settings ranged from activities of individuals via tutoring situations, and small-group work to (teacher-led) classroom discussions. Researchers' aims have included the relationships between affect, creativity and constructing (Williams 2002, 2011), issues of social interaction when co-constructing, the role of technological tools in knowledge construction, questions of cognition relating to justifications and conceptual change, and others. This has rendered the use of the RBC-model a well-validated methodological and analytical tool for research. It has also led to instances where the micro-design of activities was improved on the basis of the RBC-microanalysis of student's learning processes (e.g., Kouropatov and Dreyfus 2011).

In the course of these research studies, it became apparent that the RBC-model in fact carries a dual role as a methodological tool and as a theory in development. This was first pointed out by Hershkowitz et al. (2001): "Our definition is thus a product of our oscillating between our theoretical perspective on abstraction and experimental observations of actions (experimental data)" (p. 202). Hershkowitz (2009) discussed this phenomenon in some detail and named it the contour lines (boundaries) between the theoretical framework and the methods and methodological tools within the same research. She argued that these boundaries may be flexible and even a bit vague in the sense that the same scheme or model, in this case the RBC-model, may serve as a theoretical framework in one piece of research, as a methodological tool in a second one, and as both of them in a third piece of research.

This apparently came about as follows: The researchers approached the problem of investigating the construction of abstract mathematical knowledge. They began with a first hypothesis for a scheme or a model, using both theoretical considerations and the analysis of considerable amounts of data. In this undertaking, they were led by the need to give theoretical expression to the specific characteristics of their data, which pointed to constructing of knowledge by means of mathematical thinking. In the process, they took into account and incorporated elements of existing theories. Abstracting, for example, was taken as human activity of mathematization, specifically vertical mathematization. They realized the importance of contextual factors and described illustrative examples in different contexts.

At that stage, a circular situation arose where theory stemmed from the analysis of data, and the analysed data served as evidence for validating the theory. The researchers were quite aware of this situation and explained: "This definition [of abstraction] is a result of the dialectical bottom-up approach described above… a product of our oscillating between our theoretical perspective on abstraction and experimental observations of students' actions, actions we judged to be evidence of abstracting" (Hershkowitz et al. 2001, p. 202). It is clear that for analysing the above actions, we had to use some basic methodologies, which fit transcript analyses of an individual and the more complicated analysis of cognitive and interactive work within pairs and groups. The three epistemic actions, recognizing, building-with and constructing, and the dynamically nested relationships between them were hypothesized as the main building blocks of the model, and at the same time used as the lens and compass to describe and interpret the data analyses themselves. Such a situation held for the first steps towards the validation of the model as a theoretical framework.

Further research made it clear that the RBC+C model for AiC is an appropriate theoretical tool and methodology to describe and provide insight into processes of abstraction and consolidation in a wide range of situations. About ten years of research and more than 50 research publications, contributed by more than a few people, separate the 'birth' of the AiC framework and the RBC+C model, as an empirically based theoretical framework, from recent publications that use this model as one of two or more 'conceptual frameworks' (Bikner-Ahsbahs et al. 2010; Weiss 2010; Wood et al. 2006). These studies show some maturation of the model as a theoretical framework and as a methodology. In each case, the researchers needed two or more conceptual frameworks in their study and AiC was one of them. The RBC+C model is then not any more the focus of the study but exemplifies the flexible contour lines between a model as theory and a model as methodological tool: the model aims to serve as a framework for describing, analysing and interpreting a human mental activity and at the same time is appropriate for exploring individual student mental activity as well as for exploring collective mental activity that is distributed in a group or a classroom among different individuals.

The model with its three epistemic actions, has a very general nature, general in the sense that it can be used in many and varied contexts. The nested relationships among the epistemic actions of the RBC+C model are global and the three actions of the model are observable and can be identified. Therefore the model lends itself easily to be adapted and to contribute to research in many different contexts of constructing abstract knowledge.

However, the notion of collective abstracting raises many questions, such as: What can we learn from this kind of research about abstracting, or more generally about learning processes and knowledge constructing in classrooms? What can we say about the individual students in the classroom and the classroom community as a community that consists of all the individuals who belong to this community? Do we have a methodology/methodological tool by which we will be able to conduct the kind of research that gives some answers to such questions?

Looking back on the development of AiC over the years, one may discern a trend from investigating an individual learner or dyad with an interviewer in a laboratory setting via investigating focus groups in a working classroom, to investigating students' knowledge construction, and shifts of the constructed knowledge in a working mathematical classroom. The first phase served to develop the AiC framework and the RBC+C model, whereby the RBC+C model was used in two parallel roles: as a methodological tool for analysing the data and for validating the theory. In the second phase, the RBC+C model was applied for analysing students' processes of abstraction as they worked in a group in a working classroom. This is a big challenge because of the many variables that play a role in a whole class situation and the potentially messy data. Hence we had to deal with the issue of the shared knowledge of a group of individual students as they construct and consolidate it in the mathematical classroom. In the third phase, which is an even bigger challenge, we aim to develop a methodology, based on the RBC model and other methodological tools, in order to coordinate analyses of the individual, the group in the classroom and the classroom collective in a working mathematical classroom. Tabach et al. (2014) and Hershkowitz et al. (2014) have recently presented first results from this line of research.

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Digital Technology in Mathematics Education: Why It Works (Or Doesn't)

Paul Drijvers

Abstract The integration of digital technology confronts teachers, educators and researchers with many questions. What is the potential of ICT for learning and teaching, and which factors are decisive in making it work in the mathematics classroom? To investigate these questions, six cases from leading studies in the field are described, and decisive success factors are identified. This leads to the conclusion that crucial factors for the success of digital technology in mathematics education include the design of the digital tool and corresponding tasks exploiting the tool's pedagogical potential, the role of the teacher and the educational context.

Keywords Didactical function · Digital technology · Instrumentation

Introduction

For over two decades, many stakeholders have highlighted the potential of digital technologies for mathematics education. The U.S. National Council of Teachers of Mathematics, for example, in its position statement claims that "Technology is an essential tool for learning mathematics in the 21st century, and all schools must ensure that all their students have access to technology" (NCTM 2008). ICMI devoted two studies to the integration of ICT in mathematics education, the second one expressing that "…digital technologies were becoming ever more ubiquitous and their influence touching most, if not all, education systems" (Hoyles and Lagrange 2010, p. 2).

However, the integration of digital technology still confronts teachers, educators and researchers with many questions. What exactly is the potential of ICT for learning and teaching, how to exploit this potential in mathematics education, does digital technology really work, why does it work, which factors are decisive in

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making it work or preventing it from working? What does a quarter of a century of educational research and development have to offer here?

Of course, these questions are not clearly articulated. What do we mean by "it works"? Does this mean that the use of digital technology improves student learning, invites deeper learning, motivated learning, more efficient or more effective learning? Does it mean that ICT empowers teachers to better teach mathematics? And, concerning the effect of educational research, do studies on digital technology "work" in the sense that they provide answers to these questions, or do they just help the researcher to better understand the phenomenon, and as such contribute only indirectly to improving mathematics education? My interpretation of "why it works" in the title of this contribution includes both learning and teaching, and also refers to learning on the part of the researcher.

In this paper I will explore the question of "why digital technology works or does not" by briefly revisiting a number of leading studies in the field, that are paradigmatic for a theme, approach, method, or type of results. For each of these studies, the focus is on what they offer on identifying decisive factors for learning, teaching and research progress. As such, this contribution reports on a concise and somewhat personal journey through—or a helicopter flight over—the landscape of research studies on technology in mathematics education.

Framework for Case Description

How to decide which studies to include in this retrospective and even somewhat historical paper? Even if somewhat subjective and personal arguments cannot be completely ignored, the case selection is based on a number of criteria. A first criterion for including a study or a set of studies is that it really contributes to the field, by providing a new perspective, a new direction or is paradigmatic for a new approach to the topic. An indication for this is that the study is frequently quoted and has led to follow-up studies. A second criterion for inclusion is that the study under consideration contributes to theoretical development in the field of integrating technology in mathematics education, and as such promotes thought in this domain. A third and final criterion for the set of cases presented in this paper as a whole, is variation. Variation does not only apply to theoretical perspectives, but also to the mathematical topic addressed in the study, the type of technological tools used, and the pedagogical functionality of the digital technology. Concerning this functionality, we use an adapted version of the model by Drijvers et al. (2010a) which distinguishes three main didactical functionalities for digital technology: (1) the tool function for doing mathematics, which refers to outsourcing work that could also be done by hand, (2) the function of learning environment for practicing skills, and (3) the function of learning environment for fostering the development of conceptual understanding (see Fig. 1). Even if these three functionalities are neither exhaustive nor mutually exclusive, they may help to position the pedagogical type of use of the technology involved. In general, the third function is the most challenging one to exploit.

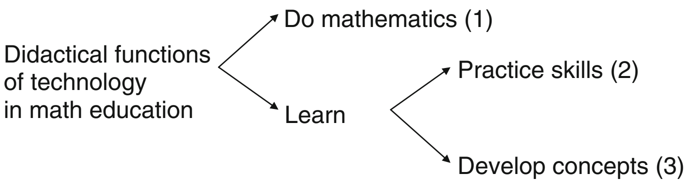

Fig. 1 Didactical functions of technology in mathematics education

How to discuss the selected studies in a short frame in a way that does justice to them and in the meanwhile serves the purpose of this paper? First, a global description of each case will be presented, including the mathematical topic, the digital tool and the type of tool use. Next, I will explain what is crucial and new in the study, and why I decided to include it. Then the theoretical perspective is addressed. Each case description is closed by a reflection on whether digital technology worked well for the student, the teacher or the researcher, and which factors may explain the success or failure.

Case Descriptions

Case 1 Concept-First Resequencing by Heid (1988)

The first case description concerns a study reported by Heid (1988), which is considered as one of the first leading studies into the use of digital technology in mathematics education. The study addresses the resequencing of a calculus course for first-year university students in business, architecture and life sciences using computer algebra, table tools and graphing tools that were used for concept development (branch (3) in Fig. 1). The digital technology allowed for a 'concept-first' approach, which means that calculus concepts were extensively taught, whereas the computational skills were treated only briefly at the end of the course. The results were remarkable in that the students in the experimental group, who attended the resequenced, technology-intensive course, outperformed the control group, who attended a traditional course, on conceptual tasks in the final test, and also did nearly as well on the computational tasks that had to be carried out by hand. The subjects in the experimental group reported that the use of the computer took over the calculational work, made them feel confident about their work and helped them to concentrate on the global problem-solving process.

One of the reasons to discuss the Heid study here is that it is paradigmatic in its approach in that its results form a first 'proof of existence': indeed, it seems possible to use digital technology as a lever to reorganize a course and to successfully apply a concept-first approach, using digital technology in the pedagogical function of enhancing concept development, without a loss of student achievement on by-hand skills.

From a theoretical perspective, Heid's notion of resequencing seems closely related to Pea's distinction of ICT as amplifier and as (re-)organizer (Pea 1987). The former refers to the amplification of possibilities, for example by investigating many cases of similar situations at high speed. The latter refers to the ICT tool functioning as organizer or reorganizer, thereby affecting the organization and the character of the curriculum. In the light of that time's thinking on the role of digital tools to empower children to make their own constructions (Papert 1980), the organizing function of digital technology was often considered more interesting than the amplification.

So did technology 'work' in this case? Yes, it did at the level of learning: the final test results of the experimental group turned out to be very satisfying. And yes, it also worked at a more theoretical level, as the notions of resequencing and concept-first approach were operationalised and made concrete. Now why did it work, which factors might explain these positive results? Even if nowadays we would not consider the digital technology available in 1988 as very sophisticated, I would guess that at the time the approach was new and motivating to the students, and the representations offered by the technology did indeed invite conceptual development. Decisive, however, I believe was the fact that the researcher herself designed and delivered the resequenced course. I conjecture that she was very aware of the opportunities and constraints of the digital technology, and was skilled in carefully designing activities in which the opportunities were exploited, and in teaching the course in a way that benefitted from this. Whether the course, if delivered by another teacher, would have been equally successful, is something we will never know.

Case 2 Handheld Graphing Technology

The second case description concerns the rise of handheld graphing technology in the 1990s. For several reasons, graphing calculators became quite popular among students, teachers and educators at that time (for an overview, see Trouche and Drijvers 2010). Teaching materials were designed that made extensive use of these devices and researchers investigated the benefits of this type of technology-rich activities (Burrill et al. 2002). Very much in line with the work by Heid (see Case 1), the focus of much of this work is on the pedagogical function of concept development. The main idea seems to be that students' curiosity and motivation can be stimulated by the confrontation with dynamic phenomena that invite mathematical reasoning, in many cases concerning the relationships between multiple representations of the same mathematical object. In many cases this mathematical object is a function, but examples involving other topics, such as statistics, can also be found.

As an example, Fig. 2 shows two graphing calculator screens which students set up to explore the effect of changes in the formula of the linear functions Y1 and Y2 on the graph of the product function Y3. This naturally leads to questions about properties of the product function and the relationship with properties of the two components (Doorman et al. 1994).

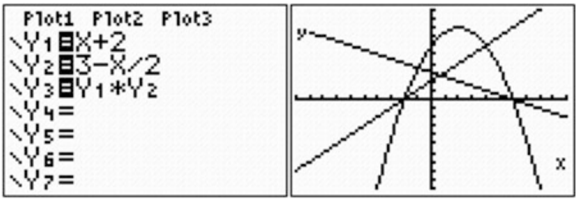

Fig. 2 Exploring the product of two linear functions

A paradigmatic study in this field is done by Doerr and Zangor (2000). The researchers report on a small-scale qualitative study, in which 15–17 year old pre-calculus students study the concept of function using a graphing calculator, with a focus on the pedagogical tool functionality of concept development (branch (3) in Fig. 1). The authors identify five modes of tool use, namely computation, transformation, data collection and analysis, visualisation, and checking. The results show that the teacher was crucial in establishing and reinforcing these modes of tool use, for example by setting up whole-class discussions 'around' the projected screen of the graphing calculator, to develop shared meaning and avoid a too individual development of tool use and mathematical insight. The researchers stress that using digital technology in mathematics teaching is not independent from the educational context and the mathematical practices in the classroom in particular.

The main reason to discuss the Doerr and Zangor study here is that it highlights the importance of the educational context in studies on the effect of digital technology, and the crucial role of the teacher in particular. The relevance of the educational context has later been elaborated in the notion of Pedagogical Map by Pierce and Stacey (2010). Concerning the teacher, she establishes a culture of discussing graphing calculator output in a format that is close to what is called a 'Discuss-the-Screen orchestration' in Drijvers et al. (2010b) and by these means contributes to the co-construction of a shared repertoire of ways to use the graphing device.

From a theoretical perspective, Doerr and Zangor use frameworks on learning as the co-construction of meaning, and that the "features of a tool are not something in and of themselves, but rather are constituted by the actions and activities of people" (p. 146). Even if this may sound somewhat trivial nowadays, during the period of initial enthusiasm these were important insights with consequences for the role of the teacher, who led the process of sharing and co-construction, particularly in the case of personal, private technology.

So did technology 'work' in this case? Doerr and Zangor did not assess learning outcomes, but it seems that the students developed a rich and meaningful repertoire of ways to use the graphing calculator for their mathematical work. Why did this work, which factors might explain these findings? My interpretation is that the use of digital tools for exploratory activities which target conceptual development is not self-evident, as it is hard for students, without the mathematical background that we as teachers have, to 'see' the mathematics behind the phenomena under consideration. It is here where the teacher comes in, and where the study becomes very informative for both teachers and researchers. In this case, I believe that the fact that the teacher herself was skilled in using the graphing calculator, was aware of its limitations, and was willing to explicitly pay attention to the co-construction of a shared and meaningful repertoire of tool techniques explains the results. As in the Heid study described in Case 1, the role of the teacher seems to be an important factor. The issue of how to deal with private, handheld technology is very relevant nowadays, as many students have smart phones with sophisticated mathematical applications, and again, teachers are faced with the danger of too individually constructed techniques and insights.

Case 3 Instrumental Genesis

By the end of the previous century, French researchers who were working on the integration of computer algebra and dynamic geometry in secondary mathematics education felt the need to go beyond the then current theoretical views. Even if they still experimented with explorative tasks, such as finding the number of zeros at the end of $n!$ (Trouche and Drijvers 2010), a theoretical perspective was needed that would do justice to the complex interaction between techniques to use the digital technology, conventional paper-and-pencil work and conceptual understanding. This led to the development of the instrumental genesis framework, or the instrumental approach to tool use (Artigue 2002; Guin and Trouche 1999; Lagrange 2000). Even if there are different streams within instrumentation theory (Monaghan 2005), it is widely recognized that the core of this approach is the idea that the co-emergence of mental schemes and tool techniques while working with digital technology is essential for learning. This co-emergence is the process of instrumental genesis. The tool techniques involved have both a pragmatic meaning (they allow the student to use the tool for the intended task) and an epistemic meaning, in that they contribute to the students' understanding. Rather than exploration, the reconciliation of digital tool use, paper-and-pencil use, and conceptual understanding is stressed (Kieran and Drijvers 2006).

A paradigmatic study in this field is the one by Drijvers (2003) on the use of handheld computer algebra for the learning of the concept of parameter. Four classes of 14–15 year old students worked on activities using a handheld computer algebra device both in its role of mathematical tool and for conceptual development (branches (1) and (3) in Fig. 1) to develop the notion of parameter as a 'super-variable' that defines classes of functions and that can, depending on the situation, play the different roles that 'ordinary' variables play as well. The results of the study include detailed analyses and descriptions of techniques that students use, and the corresponding expected mental scheme development. Figure 3 provides a schematic summary of such an analysis for the case of solving parametric equations in a computer algebra environment (Drijvers et al. 2012).

The main reason to discuss this study here is that by providing elaborated examples it contributes to a concrete and operationalised view on the schemes and techniques that are at the heart of the instrumental approach. The study shows that the instrumental approach is a fruitful perspective that can provide tangible guidelines for both the design of student materials and the analysis of student behaviour.

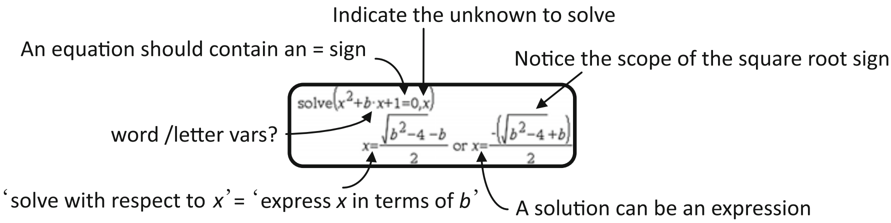

Fig. 3 Conceptual elements related to the application of the solve command

From a theoretical perspective, apart from the concretisation of the notions of schemes and techniques, the author integrated this with a more general view on mathematics education, namely the theory of realistic mathematics education (Freudenthal 1991). The two perspectives seemed to be complementary and both provided relevant guidelines for design and analysis.

So did technology 'work' in this case? No and yes. The conclusions on the learning effects of the intervention are not very clear. Even if the students learned much about the concept of parameter, their work still showed weaknesses both in the use of the tool and in the understanding of the mathematics. This suggests an incomplete instrumental genesis. Factors that may explain these findings are (1) the difficulty of the mathematical subject for students of this age, (2) the complexity of the computer algebra tool, and (3) the efforts and skills needed by the teachers to not only go through their personal process of instrumental genesis, but also to facilitate the students' instrumental genesis by their way of teaching. The latter aspect was addressed more explicitly later in the notion of instrumental orchestration (Trouche 2004; Drijvers and Trouche 2008). The study did work in the sense that it contributed to the researchers' understanding of the complexity of integrating sophisticated digital technologies in teaching relatively young students. The close intertwinement of the students' cognitive schemes and the techniques for using the digital technology is identified as a decisive factor in the learning outcomes of technology-rich mathematics education.

Case 4 Online Applications

With the growing availability and bandwidth of internet, researchers became interested in the potential of online interactive applications or applets for mathematics education. The advantages of online content include access without local software installation, ease of distribution and updating for developers, and permanent availability for users as long as the internet is accessible.

Many studies investigate this potential. For example, Boon (2009) explores the opportunities for teaching 3D geometry using online applets. Doorman et al. (2012) describe a teaching experiment in grade 8 focusing on the concept of function using

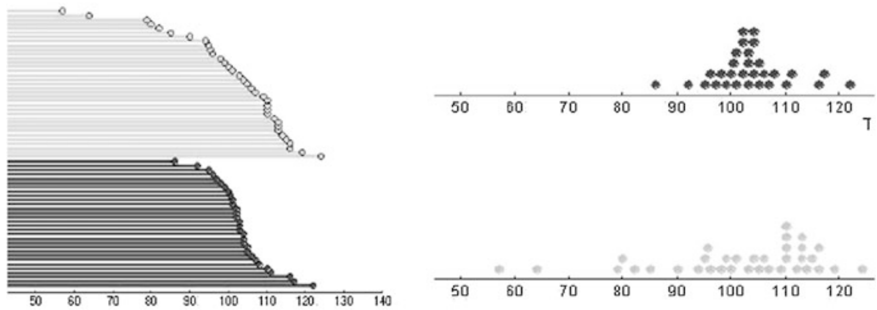

Fig. 4 Applets for investigating a small set of statistical data

an applet called Algebra Arrows[1] for building chains of operations. Apart from an instrumental perspective (see Case 3), the theoretical framework includes domain-specific theories on reification, realistic mathematics education and emergent modelling. The applet is used for concept development (branch (3) in Fig. 1). A third example is the study by Bokhove, who focuses on acquiring, practicing and assessing algebraic skills (Bokhove 2011; Bokhove and Drijvers 2012). His teaching experiments took place in grade 12 classes and made use of applets that offer means to manipulate algebraic expressions and equations.[2] The theoretical framework in this case included notions from algebra pedagogy such as symbol sense, which is expected to support skill mastery, but also elements from educational science on assessment and on feedback. In contrast to the studies described so far, the role of the digital tool in Bokhove's work includes the environment to practice skills (branch (2) in Fig. 1), which might be the easiest role, even if the design of appropriate feedback is an issue to tackle.

As a paradigmatic design research study in this field, let us now describe the work done by Bakker in somewhat more detail (Bakker 2004; Bakker and Gravemeijer 2006; Bakker and Hoffmann 2005). Bakker investigated early statistical reasoning of students in grades 7 and 8. In one of the tasks, students investigate data from life spans of two brands of batteries while using applets to design and explore useful representations and symbolizations (see Fig. 4). Clearly, the digital tools' pedagogical functionality is on concept development once more (branch (3) in Fig. 1). The design of the hypothetical learning trajectory and the student materials was informed by the development of statistics in history. In his analysis of student data, Bakker uses Peirce's (1931–1935) notions of diagrammatic reasoning and hypostatic abstraction to underpin his conclusion that the teaching sequence, including the role of digital tools, invited students' reasoning about a frequency distribution as an object-like entity, as became manifest when they started to speak about the 'bump' to describe the drawings at Fig. 4's right hand side.

[1]See http://www.fisme.science.uu.nl/tooluse/en/.

[2]See http://www.algebrametinzicht.nl/.

The main reasons to discuss Bakker's work here are not only the originality of the dedicated digital tools which meet new ideas on statistical reasoning and statistics education, and which were designed in collaboration with others (Cobb et al. 2003), but also the rich relationships with the different resources and approaches, such as the historical perspective, to inform the design.

From a theoretical perspective, it is interesting to notice that even if technology plays an important role in Bakker's study, the design and analysis are driven by theoretical perspectives from outside the frame of research on the use of technology in mathematics education, but rather from the world of mathematics pedagogy and beyond. I believe that this is a meaningful and promising approach: on the one hand, as researchers we should benefit from specific results and theories from studies on the use of digital tools in mathematics education. On the other hand, we should not forget to involve theories on mathematics education and educational science in general.

So did technology 'work' in Bakker's case? Yes, it did in the sense that the author clearly reports on conceptual development by the students involved in the study. Why did this work, which decisive factors might explain these findings? I believe that an important lesson to be learnt from this study is that design research on the use of digital technology in mathematics education should not limit itself to the study of the tools alone, but should include the tasks, and their embedding in teaching as a whole, in order to understand what works and why it works. In this case, I would guess it is the combination of the digital tools, the tasks and activities, but also the whole-class discussions, the paper-and-pencil work, the established mathematical practices, in short the educational context as a whole, that explains the result. A second lesson to learn for us as researchers is that a theoretical framework which integrates different perspectives can be very powerful for generating interesting and relevant research results.

Case 5 Mobile Mathematics

Research on the use of mobile technology in mathematics education is in its early stages but its importance is rapidly growing. It is evident that mobile technology and smart phones in particular are very popular among students and more and more wide-spread. Wireless Internet access allows for the use of mobile applications (also called midlets, Mobile Information Device applications), SMS and email services offer communication and collaboration opportunities, GPS facilities allow for geographical and geometrical activities and the tool's mobile and handheld characteristics invite out-of-school activities, for example the gathering of real-life data that inform biology or chemistry lessons (Daher 2010).

As a paradigmatic example, I now address the MobileMath pilot study carried out by Wijers et al. (2010). In this study, the tool consisted of a mobile phone with GPS facilities and a 'native' application, designed for the purpose of this game, which generated the view on the game situation and arranged communication with other teams' devices. The mathematical topic involved is geometry: teams of Grades 7 and 8 students used the GPS and the application to play an outdoor game on constructing parallelograms (including rectangles and squares), and could

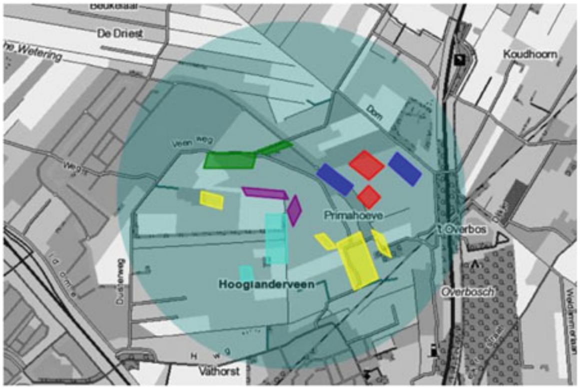

Fig. 5 Map of students' parallelogram constructions using GPS

eventually destroy other groups' geometrical shapes. This so-called MobileMath game aims at making students experience properties of geometrical figures in a lively, embodied game context. While playing the game, students look at the map to imagine where they want to make a shape, walk to the location for the first vertex to enter this location in the mobile device, which generates a dot on the map, walk to the location of the second vertex of their imagined shape which provides a line on the screen connecting the first vertex with the current (moving) location, etc. The map in Fig. 5 shows some student constructions. The deconstruction option brought extra challenge and competition into the game. From the data the researchers conclude that MobileMath adds a geometrical dimension to the world, transforming it into a game board. MobileMath also invites mathematical activity, such as the (re) discovery and use of characteristics of squares, rectangles and parallelograms, and taking notice of geometrical aspects of the world.

One reason to discuss this study here is that the digital tool—the modern smart phone with GPS facilities rather than an 'old school' computer—acts in multiple ways, and its use includes all branches of the diagram displayed in Fig. 1. The device enables the exploration of properties of quadrilaterals [branch (3)]. It also allows for practicing the construction of parallelograms, which meets branch (2). And finally, the tool also functions as an environment to outsource the mathematical work, in this case the drawing of the shapes, to, [branch (1)].

As seems to be the case in other studies on the integration of mobile technology in mathematics education, the theoretical perspective used by Wijers et al. (2010) is different from the frameworks common in most research on technology in mathematics education. It is closely associated with frameworks from studies on serious gaming, and focuses (1) on student engagement and (2) on task authenticity. Enhancing student engagement is seen as an important potential of educational games. In the MobileMath study, student engagement is stimulated by the game's hybrid reality character: on the mobile device's screen, students see the map of the reality in which they are walking, as well as the virtual geometrical shapes they are creating. Hybrid reality games are seen as beneficial for student engagement. In addition to this, the authors refer to Prensky (2001) for a model on heuristics for the design of engaging games, which include clear rules and goals, outcome and feedback, conflict, challenge and competition, and interaction. Concerning task authenticity, the authors claim that the effectiveness of learning activities can be enhanced if the tasks are authentic and realistic. In line with the framework of Realistic Mathematics Education, realistic means that problem situations presented in learning activities should be 'experientially' real to students and have meaningful, authentic problem situations as starting points, so that students experience the game's activity as making sense.

So did the digital technology 'work' in this case? As far as engagement and authenticity are concerned, the answer is 'yes'. The researchers report that the students were engaged in the game and experienced it as challenging. Apparently, the game factor, in combination with the possible attractiveness of the digital device, works out well. A second factor might be the outdoor and physical character of the game, which students may experience as a welcome change from regular classroom teaching. What is not clear yet, however, is whether these effects will persist if this type of activity were to become more common. Also, the study presented here has a small-scale pilot character and would certainly need further replication.

Case 6 Teachers' Practices and Professional Development

If we recapitulate the previous cases, in all but the last one the teachers' practices and experiences were identified as an important factor explaining why digital technology 'worked' or why it did not. Therefore, this final case focuses on the role of the teacher, teaching practices and teachers' professional development.

One of the first studies focusing on teachers' practices and professional development was the one by Ruthven and Hennessy (2002). In this study and in subsequent work (e.g., Ruthven 2007) crucial factors are identified that affect teachers integrating digital technology in their teaching. In relation to the instrumental genesis model, Trouche developed the notion of instrumental orchestration to stress the relevance of teaching practices (Trouche 2004). Case studies based on these models describe teachers' practices in relation to their opinions and beliefs (Drijvers et al. 2010b; Drijvers 2012; Pierce and Ball 2009). Another model on teachers' professional knowledge is Technological Pedagogical Content Knowledge (TPACK), which became widespread but is also criticized (Graham 2011; Koehler et al. 2007; Voogt et al. 2012).

In addressing the questions of how to prepare teachers for technology-rich teaching and how to enhance their professional development in this field, in line with the work done by Wenger (1998) on communities of practice, it is suggested that the participation in a community of teachers who co-design and use resources for teaching, can contribute to this (e.g., see Fuglestad 2007; Jaworski 2006). Digital technology in such an enterprise acts on two levels: first, the professional development concerns its use in mathematics education, and second, digital technology may support the community's work by offering online and virtual facilities for exchange. Digital technology is both the subject at stake and the vehicle to address it. Efforts have been done to exploit digital technology's potential for teachers' professional development by designing online courses.[3]

As a paradigmatic design research study in this field, let us now describe the work done by Sabra (2011) in somewhat more detail. In his PhD dissertation, Sabra describes two case studies of teachers' collaborative process of professional development in detail. In the first case ten teachers in the same school collaborate on the design of a final assessment training session and a mathematics investigation task while integrating the use of TI Nspire in their teaching. The second case study concerns a project in which eleven teachers, all members of the Sesamath community from all over France, collaboratively design resources on the concept of function that are part of the course manual. The analysis shows that the two communities develop in quite different ways, but that in both developments some critical incidents—called documentary incidents in the thesis—are decisive. The digital tools in this case include web facilities for collaborative work, file exchange and communication; the role they play for the participating teachers is best characterized by branch (1) in Fig. 1, the role of a tool for doing mathematics, or rather a tool for collaborating on the design of mathematical resources.

The main reason to discuss this study here is that its rich data including interviews, blogs and observations and its fine-grained data analysis provide a detailed insight in how communities of teachers may work (or may not) and how technology may support this.

From a theoretical perspective, Sabra uses the notion of documentational genesis as a main concept. Figure 6, taken from Gueudet and Trouche (2009), shows how this is analogue to the notion of instrumental genesis, but now addressing the level of teachers using and designing digital resources. The interesting point here, in my opinion, is that a similar framework is applied to and elaborated for different situations and different levels of technology integration.

So did technology 'work' in the Sabra study? Maybe the answer is different for the two cases that are described. In the case of the team of teachers within the same school, it seems that the digital technology does not have so much to offer, and that the professional interest of the community members does not invite a real engagement in an effective collaboration. As a result, one can wonder whether the targeted professional development really took place, and whether the community really contributed to it. In the second case of the teachers all over France, the analysis shows a very lively process of collaboration, which is clearly afforded by the digital technology and would not have been possible without it. Similar to the other cases described in this paper, it seems that decisive factors that explain the phenomena go beyond the straightforward point of the available technology. My impression is that for a school team of teachers, collaboration is far from self-evident, whereas teachers who volunteer for a role in the Sesamath project share a

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[3]E.g., see http://www.edumatics.eu/.

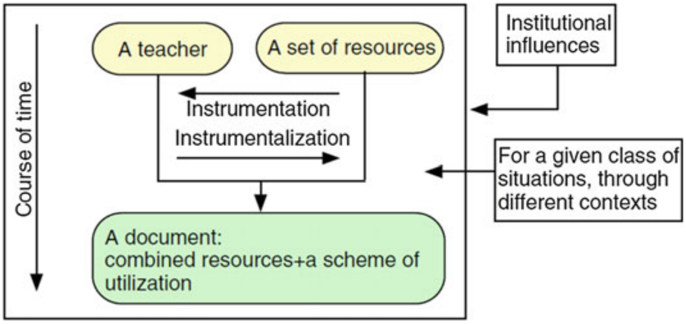

Fig. 6 Schematic representation of a documentational genesis (Gueudet and Trouche 2009)

professional interest to engage in a virtual community and in a shared process of distant collaboration. This, I would conjecture, might be the main explanation for the different results in the two cases Sabra describes.

Conclusion and Discussion

Conclusion

The—slightly provocative—question raised in the title of the paper is why digital technology in mathematics education 'works' or does not. The underlying aim was to identify factors that promote or hinder the successful integration of digital technology in mathematics education. The analysis of the six cases described in this paper show that the integration of technology in mathematics education is a subtle question, and that success and failure occur at levels of learning, teaching and research. In spite of this complexity, three factors emerge as decisive and crucial: the design, the role of the teacher, and the educational context.

The first factor concerns *design*. Cases 1, 3, and 4 reveal the crucial role of design. This concerns not only the design of the digital technology involved, but also the design of corresponding tasks and activities, and the design of lessons and teaching in general, three design levels that are of course interrelated. In terms of the instrumental genesis model, the criterion for appropriate design is that it enhances the co-emergence of technical mastery to use the digital technology for solving mathematical tasks, and the genesis of mental schemes that include the conceptual understanding of the mathematics at stake. As a prerequisite, the pedagogical or didactical functionality in which the digital tool is incorporated (see Fig. 1) should match with the tool's characteristics and affordances. Finally, even if the digital technology´s affordances and constraints are important design factors, the main guidelines and design heuristics should come from pedagogical and didactical considerations rather than being guided by the technology's limitations or properties.

The second factor concerns the role of the *teacher*, which is crucial in cases 1, 2, and 6. The integration of technology in mathematics education is not a panacea that reduces the importance of the teacher. Rather, the teacher has to orchestrate learning, for example by synthesizing the results of technology-rich activities, highlighting fruitful tool techniques, and relating the experiences within the technological environment to paper-and-pencil skills or to other mathematical activities. To be able to do so, a process of professional development is required, which includes the teacher's own instrumental genesis, or, in terms of the TPACK model, the development of his technological and pedagogical content knowledge. Case 6 suggests that technology can help the teacher to advance on this, together with colleagues in technology-supported collaboration. What seems to be an open question is how the role of the teacher changes if we consider the use of technology in out-of-school learning, gaming, and other forms of informal education (see case 5).

The third and final factor concerns the *educational context*, which includes mathematical practices and the elements of the Pedagogical Map designed by Pierce and Stacey (2010). Case 2 reveals how important it is that the use of digital technology is embedded in an educational context that is coherent and in which the work with technology is integrated in a natural way. Case 5, the MobileMath example, shows that taking into account the educational context includes attention for important aspects such as student motivation and engagement. Another factor that is not so much elaborated in the case descriptions but is important to mention here, is assessment, which should be in line with the students' activities with technology; not doing so would suggest that in the end the use of digital technology is not important. Finally, the use of digital technology may lead to an extension of the educational context towards out-of-school settings, as exemplified in case 5.

The three factors identified above may seem very trivial, and to a certain extend they are quite straightforward indeed; however, their importance, I believe, can hardly be overestimated and to really take them into account in educational practice is far from trivial.

Discussion

Let me first acknowledge that the study presented here clearly has its limitations. The discussion of the studies addressed cannot be but somewhat superficial in the frame of this paper. Also, the number of studies is small, and the choice of the studies included is not neutral. This being said, I do believe the article provides a—very rough—map of the landscape of research studies on technology in mathematics education and reveals some trends in the domain over the previous decades.

So what trends can be seen in retrospective? Globally speaking, a first trend to identify is that from optimism on student learning in the early studies towards a more realistic and nuanced view, the latter acknowledging the subtlety of the relationships between the use of digital technology, the student's thinking, and his paper-and-pencil work. A second trend is the focus not only on learning but also on teaching. The importance of the teacher is widely recognized and models such as TPACK, instrumental orchestration and the pedagogical map help to understand what is different in teaching with technology and to investigate how teachers can engage in a process of professional development. The third and final trend I would like to mention here concerns theoretical development. Whereas many early studies mainly use theoretical views that are specific for and dedicated to the use of digital technology (e.g., Pea's notions of amplifier and reorganizer in the Heid study), recent studies often include more general theories on mathematics education or learning in general, and also combine different theoretical perspectives (e.g., see the work by Bakker, using Pierce, RME, and other theoretical views).

To close off this discussion, I would like to express my strong belief that these theoretical developments are crucial for the advancements in the field. The studies addressed in this paper show strong relationships between the theoretical frameworks, the digital tools and the mathematical topics (Kieran and Drijvers 2012). We now have a myriad of theoretical approaches available in our work, including very specific theories on the use of technology in mathematics education, domain-specific instruction theories, and very general views on teaching and learning. One of the challenges in our work, therefore, is to combine and contrast the lenses each of these approaches offer (Drijvers et al. 2012). The notion of networking theories (Bikner-Ahsbahs and Prediger 2010) provides a good starting point that may help to better understand the role of digital technology in mathematics education and, as a consequence, to contribute to the learning and teaching of the topic.

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Mathematical Thinking Styles in School and Across Cultures

Rita Borromeo Ferri

Abstract A mathematical thinking style is the way in which an individual prefers to present, to understand and to think through, mathematical facts and connections by certain internal imaginations and/or externalized representations. In which way mathematical thinking styles (analytic, visual and integrated) are influence factors on the learning and teaching of mathematics is described on the basis of selected qualitative empirical studies from primary up to secondary school. Within the current MaTHSCu-project the styles are measured quantitatively by comparing mathematical thinking styles in eastern and western cultures. This study is introduced and first results are shown. Finally conclusions and implications for school are drawn.

Introduction

This citation comes from Martin Wagenschein (1896–1988), who was a pedagogue, math and physics teacher and a lecturer at several universities in Germany. Parts of his work also influenced the discussion in mathematics education in Germany. This citation is an appropriate starting point for thinking about *preferred* ways to learn and understand mathematics for one's own and as from researcher's perspective as well. How do you *like* to learn and understand mathematics? This is quite a simple question—but offers a lot of interesting answers, if pupils and students from primary up to secondary and vocational schools and university were asked:

I experienced myself, how mathematics can be opened by one teacher and closed by another. (Wagenschein 1983).

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> I understand mathematics the best way, when Mrs. D. is drawing pictures on the table, because I need these pictures also in my mind. (David, 10 years, Grade 4)
> I like to learn mathematics with numbers and symbols. Sketches do not help me really in my process of understanding. (Gloria, 16 years, Grade 10)
> My previous teacher explained fast and much and did not make any drawings… My new teacher always makes a drawing and now I understand how to come to the result, not like only by formulae and calculation. (Sarah, 15 years, Grade 9)

There are kinds of explanations which cause young people to understand mathematical methods well and others through which they only understand a little. Some individuals like pictures and visualizations or some prefer formulae and variables and again others like something in between pictures and formulae. Having a good or bad understanding of mathematics is not an unusual matter of fact happening during school time and is influenced by a lot of factors. Not only teachers also many psychologists and pedagogues still share the opinion that success and failure of learning are exclusively caused by individually different learning abilities. Similarly, it remains still unanswered why the same pupil produces bad results in a multiple-choice task in mathematics while within a math-project he or she produces extraordinary results. Mathematical abilities are probably the first explanation, which comes in mind. But mathematical abilities are not the whole answer to these phenomena. Another, meantime well-funded explanation, is based on *mathematical thinking styles* (*visual, analytic and integrated thinking style*), which are preferences for using our mathematical abilities.

In the following, the theory of *M*athematical *T*hinking *S*tyles (*MTS*) is described theoretically and on the basis of qualitative and quantitative empirical studies. These demonstrate how mathematical thinking styles influence the teaching and learning of mathematics. The MaTHSCu-Project (Mathematical Thinking Styles in School and Across Cultures (since 2012); Project leader: Rita Borromeo Ferri) focuses on the question, if there are differences in mathematical thinking styles of 15 year old pupils and their math teachers in eastern and western cultures (South Korea,[1] Japan[2] and Germany).

Theoretical Background of Mathematical Thinking Styles

The term mathematical thinking style is characterised as follows:

> A mathematical thinking style is the way in which an individual prefers to present, to understand and to think through, mathematical facts and connections by certain internal imaginations and/or externalized representations. Hence, a mathematical style is based on two components: (1) internal imaginations and externalized representations, (2) on the wholist respectively the dissecting way of proceeding. (Borromeo Ferri 2004, 2010)

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[2]Many thanks to Prof. Toshikazu Ikeda from the Yokohama National University, Japan.

A central characteristic of the construct mathematical thinking style is the distinction between abilities and preferences as mentioned before. Mathematical thinking styles are about how a person *likes* to understand and learn mathematics and not about how good this person understands mathematics. This approach is based on the theory of thinking styles of Sternberg (1997). So in the sense of Sternberg (1997), “A style is a way of thinking. It is not an ability, but rather, a preferred way of using the abilities one has.” That means that thinking styles are not viewed as being unchangeable, but they may change depending on time, environment and life demands. Sternberg and Wagner (1991) created a Thinking Styles Inventory for testing 13 different thinking styles based in Sternbergs (1997) theory of mental-self-government. These styles were not mathematical thinking styles, but he used this test in schools, universities and other professional areas. Besides Sternberg’s theory of thinking styles also Riding and Rayner (1998) and Riding (2001) have to be mentioned, who dealt with cognitive style dimensions (verbalizer-imager, wholist-anayltics) and questions how information can be differently understood on an internal and external level. For Riding and Rayner (1998, 8) a “cognitive style is seen as a preferred and habitual approach to both organizing and representing information.” So one part of the characterization of mathematical thinking styles make clear, that a preferred way of learning and understanding mathematics can also be distinguished in the way of proceeding. This means a task can be solved in a dissecting or in a wholistic way in combination to several modes of representation for example analytically or visually. The named representations were from great interest for gaining the theory of mathematical thinking styles. In the literature one can find a lot of classifications of thinking.

In 1892, in Germany, Klein constructed a typology of three different thinking styles. This classification was based on observations in cooperation with other mathematicians and not based on empirical studies:

1. The philosopher who constructs on the basis of concepts
2. The analyst who essentially “operates” with a formula
3. The geometer whose starting point is a visual one (“Anschauung”)

(quoted from Tobies 1987, 44) (Original in German, translated by the author)

A similar typology, restricted to visual and analytic thinking styles, is found in Hadamard (1945). Unlike Hadamard, but similar to Klein, Burton (1995, 95) identified three, and not two, styles of thinking: visual, analytic and conceptual thinking style. Furthermore Skemp (1987) distinguished between verbal and algebraic symbols for understanding mathematics and many other researchers are dealing with the question, which kind of representation could be effectively on learning mathematics. The analysis of the typologies or classifications of thinking and of how they were evolved illustrated, that they were not reconstructed with pupils at school. So in the first qualitative study (Borromeo Ferri and Kaiser 2003; Borromeo Ferri 2004) the goal was to reconstruct visual, analytic and conceptual thinking styles with pupils from Grade 9 and 10 during their pair-problem solving process. The design of the study was very complex for grasping the construct of the

	Internaly oriented types			Externaly oriented Types				
				congruent			incongruent	
1) / 2)	picture	symbolic	mixed	picture-picture	mixed	Symbolic-symbolic	Picture/symbolic	Symbolic/picture
Wholists								
Combiners								
Dissecters								

Fig. 1 Model to describe the construct mathematical thinking styles and their different kinds of mathematical thinking styles

construct "style" (preference) itself and the representation (visual, analytic, conceptual) and the way of proceeding (wholist, dissecting) out of the data, stimulated recall and interview. The aim was not only to reconstruct these preferences, but to find explanations, what it means to be a visual or an analytic thinker. So Grounded Theory (Straus and Corbin 1990) was the appropriate method for analysing the data. The results of the study showed, that the conceptual thinking style could not be reconstructed, but the following table, makes the two components of the characterisation of mathematical thinking styles again clear:

Some explanation of this model: Component (1) includes internal imaginations and externalized representations. Through this, I define internal types as individuals who mainly assimilate facts internally and who do not see the necessity for representations (except if they serve as means of communication). External types, however, make external representations. If their internal imaginations match with the externalized representations (e.g. picture-picture) they are called congruent, if these do not match (e.g. picture-symbolic), they are called incongruent. Component (2) examines the process of solving the task which can be understood in a wholist way (task is exploited from the whole to parts of it), in a dissecting way (task is exploited from parts to the whole) and in ways combining these two "pure" ways (Fig. 1).

Based on this an empirical grounded description of the characteristics of the visual, analytic and integrated thinking style could be developed:

- Visual thinking style: Visual thinkers show preferences for distinctive internal pictorial imaginations and externalized pictorial representations as well as preferences for the understanding of mathematical facts and connections through holistic representations. The internal imaginations are mainly effected by strong associations with experienced situations.
- Analytical thinking style: Analytic thinkers show preferences for internal formal imaginations and for externalized formal representations. They are able to comprehend mathematical facts preferably through existing symbolic or verbal representations and prefer to proceed rather in a sequence of steps.
- Integrated thinking style: These persons combine visual and analytic ways of thinking and are able to switch flexibly between different representations or ways of proceeding.

Mathematical Thinking Styles have principles, they

- are not mathematical abilities, but preferences how these abilities like to be used;
- are attributes of the personality, because preferences are connected with positive affects;
- are not mathematical problem solving strategies, because strategies are on an higher level of consciousness;
- are partly influences by (mathematical) socialisation, which means, that parents or in most cases teachers give guidelines how mathematics has to be learned and represented during lessons or tests, for example with or without visualisation, pictorial sketches etc.

The Influence on the Teaching and Learning of Mathematics–Results of Empirical Studies

In this chapter results of three empirical studies will demonstrate at first, in which way mathematical thinking styles influence the teaching and learning of mathematics. Also the mentioned comparative study within the MaTHSCu-Project is described and so is highlighting the cultural perspective.

Mathematical Thinking Styles in Primary School

After reconstructing mathematical thinking styles in grade 9 and 10 for creating the theory, the question aroused, if these styles could also be reconstructed with primary kids.[3] On the one hand the relevance for learning elementary mathematical ideas, ways of thinking and algorithms in primary school was evident, but on the other hand the conceptual development could be not enough to grasp different styles. So an explorative qualitative study was done with two classes in at the beginning of grade 4 (sample: 40 pupils and their teachers). For this study a questionnaire (open items) was developed and used, which firstly should give some ideas for preferences for certain styles. Both classes were then observed for two weeks during mathematics and science lessons. Four kids from each class solved mathematical problems on their own and were videotaped and also interviewed afterwards. The combination of these instruments and the analysis of the several kinds of data made it possible to reconstruct different mathematical thinking styles. Having in mind, that in primary school mathematical hands-on materials and several kinds of representation-modes of numbers are used for constructing

[3]In Germany primary school is from grade 1 to grade 4, ages from 6 to 10 years.

amongst others the number sense, it was interesting to see personal preferences at all. For getting a short insight, mathematical thinking styles of two children, Hanna (8 years) and Jens (9 years) are briefly described. Both solved the following problem:

> A cochlea is at the bottom of a 20 m deep standpipe. She wants to crawl to the top. During the day she crawls 5 m up, but in the night she glides 2 m down. At which day does the cochlea arrive at the top?

This problem can be solved in a wholistic and dissecting way in combination with using only numbers or also with a sketch for drawing the crawling of the cochlea up and down and of course without writing something down. Seven of the eight videotaped children got the right solution (on the 6th days in the evening).

Hanna (8 years) could be reconstructed as an analytic thinker on the basis of the questionnaire, the observations, problem solving processes and the interview. She likes numbers and operating with number and also when her teacher is writing calculations on the board. Hanna solved the problem without writing something down, so very internal oriented, although her formal-dissecting way became apparent: "…three metres, early in the morning, second day. … seven metres …no, eight metres, late in the evening at the second day.." Hanna did not speak about the cochlea; she had numbers and calculation processes in her mind. Jens (9 years) could be reconstructed as a visual thinker on the basis of the questionnaire, the observations, problem solving process and the interview. In the questionnaire he marked

> "I like to understand mathematics, if my teacher makes drawings and pictures on the board". In the interview Jens confirmed this way of understanding and learning mathematics again. For solving the problem he makes a sketch. During the solving process he said: "Yeah I see the cochlea crawling up and down and I need to draw it down."

Summarizing this explorative study in primary school, it was in particular of high interest to see *different* preferences for understand and learning mathematics at that age and so supports the principle of a mathematical thinking style as an attribute of personality.

Influence of Mathematical Thinking Styles on the Mathematical Modelling Behaviour

The aim of the project COM^2 ("Cognitive-psychological analysis of modelling processes in mathematics lessons"; 2005–2010), directed by the author, was to analyse teachers' and students' actions, ways of thinking, in sum, their behaviour while working on modelling problems in mathematics lessons from a cognitive perspective (see Borromeo Ferri 2010, 2011). For that aim, mathematical thinking styles were used as "theoretical glasses" for analysing this behaviour and to interpret this. Without going deeply in the discussion of mathematical modelling, the central research questions of the COM^2-project were:

- How do grade 10 pupils solve modelling tasks, and what influences do the mathematical thinking styles of the learners have on their modelling processes in reality-oriented mathematics lessons?
- How do mathematical thinking styles of teachers influence their way of dealing with mathematical modelling problems in the classroom? Are there differences with respect to the various phases of the modelling cycle (real situation, situation model, real model, mathematical model, mathematical results, real results)?

The design of this qualitative study was highly complex, because both teachers and pupils were in the focus. Three grade 10 classes from different Gymnasien (German Grammar Schools) were chosen. The sample was comprised of 65 pupils and 3 teachers (one male, two female). Each individual in a class had to complete a questionnaire on mathematical thinking styles, which has been developed on the basis of the Ph.D. thesis Borromeo Ferri (2004). The pupils were given three different modelling tasks, one per lesson (three lessons altogether). In each of the classes one group was videotaped during the whole lesson. Focused interviews were conducted with the teachers to reconstruct in each case his or her mathematical thinking style (the male teacher was analytic thinker, one female teacher was a visual thinker and the other female teacher was an integrated thinker). As one appropriate method within the field of qualitative research I used Grounded Theory (Strauss and Corbin 1990) because one central aim is the possibility to generate a theory on a code-based procedure.

As a quantitative result, when looking at the statement given from visual and analytic thinkers during their modelling processes altogether 87 verbal statements of analytic thinkers in the realm of mathematics and only 48 statements in reality where their preference. The spread of the visual thinkers is not so high (mathematics: 65; reality: 73), their preference on reality becomes nevertheless apparent. On the basis of deep qualitative analysis different kinds of modelling behaviour became visible, which are summarized in their central characteristics:

- Analytic thinkers usually change to the mathematical model immediately and return to the real model only afterwards when the need arises to understand the task better. They work mainly in a formalistic manner and are better at “perceiving” the mathematical aspects of a given real situation.
- Visual thinkers mostly imagine the situation in pictures and use pictographic drawings. Their argumentation during the modelling process is mostly very vividly even they work within the mathematical model. They often follow the normative modelling cycle.

But also the mathematical thinking style of the three investigated teachers had great influence on their teaching behavior. To make this more concrete on the basis of the analysis: A teacher’s mathematical thinking style can be reconstructed and manifests itself during individual pupil-teacher conversations, as well as during discussions of solutions, and while imparting knowledge of mathematical facts.

Very interesting was that teachers who differ in their mathematical thinking styles have preferences for focusing on different parts of the modelling cycle, while discussing the solutions of the problems and while helping students during their modelling processes. For pupils who share the mathematical thinking style of the teacher will have a better way of understanding, because both are talking in the same “mathematical language”. If there is a mismatch between teacher’ and pupils’ style this also can have consequences for the learning processes and at least in learners’ performance. It is obvious for teachers to reflect about their own mathematical thinking style to be flexible in their way of teaching. In the COM^2-project we showed all teachers some clips of their lessons and did a stimulated-recall asking them, if they recognized personal “teaching-patterns”. But mostly teachers were not aware of their behaviour during modelling activities in the classroom, and were astonished about their preferences for certain parts of the modelling process, connected to their mathematical thinking style.

Mr. Peters for example was reconstructed as an analytic thinker on the basis of the interview and observations. In the stimulated-recall he was asked:

Interviewer: Do you think that you have a preference for formalising?

Mr. Peters: “I didn’t think about that yet, for me that is mathematics, yes, I am doing mathematics. But yes, I like formalising mathematics.”

Mrs. Heidkamp especially recognized her strong preference for visual thinking (she was reconstructed as a visual thinker) after looking the video-clips.

Interviewer: You recognized that you are a visual thinker. Do you have experiences concerning situations “speaking not the same language” with some pupils?

Mrs. Heidkamp: “Yes, I had a girl who came from another school in my math class. After a while she came to me and told me that she is not able to understand me. She did not understand me! Now I think, that she means my explanations, my mathematical explanations, perhaps they were too visual for her or not concrete enough.”

The situation Mrs. Heidkamp is reporting about is a wonderful example for this mismatch between teachers’ and pupils’ mathematical thinking styles mentioned earlier. Consequences concerning mathematical performance are obvious, because this mismatch induces even though unconsciously impressions of weak mathematical abilities of pupils. Already Sternberg and Zhang (2001, 204) pointed out:

> Findings from a third study indicated that teachers inadvertently favored those students whose thinking styles that were similar to their own.

Mathematical Thinking Styles in School and Across Cultures (MaTHSCu)

The described studies using the theory of mathematical thinking styles for looking at thinking, teaching and learning processes were all qualitative. So a lack of research was the construction of appropriate scales for mathematical thinking styles, which was directly connected with the open question, if the construct of mathematical thinking style can be determined quantitatively. This was one of the central goals of the MaTHSCu-Project.[4]

When Sternberg started working on his Thinking Styles Inventory, he made clear that "styles do indeed appear to be largely distinct from intelligence or aptitudes." (Sternberg and Grigorenko 1997, 708) Anyhow Sternberg emphasized these distinctions for his thirteen different styles this is also applied for mathematical thinking styles. But the areas of conflict of style with abilities, intelligence and aptitudes were considered when developing a psychometric test for mathematical thinking styles for pupils and teachers.

In doing so the aim was to find answers for a lot of research questions, which could not attended so far on a general level such as:

- Are there differences in the stylistic patterns of boys and girls?
- Are there correlations between mathematical thinking styles and beliefs?
- Are there correlations between preferences for certain mathematical thinking styles and mathematical performance?
- Are there cultural differences in mathematical thinking styles, in particular between eastern and western cultures?

Before the focus will be on the last question, some aspects of the construction of the mathematical thinking style scales[5] are described.

On the basis of the theory of mathematical thinking styles (have a look at Fig. 1 again) 27 items could be developed for pupils (for grade 9 and 10) and specified for teachers. The "thinking-style-scale" comprised four different sub-scales rated with likert-scale from 1 to 4, which means from [1] strongly agree to [4] strongly disagree:

(1) visual (5 Items) (2) analytic/formal (5 items) (kinds representation)
(3) wholistic (4 Items) (4) dissecting (5 items) (ways of proceeding)
Additionally the following two sub-scales will be correlated for generating the stylistic patters: (5) internal (4 items) (6) external (4 items) (types of assimilating information)

Examples for items for the subscale "analytic thinking" are: Variables and formulae are helpful for me to understand mathematics; I like to use a formula, when I have to solve a mathematical problem. For estimating a four-step interval-scale is used. All items were piloted several times with pupils, teachers and students. The data was analysed with SPSS. After the final pilot study the scales had a good till satisfied reliability[6] (cronbachs α): visual (.77), analytic (.90), wholist (.80), dissecting (.60), internal (.65), external (.77). Besides these scales, also four problem solving tasks (open format) were integrated in the test and therefore a coding

[4]Funded by the Central Research Foundation of University of Kassel (ZFF).

[5]See the full version of the scales of mathematical thinking styles in the appendix as well as the open tasks.

[6]Results of the students' questionnaire.

manual was developed concerning the way of representation and the way of proceeding accompanied with items who asked after the kind of associations or ideas pupils had directly after reading the problem and after they had solved the problem. Furthermore scales from PISA (PISA-Consortium 2006) were integrated in the test, in particular scales of beliefs, self-efficacy, motivation, emotion and concerning exercising mathematics. The questionnaire of the teachers includes besides the scales of mathematical thinking styles also scales of beliefs and how they exercise mathematics. Furthermore a semi-structured interview will give more informations about several teachers.

A further central goal of MaTHSCu-project is the comparison of mathematical thinking styles of 15 year old pupils and their math teachers in several cultures. In this paper mathematical thinking styles of western cultures (Germany) and eastern cultures (South Korea, Japan, Singapore, Taiwan and China; see Biggs 1996, 46). In particular results in the area of the culture comparative research area of psychology, which often compared thinking processes of individuals in China versus USA, pointed out over and over again the preference of Asian people for seeing situations very holistically. On the contrary western individuals have preferences for analytic perspectives (Nisbett 2003, Masuda and Nisbett 2001). Besides Schwank (1996) in particular Cai (1995, 1998, 2002) conducted studies concerning mathematical thinking in eastern and western cultures using routine and problem solving tasks in an open format. As central results Cai emphasised, that individuals from the USA often used pictorial and Chinese individuals rather numeric or symbolic solving processes. As an open question Cai (2002, 281) is asking: “Is it possible that these Chinese students might have used visualization mentally, but hey expressed their solutions in non-visual forms (e.g. algebraic equations)? On the other hand, is it possible that U.S. students have just used drawing strategies because teachers told them so and they did not necessarily think visually?” For answering these questions Cai did not enforced further studies for investigating these phenomena. Even though in the actual discussion are no findings, in which way individual preferences of mathematical thinking styles could be measured in in eastern and western cultures. Although there are differences between both cultures concerning education as well as learning and teaching which are deeply fixed, but mathematical thinking styles are individual preferences and so can be independent of the cultural background. Looking at the societal development of both cultures (as a western culture the Greece stressed individual freedom, “so Chinese see themselves as a part of manifold networks” Kühnen 2003, 14) the conception of education become apparent. Central terms like integration and harmony characterises the teaching and learning situation of the East Asian philosophy (Leung 2001, 44). Vollstedt (2011a, b) reconstructed in her qualitative comparative study of pupils from Hongkong and Germany different types of sense-construction of mathematics, which reflected the described cultural background. So, all these studies highlighted important, interesting and different aspects of both cultures concerning learning and teaching mathematics. Investigating mathematical thinking styles of teachers and students will give again further findings.

Korrelationen

Land			MittelBildlich	MittelFormal
Germany	visual	Korrelation nach Pearson	1	-.278*
		Signifikanz (2-seitig)		.042
		N	54	54
	analytic	Korrelation nach Pearson	-.278*	1
		Signifikanz (2-seitig)	.042	
		N	54	54
Japan	visual	Korrelation nach Pearson	1	.220**
		Signifikanz (2-seitig)		.000
		N	328	328
	analytic	Korrelation nach Pearson	.220**	1
		Signifikanz (2-seitig)	.000	
		N	328	328
Korea	visual	Korrelation nach Pearson	1	.435**
		Signifikanz (2-seitig)		.000
		N	526	524
	analytic	Korrelation nach Pearson	.435**	1
		Signifikanz (2-seitig)	.000	
		N	524	524

*. Die Korrelation ist auf dem Niveau von 0,05 (2-seitig) signifikant.

**. Die Korrelation ist auf dem Niveau von 0,01 (2-seitig) signifikant.

Fig. 2 Correlations and means of visual and analytic thinking styles across countries

Some central results are shown in the following. First of all, the questionnaire, which was translated from English to Korean and Japanese language, was reliable in all their scales in all countries, especially concerning the mathematical thinking styles scales. So it becomes evident, that the construct of mathematical thinking style can be measured or grasped not only with German pupils from the pilot study. The sample of the Korean, Japanese and German students (15 year old) was N = 907 and the teacher sample was N = 20. Comparing the “structure of mathematical thinking styles” in the three countries, South Korea and Japan are very similar, which means that pupils showed a strong preference for the integrated thinking style (Fig. 2).

In Germany the preference for the analytic and/or visual thinking style became very clear, because of the negative significant correlation (−278**).

A further interesting result became evident concerning the correlations between different mathematical thinking styles and performance of the students (mark). Those students with a preference for formal representation have the best mark. These correlations are significantly in the countries: German: .53; Japan: .14; South Korea: .29. This result should be discussed, because marks in the school are of course influenced by the curriculum, learning materials and the teacher personality and so on. Mostly school books are very formal oriented and this is a perfect match for pupils with a preference for the analytic thinking styles. This also means that those students have an advantage, because they perhaps better fit in this formal way of learning and teaching mathematics than the visual thinkers. But if we want to know more about the performance in mathematics and the power of different styles, a new test should be developed, which is independent from school marks at all.

Another result shows that learners with a high self-efficacy tend to be internal types: Germany: .51; Japan: .39; South Korea: .49. These correlations were significant in all three countries. This phenomenon can be interpreted differently. Internal types are characterized as individuals who like to work on tasks mainly in their "head" and do not really want to write down something. Mostly those persons try to solve mathematical problems by themselves and not with others. Why there is such a strong correlation between preferences for internalization and self-efficacy is a new and interesting phenomenon coming out of the data. With interviews this phenomenon will be explained more deeply.

Summary and Conclusion

In this paper the theory of mathematical thinking styles was aimed to show and how different mathematical thinking styles (visual, analytic and integrated) could be reconstructed in several qualitative studies. Also in primary school preferences for mathematical thinking styles of pupils became visible and in the COM^2-project the influence of these styles on the modelling behaviour of pupils and teachers were explicit. Finally the current quantitative oriented and comparative study shows some first results concerning mathematical thinking styles as one influence factor on the teaching and learning of mathematics in a cultural context. There are a lot of open questions left from these first results and there were new phenomenon coming out of the data. The data analysis of the MaTHSCu-Project is still in progress and further interesting results are expected.

Furthermore, the presented results obtained up to present indicate a highly didactical relevance of these kinds of studies: Its significance for mathematics lessons is obvious. Pupils who are not sharing the mathematical thinking style with their teacher may have problems of understanding, but if the teacher is conscious of his own style and arranges mathematical facts in different ways, problems of understanding could be prevented. These results correlate with results from other empirical studies (Sternberg and Zhang 2001)

Therefore, it is necessary that teachers become conscious about their own mathematical thinking style, on the one hand in order to guarantee equality of chances among pupils, and on the other hand to develop their own mathematical potentials. Doing this and so coming back to the citation of Wagenschein at the beginning, mathematics would not be closed for him from teacher to teacher.

Appendix

(Scales of mathematical thinking style inventory developed by Rita Borromeo Ferri; University of Kassel, Germany in 2012)

Explanations: the first 5 items are visual/pictorial; item in red color is a neutral item and does not belong to the scale; then 5 items for analytic/formal

YOUR PREFERRED WAYS OF MATHEMATICAL THINKING

Q1) Every individual likes to learn and understand mathematics in his or her preferred way. Some like drawing sketches or strongly need pictures in their mind, others like formulas and variables. Which way of mathematical thinking do you prefer most? *(Please check only one box in each row.)*

	strongly agree (1)	agree (2)	disagree (3)	strongly disagree (4)
I prefer visual explanations of my teacher or sketches on the board.	☐	☐	☐	☐
While working on math problems, I mostly have visual images and I like to draw sketches.	☐	☐	☐	☐
I like geometry, because a drawn figure can also be a possible solution.	☐	☐	☐	☐
An adequate picture concerning to a formula would help me to understand it.	☐	☐	☐	☐
Visual images are helpful for me to understand mathematics.	☐	☐	☐	☐
Working on a lot of examples is my preferred way for understanding a mathematical theory.	☐	☐	☐	☐
Variables and formulae are helpful for me to understand mathematics.	☐	☐	☐	☐
I prefer variables and formulae for solving mathematical problems.	☐	☐	☐	☐
I like to use a formula, when I have to solve a mathematical problem.	☐	☐	☐	☐
Explanations of my teachers dealing with variables and formulae fit with my preferred way to think about mathematics.	☐	☐	☐	☐
I like to learn and to retain formulae.	☐	☐	☐	☐

Scales of the way of proceeding: wholistic (4 Items) and dissecting (5 items); item in red color is a neutral item and does not belong to the scale

How do you solve mathematical tasks? Do you prefer to solve a task step by step or do you focus on the whole problem?
How do you like to understand mathematical topics? Do you firstly like to know several parts of the topic for getting then an overview or do you prefer to know the whole topic at first and then get to know about several parts?
(Please check only one box in each row.)

	strongly agree (1)	agree (2)	disagree (3)	strongly disagree (4)
I like to solve mathematical problems step after step.	☐	☐	☐	☐
I prefer mathematical problems which are divided in several parts.	☐	☐	☐	☐
I like mathematical problems where I need to pay attention to detail.	☐	☐	☐	☐
I prefer to break down a mathematical problem into many smaller ones that I can solve, without looking at the problem as a whole.	☐	☐	☐	☐
I firstly like to know several parts of a mathematical topic and then getting knowledge about the whole topic.	☐	☐	☐	☐
We often work on difficult and tricky mathematical problems during math lessons.	☐	☐	☐	☐
While working on a mathematical problem I like to see how what I do fits into a general picture.	☐	☐	☐	☐
I like to work on mathematical problems where I can focus on general issues, rather than on specifics.	☐	☐	☐	☐
At first I prefer to know more about the whole mathematical topic and then about several parts of it.	☐	☐	☐	☐

Sub-scales for generating the stylistic patters: internal (4 items) and external (4 items) (types of assimilating information); item in red color is a neutral item and does not belong to the scale

Q3) Do you like to work on mathematical problems alone or do you prefer group work? Do you like to write down calculations or informations directly on a sheet of paper or do you rather like to calculate and think in your mind? *(Please check only one box in each row.)*

	strongly agree (1)	agree (2)	disagree (3)	strongly disagree (4)
Mostly I like to think through mathematical problems in my mind, without writing something down.	☐	☐	☐	☐
When I have to solve a mathematical problem, I like to work it out by myself.	☐	☐	☐	☐
I prefer to search informations for solving a mathematical problem on my own, rather than ask for it.	☐	☐	☐	☐
I like to work alone on a mathematical problem.	☐	☐	☐	☐
During math lessons we often work in groups.	☐	☐	☐	☐
I like to think through mathematical problems by writing or drawing all informations down, rather than doing this only in my mind.	☐	☐	☐	☐
I like to interact with others while solving a mathematical problem.	☐	☐	☐	☐
When working on a mathematical problem, I prefer to ask others for informations and ideas, rather than doing this alone.	☐	☐	☐	☐
I prefer to work in a group when solving a mathematical problem.	☐	☐	☐	☐

Open tasks within the questionnaire with additional questions

Working on mathematical problems:

Please work on the following four mathematical problems with your preferred way of thinking.

Please follow these rules:

- ***Read the task.***
- ***Response directly the questions in box A***
- ***Then work on the problem and then response the questions in box B***
- ***Please check only one box in each row.***

Problem 1)

Birthday party

Eight persons are gathering at a birthday party. Everybody wants to clink his glass exactly once with each other. How often will the glasses be clinked?

Box A

While reading the task...

	strongly agree (1)	**agree (2)**	**disagree (3)**	**strongly disagree (4)**
...I imagined people who clinked their glasses.	☐	☐	☐	☐
... I try to search a formula.	☐	☐	☐	☐
... the context of the task was not important for me, because I focused on the numbers within the task.	☐	☐	☐	☐

Your way of solution and your solution:

Box B) What had helped you solving the problem?

	strongly agree (1)	agree (2)	disagree (3)	strongly disagree (4)
With the help of my drawings it was easier to solve the problem.	☐	☐	☐	☐
I could solve this problem without any drawings.	☐	☐	☐	☐
The data in the problem were sufficient for me to solve the problem.	☐	☐	☐	☐
I made a drawing or a sketch, because my teacher often told me to do so while I'm working on mathematical problem.	☐	☐	☐	☐
It was funny to work on this problem.	☐	☐	☐	☐
I was afraid.	☐	☐	☐	☐

Problem 2)

Marbles Caroline has a bag with 24 marbles. She gives the half of the marbles to Herbert and then one third of the marbles, which are in the bag to Peter. How many marbles has Caroline for herself?

Box A
While reading the task…

	strongly agree (1)	agree (2)	disagree (3)	strongly disagree (4)
…I imagined people who changed marbles.	☐	☐	☐	☐
… I try to search a formula.	☐	☐	☐	☐
… the context of the task was not important for me, because I focused on the numbers within the task.	☐	☐	☐	☐

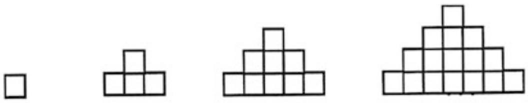
Your way of solution and your solution:

Box B) What had helped you solving the problem?

	strongly agree (1)	agree (2)	disagree (3)	strongly disagree (4)
With the help of my drawings it was easier to solve the problem.	☐	☐	☐	☐
I could solve this problem without any drawings.	☐	☐	☐	☐
The data in the problem were sufficient for me to solve the problem.	☐	☐	☐	☐
I made a drawing or a sketch, because my teacher often told me to do so while I'm working on mathematical problem.	☐	☐	☐	☐
It was funny to work on this problem.	☐	☐	☐	☐
I was afraid.	☐	☐	☐	☐

Problem 3)

Figures



(Figure 1) 1 Block | (Figure 2) 1+3=4 Blocks | (Figure 3) 1+3+5=9 Blocks | (Figure 4) 1+3+5+7=16 Blocks

a) How many blocks are in the 6th figure? Explain how you found your answer.
b) How many blocks are there in the 50th figure? Explain how you found your answer.

Box A
While reading the task…

	strongly agree (1)	agree (2)	disagree (3)	strongly disagree (4)
…I had pictures of the figures in my mind.	☐	☐	☐	☐
… I try to search a formula.	☐	☐	☐	☐

Your way of solution and your solution for a)

Box B) What had helped you solving the problem?

	strongly agree (1)	agree (2)	disagree (3)	strongly disagree (4)
With the help of my drawings it was easier to solve the problem a)	☐	☐	☐	☐
With the help of my drawings it was easier to solve the problem b)	☐	☐	☐	☐
I could solve this problem a) without any drawings.	☐	☐	☐	☐
I could solve this problem b) without any drawings.	☐	☐	☐	☐
I made a drawing or a sketch, because my teacher often told me to do so while I'm working on mathematical problem.	☐	☐	☐	☐

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Learning to See: The Viewpoint of the Blind

Lourdes Figueiras and Abraham Arcavi

Abstract Visualization goes beyond "seeing". On the one hand, it includes other sensorial perceptions, relationships with previous experiences and knowledge, verbalization and more. On the other hand, visualization can develop also in the absence of vision. On the basis of these premises, we attempt to revise the processes of visualization in mathematics education by (a) analyzing learning and teaching of mathematics by blind students with an expert blind mathematics teacher, and (b) simulating blindness with mathematics teachers with normal vision.

Keywords Visualization · Blind

Introduction

The general goal of this work is to study how certain processes of mathematical knowledge construction rely on the use of sensorial/perceptual resources available to human beings. One of these resources which has gained increased attention in the mathematics education community in the last two decades is visualization.

Attending to the blind in order to study visualization is a relatively new avenue of research. In his book on visualization, Rivera (2011) devotes a chapter to the blind and draws implications for mathematical reasoning. Similarly, Lulu Healy and her research team in Brazil (see, for example, the regular lecture in this Conference) are studying the blind. One main and general conclusion seems to arise from these and other studies, as well as from self-reports by blind mathematicians, (see, for example, Jackson 2002): there is much more to the visualization than the sense of vision, and impaired vision does not necessarily preclude our faculties to visualize.

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Our aim is to reflect upon the characteristics of visualization in connection with the spectrum of sensorial and other resources we employ, how we employ them and what are those that we could have employed but we don't.

For this purpose, we have analyzed some regular upper secondary mathematical classes conducted by a blind teacher at a special school for the blind. This data was not collected as part of an experimental program, but rather as documentation of the practices of an expert teacher, specialized in mathematics education for the blind. The aim of our analysis is neither to further studying the processes of learning mathematics by the blind per se nor to develop didactical aids for them. Rather, we aim at reflecting on the processes of visualization in the construction of mathematical knowledge within different mathematical topics: geometry (the cone and its sections) and quadratic functions and their graphs. Inspired by those two lessons, we designed two mathematical tasks and tried them with pre- and in-service teachers and with researchers in mathematics education (all with normal sight) in which they covered their eyes while performing.

In this presentation, we do not pursue theory building, neither have we described a methodologically sound empirical study. Rather, we report on our observations and reflections from both settings which helped us to pinpoint interesting phenomena (some of which we were not aware of) and which inspired the raising and formulation of research questions to explore further.

The Episode of the Cone

The excerpt transcribed below is taken from a video-taped math class in which there are three students (two totally blind and one with residual vision), and the totally blind teacher mentioned above. They manipulate a wood model of a cone made of five pieces (see Fig. 1) in order to learn to recognize the conic sections.

The section obtained by cutting the cone with a plane parallel to its base allows the teacher to refer to the truncated cone. His aim is to guide the students to visualize the truncated cone as a solid of revolution generated by a trapezium (i.e. a trapezoid). As a preparation step, the teacher appeals to the students' prior knowledge and asks one of them -totally blind- to build a right-angled triangle and to convince his mates that its rotation around one of its legs generates a cone (as a solid of revolution).

Fig. 1 A wooden model for conic sections

Teacher: So, you say that with a right-angled triangle you get a cone?
Johnny: Yes
Teacher: Let's see if you convince Mercedes, because I think that Mercedes is not at all convinced
Johnny: I stand up
Teacher: … and I doubt that Manolo is convinced
Johnny: [In front of Manolo, who has residual vision] with a right-angled triangle, when rotating it, its trail reproduces a cone. One is convinced…
Johnny: [In front of Mercedes, holding one of her hands—see Fig. 2] Look Mercedes, imagine that here we have the triangle, upright, and here we have a rotation axis. By rotating it, the trail it leaves to us, if we fill it up with matter, it would reproduce a cone, when the round is completed. Another one is convinced.

Let us compare how Johnny speaks to his two mates, keeping in mind that Manolo has residual eyesight and that both, Johnny and Mercedes, are totally blind In both instances, his goal is to facilitate to his listener the visualization of a cone as a revolution solid generated by a rotating right-angled triangle, and he uses the metaphor for the cone as the trail left by the triangle. In the demonstration for Manolo, the explanation is brief and several elements remain implicit, for example, the position of the triangle and its rotation axis. We infer that Johnny assumes that Manolo does not need explicit references to locate the figure in space or its rotation axis. This is not the case when he speaks to Mercedes. In the absence of vision much less can be assumed or remain implicit, and thus he makes many elements explicit by means of a verbal description. Moreover, when speaking to Manolo, the metaphor of a trail is not explicitly related to the mathematical content of the conversation, whereas in the case of Mercedes, there is a clear reference to the cone as a three dimensional figure—not just a surface—by stating that matter could be used to fill the space in which the triangle completed a round around its axis.

Moreover, we note that the explanation to Manolo is impersonal and lacks argumentative structure. In contrast, in the explanation to Mercedes, a perlocutionary effect is at play from the beginning (when she is asked to evoke previous

Fig. 2 Johnny and Mercedes rotating a paper right-angled triangle

images) to the end (when the physical experience of filling the space with matter is proposed to be done in collaboration by holding hands and talking about it—see Fig. 2). The validity of the conclusion that the generated volume is a cone is related to enacting that experience ("the trail it leaves to us, if we fill it up with matter, it would reproduce a cone") and it has an argumentative structure.

It would seem that when speaking to someone who can see -with the eyes-, the speaker takes for granted that vision makes obvious many things and thus making the reference to them superfluous. The implicit assumption that develops is that what it may be obvious for oneself, it will be for others as well. There is an important possible conclusion for the mathematics education of students with normal eyesight: by making fully explicit those issues considered to be "obvious" and thus not mentioned, one could help the building of sound knowledge. Moreover, such explicitness may enable to pinpoint possible points of miscommunication between two people (teacher and student, or student and student) who are looking at the same object and yet possibly seeing different things while being completely unaware of that. This may carry important morals for the integration of visualization into the practices of the mathematics classroom. First of all, it seems to be clear that, poor references to position and spatial relationships between people who are not blind should be enhanced by verbal descriptions as if it were assumed (contrary to what would seem natural) that our interlocutor is not seeing them.

In order to further analyze this type of situations, we conducted the following experience with two blindfolded teachers: We gave them a wooden stick with an attached right-angled triangle through one of its legs. The task had two parts. Firstly, both of them had to describe the object they received, and secondly, they had to describe and explain which solid would be obtained by rotating the stick, and thus rotating the attached figure. However, in the second part of the assignment the first teacher was just requested to explain, whereas the second was requested to explain it to a blind person.

We observed similar differences between the two explanations of these teachers to those given by Johnny, first to Manolo and then to Mercedes, respectively. The first explanation was considerably briefer and concluded with the teacher raising his hand and rotating the stick in order to "show" to some colleagues who observed the experience.

However, in the explanation of the second teacher he did several other things, especially in trying to provide an accurate characterization of the geometrical object attached to the wooden stick: (a) he leaned the triangle on the table in order to emphasize the existence of a right angle; (b) he made an explicit attempt to compare the lengths of the sides of the triangle by sliding two fingers along the sides at the same 'sliding speed' and then comparing sliding times; (c) he verbally described the position of the rotation axis; (d) he produced the metaphor of the cone as a funnel when the description of the movement came to a end. During these descriptions, there were a lot of redundancies.

Despite their differences, there are interesting parallels between Johnny's two explanations and the explanations of the two blindfolded teachers regarding the implicitness of the characteristics of what we see when we see it, and on the need to get a hold of many other resources to make the implicit explicit.

The Episode of the Parabola

Let us go back again to the same mathematics classroom of blind students in order to analyse this time some excerpts from a lesson on the quadratic function. In the conversation transcribed below, the same teacher (who is totally blind) guides the students to draw the graph of the parabola $y = 2x^2 + 4x$. As the extract is longer than the previous one, we have divided it into three units according to the intentions and the actions carried out by the teacher. In the first unit, the teacher elicits students' previous knowledge for plotting the graph. In the second unit, he guides them towards the conclusion that this is a parabola with two roots by reasoning by contradiction. In the third unit, he induces students to manipulate the algebraic equation of the parabola to determine the coordinates of the intersection points with the axes and of the vertex in order to sketch the graph.

Unit one: elements for sketching the graph

Teacher: Ah! A parabola. And what shall we do to draw a parabola?
Manolo: We will draw the roots, the vertex
Teacher: First of all. This one. Goes up or down?
Manolo: It goes up?
Teacher: It goes up. OK. And why?
Manolo: Because… The coefficient of x squared is positive.
Teacher: The coefficient of x squared is positive. You say the roots, do you? Then let's go for it, let's see the roots.
Teacher: zero…
Manolo: and zero
Teacher: Really? Does it have only one root? That is amazing. Sure?
Manolo: And minus two.
Teacher: And minus two?
Manolo: It makes zero and four
Teacher: Zero and four. Zero is clear. If I substitute zero for x we get zero. And if I substitute four, we don't
Manolo: minus four.
Teacher: Minus four. Let's see. Two multiplied by minus four squared. How much is it?
[the students whisper]

Unit two: reasoning by contradiction

Teacher: How do we calculate the roots? It has only one, therefore it meets *x*-axis, or does it?
Johnny: Yes
Teacher: And in how many points?
Manolo: zero, zero
Teacher: Ah, Only in one point. And this point, what could it be then?
Manolo: The vertex
Teacher: It would be the vertex, and if the vertex is in zero, zero, the graphic, how could it be?
Johnny: the top part….
Teacher: the graphic would be symmetric, perhaps?
Students: Yes
Teacher: Yes? Therefore the right….
Johnny: side…
Teacher: …would be equal to the…
Johnny: …left side
Teacher: and what's the name of those functions having the right side equal to their left side?
Students: even
Teacher: What?
Students: even
Teacher: And what happens with the formulae of even functions?
Johnny: That we get the same if we substitute x or minus x
Teacher: Let's see if this is true. Wherever there is an x, I put a minus x to see that whenever there is a minus x, the left side equals the right side whenever there is an x.
Manolo: …
Johnny: four x
Teacher: Then this is not even, therefore I am afraid that this is neither…
Johnny: symmetric
Teacher: and as a consequence the vertex cannot be at zero zero. One root is certainly zero. If I substitute x for zero, y is zero. But, which is the other root?

Unit three: algebraic solution of the equation

Teacher: Either we apply the formula [of the quadratic equation] or we decompose that.
Johnny: minus two
Manolo: One point would be zero and the second one…
Johnny: Look, Manolo, take out x as a common factor.
Teacher: And even something more can be taken out.
Manolo: Minus two.
Teacher: Then let's go for it, we write y equals two x… do you follow me?
Manolo: x plus two

Teacher: *x* plus two. For which values of *x* does it become zero?
Johnny: for zero and minus two
Teacher: zero and minus two. Therefore the roots are…
Johnny: zero and minus two
Teacher: And, what are the roots of a polynomial, geometrically, in the graph?
Manolo: The places where it meets the *x*-axis.
Teacher: The places where it meets the *x*-axis. It remains at that point. Zero and minus two. We know already that this is a parabola, it is second degree, that it is positive, it goes up, the roots are zero and minus two. Well. Anything else to get a full sketch?
Johnny: That the vertex will be in minus one because…
Teacher: That the vertex will be in minus one because…
Johnny: …because it is the middle point between minus two and zero
Teacher: The middle point between the two roots. Because there we will see the axis of symmetry of the parabola. Very well. Minus one, what?
Johnny: Minus two
Teacher: Minus one, minus two. Let's see. Two multiplied by minus one squared equals two. Plus 4 minus one minus four equals minus two. How do we denote the vertex? Capital V, do we? Minus one, two.

We first note that the conversation does not lose consistency or mathematical meaning if the second unit is suppressed. Moreover, the first and third units concatenated could be considered paradigmatic of many math classes. However, the second unit is precisely what captured our attention. The teacher decided not to correct the student's calculation mistake (see the end of the first unit) instead he pursues the reasoning that, if this parabola has a root at (0, 0), then it must have another one. Probably, most of us would have solved the quadratic equation directly in order to deduce the existence of a second root, and, at the same time, find the coordinates of the point of intersection with the *x*-axis explicitly. This is indeed what this teacher will do later, as it is shown in the third unit of analysis. So, why does the teacher invest time in pursuing the argument shown in the second unit? Which may his intention be?

All blind mathematicians and mathematics teachers agree that without vision it is extremely easy to make mistakes when recording information in writing. This is due to three essential limitations of the Braille system: (a) the use of various signs to construct a single mathematical symbol induces frequent errors and there may be confusion about the meanings of blank spaces, (b) the "linearity" of writing makes the creation of notations like exponents or subscripts a complicated endeavour, and (c) the lack of resources such as deleting, annotating an equation, and so on, imply much work load.

Studies on the learning of algebra have shown that translating word problems into equations constrains in many cases the use of useful representations and the creativity of the solution (e.g. Friedlander and Tabach 2010). Indeed, it is recommended as good practice to ask students to solve such problems without using equations in order to emphasize the importance of nurturing a certain "algebraic

Table 1 Comparison of steps followed to sketch $y = 2x^2 + 4x$

Some common steps in order to begin to sketch the graph of the parabola $y = 2x^2 + 4x$	Steps followed by the blind in order to begin to sketch the graph of the parabola $y = 2x^2 + 4x$
	Detect the intersection point at (0, 0)
	Reason by contradiction regarding the existence of a second root:
	If there is only one root, it must be the vertex (relies on previous knowledge)
	If the vertex is at (0,0), the graph is symmetric with respect to the y-axis
	Check if $f(x) = f(-x)$
	If the function is symmetric, then it is even
	The function is not even, therefore this root is not unique.
Solve the equation $y = 2x(x + 2)$	Solve the equation $y = 2x(x + 2)$
Find the two roots $x = 0$ and $x = -2$	Find the two roots $x = 0$ and $x = -2$
Conclude that there are intersection points at (0, 0) and (−2, 0)	Conclude that there are intersection points at (0, 0) and (−2, 0)

sense". In many cases, students are reluctant to set equations aside, especially when they feel comfortable enough with solution techniques. Instead, the blind resort to algebra only when it is strictly necessary. Using other forms of reasoning, and developing some creative imagination is a must if they do not want to be immersed in cumbersome calculations performed with the aid of the Braille system. Table 1 compares the most common way of representing a parabola with what happened in the episode above, considering the mathematical content involved.

Further analysis of this excerpt leads us to highlight another essential point of our discussion. In the absence of vision, the teacher chooses to guide the students to use logical reasoning that compels them to invoke images generated beforehand (a parabola that has a unique intersection point with the x-axis at (0,0) has its vertex necessarily at that point). Probably this image is related to others, like parabolas with the vertex on the y-axis and no points of intersection with the x-axis. In addition, students are requested to connect some concepts and definitions (symmetry, even functions) with images, and follow an argument by contradiction. The teacher's decision to delay to solve the equation in order to obtain the roots of the equation has redirected students' mathematical reasoning to rely on their images of a parabola as the graph of a quadratic equation.

As we did in the episode of the cone, we also tried a task related to the graph of a quadratic function in the context of professional development of teachers. This time, we provided the blindfolded participants with a relief plot of the parabola $y = 2x^2 + 4x$. They were told that they are given the graph of a function in the Cartesian plane in the upright position and their task was to establish what graph is it and then to attempt to find its equation. The plot was printed on a special material, the thickness of the axes differed considerably from the thickness of the graph of the parabola, and the lattice of integer values was also marked. The aim of our experiment was to detect which properties were detected and invoked by touching, as well as to use the experience to reflect with the teachers on their own practices. Findings in repeated experiments with different groups of teachers allowed us to establish three main types of reactions:

Difficulties to get started: Some teachers with a good knowledge of mathematics had difficulties to make sense of the graph only by touching and expressed their difficulties to distinguish the lattices from the axes, and the axes from the graph. Some of them attributed this to the lack of a fine tuned sense of touch. However, if they overheard others talking and exchanging information of what they were finding, they were able to redirect themselves and started to make better sense of what they experienced.

Attention to local properties: A common and rather unsurprising outcome of the experience was that the amount of time taken to identify the curve to be a parabola was much longer than it would have been by using eyesight. If one knows what a quadratic function is and how the parabola represents it in the Cartesian plane, only a quick glimpse suffices to identify it globally as such. By means of some mental or written work, one would be able to establish its symbolic representation using clues from the graph (position, location of the roots and vertex, intersection with the y-axis). The most interesting observations for us refer to the first stage when that global glimpse that is enabled by our eyesight is not possible. In this case, we saw that there are still some features of globality that people attempted to grasp (running their fingers at once from end to end of the curve) in order to get an overall feeling of the shape and to try to fit it with mental images. However this was interspersed with many attempts to detect local features (turning points, intersections, "sense of curvature"). This interplay between the local and the global features of a graph in order to identify it was interesting and productive. For example, one of the teachers explained that sometimes, while she was sliding down her fingers along the descending branch of the parabola, she felt the horizontal axis and related these two objects (x-axis and curve). Then she realized that the amplitude of the angle with which the curve leaves the axis was an important clue to imagine the equation of the graph, something she did not realized when watching parabolas. This and similar instances regarding the need to engage all local and global information available were an eye opener for us: the gift of vision by virtue of its immediacy saves us from the task to note how local and global features interlock in order to conform a unit which we "see" at once.

Attention to the analytical steps needed for the plot: A subgroup of the teachers dedicated enough time to find out the coordinates of x-intercepts by estimating distances with their fingers, and some of them did the same to find out the vertex of the parabola. One of the teachers concluded that he could not be sure that the represented graph was not the graph of a polynomial of fourth degree or even higher (actually this distinction is equally undetectable by seeing the graphs). Many of them concluded that the graph represented the curve $y = x^2 + 2x$, because they had only taken into consideration the points where the curve met the axes. All of them made explicit that they knew that they need the vertex to obtain the equation, but trying to memorize simple calculations and coordinates was extremely difficult; moreover, in their own view, this was probably the reason why they had not checked the coordinates of the vertex. At this stage, when the main task was to produce the symbolic equation of the function, it became apparent how importantly related are the sense of vision and short term memory. When one derives a symbolic formula on paper, one relies heavily on the sense of vision and its power to unload memory from abundant information, a quick glance at previous steps suffices when one is engaged in a symbolic calculation. The appreciation of the effortless way in which vision unloads our working memory becomes very apparent when one has to perform the same calculation on one's head only due to absence of the sense of vision.

Conclusions

We observed that, in the absence of vision, people cope with mathematical tasks by involving a rich variety of resources and mathematical reasoning processes. In contrast, when facing similar tasks, people with normal eyesight make use of only a small subset of the resources they have available. In the presence of vision, the presence of images may make verbal descriptions superfluous: one sees something and thus there is no need to describe it in words because the image speaks for itself. Note the widespread expressions in the mathematics education literature related to visualization such as "proofs without words" or "one picture is worth a thousand words", which implicitly confirm that one can do without verbal descriptions. However, the experience of understanding some very clever visual proofs show that whereas the images may lack words and symbols, there may be many words and sentences needed to unpack the proof and to make sense of it. In our observations, we noted the richness of the oral descriptions and the resourcefulness in the use of mathematical knowledge, when vision is not available. Such richness can be a source of inspiration for mathematics education of people with normal eyesight. This can be of particular importance because verbal communication regarding mathematical concepts usually involves implicit elements of which one does not talk about. Following our observations, we would like to suggest that such implicitness is due to our reliance on the sense of vision. It is assumed that the visual image communicates information that is not necessary to describe. Moreover, sometimes we also generalize our experience and assume that people who see are able to detect in the image some properties that are not explicit. We suggest that the explicitness in the verbal descriptions by the blind or for the blind can be adopted for all, even if in some cases it may sound redundant. This practice not only may help in avoiding misunderstandings but it may also support communication skills and clarity of formulations.

Regarding verbal communication, it t is also noteworthy that blind people pay careful attention to what is said by others. This was also very apparent with blindfolded teachers who were at lost when they had to identify the parabola and the talk by others in the same situation were a powerful resource to make progress. The lack of vision seems to promote attentive listening as yet another resource for knowledge construction. In addition to verbalization, knowledge construction and mathematical reasoning of blind people is supported by other resources, such as haptic perception, which could easily be incorporated as resources for general eyesight settings (for example, touch and speed of movement could be a way of apprehending length of segments). The perception by touch provides access to spatial details which may not be easily perceived otherwise and its use in the construction of mathematical knowledge by people with normal eyesight has not been explored. The global immediacy of vision may prevent us from interweaving local and global characteristics of approaching mathematical objects such as graphs of functions. Our experiences with the blindfolded teachers showed how the sense of touch allows them to integrate local and global perception in order to create a mental image of the graph they were trying to identify.

There are several difficulties for the blind people in creating and following chains of symbolic developments, like solving an equation. On the one hand, the Braille system may not support well the editing and correction of errors in symbolic derivations, and, on the other hand, these derivations which may be highly demanding to perform, even partially, in our head. Thus the blind may tend to develop alternative ways to eschew these difficulties, as it was the case of the blind teacher when he tried to foster, by means of a logical argument, the rejection of the the initial temptation of assuming that (0,0) was the vertex, instead of going right away to a symbolic calculation for that purpose.

Thus, another important realization, especially for the participant teachers, was the appreciation of the enormous power which we are granted by being able to unload our working memories into written displays which can be quickly scanned by our sense of vision, which plays a key role as a memory enhancer. These also led to the realization of a somehow paradoxical situation: those mathematical areas less associated with images (algebra, analysis) rely more on vision in order to unload memory than those which are richer in images. This is probably the reason why the areas in which most blind mathematicians work is geometry, or geometry related. This leads to propose that more research and development efforts in visualization in mathematics education should be devoted to its role in such areas as algebra.

Finally, we would like to suggest that the possible differences in the construction of mathematical knowledge that exist between the blind and the non blind people should be regarded as an issue of deployment and implementation of resources for learning and not merely in terms of the presence of the absence of the sense of vision. And thus, there is much to learn from the blind in this respect. We, the non blind, are rarely encouraged to perform the spectrum of potentially rich actions of a blind. Thus it makes all the sense to explore further how the development of reasoning and knowledge construction under visual impairment can broaden our own learning processes in mathematics, and that is what we attempted to do in this presentation.

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Issues and Concerns About the Integration of ICT into the Teaching and Learning of Mathematics in Africa: Botswana Case

Kgomotso Gertrude Garegae

Abstract This paper discusses challenges that developing countries especially African countries, face when trying to integrate ICT into the school curriculum particularly the mathematics curriculum. The much taken for granted belief that the availability of ICT gadgets in schools guarantees the ICT integration in specific subjects is challenged. Issues such as teachers' lack of relevant skills, shortage of teaching tools and unavailability of support staff act as impediment to ICT accessibility in classrooms. The paper describes the development of infrastructure in Botswana and experiences pertaining to school curriculum and argues that proper preparation for a smooth implementation of ICT infusion and integration is necessary.

Keywords ICT integration in Africa · ICT availability and accessibility · ICT and the mathematics curriculum · ICT integration in Botswana schools

Introduction

The importance and use of Information Communication Technologies (ICTs) in the educational system including the school curriculum is well debated in the literature. It is believed that ICT can be used, among other things, for pedagogical purposes (Finger et al. 2007; Guven 2008; Nabbout and Basha 2000). ICT in teaching and learning of school subjects, particularly mathematics, is critical for improving the quality of performance and classroom experiences (Nabbout and Basha 2000). A plethora of literature focuses on the use of ICT in teaching mathematics because of the belief that it is difficult, abstract, and that it is the most failed subject the world over. Thus, ICT is believed to have the potential to revitalise mathematics by

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arousing students' interest and making learning meaningful (Lapp 1999; Herbert and Pierce 2008) because of its ability to transform the teacher-centred approach into a dynamic environment where concepts are presented in various ways such as equation, line graph, and an animation mode (National Council of Teachers of Mathematics, NCTM 2000). Since mathematics plays a crucial role in every society, learners' understanding of mathematical skills and concepts, and their ability to apply these has become indispensable.

Mathematics is essential for learning science, which on its own right, generates useful knowledge necessary for the advancement of human needs and wants. And good performance in mathematics is an indication of a promising economic growth (Ogunniyi 1995). It is for this reason that both developed and developing countries aim at harnessing the potential in ICT to improve the mathematics performance. African countries including Botswana also embarked in this journey. Using Botswana as an example, this paper discusses challenges that African countries often encounter in the struggle to integrate ICT into the mathematics curriculum.

Background

Location

Botswana is a landlocked country bordered by South Africa in the South, Namibia in the West, Zambia in the North, and Zimbabwe in the East. It is a small country of about 580 km^2, the size of France and about 84 % of the terrain is covered by the Kalahari (*Kgalagadi*) Desert. Botswana obtained its independence in 1966 from British government. According to the 2011 population and housing census, the country has a population of about two million.

The Education System

Botswana education has inherited its organization from the British system and has the structure: 7:3:2:4:3—seven years of primary, three years of junior secondary, two years of senior secondary education, four years of University degree, and three years of College Diploma.

Currently, universal basic education runs from primary to junior secondary school education, covering a period of ten years. Learners are automatically promoted from primary to junior secondary level without the need to pass Primary School Leaving Examinations. However transition from junior to senior secondary schools is through merit. Students have to obtain a passing grade in national examinations (Junior Certificate Examinations) sat at the end of junior secondary school. Education has been free for about two decades up until three years ago where parents are requested to contribute maintenance fee. Schools provide food for learners. Unlike at primary level where learners are provided with breakfast only, students at secondary schools are given both breakfast and lunch. The provision of meals at school mitigates a tendency for students with less fortunate backgrounds to abscond. There are about 670 primary schools, 206 junior and 32 senior secondary schools. Pre-primary education has not yet been factored into the public system; it is offered by private sectors albeit at a smaller scale. Figure 1 shows the structure of the education system.

The core subjects at junior secondary level consist of Setswana, Mathematics, Integrated science, Agriculture, Social Studies, Design and Technology and Moral Education. Optional subjects include home-economics, commerce, Religious Education, a third language, Physical Education, Art, Accounts/bookkeeping, office skills, and music. Mathematics is regarded as an important subject. It is taught throughout all levels of schooling. The Botswana General Certificate of Secondary Education (BGCSE) is offered since 1997 after the localisation of Cambridge O level School Certificate (COSC). Its localisation was based on the need to focus the syllabus on industrialised economy and away from agro-based economy. A non-examinable Computer Awareness Course is offered to all students at secondary levels but not at primary level.

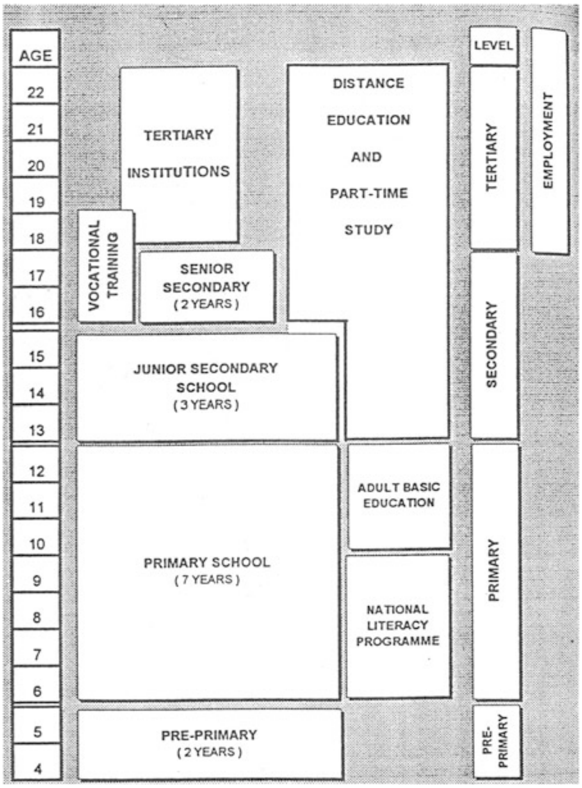

Fig. 1 Botswana education structure

Teacher training institutions include one university (UB), two colleges of education for junior secondary teachers and three colleges of education for primary teachers. UB runs a degree programme and colleges of education run diploma programmes.

ICT Policy and Its Implementation

Botswana like other African countries responded to the global trend of transforming the country into knowledge driven society by having formulated the ICT policy in 1998, which was revised in 2006. The policy was made to be responsive to the country's manifesto Vision 2016 which states that by 2016 "Botswana will have entered the information age on an equal footing with other nations. The country will have sought and acquired the best available information technology, and have become a regional leader in the provision and dissemination of information" (Republic of Botswana 1997a: 5). The seven sections of the policy include (1) connecting the communities, (2) government on-line, (3) Thuto Net (school connectivity) programme, (4) e-health, (5) ICT and economic diversification, (6) connecting Botswana, and (7) connectivity laws and policies. The government aimed mainly at providing enabling infrastructure for better implementation of the policy with emphasis on connecting rural communities as an effort to bridge the gap between urban-rural disparities. This motivated a robust move by parastatals to electrify the rural villages, a project embarked by Botswana Power Corporation (BPC) since 2007, and making it connected by the Botswana Telecommunication Corporation (BTC) and other service providers. Of particular interest are the *Nteletsa I* and *Nteletsa II* government initiatives in which BTC joined efforts with MASCOM to develop telecommunication infrastructure throughout the country, crossing the Kalahari (*Kgalagadi*) Desert into southern and eastern villages where most of the population reside. Figure 2 shows the network coverage over the whole country at the end of *Nteletsa II* project.

It is because of this good infrastructure that Botswana was ranked 6th in Africa on the ICT global state, leapfrogging Senegal, Morocco, Namibia, and Gambia in ICT competitiveness (Otieto 2012).

Through the concerted effort of the government, BTC, and the Botswana post, telecentres were established in rural areas where services such as data and internet (Bostnet) transmission, fax, photocopying, and word processing are provided. Libraries act as ICT centres where the public learn how to use computers for internet and access to government services.

According to an anonymous BTC officer, the main challenge that the government is facing with these centres is under-usage. In villages, very few people have the knowhow of utilizing these gadgets. Another challenge is that fewer channels in rural areas cause slow internet access. Insufficient supply of power which causes

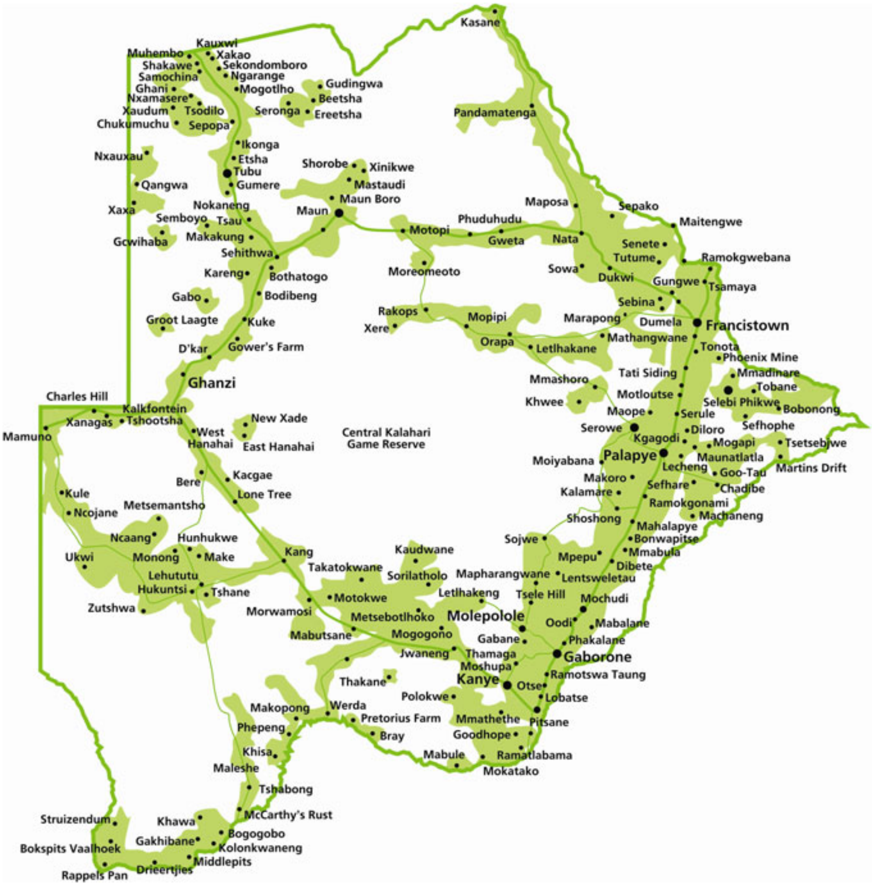

Fig. 2 Network coverage

frequent power cuts, is also a nagging issue for the country. Nonetheless, many Batswana throughout the country enjoy the availability of BTC BeMobile and landline coverage as can be seen from Fig. 3.

ICT Integration in Botswana Education System

The *ThutoNet* section of the 2006 ICT policy focused on the school connectivity, and it assumed that the availability of schoolroom computers is a guarantee for both computer literacy and ICT subject integration. The policy aimed at, among other things, providing all schools with modern PCs and internet access. The realisation of this goal was marked by the school computerisation project through which

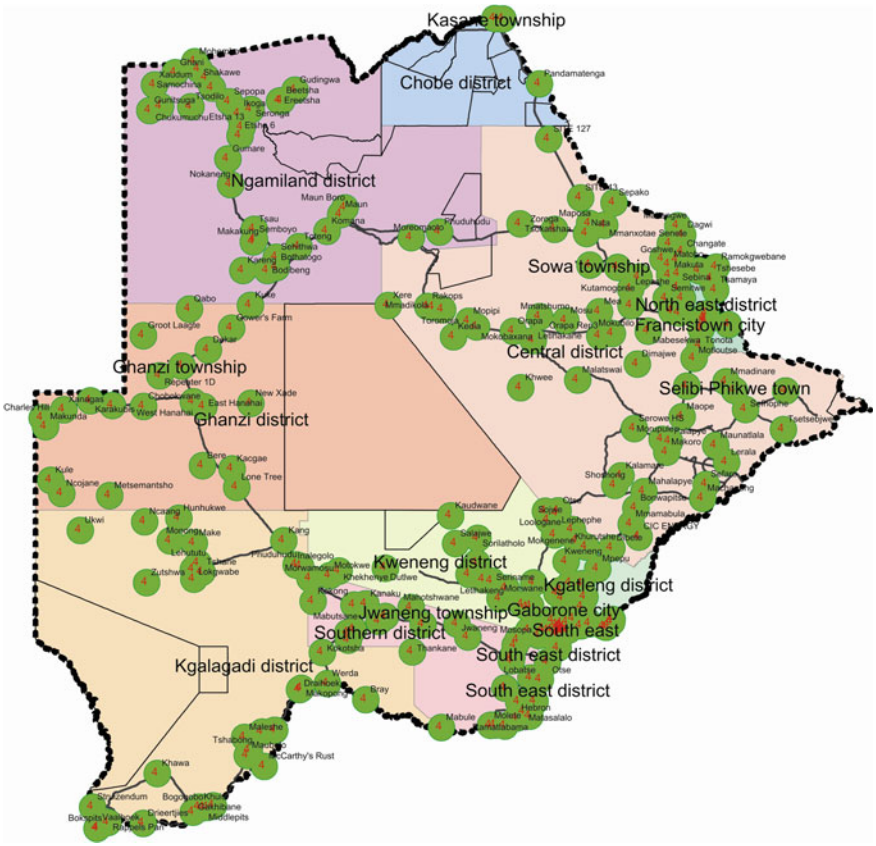

Fig. 3 BeMobile (cellular phone and BTC landline) coverage

206 junior and 27 senior secondary schools were equipped with computer labs furnished by 20 desk top computers, network printer, multimedia projector and other associated accessories (Kgokgwe 2003). The rationale for the computerisation project is that "Botswana [should] have a quality of education that is able to adapt to the changing needs of the country as the world around us changes. [And that] … schools [should] have access to computer, and to computer-based communications such as the internet (Republic of Botswana 1997a:5). It was believed that such a move will aid in preparing young people for competiveness in the global market. It is also worthwhile to note that the *ThutoNet* was not necessarily created for ICT subject integration but rather for using schools as centres or cafés for communities (Garegae and Moalosi 2011). In addition to 20 PCs, every school has at least two computers in teachers' general office for typing tests and examinations and other managerial purposes.

Computer Awareness (CA) Programme

Computer Awareness programme was introduced into both junior and senior secondary schools curricula as a way of making the school going children computer literate. The rationale for introducing this programme is that:

> Computers are becoming more and more common in all aspects of life. They are simply tools that help people to be more productive. More and more jobs require applicants to be familiar with computers. Botswana, like other countries, has recognised the need to increase the technological background of its people to better compete in world markets (Republic of Botswana 1997b:i).

In this programme, students were introduced to computer as a tool, and were being familiarised with machines to curtail possible future development of technophobia. It is expected that students will in the future pursue computer studies without being intimidated by machines. Figure 4 show a section of computer lab.

The Computer Awareness programme is divided into modules such as computer skills, keyboarding skills, productivity tools, word processing, spreadsheet, databases, and graphics.

It is expected that at the end of the junior secondary school CA programme, learners should be able to:

- describe and practise computer care and safety.
- describe the effect computers have had and will continue to have on individuals, the community and the world.
- name and describe the function of each of the hardware components of a computer system.
- differentiate between hardware and software.
- demonstrate basic file management capabilities, including understanding the nature of a file, opening, closing, saving, naming/renaming and printing files.

Fig. 4 Computer lab in junior secondary school

- demonstrate basic keyboarding skills.
- use a simple word processing program.
- use a simple spreadsheet program.
- use a simple database program.
- use a simple graphics program.
- differentiate types of application software, including productivity tools, games, simulations, tutorials, and testing software.
- work with content-specific software as part of their own learning process (Republic of Botswana 1997b:iv).

Teachers teaching this awareness course took Computer Education as a minor in addition to their teaching subject at their initial college training. This arrangement, however, proved to be wanting because these teachers are generally not competent enough to integrate ICT into their taught subject. They handled only basic knowledge of manipulating PC, Microsoft word and excel (Kgokgwe 2003).

Computer Studies

An examinable subject called Computer Studies is offered at senior secondary. Unlike CA, Computer Studies is not taken by all students but by a few who study Pure Sciences. The subject is marked at Cambridge and therefore optional. Teachers who are competent to teach O'level Computer Studies programme at Senior Secondary Schools are scarce. However, the University of Botswana started producing degree holders for the past six years and this has alleviated the situation.

Education TV Channel

The government of Botswana in partnership with Japan Broadcasting Corporation (NHK) established the Botswana Educational Television (BETV) with the aim of improving access to education by supplementing the school curriculum (Botswana Daily News 2011:3) and non-formal education (Baputaki 2009). The P22 million project was commissioned in 2006 and the need to broadcast mathematics and science was echoed by citizens during nation-wide consultations. The television channel was launched in June 2011 and it started operating by November of the same year "piggybacking on Btv with a two-hour slot from Mondays to Saturdays (11 a.m.–1 p.m.)" (Moeng 2011). BETV also augments the long standing educational radio programmes used for distance education.

ICT in the Mathematics Curriculum

Performance and Prevalent Teaching Styles in Mathematics

In Botswana mathematics is a subject that students learn from primary to senior secondary school. It is considered a very important subject such that the University of Botswana (UB), uses it as a gate keeper. For students to be admitted into UB, they should have obtained a minimum of Grade C in mathematics at BGCSE national examinations. This state of affairs narrows the transition rate from secondary to tertiary level since mathematics is among the most dreaded and failed subjects at both examinations taken at the end of junior and senior secondary school levels. The TIMSS reports of TIMSS 2003 and 2007 studies confirm the trend of inferior performance in primary and secondary schools in Botswana; Botswana was one standard deviation below the international mean. Research on classroom practice reveals that instruction is teacher-centered, and that there is shortage of resources (Garegae 2001; Taole and Chakalisa 1995; Prophet and Rowel 1990). And also, the international study on teacher development, TEDS-M 2008, indicated that student teachers' beliefs on mathematics instruction favoured teacher-centred approaches. Figure 5 show some preliminary results of how Botswana compares with Taipei and Chile concerning preferred instructional styles.

The majority of Botswana (secondary) student teachers believe that the learning of mathematics should be teacher directed as opposed to the student centred approach.

Similarly, Botswana educators favoured teacher directed instruction more than their Chinese Taipei and Chile counterparts (see Fig. 6).

Indeed, the integration of ICT in the teaching of mathematics may salvage the situation.

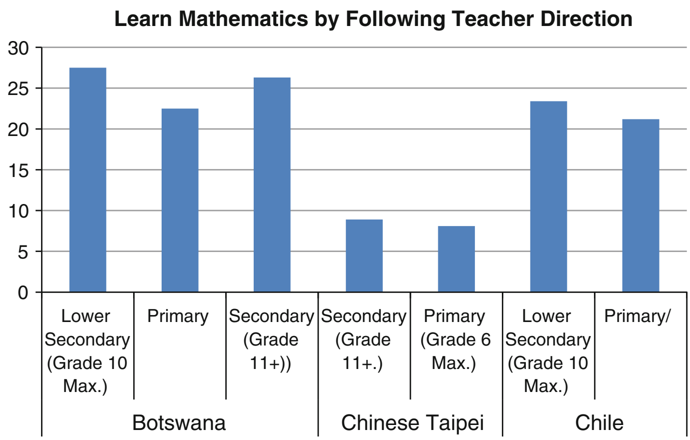

Fig. 5 Student teachers' endorsement of teacher-centred approach

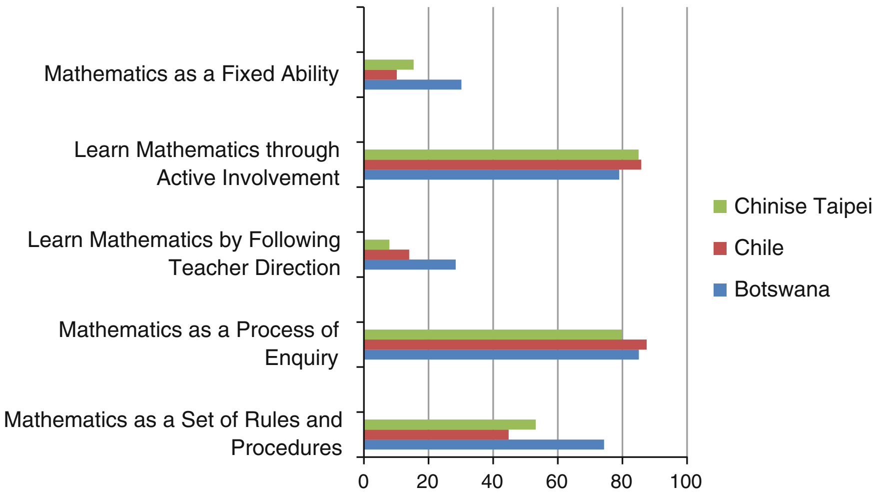

Fig. 6 Educators' endorsement of teacher directed approach

Use of Calculators in Mathematics Instruction

Non-programmable scientific calculators were introduced at the junior secondary level in 1996. At senior secondary level, non-programmable scientific calculators were long used under COSC. Although calculators were introduced in schools, it seems students do not use them in learning mathematics; they are kept in store-rooms for safekeeping against vandalism (Garegae-Garekwe 1999). Instead, these gadgets are used for tests and examinations not for exploration of mathematical concepts. This scenario is unfortunate but it is perpetuated by lack of policy guidelines on using calculators in schools. The TIMSS reports (Republic of Botswana 2005, 2009) observed that accessibility to calculators and computers did not make any significant difference in students' performance. This absence of the significant differences between students with and without calculators maybe attributed to the extent to which they are used in classrooms and the purpose of their use. The use of computers in subject integration is non-existent.

Issues and Concerns About ICT Integration into Mathematics

Botswana, through Connecting Communities initiative, made a marked stride in providing internet networks for voice and data transmission for rural and urban people and the government is commended on the investment on the infrastructure. However, there are some constraints that counteract the realization of the envisaged

ICT subject integration in schools, resulting in some issues and concerns with regard the implementation of ICT integration in the teaching and learning of mathematics.

Human Resource

Teacher Readiness and ICT Competence

It seems in 1996 when non-programmable calculators were introduced into junior secondary school mathematics syllabus, practicing teachers were not equipped for using them. As a result, most teachers did not embrace calculators in their instructional practices. Although teachers had used calculators in their learning of mathematics, they had no knowledge of how these gadgets are used in teaching (Ballheim 1999) and they feared loss of control in their classrooms. The introduction of ICT into the school curriculum should be founded on competent and confident teachers because its proper implementation depends entirely on their competence. In Botswana, and most of African countries, this issue was (and still is) omitted.

Teacher Beliefs and Perceptions

Teachers' beliefs and perceptions on the use of ICT in mathematics instruction influence their decisions to accept or reject its integration. There is a plethora of the literature about teachers refusing to use calculators due to fear that these would weaken students' arithmetic skills. This calls for robust in-service programme that will educate as well as challenge teachers' belief system about the effect of ICT in mathematics instruction.

Teacher Support Staff

Technicians who are conversant with the machine, software and the subject matter are needed for a smooth functioning of ICT integration. These technicians should be in a position to assist students outside class time. In Botswana, the training of technicians has been planned for but has not yet been implemented. Therefore, there are no such personnel in schools to provide the necessary expertise.

Limited Teaching and Learning Resources

Unavailability of Relevant Software in School

Software relevant to assist in teaching mathematics (or any school subject) are not yet bought for schools despite the fact that the 2008 junior secondary school mathematics syllabus (Republic of Botswana 2008) advocates the use of computers to provide multiple representations and animation of mathematical concepts. The unavailability of such resources frustrates teachers who would otherwise explore ICT potential in teaching.

Inadequate and Absence of Computers in Classrooms

Perhaps one computer laboratory in each school is not enough. An ideal ICT infusion environment would be having a computer (and other ICTs, e.g. overhead projector) in each classroom so that teachers may use them as tools for illustration and clarification.

Moreover, 20 PCs are not enough for a class of over 40 students. In a CA course, students are required to share a computer and this may not give them enough experience.

Environment

Class size in Botswana is between 40 and 45 students. In some cases, the numbers are higher. These are large numbers to afford an ICT rich environment where learners explore, investigate, and critique each other.

Also, classrooms are congested with iron desks leaving no space for learners to move around. Figure 7 shows a part of the classroom and lack of space therein cannot escape the reader's attention.

School Organization and Implementation Strategies

The government equipped schools with computer labs and computers in preparation for Computer Awareness course. With this understanding, such labs became the prerogative of CA teachers and were not accessible to other teaching staff, and the student body at large. Therefore, teaching mathematics using this facility is practically not feasible. This kind of arrangement, which is influenced by knowledge compartmentalisation ideology (Koosimile and Suping 2011) predisposes the under-utilisation of ICT gadgets in schools. That CA is non-examinable has

Fig. 7 Example of an overcrowded classroom

consequences on its implementation by school. In his report of 2003, the officer in charge of IT Unit at the Ministry of Education and Skills Development indicated that some heads of school did not include CA in their school timetables because they considered doing so as a waste of time. The same sentiment is being observed even today. CA is taken for granted and its teaching leaves much to be desired especially in urban areas where most students are familiar with the basics.

The Context: The Educational System

Centralised National Examinations

One of the factors that exacerbate teachers' resistance to ICT integration is the centralised national examinations. Teachers aim to complete the syllabus targeting what the curriculum emphasises and the topics that are usually included in examinations so they can prepare their students for examinations (Garegae 2005; Chakalisa et al. 2000). Because teachers' promotion is based on their student subject pass rates and that schools are ranked according to their overall pass rate, teaching for examinations phenomenon is wide spread. It is well understood in the whole world that technology influences what is taught, how it is taught, how it is assessed, and the milieu in which it is taught (Garegae 2003). Therefore, national examinations deny teachers freedom to maneuver or orchestrate classroom activities the way they would like and know best. The influence that national examinations have on teaching in general and in particular on ICT integration should not be underestimated.

Sustainability

The question of sustainability is pertinent to a developing country like Botswana where ICT gadgets are expensive. For instance, a huge sum of about $7 million was used in the school computerization project. More money is needed to furnish more schools. In addition maintenance also demands a lot of money. This expenditure is heavy for Botswana whose economy backbone is diamonds which at this time of global economy melt down, are scarcely bought.

Yet another sustainability concern is that ICT industry develops at a rapid speed. Almost within six months new software are made. Developing countries are likely to fall by the wayside of the journey because of the current economic recession and their dependence on donors.

Concluding Remarks: Looking Ahead!

This paper described Botswana experiences of ICT integration in school curriculum with more emphasis on mathematics. It highlighted challenges that schools face in trying to use ICT in schools. It seems Botswana and other African countries will continue to lag behind affluent states in ICT integration and their hope on ICT as a catalyst for development in their states would faint. The more African countries try to embrace digital innovations, the wider the ICT gap becomes. While the continent of Africa still grapples with rudimentary concerns such as shortage of teachers and classrooms, inadequate quality of teachers, lack of technicians, poverty, etc., developed countries deal with more advanced issues of improving practice rather than teething problems.

However, Botswana and other developing countries should not lose hope. They should slowly but surely continue the struggle until they achieve their goal. Some suggestions below may contribute to the development of a healthy ICT integration. (See Garegae and Moalosi 2011 for detailed discussion).

1. Formulation of ICT in Education Policy
2. Creation of guidelines for classroom use
3. Taking affirmative action in training of teachers
4. Taking a decisive step in training technicians
5. Conducting research on teacher readiness.

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Learning Mathematics in Secondary School: The Case of Mathematical Modelling Enabled by Technology

Jonaki B. Ghosh

Abstract This paper describes a study which was undertaken to investigate the impact of teaching mathematics using mathematical modelling and applications at senior secondary school level in India. While traditionally the emphasis in mathematics teaching in India is on the development of procedural skills, the study shows that the use of modelling and applications enabled by technology enhanced student's understanding of concepts and led them to explore mathematical ideas beyond their level. Using this approach, a balanced use of technology and paper pencil skills led to a deeper understanding of the subject.

Keywords Mathematical modelling · Technology · Visualization · Exploration · Paper pencil skills

Introduction

Increasingly, many countries across the world have witnessed a growing collection of didactical research on including mathematical applications and modelling in secondary school and this has impacted the mathematics curricula in these countries. The benefits of integrating applications and modelling in the curriculum are manifold. They help to motivate students and create a context for applying mathematical theory. They help students to learn new mathematical content and see that mathematics can be fruitfully applied to solve real problems. Above all they help to highlight the relevance of mathematics as a discipline and thus contribute to sustaining the student's interest in the subject.

It is possible to identify a number of different approaches and perspectives in mathematics education research on the teaching and learning of modelling. Kaiser and Sriraman (2006) report about the historical development of different research

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perspectives and identify seven main perspectives describing the current trends in the research field (Blomhøj 2008, p. 2).

One of the perspectives among them is the Educational Perspective. *The main idea of the educational perspective is to integrate models and modelling in the teaching of mathematics both as means for learning mathematics and as an important competency in its own right* (Blomhøj 2008, p. 5).

Blomhøj (2004) identifies two main arguments for teaching mathematical modelling at the secondary level. These are

1. *Mathematical modelling bridges the gap between students' real life experiences and mathematics…*
2. *In the development of highly technological societies, competences for setting up, analysing, and criticising mathematical models are of crucial importance.* (Blomhøj 2008, p. 6)

This paper attempts to investigate the impact of integrating mathematical modelling and applications into a topic of the senior secondary mathematics curriculum of India. As suggested by the educational perspective, it aims to use modelling to provide the student with the opportunity to learn new mathematics. Research has also shown that the use of technological tools can greatly enhance student's explorations and investigations. Herwaarden and van Gielden (2002) gives a detailed description of a course in calculus and linear algebra for first year university students where integration of computer algebra with paper and pencil methods helped to enhance conceptual understanding. Arnold (2004) describes computer algebra system (CAS) as "the ultimate mathematical investigative assistant" (p. 19) which allows the student to engage in "purposeful and strategic investigation of problems" (p. 21). Heid (2001) describes CAS as a cognitive technology which makes higher level mathematical processes accessible to students. According to her, CAS plays a dual role: that of an amplifier and a reorganizer. On one hand, CAS plays the role of an 'amplifier' by making it possible to generate a larger number and greater range of examples and thus can be used to extend the curriculum. On the other hand, it serves the role of a 'reorganizer' by changing the fundamental nature and arrangement of the curriculum. Lagrange (1999) suggests that CAS enables more productive balances between conceptual understanding and technical work and this supports the instrumental approach. In fact the pedagogical affordances of the use of CAS can also be extended to other technology tools available for mathematics instruction.

The Senior Secondary Mathematics Curriculum in India

In the Indian school curriculum, mathematics is largely taught as an abstract subject in the traditional 'chalk and board' manner. Topics are often taught without any substantial reference to their applications to real life problems, thus making the subject pedantic. Traditionally, the emphasis has been on the development of manipulative skills and the school assessment also tests the same. The teaching and learning of mathematics at the senior secondary school is driven by preparation for the school leaving examinations (at the end of year 12) which often determines a student's future. It is at this stage that the student has to make a choice as to whether she will opt for the science, commerce or humanities stream and her performance in the school leaving examination holds the key to her future in terms of career opportunities. In this regard the school leaving examination at the end of year 12 is indeed a high stakes examination and its impact looms large over the curriculum at the senior secondary stage.

The National Curriculum Framework (NCF, 2005) of India, in its position paper on teaching of mathematics, begins by stating that the primary goal of mathematics education is the *mathematisation of the child's thought processes*. It recommends that mathematics teaching at all levels be made more 'activity oriented'. While this has been the primary driving force for revisiting and revamping the elementary school mathematics curriculum, it has had little impact on the senior secondary curriculum. At this level, the textbooks, which happen to be the only resource for teachers and students, are usually examination oriented. Every chapter begins with a brief introductory note which sometimes includes a historical background of the development of the field, and then introduces the basic concepts of the topic. The chapter is further divided into sections and sub-sections which deal with definitions, theorems, results, examples and exercises. The mathematics curriculum in grade 12 is dominated by differential and integral calculus accounting for almost half of the content. Other topics include Matrices and determinants, vector algebra, three dimensional geometry, linear programming and probability. Manipulative and computational aspects of these topics, rather than visualization and exploration of concepts and ideas, dominate mathematics at this stage. Topics like sets, relations, logic, sequences and series, linear inequalities and combinatorics are introduced in grade 11, but only at a superficial level. The syllabus of grade 12 does not include these topics and hence there is no room for delving deeper into these areas. The NCF 2005 recommends that curriculum designers reconsider the distribution of content between grades 11 and 12.

The NCF 2005 strongly recommends the inclusion of modelling activities in the curriculum. The document suggests that this can be done in the form of investigatory exercises or projects which will enable the student to see the relevance of the mathematics taught at school. Such projects need to be designed in a manner so as to cover the depth and breadth of the topics taught while at the same time provide the student with ample scope to explore and apply important mathematical concepts and ideas in solving problems. Another important recommendation made by the NCF is the setting up of mathematics laboratories in schools. A mathematics laboratory can provide students with the opportunity to 'discover' mathematics through exploration and visualization of concepts and ideas and mathematical modelling activities can be integrated into the curriculum through the mathematics laboratory.

The Study

The study described in this paper is based on the recommendations of the National Curriculum Framework 2005. It aims at exploring how learning is impacted by integrating mathematical modelling and applications into the senior secondary mathematics curriculum. In this study, a module titled *Learning Mathematics through Mathematical Modelling and Applications* was developed by the author and 30 students of grade 12, selected from two schools of Delhi, participated in the study. The module was based on the topic *Matrices and Solutions of Systems of Equations* which is a part of the grade 12 syllabus. The module was spread over 16 hours, that is, eight sessions of two hours each, conducted on four consecutive school days. Graphics calculators and Mathematica were used as the primary vehicles of exploration. Throughout the module student's paper pencil work was recorded. At the end of the module students were required to respond to a short questionnaire and give a written feedback describing their learning experience in the module. The objective of the study was to reflect upon the following research questions:

- How does the integration of applications and modelling activities impact student's understanding of mathematical concepts?
- Does introducing mathematical modelling activities, enabled by technology, help students access higher level mathematical concepts?
- How does technology driven mathematical modelling activities affect student's perception of paper and pencil tasks?
- Does the integration of modelling and applications help to sustain and enhance student's interest in the subject?

Educational Setting

All 30 students who took part in the study were from two Delhi schools which follow the curriculum prescribed by the Central Board of Secondary Education (CBSE), a national board for school education in India. The CBSE (2009) does not prescribe the use of technology for teaching mathematics nor does it permit its use in examinations. Individual schools however have the freedom to integrate technology in their classrooms. The topic, Matrices and Determinants comprises of the subtopics on types of matrices, matrix operations, inverse of a matrix, computing determinants and their properties and solutions of systems of equations. The emphasis is on manipulation and on developing computational skills in dealing with matrices and determinants. The exercises at the end of a chapter also test computational skills. Two sample exercises from the textbook are

$$\text{If } A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}, \text{ show that } A^2 - 5A + 7I = O$$

Using properties of determinants prove that

$$\begin{vmatrix} a^2+1 & b^2 & c^2 \\ a^2 & b^2+1 & c^2 \\ a^2 & b^2 & c^2+1 \end{vmatrix} = 1 + a^2 + b^2 + c^2$$

The 30 students who were a part of the study had been taught matrices in a traditional manner in their regular class. However the manner in which the content was presented did not focus on the need and use of matrices in solving or modelling practical problems. In the module designed by the author, the emphasis was on integrating applications and mathematical models based on matrices to introduce concepts and procedures thereby highlighting the relevance of matrices as a tool for solving problems.

This section includes a detailed description of the laboratory module in which 30 grade 12 students explored the topic of Matrices and Determinants through applications and mathematical models enabled by technology. In some of the sessions students were given worksheets which required them to explore a problem or a mathematical model and record their solutions or observations. During the module students were given access to graphics calculators for computational work. While solving systems of equations in three unknowns, Mathematica was used to plot the planes to help visualize the solution. Throughout the module students were encouraged to do certain procedures, such as, multiplying 2 by 2 matrices, finding determinants of 3 by 3 matrices or solving a system of equations in matrix form, by hand. The objective was to maintain a balance between technology enabled explorations and procedural skills. The graphics calculator helped to trivialize tedious calculations thus enabling students to focus on exploring the models and interpreting the solutions. The applications or mathematical models discussed in the module were taken from Genetics, Cryptography and the PageRank Algorithm. Before transacting the module a short workshop session on the graphics calculator (Casio FX 9860) was conducted to familiarize the students with some of the basic features of the calculator.

Session 1: Introduction to Matrices

Students were introduced to the concept of a matrix using the following two examples:

Example 1

The results of a chess championship between four players A, B, C, D are as follows:

A defeated B and C; B defeated D; D defeated A; C defeated B and D.

$$M = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

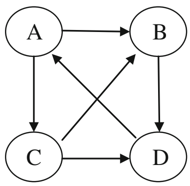

Fig. 1 Network diagram and incidence matrix representing the results of a tournament

This information may be represented in the form of the network diagram shown in Fig. 1. The arrows point from winners to losers. The information regarding wins and losses can also be arranged in the form of a matrix using 0's and 1's.

In the matrix M, rows 1, 2, 3 and 4 as well as columns 1, 2, 3 and 4 denote the players A, B, C and D respectively. This is a first order dominance matrix where 1's denote the victories and 0's the losses. Thus, 1 in position (1, 2) i.e. in row 1 and column 2, represents the fact that player A has defeated player B.

Example 2

In this example students were introduced to Autosomal inheritance, where the inherited trait under consideration (say petal color in a certain plant) is assumed to be governed by a set of two genes, denoted by **A** (red color) and **a** (white color). The three possible genotypes are **AA**, **Aa** and **aa** where **AA** produces red flowers, **Aa** produces pink flowers and **aa** produces white flowers. Every individual inherits one gene from each parent plant with equal probability. Thus, if the parent pairing is **AA-Aa**, the offspring will inherit an '**A**' gene from the first parent and either an '**A**' or an '**a**' (with equal probability) from the second parent. Thus the offspring is likely to inherit a genotype of **AA** or **Aa** with probability $\frac{1}{2}$ each.

Students were asked to list all possible parent pairings along with the probabilities of the resulting offspring combinations which led to the *genotype probability matrix* as shown in Table 1. This example was an interesting way of recalling the concept of probability and highlighting the fact that the entries of a matrix could also be probabilities.

By the end of this session students were introduced to a matrix as a mathematical model for organising information in rows and columns. They were also acquainted with different practical situations in which matrices were used to represent information.

Table 1 The genotype probability matrix

Parent pairings							
Offspring outcomes		AA-AA	AA-Aa	AA-aa	Aa-Aa	Aa-aa	aa-aa
	AA	1	1/2	0	1/4	0	0
	Aa	0	1/2	1	1/2	1/2	0
	aa	0	0	0	1/4	1/2	1

Session 2: Matrix Operations

The second session began with the definition of a matrix, the notion of order of a matrix, symbolic representation in the form $A = [a_{ij}]_{m \times n}$ and different types of matrices (square matrix, row and column matrices, zero and identity matrices) examples of which were taken from the above matrix models as well as other examples. The operations of matrices were also introduced with the help of the above models. For example, matrix multiplication was introduced using a garment manufacturing example where the cost incurred by each of three factories for manufacturing three types of garments was obtained by multiplying the production matrix P to the cost matrix C.

$$\text{PC} = \begin{bmatrix} 3000 & 1000 & 1500 \\ 5000 & 1650 & 2000 \\ 3500 & 1450 & 1000 \end{bmatrix} \begin{bmatrix} 1500 \\ 800 \\ 500 \end{bmatrix} = \begin{bmatrix} 3000 \times 1500 + 1000 \times 800 + 1500 \times 500 \\ 5000 \times 1500 + 1650 \times 800 + 2000 \times 500 \\ 3500 \times 1500 + 1450 \times 800 + 1000 \times 500 \end{bmatrix}$$

This example helped to highlight the necessity of multiplying the matrices row by column. Students were then made to explore and observe the properties of matrix multiplication by working out products of matrices of different orders. By doing this they realized that matrices of different orders could be multiplied *if and only if the number of columns of the matrix on the left is equal to the number of rows of the matrix on the right.* Students used the graphics calculator for computing products of matrices of larger orders.

For squaring a matrix they revisited the chess tournament example and calculated M^2. The author asked them to find the relationship between M^2 and the results of the tournament. After some facilitation students figured out that each row of M^2 denotes the second order victories of each player over the others. E.g the number 1 in row 1 of M^2 indicates that player A has had one second order victory over B (i.e. A defeated C and C defeated B). After this, students were required to combine the results of the first order victories and the second order victories by evaluating the expression $M + \frac{1}{2}M^2$. This exercise was a way to get them to use the operations of addition, multiplication as well as scalar multiplication. The row sums of the resultant matrix were used to rank the teams.

By the end of this session students were comfortable with performing matrix operations by hand as well as on the graphics calculator. There was no compromise on procedural skills as most students preferred to solve the exercises by hand and then verify their solutions using the graphics calculator.

Sessions 3 and 4: Investigations Based on Matrix Operations

In this session the transpose of a matrix and its properties was introduced. To verify the property that the transpose of the product of two matrices is the product of the transposes taken in the reverse order, students were made to revisit the garment manufacturing example in which they worked out $(PC)^{\mathrm{T}}$ and verified that it is equal to $C^{\mathrm{T}}\,P^{\mathrm{T}}$.

Having completed the above exercise, students explored the example on Autosomal inheritance. The problem was to create a model which could predict the genotype distribution of a plant population after any number of generations under specific breeding programs. For formulating the problem it was assumed that a_n, b_n and c_n are the fraction of plants of genotypes **AA**, **Aa** and **aa** in the nth generation of the plant population where $n = 0, 1, 2\ldots$ and $a_n + b_n + c_n = 1\ \forall\ n$. a_0, b_0, c_0 were used to denote the initial distribution of genotypes **AA**, **Aa** and **aa** respectively in the population. Students were first introduced to the case when all the plants are fertilized with type **AA**. The genotype distributions a_n, b_n and c_n were represented by the equations:

$$a_n = a_{n-1} + \frac{1}{2}b_{n-1}, \quad b_n = \frac{1}{2}b_{n-1} + c_{n-1}, \quad c_n = 0. \tag{1}$$

Students were then introduced to write the above equations in matrix notation as $x^{(n)} = Mx^{(n-1)}$, where $n = 1, 2,\ldots$ and

$$X^{(\mathrm{n})} = \begin{bmatrix} a_n \\ b_n \\ c_n \end{bmatrix} \text{ and } \mathrm{x}^{(\mathrm{n}-1)} = \begin{bmatrix} a_{n-1} \\ b_{n-1} \\ c_{n-1} \end{bmatrix} \text{ and } M = \begin{bmatrix} 1 & 1/2 & 0 \\ 0 & 1/2 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

They verified using matrix multiplication that $x^{(n)}$ when equated to the product $Mx^{(n-1)}$ actually led to the Eq. (1). They used the equation $x^{(n)} = Mx^{(n-1)}$ to arrive at $x^{(n)} = M^n x^{(0)}$ where $x^{(0)}$ is the matrix of the initial genotype distribution. Students were able to do this easily by repeatedly putting n as 1, 2, 3 etc. This matrix equation was used to predict the distribution of genotypes in the nth generation for a given initial distribution $x^{(0)}$.

The next exercise required them to use graphics calculators and the equation $x^{(n)} = M^n x^{(0)}$ to find the genotype distribution of a plant population in the first, second, third, fourth and fifth generations when the initial distributions $x^{(0)}$were given as

$$\begin{bmatrix} 0.5 \\ 0.3 \\ 0.2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0.5 \\ 0.5 \end{bmatrix}, \begin{bmatrix} 0.2 \\ 0.25 \\ 0.55 \end{bmatrix}$$

Figure 2 gives the screen shots of their calculations.

After completing the calculations the students interpreted the solutions in terms of the proportion of the three genotypes in the population after 5 generations. Some students began to calculate for the 6th and 7th generations and then calculated for the 10th generation. They concluded that in the long run after a certain number of

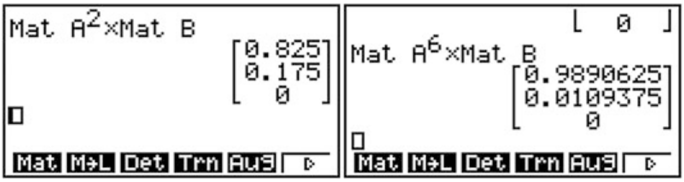

Fig. 2 Graphics calculator screenshots showing matrix calculations for finding the steady state genotype distribution

generations, the proportions of the three genotypes steady out to the matrix $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and that this is independent of the initial distribution. They then changed matrix M to explore the case when each plant is fertilised with type **Aa**. Here the steady state distribution was obtained as $\begin{bmatrix} 0.25 \\ 0.5 \\ 0.25 \end{bmatrix}$. Thus the calculator took over the computational part of the modelling process and enabled students to extend their investigations. They explored the steady state distribution in other breeding situations such as:

- If each plant was fertilised with a plant of its own genotype.
- If alternate generations of plants were fertilised with genotypes AA and Aa respectively.

Sessions 5 and 6: Solutions of Systems of Equations

In this session of the module, students used Mathematica to visualize the solutions of systems of equations in three unknowns by plotting the planes representing the equations in three unknowns. They used the graphics calculator for solving the equations. They were encouraged to work in pairs or in groups of three and each group was given a worksheet with five systems of equations in the unknowns x, y and z as follows.

$$\begin{array}{ccc} x+y+z=2 & x+y+z=1 & x+y+z=1 \\ x+3y-z=1 & x+4y+9z=3 & x+y+z=7 \\ -x+4y+9z=3 & 2x-y-6z=0 & 2x+2y+2z=25 \\ \text{(i)} & \text{(ii)} & \text{(iii)} \end{array}$$

$$\begin{array}{cc} x-2y+3z=1 & x+y+z=1 \\ 2x-y+2z=3 & x+y+z=12 \\ x+y-z=4 & 8x+y-6z=0 \\ \text{(iv)} & \text{(v)} \end{array}$$

Students were introduced to the method of solving systems of equations by reducing them to triangular form using elementary row operations. The process of reducing the system (i) to triangular form was demonstrated on the graphics calculator. The system (i) had a *unique solution* $x = \frac{3}{2}$, $y = 0$ and $z = \frac{1}{2}$. Figure 3 shows the screen shots of the row operations performed on their graphics calculator.

The students also verified their solution using Mathematica's **Solve** command. This was repeated for all five systems of equations. After solving the system of equations (ii) the last equation was read as 0.z = 0. This was interpreted as a case of *infinitely many solutions* where z may be treated as a free variable. For the systems of equations (iii), (iv) and (v) row reduction led to an equation of the form 0.z = k which was interpreted as inconsistency. Students then used Mathematica's **Plot3D** command to plot the planes for all five systems. The plot for (i) showed three planes meeting at a point. The output for (ii) revealed three planes meeting in a line. See Fig. 4. For (i) students interpreted that the point where the three planes meet must be the unique solution having the coordinates $x = \frac{3}{2}$, $y = 0$ and $z = \frac{1}{2}$. For (ii) they made the guess that since any point on the line (where the three planes meet) is a solution of the system, this was identified as a case of infinitely many solutions.

The Mathematica outputs for (iii), (iv) and (v) revealed three different situations of inconsistency. In (iii) the planes were parallel. Since the planes do not meet, students concluded that the system was inconsistent. The plot of (iv) revealed that

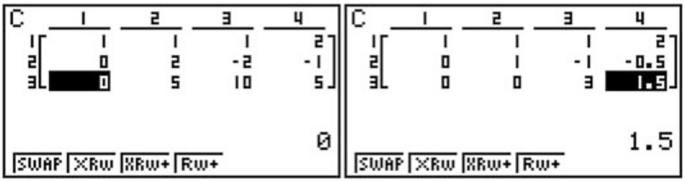

Fig. 3 Graphics calculator screenshots for solving a system of equations using elementary row operations

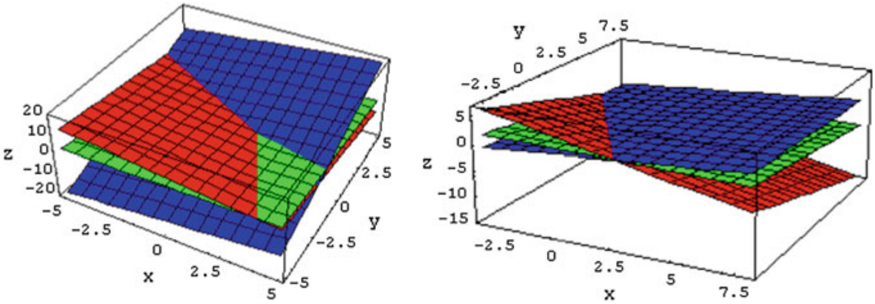

Fig. 4 Mathematica plots of the planes representing systems of equations (i) and (ii)

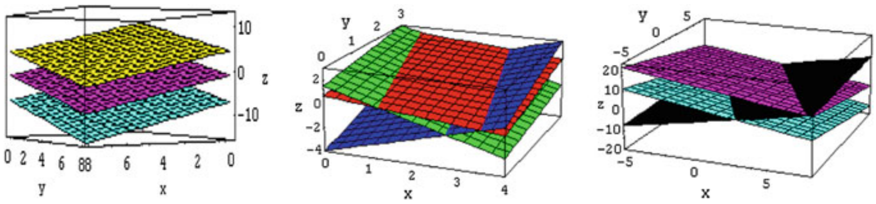

Fig. 5 Mathematica plots of systems of equations (iii), (iv) and (v) showing three different situations of inconsistency

the *three planes intersected pair wise in three non-planar parallel lines*. Since all three planes do not intersect in a single point or line, students concluded that the three equations are inconsistent. For (v) students concluded that *two parallel planes were intersected in two parallel lines by a third non-parallel plane*. Since all three planes do not intersect in a single point or line the three equations are inconsistent. Visualizing the systems of equations in the form of planes in three dimensions led to a graphical insight which would not have been possible without the 3-D graphing feature of Mathematica (Fig. 5).

The PageRank Algorithm

As a practical example of arriving at systems of equations in more than two unknowns students were introduced to the Google PageRank Algorithm. The pagerank of a webpage is calculated using the following ("PageRank," n.d., para. 18)

$$PR(A) = 1 - d + d\left(\frac{PR(T_1)}{C(T_1)} + \frac{PR(T_2)}{C(T_2)} + \cdots + \frac{PR(T_n)}{C(T_n)}\right) \quad (2)$$

where PR denotes the page rank of a page, T_i are pages which link to A, $C(T_i)$ are the outbound links from page T_i, and d is the damping factor (usually taken as 0.85).

Students were asked to apply the formula to a simple website comprising of two pages A and B as shown in Fig. 6. The arrows indicate that there is a link from page A to page B and vice versa.

Considering d = 0.5 and using (2) they obtained the following equations.

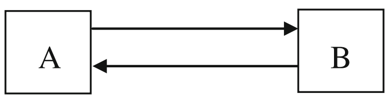

Fig. 6 A diagram representing a website consisting of two pages

$$\begin{aligned} \mathrm{PR(A)} - 0.5\,\mathrm{PR(B)} &= 0.5 \\ -0.5\,\mathrm{PR(A)} + \mathrm{PR(B)} &= 0.5 \end{aligned} \tag{3}$$

In matrix form, (3) can be written as MX = B, where

$$M = \begin{bmatrix} 1 & -0.5 \\ -0.5 & 1 \end{bmatrix}, \quad X = \begin{bmatrix} PR(A) \\ PR(B) \end{bmatrix}, \quad B = \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}$$

To obtain the page ranks of pages A and B it was required to solve the matrix equation MX = B. At this point the idea of an inverse of a matrix was introduced. Students were told that if they could find a matrix N such that NM = I, the identity matrix, then by multiplying both sides of the equation MX = B by N one could obtain

N(MX) = NB leading to (NM)X = NB or IX = NB.

Thus X = NB. Since N is the inverse of M we can write N as M^{-1}. Hence $X = M^{-1}B$.

The process of finding the inverse of M was however not discussed here. Students used the graphics calculator for solving the equation $X = M^{-1}B$ (Fig. 7) and arrived at the solution of the matrix equation as the pagerank vector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, which implied that PR(A) = PR(B) = 1.

The next exercise required the students to consider the website in Fig. 8, comprising of the four pages A, B, C and D and to find the pageranks of the pages using d = 0.85 as the damping factor.

Considering PR(A) = x, PR(B) = y, PR(C) = z and PR(D) = w, students used (2) and obtained four equations in four unknowns x, y, z and w.

$$\begin{aligned} &x - 0.85\,z = 0.15, -0.425\,x + y = 0.5, -0.425\,x - 0.85\,y - 0.085\,w = 0.15, \\ &w = 0.15 \end{aligned} \tag{4}$$

They solved the equations by reducing the augmented matrix of (4) to triangular form.

The primary objective of this session was to help students visualize solutions of equations in three unknowns and interpret them graphically. An attempt was made to ensure that there was no compromise on the 'by hand' skills as students were also

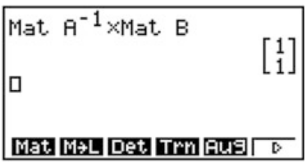

Fig. 7 Graphics calculator screenshot for computing the pagerank vector

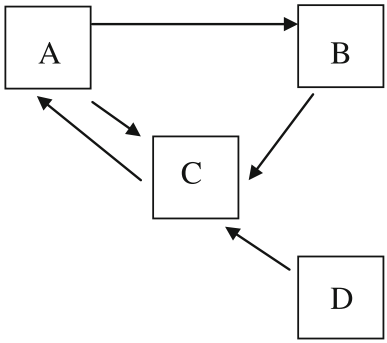

Fig. 8 A website consisting of four pages

required to work out the solutions manually by reducing the system of equations to triangular form. The Mathematica plots gave a physical meaning to their by-hand solutions. Introducing them to the PageRank algorithm aroused their interest since all students are Google surfers and importance of webpages was something they were interested in. It also led to the need for solving a system of equations.

Sessions 7 and 8: Exploring the Inverse of a Matrix

In the last session of the module students were introduced to the idea of the inverse of a matrix and were made to find the inverse of any 2 by 2 matrix A = $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ by solving the matrix equation $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x & y \\ z & w \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Many students asked whether there was a practical use of the inverse of a matrix.

Secret Codes Using Matrices: The Hill Cipher Method

At this point they were introduced to the Hill Cipher method (Eisenberg 1999), an application of matrices to cryptography, the science of making and breaking codes. Ciphers are methods for transforming a given message, the plaintext, into a new form that is unintelligible to anyone who does not know the key (the transformation used to convert the plain text). In a cipher the key transforms the plaintext letters to other characters. The secret rule, that is, the inverse key, is required to reverse the transformation in order to recover the original message. The students were given 29 characters and their numerical values as shown in Table 2.

The encoding matrix was chosen as $\begin{bmatrix} 1 & 4 \\ 2 & 9 \end{bmatrix}$. Students were then introduced to the encoding process (converting the plaintext to the ciphertext) as follows:

Table 2 The substitution table for the Hill Cipher method

A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
P	Q	R	S	T	U	V	W	X	Y	Z	.	_	?	
15	16	17	18	19	20	21	22	23	24	25	26	27	28	

Step 1. Let the plaintext message be **MATH_IS_FUN**.

Step 2. Convert it to its substitution values from the substitution table and group them in pairs

12 019727 8 18 27 5 2013 26

Each pair will form a column of the message matrix M.

Step 3. Compute the product EM

$$EM = \begin{bmatrix} 1 & 4 \\ 2 & 9 \end{bmatrix} \begin{bmatrix} 12 & 19 & 27 & 18 & 5 & 13 \\ 0 & 7 & 8 & 27 & 20 & 26 \end{bmatrix}$$
$$= \begin{bmatrix} 12 & 47 & 59 & 126 & 85 & 117 \\ 24 & 101 & 126 & 279 & 190 & 260 \end{bmatrix}$$

Step 4. Reduce the product modulo 29 to obtain the Hill-2-cipher values. This is called a Hill-2-cipher since the encoding matrix is a 2 by 2 matrix. Students required help in understanding the concept of reducing a number modulo 29. The meaning of the congruence relation a ≡ b (mod c) was explained using various examples.

$$EM = \begin{bmatrix} 12 & 47 & 59 & 126 & 85 & 117 \\ 24 & 101 & 126 & 279 & 190 & 260 \end{bmatrix}$$
$$= \begin{bmatrix} 12 & 18 & 1 & 10 & 27 & 1 \\ 24 & 14 & 10 & 18 & 16 & 28 \end{bmatrix} \pmod{29}$$

Step 5. The encrypted message is **MYSOBKS_QB?**

Some screenshots of the encoding process is shown in Fig. 9.
The decoding process was introduced as follows:

Step 1. The ciphertext message reaches the receiver as **MYSOBKS_QB?**

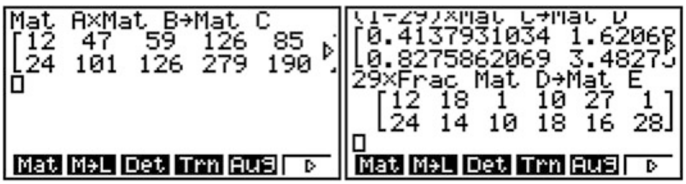

Fig. 9 Graphics calculator screenshots for the encoding process of the Hill Cipher method

Step 2. Convert the characters to their respective Hill-2-cipher values and make pairs.

12 24 18 14 1 10 10 18 27 16 1 28

Each pair will form a column of a matrix N.

Step 3. Compute the product $E^{-1}N$

$$E^{-1}N = \begin{bmatrix} 9 & -4 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 12 & 18 & 1 & 10 & 27 & 1 \\ 24 & 14 & 10 & 18 & 16 & 28 \end{bmatrix}$$

$$= \begin{bmatrix} 12 & 106 & -31 & 18 & 179 & -103 \\ 0 & -22 & 8 & -2 & -38 & 26 \end{bmatrix}$$

Step 4. Reduce the product modulo 29 to obtain the substitution values.

$$E^{-1}N = \begin{bmatrix} 12 & 106 & -31 & 18 & 179 & 103 \\ 0 & -22 & 8 & -2 & -38 & 26 \end{bmatrix}$$

$$= \begin{bmatrix} 12 & 19 & 27 & 18 & 5 & 13 \\ 0 & 7 & 8 & 27 & 20 & 26 \end{bmatrix}$$

When the values are written column wise they become

12 0 19 7 27 8 18 27 5 20 13 26

which translates to **MATH_IS_FUN**.

Students found this technique very exciting. They wanted to try out the more examples and were divided into groups. Each group had to encode a message and post it on the whiteboard after declaring the encoding matrix. The ‘receivers’ had to use the inverse matrix to decode the message. Students were asked to choose encoding matrices with determinant equal to 1. Though it was a challenge, students were able to come up with 3 by 3 matrices with determinant 1. Two groups used 3 by 3 matrices as encoding matrices while the other three groups chose 2 by 2 matrices.

All groups were able to decode the messages successfully. But the natural question which arose was what happens in the decoding process if the encoding matrix had a determinant other than 1. Also the other question which arose was, is it possible for a person (other than receiver or sender) to crack the code, that is, figure out the encoding matrix if the plaintext and ciphertext are known. Both had to be addressed in a separate special class which was not a part of the module. Here students had to be introduced to the following ideas.

In Z_n, the set of integers modulo n, every element has an inverse in the set provided n is prime. Hence the technique required that the number of characters be prime. If each element in Z_n did not have an inverse within Z_n then decoding would not be possible when the encoding matrix had a determinant which did not have an inverse within the set.

The Cracking theorem
Suppose the length m of the alphabet is a prime. Let P and C be the plaintext matrix and ciphertext matrices respectively.

$$p = [\vec{p}_1 \quad \vec{p}_2 \quad \vec{p}_3 \quad \ldots \quad \vec{p}_n] \quad C = [\vec{c}_1 \quad \vec{c}_2 \quad \vec{c}_3 \quad \ldots \quad \vec{c}_n]$$

Then the elementary row transformations that reduce C^T to the identity matrix I also reduce the matrix P^T to $(A^{-1})^T$.

This project gave students a flavor of the practical use of manipulating matrices and their inverses. Also students had to be familiarized with some concepts in number theory which were beyond the scope of the curriculum. This application of matrix theory to a cryptographic technique gave students access to higher level mathematical concepts. The computational aspects of multiplying matrices of higher orders, inverting matrices as well as reducing them modulo 29, were performed on the graphics calculator. Since students were able to 'outsource' the computations to the calculator they were free to think about the mathematical aspects of the Hill cipher technique.

Results of the Study

At the end of the module students were asked to respond to a short questionnaire of 12 items by entering a number from 1 to 5, where the numbers indicated the following.

1-Strongly disagree, 2-Disagree, 3-Not sure, 4-Agree, 5-Strongly agree

The data of student's responses are shown in Table 3. They were also asked to give a written feedback in terms of specific comments describing their impressions regarding how the module helped (or did not help) them.

Table 4 shows the sample means and standard deviations for agreement scores (1–5) for selected items of the questionnaire, shown in Table 3. These have been used to estimate the population mean scores by calculating 95 % confidence intervals. A mean score of 3.75 and above indicates agreement with the statement given in a particular item. For items 1, 2, 3, 5, 6 and 8 the interval estimates indicate strong agreement and the interval estimate for item 4 indicates that respondents are not sure about the statement. These estimates show that majority of students agree that the module is more interesting than traditional classroom teaching (item 1) and that it helped them to see the relevance of mathematics to realistic problems (item 2). One student commented, "The way the concepts were taught with applications was the best part of this module" and another wrote, "I particularly found the Hill Cipher method very useful and interesting. Including such models based on matrices made the topic very interesting". Majority of students agreed that such modules should become a part of the regular curriculum (item 3) although they had reservations about including questions based on modelling and applications in the examinations (item 4). This was probably due to the fact that they had been

Table 3 Responses of grade 12 students (N = 30) to questionnaire

Item No.	Item	SA	A	NS	D	SD
1.	I found this module on matrices and determinants more interesting than the traditional classroom teaching	7	16	7	–	–
2.	This module has helped me to see the relevance of mathematics to real life	12	16	2	–	–
3.	Such modules should become a part of regular school curriculum	21	7	2	–	–
4.	Questions based on modelling and applications should be included in examinations	2	5	16	7	–
5.	The inclusion of applications and modelling in this module played a key role in my understanding of the concepts	9	18	3	–	–
6.	My interest in mathematics will increase if topics are taught by including modelling and applications	21	7	2	–	–
7.	Mathematics as a subject will seem less abstract if topics are taught with the help of modelling and applications	11	11	6	–	2
8.	I feel that in this module technology helped me to visualize and explore concepts and thus gain a deeper insight into the subject	9	18	3	–	–
9.	I feel that technology played a key role in trivializing calculations so that more time could be spent on exploring the mathematical models	6	14	7	3	–
10.	My confidence level in solving problems related to matrices has increased after going through this module	5	16	7	2	–
11.	The inclusion of modelling and applications may have a detrimental (harmful) effect on my paper and pencil skills	–	–	11	5	14
12.	I feel technology should be used to solve problems which cannot be done by hand	18	5	2	5	–

introduced to this approach for the first time and it would take them more time to be comfortable enough to be able to attempt problems based on applications in examinations. Students also agreed that the applications played a key role in helping them understand concepts (item 5) and that their interest in mathematics would increase if such modules were included in other topics of the curriculum (item 6). A student commented, “The module helped to understand why we are using certain formulae and procedures. The regular classes do not deal with these things.” Another reflected, “this module helped to clear my concepts and doubts.” They disagreed that focus on modelling and applications can have a harmful effect on their manipulative skills (item 11) and also agreed that technology should be used to solve problems which are difficult to solve by hand.

Table 4 95 % confidence interval estimates for population mean of students responses

Item No.	Sample mean (N = 30)	Sample standard deviation	Population mean (95 % confidence interval)	
			Upper bound	Lower bound
1	4.00	0.69	4.25	3.75
2	4.33	0.61	4.55	4.12
3	4.63	0.61	4.84	4.41
4	3.07	0.83	3.36	2.77
5	4.20	0.61	4.42	3.98
6	4.63	0.61	4.85	4.41
8	4.20	0.61	4.42	3.98
11	1.90	0.92	2.23	1.57
12	4.20	1.16	4.61	3.70

The responses based on the role of technology were also insightful. Students felt that technology contributed to a deeper understanding of concepts by aiding visualization and exploration (item 8) and that it helped to trivialize calculations thereby allowing more time for exploration (item 9). They also agreed that technology should be used to solve problems which cannot be solved by hand (item 12). One student observed, "The use of graphics calculators helped a lot. We focused on the concepts and problems but the calculator did all the lengthy calculations. The calculator made the sessions more interactive". Commenting on the use of Mathematica, one student wrote "Mathematica helped us to actually see the solutions of equations in three unknowns. The methods taught in class only helped us find the solutions but I could never understand what they looked like. Now I know, thanks to the 3-D graphing feature of Mathematica."

Conclusion

This paper describes a module titled *Learning Mathematics through Mathematical Modelling and Applications* which was undertaken by 30 students of grade 12. The module was based on the topic *Matrices and Solutions of Systems of Equations* and was spread over 8 two hour sessions. In each session, students were introduced to concepts of the topic through mathematical models and applications. After an introduction to the mathematical theory on which the models were based, students were required to explore the models through tasks and problems which were posed to them in worksheets. Their explorations were facilitated by the researcher and they were given access to graphics calculators throughout the module. In many cases it was found that students preferred to do the calculations by hand and then verify their answer on the calculator. However in some explorations, such as finding the genotype distribution of a plant population after several generations, students were required to raise a matrix to higher powers. Here the calculator helped to trivialize the computations and enabled the students to focus on exploring the problem which led to a deeper insight. As Herwaarden and van Gielden (2002) study suggests the integration of technology helped to create a connection between paper-pencil methods and calculator manipulations. In the Hill Cipher method, the lengthy calculations such as multiplying matrices of large orders and reducing matrices modulo 29, was 'outsourced' to the calculator. Throughout the module the graphics calculator served as a 'mathematical investigation assistant', as proposed by Arnold (2004), giving students control over what they were learning. While exploring the solutions of three equations in three unknowns, paper pencil methods helped students to compute the solutions while Mathematica gave a physical meaning to their computations. Thus, as suggested by Lagrange (1999), Mathematica helped to balance 'by hand' calculations and conceptual understanding. The exploration of the mathematical applications such as the PageRank algorithm and Hill cipher method provided the students with new contexts where they could apply the theory of matrices. They also learnt new mathematical content, some of which was far beyond the grade 12 level. For example, the Hill cipher method led to the discussion of the some ideas in number theory and the Cracking Theorem which was beyond the curriculum. The technology enabled explorations of the mathematical applications gave students access to higher level mathematical concepts and this supports Heid's (2001) theory that technology acts as an 'amplifier'.

It may be concluded that integrating mathematical modelling and applications for teaching the topic of Matrices led to a very satisfying combination of technology use and 'by-hand' skills. The study supports Lindsay (1995) that, when properly used, technology enabled explorations can augment learning and provide a rich and motivating environment to explore mathematics. It also supports the educational perspective of mathematics education research on teaching and learning of modelling which asserts that including modelling in the curriculum helps to highlight the relevance of mathematics as a discipline. Through the module described in this paper students not only learnt new mathematical content but also obtained a glimpse of how mathematics can be applied to solve realistic problems.

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Doing Mathematics in Teacher Preparation: Giving Space and Time to Think, Reflect, Share and Feel

Frédéric Gourdeau

Abstract Describing the minimal mathematical content knowledge needed for secondary school teachers is not the most useful way to approach the mathematical preparation of teachers. Rather, focusing on the doing of mathematics, on the quality of their engagement with mathematics, is crucial. In doing so, I argue that the role of mathematicians in the mathematical preparation of teachers is not reduced but rather enhanced: it is work of a different nature than is often argued, leaning more on the expertise of the mathematician in the doing of mathematics than on his or her knowledge of the facts of mathematics. This talk is based on work done at Université Laval (Québec, Canada) for the past 17 years, mostly with Bernard R. Hodgson. We have developed a series of courses for teachers (in pre-service training mostly) which aim to engage them fully in the doing of mathematics. What do we mean by engaging them fully in the doing of mathematics? How do we try to achieve this? We will describe our approach through some examples, outlining important aspects which need to be taken into account: for instance, allowing genuine exploration of mathematical problems, working on the communication of ones' understanding (paying attention to words, definitions and statements) and learning to identify mathematical processes. These aspects are generally largely independent of the actual mathematical topics at hand and, in that sense, the main objectives pursued in each of these courses have little to do with the precise mathematical content. Even though our reflections are based on a specific curriculum in a specific setting, we believe that some of the reflections we wish to share will have resonance with many outside Canada.

Keywords Teacher education · Doing of mathematics · Mathematics preparation · Mathematicians' role

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Introduction

This paper is based on work done at Université Laval (Québec, Canada) for the past 17 years, mostly with Bernard R. Hodgson. It is also based on many conversations with colleagues in mathematics education through the Canadian Mathematics Education Study Group (CMESG), a group which includes mathematicians and mathematics educators and which has served as an inspiration for many educational initiatives in Canada. The spirit of collaboration which characterizes CMESG lives in our approach to mathematics education, and this paper is offered with this same approach, that of sharing and reflecting about mathematics education.

At Université Laval, we have developed a series of courses for mathematics teachers (in pre-service education) which aim to engage them fully in the doing of mathematics. We give mathematics courses (as opposed to Mathematics Education courses—more precisely *didactique des mathématiques*) and hold positions at the Department of mathematics and statistics.

What do we mean by engaging them fully in the doing of mathematics? How do we try to achieve this? And why is this part of mathematics courses and not only mathematics education? The aim of this paper is to answer these questions: the title of the paper gives some of the crucial elements of our answer.

The paper is organized as follows. After a brief description of some important aspects of the pre-service teacher program, we describe activities which are part of mathematics courses in pre-service education of secondary school mathematics teachers at our institution. We present these activities along with their objectives. We conclude with a reflection on our role as mathematicians in this type of work.

Context

The B. Ed. Program in Quebec (Canada)

In Canada, Education is a provincial responsibility and therefore the description which follows only applies to the province of Quebec. Students enter university after 13 years of study (a K-11 system followed by 2 years in a separate college system—the CEGEPs). Most university degrees are 3 years.

Students who wish to become teachers must enroll in a four-year degree in education (bachelor degree in education). One of these programs is the *Baccalauréat en enseignement au secondaire—mathématiques* which is the only program enabling one to teach mathematics at the secondary school level (years 7 to 11 of the K-11 system). Hereafter, this is the program we will be considering: it will be abbreviated as B. Ed. Math.

Upon entering the B. Ed. Math. program, students have completed a basic linear algebra course as well as differential and integral calculus. They have done few proofs and rarely had to come up with a proof of their own.

The B. Ed. Math. program at *Université Laval* is a four-year integrated degree. A year comprises two fifteen-week semesters, and generally students have courses or other learning activities for a total of 15 units for each semester (the B. Ed. Degree having 120 units). Most courses are worth 3 units.

The B. Ed. Math. program comprises 14 courses in mathematics and statistics (42 units) taught by the Department of Mathematics and Statistics and 3 courses in *Didactique des mathématiques* (Mathematics Education) which are taught by colleagues at the Faculty of Education. Most other courses in the program are also under the responsibility of the Faculty of Education (with some exceptions) and there are important internship activities.

Of the 14 courses in mathematics and statistics, 8 are also part of the undergraduate degree in mathematics. These are Euclidean geometry, Elements of mathematics, Discrete mathematics, Linear algebra, Calculus of functions of several variables, Real analysis, Number theory, and Probability and statistics.

This paper deals with the other courses of mathematics and statistics. These courses are taken only by students enrolled in the B. Ed. Program, and we will refer to these as the *B.Ed. specific courses*. The six B. Ed. specific courses are regular one-semester course, with 45–60 h total contact time (in class), and somewhere between 20 and 45 students enrolled in a course.

The Students

Some words about our students are important to set the tone. Prior to entering university, by enlarge, they have had positive experiences in mathematics and hold fairly conventional views on the teaching of mathematics. They have been reasonably successful as students, have chosen to do mathematics and are generally not on a revolutionary path to change education!

Courses such as Euclidean geometry, Elements of mathematics, Discrete mathematics and Real analysis place a lot of importance on proofs, and this is difficult for most students when they begin their university education. They are not well prepared for that, being used to evaluations which place the emphasis on algorithmic mastery (though a recent reform in the secondary school curriculum places more importance on problem solving and communication, still some distance from proofs and rigor). Some experience real difficulties in mathematics for the first time. They try to learn proofs, not understand them, and are dismayed at the importance placed on them. The confidence of many in their mathematical abilities is affected.

The B. Ed. specific courses are being offered from the fourth semester onwards and therefore after the initial year. By the time they come into our B. Ed. specific courses, they have learnt many proofs, have some idea of rigor, but have not had the chance to reflect about mathematics itself.

The Objective of B. Ed. Specific Courses

Even though each course has its own set of objectives, these courses share a common purpose of engaging students fully in the doing of mathematics. Engaging them fully implies that they should: think for themselves; reflect on their engagement with mathematics; reflect on mathematics itself; share with others about their understanding of mathematics; be emotionally engaged and be aware of their emotions.

This is a partial list but it is already very ambitious. How do we think that this may be achieved, even partly? Let us give some sample activities from our courses.

Problem Solving

In North America, problem solving in mathematics is often associated with the William Lowell Putnam Mathematical Competition, an elite competition where the emphasis is solidly placed on performance, or on other such competitions. A *problem* in this context is a statement which can be answered within a very limited time-frame: it is unambiguous (or aims to be), and is definitely not aimed at encouraging exploration or one's own original take on the subject (there is a good answer and one aims to obtain it). A problem solving course is often understood as a course which aims to provide student with strategies, heuristics into problem solving, and also tools and techniques. [Some books go beyond that, obviously, and an excellent book for this type of course, richer than most, is Zeitz (2006)].

For those studying to be teachers, the personal mastery of tools and techniques and the personal expertise at problem solving is not what is most important. Of course, this is not to say that this is of no importance at all, but rather that other aspects are more important, like the ability to engage their students in problem solving. Understanding the complexity of problem solving, having a vocabulary to describe its processes and phases and being able to identify them is then very important. (It may not help you do better in problem solving, but it will help you describe it and help others do it.)

With this partial analysis in mind, the nature of the work in problem solving done by the B. Ed. students has been structured as follows.

The Problem Solving Portfolio

The first part of their work is a personal problem solving portfolio (which is referred to as *journal de résolution*). It is based on their individual work with *Thinking mathematically* (Mason et al. 1982). The work spans 8 weeks, and students hand in a manuscript of somewhere between 50 and 200 pages.

In their work, students resolve problems of their choosing: there is a very nice collection in the book. Let us give two examples of such problems, where we have reformulated the second slightly to make the description self-contained.

Diagonals of a Rectangle: On squared paper, draw a rectangle 3 squares by 5 squares, and draw in a diagonal. How many grid squares are touched by the diagonal?

Ins and Outs: Imagine a long thin strip of paper stretched out in front of you, left to right. Imagine taking the ends in your hands and placing the right hand end on top of the left. Now press the strip flat so that it is folded in half and has a crease. Unfold it and observe that some of the creases are IN and some are OUT. For example, three folds produce the sequence: in in out in in out out. What sequence would arise from 10 folds (if that many were possible)?

These are the initial question proposed in the book. However, the initial question is meant to be a starting point. Readers are asked to generalize, to propose and solve extensions, etc. There is no end to what can be investigated. For instance, in the paper folding problem, one student decided to find out how to predict if the *k*th crease was in or out according to the value of k: I saw that extension only once. Students can work for many hours, over a period of a few days to a few weeks, on any given problem.

As they do, they are instructed to write down their thoughts, to leave traces of their reasoning. They are gradually led to an analysis of the phases and processes of problem solving. They are led to comment on their joy and frustration, to add key words outlining their emotional or cognitive state. The instructions given for this work include:

- Have fun, work, think and try problems which may seem hard to you.
- Do not finish some problems; push yourself on some others.
- If you have an idea in a restaurant and work on a paper mat, don't hesitate to include it as part of your *journal de bord*.
- Do not erase or recopy so it's neat and tidy; do not hide what you really did.

Personal Reflection on Problem Solving

The students are asked to reflect on their work. This is done after each problem that they solve, in part or totally (according to their own judgment): what was most useful in the (partial) resolution of the problem? Answers may include the introduction of a notation or a table for regrouping data, a mistake which helped them understand what needed to be done to solve the problem, a conversation with someone, trying to explain the problem to a parent, a week away from the problem, etc.

Towards the end of their work on their *journal de bord*, they are instructed to answer the following questions.

- What are your strengths and weaknesses in problem solving?
- Which phases or processes are easier or more natural for you?
- What types of problem attract you and why?
- Are there some phases or processes which seem more important to you? Explain.
- What do you think is most relevant with respect to the teaching of mathematics in a secondary school setting?

Sharing and Reflecting as a Team

Once their personal work is completed, they work in teams of four. Each member contributes the work they have done on one problem and which they feel is particularly representative of the problem solving process. Each member of the team reads and comments the work done by the other three members. Finally, as a team, they write a joint conclusion to the work in which they are asked to explain what they view as the most important aspects of problem solving in itself and in view of mathematics teaching at the secondary level. This part of the work is largely about communication and reflection.

What Does This Is Achieve?

As students work on their problem solving portfolio, they are invited to ask questions in class about any aspect of the work. This gives us material to engage in discussion about a range of topics, including reflecting on the doing of mathematics as well as understanding aspects and processes involved in creating mathematical knowledge and in validating its truth. Here are some of the aspects which we look at in relation to this problem solving work: notice how many of these are part of the area of expertise of a mathematician.

- Being at a loss, feeling unsafe, needing to start but not knowing how
- Exemplifying and generalizing: the continuous tension between the two processes
- Exemplifying and gaining intuition
- Exemplifying, spotting patterns and the inductive process
- The importance of multiple representations: drawing, physical experimentation, notation, algebra, graphs, etc.
- Emitting conjectures: their importance and role
- Justifying and proving: when do we have a proof?
- Mistakes:

  - Being wrong, so wrong that we want to deny it
  - Silly mistakes—are they?
  - Their importance in understanding what might be

- Why didn't I think of this: it is so simple! Or seeing simply the complex doesn't mean the complex was simple, or simple to see.
- Taking our time, letting ideas mature
- Communicating: to myself, clarifying my thoughts; to others, using appropriate language and notation
- Understanding and being understood; writing mathematics and being understood
- Efficiency and clarity: allies and foes.

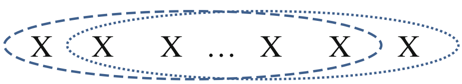
Working for many hours on mathematics, searching for solution, formulating your hypotheses: students engage in this with a mentor. They come to me for advice, encouragement; share their joy when they make progress, vent their frustration. Grading this requires many hours but this is part of the informal contract between them and me: they work a lot, and so do I, reading their solutions, adding remarks and paying attention to them. A lot of this, I do as a mathematician.

Proofs in Mathematics

One of the first themes we explore in a B. Ed. specific course is that of proofs. Students are presented with some paradoxes: classical from Zeno of Elea and others, geometrical from Lewis Carroll, some from Movshovitz-Hadar and Webb (1998)—a great teaching resource—as well as examples of interesting visual or geometrical proofs (Nelsen 1997). Alongside these, we revisit classical types of proof: direct, by contradiction, by induction, using Fermat's method of infinite descent, etc.

Interestingly, when they are presented with a false proof, students will often argue that a certain type of reasoning is not valid while it is one that they readily accept in other circumstances. There are led to reconsider what they accepted as valid. One nice example is the following proof that all horses are the same colour.

Let us prove the statement P(n) which states that in any set of n horses, all horses are the same colour. The proof is done by induction. Clearly, P(1) is true. Suppose now that P(k) is true for some k and consider a set of k + 1 horses. Form a set with the k first horses: these are the same colour as P(k) is assumed to be true. Now consider all horses except the first: these are also the same colour as P(k) is assumed to be true. This gives the diagram where X stands for a horse:



As we see, if the first k horses are the same colour, and the last k horses are the same colour, then all k + 1 horses must be the same colour. Hence P(k + 1) is true. The proof by induction is complete: all horses are the same colour.

In discussing this, some students argue that one cannot do a proof by induction on sets of horses: these are not mathematical objects. Some argue that we cannot assume P(k) as it is not true. Many are baffled. We discuss these objections and revisit the concept of a proof by induction: students are asked to explain how it works, to find everyday images to explain it to a non-mathematician. They reflect about proofs by induction.

The general objective of this part of the course is to get students, prospective teachers, to revisit and better understand what constitutes a proof, after they have themselves been exposed to proof in many courses. We discuss the nature of proofs including their social nature (Delahaye 2008; Gourdeau 2008a). We discuss the usefulness of certain types of arguments (proofs) in teaching. We see that some types of reasoning may be flawed and generally aim at becoming better at deciding if an argument constitutes a proof.

They gradually accept to position themselves as able to judge if a statement has been proven or not. They become producers of mathematical truth, and start to think in this way. As teachers, this is something they will do constantly, accepting or rejecting mathematical ideas and statements offered by their students or by colleagues. This is something mathematicians do.

Exploring and Developing Mathematics in Class

In class, we may work as a group or in small teams, or discuss in a whole group after some work done outside the classroom. This might involve to justify or to prove statements in different ways, according to suggestions. I will follow leads, write on the board according to suggestions and often ask what, if anything, is missing, or wrong. I might not comment on their suggestions and invite them to think about these. I might also complete and explain some aspects, without necessarily clarifying everything in writing; I will then invite them to write a complete and correct version of a proof or justification.

Generally, we will not discuss this any further until they have done some work by themselves. Any further explanation or clarification will be based on what they have done. They should be able to judge if a line of reasoning is correct, if an explanation is a proof, as I wrote earlier. This, of course, is difficult.

Another difficulty is that of writing mathematics. They are used to taking notes, using a defined notation without questioning it, and using the same or similar notation in exercises. They are not used to defining their notation or choosing what notation to adopt. Let us see two examples.

Determining the volume of a sphere. Using elementary means (and not integral calculus), we work at obtaining the formula for the volume of the sphere (Lang 1984). This goes via the volume of cylinders and summing an infinite series using geometrical/visual arguments. There are approximations and error terms. Students generally understand the arguments which we go through informally in class. However, writing down the argument using mathematical notation is very difficult for them.

On being the right size. In a famous essay, Haldane (1926) explains why larger animal are more complex, basing the arguments on elegant and simple consideration about shapes, areas and volume. Many of these are easy to express mathematically as they deal with proportions and the influence of scaling on these. Working from this text and from Gourdeau (2008b), students are asked to mathematize the arguments, expressing them with correct mathematical notation.

To develop the notation to write about a topic is not something students seem to have experienced much, yet it helps understand the mathematical language itself and standards we have in mathematics (for instance, using different variables for different values). The emphasis is not on adopting a standard notation (something which we know is very useful as mathematicians) but rather on understanding the difficulties of writing mathematically precise statements. This involves being able to read mathematical statements written by ourselves and by others, and producing them. As teachers, they will need to produce and read written material dealing with mathematics, to talk about mathematics: our university classrooms can be a good place to explore and practice these skills.

Mathematicians in Teacher Education

Being an active member of the Canadian Mathematics Education Study Group (CMESG) since 1997, I have had the opportunity of working with researchers in mathematics education since the beginning of my career (1996). Building on work done by my colleagues Bernard R. Hodgson and Charles Cassidy, and on the conversations and working groups I joined at the CMESG annual meetings, I gradually modified some of the B. Ed. specific courses which I have been teaching and the activities presented come from these.

As a mathematician, I often wished for someone in mathematics education to come in my classes so that we could reflect together and help improve the mathematical preparation of teachers. The opportunity presented itself in 2010. I was asked to present some of our work at Université Laval at a colloquium which was being held in 2011. I did not immediately agree and rather asked the organisers if we could take this opportunity to have a different type of report: one that would come from collaborative work with someone in maths education.

It is in this context that during the academic year 2010–2011, Jérôme Proulx (researcher in mathematics education at UQAM) and I undertook some collaborative work. Jérôme came to observe some of my classes, and we engaged in a discussion on the mathematical training of pre-service mathematics teachers. We were joined by Jean-François Maheux (also a researcher in mathematics education at UQAM), Bernard Hodgson (mathematician) and Jérôme Soucy (teaching instructor with a masters in mathematics), both at Université Laval. This enabled us to engage in a fruitful conversation leading to Gourdeau and Proulx, with Hodgson and Maheux (2012).

In this collaborative endeavour, I was struck by the importance given by Jérôme to some of the content of the lectures and to the nature of the conversation with students in class. Even if I have been exposed to quite a bit of research and discussions in mathematics education, I would not have proposed that this was very important.

For example, after a regular lecture in a standard mathematics course (Euclidean geometry), Jérôme noted how I explained what one could or could not do in a proof, and how I proceeded to amending a definition at the end of a proof so that it would apply to all cases and not only those we had first thought of. He also noticed how a problem was partly solved backwards, starting from the desired conclusion. He also drew attention to generic examples: as this seemed pertinent at the time, I had explained how a generic example for a proof of a theorem could be deemed as sufficient or not depending on what they saw the example as representing. This discourse of the mathematician on the doing of mathematics struck a chord with him.

In our initial conversation, we were struggling with the notion of advanced mathematics courses, some research questioning their use for teachers. Upon reflection, we agreed that it might not be the nature of the mathematical concepts and the fact that they were advanced which posed problems but rather the way in which mathematics was done and presented in some advanced mathematics courses. These are often given as a traditional lecture course, and mathematics is presented as a highly structured set of truths (e.g. definitions, lemmas, theorems) that one should learn and, possibly, understand. Little time is spent on learning how to develop mathematical knowledge. One does not learn from mistakes and see how to go from initial attempts to a better theory. And therefore I would not like to coin the content we deal with in B. Ed. specific courses as advanced or not: I would like to claim that the way in which it is done is what makes it relevant (or not).

As an example, we look at infinity: this is certainly not elementary. We look at paradoxes and then revisit the denumerable and the non-denumerable. We show that there is a bijection between the unit interval and the two-dimensional square, and therefore with the n-dimensional hypercube, for any n. Why do we do this? It is not part of the secondary school curriculum and it may be deemed too advanced or too abstract. However, it is beautiful and, at the same time, disconcerting. Some of the results are counterintuitive: when you first encounter them, you might want to say that some of the results we succeed in proving are false. It is a beautiful illustration of how discoveries reshape our (intellectual) world. And it is also a magnificent example of what a creative mind can do: taming infinity. It is a nice example of what research can be and of what creativity and imagination can do in mathematics.

A study of Burton (2004) with mathematicians led to some conclusions which resonate with me. The way in which we teach mathematics courses is often far from the way in which we do, learn or discover mathematics. It is therefore somewhat liberating to hear that our expertise as mathematicians might be useful. Teaching about how we do mathematics is of value. Doing mathematics with students, leading them to do mathematics is of great importance.

Conclusion

In this paper, mathematics for pre-service teachers was presented without precise reference to content. For mathematicians and mathematics educator to become partners in this endeavour, I believe that focusing on the doing of mathematics by the students, on the quality of their engagement with mathematics, can be very helpful. As I have just explained, in doing so, I believe that the role of mathematicians in the mathematical preparation of teachers is not reduced but rather enhanced: it is work of a different nature than is often argued, leaning more on the expertise of the mathematician in the doing of mathematics than on his or her knowledge of the facts of mathematics.

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Resources at the Core of Mathematics Teachers' Work

Ghislaine Gueudet

Abstract Mathematics teachers work with resources in class and out of class. Textbooks, in particular, hold a central place in this material. Nevertheless, the available resources evolve, with an increasing amount of online resources: software, lesson plans, classroom videos etc. This important change led us to propose a study of mathematics teachers documentation. Mathematics teachers select resources, combine them, use them, revise them, amongst others. Teachers' documentation is both this work and its outcome. Teachers' documentation work is central to their professional activity; it influences the professional activity, which evolves along what we call professional geneses. In this conference, I introduce a specific perspective on teachers resources, which enlightens in particular the changes caused by the generalized use of Internet resources.

Keywords Communities · Documentation · Internet · Professional development · Resources

Introduction

Mathematics teachers work with resources (Adler 2000), in class and out of class. This fact has been considered by many researchers focusing on curriculum material (Remillard et al. 2009). Textbooks, in particular, hold a central place in this material (Pepin 2009). Nevertheless, the available resources evolve, with an increasing amount of online resources: software, lesson plans, classroom videos etc.

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This important change suggests possible connections between the study of traditional curriculum material and the study of technology in mathematics education. It also enlightens the evolving role of the teacher; he/she can no longer be considered as a passive resource user, he/she becomes a potential designer of her own teaching resources.

These reasons led us (Gueudet and Trouche 2012) to propose a study of mathematics teachers documentation. Mathematics teachers select resources, combine them, use them, revise them, amongst others. Teachers' documentation is both this work and its outcome. Teachers' documentation work is central to their professional activity; it influences the professional activity which evolves along what we call professional geneses.

In this conference, I introduce a specific perspective on teachers resources which enlightens in particular the changes caused by digitalisation. I present a theoretical approach for the study of teachers documentation work, conceptualising the articulation between documentation and professional development. I illustrate this approach with case studies from several research projects, considering the documentation work of primary and secondary school teachers, and its consequences for their professional development. A specific interest is devoted to teachers communities and their work with resources.

Mathematics Teachers Resources

Teaching mathematics requires various resources; I do not intend here to propose a full description of these resources. My aim in this section is, on the one hand, to introduce a specific perspective on resources, inspired by the work of Adler (2000); on the other hand, to analyse the current evolutions yielded by the Internet in mathematics teachers resources.

Conceptualisation of Resources

Resources for mathematics teachers exceed curriculum material. The origin of this statement, in our work, comes from our interest for Internet resources (Gueudet and Trouche 2009). Online exercises, a digital textbook are resources for teachers. These exercises, this textbook, are offered on a website, which is likely to include a forum for its users. Discussions on this forum can also constitute resources: for example, a teacher can declare that she especially appreciates a given activity in the textbook, and this comment will act as an advice for another teacher reading it. This statement leads to consider that any discussion between teachers also can constitute a resource; same for the discussions with students.

For other reasons, Adler (2000) has similarly encountered the need for a concept of resources in mathematics teaching which focuses more on how a given object intervenes in teaching than on the nature, the features of this object.

> It is possible to think about resource as the verb resource, to source again or differently. This turn is provocative. The purpose is to draw attention to resources and their use, to question taken-for-granted meanings. (Adler 2000, p. 207).

Adler distinguishes between human resources (the mathematics teacher herself, her knowledge), material resources and socio-cultural resources (such as language and time). With such a perspective, what matters is not the intention which prevailed, when the resource has been initially elaborated, but its intervention in mathematics teaching, its use by a teacher for teaching objectives. A movie theatre can become a resource for the teaching of mathematics, if the teacher proposes to her pupils to count the seats, for example.

While our study mostly concerns curriculum material, its current evolutions and the consequences of these evolutions, we retain nevertheless Adler's perspective, focusing on "what intervenes in the teaching of mathematics". With this perspective, I will focus on the changes brought by the Internet in the resources available for the teaching of mathematics.

Internet and Resources Evolutions

The Internet naturally introduces evolutions in the available resources. Teachers looking for resources on the Internet have to choose amongst a profusion of websites, files and software proposed for downloading. The evolutions are not limited to the number, or the kind of available resources, they also concern, in particular, their design modes, and their quality assessment.

Design Modes

The process of designing textbooks (traditional paper textbooks) can be very different, from one country to another, according to the national policy. In some countries, textbooks are published by private publishers, with no control of their content. In others, a national authority controls the book before its publication; in some cases, for each grade level a unique, official textbook exists, written and published by the institution. Whatever the national policy is, a teacher wishing to write a textbook faces a difficult and long work; a user of a textbook can not modify its content, according to his own needs or perspectives.

The situation, with websites, is completely different. Each teacher equipped with a computer connected to the Internet can easily develop teaching resources and publish them on this website for the use of other colleagues.

Naturally, the official institution can also develop and propose Internet resources. In Mexico, the Enciclomedia website has been produced in the context of a national project (Trigueros and Lozano 2007; Trouche et al. 2013). The educational authorities formed teams of specialists in education to transform classical textbooks in online resources with an aim of teacher education. This kind of design can be considered as “top-down”: specialists, officially recognized by the ministry, conceive resources for teachers. These teachers are considered as users, who will develop their own practice by the use of these resources.

Simultaneously, Internet opens the possibility of new design modes. In particular, communities of teachers use these possibilities to develop and broadcast Internet resources. The Geogebra community (Lavicza et al. 2010) gathers teachers and researchers all over the world, designing resources, organizing training sessions, conferences around this free dynamic geometry software. In France, the website of the Sesamath association offers various kinds of resources in mathematics for grade 6–10; it records more than 1.3 million connections each month. The association comprises around 70 members, who develop online interactive exercises and different kinds of software. Teachers work with the association members to write free online textbooks. A specific website, Sesaprof (Fig. 1), is devoted to Sesamath resources users. They can formulate their remarks, suggestions, or even inscribe themselves in design groups, using an online platform to develop new exercises.

Fig. 1 Sesaprof, a website for Sesamath resources users

Alongside teachers associations, groups gathering teachers and researchers also use digital networks to design resources. The National Centre for Excellence in the Teaching of Mathematics in UK is a noticeable example. The NCETM offers both resources and a sustainable national infrastructure for mathematics-specific Continuing Professional Development. In 2012, the NCETM has reached nearly 60,000 registered users. Interestingly the numbers of users are increasing by approx 100 per day and users are staying longer suggesting more in depth interaction. Research works demonstrate that the NCETM impacts the way teachers work and interact together (Hoyles 2010). The NCETM provides and signposts resources to teachers and supports mathematics education networks, which can include universities and the whole range of professional development providers throughout England. At the same time, the National Centre encourages schools and colleges to learn from their own practice through collaboration among staff and by sharing good practice locally, regionally and nationally. These collaborations can take place face-to-face at national and regional events and in local network meetings, facilitated by a team of NCETM staff with expertise in different phases of education from primary to further education, spread across the nine regions of England, or virtually, through interactions on the NCETM portal. The NCTEM claims, on its website, that it provides "high quality" resources. This claim raises the issue of quality for Internet resources for mathematics and of the assessment of this quality.

Quality Assessment

As mentioned above, in France the textbook market is free; there is no control of the validity of the textbooks content. Nevertheless, for digital resources, an official, national label exists (RIP, for Recognized of Pedagogical Interest). The quality of online resources seems to raise specific questions; this is a direct consequence of the new design modes evoked above, in particular the possibility for any individual to publish resources.

Defining the quality of an online resource, for the teaching of mathematics, is not straightforward. Which criteria can guarantee this quality? The expertise of the authors can not intervene as criteria, or this would discard the new, bottom-up design possibilities. Quality criteria have to take into account the mathematical content, the didactical aspects, the ergonomic dimension (Trouche et al. 2013). Nevertheless, these dimensions do not ensure the ease of appropriation by the user (a resource that nobody uses can hardly be considered as a high-quality resource); at the same time, the number of users can not account as guaranteeing quality. Another question is: who will assess the quality? Once again, preserving the bottom-up design possibilities pleads for avoiding the assessment by experts, or by an institutional authority.

In the Intergeo project, resources are offered on a platform, to support the use of dynamic geometry software (Kortenkamp et al. 2010). An important aspect of the project is the assessment of the quality of these resources (Trgalovà et al. 2010, p. 1162). Any registered user can propose a resource. This choice creates a large repertoire of resources, but makes at the same time the issue of quality assessment essential. In Intergeo, the main tool for quality assessment is a questionnaire to be fulfilled by the users. Given the aim of the Intergeo project, it focuses on the exploitation of the potential of the geometry software. The answers to the questionnaire are automatically collected and treated, and this treatment leads to a label (a number of stars) associated to the resource on the website.

The profusion of resources calls for quality assessment; but the new modes of design require new modes of quality assessment. An important part has to be played by the users, in the assessment of resources quality, in particular assessing the possibilities of appropriation of a resource requires to ask users.

These new phenomena indicate the need for specific research on the topic of mathematics teachers resources and their use. Moreover, the deep changes require a specific approach, both on a theoretical and on a methodological level; I present it in the next section.

Resources Use and Teachers' Professional Development

In this section, I present a theoretical approach, developed for the study of teachers' interactions with resources, and of the consequences of these interactions, in particular in terms of professional development. Specific methods for collecting data are also associated with this approach. I finally expose a case study, concerning a primary school teacher, to illustrate the approach use and its possible outcomes.

The Documentational Approach: Theory and Methodology

As mentioned above, I refer here to the conceptualisation of resources introduced by Adler (2000), focusing on the interactions between teachers and resources intervening in their professional activity.

Our specific interest in digital resources also leads to consider research about technology. The interactions with technological tools and their consequences for knowledge evolutions have been extensively studied in the case of students using various kinds of software on computers or calculators. These processes have been in particular conceptualized within the instrumental approach (Guin et al. 2005), grounded in cognitive ergonomics (Rabardel 1995). Rabardel distinguishes an artefact, available for a given user, and an instrument, which is developed by the user, starting from this artefact, in the course of his situated action. These development processes, the instrumental geneses, are grounded, for a given subject, in the appropriation and the transformation of the artefact, for a given class of situations, through a variety of usage contexts. Through this variety of contexts, utilisation schemes of the artefact are constituted.

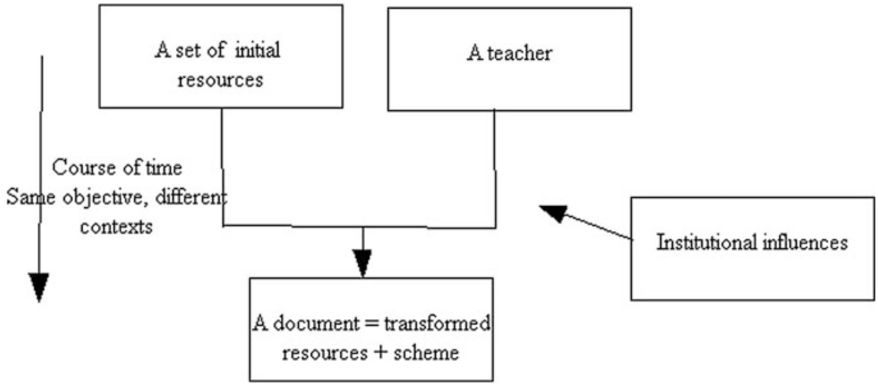

Fig. 2 A documentational genesis

A scheme (Vergnaud 1998) is an invariant organization of the activity, which comprises in particular rules of action and is structured by operational invariants developed in the course of this targeted activity, in various contexts met for the same class of situations, and which pilot the activity. This definition can be represented by the equation: instrument = artefact + scheme.

Combining these different theoretical references, we introduced (Gueudet and Trouche 2009) a distinction between available resources and a document developed by the teacher in the course of her interactions with these resources for a given objective.

This perspective—the documentational approach—shares some features of the instrumental approach (but at the same time, focuses much more on what is conceived by the teacher). The definition of a document can be represented by the equation: document = resources + scheme. The corresponding development process is called a documentational genesis (Fig. 2). The geneses are long-term processes: schemes are developed across various contexts, encountered for a given class of situations. A class of situation, for a teacher, is a set of professional activities with a similar aim. For example, "preparing and setting up the introduction of functions", "preparing and setting up applications exercises in algebra" are classes of situations, that the teacher encounters in different contexts (different classes, different years). These geneses are naturally influenced by the institutional environment.

Resources intervene in all aspects of the teacher's work. What we call the documentation work: collecting resources, transforming them, setting them up in class etc. is central in the teachers professional activity. Thereof the documentational geneses hold a central place within teachers professional development. During their career, teachers constitute an evolving and structured system of resources. Alike Ruthven (2012), we consider that this system is a determining feature of teacher's work. Moreover, in our approach we claim that this system is associated with a system of documents (or documentation system), composed of the system of resources, associated with the corresponding professional knowledge.

The study of teachers documentation work requires a specific methodology. This documentation work happens indeed in many places, in-class and out-of-class, in school and out-of school. It involves, for any single teacher, different groups: colleagues of the same school, parents, administrative staff, but also potentially online communities etc. The geneses are long-term processes, associating stabilities and evolutions; the teachers interact with multiple resources.

For this reason, we designed a methodology, entitled "reflexive investigation methodology" (Gueudet and Trouche 2012). The aim of this methodology is to follow, as completely as possible, a teacher's documentation work. This aim requires an active collaboration of the teacher followed: only the teacher herself has a complete access to her activity and resources.

For a given year, the follow-up lasts at least three weeks. During these three weeks:

- the teachers fills in a logbook, describing her professional activity (in-class and out-of-class), the resources used and produced, the agents involved;
- a lesson is observed and videotaped;
- interviews of the teacher by the researcher are organised;
- the researcher collects as far as possible all the teachers' material resources: files, e-mails on the computer; written notes, students sheets etc.

To analyse these data, we associate quantitative and qualitative exploitation. We notice the different activities mentioned, and the time devoted to each activity; the resources intervening, the corresponding activities; the groups where the teacher is involved, the exchanges of resources with these groups. We present here a case study issued from a research project using this methodology.

A Case Study at Primary School: Integration of the Virtual Abacus

The aim of the research project where this study took place was to follow the integration of various software by primary school teachers, in their teaching of mathematics. In this particular case, the software involved was a virtual abacus. We followed its integration by Carlos, an experienced primary school teacher (Poisard et al. 2011).

Carlos took part during two years to a research and development group, with five other colleagues and three researchers. Several resources were presented and discussed in the group; he retained the virtual abacus, and still uses it very regularly, mostly with grade 3 classes. Naturally, his participation in the group is an important factor, explaining his integration of the abacus. But other factors intervened as well: his resource system; the institutional incentives, and his professional knowledge about the teaching of numeration.

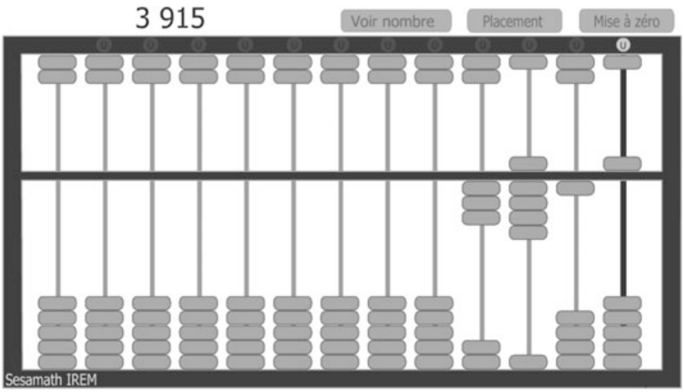

Fig. 3 The virtual abacus, displaying 3915

In his resource system, his usual textbook has been determining. This textbook proposes indeed an activity with material abaci. Carlos found interesting the mathematics content of this activity; but he was not able to organize it in class. Trying to use material abaci with his pupils, he observed that it was for him almost impossible to check whether they displayed the correct number or not: if a pupil touches the abacus, the beads move! On the virtual abacus (Fig. 3), there is no such risk. Moreover, the button "display the number" shows, in digits, the number represented by the position of the beads.

The institutional conditions represent another important factor for this integration. In France at primary school, teachers have to assess skills regarding the use of the computer, and of Internet; they deliver to the pupils a certificate. Working with the virtual abacus in the computer lab provides Carlos with the opportunity of assessing some of these skills (starting a computer, opening a software etc.).

The virtual abacus articulates with Carlos' resources, and is consistent with institutional expectations. Nevertheless, these factors are not sufficient to explain its integration by Carlos: his professional knowledge also played a major role. This knowledge has several aspects: didactical knowledge about the teaching and learning of numeration. The objective of working on "grouping and exchanging" (five lower beads against one upper, two upper beads on one rod against one lower bead on the rod on the left etc.) is central for him, since he noticed students' difficulties with the meaning of numbers' writing, along his teaching experience, and in particular while teaching division. Carlos also developed professional knowledge about organizing group work in the classroom; the computer lab has only eight computers for 20 pupils, but this is not an obstacle for him. Carlos has also developed knowledge about introducing new tools in the classroom and working on writing directions for use, with the calculator in particular. He introduces the abacus to his class in a similar way: he first gives material abaci, then proposes a work in the computer lab, where the pupils have to discover how it works: the meaning of each bead, of the rods, and the role of the central beam. The class discusses different interpretations, before a collective writing of directions for use.

Following Carlos' work, we observed a genesis. Carlos developed a document involving resources: the virtual abacus, but also the classroom textbook, the written directions for use… This document also comprises professional knowledge. The already existing knowledge guided Carlos' choices in an instrumentalisation process; but the use of the virtual abacus also lead to the development by Carlos of new knowledge about the abacus itself and its use, and about the teaching and learning of numeration.

The documentational approach can be used to study the work of individual teachers. It permits, as in the case presented above, to follow their integration of specific resources. It also allows, more generally, a study of their professional development via this focus on resources: resources evolutions, along the time, can be considered as indicators of professional evolutions. Moreover, the documentational approach can be used to study professional development within groups of teachers; we present it in the next section.

Collective Documentation Work and Teacher Education

The digital means permit new forms of teacher collaboration. Teachers' documentation work can be individual or collective when teachers work together to prepare a lesson, an exam text etc. Teachers communities of practice (Wenger 1998) in particular have shared repertoires of resources: the community resources system. In the context of teacher education, collective documentation work can be used with an objective of teacher professional development. We present here the example of a study about such a teachers education program, the Pairform@nce program in France.

Presentation of the Pairform@nce Program

The program Pairform@nce (organised by the national ministry of education in France) offers in-service training paths, aiming to provide sustainable ICT integration for all school levels and all topics. These paths are templates for training programs which might be implemented across the whole country. These training programs are a blend of face-to-face and distance learning. They are grounded in the collective design of lessons by teams of trainees.

There are thus three sets of actors in Pairform@nce: the trainees, following a training program; the trainers, setting up and managing the training; and the designers of the path which frames the training. Naturally, the students who take part in lessons designed by teachers following the training program are also important actors.

In the context of a design and research project, our team has designed several training paths, in particular a path about the use of dynamic geometry for inquiry-based mathematics learning (Gueudet and Trouche 2011). In France, the official curriculum for middle school suggests the use of inquiry in class. It also suggests the use of dynamic geometry software (DGS). Nevertheless, in class DGS is not much used and investigation is not often organised. The training path "Investigation with dynamic geometry at middle school" thus faces a double challenge.

The aim of the path is that the trainees design and test a lesson giving real responsibilities to the pupils, regarding both the use of dynamic geometry and geometry itself. The training takes place over 13 weeks (excluding holidays); it comprises three face-to-face workshops of one day each. Between these face-to-face days, continuing work is done, using e-mail and the distance training platform (Fig. 4).

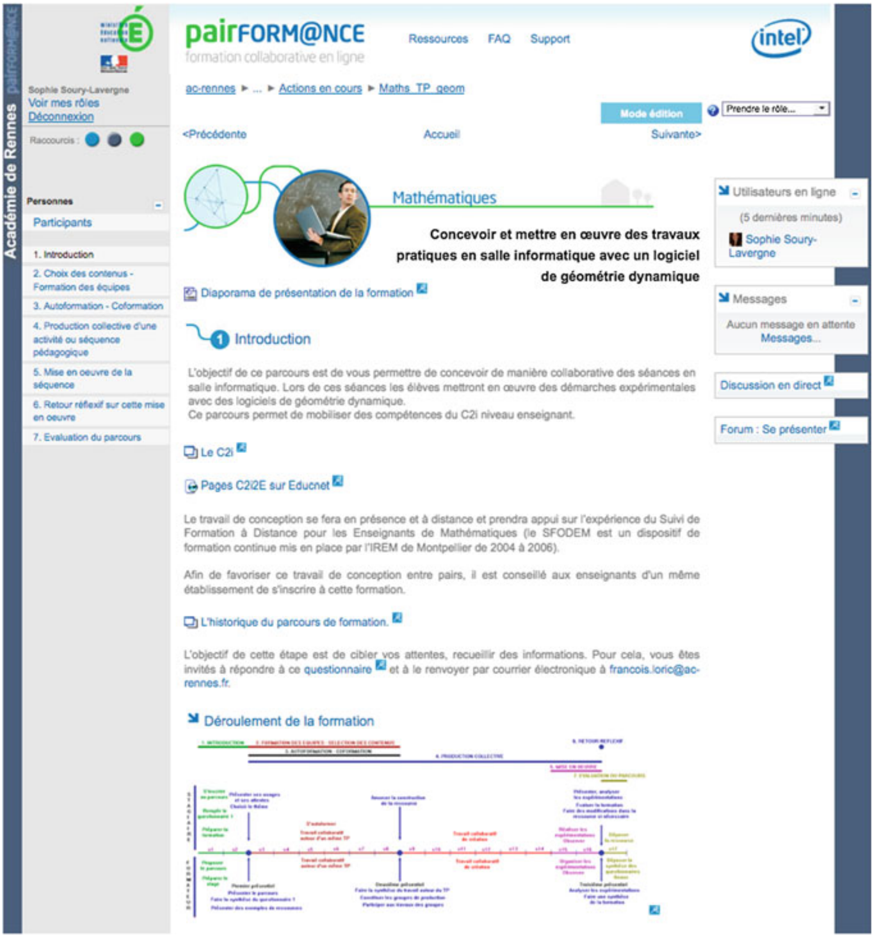

Fig. 4 "Investigation with dynamic geometry at middle school", a training path on the national Pairform@nce platform

We have implemented this path in several training sessions and followed the work of the teams of trainees. We proposed questionnaires to the trainees and followed moreover all their discussions, during the workshops in presence or on the platform.

A Team of Trainees Working with Dynamic Geometry

We consider that the elaboration of lessons by the teams belong to the trainees documentation work. Many resources and professional knowledge are involved in this work. We have selected here the work of a team of trainees to illustrate the use of our theoretical frame, and to investigate the possible 'impact' of such as a training that we conceptualize in terms of geneses.

The team was composed of four teachers (Mary, Fanny, Clara and Georges), all of them teaching in different schools but in the same city; they did not know each other before the beginning of the training. Each of them had grade 9 classes; they quickly decided, during the first workshop, that they will design a lesson for this level, all the more considering the fact that grade 9 students had laptops borrowed for one year from the local authorities. These laptops were equipped with GeoGebra, thus the team picked this software for their lesson preparation.

The scenario they designed was concerning the "angle at the centre" theorem; its aim was to introduce two related results ("in a circle, the size of an inscribed angle is half of the size of the corresponding angle at the centre"; "in a circle, the angles subtended over the same arc are of the same size"). Two hours in class were dedicated to an inquiry-based activity about these results. During the first hour, the students worked in pairs on the computer; during the second hour a synthesis of the obtained results was made, and the teacher wrote the theorems and proposed application exercises. For the first hour, the students had a worksheet to fill, and a GeoGebra file to complete (Fig. 5); the mathematical situation was presented as a soccer training.

On the GeoGebra file, the goal is represented by a segment [AB], and P is the point to shoot penalties. The students had to display the measure of the shooting angle, for a player placed on P. Then they had to build a point E, belonging to the arc of circle of centre P with extremities A and B (and naturally on the field, outside of the goals). The size of the shooting angle, for a player placed on point E, should be displayed. E had to be dragged along the circle, and the students were supposed to observe the modification of the angles size, and to compare the size of both angles. All these successive steps have been described on a worksheet given to the students. This kind of student's activity, consisting in dragging a figure to observe invariant properties or measures is one of the most widespread use of dynamic geometry among teachers, although it has been shown that it does not take the best advantage of the dynamic geometry added-value for learning (Ruthven et al. 2005).

This lesson was the object of a first trial by Fanny in her class; Clara observed it and took notes. Clara wrote down some students difficulties (notes available on the

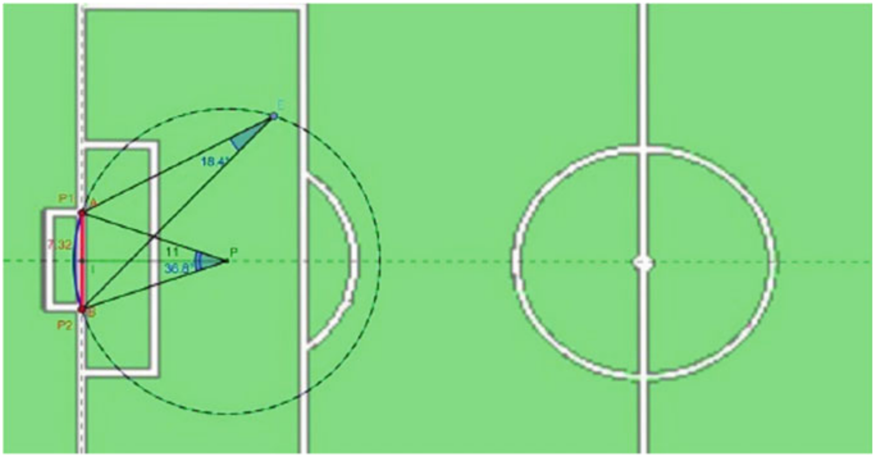

Fig. 5 The GeoGebra file in the team's lesson. The points *A*, *B* and *P* were given, with fixed positions. Point *E* can move on an arc of circle with extremities *A* and *B*

platform): some with the GeoGebra figure, in particular a wrong construction of the circle; some with the vocabulary, for example "shooting angle" does not make sense for some students. After this observation, Clara proposed a sheet to the team with different written hints (eleven different hints, according to the difficulties encountered). She tried these hints in her own class; it turned out to be very complicated and difficult to choose the appropriate hint. Georges has observed this session, and has reported the difficulties to the group. During the third workshop, the team worked on a better choice of possible hints.

For this team, it was important to make the lesson part of the usual curriculum: these theorems had to be introduced, the "inquiry-based" lesson took place within the usual teaching. Several observations indicate that the members of the team seemed to consider of most importance to stay within the official curriculum objectives, even for inquiry. Questioned by the trainers about their choice, they have confirmed that they wanted to avoid loosing time, working on the computer, with an inquiry-based approach. This shared knowledge of the team has influenced the choice of theme for the lesson. We infer that, during the training, they started to develop a document for the class of situations "organising inquiry in geometry". This document has included resources (GeoGebra in particular), but also professional knowledge like "the official curriculum has to be completed, within a short time", and "the inquiry-with-ICT lessons must correspond to objectives of the usual teaching".

Another choice of the team was to start with a "real life" situation. It seemed especially important for Clara and Georges, according to the 'distant discussions' via the platform or email. Georges built the GeoGebra file to be given to the students. In one of his emails to the team (they preferred to use email than to write on the available forum on the platform), he declared.

"I have found a nice soccer field image, and I know how to insert it in GeoGebra!" and Clara answered with congratulations: it clearly indicates their will to make the situation looking as "real" as possible.

Clara also added comments on a first version of the student's sheet (uploaded on the platform), proposed by Mary for the "soccer situation". While Mary asked the students to write the size of "angle AM(correct?)B, in degrees", Clara crossed this sentence out and commented: "We discussed it and decided not to name the angles, to have a more concrete activity". Indeed, on the final version of the students' sheet, the sentences were: "size of the shooting angle, if the player is on the penalty point" and "size of the shooting angle, if the player is on the circle". We interpret the corresponding process as follows. Clara and George had a shared conviction about "real life" situation fostering inquiry. This conviction has influenced the team's choices. We do not claim that Mary and Fanny now share this conviction. But their use of the "soccer" lesson might lead them to develop it (perhaps not the first time they use it, but along several uses).

The choice of possible hints was also an important aspect of the team's documentation work. Clara's observation of Fanny's class leaded her to propose these hints. Clara's own trial of the "soccer" lesson occurred only a few days after Fanny's, and the members of the team did not have time to comment on the proposed hints before she used them; these hints were only discussed during the third workshop.

All the members of the team agreed about the idea of planning hints in case where students would encounter difficulties. They seemed to share the conviction that: "for an inquiry lesson, the teacher must intervene when students encounter too many difficulties. But these interventions must be carefully prepared to avoid saying too much about the answer". But they also said that eleven different hints were too much, and too difficult to manage for the teacher; so they started to make choices amongst Clara's propositions. In particular, the hints concerning the use of language and the mathematical meaning of the everyday language were considered as important by all the members. Such hints were strongly connected with the previous choice of "real life" situations, which yielded a need for mathematical modelling and the connected potential difficulties. We hypothesize that the document developed at the beginning by members of this team included operational invariants like "the mathematical interpretation of the non-mathematical language is difficult for students" and "the language difficulties have to be anticipated, while planning an inquiry-based lesson". This team also seemed to have developed an operational invariant that could be expressed as "an inquiry-based activity should start with a mathematical modelling".

Naturally, additional observation is required to assert that this new knowledge influences the teachers practices, even after the end of the teacher education program. Research about teacher education grounded in teacher collaboration (Jaworski 2008; Winsløw 2012) often concern programs that last at least two years. We worked during a few months; the impact of such a relatively short training is probably more limited.

Conclusion

Our aim in this conference lecture was to give an overview of our work in progress. The deep changes in teachers work introduced by digitalisation require a new theoretical perspective for the study of this work and of its consequences for professional development. We have presented this approach and its use in two research projects. More examples, theoretical work, and other perspectives can be found in Gueudet et al. (2012). The research field of teachers resources remains open; new directions are now investigated:

- About the resources themselves, their design, their analysis. Research in mathematics education does not, for the moment, provide accessible tools for researchers or teachers intending to analyse a mathematical website for example;
- About the use of resources, and the link between the authors intentions and the actual use by teachers, by students. The notion of instrumental orchestration, introduced by Trouche (2004), is in particular elaborated and refined to include the use of resources (Drijvers et al. 2010; Drijvers 2012);
- Only a few studies consider the use of resources by students—and none of it considers the use of websites by students and its consequences in class, as far as we know.

These are only a few examples of possible research directions. Moreover, beyond research on resources, we consider more generally that research in mathematics education has to take into account more and more the resources aspect, whatever the research question is.

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The Mathematics Education Reform Movement in Indonesia

Sutarto Hadi

Abstract The reform of mathematics education in Indonesia started in the mid-seventies. The reform movement reported in this lecture is the second attempt after the first movement to reform traditional mathematics to modern mathematics (1975–1990) was a complete failure. Several mathematicians have dedicated their expertise and experiences to rebuild mathematics education from the remnant of modern mathematics. Their concerns are focused particularly with the weakest group of students. After a long consideration they came to the decision to implement the theory of Realistic Mathematics Education (RME) as a basic concept for developing the local theory of mathematics teaching and learning. They have the view that RME could be a vehicle for improving mathematics teaching and learning and at the same time be a tool for social transformation. The process began with four teacher education institutes and 12 pilot schools. RME has since expanded to 23 universities that supervise over 300 schools and has trained thousands of teachers. In this process of mathematics education reform the theory of RME has been transformed into PMRI, the Indonesian version of RME, and has been widely accepted as a movement to reform mathematics education.

Keywords Mathematics learning · RME · Teacher education · PMRI

Introduction

Why do we need to give our children mathematics education? Is it to create a group of great mathematicians? If it is, then we do not need to think of a reform movement to improve the current teaching practice. Every year, a number of Indonesians receive PhD degrees in mathematics from universities in Indonesia and overseas.

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Many Indonesian participants in the Math Olympiad return home with gold medals. The concern is not with this small percentage of high achieving students, but with the larger majority of the student population. The gap between the highest and the weakest mathematics students is very wide in Indonesia.

The aim of mathematics teaching in Indonesia is twofold. The first aim is to prepare students to be able to face the changing world through practical work based upon a rational, critical, cautious, and honest attitude, and logical, efficient and effective reasoning. The second aim is to prepare students to be able to use mathematics and mathematical reasoning in their life and study (Hadi 2002). These aims are not easy to realise. Most students fear mathematics and are math phobic. They tend to skip class, and are happy when their mathematics teacher is not able to come to class. This implies that there is a low quality of mathematics education and student achievement.

A group of mathematicians were deeply worried by this situation. They wanted to reform mathematics education in their country. Their concerns are based on two ideas. Firstly, they realised that Indonesia needed a larger body of mathematically literate citizens for the country to develop and prosper. Secondly, they foresaw that mathematics education that aimed at developing students' understanding and reasoning could help the country to become a democratic society (Sembiring et al. 2010a).

They researched mathematics education in different countries and chose to develop an Indonesian form of Realistic Mathematics Education (RME). They decided to create a local version of RME. Why local? Because, past experiences (notably the implementation of modern mathematics) has shown that it is not enough to import and disseminate what worked in another country. Also, the group understood that a top-down reform had a low chance of success. In their view, mathematics education reform needed to be bottom-up and start from the specific situation in Indonesia. This led to the development of Realistic Mathematics Education in Indonesia or PMRI (Sembiring et al. 2010a).

PMRI is defined as a domain-specific instructional theory, which offers guidelines for instruction that aims at supporting students in constructing or reinventing mathematics in problem-centred interactive activities. It refers to Freudenthal's concept of mathematics as a human activity. According to Freudenthal (1973), students should engage in mathematical exploration and should be given the opportunity to reinvent mathematics using well-chosen tasks, with the help of the teacher. Students are not merely being taught mathematics as a ready-made product. This point of departure, for some decades, formed the basis of design research in the Netherlands and later on in Indonesia. This research resulted in a range of local instructional theories, and domain-specific instructional theories, known as the RME theory (Gravemeijer 2010; Widjaja et al. 2010).

The challenges faced by mathematics educators in Indonesia in transforming teacher-centred instruction to problem-centred interactive instruction also include teacher preparation and training. The implementation of RME in Indonesia was a complex process because it related to changing teachers' beliefs, the implementation of new methods, and the use of new materials. Often, the introduction of a new approach faces challenges from teachers who are already settled in their own approach. Therefore, from the beginning, the PMRI team has exercised a strategy to make teachers gradually come to understand, and become skilful and competent in the use of new methods. The grounding principle in PMRI is the bottom-up approach. This principle is accompanied by other principles of learning through modelling, ownership, and co-creating (Sembiring and Hoogland 2008). That is, PMRI is not only dealing with developing local instructional theory on mathematics teaching and learning, but it is also developing an effective professional development program.

The Reform Strategy

The preparation for PMRI implementation in Indonesia was started in 1998 when six very talented young Masters students were sent to the Netherlands to study for a PhD in mathematics education. Keuper-Makkink (2010) noted that preparing experts on RME was the first step for the PMRI movement. The next step was to gain support from larger audiences, especially policy makers in the Ministry of National Education (MoNE). A proposal to start a Dutch project to improve mathematics education found little support. The opposition was understandable because Indonesia was taking on too many projects. During a visit of Indonesian experts to the PhD-candidates in Netherlands they met Dutch mathematics education experts. They came to the conclusion that the programme must be authentically Indonesian, while Dutch experts could assist with setting up workshops. These workshops would contain mostly hands-on activities, not top-down instruction, but especially focusing on bottom-up activities. Every workshop would be preceded by a trial in the schools and continued by classroom practices. Classroom research was obligatory.

The initiative was supported by the Directorate General of Higher Education, MoNE. Dutch consultants, based at the Freudenthal Institute and APS (Dutch National Centre for School Improvement), were involved in the development starting in 2001. Both Dutch institutes initially invested support in the movement by offering conceptual RME knowledge, knowledge on learning from pilot projects, and knowledge about strategies for dissemination and implementation (Hoven 2010).

In 2006, PMRI won a large grant from the Dutch Government. It offered the PMRI team the opportunity to design a support project for expanding the PMRI movement. The basic principles of the movement were identified, strengthened, and coined in the project plan (Hoven 2010). Those principles are (Sembiring and Hoogland 2008; Sembiring et al. 2008):

- bottom up implementation
- materials and framework based on and developed through classroom research

- teachers being actively involved in design investigation and developing associated materials
- day-by-day implementation strategies that enable students to become more active thinkers
- development of contexts and teaching materials that are directly linked to school environment and interest of students.

The bottom up strategy means that although the initiative was first taken by PMRI team; schools should play an active role. The process would be initiated when each individual within the organisation had the same view about the innovation and would contribute their part to the project. Since the main concern was improving teaching and learning, innovation might be initiated from classroom experiments. These have not only provided the base for the development and refinement of the PMRI theory but have also informed those involved in the development of teacher workshop and learning materials (student books and teacher guides). In fact, teachers played a very active role in PMRI workshops. PMRI materials are mostly written by teachers.

The Teachers' Role in PMRI Workshops

The nature of educational reform in a large country, like Indonesia, is different from that of a small scale project. In most cases, the activities that make small scale programmes successful are not helpful when used on a larger scale. This may be the origin of the complex problems that are encountered in a large project such as this: the number of schools, teachers and the vast area that needs to be covered. Large scale projects require different interventions than small scale projects. So, there was a need to develop a model that was appropriate for the large scale implementation of PMRI.

The main concern when planning for dissemination to the wider audience was the limitation of human resources. A strategy chosen to resolve this limitation was developing stratified workshops (local and national levels) for teachers and mathematics educators (university lecturers), while at the same time selecting talented teachers/lecturers to be partners in supporting activities like the workshop programme and development of materials. There were several task-forces for each intended goal. There was a task-force for design research, for learning materials development, and for quality assurance.

In every PMRI workshop, teachers played an active role in planning, executing, and reflecting. There had been a shift from teachers as objects to teachers as subjects. Workshops consisted of theory and practice, doing mathematics, and modelling lessons. At the end of each session teachers were asked to reflect on what they knew, their perception of mathematics, how students learn mathematics, and how mathematics should be taught to students.

Since its introduction, thousands of primary schools teachers have been trained with PMRI. Most of them were able to shift from a mechanistic way of teaching to problem-based, interactive instruction. The implementation has been supported by a series of workshops: start-up workshops, follow-up workshops, and quality boost programmes (Hadi et al. 2010a; Haan et al. 2010). However, it was realised that workshops did not stand alone in the professional development programme. Only one strategy would not be sufficient (Louck-Horsley et al. 2010). The PMRI workshops were put into teachers' own settings. Several sessions in the workshops were moved into the classroom. Classroom practices became a critical component in the programme. Teachers were provided with the opportunity to practice and observe what the theory looks like in a real situation on a daily basis. There were pre-workshop and post-workshop activities. During pre-workshop activities facilitators came to schools and observed lessons. This was followed with a post-observation that included redesigning the lesson that had been taught. During post-workshop activities facilitators and participants visited one of the participant's schools to observe a lesson (Fig. 1). The school visit was then conducted regularly in the following months by facilitators of local PMRI centre. In this way, teachers received continued time and support for reflection, collaboration, and learning (Hadi et al. 2011).

A case study conducted by Haan et al. (2010) found that PMRI workshops had achieved their intended goal for hands-on activities, doing mathematics, reflection after each activity, and connection to daily practice. PMRI workshops fulfilled most of the conditions for an effective workshop. Moreover, the workshops met the expectations of the greater majority of the participants; during the workshops, there was a slight change in the attitude toward PMRI. At the start of the workshop, most

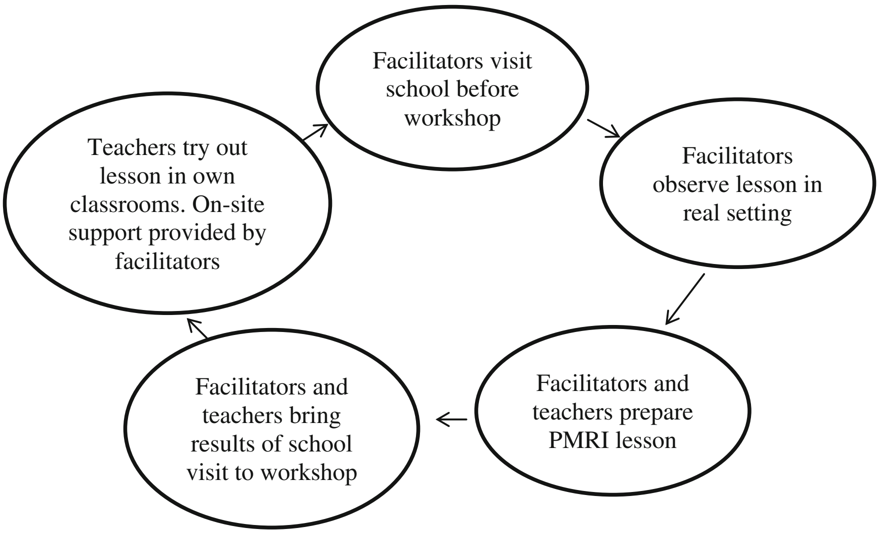


Fig. 1 The full cycle of activity during PMRI workshop

participants indicated they wanted to know more about PMRI; at the end of the workshop, the larger part of the participants declared they wanted to start implementing PMRI.

Developing PMRI Learning Materials

Another key factor in transforming from a pilot project to a large scale innovation is the availability of learning materials. In the first phase of the project it was clear that materials had to be developed. Teachers who were involved in the PMRI workshop repeatedly asked for them. Starting in 2001, a first version of the materials was made by the four early adapting universities. The first version was based on the national curriculum and the knowledge of the learning process of the children at the time. The materials were tried out in the 12 pilot schools (Amin et al. 2010). This first version was used to help teachers implement the concept of PMRI in a lesson. After gaining experience from practising, the PMRI team realised that further development of classroom learning materials was an essential ingredient to institutionalise PMRI, both in teacher education institutes as well as in the associated pilot schools.

For the purpose of developing learning materials, a task force was created. The leaders of the task force, supported by Dutch consultants, were assigned to develop a workshop for prospective authors and form a team capable designing a complete set of learning materials for Grades 1–6 (Amin et al. 2010). Members of the task force were mathematics educators and teachers. Since they developed something that was different from the ordinary textbooks in the country they looked for inspiration from RME materials in other countries, for instance the Dutch textbook series *Pluspunt* and *Wereld in getallen* or American textbook series like *Mathematics in Context* and *Context for Learning Mathematics* (Amin et al. 2010) (Fig. 2).

The Design of Standards for PMRI

Since its first initiation, PMRI has passed through several phases. The first was the preparation (initiation) phase (1998–2002), the second was the pilot phase (2003–2005), the third was the development (implementation) phase, and the fourth was the maturation (institutionalisation) phase (2010–present). After 14 years, a vast body of knowledge has been acquired on PMRI and on what is considered good PMRI education in Indonesia. The current phase requires that PMRI standards be upheld to anticipate the increasing number of schools and universities wanting to join the PMRI movement. Within the PMRI movement, there has been a strong belief in a bottom-up approach, therefore new universities in new regions could

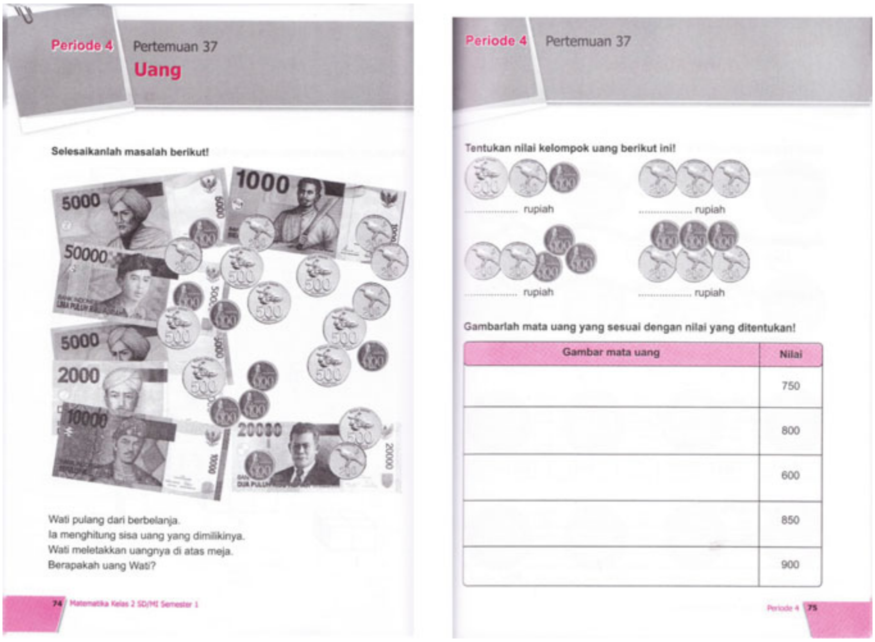


Fig. 2 Example of pages from the learning materials for grade 2

adapt materials and ideas of PMRI to fit their local cultures, needs, and circumstances.

For the purpose of maintaining the quality and integrity of the PMRI concept, a set of standards has been developed. These are standards for PMRI teachers, lessons, learning materials, lecturers, workshops, and local centres (Hadi et al. 2010b).

Standards for a PMRI teacher:

- has a repertoire of mathematics and PMRI didactics to develop a rich learning environment;
- coaches students to think, discuss, and negotiate to stimulate initiative and creativity;
- guides and encourages students to express ideas and find their own strategies;
- manages class activities in such a way as to support students' cooperation and discussion for the purpose of knowledge construction;
- together with the students, summarises mathematics facts, concepts, and principles through a process of reflection and confirmation.

Standards for a PMRI Lesson:

- fulfils the accomplishment of competencies as mentioned in the curriculum;
- starts with a realistic problem to motivate and help students learn mathematics;
- gives students opportunities to explore and discuss given problems so that they can learn from each other and to promote mathematics concept construction;

- interconnects mathematics concepts to make a meaningful lesson and intertwine knowledge;
- ends with a confirmation and reflection to summarise learned mathematical facts, concepts, and principles, and is followed by exercises to strengthen student understanding.

Standards for PMRI learning material:

- are in line with the curriculum;
- use realistic problems to motivate students and to help students learn mathematics;
- intertwine mathematics concepts from different domain to give opportunities for students to learn a meaningful and integrated mathematics;
- contain enrichment materials to accommodate different ways and levels of students' thinking;
- are presented in such a way to encourage students to think critically, creatively, and innovatively and to stimulate students' interaction and cooperation.

Standards for a PMRI lecturer:

- uses PMRI principles during the courses to help student-teachers experience and understand PMRI;
- teaches in a way that supports interactivity in the classroom as a reflection of the principles of PMRI teaching;
- observes PMRI classrooms to collect data and information that can be integrated in the courses at the university and can be used as a basis for research to develop PMRI;
- supports teachers who implement PMRI in School;
- conducts research and produces publications about PMRI.

Standards for a PMRI workshop:

- activities are process-oriented which can support the participants to understand PMRI ideas and are product-oriented aiming at providing materials that can be used in school;
- facilitates participants to experience the PMRI characteristics themselves in order to build their knowledge and skills;
- contents are in line with curriculum demand and the internal and external condition of schools, and envision an ideal situation in order to enhance the adaptability of PMRI in schools;
- participants reflect on the relationship between the activities, mathematical concepts, and PMRI theories;
- empowers and builds the confidence of the participants to sustain the implementation of PMRI in schools.

Standards for a Local PMRI Centre (LPC):

- is an organisation for lecturers, teachers, and student-teachers to do research and develop PMRI;

- is an information and consultation centre about PMRI that provides information, books, teaching materials, teaching media, agendas for professional development, workshops and training, journals, magazines, and videos;
- is a training centre that offers attractive and well-organized training on PMRI that focusses on the process and content;
- is a communication centre that creates cooperation between partner schools, teacher training colleges, other LPCs, and national and international centres;
- is an organisation that is legalised by the rector of the university as a semi-independent organisation with an office and staff.

As an umbrella above these standards, some more general principles and characteristics were formulated.

PMRI Principles

- Guided reinvention and progressive mathematisation
- Didactical phenomenology

PMRI Characteristics

- Use of contexts for phenomenologist exploration
- Use of models for mathematical concept construction
- Use of students' creations and contributions
- Student activity and interactivity in the learning process
- Intertwining mathematics concepts, aspects, and units
- Use of typical characteristics of Indonesian nature and cultures.

Examples from Practice

Thousands of primary school teachers from at least 23 cities in Indonesia use PMRI in their mathematics lessons. The next two examples will show how schools have implemented PMRI. The first example is taken from the mathematics lesson at Grade 3 (8 years old) of Mrs Dewi Mustikawati from Al Hikmah Primary School in Surabaya. Her lesson was about fractions (Figs. 3, 4 and 5).

A day before the lesson Mrs Dewi asked her students to bring the learning materials for the lesson.

Mrs Dewi: "Students, tomorrow we are going to have lunch together. You are going to be divided into groups and each group will bring their own bread, knives, and jam"

Students imagined how cheerful the lesson would be.

A student: "I don't like jam, Mom. May I bring such butter or sugar?"

Mrs Dewi: "That's alright you may also bring cheese or whatever you like."

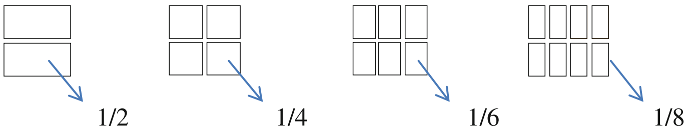
On the day of the lesson Mrs Dewi divided her students into 5 groups. She arranged the activities for each group as following:

Fig. 3 Students exhibit their work

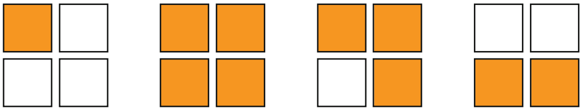
Fig. 4 A student writes a fraction that represents sliced bread spread with jam

Fig. 5 Students celebrate their lesson by eating their bread

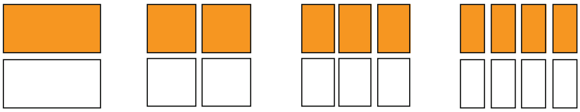
Activity-1: Four loaves of bread were sliced differently.



Activity-2: Four loaves of sliced bread were spread with jam.



Activity-3: Four loaves of bread were sliced spread with jam that show the same fractions.

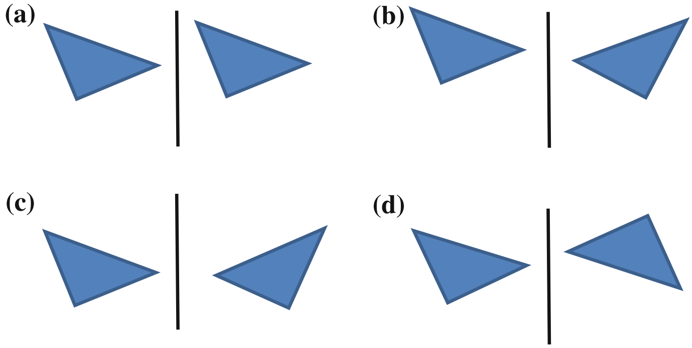


Using these simple activities Mrs Dewi gave students a stimulant to comprehend fractions as division (in activity-1), fractions as part of a whole (in activity-1 and activity-2), comparing fractions with the same denominator (activity-2), putting fractions with the same denominator in the right order (activity-2), and equivalent fractions and simplifying fractions (activity-3).

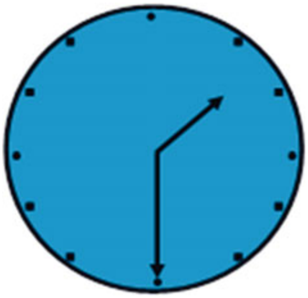
Mrs Dewi was aware that she rarely found a situation where students were so excited and looking forward to their mathematics lesson. PMRI made the mathematics lesson real, and she saw her students act as if they were playing, but they actually learned mathematics from the first minute of the lesson. They did not only easily digest the mathematical concepts, but also enhanced their understanding, since they experienced math with hands-on activities. However, Mrs Dewi realised a challenge she faced in the PMRI lesson, that was to reconcile the limitation of pacing time and the content of the curriculum. Nevertheless, this challenge could be solved using a good design of contextual problems that promote intertwining of the learning strands. These facts convinced her that by having good contextual problems, she would be able to link among units in curriculum, and she did not need to explain mathematics to students from page to page.

The second example is the PMRI lesson of Grade 4 (9 years old) on the topic of reflection conducted by Mr Yusri Zani from Antasan Besar 7 Primary School in Banjarmasin. Mr Yusri found that his students faced difficulty in comprehending the concept of symmetry. He knew that students every day looked in the mirror, but never realised its connection to mathematics. He used this fact to design his lesson. He started with the following problems.

1. Which pictures below show the correct reflection?



2. The following picture is a shadow of a clock. What time is it?



In the first problem, several students chose option “a” as the answer, while for the second problem most students answered that the time is 01:30 P.M. Having these facts Mr Yusri designed his lesson. The materials needed for the lesson were a squared-shape mirror (dimension of 10×10 cm^2) and a worksheet (grid paper). Students were divided into several groups. Each group got a 10×10 cm^2 dimension mirror and a worksheet (Fig. 6). Each group did the following activity.

1. Put the mirror on the thick line of the worksheet (grid paper).
2. Put a dice on the grid paper in front of the mirror, and look at the shadow in the mirror! What do you notice about the shape and size of the shadow?
3. How far is the distance of dice to the mirror and how far is the distance of the shadow to the mirror?
4. Look at the side of the dice in front of the mirror and the dice side in the shadow. What about the direction of the dice through the shadow? Turn the dice that the opposite side faces the mirror. What about the side of the dice shadow?
5. Do the same things for other dice. Is the result the same?
6. Write down your conclusion about the characteristics of reflection based on the experiment you have done.

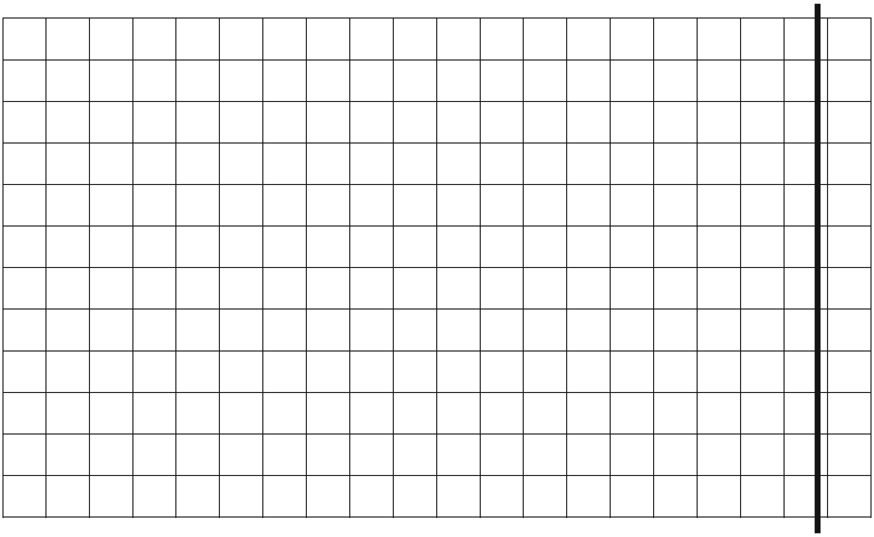


Fig. 6 Student worksheet for lesson about reflection

In Retrospect

The overall goal of the PMRI movement is to improve the learning results in mathematics of school age children in Indonesia. All children should acquire a reasonable amount of knowledge and skills in mathematics during their elementary school years and first years of secondary education. The learning of mathematics must be an inspiring and meaningful activity for all children, must be taught at each child's own level, and must lead all children to a practical knowledge base that will help them cope with quantitative situations in the world around them. For some children, learning mathematics at a young age must also function as on introduction to the more formal world of science and academic mathematics ahead of them. (Sembiring et al. 2010b, p. 189).

PMRI has been proven to be an approach that can accomplish this. It works. However, what worked in the selected pilot schools is not automatically implementable on a large scale. The implementation and institutionalisation of PMRI all over Indonesia is still an enormous endeavour. It can only be accomplished with the hard and enduring efforts of many: teachers, parents, principals, teacher educators, mathematicians, publishers, journalists, policy makers, politicians, and many more. (Sembiring et al. 2010b, p. 189).

For the coming years the following concrete issues will be addressed and be worked upon (Sembiring et al. 2010b, p. 189).

- Expanding a school-based system of professional development of teachers on the subject of PMRI. Mathematics and language are the key subjects for further development.

- Increasing the capacity of universities to educate prospective teachers with a conceptual and practical base of PMRI. Teachers are among the most crucial factors in the improvement of mathematics education.
- Creating a research agenda on PMRI and conducting design research in the classrooms, PMRI must become an instruction theory with a sound scientific basis in order to make evidence-informed choices.
- Creating assessment materials that reflect the concept of PMRI. These concepts are in line with worldwide developments in mathematics education. See, for instance, PISA (OECD 2006).
- Working on the public relations of PMRI through bulletins, newspaper articles, TV, etc.
- Creating a text book series of PMRI learning materials from Grades 1 to 6, as an example of PMRI practice and as a starting point for further local adaptation and development.
- Through the accomplishment of the above items, reaching an increasing number of schools in an increasing number of regions and cities in Indonesia, by striking a balance between bottom-up conceptual development and top-down facilitation and support.

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Emotions in Problem Solving

Markku S. Hannula

Abstract Emotions are important part of non-routine problem solving. A positive disposition to mathematics has a reciprocal relationship with achievement, both enhancing the other over time. In the process of solitary problem solving, emotions have a significant role in self-regulation, focusing attention and biasing cognitive processes. In social context, additional functions of emotions become apparent, such as interpersonal relations and social coordination of collaborative action. An illustrative case study presents the role of emotions in the problem solving process of one 10-year old Finnish student when he is solving an open problem of geometrical solids. The importance of emotions should be acknowledged also in teaching. Tasks should provide optimal challenge and feeling of control. The teacher can model the appropriate enthusiasm and emotion regulation. Joking and talking with a peer are important coping strategies for students.

Keywords Emotion · Problem solving · Coping

Introduction

Problem solving competence is given high priority in curriculum documents (OECD 2003) and especially in mathematics, problem solving is considered essential (Schoenfeld 1992). In this presentation we do not consider mathematical routine tasks as problems. Instead, a mathematical task is a problem for a person only if he or she does not immediately know how to solve it.

The role of affective elements in mathematical problem solving has been widely acknowledged. Already Polya (1957) addresses determination and hope (p. 93) in his short dictionary of heuristics, mentioning also the necessity to become familiar with all emotions related to the problem solving process. More explicitly the role of

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affective variables was elaborated in the 1980s, when Mason et al. (1982), Schoenfeld (1985), McLeod (1988), Goldin (1988, 2000) and Cobb et al. (1989) all gave a significant role for affect in their analysis of mathematical problem solving.

The first part of this paper will review the affective aspects of non-routine problem solving behaviour. This discussion will be divided into two sections, the first focusing on students' relatively stable emotional traits that may influence the problem solving behaviour and the other part on the continuously changing emotional states that occur during the problem solving process. The second part of the presentation will provide an illustrative example of one student's emotions and coping while engaged in problem solving in classroom setting. The last part of the paper will focus on the different ways in which the teacher can enhance the desirable emotional climate in a problem solving lesson.

In the literature, there are several definitions for emotions stemming from three distinct traditions. The Darwinian research tradition focuses on the evolutionary and biological foundation of human emotions. Freudian tradition focuses on the role of emotions in psychopathology. The cognitivist tradition focuses on the interplay of emotions and cognitions. Despite the separate histories, these traditions agree upon some aspects. There is a general agreement that emotions consist of three processes: physiological processes that regulate the body, subjective experience that regulates behavior and expressive processes that regulate social coordination. Moreover, most emotion theories agree that emotions are closely related to personal goals, that they have an important role in human coping and adaptation. Furthermore, they perceive a physiological reaction to be the distinguishing feature between emotions and cognition. Emotion theories are different in the number of emotions they identify, the degree of consciousness they attribute to emotions, and in how they perceive the relation between emotion and cognition (Hannula 2004, p. 26–27).

Some emotion theories (e.g. Lazarus 1991) identify a large number of different emotions based on the different social scenarios and cognitive appraisals related to the emotion, while some other emotion theories (e.g. Buck 1999; Ekman and Friesen 1971; Ekman 1992) identify a small number of basic emotions that differ in their physiology, and the different cognitive appraisals and social scenarios are seen as external (though closely related) to the emotion. Moreover, some theorists have attempted to organize emotions based on the dimensions of arousal and valence (e.g. Lang 1995).

In this paper, we identify different emotions along the tradition of basic emotions. These basic emotions include at least happiness, sadness, fear, anger, and disgust (Buck 1999; Power and Dalgleish 1997, p. 110). Further basic emotions include some of the self-regulative emotions, where best candidates seem to be surprise, curiosity and confusion (Lehman et al. 2008; Goldin 2000). Furthermore, the list should include also further social emotions such as shame (Pekrun and Stephens 2010), which has a characteristic physiological expression (blushing), and attachment and submission (Buck 1999), which have important influence on interpersonal relationships. Other emotions (e.g., gratitude, pride, contempt etc.) are seen as basic emotions in a characteristic situation or with a characteristic target or, in some cases, as blends of two or more basic emotions. For example, anxiety can be considered as fear for failing on what one is doing, and hopelessness can be considered as anticipatory sadness when failure is seen to be inevitable. The more complex emotions are based on the basic emotions and characterised largely by the related cognitions (Buck 1999; Power and Dalgleish 1997). Although we adopt the approach of basic emotions, we use more specific vocabulary for these emotions when the social scenario or cognitive appraisal plays a significant role. Although some researchers define moods as a separate concept, we do not discuss them separately, because we consider them here as low-intensity emotions (Pekrun and Stephens 2010).

In this paper, we adopt the view by Damasio (1999) regarding three stages of consciousness of emotions:

> I separate three stages of processing along a continuum: *a state of emotion*, which can be triggered and executed nonconsciously; *a state of feeling*, which can be represented nonconsciously; and *a state of feeling-made-conscious*, i.e., known to the organism having both emotion and feeling. (p. 37; as cited in Schlöglmann 2003)

Student Disposition and Mathematical Competence

It is widely acknowledged that the problem solver's overall disposition (attitudes, beliefs, values, motivational orientations) influences how successful the solver will be in their attempts to solve the problem. There are several different theories that focus on different aspects of the disposition. Usually these studies have focused on the relationship between a specific affective trait (anxiety, attitude, motivation or beliefs) and achievement in mathematics. However, there is good reason to assume that these findings apply in a similar way also to the more specific relation between these affective traits and achievement in problem solving.

Perhaps the first approach to explore the connection between disposition and mathematical performance were the numerous studies on mathematics anxiety and its negative correlation with mathematics performance (for a meta-analysis, see Hembree 1990). Similar relationship was confirmed between attitude towards and achievement in mathematics (McLeod 1992).

Motivation research has several theoretical approaches and use of terminology is sometimes confusing (Murphy and Alexander 2000; Niemivirta 2004, 10; Pintrich 1994). However, there is conclusive empirical evidence for a positive correlation between motivation and performance (e.g. Middleton and Spanias 1999). The more value the student gives to solving a particular problem, the more persistent the student is in his or her effort. More specifically, it has been confirmed that intrinsic motivation (interest in the task) is more productive than extrinsic motivation due to rewards (Middleton and Spanias 1999). In the achievement goal theory, a positive relationship has been found between mastery goal orientation and achievement (Friedel et al. 2007; Midgley et al. 1998). Results concerning performance goal orientation and achievement have been less consistent. Some have identified negative learning behavior, while other results indicate performance orientation to lead to positive learning behavior and achievement (Freeman 2004; Midgley et al. 1998).

Bandura's theory of self-efficacy is looking at the connection between self-efficacy beliefs and actual performance in a specific domain (Bandura and Schunk 1981). Numerous studies have confirmed the high correlation between the two (e.g. Lee 2009).

Although survey studies indicate a clear correlation between mathematics-related affect and achievement, it has been more difficult to confirm the direction of causality. Ma and Kishor (1997a, b) and Ma (1999) have summarized much of that research in their meta-analyses. In one of these studies, they synthesized 113 survey studies of the relationship between attitude towards (= liking) mathematics and achievement in mathematics. The causal direction of the relationship was found to be stronger from attitude to achievement (Ma and Kishor 1997a). However, there has been criticism of studies that do not use a longitudinal design (see Ma and Xu 2004). Minato and Kamada (1996) reviewed eight studies that had used a cross-lagged panel correlation technique (a longitudinal design) in order to synthesize findings on the causal relationship between attitude towards mathematics and achievement in mathematics. In most of the studies, there was no predominance of either attitude or achievement. In the few instances that predominance was found, the causal direction was from attitude to achievement. However, Ma and Xu (2004) found a contrasting result with a larger and more representative sample. According to their study, the dominant causal relationship is from achievement to attitude. Taken together these studies suggest a reciprocal rather than unidirectional causality between achievement and affect. Such reciprocal relationship has been identified between self-efficacy and achievement in mathematics across countries (Williams and Williams 2010).

There are also studies that address more specifically how student's beliefs influence the choices they make. Schoenfeld (1985) concluded that students' beliefs were an important determinant of their problem solving success. He observed that most students were likely to give up if the problem was not solved in five minutes, and concluded that students possessed a belief that mathematics problems can be solved in five minutes or less.

As a summary we can conclude that there is strong evidence for the correlation between student disposition towards mathematics and their actual performance in mathematics. High achievement is related to liking mathematics and determination to do well. Moreover, the relationship is reciprocal, indicating that changes in either can lead to changes in the other. Moreover, beliefs about mathematics and problem solving may direct the problem solver to explore venues that support innovative solutions or they may suggest staying on more familiar paths.

Emotional States and Problem Solving

Another perspective looks more specifically at the different choices the student makes throughout the problem solving process and the role of emotions in it. Regardless of the overall disposition, all problem solvers encounter positive and negative emotions that influence their solution process (e.g. Schoenfeld 1985; McLeod 1992; Goldin 2000). In fact, emotions are an essential part of the problem solver's self-regulation (Goldin 2000; Hannula 2006; Carlson and Bloom 2005; Malmivuori 2006). Moreover, problem solving takes place by social beings in the complexity of the learning environment where multiple goals need to be addressed (Hannula 2006; Goldin et al. 2011).

Next, we will review some studies that have explored which emotions are present in the learning context and more specifically in problem solving. It has been suggested that six basic emotions (anger, sadness, fear, disgust, happiness, and surprise; Ekman 1972, 1992) would be rare in learning context (Vogel-Walcutt et al. 2012). However, as these basic emotions relate to social coordination, they should be more frequent in collaborative learning settings. Pekrun and Stephens (2010) identify a number of emotions that appear frequently during the learning process and three dimensions in their taxonomy: valence, activation, and object focus. The variety of emotions in their twelve categories can be alternatively categorized under basic emotions: the negative activating emotions represent three different basic emotions: anger (anger, frustration), fear (anxiety), and shame (shame) while all deactivating negative emotions (boredom, hopelessness, sadness, and disappointment) are variants of sadness. The activating positive emotions (enjoyment, hope, anticipatory joy, joy, pride, and gratitude) are all variants of the basic emotion of happiness while the positive deactivating emotions (relaxation, anticipatory relief, contentment, and relief) seem not to be basic emotions, but rather lack of any emotions or removal of a negative emotion (relief).

Emotions that have been identified to appear frequently during solitary problem solving (logic problems) are curiosity, boredom, frustration, confusion and happiness while occurrences of anxiety, contempt, eureka, anger, disgust, fear, sadness, and surprise were found to be rare (Lehman et al. 2008). However, their study design included an incorrect feedback to 25 % of the responses, which might explain the high frequency of curiosity, frustration and confusion. In solitary mathematical problem solving, emotions identified to have importance in the self-regulative processes include curiosity, puzzlement, bewilderment, frustration, pleasure, elation, satisfaction, anxiety, and despair (DeBellis and Goldin 2006). When mathematical problem solving has been studied in the mathematics homework context, researchers have identified the following emotions to appear in the mother-child interaction: tension, distress/dismay, frustration, sadness, boredom/apathy, anger/disgust, contempt, positive interest, affection/caring, joy/pleasure, humor, and pride (Else-Quest et al. 2008). If we analyze the emotions in these lists according to a basic emotions framework, we can expect to see most of the basic emotions, when observing problem solving: curiosity to begin with and happiness if the problem is solved. In case the student struggles with the problem or cannot solve the problem there might be sadness, confusion, fear, anger or disgust, depending the attribution the student gives for the lack of progress.

There is a general agreement that emotions are functional and that they have an important role in human adaptation to different situations and learning. This applies also to mathematical problem solving. However, dynamic theories about the role of emotions in the process of problem solving are at the moment at their preliminary stages (Lehman et al. 2008; Goldin et al. 2011).

Emotions serve three fundamentally different functions in human self-regulation: physiological, psychological and social. The first function is physiological adaptation, where the most clear example is the ‘fight of flight’-response to surprising threatening stimulus. The emotion triggers release of adrenaline, which prepares the body physically to fight or alternatively to escape. (Power and Dalgleish 1997) This functional aspect of emotions is relevant to learning situations in the sense that most emotions have physiological reactions as a side effect. For example, fear may influence the physiology in a way that is detrimental for optimal cognitive functioning in a test situation.

The second role of emotions is in the psychological self-regulation through influence on cognitive processing. Just as fear or anger have clear consequences in the physiological adaptation; surprise and curiosity have clear influence on attentional processes and memory. It is now well established that emotions direct attention and bias cognitive processing. For example, fear (anxiety) directs attention towards threatening information and sadness (depression) biases memory towards a less optimistic view of the past (Power and Dalgleish 1997; Linnenbrink and Pintrich 2004).

It has been suggested (Forgas 2008) that the positive emotions would promote the more inductive, bottom-up thinking while the negative emotions would promote the more deductive, top-down thinking. Although there is not yet sufficient evidence to conclude that to be case for all positive and negative emotions, it seems that positive emotions would facilitate the creative aspects of problem solving, while the negative emotions would facilitate reliable memory retrieval and performance of routines, which also are essential in certain phases of problem solving (Pekrun and Stephens 2010). More intensive levels of mathematics anxiety seem to be exclusively detrimental for problem solving and the suggested mechanism is the overloading of working memory as the subject is preoccupied with one’s math fears and anxieties (Ashcraft and Krause 2007; Rubinsten and Tannock 2010).

Moreover, emotions not only bias memory retrieval, but also function as a ‘fixative’ in the storage to long term memory. A recent study has identified that activity in the amygdala during an Aha! experience is a strong predictor of which solutions will remain in long-term memory (Ludmer et al. 2011).

Emotions serve also a third adapting function in the social coordination of a group, which brings forth new types of emotions. The close relation between emotions and the social interaction is well acknowledged (e.g. Forgas 2008; Pekrun and Stephens 2010). Previously Cobb et al. (1989) identified that students’ emotions are related to two types of problems in collaborative problem: mathematical problems and cooperation problems. Hannula (2005) identified three different social functions for emotions in collaborative problem solving:

1. emotions concerning interpersonal relationship needs and goals (e.g. sadness due to exclusion),
2. emotions concerning individual learning goals when their cause is attributed to peers (e.g. gratitude for help), and
3. emotions concerning social coordination of individual goals (e.g. anger when own idea has been rejected by others).

The interpersonal relationship needs and goals are not specific to mathematics or even learning, but they should not be ignored when we analyze student emotions in the classroom. The emotions that relate to individual learning goals, on the other hand, are rather straightforward extensions of self-regulative emotions in solitary problem solving. The third category deserves further elaboration.

In the coordination of collaborative behavior, emotions can be expressed and interpreted unconsciously. Emotions may also be used consciously in power games or as means to solve communication problems. Furthermore, emotions may be interpreted consciously, when they become subject to reflection and re-evaluation. Emotional communication in the coordination of collaborative problem solving behavior can be very powerful, one example being shared cognitive intimacy (Hannula 2005, c.f. Williams 2002):

> In shared cognitive intimacy, students enter an intimate interaction with each other and with a task. This intimacy is indicated by one student frequently continuing or completing the other student's utterances and occasionally by both speaking in unison. It is an example of a situation where students can achieve their cognitive and social goals simultaneously. This dual intimacy with peers and mathematics is rewarding for the students and, furthermore, it can be an extremely useful tool for enhancing the classroom climate. One problem with this kind of intimacy is that sometimes it may exclude other students. (Hannula 2005, p. 35)

The same study identified also a number of strategies that a student may use to 'save their face' when problem solving does not seem successful (Hannula 2005). Of these strategies devaluing of the task and expressing lack of effort may be detrimental for the development of productive social norms for collaboration.

A study on mother and child emotions during mathematics homework supports the hypothesis that emotions are both an influence on and an outcome of mathematics performance (Else-Quest et al. 2008). Moreover, it provides some interesting qualitative data on the dynamics of emotions while doing homework, suggesting a reciprocal relation where either mother or child may initiate important changes in the emotional state of the other.

Emotions and Problem Solving in the School Context

So far, most research on mathematics-related affect has been done using surveys, which has provided strong evidence for the theories regarding trait-type emotions (although good longitudinal studies are too few). Research on the role of emotions in cognitive processes has mainly been conducted qualitatively, and while such studies are good for identifying important concepts and their relations, they seldom are rigorous enough for testing theories. Quantitative studies, on the other hand, are mainly conducted in laboratory settings and the ecological validity of research findings must be tested in realistic school settings (Pekrun and Stephens 2010).

Perhaps the first to report the role of emotions both in the cognitive and social domain were Cobb et al. (1989). Their study challenged the previous observations of emotionally neutral or 'flat' classrooms, illustrating instead a lively classroom:

> Children frequently jumped up and down, hugged each other, and rushed off to tell the teacher when they solved a particularly challenging problem. Significantly, the positive emotional acts occurred when the children completed personally challenging tasks or constructed mathematical relationships. (Cobb et al. 1989, p. 61)

In their framework, social norms provide a framework for interpreting the individual emotions (Cobb et al. 1989).

Another approach has been to report case studies of student's emotions while solving problems in a mathematics classroom (e.g. Evans et al. 2006; Hannula 2003; Op 't Eynde and Hannula 2006; Williams 2002).

However, Goldin et al. (2011) have suggested that there is a need for 'mid-level' concepts between in-depth qualitative studies and 'high-level' concepts like norms. In a qualitative study of an US inner-city middle school, they identified a number of behavioral patterns that integrate students' affective and social interactions, which they call *engagement structures*. Each engagement structure is characterized by ten aspects, which include goal, behavioral and emotional pattern, self-talk and interaction with beliefs. These engagement structures could be also described as behavioral scripts that relate to a specific goal or desire. (Goldin et al. 2011)

Emotions are ubiquitous in human interaction and, therefore, it is extremely challenging to extract the influence of teacher's emotional support in research design. However, in computer-based learning environments it is easier to study the influence of emotional or motivational support. Kim and Hodges (2012) found out that even a minimal six minute video aimed at improving student emotion regulation was able to produce significant influence on university students' academic emotions in an on-line remedial mathematics course.

As said earlier, emotions serve the three purposes of physiological adaptation, psychological regulation, and social coordination. Although emotions are functional for the human species, not all emotional reactions are functional in classroom context. For example, anger provides functional physiological adaptation to overcoming obstacles when solving a physical problem (e.g. moving a heavy object). Yet, it is usually less functional when solving cognitive challenges collaboratively. Moreover, the physiological aspect of the emotion (e.g. adrenaline or endorphins) may have effects beyond the duration of the emotional event and such emotional residual may interfere with following emotional episodes. Such unwanted side effects of emotional reactions could be considered as emotional 'noise' confusing smooth emotional communication.

Therefore, especially in the school context, emotions need to be regulated (De Corte et al. 2011; Fried 2012). Emotion regulation refers to

> the ways individuals influence which emotions they have, when they have them, and how they experience and express these emotions. (Gross 1998, p. 275)

There are different ways to categorize emotion regulation strategies. The target of emotion regulation can be attention, emotion-relevant knowledge, or body manifestations of emotion, their psychological function may be oriented towards needs, persons or goals, and both antecedent and response-focused strategies are possible (Fried 2012). Emotion regulation that mainly focuses on reducing negative emotions or their effects is often called coping. The coping strategies that Flemish high-school students report using when facing difficult mathematics test, homework or lesson are (in order of frequency) active coping (i.e. effort), joking and acceptance, social-emotional coping (i.e. seeking social support), abandoning and negation, religion and—rarely—alcohol and drug use (De Corte et al. 2011). In addition to coping, it is important to consider the emotional regulation that occurs before emotional reaction through choice of goals and selective attention.

An 10-Year Old Student Solving an Open Problem

In this section, we shall analyze one student's emotions in the course of one lesson. The case will illustrate different functions of emotions.

The Context

The analysis is based on a video recording of a research class that participates in a project, where one mathematics lesson each month is used to solve an open-ended problem and the participating eight teachers meet once a month with project research team to discuss using open-ended problems in teaching (Näveri et al. 2011). This analysis will focus on one particular 10-year old male student (Tomi[1]) and his interaction with his teacher and his two peers sitting next to him (Arto and Eetu). One video camera was used for recording the overall progress of six students. We do not have a full coverage of Tomi's behavior during this lesson. However, due to his position in the middle of the group, he appears on the video most of the time.

[1]All names are pseudonyms

The Analytical Framework

For the analysis of this case, we shall use the emotion coding scheme by Else-Quest et al. (2008), which was developed to analyze mother-child interaction during homework. This coding scheme is based on two emotion coding schemes: Ekman's FACS system (Ekman et al. 2002), which identifies six basic emotions from facial expressions and Gottman's SPAFF system (Levenson and Gottman 1983), which identifies 16 emotions in dyadic interactions based on facial expressions as well as other verbal and nonverbal expressions of emotions. This coding scheme was chosen for this analysis, as it has been tested and found useful in context where both self-regulative and social aspects of emotions are essential. The coding scheme includes 13 emotions: tension, distress/dismay, frustration, sadness, boredom/anxiety, anger/disgust, contempt, positive interest, affection/caring, joy/pleasure, humor, pride, and off-task. The coding scheme provides a number of markers for each emotion. Table 1 provides markers for two of the emotions. For details, see Else-Quest et al. (2008).

The Case

The open problem of the lesson was to construct models of different solids using given manipulatives (peas and cocktail sticks) and to record the number of edges and vertices of each solid. In the first phase, the students were asked to find out different solids that they can construct using no more than 12 sticks. When student could not find any further solids with 12 sticks, they were permitted to use more sticks (14, 16, etc.).

The lesson began by the teacher giving instructions to the class and showing one exemplary model of a shape to the students. Each student was given their own pile of sticks, and Tomi, Arto, and Eetu received one cup of peas to share, located on Tomi's desk. In the transcript, the time starts to run from the moment students have received the peas and sticks, and they begin their construction.

Time

00:00 {Tomi, Arto, and Eetu all start building a cube as their first solid.}[2]

01:39 {Tomi accidentally hurts his hand into a sharp stick.} Ow, sshiit! {He pulls his hand away in a reflex-like motion. Emotion: distress/dismay.}

Here, Tomi experienced pain and cursed because of it, indicating that he did experience anger. However, no further consequences could be observed. It is likely that Tomi was little more careful when attaching the next pea, but the incident did not distract him from his work nor did other students react to it. However, a similar

[2]Curly brackets {} indicate observations and interpretations based on the video.

Table 1 Two examples of emotions and their markers used for the analysis

Emotion	Markers
Joy/pleasure	High-fives, smiling (lip corners up, raised cheeks, outer brow down), exclamations (e.g., "Wow!" or "Cool!")
Pride	Sitting upright,"showing off", similar facial expression as joy/pleasure, but antecedent event is achievement

incident happened later in the lesson, when he encountered a rotten pea. Then his face indicated disgust, and his work was interrupted until he had cleaned the rotten pea away. The incident also attracted the attention of Arto and Eetu, who interrupted their own work to examine the cause of disgust.

These two incidents illustrate how the 'primitive', self-protective role of emotions can intervene problem solving. In both cases they were given priority (the first one was like a reflex), but at least for Tomi, the distraction was short. However, students who are more timid and prone to anxiety, might not overcome the emotional distraction as easily as Tomi.

01:45	{Tomi is still building the cube. Emotion (momentarily): tension. After that he speaks some off-topic with Arto and Eetu.}
02:38	{The cube is more than half ready. Tomi leans close to it and points at each of the sticks (vertexes) and also to each vertex that has not yet been constructed. Arto and Eetu speak off-topic past Tomi, who ignores it. Emotion: positive interest.}
02:44 Tomi	This will be exactly 12! {The tone of voice indicates excitement, perhaps also pride.}

In this episode Tomi was effectively regulating his attention when focusing on the task and he expressed elation when realizing that the construction fulfills the requirements of the task. Specific to this episode was that these emotions related to the individual goals of Tomi and perhaps for that reason they did not well fit the markers of the coding scheme. The tension in the beginning was not of unpleasant nature; rather it just reflected the intensity of concentration. Later, Tomi expressed his excitement verbally, perhaps wishing to share the emotion with peers, but they were focusing on their individual work and they did not react.

03:07 Tomi	{Arto is picking pea from the cup, as he has done several times.} Why are you taking mine! {Turns his face to look Arto directly in the eyes, at the same time moving his hand on the desk sharply towards Arto, perhaps as to mark 'his' space. Then moves his hand towards Arto's hand, as if trying to stop him from taking peas. Clearly, he is challenging Arto, although the specific emotion is not clear.}
03:10 Arto	Because we don't have [own][3]

[3]Square brackets [] indicate overlapping talk.

03:11	Tomi	[Ah, right.] We don't have. {Covers his face with both hands.}
03:13	Arto	What are you freaking out! {Smiling}
03:16	Eetu	{Tomi takes hand off his face, is blushing, smiles meekly. Emotion: shame.} For real, Tomi! Laughable. Someone's insane here. One freaks out.'Why are you taking mine!' {pitched voice}. {Arto smiles widely and giggles, also Tomi laughs shyly, still blushing}
03:26	Tomi	I forgot that we'd have our own {in a meek voice}
03:30	Eetu	You forgot that we'd have our own?
03:32	Tomi	{Inaudible}

This second episode was clearly about social coordination, use of shared resources but also about the social relationships and reputation. First, Tomi reacted to defend his 'rightful territory'. However, as he realized that peas were a shared resource he was very embarrassed, which is in this case very expressive: he gave rapidly up his challenge, covered his face, blushed, and even the tone of his voice changed. Such a reaction was perhaps the most reliable apology to make, as blushing cannot be faked. Eetu was emphasizing the incident, making fun of Tomi, who in this situation was vulnerable. However, all three laugh at the end, indicating that Tomi did not feel the attack too serious. This incident was also noticed by two other students who were seen on the video, turning to look at the three boys a little bit worried first, and at least one of them was smiling at Eetu's ridiculing of Tomi. Also in this case the coding scheme was insufficient, providing a category for neither of the main emotions, although the challenge could be classified under tension.

Also this incident was passed with little effect on students' problem solving. There was clearly some emotional residual for Tomi, and possibly this incident influenced his goals in the future episodes (he was ambitious, wanting to produce something special).

All three boys finished their cubes, Tomi made swiftly also a tetrahedron.

03:36	Tomi	{to Arto} What are you doing? {Looks at the constructions of Arto and Eetu.}
04:21	Arto	For real, don't do exactly same as me
04:26	Tomi	I don't even know what you intend to do
05:00	Tomi	You are doing the same as me. All these [cubes] are identical
05:12	Tomi	{Builds a'house' of the cube and the tetrahedron. Emotion: Humour.} Haha haha, look! Ha! {Arto looks at the'house' and smiles.}
05:21	Tomi	What to do now? {Emotion: boredom.}
05:44	Tomi	What would I do now? {Tomi leans back, slouching on his chair, soon sits again more upright, then puffs air out with a sound and fidgets with sticks. Emotion: boredom.}

06:04 Tomi	{to teacher} I don't get it. {Emotion: distress/dismay. Teacher's response is inaudible.}
06:08 Tomi	Can they be something totally own? {Emotion: Joy/pleasure. Tomi begins to construct a new model.}
Teacher	They can be your own. As long as there are not more than 12 sticks

Tomi had difficulties to continue after the two first solids. He realized that they had done identical cubes and Arto warned him from doing the same as he. Tomi used joking as a coping strategy, but symptoms of boredom become soon clearly visible. He did not like the situation, and seeked support from his teacher. Something that the teacher said gave him an idea to work with.

For the sake of brevity and to keep the red line visible, only part of Tomi's interactions is reported in the next section. Tomi built a 'flat' shape of two attached squares.

08:30 Tomi	Teacher! See, ladders! {Shows his construction to teacher, whose reaction cannot be seen, plausibly he is occupied with some other students. Arto is looking at Tomi's 'ladders' and both boys are laughing. Emotion: humour.}
08:40 Eetu	What is that?
08:42 Tomi	{Emotion: changes rapidly to neutral.} I don't know. This just came out like this
08:50 Tomi	{To teacher, showing his'ladders', smiling. However, emotion is not joy/pleasure.} Does this count?
08:54 Teacher	That's not a solid yet. It's on a plane, in a way. Think if you could continue it.{Tomi makes a new type of construction.}
09:40 Tomi	Look! {Emotion: joy/pleasure.}
09:47 Tomi	Well, teacher! Is this one?
09:52 Teacher	See, it is still slightly open here. {Pointing.}
09:53 Tomi	Hmph! {Emotion: frustration.}{Tomi adds one stick to his construction, ending up with a model that is not a polyhedron. It is relatively complex model and it is ambiguous which solid it would represent. However, it could be interpreted as a model of a solid whose some surfaces are curved.}
10:02 Tomi	Well is this now? {Emotion: Joy/pleasure. Teacher is with another student.}
10:27 Tomi	{to Arto} Hey look! {Expressed amazement}. Diamond. {Emotion: joy/pleasure.}
11:14 Tomi	Teacher! Come and look. I will show
11:18 Tomi	It's a diamond. A perfect diamond. {Funny voice.}
11:28	{Teacher arrives. Tomi's emotion: Looking for reinforcement?}
11:32	Teacher: Is it a solid now? {The teacher does not accept the solution and his nonverbal communication seems to converse that to Tomi.}

11:33 Tomi	No.
11:34 Teacher	Think! {Turns and leaves.}
11:35 Tomi	{Twists the model a little.} Now it is! {Smiling. Emotion not clear. Teacher turns to look, smiles gently, his expression communicating something like "you can't fool me".}
11:45 Tomi	I don't want to demolish this. {Pulls the model apart. Emotion: sadness.}
11:47 Tomi	A dog! {Arto gasps, pretending to be amazed.}
11:48 Eetu	Where's a dog? {First Arto, then also Tomi and Eetu start laughing. It is not clear, but Tomi might be slightly blushing again. Emotion: Humour.}
12:20 Tomi	Now I got it!
12:45 Tomi	Hey teacher, this is certainly a solid! {Hits his desk with both hands, fists clenched. Emotion: frustration.}{Tomi makes a new shape, which also has'curved' surfaces.}
13:52 Tomi	Is this, then? And this? {Emotion is not clear. Although he gives a social smile, the emotional feeling is more like sadness.}
13:58 Teacher	You really got [interesting.]
13:59 Tomi	[And look!] This is a house. No, a tent
14:06 Teacher	It has to be closed from all directions. {Turns and leaves.}
14:10 Tomi	Now I will close it from all directions. I will cram if need be. {Emotion: Anger/disgust.}

Here it became apparent that Tomi considered his teacher as the criterion for acceptable solutions. Although he himself got pleasure from inventing his models, his joy proved to be repeatedly premature as his teacher's judgment rejected the attempts. Joking was the all-around coping strategy for Tomi. Although Tomi showed great resilience, his negative emotions grew stronger as rejections were repeated. The expressions of anger and sadness and the related self-talk were not directed at anyone. Yet, there was a feeling of display attached to them. Perhaps the display of negative emotions in a theatrical fashion was a way to regulate the expression of emotions to be socially acceptable. Moreover, Tomi's emotional communication with his teacher seemed to develop towards increasingly angling for sympathy.

This analysis provides an example of the rich variety of different functions of emotions in mathematical problem solving in classroom setting. More specifically, the task was open and the duration of problem solving was long, here we have analyzed less than half of the time Tomi spent working on the task. One thing that is apparent here is the presence of emotions and student's capacity to regulate them in a productive manner.

Conclusions and Implications for Teaching

The main lesson to learn from the research on emotions in problem solving is that emotions are an essential part of problem solving. Some emotions direct attention and intuition and are functional, perhaps necessary, in the process of successful problem solving. Emotions play an important part also in learning from the experience of both successful and unsuccessful problem solving. Moreover, emotions influence the formation and development of mathematics-related motivation, attitudes, and beliefs. As some emotions seem to be more beneficial to learning outcomes than others, teachers and curriculum developers need to pay attention to the student emotions.

What are the characteristics of a classroom that promotes optimal emotional climate? It has been shown that the teacher enthusiasm (Frenzel et al. 2009) and the chosen method of teaching (Schukajlow et al. 2011) can have an influence on student emotions.

Several schools have implemented programs to enhance students' social and emotional learning and aim to promote a healthy learning environment. Specific goals for the program are that students acquire core competencies to recognize and manage emotions, set and achieve positive goals, appreciate the perspectives of others, establish and maintain positive relationships, make responsible decisions, and handle interpersonal situations constructively. According to a meta-analysis, universal school-based social-emotional development programs have beneficial effects on positive social behaviour, problem behaviours, and academic performance (Durlak et al. 2011).

When students are engaged and face an optimal challenge, they can experience flow (Csikszentmihalyi and Csikszentmihalyi 1992; Williams 2002). Unfortunately present school seems to rarely provide experiences of flow. Quite the contrary, a specific and persistent problem is that classrooms are often emotionally flat, and boredom (i.e. temporary feelings of low-arousal and unpleasant emotions induced by environmental factors) is one of the most frequently experienced emotions. (Pekrun et al. 2010; Nett et al. 2011; Vogel-Walcutt et al. 2012). Classroom environment seems to be a critical determinant of students' boredom experiences indicating that instructional approaches can influence the amount of boredom in the class (Nett et al. 2011). Specifically, subjective experience of control decreases the level of boredom (Pekrun et al. 2010).

Pekrun and Stephens (2010) claim that except for research on test anxiety, there is little research concerning the effects of the task and the environment on academic emotions. Based on Pekrun's control-value theory and results regarding test anxiety they suggest the following characteristics to have positive effects on student emotions:

- students perceive value in the task and have a feeling of control,
- there is a match between task demands and student competence,
- tasks and learning environments meet individual needs,
- teacher is enthusiastic,

- student autonomy is supported,
- emphasis on mastery goals,
- positive feedback, and
- positive consequences of achievement.

Because of the functional aspect of emotions in self-regulation and learning, there should be space for emotions in the classroom. Even the negative emotions related to failures seem to have an important role in creating intuitions (Immordino-Yang and Faeth 2010). However, sometimes emotions need to be regulated. There are three possible perspectives to emotion regulation in the classroom: teacher regulation of student emotions; student and teacher self-regulation of own emotions and students' regulation of peer emotions (Fried 2012).

The teacher can influence the students' emotion regulation through modelling emotion regulation strategies. Through teacher modelling students learn to recognize and understand the role of different emotions in problem solving and they also learn about the different ways to control the experience and expression of these emotions. The teacher may also provide more direct support through controlling student emotions, although a teaching style where students are encouraged to think for themselves (autonomy supportive) is more effective in helping students develop their emotion regulation. (Fried 2012)

Perhaps more important than focusing directly on students' emotion regulation, is to develop such social norms in the classroom that encourage students to regulate their own and peer emotions. Also joking and seeking peer support serve important function in coping with frustration or other negative emotions (De Corte et al. 2011). Firstly, expressive environments have been found to support development of emotion regulation strategies. If such expressive environments have also a feeling of community, where students feel belonging to the classroom, they can more easily assimilate external regulation they observe into the self (Fried 2012).

Cobb et al. (1989) also emphasized the relationship between social norms and emotions. In the classroom they observed, engagement in mathematical activity was the goal and therefore, even weaker students experienced and expressed positive emotions as they participated in group activities and whole class discussions. More specifically, they did not observe a single event over the semester, where a student would have become frustrated and given up the task.

Hannula (2006) suggested that open approach (e.g. Nohda 2000) would provide opportunities to meet student needs of autonomy, competence and belonging. An open approach would fulfil many of the criteria suggested by Pekrun and Stephens (2010). In the ongoing research project (Näveri et al. 2011) we explore the overall effect of open problems for mathematics related affect and achievement. It is clear already from our preliminary analysis that open problems bring a lot of emotions in the classroom, including flow. Moreover, we have identified that in a good open task there are different levels of complexity that the students can choose from.

One key element is to develop social norms in the class that ensure safe space for explorations and a feeling of playfulness in the lesson. The appropriate climate of a lesson can be further supported through presenting problems in a humorous form, and introducing them to the class in a humorous light (Shmakov and Hannula 2010). The modes of working with the problems should provide opportunities for students to reach their social goals of belongingness and the teacher should be prepared to provide emotion scaffolding for those students who are not yet able to cope with moments of frustration. Finding the solution should provide a feeling of accomplishment and pride for all students. This can be best achieved through focus on the process, and by highlighting important variation in ideas and perceiving unsuccessful attempts as important learning opportunities.

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Hands that See, Hands that Speak: Investigating Relationships Between Sensory Activity, Forms of Communicating and Mathematical Cognition

Lulu Healy

Abstract This contribution explores the role of the body's senses in the constitution of mathematical practices. It examines the mathematics activities of learners with disabilities, with the idea being that by identifying the differences and similarities in the practices of those whose knowledge of the world is mediated through different sensory channels, we might not only become better able to respond to their particular needs, but also to build more robust understandings of the relationships between experience and cognition more generally. To focus on connections between perceptual activities, material and semiotic resources and mathematical meanings, the discussion concentrates on the mathematical practices of learners who see with their hands or who speak with their hands. This discussion centres around two examples from our research with blind learners and deaf learners and, in particular, analyses the multiple roles played by their hands in mathematical activities.

Keywords Blind mathematics learners · Deaf mathematics learners · Embodied cognition · Gestures

Introduction

Can listening to learners' hands help us understand mathematical cognition? The search for answers to this question, and perhaps even its very posing, requires some reflections about how our bodies are involved in thinking mathematically and about the relationships between thinking and doing. In relation to these questions, recent years have seen a growing interest in the embodied nature of human cognition, with increasing evidence and support for idea that the ways in which we think should not (or cannot) be separated from the ways in which we act and that both doing and

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imagining have their basis in our bodies, its physical capacities and its location in space and time (Gallese and Lakoff 2005; Barsalou 2008, 2009). Embodied approaches emphasising how the meanings assigned to abstract symbols are shaped by aspects of the body's sensory and motor systems and its interactions in its environments have also begun to permeate research in mathematics education (Nemirovsky and Borba 2004; Lakoff and Núñez 2000; Radford 2009; Roth 2010). Indeed the legitimacy of the opening question and the current attention to gestures in mathematical activity represent indications of the arrival of embodiment on the scene (see for example, the special issue of *Educational Studies in Mathematics*, Radford et al. 2009).

Yet, while the argument that our mathematical understandings are structured by our bodies' encounters and interactions with actual and virtual worlds might be gaining force, it is also important not to forget either that mathematics as discipline of knowledge is also a cultural affair, defined by the set of the artefacts and practices created in the historical trajectory of its construction, or that learning mathematics occurs in settings which too are associated with particular forms of social practices. In this context, it can hardly be controversial to claim that we develop and that we learn by interacting within the various biological, social and cultural systems that make up the world as we experience it. Individuals construct their own meanings for the mathematics they encounter which depend upon the ways and means through which they come into contact with the knowledge culturally labelled as mathematics, as well as upon their individual resources. Attention to how mathematics is encountered emphasises its social, cultural and political nature—as it cannot be assumed that all learners have the same opportunities to participate in the same learning activities.

In our work, which involves students with disabilities,[1] this is especially the case. Rather little attention has yet been given either to the particular ways in which students with different kinds of disabilities make sense of the mathematical artefacts which compose school mathematics or to the material and semiotic tools which best support their participation in mathematical activity. Our work with these students is in part motivated by our belief that if we understand the differences and similarities in the mathematical practices of those with and without disabilities, and particularly those with different physical (sensory or motor) means of experiencing the world, then we may build a more robust understanding of the relationships between experience and cognition more generally. The approach that we adopt attempts to combine the premise of embodied cognition, that mathematical (like all) thinking has its basis in our bodily capacities, with Vygotsky's ideas about how the inclusion of tools, be they material or semiotic in nature, in the process of activity alters its entire structure and flow (Vygotsky 1981).

The two approaches appear to us as complementary, as the Vygotskian notion of tool mediation actually has its roots in his work with differently-abled individuals

[1]The research reported in this paper is being carried out together with other members of the research programme *Rumo à Educação Matemática Inclusiva*, PROESP-CAPES, No. 23038.019444/2009–33).

(Vygotsky 1997), and his view that bodily organs can be thought of as tools, "*the eye, like the ear, is an instrument that can be substituted by another*" (p. 83). In his perspective, such a substitution is expected to cause a profound restructuration of the intellect. Here, it is important to stress that Vygotsky believed that the adequate substitution of a missing or disabled tool would enable those with sensory impairments to attain as highly as their hearing and sighted counterparts, but that the paths by which they would do so might be quite different. That is, in adopting this position, rather than seeing difference as equated to a state of deficiency, difference can be treated as just that, difference.

The rest of this paper returns to a modified version of the question posed at its beginning. It aims to examine what might be learnt from listening to the hands of blind learners and deaf learners as they engage in mathematics learning activities. Before presenting two examples from our ongoing work, it begins with an outline, in the context of the theoretical approach briefly introduced above, of why gestures have come to assume such a central role in our research.

Hands, Gestures and Embodiment

Rotman (2009) argues that gestures have tended to be seen rather as the poor relations of the spoken word and as inferior too to visual images: "*one encounters everywhere a dismissive attitude if not hostility to the notion that gesture be taken seriously*". However, he also suggests that current attention to the embodied nature of being is calling such a view into question. Defining a gesture as "*any body-movement that can be identified, repeated, and assigned significance or affect as a sign, a function, or an experience*", he divides gestures along three modes of embodiment. In the semiotic mode, the gesture represents a form of conveying meaning, significance or affect. In an instrumental gesture, the body becomes akin to a machine, performing a goal-related action to produce a particular effect. In the third, immersive mode, the gesture indicates the body's experience of, or participation in, forms of cultural activity. Rotman's view of gesture is more inclusive than the definition more usually found in the mathematics education literature, in which McNeill defines gestures as "*movements of the arms and hands … closely synchronized with the flow of speech*" (McNeill 1992; p. 11). In contrast to Rotman, McNeill appears to privilege mainly gesture as sign, stressing the speech-gesture relationships and, perhaps because of this, many of those interested in gesture research limit themselves to "*the study of hand and arm movements that are interpreted by others as part of what a person says*" (Roth 2001; p. 368). In McNeill's definition, then, gestures are to be treated as part of the same semiotic system as speech or, more precisely, as "*elements of a single integrated process of utterance formation in which there is a synthesis of opposite modes of thought—global-synthetic and instantaneous imagery with linear segmented temporally extended verbalization*" (ibid. p. 35). In the context of our research with deaf students and blind students, we have some difficulties with this view. The first question that it raises for us is if we are to see gesture as the visual manifestation of imagistic aspects of cognition, then can we expect gestures to be used in the same ways by the blind and the sighted? And the second is what this means in the case of those who speak with their hands rather than with their mouths.

In the first case, we might ask if global-synthetic, instantaneous imagery comes about at all for those who are blind, since when hands are used as substitute eyes the process of seeing can no longer be described as synthetic and global: touch permits a gradual analysis from parts to the whole. Does "seeing" in this case become more sequential?

In the second case, the separation is perhaps even more problematic, since Sign[2] languages are in themselves visual-gestual languages and of a rather different nature to sequential-auditory spoken languages. Rotman (2009) goes as far as to offer the suppression of the use of Sign languages in the education of deaf learners as the strongest evidence on offer of how gestures have been traditionally placed as lower-level or more primitive in relation to spoken language. It is true that there have been times in the historical trajectory associated with the education of deaf learners during which oral methods have been championed at the expense of Sign languages. Without going into all the details, two important moments of this history will perhaps suffice to synthesise the split between the two language forms. The first occurred in 1880, at the Congress of Milan, where hearing participants "voted to proclaim that the German oral method should be the official method used in the schools of many nations" (Lang 2003, p. 15). Deaf people were excluded from this vote. It took until the 1960s and the scientific recognition of American Sign Language (and consequently the Sign languages of other countries throughout the world) as a true and natural language (Stokoe 1960/2005) before this dominance began to be challenged. It is now generally accepted both that Sign languages exhibit the fundamental properties that exist in all natural languages and are just as rich and complex as any oral one (Kilma and Bellugi 1979), with lexicons, grammar, syntax and morphological rules, and that the manual modality is as good a medium for language as the oral modality (see, for example, Goldin-Meadow 2003). But where does this recognition leave gestures?

Certainly, gestures accompany Sign languages just as they accompany spoken languages, but McNeill's sequential/imagistic split is not easy to apply, since phrases in Sign languages are not linearly segmented in the same way as spoken languages. One way out would be to label gestures those hand and arm movements which accompany speech but are not officially recognised parts of any language. Once again, this would limit gestures to complements of speech acts.

If we return to Rotman's alternative view however, instead of treating gestures as some kind of semiotic supplement to speech, speech itself is treated as a species of gesture, sometimes perceived by auditory rather than visual means, but a gesture

[2]To distinguish between the use of the world "sign" as the generalized manifestation of a signifier-signified relationship and the Signs that compose the manually-expressed languages of the Deaf, a capital "S" will be used in the second case.

nevertheless. In subordinating speech, he underlines the embodied nature of language, but he is by no means the first to do so. In the 18th century, the sensualist Condillac was describing signs as "transformed sensations" and suggesting that the transformation of embodied experiences into shared material signs holds the key to human knowledge:

> Condillac called signs "sensations transformées", transformed sensations, by which he meant the entire complex of affect, desire, sensory perception, and motor action that make up what nowadays we might call "embodied experience" … In his view … [t]he formation of a symbol is a defining moment in the fabrication of shared knowledge because it allows the participants to focus upon and re-invoke previously shared experiences and to plan and conduct shared activities in their wake (LeBaron and Streeck 2000 pp. 118–119).

For the phenomenologist Merleau-Ponty, like Rotman, speech is gesture: "*the spoken word is a gesture, and its meaning, a world*" (Merleau-Ponty 1945/1962, p. 215). In common with Condillac, for him the importance is not so much the form in which a sign representing experience is expressed, it is what this sign incites, invokes, that is, how it is felt by the interlocutors.

> I do not see anger or a threatening attitude as a psychic fact hidden behind the gesture, I read anger in it. The gesture does not make me think of anger, it is the anger itself […] The sense of gestures is not given, it is understood, that is, recaptured in an act on the spectator's part […] The communication or comprehension of gestures comes about through the reciprocity of my intentions and the gestures of others, of my gestures and intentions discernible in the conduct of other people. It is as if the other person's intention inhabited my body and mine his. (pp. 214–215)

Hence, Condillac, Merleau-Ponty and Rotman all appear to agree that the signs through which we communicate experiences have their bases in the body. These positions can be connected to more contemporary writing about embodied cognition as well as to tenets of socio-cultural approaches.

In relation to embodied cognition, Merleau-Ponty's view that the sign *is* the experience can be associated with recent research into the multimodality of human cognition and the idea that both doing something and imagining doing the same something involve a shared neural substrate. Gallese and Lakoff (2005) offer neurological evidence to support this position, arguing that circuitry across brain regions link different sensorial modalities "*infusing each with the properties of others*" (p. 456). In the case of mathematics learning, for some at least, accepting that doing and imagining involve the same cognitive resources, might go somewhat against the grain. Mathematical imaginings are frequently revered as the means of engaging with the abstract, while doing is seen as a less sophisticated interaction with concrete situations. It also raises questions about what we understand as mathematical concepts. If cognition is multimodal and if imagining involves reliving—and re-feeling—previous doings, then concepts cannot be seen as mental representations in which the abstract, logical universal properties of an object are stored in a somehow transcendental form stripped of the particularities of the settings in which it was encountered.

Instead, as Rosch (1999) has it, *"[C]oncepts and categories do not represent the world in the mind; they are a participating part of the mind-world whole*" (p. 70). Concepts in this perspective resemble what Barsalou (2009) defines as *simulators,* the distributed neural systems of the multimodal content associated with a particular category (p. 1282). According to him, these simulators are self-extending systems that enable us to interpret new instances as representatives of a type (Barsalou 2003). In this viewpoint, understanding anything—something immediately perceivable by our sensory apparatus or something that only manifests itself only through semiotic sign systems—involves activating and reliving, that is, simulating, any part of the previous experiences that have come to be allied with it. In this sense, when we come across any object, or a representation of an object, the plurality of perceptions associated with it are reactivated, leading Roth and Thom (2009) to suggest that we experience a single instance associated with any concept as its whole.

Thus far, this description may seem to privilege only the web of relations which exist between an individual subject (body and mind) and manifestations of a class of objects in actual or virtual worlds. However, as the above quotes indicate, Condillac and Merleau-Ponty believed that we are essentially social, not individual beings. They stress our attempts to communicate our feelings and interpretations so that others can understand (experience) them and they signal the role of semiotic tools for achieving this. To a certain degree, this takes us into the realm of Vygotsky's socio-cultural perspective, with its emphasis on our need and ability as human beings to both create and use artefacts and to encourage their appropriation by subsequent generations (Cole and Wertsch 1996). Gestures, be they idiosyncratic body movements or representatives of organised language systems, can hence be understood as bodily-based semiotic means of mediation. Through them, we both feel our own interpretations and communicate them to others.

Given that the students with whom we work lack access to one or other sensory field, it seems reasonable to ask what the gestures they produce tell us about their experiences of mathematical objects. It is to this that we now turn. The main focus in the examples that follow is on the production of gestures intended to some extent as shared signs.

Mathematical Gestures of Blind Students and Deaf Students

The two examples are both drawn from our ongoing programme of research which aims to (1) investigate forms of accessing and expressing mathematics which respect the diverse needs of all our students; (2) contribute to the development of teaching strategies which recognise this diversity; and (3) explore the relationships between sensory experience and mathematical knowledge. The research strategy used in this project is based on establishing collaborative partnerships between school- and university-based participants. The methodology used is a kind of action research, co-generative inquiry, in which all participants co-generate knowledge through a process of collaborative communication (Greenwood and Levin 2000).

Hence, the project involves researchers and practicing mathematics teachers working together to design learning activities that might be used to contribute, in the long run, to the development of a more inclusive school mathematics and which, in the short run, the teachers can implement with their own students.

Example 1 Exploring symmetry and reflection with deaf and hearing students.

The first example[3] involves a group of eight 7th grade students, composed of five deaf and three hearing students. It is drawn from a sequence of activities involving the investigation of the transformation reflection. The example has been selected to illustrate our attempts to understand the challenges associated with learning mathematics in Libras, the Sign language used by the deaf community in Brazil. In particular, the example explores how deaf mathematics learners express mathematical objects and their properties and relations in their visual-gestual language. The students were all from the same class of a mainstream school in the municipal of Barueri, São Paulo. This school is known as one in which teaching is conducted in both Portuguese and Libras (although only Portuguese and not Libras, is actually studied as a school discipline). Not all the teachers in the school are bilingual, but interpreters are present to assist in the classes of the monolingual teachers. Reflection and symmetry were chosen as objects of study in accordance with the mathematics curriculum of the school.

The first step in designing the activities had been to search the literature for previous studies involving the same mathematical object. The literature concerning mathematics learning of deaf students tends to emphasise their performance in relation to arithmetic tasks and to focus on students working individually or with other deaf students (see, for example, Bull 2008; Kelly et al. 2003; Nunes and Moreno 1998; Nunes 2004; Pagliaro 2006). Indeed, in general, we were not able to find studies specifically investigating the relationships between the Bilingual approach adopted in the school we worked and the learning of geometry. As far as the bilingual approach is concerned, most research attention has been given to literacy and to the complexity of applying the principles of linguistic interdependence derived from studies of second (spoken) language learning to deaf learners. In this respect, it has been argued that to understand deaf literacy learners, it is necessary to take into account the set of sensory modalities available to them, to ensure they have the opportunity to appropriate and manipulate all possible meditational means at their disposal (Mayer and Akamatsu 2003). This was hence a principle design concern and one of the reasons that we decided to structure the learning activities in our study around a digital microworld, *Transtaruga,*[4] which offered different modes of interacting with geometrical ideas (visual, dynamic and

[3]This research was carried out in collaboration with the researchers Heliel Ferreira dos Santos (also the students' mathematics teacher), Solange Hassan Ahmad Ali Fernandes, Fabiane Guimarães Vieira Marcondes and Kauan Espósito da Conceição.

[4]Borrowing "*trans*" from the Portuguese word for transformation and "*taruga*" from tartaruga or turtle in Portuguese.

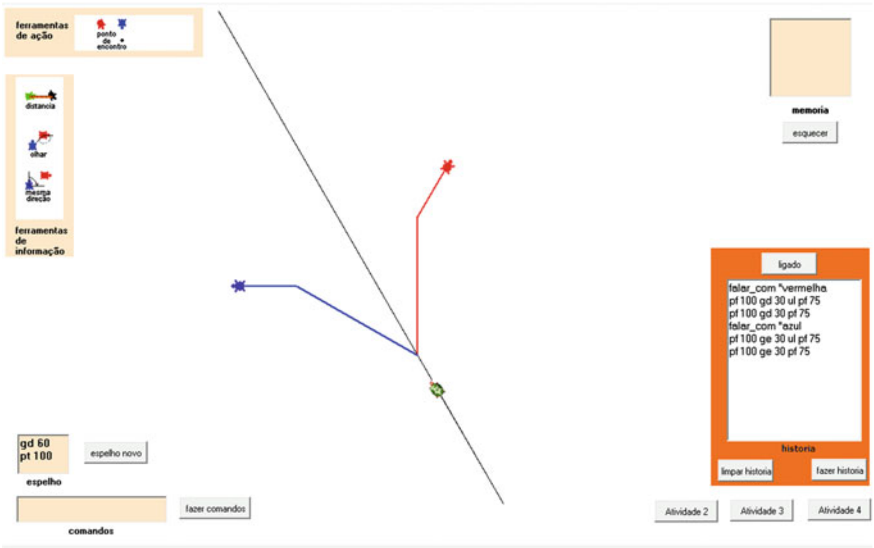

Fig. 1 Symmetrical trajectories in the Transtaruga microworld

symbolic). *Transtaruga* is a multiple-turtle geometry microworld written in the Imagine version of the programming language Logo. It inherited many features from two earlier *Turtle Mirrors* microworlds used in previous research with hearing students (Hoyles and Healy 1997; Healy 2002). Figure 1 presents a page from this microworld.

None of the eight students who participated in this study had had any previous experience with the Logo programming language and, in fact, digital resources had never featured in their mathematics lessons before. They were divided into pairs to work on the activities, with each pair positioned in front of one of four laptops, arranged in a semicircle so that, during whole group discussions, the Signs of those using Sign language could be easily seen by all the participants. Two pairs were comprised of only deaf students, one pair was made up of two hearing students and the fourth pair had one deaf and one hearing member. The hearing student in this pair spoke some Libras and the deaf student was partially oralised. Each laptop had a webcam and it was thus possible to capture not only the four screens, but also the faces of the students and the discussions between them while they worked on the activity. In addition, three other video cameras were positioned to capture all the interactions of the pairs using Libras as a communication form. Clearly the presence of all this equipment along with the researchers operating it, made this learning setting rather different from their usual classroom, however, it was necessary in order not to lose the discussions between the deaf students. A total of five researchers participated in the research session from which the episodes that follow were drawn, one of whom was the students' mathematics teacher. He spoke some Libras, but was not completely fluent and hence an interpreter was also present. All the researchers participated actively in the sessions, with two, the class teacher Heliel and myself assuming the main teaching roles.

In the first session, the students were introduced to a limited subset of turtle geometry commands: **pf** (forward, *para frente*), **pt** (backwards, *para trás*), **gd** (right, *girar direita*), **ge** (left, *girar esquerda*). **ul** (pendown, *usar lápis*) and **un** (penup or *usar nada*) and shown how to activate different turtles by clicking on them. Four additional microworld tools were also presented: **ponto de encontro** (meeting-point), which constructed a point (turtle) where two turtles intersect; **distância** (distance), which measured the distance between two turtles, displaying the value in the memory box (*memória*); **olhar** (towards), which measured the angle one turtle should turn to point towards another, again with the result displayed in the memory box; and **mesma direção** (same heading), which measured the angle one turtle should turn so as to have the same heading as another. These four tools were represented as icons which displayed the tools' name but also by means of a visual image which aimed to summarise function of the tool. This design decision was motivated by the fact that the deaf students had some difficulties with written Portuguese and tended to avoid reading any written text. A history box was also available on screen so that the students could build up a symbolic record of the commands that they used in the negotiation of the different tasks.

The first task given to the students involved constructing a symmetrical trajectory—that is, given the set of commands used to construct a trajectory of the red turtle, the students' task was to construct a trajectory for the blue turtle which would leave a trace symmetrical to that of the red turtle in relation to the axis of symmetry. Both the red and blue turtles started from the same location. Previous research with hearing students suggested that this task is very accessible to the majority of students (Healy and Hoyles 1997), who rapidly perceive that symmetrical traces can be obtained by using the same commands to move both turtles forward and back, while swopping the directions in the turning command.

When we attempted to explain this task to our student group, an unpredicted problem emerged almost immediately. Neither the class-teacher nor the interpreter knew the Sign for "symmetry" in Libras. The interpreter's first instinct was to substitute the word "symmetric" with a Sign representing "equal", but this substitution was rejected on the grounds that it would emphasise only the congruency relationship (common to all the isometric transformations). One option at this point would have been to finger spell the word, that is, to use the Libras signs for the letters that compose the word in written Portuguese. In itself, this does not represent a very satisfactory solution as simply spelling the word could not be expected to help the students understand what was required of them, since it would communicate nothing of the meaning of the word in question. A third option was chosen: that of re-introducing the task, suggesting that the students imagine that the line representing the axis of symmetry is like a mirror, and their task is to produce the image of the trajectory of the red turtle in this mirror. Our belief was that this explanation would serve equally for the hearing and deaf students—as the reflection transformation is frequently introduced to learners in relation to activities with mirrors. Perhaps it is worth pausing to comment on the embodied nature of this explanation: its intention is to invite the students to simulate their previous experiences of looking into mirrors, and to use associated feelings to imagine what they would see. Doing and the imagining of the doing are treated as alike in this invitation.

To a certain extent, the need for this association between mirrors and symmetry had already been anticipated in the design of the microworld, even though the absence of the Sign for symmetry had not been predicted, with the turtle responsible for producing the axis of symmetry named *espelho* (mirror). In the event, this association did indeed appear to make the task requirements clear to all four student-pairs. The trajectory to be reflected is shown in Fig. 2.

All the students seemed confident as to where the blue trace should appear on the screen, but we noticed two differences in the strategies used by the pairs composed only of deaf students in comparison to the other two pairs. First, the deaf students were more likely to confuse the functions of the Logo drawing commands. This difficulty is relatively easy to understand. The choice of the two letters that compose the Portuguese version of the forward command (**`pf`**) is not arbitrary, but an abbreviation of ***para frente***.

The hearing students sense the non-arbitrariness of these commands, the sounds associated with the letters representing an indication of the action they serve. This was not the case for the deaf students, which made it harder for them to remember which command to use when. The result being that the process of appropriating the functions of the tools, or we might say, the meaning of the written "pf" gesture, took longer for the deaf students.

The second difference is to a certain extent related to the first and concerns where the students directed their attention. The students in the two deaf-only groups initially paid only limited attention to the commands displayed in the history box (*história*) and focussed their attention mostly on the part of the screen displaying the visual turtle traces. When they did look at the information displayed in this box, they seemed to be drawn more to the numbers than to the letters representing commands. In contrast, the groups with at least one hearing student tended to move their attention equally between the history box and the visual traces. One interpretation of this difference in attention would be to suggest that the deaf students

Fig. 2 The first image to reflect

were less aware of the dependence of the blue turtles trace on the commands used by the red. However, on a closer look, this would not seem to be entirely the case, the numbers used as inputs to the commands used by the deaf-only groups were all numbers associated with commands given to the red turtle. Because the deaf students took longer to understand which command was associated with which movements, at times this made it look as if they were using rather random strategies of assigning any of the numbers with any of the commands. Perhaps too their initial difficulties in assigning letters to movements also discouraged them to look for regularities in the symbolic records of the two turtles' trajectories.

If we look only at their interaction with the symbolic code, without looking at how they attempted to visually decode the necessary movements, however, we gain rather a biased picture of their solution strategies. A closer look at the work of Pedro and Daniel shows that, they knew exactly what they wanted to produce. The symmetrical gesture produced by Pedro to indicate the desired relationship between the two trajectories provides evidence of this (Fig. 3).

Figure 3 shows Pedro's hands only in their final configuration. In fact, the gesture as a whole was a dynamic one in which his hands traced out a symmetrical movement. His gesture in this case resembles the specific image that they are trying to construct, but the way Pedro produced it, indicates how he sensed symmetry, perhaps even the gesture was being offered to his partner as a Sign for symmetry—or, following Roth and Thom (2009), might indicate that he was experiencing this single instance as representative of symmetry as a whole.

It is also the case that the relationship between turns in symmetrical tasks was signalled in the gesture; Pedro bodily enacts a process of rotating his hands in opposite directions. He may not yet be conscious of doing so, the gesture might be felt as the whole movements and not broken down into constituent parts. One thing that we can be sure of is at this point the movement is by no means connected to the Logo language which potentially provides another means of expressing it.

Fig. 3 A gesture for symmetry

Despite knowing the trajectory that the blue turtle should produce, it was their unfamiliarity with these commands, and in particular a mixing up the commands **pt** and **pf** that eventually led Pedro and Daniel to abandon the given task and attempt to construct a different pair of symmetrical paths. This decision serves as further evidence that they were thinking beyond the specific case. A major concern of theirs continued to be the appropriation of the functions of the Logo commands, and this time, as they discussed the commands to be used, Daniel uses his fingers to represent the two turtles they were communicating with (Fig. 4). The pair decided to move the turtles forward 100 and they agree to try **pt 100** (Fig. 5). During this conversation Daniel was using his right hand simultaneously to both Sign and to represent the turtle, but this did not seem to be a problem for him.

When the command **pt 100** was typed, they both saw the associated turtle movement and Daniel also acted out its backward movement. Realizing that they had used the wrong command, Daniel made a kind of rubbing-out gesture (Fig. 6).

Fig. 4 Daniel's fingers become turtles

Fig. 5 **pt** is suggested to move forward

It was as if he was somehow guarding a visual image of the conversation and then rubbing out this image when a mistake was made. We saw Daniel use this same gesture on other occasions, such as when he was performing calculations. Whatever this gesture implies about the mental processing of Sign language, the gesture was clearly understood by both Pedro and Daniel to mean they should start again. Accompanying the movement of the turtles with their fingers seemed to have been useful in helping the pair to decide which command should be used when and, in Fig. 7, Daniel is able to feel and to articulate to Pedro that the turns should be made in opposite directions. This time the relationship between these turns is the explicit focus of attention Daniel's gestures, rather than part of what might be described as the more composite gesture originally made by Pedro.

While it took Pedro and Daniel much longer to appropriate the function of the written Logo commands and to explicitly articulate the relationships between the turns made in symmetrical paths, physically in gestures as well as in the Logo

Fig. 6 Daniel "rubs out" the exchange

Fig. 7 Feeling opposite turns

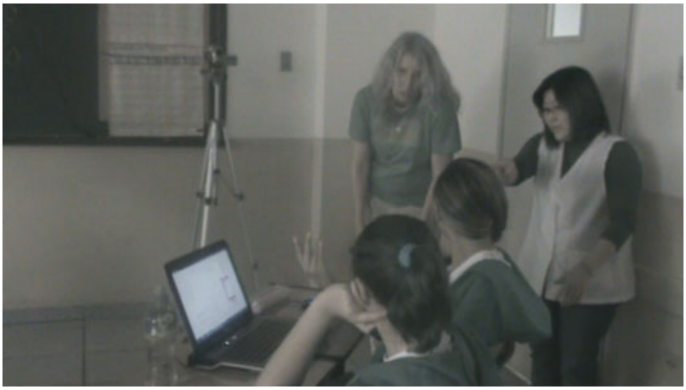

Fig. 8 Lara's suggestion

languages of right and left turns, their efforts to make sense of the activity were eventually so successful that they took responsibility for explaining this relationship to Aline and Lara. At the end of this explanation, not only did all the students seem to be confident about producing symmetrical trajectories, the deaf students wanted to negotiate a Sign to represent the process of reflecting a turtle path. Figure 8 shows Lara's sign which involved the use of a flipping movement of one hand in a way reminiscent of the flipping of space over an axis of symmetry.

In this process of negotiating a shared Sign, which is a common activity for these students and for members of the deaf community in Brazil in general, both suggestions incorporate bodily movements which represent what reflection means to the students in the context of their work with Turtle Geometry. Embedded in the gestures are aspects of the reflection process that the students want particularly to stress. While it might not be clear in the case of Pedro's symmetrical gesture whether he was thinking only of a particular image, or whether the instance represented the whole, in the negotiation of the Sign for reflection there can be no doubt. The students were not representing specific cases, they were offering gestures in which aspects of the process of reflecting were abstracted in the production of a dynamic Sign intended to bring this process to mind.

Example 2 Blind students sharing their feelings of area.

The second example involves an episode from a sequence of learning situations undertaken with a group of four blind learners who attended a mainstream school, which was part of the public education system of the State of São Paulo. The learners, whose ages were between 14 and 18 years, were first-year high school students. The data reported here come from a sequence of activities associated with the study of volume, area and perimeter. The activities were designed to include tools intended to favour tactile exploration of the mathematical objects in question and were implemented over four research sessions, each approximately 90 min in duration. Activities which took places during the first two session contributed to

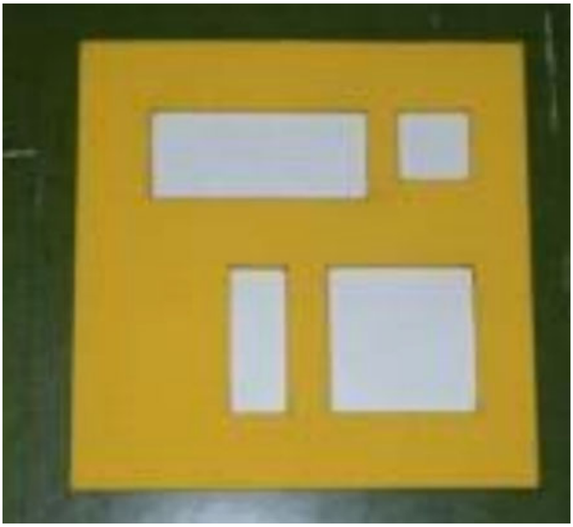

Fig. 9 Impressions of rectangles

understanding the interactions in the episode presented here (more details on the students' interactions on the tasks in this learning sequence are available in Fernandes and Healy (2010) and Healy and Fernandes (2014)). All the activities took place in a room which houses the resources specifically designed for the blind students and substituted their regular mathematics lessons.

Before they began to work on the activities, the students had explained to us that it was rare for them to interact with representations of geometrical shapes, and the tendency was for measures of shapes to be given to them to operate with, rather than being expected to perform the measurements for themselves.

For this reason, we[5] wanted to make sure that they had access to tactile materials that would make this possible. During the first session, they were given a board with the impressions of four rectangles, which could be filled either with wooden unit cubes or with rectangular and triangular shapes in foam rubber (Fig. 9). In this session, the students explored the areas of rectangles and triangles. In particular, they experienced how the area of a triangle could be perceived as half the area of a rectangle with the same height and base.

In the second session, the students worked on determining the areas of the plane figures represented in foldable cardboard. During this session, adapted rulers, in which the number marks were raised so they could be read tactilely, were available, as were the wooden-unit cubes.

The episode which follows occurred during the second session, when the students were working on finding the areas of plane figures and in particular relates to

[5]This work was carried out together with Solange Hassan Ahmad Ali Fernandes.

their attempts to determine the area a parallelogram with a length of 12 cm and a height of 7 cm. Each student received his own cardboard representation of this parallelogram.

The four previous shapes that they had worked with were triangles. The area of the third shape was easy for them to work-out, as it was a right-angled triangle, the area for which they already knew how to obtain from their activities in the first session. They were shown how other triangles could be divided into two right-angled triangles, either physically by folding along a line perpendicular to one of the sides and which passed through the vertex, or virtually, using the ruler to mark this same line segment and then measuring the height and base of both the resulting triangles. Using these methods they successfully calculated the areas of three non-right-angled triangles.

After their initial explorations of the new parallelogram shape, it looked as if the students were about to agree that its area was 42 cm^2. This measure had been obtained by measuring the length of the parallelogram (which they called the "base"), its height (correctly located, by measuring the perpendicular distance from the base to the vertex on the opposite side, as they had done with the triangles), and then dividing by 2, also using the strategy that had worked for the triangles. Just as they were about to confirm to the researchers that this was the area on which they were all agreed, first Fabio, then, Caio began to have doubts about whether or not they ought to be dividing by 2.

Fabio seemed to be thinking about straightening the two parallel sides, measuring 8 cm, joined to the base, so that they were perpendicular to it, forming a rectangle, with what he called "*a better-behaved area*", which he believed would have an area of 8 times 12. His mentioning of a rectangle, led Caio also to question the appropriateness of dividing by 2 in the case of the figure they were now exploring, although he didn't agree that the measure of 8 cm should be used. His view was that the area of the as yet unnamed shape could be determined by multiplying 12 by 7.

As Leandro and Marcos were attempting to defend the method which included dividing by 2, Caio and Fabio had become convinced that this was no longer an appropriate option. To persuade the other two, Caio did not discount the idea that there was something about this shape that resembled the triangles, but rather used this as the basis to argue that they were now investigating a different kind of shape "*It seems like a triangle, but it isn't a triangle, it's something close to a rhombus or something like that*". Taking up Caio's position, and perhaps also the inclusion in the conversation of another kind of quadrilateral, Fabio went on to state "*We are not working with a triangle anymore; we are working with a whole figure.*" This comment is particularly interesting. Again, his words suggest the simulation of a previous experience. This time, the experience in question is one in which all four of students had participated in the first research session—that is, the treating of a triangle as half an impression of a rectangle in the material presented in Fig. 9. It was as if, by choosing to describe the four-sided shape as a "whole-figure", Fabio was bringing to the present aspects of a previous activity, in which, he knew, all the others had shared and offering this verbal gesture as a sign which might help the others to appreciate his point of view. He does seem to have managed to communicate the idea, as, when Caio attempted to defend his value for the area of the parallelogram, he used exactly the same combination of words once more "*12 times 7, which is base times height, and since it's a whole figure you don't divide by 2*". Moreover, from this point on, none of the students adopted the method they had used for a three-sided shape when working with quadrilaterals.

Shared Signs as Embodied Abstractions

The two examples presented in this paper are offered with the aim of exploring the mathematical sense-making activities that occur when students speak and see with their hands. In both examples, there are indications of how the particularities associated with the sensory channels through which these learners experience the world, contribute to the ways in which they appropriate knowledge. For example, for the blind students, using hands as seeing tools in the absence of eyes appears to lead to particular views of geometrical figures, emphasised by the dynamic gestures that emerge during this kind of seeing process. While in the case of the deaf students, the gesture in which Daniel seems to rub out an ongoing argument in order that it might be reconstructed from scratch, offers some evidence that reasoning is in some sense a visual process for those who communicate in Libras.

Yet despite the obvious differences between the groups of students who participated in the different learning scenarios, their sense-making activities are also characterised by a number of similarities and, in particular, by the efforts of the students to create sharable signs intended to re-invoke, both by themselves and by the others present, previously shared experiences. That is, their aim was to permit that all those involved in the interactions might feel what they were feeling. In both cases, the signs offered resulted from what we have elsewhere termed *embodied abstractions* (Healy and Fernandes 2011). We call them abstractions because the gestures offered as shared signs not only expressed the learner's own sense of an experience of a specific mathematical situation by relating it to some other, they also imply some conscious appreciation by learners of the generalised relationships implied in their expressions. The shared signs might be expressed in manual or verbal forms—in this paper both are considered as gestures—and it is important to stress that one form cannot be considered as more or less embodied, nor more or less abstract than the other. Regardless of their form, the gestures expressed in both example episodes were not only material manifestations, traces, of aspects of physical activities, they also had a semiotic role, carrying a sense of the mathematical understanding that had come to be associated with the activities of those who made them.

Another commonality that can be identified in both examples is that although the different gestures were expressed by particular individuals, the abstractions that they expressed should be considered as social rather than individual acts—the imagined, or actual, actions they conveyed, involved the reactivation of perceptive, motor and introspective states associated with such objects in previous activities. In this sense, gestures were outward signs of *imagined re-enactments* of previous doings with the things and the other people involved in the activity in question (Healy and Fernandes 2014). In the case of both the gesture offered by the deaf students to express the word "reflection" in Libras and the use of the expression "whole-figure" by the blind students to indicate a significant difference in determining areas of triangles as compared to quadrilaterals, there was evidence of a conscious attempt to choose a gesture so that it might re-invoke the multimodal content associated with this concept by the others—or, putting it another way, so it could be felt by the others. What the gestures of these students suggest is that in order to best communicate the meanings that we hold for particular mathematical objects and activities, we need to look for ways of permitting that others sense how we feel these objects. Perhaps, this is what Merleau-Ponty meant when he talked of how when we interpret the gestures of others, their intentions come to inhabit our bodies, and reciprocally, our intentions inhabit theirs. Perhaps even we might suggest that as teachers of mathematics we should seek to do precisely that: permit that others feel mathematics as we do, whilst simultaneously attempting to appropriate their feelings.

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Freudenthal's Work Continues

Marja Van den Heuvel-Panhuizen

Abstract In this paper I address a number of projects on elementary mathematics education carried out at the Freudenthal Institute. The focus is on (i) using picture books to support kindergartners' development of mathematical understanding, (ii) revealing mathematical potential of special needs students, and (iii) conducting textbook analyses to disclose the learning opportunities that textbooks offer. I discuss how these projects are grounded in the foundational work of Freudenthal and his collaborators in the past and how our work will be continued.

Keywords Picture books · Special education · Subtraction · Textbook analysis · Didactics of mathematics

The Title Explained

The title of this paper suggests that something is going on in Utrecht, which is indeed the case. Part of the Freudenthal Institute will move. In fact, this is the umpteenth removal of the institute in its moving history. The institute was established on January 26, 1971 as IOWO (Institute for Development of Mathematics Education), as an independent part of the State University of Utrecht with Hans Freudenthal as its first scientific director. The heart of the IOWO was the so-called Wiskobas (Mathematics in Primary School) project with Treffers (1978, 1987) as one of its leaders. On December 31, 1980, IOWO ceased to exist and was absorbed by the newly established National Institute for Curriculum Development (SLO). Only a small part of IOWO, the research part, could stay at Utrecht University and under the name OW&OC (Research of Mathematics Education and Educational Computer Center) this part became a department of the Faculty of Mathematics and Computer Science. In September 1991, 1 year after the death of Freudenthal,

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the institute was renamed *Freudenthal Institute*. Further change took place in 2005, when all science faculties of Utrecht University merged into one Faculty of Science. From that moment on the Freudenthal Institute belonged to the Department of Mathematics of the Faculty of Science.

Another real change happened in December 2006, when we merged with the researchers from physics education, chemistry education and biology education and our name was officially changed to *Freudenthal Institute for Science and Mathematics Education*. Finally, in September 2010, we, the mathematics education part, moved from our building in Utrecht Overvecht to the University campus De Uithof where the science part was located. As a result, we were now also physically one integrated institute. So far, so good.

Alas, in November 2010, the Faculty of Science announced it had to make drastic cutbacks and that it also reconsidered its mission. The latter included the decision that …

… research and development in mathematics (and science) education for early childhood, primary school, special education, and vocational education were not the core business of the Faculty of Science. This means that the heart of the Freudenthal Institute will be taken out, and move to the Faculty of Social and Behavioural Sciences. The good news is that this faculty has welcomed us. This means that our research and development projects on mathematics education for early childhood, primary school, special education, and vocational education will not stop, but will hopefully be prolonged in a newly to be established *Freudenthal Center* in which the two faculties work together. In other words, Freudenthal's work continues. To articulate this, I have taken the invitation to give a Regular Lecture at ICME 12 in Seoul as an opportunity to showcase some of my present projects that will have a new future in the Faculty of Social and Behavioural Sciences. For every project I will look back on the work of Freudenthal and his collaborators who, in one way or another, have laid the basis for these projects.

Mathematics Education in Kindergarten: The Didactical Use of Picture Books

This section addresses the role of picture books in kindergartners' learning of mathematics. The section is based on various sub-studies I did with others on this topic, each study taking a different perspective. The studies were part of the NWO (Netherlands Organisation for Scientific Research) funded PICO-ma (Picture books and COncept development MAthematics) project (Project 411-04-072) and were carried out with two PhD students and with Iliada Elia of the University of Cyprus and Alexander Robitzsch of the Federal Institute for Education Research, Innovation and Development of the Austrian School System in Salzburg, Austria. All sub-studies had in common the goal of providing insight into the power of picture books to contribute to the development of mathematical understanding in young children.

Before discussing what these studies taught us, I will return to Freudenthal and the work of his collaborators in the foundational stage of the Freudenthal Institute.

Freudenthal's and His Collaborators' Work in Kindergarten

Kindergartners and mathematics was one of the core topics at the IOWO. People like Jeanne de Gooijer-Quint, Edu Wijdeveld, Fred Goffree, and Hans ter Heege developed many activities for prompting kindergartners to mathematical reasoning: playing with a newspaper at a magician's party in the gym, making photographs with a real polaroid camera around the sandbox in the schoolyard, looking through a set of paper binoculars, making a calendar, knocking over a pile of cans, completing two jigsaw puzzles containing the same picture but on a different scale, ordering candles in a candle shop in the classroom, discussing what the different candles might cost and comparing differences in burning time.

Freudenthal (1984a, p. 7) himself wrote a position paper about developing education for kindergarten. Surprisingly, he started his paper by confessing that he was not familiar with kindergarten:

> I have never been in a kindergarten class, not even when I was a kindergartner. Also, I have never dealt systematically with kindergarten education; my knowledge about the research literature in this field is restricted to what I encountered accidentally, my experience to informal meetings with kindergartners. [translated from Dutch]

However, when Freudenthal delved into the then prevailing educational practice in kindergarten, he took offence at the meaningless paper work kindergartners had to do. Especially the atomization, the crumbled and un-integrated character of the assignments, were a real eyesore for Freudenthal. According to him, cognitive learning does not start from split-up material. The world of kindergartners is an integrated one. Therefore, instead of assignments on paper-and-pencil sheets Freudenthal argued for offering kindergartners "rich contexts" in which they could develop a first understanding of elementary mathematical structures, such as succession, repetition, cyclic-ness, and detour-ness.

Although commercial literary children's books with stories and pictures can offer children these rich contexts and the use of picture books in kindergarten was not an unknown approach in the 1980s (see, e.g., Strain 1969), Freudenthal did not mention them in his paper about mathematics education in kindergarten. In a way, this is remarkable, because Freudenthal was very interested in literature and art. During World War II, when he was prohibited from pursuing his profession and was arrested for some time and had to stay in a detention camp, he wrote poems, novels, plays and short stories, including a children's story, titled *Folie Antje* [Little Phele Ant] (Mathematisch Instituut and IOWO 1975). This story, written for his own children, is about the adventures of a young elephant and its friends. The story suffuses the spirit of *Winnie-the-Pooh* and *Alice in Wonderland*, as may be recognized in the following passage.

> Below the stone, there was a trap door with an iron ring. Hare Leap wanted to lift up the door, in which he also only succeeded when Little Phele\Ant continued whistling. Below the trap door there was a narrow steep flight of stairs. The steps were of soil and moss, and were so slippery that Little Phele Ant and Hare Leap repeatedly slipped away. There were 144 steps. The others asked Hare Leap again and again whether it would ever come to an end! Hare Leap answered time after time that they should not be impatient. There were 144 steps, on which they could bank, and they had to finish them all. [translated from Dutch] (Mathematisch Instituut and IOWO 1975, p. 79-81)

Yet it is also known of Freudenthal that he was not in favor of using a world of gnomes if you could also use a real world context (La Bastide-Van Gemert 2006). As far as I know, the only reference Freudenthal explicitly made to using picture books for mathematics education is in a paper about ratio (Freudenthal 1984b), in which he discussed that picture books are the place where children can meet, for example, Tom Thumb and the giant.

The PICO-ma Project

The rationale of the PICO-ma project that started in 2006 is the idea that stories and pictures in picture books can offer children rich contexts in which they can encounter mathematics-related problems, situations and phenomena which make sense to them. This learning of mathematics as a meaningful activity is one of the key principles of Freudenthal's (1973a, 1978a, 1983, 1991) approach to teaching mathematics. Picture books can contribute to the process of acquiring mathematics as an activity involving meanings that are historically developed and approved, and can provide children cognitive hooks to explore mathematical concepts and skills. Moreover, by means of their visual images, picture books can give support to the initial stages of reaching a symbolic level of dealing with mathematics, which requires an ongoing semiotic activity concerning the development of meaning. Through their interaction with picture books, children may be enabled to encounter problematic situations, can ask themselves questions, search for answers, consider different points of view, exchange views with others and incorporate their own findings to existing knowledge.

Our studies (Van den Heuvel-Panhuizen and Van den Boogaard 2008; Elia et al. 2010) that were set up to investigate children's spontaneous reactions when they are read a picture book, revealed that reading picture books can indeed make children cognitively active and can lead to mathematics-related utterances. In one of the studies (Van den Heuvel-Panhuizen and Van den Boogaard 2008) four 5-year-old children were individually read *Vijfde zijn* [Being Fifth] (Jandl and Junge 2000) without any questioning and probing. The story of this book is about a doctor's waiting room in which five broken toys are waiting for their turn. The toys go into the room behind the door one by one.

In total, the four children produced 432 utterances spread over a total of 22 pages, front cover, back cover and endpapers included. About half of the utterances were mathematics-related and all four children in the study were found to contribute to this result. The mathematics-related utterances could be distinguished into two different types with respect to their content: spatial orientation-related utterances and number-related utterances. The spatial orientation-related utterances (31 % of all utterances) exceeded the number-related utterances (14 %). Of this latter type, most utterances referred to resultative counting, "how many there are". A closer look at these utterances revealed that in a number of cases the children structured numbers. For example, when describing a picture in which the five toys are sitting in the waiting room, a child said "two are looking at the ceiling, and three are watching television". Within the spatial orientation-related utterances, the children spontaneously took the waiting room perspective instead of the doctor's office perspective that is taken by the author of the book. As a result, there was a discrepancy between the children's utterances and the text.

Because there is evidence that picture books vary in the amounts and kinds of mathematics-related utterances they evoke in children (Anderson et al. 2005), some picture books might have more power than others to provide children an environment in which they can learn mathematics. Therefore, we set up another sub-study (Van den Heuvel-Panhuizen and Elia 2012) to gain more knowledge about the characteristics which picture books should have to contribute to the initiation and fostering of mathematical understanding in young children. The framework with learning-supportive characteristics that was developed in this study—by carrying out a review of relevant academic and professional publications, followed by a Delphi study—makes a distinction between *what* mathematics can be found in the picture book (mathematical processes and dispositions, mathematical content domains and mathematics-related themes) and *how* the mathematics is presented (way and quality of presentation).

Apart from the power of the picture books themselves to elicit mathematical thinking in children, we also investigated possible ways of reading. In our view, the reading style that best fits the power of picture books is dialogic book reading (e.g., Whitehurst et al. 1988), but with not too many questions asked by the readers of the book. To let the books do the work, we requested the teachers to maintain a reserved attitude and not to take each aspect of the story as a starting point for an extended class discussion, since lengthy or frequent intermissions could break the flow of being in the story and consequently diminish the story's own power to contribute to the children's mathematical development. In addition, we tried to enhance the power of the books and cognitive involvement of the children by having the teachers as a role model of cognitive engagement or as a person who provokes discussion with the children that brings them to mathematical reasoning as well. Therefore, we suggested to the teachers involved in our project that they themselves react to the story and pictures in the picture books by performing behavior such as (a) asking oneself questions, (b) playing dumb, and (c) showing inquiring expressions (see Van den Heuvel-Panhuizen et al. 2009; Van den Heuvel-Panhuizen and Elia 2013).

Finally, we investigated the effect of reading picture books to kindergartners on their performance in mathematics by conducting an experimental study with a picture book reading program as an intervention (see Van den Heuvel-Panhuizen et al. 2014). In total, 384 children (4- to 6-year-olds) participated in our study: 199 children from nine kindergarten classes in the experimental group and 185 from nine kindergarten classes in a comparable control group. During 3 months, the children of the experimental group were read a collection of 24 literary picture books in which mathematical topics are unintentionally addressed by the authors of the books. The picture books touch upon issues related to number, measurement, or geometry. All children were pretested and posttested with a project test on these topics, the so-called PICO test, and a standardized mathematics and language test.

To investigate the intervention effect on the mathematics performance we performed a regression analysis where the gain score (PICO posttest score minus PICO pretest score) was used as the dependent variable and the intervention as the independent variable (Model 1). In order to find estimates of the intervention effect with the least bias, another regression analysis was applied, in which the various variables representing children's characteristics (kindergarten year, age, gender, home language, SES, urbanization level of school location, mathematics ability, and language ability) were included (Model 2). Both models revealed a significant intervention effect (Model 1: $B = .90$, $p = .01$; Model 2: $B = .76$, $p = .02$) with an explained variance of $R^2 = .703$ in Model 1 and $R^2 = .733$ in Model 2. The effect sizes, as defined by Cohen (1988), were calculated for each model in order to investigate the size of the general intervention effect. We found small effect sizes. For Model 1 the effect size was $d = .16$ and for Model 2 the effect size was $d = .13$. For the gain score in the control group we found an effect size of $d = .60$, which means that the influence of the intervention in Model 1 was 27 % (.16/.60 = .27) larger than this effect size in the control group. In Model 2 the increase in effect size was 22 %. This finding supports the assumption that picture book reading can yield significant learning outcomes in early years mathematics.

The PICO-ma project is continued in the PRIMAL (Picture book Research Into Mathematical Language) project in which the influence is investigated that picture books can have on the development of mathematics vocabularies of French, Spanish, and English speaking children. In this project, Iliada Elia and I work together with our PhD student Nathalie Martel and with Ann Dillon and Virginia Ferrari, two colleagues from the USA and Mexico, respectively.

The Mathematical Potential of Students in Special Education

This section addresses the learning of mathematics by students with special needs. In the Netherlands, about 3 % of children of primary school age are in special education (SE) schools for students with mild learning difficulties. These SE students have a severe delay in their mathematical development. At the end of special primary school, SE students' scores are between one to 4 years behind those of their peers in regular primary schools (Kraemer et al. 2009). Therefore, the more advanced topics in the primary school curriculum, such as ratios, rational numbers, measurement, geometry, combinatorics and data handling are often not taught in SE. The focus is mostly only on straightforwardly carrying out addition and subtraction problems with whole numbers and some multiplication and division.

The IMPULSE (Inquiring Mathematical Potential and Unexploited Learning of Special Education students) project which started in 2008 aims at investigating SE students' mathematical potential through the use of dynamic, ICT-based assessment approaches that offer SE students opportunities to show what they are able to do. In this project I cooperate with my PhD student Marjolijn Peltenburg and with Alexander Robitzsch. Here, I will discuss one sub-study of this project in which we investigated SE students' ability to apply alternative methods for solving subtraction problems up to 100 (Peltenburg et al. 2012). The starting point of this study was the strong belief in circles of SE educators and psychologists in the Netherlands, but also in other countries, that students who have low scores in mathematics cannot handle alternative calculation methods. The idea is that it is better to teach them only one fixed method for each number operation because otherwise they get confused.

Freudenthal's and His Collaborators' Idea of Mathematics for All and Flexibly Solving Subtraction Problems

Revealing SE students' mathematical potential is in line with the idea of 'mathematics for all', which is an inalienable aspect of Freudenthal's (1968, 1991) conceptualization of mathematics and its teaching. In many observations of children's learning processes, Freudenthal (1975, 1976, 1977, 1978b) made it clear that mathematics can be done at any level. Moreover, he showed that there is hope for underachievers (Freudenthal 1981). In this way Freudenthal guided us in revealing the mathematical potential of SE students. In particular, this also applies to Ter Heege (1980, 1981–1982) with his pioneering work on low achievers in mathematics.

Although even Freudenthal (1973b) once argued for a single solution method when a very confusing subtraction method was suggested for low achievers, he (1991, p. 76) also warned us against inflexible instruction especially for these students: "Flexibility [referring here to the use of palpable material] should be allowed, and if need be, taught rather than fought." Regarding subtraction, Freudenthal (1982, 1983, p. 107) emphasized how necessary it is to work on a broad mental constitution of mathematical concepts. Just as addition can appear as putting together and as appending, we should also address the two appearances of subtraction: taking away and finding the difference, or in his words: "explicitly taking away suffices as little for the mental constitution of subtraction as uniting explicitly given sets suffices for addition."

The IMPULSE Project

Solving subtraction problems by adding on (see details in Peltenburg et al. 2012; Van den Heuvel-Panhuizen 2012)—The methods that can be applied for carrying out subtractions up to 100 can be described from two perspectives (see Fig. 1).

From the operation perspective subtraction problems up to 100 can, for example, be solved by: direct subtraction (DS), indirect addition (IA), and indirect subtraction (IS). The number perspective describes how the numbers involved are dealt with. Roughly speaking, there are three strategies: splitting (the minuend and the subtrahend are split into tens and ones and then the tens and ones are processed separately), stringing (the minuend is kept intact and the subtrahend is decomposed in suitable parts which are subtracted each after each other from the minuend) and varying (the minuend and/or the subtrahend are changed in order to get an easier subtraction problem). Although in theory all three strategies can be combined with each of the four procedures, not all combinations are that suitable.

Connected to the debate on whether or not teaching SE students one fixed method for solving number problems, there is the controversy of whether SE students should be taught IA to solve, for example, a problem like 62 − 58 (i.e., calculating 58 + 2 = 60, 60 + 2 = 62, so the answer is 4, instead of calculating 62 − 50 = 12; 12 − 2 = 10 and finally 10 − 6 = 4).

The sub-study was set up to answer whether SE students can make spontaneous use of IA for solving subtraction problems up to 100. In total, 56 students from fourteen second-grade classes in three Dutch SE schools participated in the study. The participating students were 8–12 years old, with an average age of 10 years and 6 months (*SD* = 10.4 months). These students were 1–4 years behind in mathematics compared to their peers in regular primary school. Data were collected with

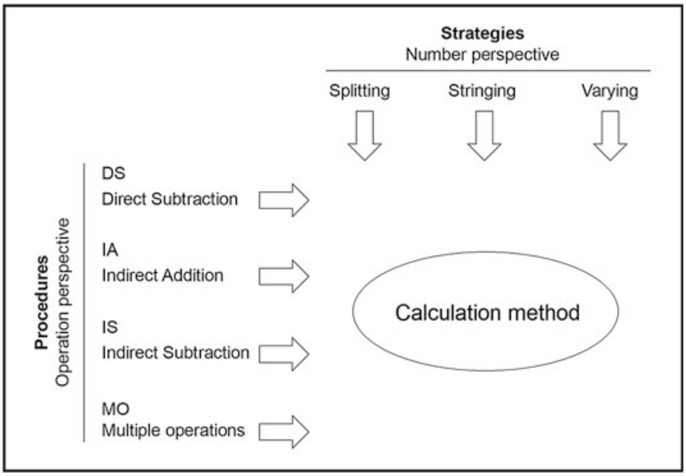


Fig. 1 Calculation methods for subtraction up to 100

an ICT-based Subtraction test in which the item characteristics were varied systematically over the fifteen items. These characteristics include number characteristics (the size of the difference between minuend and subtrahend, whether the tens have to be crossed and whether or not minuend and subtrahend are close to a ten) and format characteristics (bare number problem or context problem). The context problems either described a taking-away situation or an adding-on situation.

Our study showed that SE students: (a) are able to use IA spontaneously to solve subtraction problems, (b) are rather flexible in applying IA to solve subtraction problems, (c) are quite successful when solving subtraction problems by IA. The plan is to continue the IMPULSE project in a project in which informing teachers about the mathematical potential of students with special needs will be used to raise teachers' expectations and consequently increase their students' achievement.

Textbook Analyses to Secure Relevant Opportunities to Learn

More than any other project I am currently involved in, the META (Mathematics Education Textbook Analyses) project, discussed in this section, is grounded in the didactics of mathematics. Through analyzing textbooks for mathematics education, the project examines what mathematical content is taught in primary school and how it is taught. In the META project, I collaborate with my PhD student Marc van Zanten, who is also a mathematics teacher educator of prospective primary school teachers.

Freudenthal's and His Collaborators' Work on Textbook Analyses

Because textbooks have a crucial role in determining mathematics education, the Wiskobas group at IOWO started carrying out textbook analyses (De Moor and De Jong 1980) as early as 1975. This allowed them to give advice to teachers and school teams on choosing a textbook series. Especially in the Netherlands giving such help to teachers was (and is still) essential because of the absence of a centralized textbook design and the lack of a state authority which approves textbook series before they are put on the market. The textbook analyses by IOWO, and its successor OW&OC, resulted in a series of documents published in 1980, 1983, and 1987, which described and evaluated the available textbooks. The crowning glory of this textbook analysis work was the PhD thesis of De Jong (1986) whose study investigated the influence of the Wiskobas project on textbooks. It is interesting to mention here that, after IOWO ceased to exist, Rob de Jong became a staff member of the Department of Education of the Social Sciences Faculty of Utrecht University. Thus, in a way, he preceded us.

And what about Freudenthal's thoughts about textbook analyses? Although it is clear that he studied numerous textbooks (Freudenthal 1973a) and De Jong (1986) mentioned him as involved in his analysis, Freudenthal (1991, p. 177) said later that he had never reviewed textbooks (which in his eyes might not be the same as analyzing textbooks) and that he was reluctant to do so, because he was "wonder [ing] how much teaching experience is required to undertake the task of reviewing textbooks before or without having used them." Moreover, "[w]hether one likes it or not, textbooks are merchandise, and in the marketplace good quality is what appeals to the needs and the tastes of prospective customers." As Freudenthal admitted, "[t]his would seem to be a gloomy perspective for change". However, he also gave us hope, when continuing "were it not that needs can be stimulated and tastes can be educated."

Considering the reform movement that has taken place in the Netherlands (Van den Heuvel-Panhuizen 2001) textbook analyses have unmistakably had their desired effect. In the 1980s, the market share of textbooks with a traditional, mechanistic approach was 95 % and the textbooks with a reform-oriented approach—based on the idea of learning mathematics in context to encourage insight and understanding—had a market share of only 5 %. In 1987, the market share of these latter textbooks was around 15 %. In 1992 this had increased to almost 40, and 75 % in 1997. In 2004, the reform-oriented textbooks reached a 100 % market share. However, due to the debate that has taken place in the Netherlands after 2007, in which the reform-oriented approach is criticized in favor of a return to the traditional, mechanistic approach (Van den Heuvel-Panhuizen 2010), in their new editions some textbook series have adapted the content (more emphasis on algorithms) and teaching approach (more training of knowledge and skills). Therefore, textbook analyses are once again important to inform teachers and others about students' opportunities to learn mathematics with these textbooks.

The META Project

The focus in the first sub-study of the META project was on subtraction up to 100 (see details in Van Zanten and Van den Heuvel-Panhuizen 2014). In agreement with the idea of Freudenthal (1978a, 1983) that a "didactical phenomenology" is an indispensable precondition of educational research in mathematics, we started with what we called a "mathedidactical analysis" of the concept of subtraction (later more about this difference in terminology). This analysis, together with a literature review, resulted in an analysis framework covering three perspectives: the mathematical content, the performance expectations and the learning facilitators included in the textbooks to be analyzed.

Our textbook analysis was applied to two recently developed textbook series that, although they are from the same publisher, clearly position themselves within two contrasting approaches to mathematics education. The first textbook series, called *Rekenrijk* (RR) (Bokhove et al. 2009), is a reform-oriented textbook series.

The name refers to "rich arithmetic" and "realm of arithmetic". The second textbook series, called *Reken zeker* (RZ) (Terpstra and De Vries 2010), is a new textbook series that was presented as an alternative for the reform-oriented approach. The name of this textbook series means "arithmetic with certainty". The analyzed materials were the Grade 2 books from which we excluded the assessment lessons and the additional tasks for students who need repetition or more advanced content.

The analysis revealed that, in grade 2, RZ (1440) has more subtraction-related tasks than RR (1166). Both textbook series have more subtractions between 20 and 100, and fewer subtractions in the range up to 10 and up to 20. Although RZ has more subtraction tasks, the number of tasks involving bridging a ten in RR and RZ is about the same (approx. 600). Attention for the prerequisite knowledge for subtraction problems differs in the two textbook series. For decomposing numbers up to 10, RR has a substantial number of such tasks (107) and RZ only a very few (4). For counting backwards with tens (e.g., 46-36-36), RR has a few tasks (23), while RZ has almost none (1). When examining the content in grade 1, we found that both textbooks put more emphasis on prerequisite knowledge in grade 1 than in grade 2. But again, in grade 1 there were more such tasks found in RR (418) than in RZ (167).

The two textbooks also differ in the amount of attention for the semantic structure of subtraction-related tasks. Although both textbook series address subtraction as taking away (Fig. 2a, b), subtraction as determining the difference is only dealt with in RR (Fig. 3).

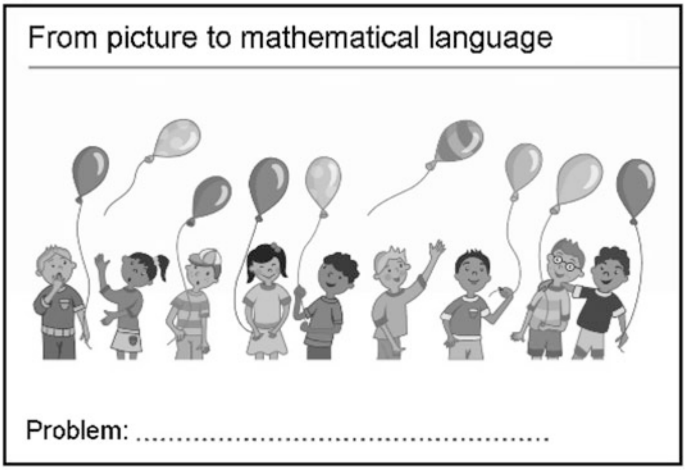
Another difference between the two textbook series is the degree in which they relate addition and subtraction to each other. RR explicitly pays attention to this (see an example in Fig. 4), whereas RZ does not. Moreover, only RR deals with subtractions in an addition format, which elicit subtraction as adding on (e.g., $3 + \ldots = 6$ and $27 + \ldots = 32$) (see Figs. 4 and 5). However, missing number subtractions (e.g., $19 = 20 - \ldots$ and $26 - \ldots = 21$) are only dealt with in RZ.

To measure the performance expectations regarding understanding, we determined which tasks explicitly offer directions or questions that undoubtedly have the intention to prompt students' reasoning. In RR, we found 111 such performance expectations. They include directions to students to explain their thinking, visualize their calculation method or choose an appropriate calculation method for a given subtraction with certain numbers. In RZ, we did not find clearly distinguishable performance expectations regarding understanding.

RR offers didactical support for 77 % of its tasks, and RZ for 23 % of its tasks. Both textbook series use models, textual instructions and analogy with easier subtractions as a form of didactical support. Contexts for meaning making and own productions are only used in RR.

Apart from not giving much didactical support, another shortcoming of RZ is the lack of a match between model and strategy (cf. Van den Heuvel-Panhuizen 2008a). RZ uses base-10 arithmetic blocks (which is a group model that fits more to a splitting strategy) to support stringing (Fig. 6a). To a certain degree, a similar inadequacy applies to RR when using a particular symbolic representation of

(a)



(b)

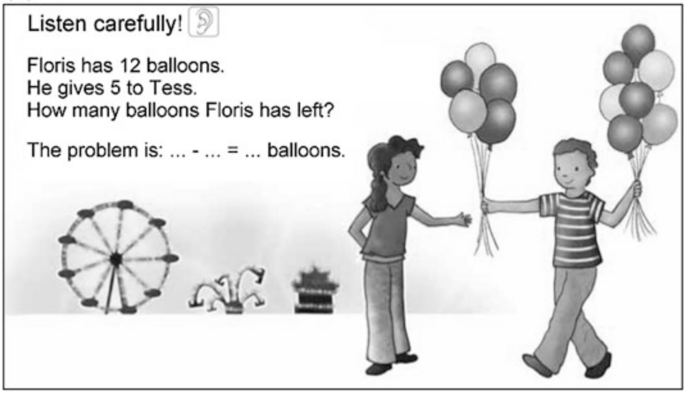


Fig. 2 **a** RR task that reflects subtraction as taking away (RR Workbook 4a-1, p. 14). **b** RZ task that reflects subtraction as taking away (RZ Book 4a, p. 20)

subtraction like …− 68 = …, which does not match with the presentation on the empty number line that refers to 73 − … = 68 or to 68 + … = 73 (Fig. 6b).

Our textbook analysis on subtraction up to 100 in grade 2 revealed that the two textbooks series really differ with respect to offering students learning opportunities. For example, it does make a difference for the students whether or not they are offered a broad mental constitution of subtraction, whether or not they are presented reflection-eliciting questions and whether or not there is a match between models and symbolic representations or models and calculation methods. Of course, what is in the textbook is not necessarily similar to what is taught in class, but following Valverde et al. (2002, p. 125), there is enough evidence that "how content is presented in textbooks (with what expectations for performance) is how it will

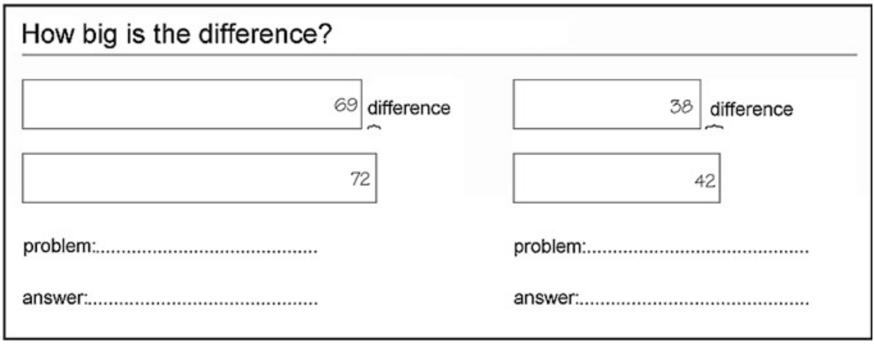
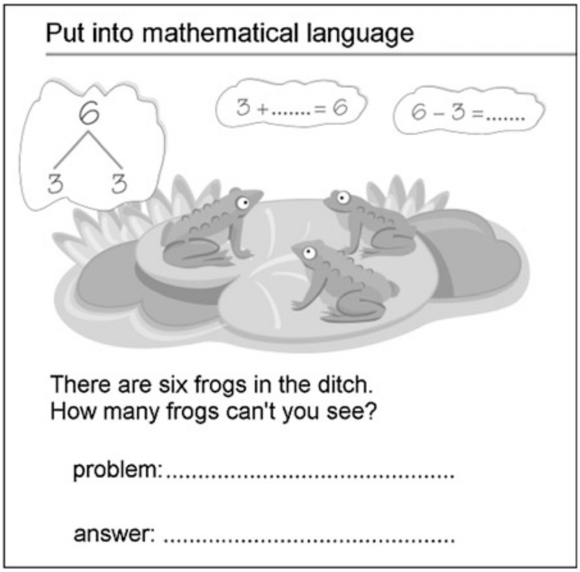


Fig. 3 RR tasks that reflects subtraction as determining the difference (RR, Workbook 4b-2, p. 58)

Fig. 4 The relationship between addition and subtraction in RR task (RR Workbook 4a-1, p. 4)



likely be taught in the classroom." Therefore, textbook analyses can give us a first inside view of how a subject is taught. As such, textbook analyses are a crucial tool that can preserve us from the danger of having a teaching practice that is not in agreement with the intended curriculum and does not offer students the desired learning opportunities. How necessary such analyses are, was shown when a textbook analysis disclosed that higher-order problem solving is lacking in Dutch mathematics textbooks (Kolovou et al. 2009).

These and other examples of textbook analyses make it clear that making known the learning opportunities that textbooks offer is equally important as examining the

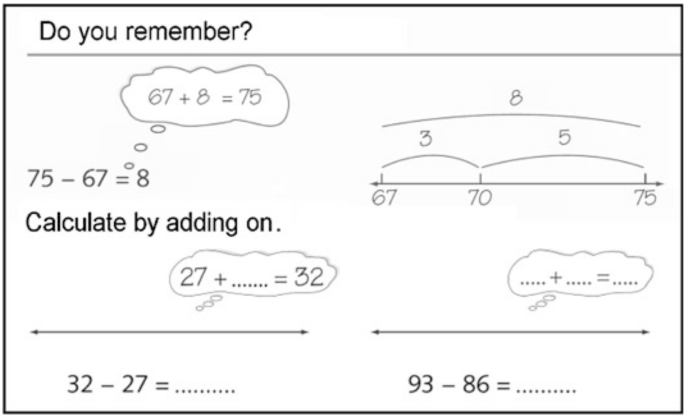


Fig. 5 Subtraction as adding on in RR task (RR Workbook 4b-2, p. 61)

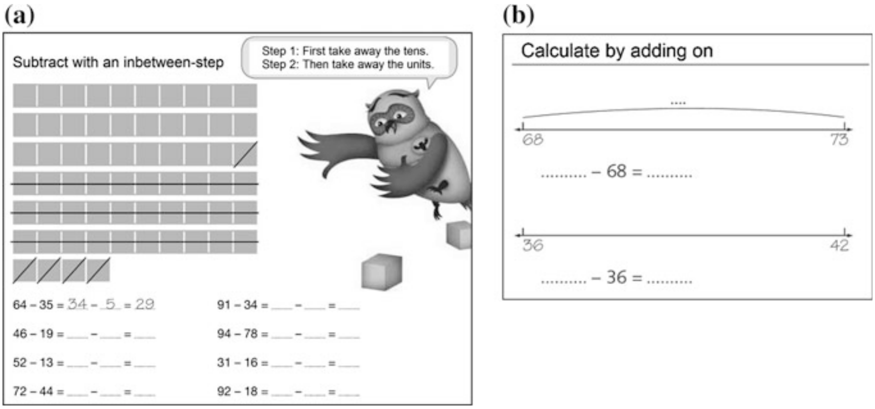


Fig. 6 **a** RZ use of blocks combined with stringing (RZ Learn-workbook 4e, p. 12). **b** RR use of empty number line (RR Workbook 4b-2, p. 78)

efficacy of textbooks. When students do not encounter particular content, we cannot expect them to learn this content. Therefore the results of textbooks analyses are relevant for all involved in education: for teachers (when using textbooks), for mathematics educators (when introducing prospective teachers to textbooks), for inspectors (when controlling the quality of education), and for textbook authors (when writing or revising textbooks). Last but not least, textbook analyses are also important for the further development of the didactics of mathematics as a scientific discipline. In textbooks, as the potentially implemented curriculum, a broad variety of operationalized didactical knowledge converges—even including didactical fallacies—which can feed our thinking and understanding of how to teach mathematics.

More Elementary Mathematics Education Research and Beyond

Besides the projects discussed above, other research projects in elementary mathematics education (including primary school and the adjacent areas of pre-school, kindergarten, and the beginning of secondary school) are presently carried out, just finished, or will soon start at the Faculty Science or the Faculty of Social and Behavioural Sciences.

In the Curious Minds project, two new studies, in cooperation with Paul Leseman of the Faculty of Social and Behavioural Sciences, are initiated. The first study is on the role of embodied cognition and representational redescription in children's understanding of phenomena in mathematics, science and technology. The second study is on how perception-action affordances of mathematics, science and technology tasks can elicit and guide children's exploration behavior towards discovering scientific principles embedded in these tasks.

Furthermore, we are looking for a continuation of the POPO (Problem Solving in Primary School) project that was aimed at investigating early algebra in primary school, i.e. the use of an online game to give primary school students experience in dealing with covarying quantities (Kolovou 2011). Together with the Faculty of Social and Behavioural Sciences a new interlinked research project is in preparation that is meant to contribute to a thorough theoretical and practical understanding of how higher-order thinking in mathematics develops and can be fostered in primary school, in particular in the subdomains of data handling, probability, and early algebra. The project will be theoretically grounded in embodiment theory and variation theory and will make use of interventions with ICT environments containing mathematical applets and with what we call 'learning movies'. Another project in which the transition from arithmetic to algebra is scrutinized is Al Jupri's PhD study. The goal of this IISPA project is to understand and improve Indonesian students' low performance in algebra (Jupri et al. 2014).

Also close to the core work at the Freudenthal Institute is Ariyadi Wijaya's PhD study into the difficulties that particularly Indonesian students experience in solving context-based mathematics tasks (Wijaya et al. 2014). In this CoMTI project, the principle of teaching mathematics-in-context is revisited by investigating this principle when it is applied in another cultural context.

In the BRXXX (Basic Number Skills with Mini-games) project, in which I work together with my PhD student Marjoke Bakker and my colleagues Sylvia van Borkulo, Hanneke Loman and Alexander Robitzsch, the focus is on students' learning of multiplication tables—or more formally expressed—the learning of multiplicative relationships of positive integers. As such, this is a classic topic, which has been investigated many times at the Freudenthal Institute in the IOWO and OW&OC times—as well as in many other places in the world—but in the BRXXX project we situate this learning in a new learning environment. The students play mini-games and are shown 'learning movies' which explain to them the ins and outs of the games and give them hints to cleverly play the games by which they are implicitly put on the track of better understanding multiplicative relationships between numbers, and will eventually be led to an improved performance. The effect of the mini-games is examined in three conditions (playing the games in a mathematics lesson context, just playing at home, and playing at home with a class discussion only afterwards). Moreover, there is a control group that plays mini-games in class on a topic not related to multiplicative relationships. Besides the use of online mini-games and the online assessment of the students' performances—which was also applied in the POPO project—the BRXXX project also reflects another new step in mathematics education research at the Freudenthal Institute. This is its large scale. In total, approximately 1500 students, who we are following over more than 2 years (of which about 250 students in special education, who are followed for only 1 year), are involved. The focus will be both on their understanding of multiplicative relationships and their appreciation of mathematics as a school subject. Some first results can be found in Bakker et al. (2011) and Van Borkulo et al. (2011).

In the SANPAD-funded COCA (Count One, Count All) project, carried out in collaboration with the University of Cape Town and the Cape Peninsula University of Technology, a learning pathway for number (LPN) in primary school has been developed (Van den Heuvel-Panhuizen et al. 2012). The LPN was inspired by the TAL teaching/learning trajectory on number (Van den Heuvel-Panhuizen 2008b) which was developed at the Freudenthal Institute. The LPN was initially meant as a support for teachers involved in the COCA project, but now it is also available for other teachers, as well as for teacher advisors, teacher educators, and researchers of mathematics education.

Finally, a few words about the recently started ICA (Improving Classroom Assessment) project, again a project building on earlier research and development activities carried out at the Freudenthal Institute, in this case especially on the assessment work by De Lange (e.g., 1987, 2007) and myself (Van den Heuvel-Panhuizen 1996, 2003). However, in contrast with this older work on assessing students' understanding of mathematics in which the focus was mostly on task design, the ICA project involves the assessment activities undertaken by the teacher. In this way the ICA project can be considered a continuation of the CATCH (Classroom Assessment as a basis for Teacher CHange) project that the Freudenthal Institute ran in the USA, and which was aimed at using classroom assessment as a means for professional development of teachers (Dekker and Feijs 2005). In the ICA project, my PhD student Michiel Veldhuis and I work together with researchers from Cito (Central Institute for Test Development in the Netherlands) and Twente University. We started with a joint survey on how primary school teachers in the Netherlands collect data about their students' learning processes so that they can make informed decisions on how to continue teaching. In the second phase of the project, the Freudenthal Institute part of the ICA project will develop jointly with teachers a collection of informative classroom techniques to improve classroom assessment. In the third phase, the effect of the improved classroom assessment on student achievement will be evaluated in an educational experiment. Soon, the ICA team of the Freudenthal Institute will be extended with Xiaoyan Zhao, my new PhD student from China with whom we will carry out a study about classroom assessment in China.

A Blueprint for Continuing Our Work

In mathematics education research carried out at the Freudenthal Institute four different perspectives are included: (i) the students' learning, (ii) the teachers' teaching and learning, (iii) the teaching/learning process and the environment—understood in the broadest sense—in which this process takes place, and (iv) the assessment to inform teachers and students (and others) about the teaching/learning process (see Fig. 7). Of course, having these different viewpoints does not apply exclusively to research at the Freudenthal Institute, but can be recognized in mathematics education research in general. Furthermore, it should be clear that making a distinction between these different viewpoints does not mean that they are dealt with in isolation. In most of our projects the different perspectives are considered in close connection with each other.

Furthermore, what all projects have, or should have, in common, is that whatever perspective is taken, research within a project should both *contribute to* the didactics of mathematics as a scientific discipline and *emerge from*, or at least be guided by, knowledge and theories generated within the didactics of mathematics. Certainly, research of mathematics education can also be informed by other scientific disciplines within the educational and learning sciences. However, in Fig. 7, I left these disciplines out, both for reasons of clarity and to focus on the *didactics of mathematics*.

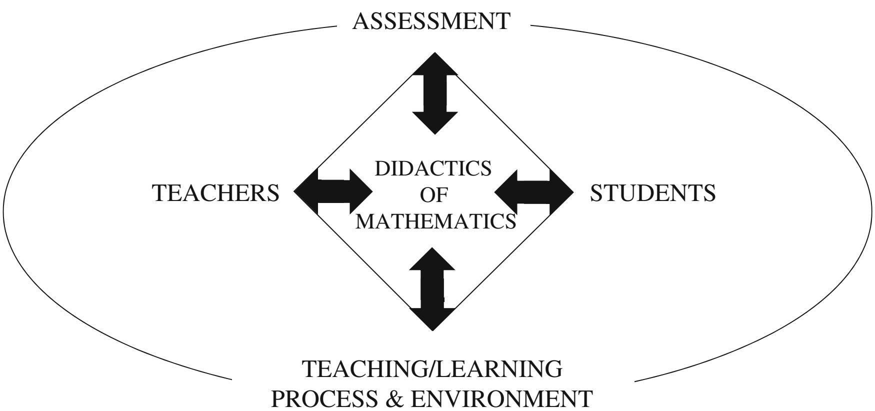


Fig. 7 Different perspectives in mathematics education research

Depending on the research questions that are at stake within the various perspectives, research in mathematics education makes use of *different research methods* (Fig. 8). Among these methods, design research approaches undeniably have a very prominent place, especially when the goal is to develop educational material. Nevertheless, other research methods such as quasi-experiments (including pretest-posttest designs and micro/macrogenetic designs), surveys (including comparative achievement studies), and document studies (including textbook and software analyses) are equally significant for the further development of the didactics of mathematics. What matters is whether the research methods fit the research questions and whether the methods guarantee robust, tenable findings—and this also applies to design research. Yet we should not forget Freudenthal's warning, expressed in a letter sent to Henry Pollak in 1977 (see La Bastide-Van Gemert 2006, p. 284): "The greatest danger is the so-called empirical work, processed with statistical nonsense methods."

But there is more. Research in mathematics education does not only require sound empirical methods as they are generally used in the social sciences. To make research results relevant for students' learning of mathematics, the application of these methods should deal with mathematics that makes sense and is worthwhile to be learned. To achieve this, the empirical methods should be nourished by analyses that are related to mathematics. These analyses form the heart of the didactics of mathematics (see Fig. 9).

The didactical analyses are aimed at revealing the nature of the mathematical content as a basis for teaching this content. By identifying the determining aspects of mathematical concepts and the relationships between concepts, knowledge is gathered about, for example, the didactical models that can help students to understand these concepts. Further analyses can strengthen this knowledge basis for research in mathematics education. Phenomenological analyses disclose possible

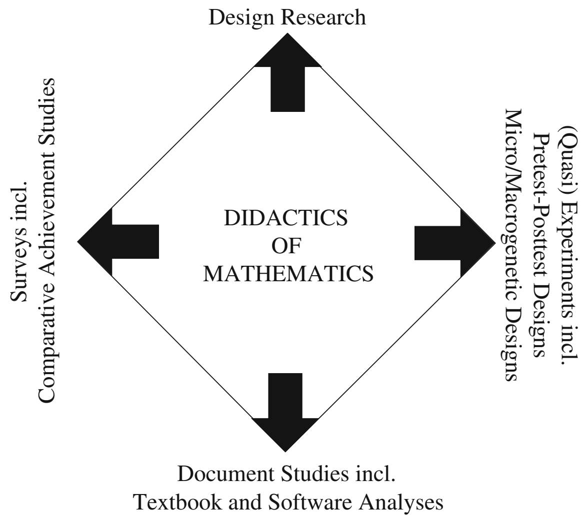


Fig. 8 Different research methods

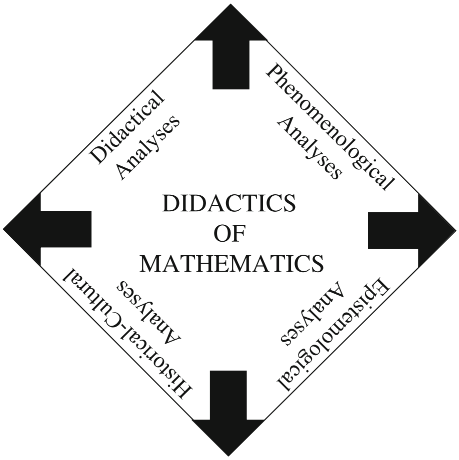


Fig. 9 Mathematics-related analyses that constitute the didactics of mathematics

manifestations of these mathematical concepts in reality and can suggest contexts in which students can meet these concepts. Epistemological analyses focus on students' learning processes and can, for example, uncover how students in a classroom interaction can make a shift in mathematical understanding. Finally, in historical-cultural analyses, we may come across various approaches to teaching mathematics in the past and in other countries through which we can gain a better understanding of how to learn mathematics and how education can contribute to it.

To avoid misinterpreting Fig. 9, I have to say that these analyses do not exclusively fit a particular research method but can be applied in combination with any of the methods mentioned in Fig. 8. Moreover, not all these analyses are required in any one research project. The focus can be on one type of analysis or a combination of different approaches.

What is essential for these analyses is that they all take mathematics as their starting point. Considering these analyses as the heart of the didactics of mathematics follows strongly the ideas of Freudenthal. Although he did not distinguish four different types of analysis, Freudenthal's DNA is firmly rooted in this heart.

In his *Preface to a Science of Mathematical Education*, Freudenthal (1978a) explained that a profoundly scrutinizing analysis of the subject matter is fundamental to educational research in mathematics. The name he chose for this analysis, which he exemplified by an analysis of the topic of ratio and proportion, was "didactical phenomenology". However, after naming it in this way, he added immediately:

> [T]he name does not matter; nor is that activity an invention of mine; more or less consciously it has been practiced by didacticians of mathematics for a long time. In various earlier books and papers I have given examples of the didactic phenomenology of mathematics, and I hope to deal with it comprehensively in another book (Freudenthal 1978a, p. 116).

That later book was *Didactical phenomenology of mathematical structures* (Freudenthal 1983) in which he gave more examples of didactical phenomenologies. Although he also included in this book a short chapter about the method, it never became very clear how such an analysis should be carried out. Another characteristic of Freudenthal's didactical phenomenology is that it does not only focus on its strict meaning of describing how "mathematical concepts, structures and ideas [can] serve to organize phenomena—from the concrete world as well as from mathematics" (Freudenthal 1983, p. 28), but that it also encompasses other ways of analyzing mathematical content for educational purposes. In fact, the mathematics-related analyses mentioned in Fig. 9 can all be recognized in Freudenthal's didactical phenomenology. For us, the challenge is to keep this heart alive and equip it with a strong methodology under the umbrella of both the Faculty of Science and the Faculty of Social and Behavioural Sciences of Utrecht University.

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Teachers Learning Together: Pedagogical Reasoning in Mathematics Teachers' Collaborative Conversations

Ilana Seidel Horn

Abstract In the United States, teaching is an isolated profession. At the same time, ambitious forms of teaching have been shown to benefit from teacher collaboration. What is it about collegial conversations that supports teachers' ongoing professional learning? In this paper, I synthesize findings from prior studies on mathematics teachers' collaborative conversations, focusing my analysis on collective pedagogical reasoning. I examine four facets of collegial conversations that support refinements in this reasoning. These facets are: interactional organization, engagement of individual teachers in a group, epistemic stance on mathematics teaching, and locally negotiated standards of representational adequacy. Together, these aspects of teacher talk differently organize opportunities for professional learning.

Keywords Professional learning · In-service teachers · Discourse analysis

Introduction

In the United States, teaching is an isolated profession. Teachers tend to work alone in their classrooms, having little interaction with colleagues. Typically, other adults in the school only visit a teacher's classroom to evaluate performance—and even then, such visits are infrequent.

At the same time, mathematics teaching that engages learners in sense making requires challenging forms of pedagogy. For example, practices like building on student thinking or effectively using high-press questioning shift teachers' attention away from the clear presentation of ideas to building students' understanding. This shift increases the uncertainty of teaching, requiring adaptation and maneuvering even among the most sophisticated of practitioners (Cohen 2011).

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My work takes these two observations as a starting point to make the claim that teachers' professional isolation works against the demanding teaching practices that are at the center of many reforms in mathematics education. By leaving teachers on their own to diagnose and respond to the inevitable puzzles of practice that arise, these desired forms of teaching may not take a firm root in many classrooms.

There are two primary reasons for this. First, many teachers will implement superficial changes that take the form of ambitious pedagogies without fulfilling their intended function. They may, for instance, have students work on a cognitively demanding task, but in implementation, turn it into a procedural one. Alternatively, teachers will find the uncertainty inherent in this kind of teaching untenable and simply abandon it for more traditional forms of practice that are better aligned with the institution of schooling.

For these reasons, I have spent the last decade examining how teachers might learn together in support of ambitious mathematics pedagogy. Educational research has consistently shown a relationship between schools or departments in which teachers work collectively and higher-than-expected student achievement (Bryk et al. 2010; Lee and Smith 1996; McLaughlin and Talbert 2001). This relationship is often coupled with more ambitious forms of teaching. The hunch in the field has been that these teacher collectives support and sustain professional learning, yet the ways in which this learning unfolds has been underspecified.

In this paper, I describe findings from several studies I have conducted with the aim of uncovering how teachers' collaborative conversations might support and sustain professional learning.

How Mathematics Teachers Learn Together

My program of research focuses on secondary mathematics teachers' conversations about their work. I have collected records of hundreds of hours of these discussions in the form of field notes, audio recordings, and videos. I analyse the learning opportunities within these interactions by looking at the *episodes of pedagogical reasoning* (EPRs), or the moments in teacher talk where there are moments of teacher-to-teacher talk where issues or questions about teaching practice are brought out and accompanied by some elaboration of reasons, explanations, or justifications. EPRs can be single turn utterances, such as, "I'm not using this worksheet because it bores the kids." More relevant to group development are multiparty EPRs, in which descriptions of practice and pedagogical reasoning are co-constructed over many turns at talk.

EPRs provide a window on teachers' learning opportunities because they are moments in which pedagogical reasoning is made publicly available to colleagues. Depending on the interactional organization of a particular group, the nature of the learning opportunities may vary. On the one end of the spectrum, where norms of questioning others' thinking are not in place, teachers may simply have an opportunity to hear the way their colleagues think without taking it on. On the other end of the spectrum, teachers may have an opportunity to critically engage with one another's thinking, considering alternative explanations in ways that support deeper understandings or even conceptual change.

In this paper, I synthesize prior analyses of the resources for teacher learning in collegial conversations. The four components I describe below are features of any teacher workgroup. They can be differently marshalled to provide more or less opportunities for the kinds of learning that supports ambitious mathematical teaching. In the following sections, I describe critical features of these conversations: interactional organization, engagement in a group, epistemic stance, and representational adequacy. Together, these allow for an analysis of the resources for learning within a teacher group.

Interactional Organization

To learn together, teachers must, at the very least, have opportunities to work together. In American secondary schools, this is a nontrivial condition. In 2002 survey conducted by Public Agenda, only 20 % of U.S. high school teachers reported that they "regularly meet to share ideas about lesson plans and methods of instruction" (Public Agenda 2002, p. 23), suggesting that norms of privacy still prevail. Given the paucity of idea sharing among teachers—which does not approach the complexity of collaborative pedagogical problem solving (Horn and Little 2010)—typical school cultures are not ripe for productive teacher conversations.

Gatherings of teachers are rare in U.S. schools, yet even these are decidedly insufficient for the development of pedagogical reasoning. My research has supported findings in earlier work saying that, to succeed in improvement, teachers need a common focus on student learning and a commitment to improvement. When examined at the level of interaction, we find that this, too, is insufficient to support teachers' professional learning. Teachers can have a stated commitment to student learning and professional improvement, yet still focus on logistical aspects of planning over a deeper focus on student thinking (Horn and Little 2010).

The kinds of conversations that provide teachers with professional learning opportunities take a distinct shape. In an analysis of a group of teachers who sustained ambitious forms of mathematics teaching for over a decade, I found that their conversations unfolded in ways that first gave emotional support and then encouraged elaboration, specification, and revision of interpretation of classroom teaching events. These conversations supported a collective interpretation of teaching, with colleagues providing multiple conjectures about the uncertain aspects of teaching and the introduction of teaching principles to interpret and eventually build a general understanding of the work (Horn 2010).

Consider the following problem statement by Tina, a new teacher, to her colleagues in the Algebra Group:

TINA: My students, I don't know where they're from, but they're doing so well, I mean they know the difference between a linear graph versus exponential. But the thing about my students is that there's kids that know a lot and then there's kids that you know, feel like they're slow learners.
CHRISTY: (*Nodding vigorously from* "*a lot*" *to* "*slow learners*")
TINA: And I'm trying to find group-worthy activities
CARRIE: (*Nods*)
TINA: where the kids who are fast learners and the kids who are slow learners, that it can close the gap.

Another new teacher, Christy, nodded as Tina explained her problem, expressing at empathy for Tina's portrait of her students' knowledge. Carrie, a more experienced teacher, then nodded as Tina described her proposed solution to the problem, finding group-worthy activities. Such gestures indicate emotional support and alignment among the teachers as they report on teaching problems.

As the conversation unfolded, Carrie posed questions that drew out a more elaborated account of Tina's problem. She then offered another interpretation:

CARRIE: I wonder if it's not just the activities you're doing but also just status.
TINA: Mhm.
CARRIE: You know? I mean even if you did give them a group-worthy task, those kids who feel like they have low status will just continue to play that *role*
TINA: (*Nodding.*)
CARRIE: (*Nodding*) because they think that that's what they're supposed to do.

Carrie's introduction of status reframed Tina's analysis of the fast and slow kids in her classroom by proposing that something besides the content of the activities may have been hindering students' participation. As the episode went on, the elaboration and re-interpretation of Tina's fast kids problem continued (Horn 2007), providing supports for her professional learning as well as other interlocutors like Christy.

What is it about this kind of interaction that supports professional learning? In a comparison of the mathematics teachers' conversations to those of statisticians and health scientists, Hall and Horn (2012) found similarities in the kinds of talk that supported new understandings about work-as-usual. Consulting interactions, such as the one above, provide an occasion for comparisons over accounts of practice that allow participants to borrow and extend methods. These kinds of consultations support teachers' ongoing learning about their work. The introduction of multiple viewpoints—is the problem the activity or student status?—and the work of reconciliation can reorganize teachers' understandings of teaching.

Engagement in a Group

The question of interactional organization becomes meaningless if teachers are not engaged in the activities of a teacher group. At a minimum, teachers find collegial conversations for the emotional support they are able to garner. Simply telling the tale of classroom trouble—griping or venting—to other adults may be emotionally beneficial. Beyond this, individual teachers may or may not be aligned with the broader purposes of the story swapping, even in the most productively organized teacher group.

The most productive groups I have studied have a clearly stated goal that has meaning to the participants. For instance, one group met regularly to work on detracking, or reconstructing the curriculum so that students are grouped heterogeneously. Because this goal involved teaching to students with a broader range of achievement levels, their work emphasized finding activities that supported multiple forms of student mathematical competence (Horn 2005, 2007, 2012). Another group aimed to increase success of first year college preparatory algebra class (Horn 2012; Horn and Kane in press). Teachers not aligned with the groups' purpose did not participate as successfully, whether because they wanted to hold onto traditional forms of teaching or because they found the work of examining students' thinking too demanding.

Engagement supports teachers in sustaining complex forms of practice. In more accomplished teacher groups, participants reported a sense of accountability through regular meetings with colleagues. In interviews, they often compared their workgroups to having an exercise partner: they held teachers accountable to what they set out to improve. Ambitious forms of mathematics teaching are difficult to sustain, particularly when students' expectations are for other kinds of teaching. Regular conversations with other teachers motivated participants to persist (Horn 2012).

Engagement is not always an easy accomplishment. In addition to alignment of individuals with the group's purpose, engagement requires a capacity for conflict (Achinstein 2002; Grossman et al. 2001). For individuals to learn they must be able to air their thinking. To this end, alternative viewpoints need to have a means to be considered, as when Carrie proposed another interpretation of Tina's problem. If teachers cannot honestly speak their minds, their ideas about teaching will not be in play and not have traction for conceptual change.

Because teaching has moral dimensions, certain questions about the role and obligation of teachers are a matter of interpretation and commitment (Bartlett 2004). Misalignments can occur between a teacher's conception of their role and the group's conception. In these instances, the group is not a productive place for that teacher's learning. Likewise, such an individual can disrupt other's learning by persistently airing alternative framings of problems and redirecting the conversations (Horn and Little 2010).

In groups where sufficient alignment exists among participants, the collaboration becomes a resource for teacher identity. On the less productive end, this collective identity can take the form of balkanization (Hargreaves 1994) in what McLaughlin and Talbert (2001) have called *traditional teacher communities*. More productively for the goals of improvement, these teachers groups support ambitious forms of practice. Participants become, for instance, "an Algebra Group teacher"; this identity marks them as taking a certain stance on teaching, embodying a set of moral and professional commitments. In these instances, teachers report in interviews the significance of their participation, and sometimes the visibility of their participation to other colleagues who may express either envy for their support or scepticism toward their ambitious goals. The group identity supports ongoing improvement efforts on the part of individuals, since reverting to traditional practice would be to deny this part of themselves.

Epistemic Stance on Mathematics Teaching

The organization of interaction and resulting teaching identities described above indicate a certain epistemic stance on the work of teaching. By *epistemic stance,* I refer to an expressed perspective on what can be known, how to know it, and why it is of value (Hall and Horn 2012). Being an Algebra Group teacher like Tina and Carrie, for instance, required some commitment to the project of detracking. Such a project entails a certain epistemic stance that presumes ridding a department of ability levelling is a worthwhile and reasonable goal. Likewise, to participate successfully in the Algebra Group meant seeking collegial discussions to collectively interpret the work of teaching. Epistemically, this indicates a view of teaching expertise as partial and in ongoing development.

At a finer grain of analysis, we find other indicators of epistemic stance in teachers' conversations. Three interrelated features of conversations highlight any teacher group's epistemic stance on the work of teaching: *framings*, *epistemic claims*, and *category systems*.

Framings refer to the ways that problems are defined through activities and interactions (Goffman 1974). Problem definition, whether explicit through talk or implicit through the organization of activity, communicates an epistemic stance. By defining an activity as serving the goal of detracking, for instance, this communicates that this work is worthwhile and feasible. At the level of conversational routines, interactional organization that helps teachers coordinate schedules versus re-interpret classroom events (as described in the previous section) communicate very different epistemic stances on what can be known, how to know it, and what is of value.

Closer in the content of teachers' talk, we find all teacher groups use *epistemic claims*, or propositions that serve as the foundation for pedagogical reasoning (Horn and Little 2010; Horn and Kane in press). *Epistemic claims* occur in teachers' talk and focus the analysis of a teaching problem on any combination of teaching, students, or content. These statements may be more or less explicit, ranging from statements like, "Being consistent with routines help students understand expectations," to smaller claims like, "Starting a new unit is a good time to start fresh." While both of these *claims* express a stance on teaching and students, neither engages issues of mathematics.

In a comparative analysis of teacher groups at different levels of teaching accomplishment, *epistemic claims* turned out to be the primary window into the content of the teachers' collective thinking about students, teaching, and mathematics (Horn and Kane in press). In addition, teachers' use of *epistemic claims* revealed conversational processes that yielded important distinctions across the groups. The more sophisticated teacher group tended to use *epistemic claims* with greater frequency, grounding their pedagogical reasoning in well-articulated stances on teaching. In addition, they tended to use *multidimensional claims*, focusing simultaneously on teaching, students, and mathematics more often than either the beginning or emergent teacher groups. This reflected the sophisticated group's more complex view of teaching they expressed and exhibited in their own classrooms. In this way, *epistemic claims* reflected epistemic stance as expressions of how one knows something in the work of teaching.

Teachers' conversational category systems classify things in the world in everyday talk. For instance, a teacher might, as Tina did, refer to "slow" or "fast" students, a "hard" or "easy" class. A component of frames, these systems model problems of practice and communicate assumptions about students, subject, and teaching (Horn 2007). In the fuller paper that Tina's problem comes from, I analyzed how teachers' conversational categories for students played out in two different mathematics teacher groups who were faced with a Mismatch Problem. That is, they felt that their students' achievement levels were not well matched to their intended school curricula. As they talked through the problem, one group of teachers maintained static categories of student ability and motivation: students *were* fast, slow, or lazy. The only viable solution to the Mismatch Problem was to lessen the demand of the curriculum to accommodate the problem as they understood it. In contrast, Tina's group saw student abilities as malleable: students were fast and slow at certain mathematical things, but these descriptors did not fix student characteristics. In this light, teachers could shift the nature of activities to allow different students' strengths to come into play, keeping the curriculum's rigor intact.

Representational Adequacy

For people to learn together, they require a common object to reflect upon and examine. When teachers work together to deepen their understanding of teaching, a persistent challenge is that their conversations are generally outside of moments of practice. This asynchronous alignment of conversation and action demands some form of representing their work to provide grounds for consultation.

Representational adequacy describes the locally determined standards for constructing portraits of practice (Hall and Horn 2012). In some teacher groups, reports like, "The lesson went well," or plans like, "We'll teach Section 7.1" sufficed for participants to feel as if they had a grasp on the discussion. In other groups, these representations of teaching were not adequate, and teachers would provide or solicit a clearer window on classroom life (Little 2003). Three conversational means I have found for this representational work are *artefacts*, *replays*, and *rehearsals.*

Artefacts of practice include curricular materials, student work, calculators, or even slogans for reform. They serve as symbols for more complex activities, and, as such, their full meaning comes out through teachers' interactions. Nonetheless, they can anchor conversations by providing a common object for teachers' conversations.

Much of the complexity of ambitious teaching practice occurs in the course of interaction. Artefacts typically cannot fix classroom dialogue as a common object (unless they are observation notes or a video tape). Because these discussions about interaction are critical to the success of ambitious teaching, workgroups often find ways to represent this facet of practice. I have found this representational work done in two ways. Retrospectively, teachers *replay* classroom events, reporting classroom dialogue, often embedded in a story. Prospectively or generally, teachers *rehearse* classroom events (Horn 2010).

The issue of representational adequacy becomes interesting in the analysis of replays and rehearsals in teachers' conversations. Like principles and artefacts, the teachers more adept at sophisticated forms of practice represent their work with greater specification and complexity. "We'll teach 7.1" would be a wholly insufficient rehearsal for a sophisticated group. Consider this excerpt of a sophisticated teacher examining a possible lesson on standard deviation. The underlined text is rehearsal talk that could be directly imported into the classroom:

> He says aloud, <u>"So standard deviation helps decide what does 'in here' mean.</u> Whether they learn/go through an algorithm or this messy thing that doesn't make much sense, or whether they just get it from their calculators, either way the connection to something real… <u>Is it in *this* area?"</u> He points to the area underneath the normal curve that would be within two standard deviations of the mean. He pauses and then continues. <u>"What's the difference between the data point and the mean? What's the square root all about? To get rid of the negative…" He looks up at his colleagues around the table and says: "Try out this algorithm." He points to the algorithm for finding standard deviation. "Try explaining each step."</u>

As he examined a piece of curriculum, the teacher's evaluation did not come from simply reading and evaluating it. Instead, he imagined it enacted in the classroom via rehearsal talk, trying to understand how it will help his students understand the key mathematical idea. In other words, the curricular artefact was not adequate for representing the classroom but required the enhancement of instructional language.

Sophisticated teacher workgroups consistently produce more complex representations. In particular, the teachers' replays and rehearsals simultaneously portray student and teacher voices in interaction (Horn and Kane in press). This multivocality indicated a stance on representational adequacy of classroom activity. Specifically, it was not enough to represent teaching through only utterances of teachers *or* students. Instead, interactions became a key point of inquiry in the workgroup conversations, requiring the reported speech of both parties. Such representations fix interactional aspects of teaching for collective interpretation.

Discussion

Taking these different pieces of analyses together, I propose that teachers' collaborative conversations stand to provide opportunities for practice-based understandings of ambitious instruction. Interactionally, teachers revisit important classroom events in consultation with their colleagues, interpreting them, considering alternative responses, and linking them to other instances of practice.

The four components of teachers' conversations described in this paper work together to support these professional learning opportunities. In the most productive examples of teacher workgroups, teachers' conversations accrue complex representations of practice linked to certain principles. The principles serve as the interpretive lens for these instances of teaching, embodying epistemic stances on what is knowable in teaching and how one should go about learning it. Routines of interaction allow for the description and elaboration of classroom practice, communicating local standards for representational adequacy. Teachers' engagement in productive groups supports their work toward ambitious goals and their sustaining complex forms of teaching practice. This engagement is important to help them air their ideas and find meaning in their colleagues' responses to their problems of practice.

Although teacher collaboration is not a regular part of teachers' work in the United States, it seems to have a role in the support of ambitious forms of mathematics teaching. By highlighting the aspects of conversations that open up opportunities for professional learning, this framework provides an analytic tool for both researchers and practitioners seeking to support productive conversations. Instructional improvement efforts would be well served by providing opportunities for teachers to consult with colleagues in ways that support ongoing learning through practice.

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Transforming Education Through Lesson Study: Thailand's Decade-Long Journey

Maitree Inprasitha

Abstract The development of teaching and the teaching profession is an issue countries around the world have been struggling to solve for many centuries. Lesson study, a Japanese way of professional development of teachers, dates back nearly 140 years, in 1872 the Meiji government invited foreign teachers to teach Japanese teachers about "whole class instruction" (Isoda 2007). Ironically, in 1999, Stigler and Hiebert brought back to the U.S. the same idea on how to present whole class instruction, "If you want to improve education, get teachers together to study the processes of teaching and learning in classrooms, and then devise ways to improve them" [Stigler 2004 cited in Fernandez and Yoshida 2004]. Although the education reform movement around the world calls for effective reform tools or even ideas like Japanese lesson study, transferring those tools/ideas to other socio-cultural setting in other countries is not easy and always complicated. Thus, education reform movements sometimes support but sometimes hinder movement of society. Taking Japan as a case study, Japan has undergone the movement of society from agricultural to industrialized, to information, and knowledge-based society during the two centuries since the late 18th century to the present. Not visible to outside people, an evolution in the approach to school has taken place in Japan, which supports the movement of society, which has not occurred in most developing countries, including Thailand. Thailand has looked to Japan for ideas and has been implementing lesson study since 2000 but with a unique approach to adaptation. Thailand's experience with lesson study has been shared with APEC member economies over the last six years and has been deemed "quite a success" in improvement of teaching and learning of mathematics.

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Introduction

Education reform around the world has moved in the same direction but has begun at different times. It sometimes has supported but other times hindered the movement of society. Japanese society has transitioned from agricultural to industrialized to information, and now a knowledge-based society over the two centuries between the late 18th century and the present. Not visible to outside people, an evolution of school approach has occurred, which supports the movement of society; such a change has not taken place in most developing countries, including Thailand (Fig. 1).

Traditionally, most mathematics classrooms in Thailand depend heavily on following the national textbooks. The textbook format consists of with introducing some definitions, principles, rules, or formula followed by some examples, and ends with some assigned exercises. Unfortunately, most exercises are closed problems, which have one and only one correct answer. These textbooks have influenced teachers' teaching style. Mathematics teachers' teaching script begins with explaining new content, provides some examples, and ends with assigning students some exercises. This method of teaching is prevalent in the classrooms, where students cannot initiate their own learning and become passive learners. Such school approach in Thailand does not support the movement of Thai society, as compared with Japan. Over four decades from 1960 to 2000, Thailand had only two major educational reform movements, in 1975 and 1999. The first Educational Act was enacted in 1999 "To reform the learning process" as a national agenda. In response, the Ministry of Education implemented a new core standard curriculum, which demands that school teachers integrate subject matter, learning process and skills, and desirable characters when implementing the curriculum. However, it is not easy for them to comply the Educational Act, and the traditional approach to teaching persists among non-affluent countries. During the last decade, Thailand has been introducing the idea of lesson study into Thai schools and sharing its progress with other APEC member economies.

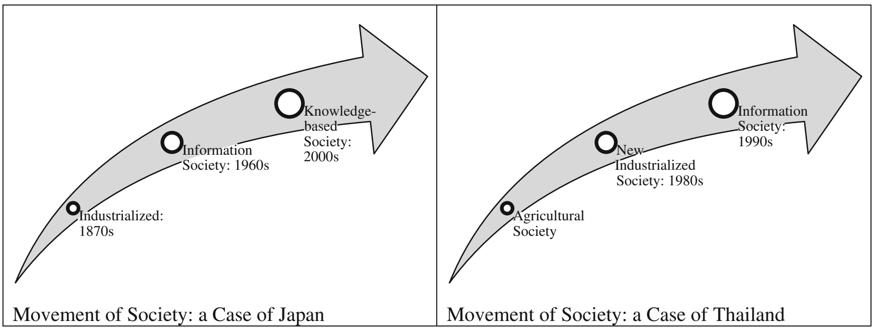

Fig. 1 Movement of society

Adaptive Innovation

The term "Lesson Study" was first translated by Makoto Yoshida from the Japanese term "Jugyo Kenkyu" and was popularized by Stigler and Hiebert (1999) in the Teaching Gap. If lesson study is a way to improve daily teaching practice or teaching profession "*Kounaikenshuu*" (*in school professional development*) is the word used to describe the continuous process of school based professional development that Japanese teachers engage in once they begin their teaching careers (Yoshida 2007). One of the most common components of *Kounaikenshuu* is Lesson Study (Stigler and Hiebert 1999). Lesson study is an innovation that was developed and implemented in Japan around 140 years ago (Isoda 2004; Shimizu 2006) and has been attracting attention around the world (Isoda and Nakamura 2010). It has been recognized and used for teacher professional development in many countries around the globe. In Thailand, lesson study started in 2002 by preparing the context for applying innovation. Several possible areas of implementation were identified and these included: (1) the teacher preparation program, (2) the graduate study program, (3) in-service teacher education, and (4) long term teaching professional development which was used with fourth year students practicing their internship in 2002 (Inprasitha 2011).

The implementation of lesson study was supplemented with the 'Open Approach' using open–ended problems in mathematical activities with fifteen 4th year student teachers teaching in seven secondary schools in Khonkaen City in 2002 academic year. Lesson study was implemented implicitly without using the term "Lesson Study" (Inprasitha 2004, 2007). This phase was called "*Incubation of the Idea*" (Fig. 2; Inprasitha 2011).

In the years 2002–2005, the open approach was expanded to two districts in Khonkaen Province. More than 800 teachers were introduced to the use of open-ended problems to help them create rich mathematical activities in their classrooms. This phase was called "Experimentation in Some Schools" (Fig. 3; Inprasitha 2011). In 2004, leading teachers were expected to have the understanding and skills in the development of lesson plans by using lesson study. In addition, they were required to implement the plans in actual classrooms, to follow up on the results, and to share their practice with

Fig. 2 "Incubation of Idea" in 2002 academic year

Fig. 3 “Experimentation in some schools” in the years 2002–2005

other teachers in their district. The project evaluation was carried out through project exhibitions and presentations. Based on the positive outcomes, the project has proved to be valuable in teacher development. All leading teachers were interested in participating in the activities in order to fully enhance their capacity. The continuous effort in expanding the knowledge to the network teachers in the other education areas was one of the evidence showing the success of this study (Inprasitha 2006).

In 2006, the Center for Research in Mathematics Education (CRME), Khon Kaen University, Thailand and Center for Research on International Cooperation in Educational Development (CRICED), University of Tsukuba, Japan proposed to APEC the project “A Collaborative Study on Innovations for Teaching and Learning Mathematics in Different Cultures among the APEC Member Economies”. This project was unique in terms of collaboration. It was an attempt between Japan as a developed country and Thailand as a developing country to seek collaboration among APEC member economies to create a collaborative framework. Lesson study was selected by the delegates (specialists) from the participating economies as the collaborative framework. Each specialist with the collaboration of his/her network in each economy has been developing some “good practices” in teaching and learning mathematics that will lead to innovations in that economy (Inprasitha et al. 2006).

During 2006–2008, the CRME started a long-term collaborative project with the Ministry of Education in “Improving Mathematical Thinking using Open Approach through Lesson Study Approach.” This phase was called “Whole School Approach” Three basic phases of lesson study were incorporated beginning with the collaborative design of a research lesson (Plan), the collaborative observation of the research lesson (Do), and the collaborative discussion and reflection on the research lesson (See). In addition, in 2008, the first year cohort of student interns in schools carried out their practice in collaboration with the in-service teachers by implementing lesson study in the project schools (Inprasitha 2011).

In 2009, lesson study and open approach had been implementing in 22 schools in the northeastern and northern parts of Thailand, a collaborative project with the Office of Basic Education and Office of Higher Education, Ministry of Education. In 2010, 60 mathematics student teacher interns from Khon Kaen University and

Fig. 4 Mathematics student teacher interns participated in school project

six mathematics student teacher interns from Chiang Mai University practiced teaching at 22 project schools (Fig. 4).

Unit of Analysis for the Lesson Study Cycle

Lesson study is a direct translation for the Japanese term Jugyo Kenkyu, which is composed of two words: Jugyo, which means lesson, and Kenkyu, which means study or research (Fernandez and Yoshida 2004). The author pioneered the introduction of lesson study and the open approach into Thai mathematics teaching circles. The term "Lesson Study" was paraphrased by him to mean "Classroom Study" in order to make it comprehensible in the Thai context. This meaning is different from the meaning used in Japan because in Japan, the unit of study is the "lesson", while in Thailand, the unit of study is "classroom". The purpose of introducing these innovations into Thai classrooms was to improve the quality of classrooms using lessons as a tool for teachers to know their classrooms better from the angles of knowing their students, understanding their ideas, realizing their own roles and recognizing the classroom culture. Therefore, to introduce innovations into classroom practice it is essential to adjust the steps or processes to fit in with the working culture of each locality. In Thailand, the application of "Lesson Study" as the main means for enhancement of the mathematics teaching profession consisted of three major steps: collaboratively design of a research lesson (Plan), collaboratively observe the research lesson (Do), and collaboratively discuss and reflect on the research lesson (Figs. 5 and 6; Inprasitha 2010, 2011).

In the case of Thailand, the development of the teaching profession under the tenets of the three steps of lesson study and the open approach was initiated in order to make way for a discussion about problems the teachers have been facing, such as teaching activities that depend heavily on lecturing, explaining and asking-answering questions tersely like "right" or "wrong", avoiding sufficient time for student participation, failing to draw out the students' ideas, or to observe their ways of thinking while engaging in problem-solving activities, etc. The author has embarked on the development of the open approach as a new method of teaching

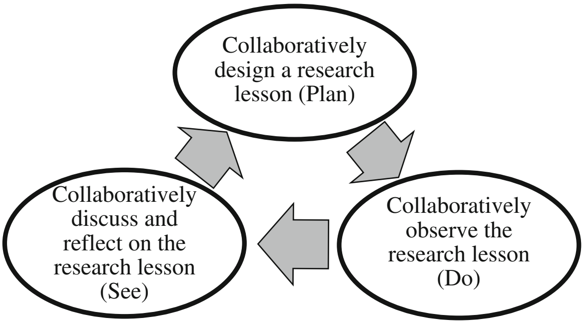

Fig. 5 Adaptive lesson study in Thailand

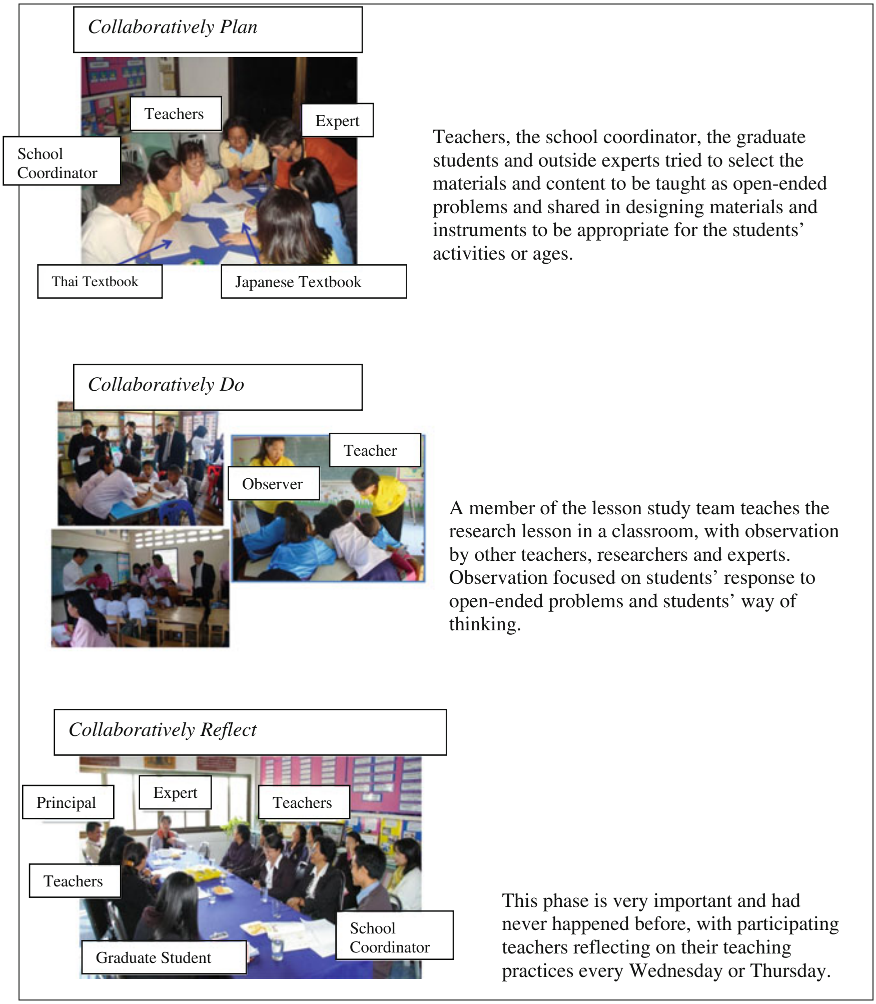

Fig. 6 Collaborative plan, do, and reflect

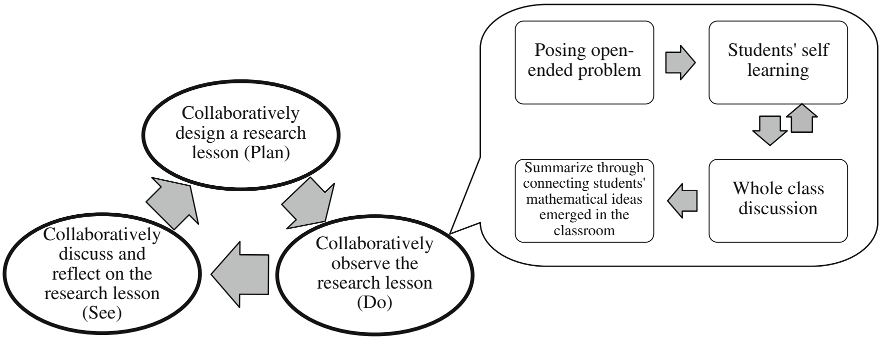

Fig. 7 Four phases of the open approach incorporated into lesson study

that emphasizes problem-solving process (Inprasitha 2010, 2011), which is integrated with lesson study. The four phases of this teaching approach are: posing of open-ended problem, students' self learning through problem solving, whole class discussion and comparison, and summary through connecting students' mathematical ideas that emerged in the classroom (Fig. 7).

How to Form Lesson Study Teams

The most difficult part of implementing lesson study in schools in Thailand is how to form lesson study teams. We do not have senior or expert teachers in schools like those of Japan. We also lack external knowledgeable persons (knowledgeable others) to support the schools. In order to have effective lesson study team in the project school, Faculty of Education, Khon Kaen University has prepared our graduate students in master degree programs in mathematics education, which was first offered in 2003 as part of the workshops organized by the Faculty during 2003–2005. We organized our workshop into small groups mixing both teachers, school principals and supervisors. The graduate students observe the working groups and then reflect on what they observed after the group members presented their work.

The graduate students provided a chance for school teachers to reflect on their traditional roles. In 2006, when we started to fully implement the idea of lesson study and the open approach, our graduate students were assigned as members of lesson study teams working closely with teachers at the school that served as their research site. Thus, each lesson study team is composed of three classroom teachers from grade 1, 2, and 3, a graduate student, one teacher from another grade (option), the principal (who mostly attended the reflection session). A team for grades 4, 5 and 6 or for grades junior 1, 2, and 3 is formed in a similar manner. Three steps of lesson study have been practiced as follows: Monday or Tuesday was set for collaboratively planning the lesson for each team. One teacher teaches according to the usual timetable during the week. Then, all teachers in that school with the school principal joined the reflection session at the end of the week, on either

Thursday or Friday. The author adapted many steps of lesson study by putting a revision step into yearly cycle. This made it possible for the three-step lesson study to take place on a weekly cycle. Thus, we can plan to do lesson study every week while still covering all the content for which teachers are responsible. This adaptive version allowed teachers to be more comfortable using innovations like lesson study and open approach in their classroom. They feel like they have outside knowledgeable persons to help them improve the classroom, rather than feeling that they are burdened with more extra work.

Khoo Kham Pittayasan School is an extended school (1st grade to 9th grade). There were 180 students and eighteen teachers during 2010 academic year. The school has been participating in the project since 2006. In 2006, the school implemented three phases of lesson study in the 1st grade, the 4th grade and the 7th grade. In 2007, they extended lesson study to six classrooms: 1st grade, 2nd grade, 4th grade, 5th grade, 7th grade and 8th grade. From 2008 to the present, they extended lesson study to nine classrooms: 1st grade, 2nd grade, 3rd grade, 4th grade, 5th grade, 6th grade, 7th grade 8th grade and 9th grade. For all 18 teachers to participate in the three phases, especially the reflection phase, the school arranged the schedule (Table 1). The school principal took leadership in the reflection phase and it is obligation for all school teachers had to participate in this phase.

Lesson Study Team Member Participation

According to the schedule members of the lesson study team could participate in the lesson study activities in weekly cycles. The author surveyed the participants to determine to what extent lesson study teams were involved in lesson study activities.

During the "Plan" phase, it involved the researcher, school coordinator, co-researchers, and participant teachers to collaboratively design a research lesson (Plan). During this phase mathematics problem activities were chosen using open-ended problems based on a Japanese mathematics textbook. The materials to be used in the classroom were then developed. This was conducted once a week. Figure 8 shows the percentage of participating members during the Planning Phase at one of the participating schools, the Khoo Kham Pittayasan School.

During the "Do" phase, the lesson study group collaboratively observed the research lesson (Do) and implemented the lesson plan of the school teacher in the classroom. In addition, the classroom teaching was observed by the research team, school coordinator, co-researchers, and other teachers. The objective of the observation focused on the students' thinking approach, and not on the teacher's teaching competency. Figure 9 shows the percentage of participating teachers in the "Do" phase at the Khoo Kham Pittayasan School.

During the "See" phase, the team collaboratively discussed and reflected on the research lesson, and examined the findings of the teaching observation for improving the research lesson. The research lesson was then revised with a view of using it again in the following year. This phase was conducted once a week.

Table 1 Lesson study schedule

Grade	"Plan"	"Do"	"See"	LS team
1st	Every Tuesday starting at 14.30	Monday, the 1st period Tuesday, the 1st period Wednesday, the 1st period Thursday, the 2nd period Friday, the 2nd period	Every Thursday starting at 14.30	1st grade teacher, junior high school science teacher, 1st grade mathematics student teacher intern
2nd		Monday, the 2nd period Tuesday, the 3rd period Wednesday, the 1st period Thursday, the 1st–2nd periods		2nd grade teacher, 2nd grade mathematics student teacher intern, 3rd grade mathematics student teacher intern
3rd		Monday, the 3rd period Wednesday, the 2nd–3rd periods Thursday, the 3rd period Friday, the 1st period		3rd grade teacher, 3rd grade mathematics student teacher intern, 2nd grade mathematics student teacher intern
4th		Tuesday, the 1st–2nd periods Thursday, the 1st–2nd periods		4th grade teacher, 4th grade mathematics student teacher intern, 5th grade mathematics student teacher intern
5th		Monday, the 1st–2nd period Wednesday, the 2nd–3rd periods		5th grade teacher, 5th grade mathematics student teacher intern, 4th grade mathematics student teacher intern
6th		Monday, the 2nd–3rd periods Tuesday, the 3rd period Wednesday, the 3rd period		6th grade teacher, 6th grade mathematics student teacher intern, 1st grade mathematics student teacher
7th		Monday, the 1st period Thursday, the 2nd–3rd periods		junior high school mathematics teacher, 7th grade mathematics student teacher intern
8th		Tuesday, the 2nd period Wednesday, the 2nd–3rd period		junior high school mathematics teacher, 7th grade mathematics student teacher intern
9th		Thursday, the 1st period Friday, the 2nd–3rd periods		junior high school mathematics teacher, 7th grade mathematics student teacher intern

A unique feature of this phase is that the school principal took leadership in running this session and this motivated all the teachers in school to attend the session. Figure 10 shows the percentage of the participating members of the lesson study group at the Khoo Kham Pittayasan School in 2010.

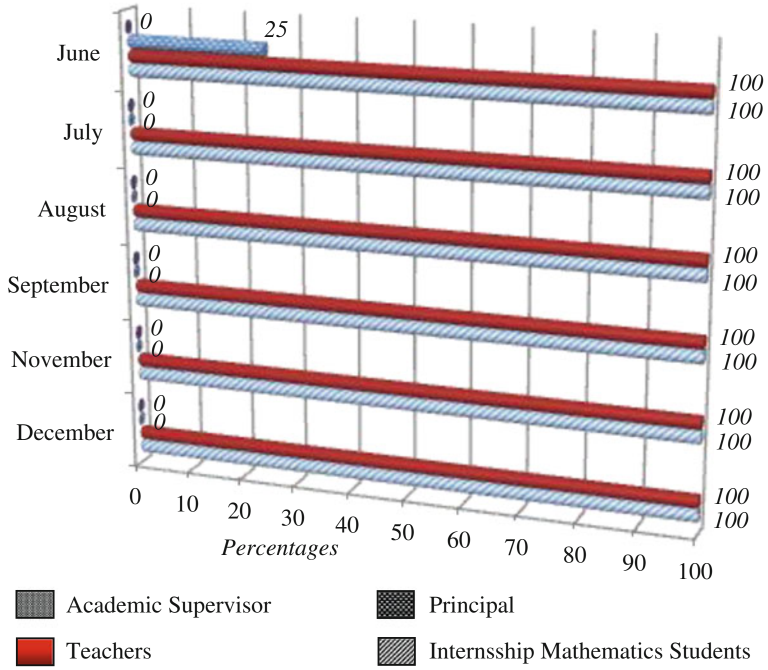

Fig. 8 Percentage of participating teachers during the “Plan” phase at Khoo Kham Pittayasan School in 2010

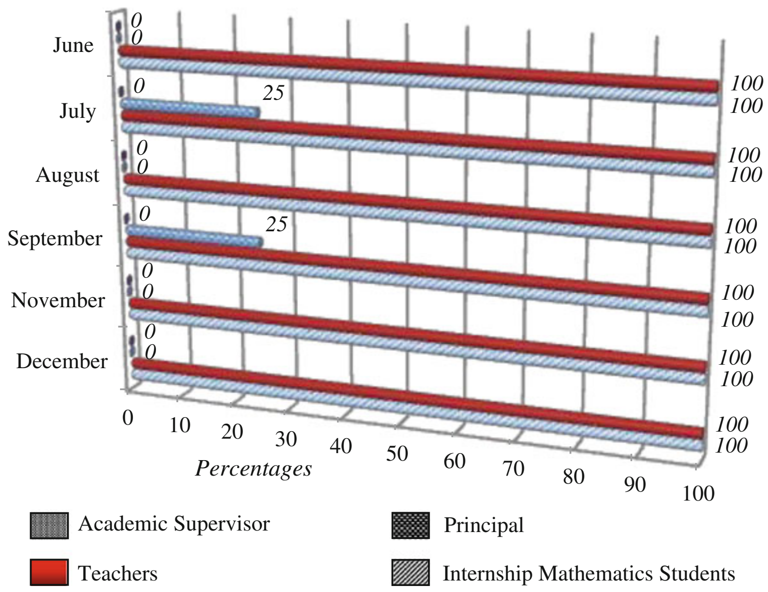

Fig. 9 Percentage of participating teachers in “Do” Phase at Khoo Kham Pittayasan School in 2010

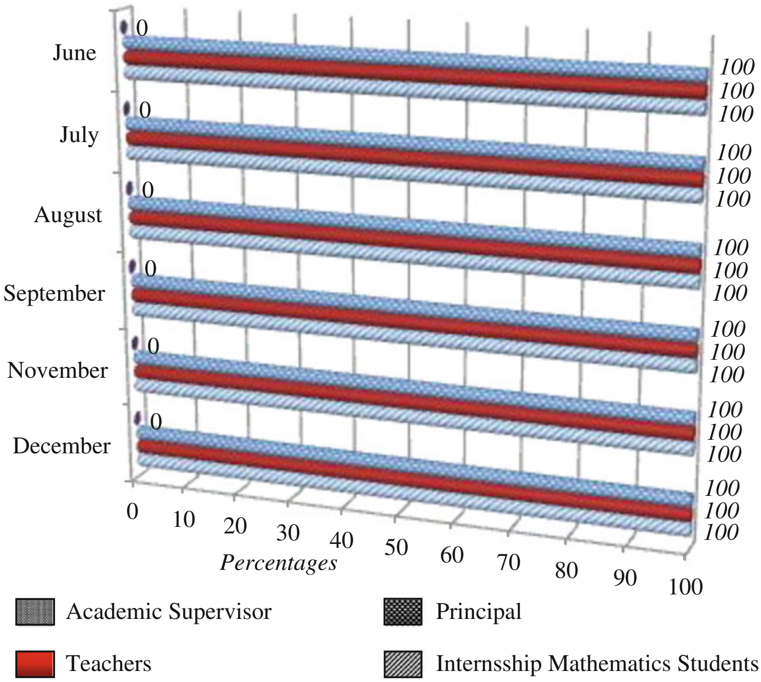

Fig. 10 Percentage of participating teachers in the "See" Phase at the Khoo Kham Pittayasan School in 2010

Eliminating the "Revision Step" from the Weekly Cycle

Most of the teachers were concerned about covering all the content specified in the curriculum. So, we implemented lesson study in weekly cycles to cover all the content they have to teach. Only the basic steps of lesson study, that is "***Collaboratively plan***", "***Collaboratively do***", "***Collaboratively reflect***", were implemented during the *weekly cycles*. Teaching the revised lessons in the weekly cycle seemed to be difficult for the teachers. When adding the fourth step; "***revise the lesson***", the cycle was changed to a yearly cycle.

Conclusion and Suggestion

The Japanese professional development known as lesson study and the organization of learning activities based on the ideas of the open approach are rather new to the conventional Thai education context and culture. So, in any attempt to apply the ideas to the development of the teaching profession in Thailand it is necessary to take these two factors into account. It has been our experience when attempting to prepare ground for the introduction of the ideas into Thailand that the Thai social and cultural contexts are very influential in shaping the mode of developing the Thai teaching profession. It can be seen clearly in the teachers' role in the mathematics learning activities. It must also be noticed that the teachers' role is one of the factors that influences the students' way of expressing their thoughts. A change from the teacher's former role that emphasizes giving lectures, doing exercises on the board as examples for the students to see, and then drawing conclusions at the end of each lesson, to a new role of organizing learning activities that emphasizes students learning through the open approach is crucial. The teachers must also avoid behaving like a knowledge provider or try to give hints for correct answers. Rather, they should act in such a way as to stimulate the students to think by switching from the normal way of asking questions with the purpose of verifying the correct answers, to a new mode of inquiry that stimulate students to reflect on their own ideas. All of these ideas and practices are new to Thai teachers. Indeed a longer period of time is needed for the development of the teaching profession and for the change in teachers' teaching behavior.

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Dialectic on the Problem Solving Approach: Illustrating Hermeneutics as the Ground Theory for Lesson Study in Mathematics Education

Masami Isoda

Abstract Lesson study is the major issue in mathematics education for developing and sharing good practice and theorize a theory for teaching and curriculum development. Hermeneutic efforts are the necessary activities for sharing objectives of the lesson study and make them meaningful for further development. This paper illustrates hermeneutic efforts with two examples for understanding the mind set for lesson study. The first example, the internet communication between classrooms in Japan and Australia, demonstrates four types of interpretation activities for hermeneutic effort: Understanding, Getting others' perspectives, Instruction from experience (self-understanding), and the hermeneutic circle. Using these concepts, we will illustrate the second example with dialectic discussion amongst students in the problem solving classroom engaged in a task involving fractions.

Introduction

From early 1980s, mathematics educators have established various theories of understanding using cognitive models. These theories have illustrated such various evidence as described by qualitative research methodologies. Those evidence-based theories well illustrated students' understanding and what really happened in the classroom. Those evidences are meaningful for social scientists who are working in the office and go to the classroom for finding something which no one in the office knows about it. In the academic society of the scientists, knowing means using specific frameworks for description and special setting for using procedures which are recognized as scientific in their society. Everyone, including myself, believes that those activities are the basic methods for academic and scientific research in mathematics education which are necessary for developing general theories for mathematics education as an academic discipline.

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On the other hand, even if Newton did not explain the free fall motion of the falling apple, children would still enjoy catching ball by their own efforts and improve their ability for baseball in their team. What is the necessity to explain such activity by cognitive models which are developed by researchers? If a child began to throw the ball along various paths, whether straight or curved, his friends and observing coach would interpret that he is challenging a new ball like a pitcher. Teachers usually observe children for understanding what they are doing for supporting their effort. Children here are engaged in activities which are related to their objectives, aims, or wishes. Through synchronizing willingness of each other, good teachers usually make efforts to set good tasks with their objectives for sharing with children, attentively observing children's thinking and trying to extend children's competencies. They usually recognize children's misconceptions before researchers have a chance to find them. How can we theorize those teachers' recognitions with their practical wisdom? Their wisdom has been called pedagogical content knowledge. How can we share and theorize it for teachers?

Before Newton, many scientists such as Galileo, had already explained the motion with constant acceleration. Through the decades, scientists read the same textbook such as 'Discourses and Mathematical Demonstrations Relating to Two New Sciences' for interpreting the nature and re-organize their theories. The same subject appears recursively in the history of mathematics and in the textbooks for mathematics education. For showing evidence, historians may interpret that they must meet similar difficulties and try to find the evidence from their historical texts.

It is easy to say that these notions of teachers and students are necessary for them and not the matters for the theory for observation because they are living and working in different societies or paradigms. However, beyond this sectionalism, how can we develop the grand theory for teaching and learning for teachers who develop their children? What does it mean to establish such theory for teaching and learning for teachers and children? How can we make it practical? Lesson study is one of the methods whereby teachers can develop a fundamental theory for teaching.

On the World Association of Lesson Study Conference 2011, Yrjö Engeström mentioned that various theories in education introduced until today did not treat the objective/aims of teaching. Instead of the activity of social scientists, Lesson Study is the activity of sharing objective for teaching among teachers, children, and observers for better practice. Then, in Lesson Study, what kind of methodology would be meaningful for sharing and synchronizing their objective? Here, I propose hermeneutics, introduced by Jahnke (1994) in the PME plenary with the title 'Objectifying the Subjective'. Indeed, to ensure being scientific, mathematics educators have been focusing on the objective evidence in the subjective object.

Hermeneutics is a general theory for interpretation of human view of all natural sciences, art and literature, and one of the ways to humanizing mathematics education through objectifying subjective understanding of others. Because it is a fundamental theory of the methodology of qualitative studies, many mathematics education researchers deeply relate to it or implicitly apply it, but there are not so many researches who openly and explicitly have applied it for mathematics education.

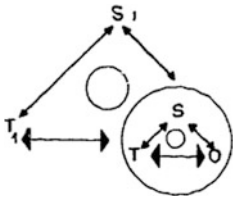
This paper is the extraction from a forthcoming paper which will explain how to design a mathematics class based on the problem solving approach. In this extraction, the author proposes and emphasises the importance of hermeneutic effort as the methodology to theorize the wisdom of lesson study. Firstly, hermeneutics will be defined by two examples which are intended to explain it.

Hermeneutics for the Theory of Understanding Others

Here, hermeneutic effort is defined and applied for analysing mathematical activity and enabling to explain it in a more subjective perspective for humanizing mathematics education.

Hermeneutics in Mathematics Education Research

In mathematics education research, major proposers of hermeneutics were Jahnke (1994) and Brown (1997). Brown clearly described its historical development and focused on current theoretical issues in relation to language for developing research methodology. Jahnke (1994) introduced hermeneutic effort as an activity of the recursive process of interpretation for explaining the role of history for education as follows;



> My principal thesis is that the concept of "hermeneutics" is suitable to describe the pedagogical interaction between synchronous and diachronous culture. …My thesis implies the claim that the historian's perspective represents an important element of an appropriate teaching culture. Seen under the aspect of method, history of mathematics, like any history, is essentially a hermeneutic effort. Theories and their creators are interpreted, and the interpreter is always aware of the hypothetical, even intuitive character of his interpretation. Interpretation itself takes place within a circular process of forming hypotheses and checking them against the text given, in the case of history of science, the objects of his process of interpretation, the scientific subjects (individuals and groups), are again hermeneuticians who interpret fields of objects. Of course, this view of scientific work will be adequate with varying precision in different times and different fields. But if we do not understand it too narrowly, this description can very well be advanced. Scientific interpretation, too, is now subject to the circular process of forming hypotheses, testing and revisiting them. He who is concerned with history, thus, has to do with a complex network of interpreters, problem fields, and interpretations (theories) which I have represented in a little diagram and which I should like to name the "twofold circle". This diagram consists

> of: Primary circle in the right bottom representing the circular relation between a scientist (s), a theory (T) and a field of objects (O) and a large secondary circle representing the historian (S1), a historical interpretation (T1) and the primary circle as his field of objects. My proposition that the teacher should know and understand something about the historian's perspective if he takes history of mathematics into the classroom, refers precisely to the problem that he/she must be aware of this twofold circle and able to move within it. Only this will enable him and his students to acquire a certain freedom against the subject matter to form hypotheses and to be ready to think oneself into other persons who have lived in another time and another culture. For me, this thinking oneself into another person and into a different world seems to be the core of an educational philosophy, providing a basis for historical contents in mathematics teaching (Jahnke 1994, pp. 154–155).

Here, from Jahnke's perspectives, the focus is on hermeneutic efforts which characterize trying to get others' perspectives. At the same time, for clarifying the meaning of hermeneutics within the diversity of the meanings of hermeneutics, here, we introduce the word 'hermeneutic effort' for understanding human activities from the perspective of hermeneutics. Hermeneutic effort is the activity to objectifying the subjective.

Hermeneutic Effort for Clarifying Humanization

Based on various theories of hermeneutics and examples of interpretation (e.g., Isoda et al. 2000; Isoda and Tsuchida 2001; Arcavi and Isoda 2007), Isoda characterized hermeneutic effort as an activity according to four principles: "Understanding," "Getting others' perspectives (the assumption of the positions of others/imaging others' minds)," "Instruction from experience (self-understanding)," and "The hermeneutic circle."

"Understanding" is one's interpretation regarding a text (or other objects). "Getting others' perspectives (the assumption of the positions of others)" means that the appropriate interpretations of a text (or other objects) is only possible through a subjective approach whereby we assume the writer's (or speaker's) position, feelings and sympathetically attempt to put ourselves into the position of another (writer or speaker). "Instruction from experience (self-understanding)" means that when one interprets assuming the position of another, one's own subjective opinion (at times, one's preconceived opinion) is reflected, in other words, one obtains an instruction about one's self with comparison of others' perspectives. "The hermeneutic circle" refers to the cycle of hypothetical interpretation and confirmation from further readings or interaction with other object such as textual interpretation whereby understanding the particulars. It broadly contributes to the whole, and overall understanding contributes to understanding the particulars, but broadly refers to the fact that a recursive or multi-layered advance in interpretation leads to a more objective interpretation: if we have some understanding, we apply it to new situations and if it is applicable, it will become more objectively correct.

In particular, "getting others' perspectives (the assumption of the positions of others)" and "instruction from experience (self-understanding)" are acts subjectively carried out through the empathy of the interpreter toward the object of understanding; accordingly the objectivity of interpretation can be stipulated in the subjectively shared act of empathy. Through this, one can see man attempting to recognize mankind as an existence able to think from another's perspective—an existence equipped with a nature that empathizes with others and comprehend human acts by such human subjectivity.

Those four principles for characterization of the hermeneutic effort were identified by the author from various hermeneutic theories and reflecting on personal interpretations of mathematics history and a hundred classroom experiments, which enabled students to perform hermeneutic efforts in mathematics on historical subject matter.[1] Here, four principles are illustrated at first and then, confirmed with some references.

An Example for Illustrating the Four Principles of Hermeneutic Effort

For illustration of the four principle of hermeneutic effort, here we read and interpret the classroom communication between Japan and Australian high school through the Internet (Isoda et al. 2006). The theme was to determine the attributes of the sums of consecutive numbers (see Figs. 1 and 2).

As Isoda et al. (2006), on the meaning of Jahnke (1994), there are two different dimensions of activities which can be noted—the students participating communication are seen as synchronous and the observing researchers interpreting their communication dialog and data are seen as isochronous. And interpreting those kinds of activities, he enhanced the role of hermeneutics. Here, we interpret this extract for illustrating the four principles of hermeneutic effort: Understanding, Getting others perspective (the assumption of the position of others), Instruction from experience (self-understanding), and The hermeneutic circle.

Each students involved in this communication conducted hermeneutic effort. First, in (B), an answer came from the Australian side that expressed three consecutive numbers as x, y and z, which differed from the Japanese customary ways of expressing algebraic expression. In (D), the Japanese side limited themselves to self-introduction, and in (E), the Australian side politely urged and showed concern for the Japanese side's failure to send an answer. In (F), the Japanese side gauged how the Australian side would respond to answer (G), which consisted of algebraic generality when sent. Simultaneously questions were submitted in (I) and (J). In (M), the Australian side explained using the Japanese side's ways of expression.

[1]A huge number of the experimental study of lesson study were done on this theory by Isoda, his lesson study team and his graduate students. Most of the books in Japanese by Masami Isoda on Worldcat are the product of it. Using original historical resources, their produced materials and classroom activities can be seen on http://math-info.criced.tsukuba.ac.jp/Forall/project/history/.

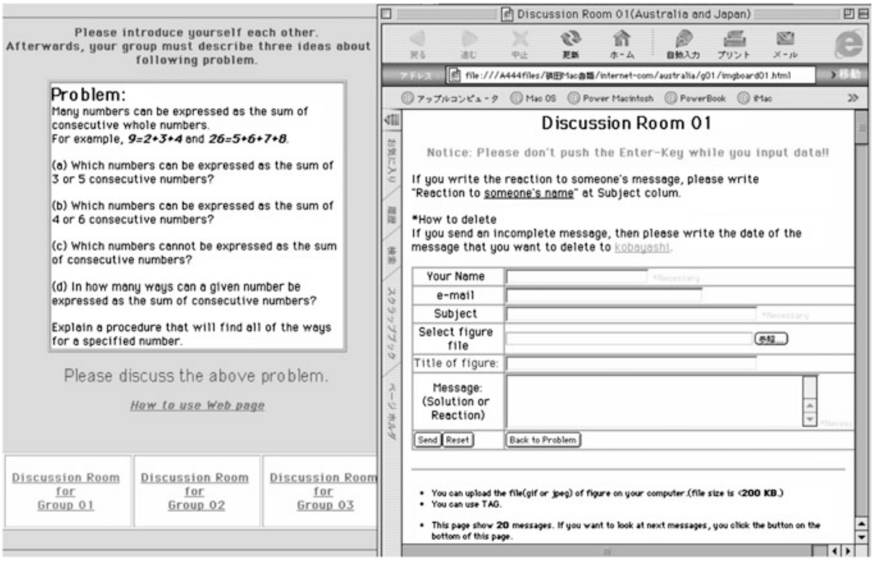


Fig. 1 A problem and discussion room

Fig. 2 The discussion between Japan and Australian

☐ Reacton and our idea to solve (a) ,(b) name :Y K [1999/10/30,22:20:51]

(F) #As you know, we are not so good at writing English. So please let us know if you don't understand.

(OMISSION)

We read your message. The answer is same as ours. But we solve it in a differnt way. I think this way is easier than yours. You used three letters. But to use only X is easier. I'll show you our way.

---- part (a) ------

(G) #In this problem, we have to think 3 consecutive numbers and 5 consecutive numbers separately.

< 3 consecutive numbers >

Let the first number be X. As three numbers are consecutive, the next number must be (X+1).
In the same way, the last number must be (X+2). So the sum of these 3 numbers is...
X+(X+1)+(X+2)=3X+3 =3(X+1)
X will be natural number. (It can be taken for only integer which includes negative numbers.)

(OMISSION)

(H) Gentlemen and Gentlemen! (You are only boys) I hope you will understand the meaning of this expression.
Actually, when X=2 3(X+1)=9, when *(OMISSION)* Anyway the answer is <u>multiple of 3 bigger than or equal to 6</u>

< 5 consecetive numbers >

(OMISSION)

(I) #Question from us (1)
About "A" consecetive numbers. When "A" is an odd number, you can express the sum as multiple of "A".
When "A" is an even number, you can't express the sum as multiple of "A". Can you tell us why?

---- part (b) ------

#We considered part (b) in the same way. I'll show you waiting for your pointing out our mistakes.

< 4 consecetive numbers >

(OMISSION)

(J) The expression 2(2X+3) means that when X increases 1, the answer increases 2.
The answer is <u>multiple of 2 bigger than or equal to 12</u>

< 6 consecetive numbers >

(OMISSION)

#Question from us (2)

(K) We considered this problem over an basic condition. It is that the "numbers" means natural numbers.
But as I discribed before, "numbers" can be taken for integer which includes negative numbers.
If "numbers" means integer, how does the answer change?

---- Message -----

(L) Are you happy? Be happy! *(OMISSION)*

☐ Internet Project name :P P [1999/11/03,07:30:13]

(M) If negative numbers were included then the answer would be the same, but include all the answers as a negative as well as the positive.

part (b)

(N) The lowest number is 10 , this is because the numbers can be represented as (x, x+1, x+2, and x+3.) This is for the addition of 4 consectutive numbers.This works out as 4(x+1.5) As with your solution for part (a), It goes up in multiples of the amount of adding consecutive numbers, in this case, 4. This meanns the values are 10. 14. 18. 22. 26, etc...

For 6 numbers...

(OMISSION)

☐ Let's think about part (c)! name :Y K [1999/11/05,21:05:34]

(O) We read your letter. Your answer of <u>#Question from us (2)</u> was perfect!
If negative numbers are included, there is no minimum value. I think we discussed enough about (a) and (b).
But, have you discussed on <u>#Question from us (1)</u> in your group?
I will tell you the answer of it in the next letter. Please think about it again before the next letter comes.
Anyway, we want to go to part(c).

(OMISSION)

(P)
This chart means 1+2+3+4+5+6
0
0 0
0 0 0
0 0 0 0
0 0 0 0 0
0 0 0 0 0 0
It is similar to a right angled isosceles triangle.
<u>Instead of counting all the points, calculate its area.</u>

☐ Internet Project name :P P [1999/11/08,

Discussion for part (c)

(Q) We like your idea, but we have another idea.
For this problem we will just focus on positive numbers, as you can get any numbers using negatives, eg. (−3)+(−2)+(−1)+(0)+(1)+(2)+(3)+(4)=8

(OMISSION)

Fig. 2 (continued)

The Japan side confirmed using the same means as the Australian side. In (N), the Japanese side expressed consecutive numbers using a diagram and explained the total, but in (O), while still being supportive, the Australian side advocated thinking of a different expression that cannot express negative numbers.

In the above communications, the students had an empathetic stance at each stage by the time the sending and receiving content from both sides were synchronized. Each side described their own *understanding*. The other side's message content indicated their understanding, the other side's mathematical ability was appraised using their mathematical expression, and an attempt was made to respond by adopting the perspective of the other side. Then, the other side was asked questions for the purpose of obtaining deep *self-understanding*. Through *the recursive activity of mutual interpretation*, communication was synchronized. For example, Australian students answer (B) was not appropriate for Japanese and thus why they considered explanation (P) which was easier to understand. It is evidence that Japanese students *were trying to get Australian students' perspectives*. Each reply itself described the content they learned and developed from others. Further instruction from their experience was clearly recorded in the comments sheets in both sides. In fact, the students themselves applied their views of mathematics through message interpretation, and as a result, their individual views of mathematics were adjusted and updated. For example, an impression from the Japanese side included, "I usually did mathematics alone, but discussing a problem as a group, in this way, has its own appeal and I think it's a good thing." This comment shows that the student's usual way of studying mathematics became clear through the mirror of this study activity. The instruction included 'leaning how to learn' for developing social norm among their communication. In another impression on the Japanese side, "It was fun that we could talk with students in far-off Australia. We could neither see them nor hear them, but the three of them certainly exist on the other side of the ocean and were thinking about the same questions as we were. Just imagining that makes me happy," said a student. The students were delighted by this synchronized communication carried out with others on the far side of the ocean. By means of mathematics communication, the student himself reappraised his own view of mathematics through their hermeneutic effort of thinking and sharing with others. At the same time, this comment clearly illustrated the *human activity where students were imaging the existence of students on other side*. Because of *trying to understand the other side* (*getting other perspectives*), Australian students kindly waited reply at (E) and, on (O) and (Q), they positively evaluated previous messages at fist and then, they replied their own ideas. The whole processes of developing synchronized communication and developing human relationship. This is a process of *hermeneutic cycle*; For example, Australian students used different characters for consecutive numbers on (B), thus, Japanese students imagined and hypothesised that Australian students may not be using symbolic-algebraic representation well. Japanese students asked numbers on (K) and Australian students simply answered (M). Japanese confirmed the hypothesis, and for easy understanding, Japanese used Pythagorean representation on (P).

What is clear is that the students and all of us who are interpreting students' activity engage in the activity of hermeneutics efforts. We feel empathy with the interpretation that assumes the position of others, as seen in the students' communication acts, with the instruction through self-understanding as evident in their impressions, and the ability to discuss this.

Additionally, this example for illustrating hermeneutic efforts illustrates the synchronized letter style communication which included sympathetic and competitive attitudes (Ishizuka et al. 2002). For example, "We like your idea but we have another idea" on (P) shows sympathetic and competitive attitude. It is not limited in this example but historically well-known on historical mathematical text such as the Method by Archimedes and the letter on probability from Pascal to Fermat. In the next chapter, we will illustrate that those kinds of communication are observed in classroom communication, and analyse it from the viewpoint of the four principles, too.

Four Principles of Hermeneutics Effort in Its Theoretical Background

The above four principles were identified by the author from taking into consideration various hermeneutic theories and reflecting on personal interpretation of mathematics history and classroom activity. As Brown (1997) described, the current meaning of hermeneutics was described by Gadamer. On the other hand, here, we are not considering current meanings of hermeneutics but hermeneutic efforts like the historian engagement which was introduced by Jahnke (1994) into mathematics education. Especially, historians, such as Schubring (2005, pp. 1–7), described hermeneutics as for their methodology and their hermeneutics is more traditional and it is far from Gadamer's Hermeneutics. Here I will identify traditional ideas from the historical development of hermeneutics for explaining the four principles as for clarifying the hermeneutic efforts.

Gadamer (1993), who led the development of hermeneutics from the 1960s and in recent years, said that the development of hermeneutics began with "seeking the will of God in the Bible," advanced with Schleiermacher, D.F. and Dilthey, W., and then, the opinions of Heidegger and Gadamer.

Schleiermacher (1905) generalized hermeneutics from Protestant biblical hermeneutics to methodological theory on literature, history and other textual interpretations: "Two contrasting maxims of understanding. (1) I understand until I encounter a contradiction or nonsense. (2) I do not understand anything that I cannot perceive and comprehend as being necessary." This is an allusion to the confirmation of understanding by necessity and non-contradiction as seen by the subjective, and to the *hermeneutic circle* whereby hypothetical understanding is preserved until the acknowledgement of contradiction, with interpretation continually occurring. "The main point of interpretation is that the person must be able to make the transition from his own mind to the mind of the author." This is an allusion to *getting other's perspective (the assumption of the other's (author's) position)*. The act of seeking to interpret empathetically by assuming the mind of the author and aligning one's own mind with that is described. "Grammatical interpretation is objective interpretation, and technical interpretation (hermeneutics) is subjective interpretation." The act of aligning the mind for getting other's perspective is totally subjective. The position of Schleiermacher based on this kind of subjective interpretation hints at the inclusion of self-understanding whereby interpretation becomes a mirror that reflects the subjective understanding itself; however, I could not find that Schleiermacher himself mentioned about this.

Dilthey (1900) considered hermeneutics to be a methodological theory of mental science that gives objectivity to interpretation based on the subjective: "Comprehension always remains merely relative, and can never be complete." In other words, the hermeneutic circle applies. "The central point of the techniques (hermeneutics) that apply to comprehension is in the interpretation of human existence implicit in the text," "The artful comprehension of the permanently emended in expression of the existence (life) is called 'interpretation'. In terms of the interpretation, the only expression capable of such objective apprehension is linguistic expression." The salient feature of Dilthey's understanding is the point of seeing living human testimony (activity) in the text. The perspective that recognizes objectivity in interpreting subject's empathetic reading of this human nature is a characteristic of hermeneutics. "By comparing myself with others, I am able for the first time to experience individuality in myself." *Dilthey described getting others (the assumption of the position of others) and instruction from experience (self-understanding) through empathy and the superposition of the mind by* "*transferring one's self into the macrocosm of the given expression of existence*."

Gadamer disagrees with the hermeneutics descended from Schleiermacher and Dilthey. Gadamer (1960) said, "The essential character of the historical spirit is not in the restoration of the past, but rather in the mediation of present existence through thinking." For Gadamer, the emphasis is on self-understanding as a manifestation of the interpreter's present existence conducted in the medium of the work of restoring the living testimony of the past: "With all comprehension (in addition to Heidegger's understanding and interpretation), the third performance opportunity arises in order to 'comprehend one's self'." This is an expression of understanding (comprehension) from Gadamer's perspective.

Gadamer enlarged the object of interpretation from text to other area. For example, with regard to conversation he states, "Placing one's self in another's position is on all occasions an element of true conversation." (Gadamer 1960; Warnke 1987) As we already mentioned and illustrated, the subject of hermeneutic effort, whereby one aligns one's mind with another's position and understands one's own subjective thinking, is not limited to the diachronic text. After Gadamer, there also appeared movements to see hermeneutics as a philosophy that applies to natural science and all sciences. This included viewing the relationship between theory and observation as a hermeneutic circle in the natural sciences.

As shown above, the four principles, "understanding," "getting others' perspectives (the assumption of the positions of others)," "instruction from experience (self-understanding)," and "the hermeneutic circle" are known on the historical development of hermeneutics even if they are not mentioned by the same terms. Instead of the discussion of difference of every philosophers' terms, here, we prefer to use traditional hermeneutics which is usually used on the historians' activity and choose those four principles of hermeneutic efforts for describing human activity on mathematics education and for designing good teaching practices.

Hermeneutic Effort in Dialectic Discussion on Problem Solving Approach in Mathematic

On the part of illustrating the significance of hermeneutic effort, the dialectic communication in the classroom are shown for demonstrating the getting others' perspectives and knowing ways of argumentation in the classroom which is designed through the teachers for constructing mutual understanding.

Japanese problem-solving approach in mathematics classes (Isoda and Shigeo 2012) is comprised of both individual solving of an unknown problem using students' previous knowledge (known or learned), as well as a whole classroom work (communication or dialectic discussion) that utilizes individual ideas used in their problem solving. In particular, the classroom work, which is aimed at using each other's thought (individual problem solving) and reorganizing through sublation that can be shared publicly based on values of mathematics such as simple, understandable, reasonable, general, easier and so on (Isoda et al. 2010). If one focuses on the dialectic communication of information between different answer-groups, one notices that this is a process of interaction between groups. If one views this as a process of each individual cognition, it becomes evident that this is a process of reviewing one's own thinking as perceived based on information from others. This chapter focuses on the difficulties of students for getting others' perspectives and analysing the argumentation planned by the teacher among children who are engaging in every dialectic argument for the correctness of their own thoughts, in a manner that makes it easy to elicit both sides.

First, the dialectic communication of fraction in classroom (Isoda 1993) will be described and ways of argumentations will be illustrated.

Case Study: Divisional (Partitive) Fractions Versus Quantitative Fractions

The 45 min class (first lesson) and the 15 min class (second lesson) at fifth grade were taught by Hideaki Suzuki (Sapporo Elementary School attached to Hokkaido

University of Education) regarding the problem of “making (creating) a $\frac{2}{3}$ m piece of tape from a 2 m piece of tape”. This exercise is known for its very low percentage of children and even higher grade students of giving a correct answer (Nohda 1981). It involves discriminating the different understandings of fraction between **divisional fraction** (fraction in partition; n parts from among m equally divided parts of the whole) which is studied at grade 3 on the 1989 curriculum and **quantity fraction** (n parts from among m equally divided parts of a unit quantity (such as ‘1 m’), where $m < n$ is also possible such as ‘$\frac{3}{2}$ m’) which studied at grade 4 based on the 1989 curriculum in Japan (Ministry of Education 1989). Both of them were learned in the previous grades. But in the case of the classroom students, the actual results of a previously implemented test showed that only one out of 38 students gave a correct answer. For those students answering incorrectly, some simply misread the question, or others simply focused on the “from 2 m” part and automatically applied the procedure of divisional fractions (Fig. 3).

The class proceeds in two groups: (1) Matsuura’s Group, those who followed the divisional fraction method of thinking, whereby they came up with an answer that was “two parts of the three equal parts of 2 m (it means $\frac{4}{3}$ m as quantity)”, and (2) Minamiyama’s Group, those who followed the quantitative fraction method of thinking, whereby they came up with an answer that was “$\frac{2}{3}$ m”. Groups were originated from each student’s individual solution. Groups exchanged each of their argument while attempting to convince the other side. Through the argumentation, students changed their idea and move their position. What followed was an overview of how a student Suzuki repeatedly said the same thing in an attempt to persuade from the perspective of the listener, whereas the student Minamiyama failed to persuade from the perspective of the listener. This overview focuses on the intervention by the teacher: Teacher chose speakers among students who raised their hands and intervened for synthesizing parallel discussion. Otherwise, the communication goes parallel without connection between different opinions and will be in discrepancies.

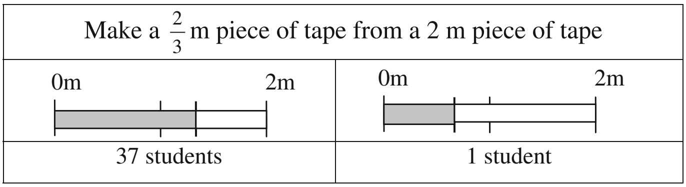


Fig. 3 The Results of pre-test

Class Overview

First class (45 min)

Scene 1 Presentation of the results of individual problem solving

When the teacher described the problem and distributed out pieces of tape (see it as 2 m each), telling students to cut the tape, complains from students such as "this is a pain" or "this is too simple." can be heard. The majority (Matsuura's Group) sees this exercise as a problem of divisional fractions and cuts the tape into two parts of the three equal segments of the 2 m (resulting in $\frac{4}{3}$ m). Only two students (Minamiyama's Group) see this exercise as a problem of quantitative fractions with the desired quantity as $\frac{2}{3}$ m, or $\frac{2}{3}$ m with a base quantity of 1 m (from the perspective of Matsuura's Group, this is one part of three equal segments of 2 m) (Table 1).

(In the following extractions, the named children appeared several times. Other children are distinguished by the names of groups.)

The teacher asks students to present the cut tape on the blackboard. Matsuura presents the answer without saying anything while Minamiyama says the following:

Minamiyama (1): Basically, it's half of this (Matsuura's tape).
Matsuura Group: No, it's two thirds!
Matsuura Group to Minamiyama's Group: Wait, I get it; I'm in Minamiyama's Group, too.
Matsuura Group: The way the teacher wrote this isn't "get $\frac{2}{3}$ of 2 m from 2 m of tape"…
Suzuki (1): Teacher, this can be interpreted either way.

The teacher asks the students to indicate their positions which they agree with their Named Magnets on the blackboard. "Matsuura Group" has 22 students, "Minamiyama Group" has 4 students, "Either Way Is Fine" has 11 students (included Suzuki), and "Undecided" has 2 students. At this point, the student Suzuki is in the "Either Way Is Fine" group (Table 2).

Table 1 The result of first selection

Matsuura group: two parts of the three equal segments of the 2 m	Minamiyama group: $\frac{2}{3}$ m
37 students	2 students

Table 2 The result of second selection

Matsuura group: two parts of the three equal segments of the 2 m	Either way is fine	Undecided	Minamiyama group: $\frac{2}{3}$ m
22 students	11 students	2 students	4 students

Scene 2 Exchanges regarding each side's position (Each others' opinion)

Either Way Is Fine Group: Maybe the problem is coming from the way of question. According to Matsuura Group, the answer is $\frac{2}{3}$ of 2 m.
Minamiyama Group: But it says "from 2 m."
Either Way Is Fine Group: So the reason for Matsuura's answer of $\frac{2}{3}$ is that since 1 or 1 m is taken from the 2 m tape.
Matsuura Group: You just divide it into three and take two of those segments.
Minamiyama Group: But it's $\frac{2}{3}$ from the 2 m tape (Note: the "m" of "$\frac{2}{3}$ m" is missing in his explanation).
Suzuki (2): This is a 2 m piece of tape, so with 2 m, you get $\frac{2}{3}$ m, right? Usually when you have a fraction, the base number is 1. Since it's $\frac{2}{3}$ m here, you have to get the base to 1. It says $\frac{2}{3}$ m, right? Since there's an "m" on it, that means $\frac{2}{3}$ of 1 m. So it's $\frac{2}{3}$ m from 2 m of tape, and Minamiyama first threw out this half (1 m), and I think you use two of the three segments of the remaining tape. If Matsuura's Group did this without the "m" in "$\frac{2}{3}$ m", I think it would be just like Matsuura's answer.
Minamiyama Group: Now there's the "m", so wouldn't Minamiyama be right?

Here, from the viewpoint of observers who knew which is correct, Suzuki (2) is speaking with a good understanding of both sides, and so that should cause Minamiyama's Group to win the discussion. Logically, this should have finished the discussion however the students are not satisfied.

Matsuura Group: Why is it that Matsuura's right if there is no "m" (in "$\frac{2}{3}$ m")?

Some of Matsuura Group do not quite understand what Suzuki is saying. Then, Teacher asked Minamiyama to explain the meaning once more.

Minamiyama (2): I thought that $\frac{3}{3}$ is equal to 1 m.
Teacher (1): One more time.
Minamiyama (3): $\frac{3}{3}$ means 1 m, right?

In spite of the fact that he, Minamiyama, will not be able to persuade Matsuura Group until he clarifies the fact that this is a quantitative fraction, the "m" quantity is consistently missing in Minamiyama's explanations from Minamiyama (1) to (3) such as "$\frac{3}{3}$ is equal to 1 m" but the divisional procedure is represented as well as the explanations of Matsuyama group. Even if the "m" is clearly written in Minamiyama's note, when he explains what he has done to other students, he equates $\frac{3}{3}$ with 1 m rather than stating "$\frac{3}{3}$ m". Minamiyama is applying divisional fractions with 1 m as the unit quantity, and is overlooking the fact that this is a quantitative fraction despite of the repeated prompting from the Teacher (1). This makes it impossible for Minamiyama to deny the "three equal parts of 2 m" idea of Matsuura's Group.

Because Minamiyama himself did not explain well, the teacher diverts the discussion away from what Minamiyama is saying in the following way and starts stirring things up.

Scene 3 Stirring things up (Teacher intervention 1)

Teacher (2): $\frac{3}{3}$ m is 1 m, no doubt about it (Note: he emphasized the "m").
Minamiyama Group: Right!
Matsuura Group: No, absolutely not.

The teacher focuses on whether or not $\frac{3}{3}$ m is 1 m for trying to find sharable ground of discussion.

Suzuki (3): Teacher, it is not related with the problem (Note: the original question), isn't it?

Suzuki (3) takes this to mean that other than the original question, anyone would think that $\frac{3}{3}$ m $= 1$ m, or in other words that the "base is 1 m". This is also a counterargument and then, they start to include the original question in teacher's new question.

Matsuura Group: You take $\frac{2}{3}$ from 2 m, right? So maybe Minamiyama's $\frac{3}{3}$ m is 2 m.

Matsuura group still assumes that their answer is correct and for persuading Minamiyama group, they begin to follow Minamiyama's thinking that $\frac{3}{3}$ m is 2 m if $\frac{2}{3}$ from 2 m but it just reflects on their interpretation. On the other hand, Minamiyama explanation continued to be expressed as a divisional procedure and failed to explain using the basis (meaning) of a quantitative fraction.

Minamiyama (4): (Pointing at the 2 m figure) This half is 1 m, and these two segments are $\frac{2}{3}$.
Matsuura Group: No mentioned $\frac{2}{3}$ of "1 m" in the original problem.
Minamiyama Group: It doesn't say create "2 m" tape. It just says is "from 2 m of tape".
Minamiyama Group: Since the original problem doesn't say to make this only from a 2 m tape, you can make it from 1 m as well.

The teacher reorganizes the conflicting arguments. Matsuura's Group sees the "base as 2 m", and the Either Way Is Fine Group sees "both 2 and 1 m can be the base". In order to summarize their viewpoints, the teacher questions Minamiyama Group as follows.

Teacher (3): Minamiyama, if your answer is $\frac{2}{3}$ m, then we would like to say that the base is 1 m. This is the reason why $\frac{3}{3}$ m is 1 m, and the base is 1 m, according to what you are trying to say, right, Minamiyama?

This questioning clarifies the ground of Minamiyama's inference as opposed to the aforementioned Matsuura Group's ground.

Scene 4 Sharing the argument

Teacher (4): Well, this is a problem, isn't it?
Matsuura Group: Since Minamiyama has left 1 m over, doesn't that mean what really remain is 1 and $\frac{2}{3}$?
Matsuura Group: So Minamiyama does not take 1, but $\frac{1}{6}$.
Teacher (5): No. Minamiyama's answer works when he's only using this (1 m). The remaining 1 m is irrelevant for him.

Many of Matsuura Group still reflect on their interpretation. Matsuura Group which "divides 2 m", interprets Minamiyama's unit quantity $\frac{1}{3}$ m as $\frac{1}{6}$ of 2 m. Accepting the teacher's statement that "the remaining 1 m is irrelevant", Suzuki stated that she is moving (to Minamiyama Group) and started to talk.

Suzuki (4): If the original problem involves making $\frac{2}{3}$ of a 2 m tape, then Matsuura's side is right, I mean, I think Matsuura's argument is easier to understand. Since you're supposed to create $\frac{2}{3}$ m from a 2 m piece of tape then it must be $\frac{2}{3}$ m. So you ignore the 1 m, and this $\frac{3}{3}$ m is also 1 m. Since you are going "from", you've got to deal with both "from" and "m". If there wasn't this "m", and if "from" was "of", then I would agree with Matsuura. (Repeating while reviewing the figure) This $\frac{2}{3}$ m means that the base is 1 m. If there wasn't an "m", then you could use any amount of "m" as the base, but since there is an "m", then 1 m must be the base.
Matsuura Group: If the problem is "create $\frac{2}{3}$ from a 2 m tape", or "create $\frac{2}{3}$ m of a 2 m tape"?
Matsuura Group and Minamiyama Group: The first one, "create $\frac{2}{3}$ from a 2 m tape", is Matsuura Group but what about the second one?
Matsuura Group: 2 m might be the base, but since its $\frac{2}{3}$ m, 1 m might be the base, too.
Teacher (6): So the second one would be strange and contradicting.

Ever since Suzuki (4)'s statement, the semantic interpretation of each group was not the same. On the other hand, the statement of Matsuura Group here has been influenced by the teacher's specification of the base amounts and the group now shares Suzuki's statement. Matsuura Group should now focus on "from" and "of" while considering questions that they come up with themselves in order to review the points they presented themselves. This awareness of contradiction then causes some members of Matsuura Group to begin sharing Minamiyama's idea that the base quantity for the case of $\frac{2}{3}$ m is 1 m, indicating that they are considering joining Minamiyama Group. Teacher (6) mentioned that "create $\frac{2}{3}$ m from a 2 m tape" do not contradict but "create $\frac{2}{3}$ m of a 2 m tape". It's already implicated for the people who have appropriate knowledge that Matsuura group is inappropriate but they do not well understand teacher's saying even if they felt it as strange in Japanese.

In order to articulate this state where ideas have changed, the teacher asks the students to move their named magnets (for the second time). The results are 16

Table 3 The result of third selection

Matsuura group: two parts of the three equal segments of the 2 m	Either way is fine	Undecided	Minamiyama group: $\frac{2}{3}$ m
16 students	No student	2 students	20 students

students in Matsuura Group, 20 students in Minamiyama Group, no student in the Either Way Is Fine Group and 2 students in the Undecided Group (Table 3).

Scene 5 Stirring things up (Teacher intervention 2)

In order to stir things up again, the teacher asked the students to forget the original question and whether or not "$\frac{3}{3}$ m is 1 m" temporarily. Some members of Matsuura group are still the opinion that "it is three equally divided parts of 1 m or 2 m". This opinion indicates that some students are still caught up in the idea of divisional fractions. The teacher asks "can we change tracks?" and continued as follows.

Scene 6 Stirring things up (Teacher intervention 3)

Teacher (7): If we have 0.5 m, then do we indicate what the length is?
Matsuura Group and Minamiyama Group: Yes, it's the same as 50 cm.
Teacher (8): Can we express this as a fraction? (Detailed discussion omitted) So is it the same as $\frac{1}{2}$ m, or is it different?
Minamiyama Group: It's the same.
Matsuura Group: Wow! (Note: this is taken to mean that they are realizing their contradiction.)
Matsuura Group: It's different.
Suzuki (5): If 0.5 m is the same as $\frac{1}{2}$, then what is $\frac{1}{2}$?
Teacher (9): And if I asked you to express $\frac{1}{2}$ m as a decimal of m, what would that be? (Note: he added 'm'.)
Matsuura Group: 0.5. (Note: 'm' is still missing.)
Matsuura Group: It might be $\frac{1}{2}$ of 2 m.

Some members of Matsuura Group now think that "if $\frac{1}{2}$ m is an invariant then maybe $\frac{3}{3}$ m is 1 m". However even now, some members of Matsuura Group still recognize the fact that $0.5\,\text{m} = \frac{1}{2}$ m, but do assert $\frac{1}{2}$ m $\neq$ 0.5 m because '$\frac{1}{2}$ m is $\frac{1}{2}$ of 2 m' is also true. These members insisted that their own explanation on the original problem is correct and therefore account their thinking on divisional fractions with a division target of 2 m as the base. Minamiyama continued his explanation.

Minamiyama (5): $0.5\,\text{m} = \frac{1}{2}$ m and $\frac{1}{2}$ m $= 0.5\,\text{m}$ are the same thing, all you're doing is reversing the order. So I think you can say that $\frac{3}{3}$ m is 1 m. But if the base changes, I'm not sure if you can still say that $\frac{1}{2}$ m $= 0.5\,\text{m}$.

It is evident now that Minamiyama himself recognized the way of thinking of Matsuura Group. It looks that both groups had now reached a state where they recognized the thinking of the other group. At the same time, Minamiyama himself

Table 4 The result of forth selection

Matsuura group: two parts of the three equal segments of the 2 m	Either way is fine	Undecided	Minamiyama group: $\frac{2}{3}$ m
14 students	No student	1 students	23 students

is dealing with the problem of how to find “what unassailable ground of discussion can be shared” with Matsuura Group which is still fixated on divisional fractions. Time runs out at this point and the teacher returns to the argument at hand about “if ‘m’ is affixed on $\frac{2}{3}$ m, whether or not 1 m is the base”, asked the students to move their magnets for the third time. At this point, Matsuura Group has 14 students, Minamiyama Group has 23 students, the Either Way Is Fine Group has no student and the Undecided Group has 1 student (Table 4).

Second class (15 min)

Scene 7 The next day class

The lesson began with the review on the explanation of what had been discussed in the previous lesson for students who were absent yesterday. The question of “from” or “of” is examined once again, with the aim of articulating the difference between interpretations that determine whether one is a member of either Matsuura Group or Minamiyama Group. However the discussion between the groups is not as heated as it was yesterday. The teacher noted the mood of the classroom and started the guiding instruction for concluding.

Teacher (10): The class seems to be in Minamiyama’s direction. Matsuura Group, do you have anything to add to the discussion?
Matsuura Group: It says “from” a 2 m tape, right? If it said “from a 1 m tape”, or if it didn’t say “from” (“of 1 m”), then Minamiyama would be right, but it does say “from”, so 2 m is the base.
Minamiyama Group: 2 m is larger than 1 m, right? So we can just forget 1 m of the 2 m for the moment, and take $\frac{2}{3}$ m from 1 m, for instance.
Minamiyama Group: Just ignore where it says “from”.
Teacher (11): So that you are saying, just “create a $\frac{2}{3}$ m tape” is the same as the original question.

The teacher asked the student Suzuki to explain the answer by focusing on the original problem from the children regarding “how the exercise changes depending on whether m is affixed or not.”

Suzuki (6): For instance, you have a blackboard and you have $\frac{2}{3}$ of a blackboard. We say this is $\frac{2}{3}$. For instance, if you have a blackboard eraser, you could say $\frac{2}{3}$ of this blackboard eraser. Understand?
Teacher (12): I know what you’re trying to say. I really do understand.

Suzuki (7): You can go with anything whatever. But it says $\frac{2}{3}$ m. Since it has an "m" on it, that "m" must be the base. We studied that it was determined by the distance from the Equator to the North Pole divided by some tens of millions, right? Before they standardized it that way, "1 m" was not always equal, right? If you use 2 m as the base, you back then against the standardization. Anyway, since m has 1 m as the base. This is the difference between when you have a given base and when you don't.

The teacher then asked other students to explain how they understood what Suzuki had explained in their own words and summarized the discussion as follows:

Teacher (13): Suzuki wants to say that since there is a unit affixed, the base is already completely settled. So that's why she feels she has to join the Minamiyama Group.

Teacher (14): We haven't heard from the Matsuura Group at all lately. Can we end this discussion now, then? Since "m" is affixed, the base is "1 m", but since we are dealing with $\frac{2}{3}$, we can change the base accordingly. It's as simple as that, isn't it? Is this fine with everyone? (The class ends at this point.)

The teacher ends the class by using Suzuki's statement as the basis of bringing the discussion to a conclusion. Although there are no longer any counterarguments from Matsuura Group, some of the members remain unconvinced. The teacher continued by reformulating the lesson using fractions of various sizes of quantity such as more than 1 m as a subject matter in order to convince those who still remain in doubt.

This overview includes excerpts focusing on the following phenomena, which are recognized as part of the argument process.

(a) At the beginning, Matsuura Group failed to share Minamiyama's way of thinking and interpreting it wrongly. This prevented connection between the two groups.
(b) Minamiyama's explanation without using the unit "m" in his words is unsuitable for persuading Matsuura Group.
(c) Suzuki's statements were consistent from the beginning but still failed to fully persuade Matsuura Group.
(d) The teacher intervention for trying to conclude the contradiction with respect to quantity was effective in persuading the class.
(e) As both sides interacted, they shifted from a state of misunderstanding each other to a state of shared understanding.
(f) Controversially, there were some students in Matsuura Group who developed the hard core on insisting that their believed conclusion was true which enabled them to insist that $0.5\,\mathrm{m} = \frac{1}{2}\,\mathrm{m}$ but $\frac{1}{2}\,\mathrm{m} \neq 0.5\,\mathrm{m}$.
(g) Some students were convinced, but others were not completely convinced by the efforts of the persuasion.

The flow of the class overview is shown in Fig. 4 on the next page.

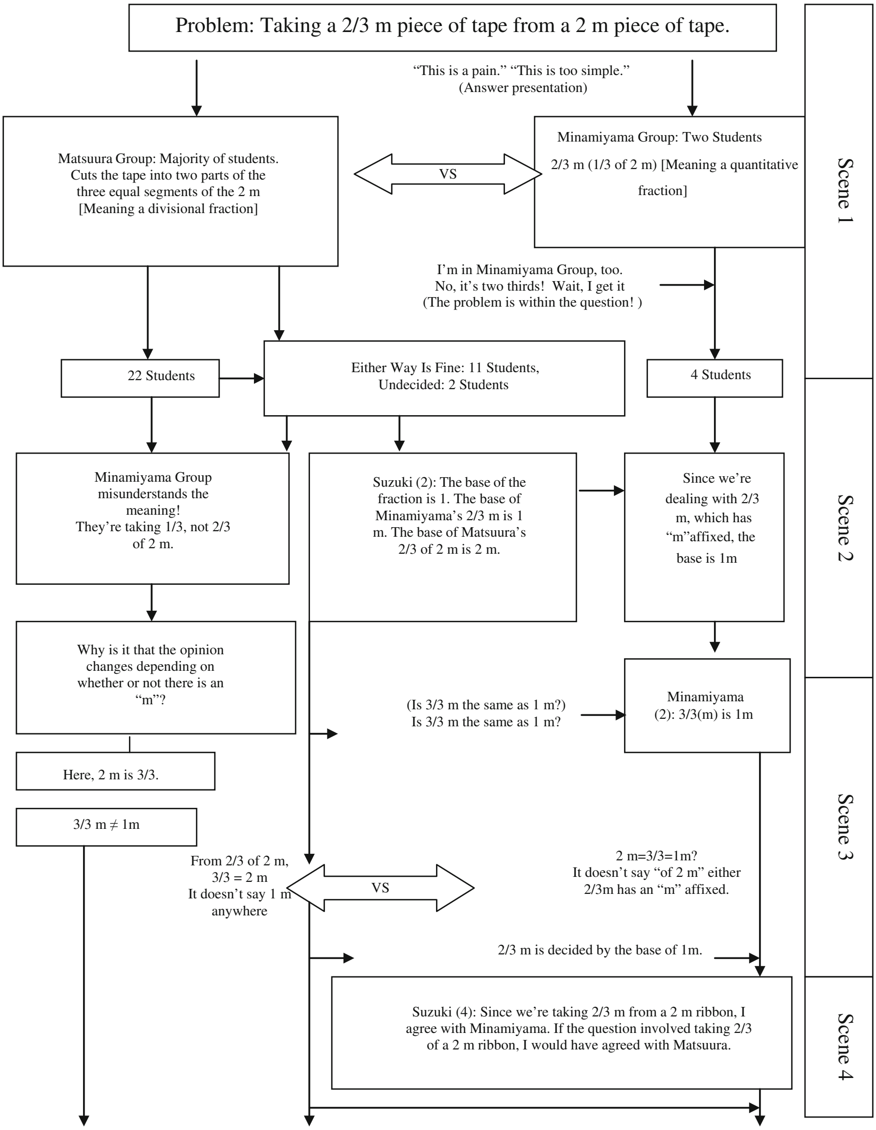


Fig. 4 The process of argumentation

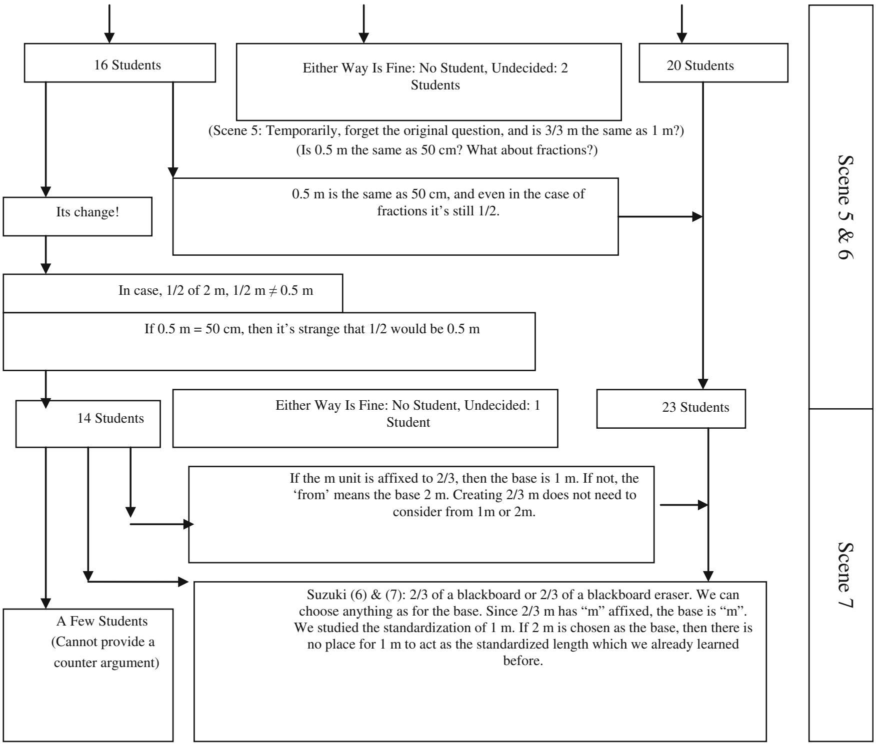


Fig. 4 (continued)

Analysing Argumentations Between Groups

Japanese problems solving approach usually goes in a whole classroom work based on individual solution and the position of each individual opinion (mathematical ideas) is recognized on categorized solution (positioned discussion). A Teacher usually planned the process of arguments based on the category and conducts argumentations in a whole classroom with categorized solutions after solving individually. The difference of individual ideas will be communicated in relation to categorized solutions which are presented by students. On this context, here, we analyze the process of argumentation with categorized groups which are shown on Fig. 4a, b.

From the view point of dialectic, the process of argumentations on Fig. 4a, b are summarized by the following structure on Fig. 5.

The process of argumentations were controlled by the teacher. He intervened in the following ways for completing dialectic;

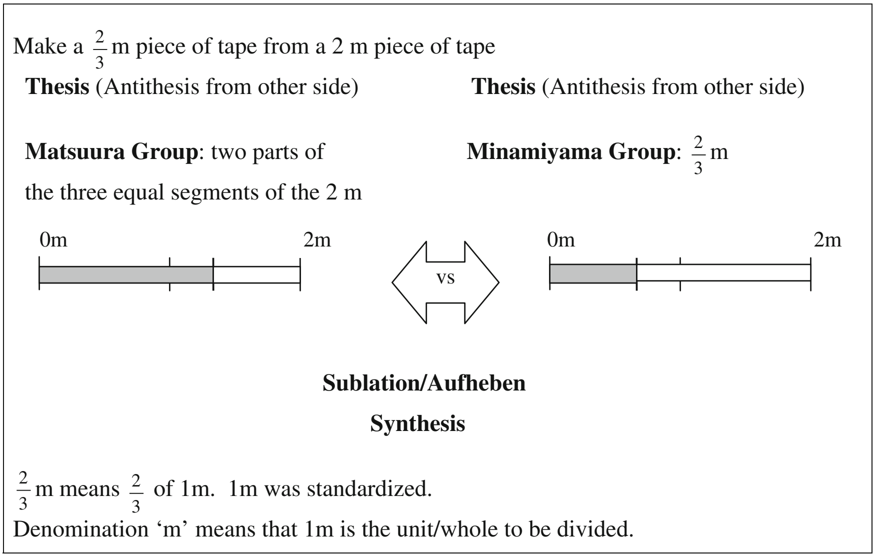


Fig. 5 The dialectic at the class

Firstly, he gave opportunity to present every argument/assertion from both sides (Scene 1),
Secondly, he allowed Suzuki who understands both sides' arguments to explain,
Thirdly, he gave a counter example $\frac{3}{3}$ m in relation to the original question (Scene 3),
Fourthly, he gave generalized counter example $\frac{1}{2}$ m (Scene 6), and
Fifthly, he supported Suzuki who explained the necessity to fix '1 m' as a unit for conclusion.
Finally, because the teacher supported Suzuki and there was no new counter argument, Matsuura Group did not give their comment further more. Teacher set remedial instruction about the quantitative fraction more than 1 m such as $\frac{3}{2}$ m: Students already learned those concepts in grade 4.

Figure 4a, b showed that students shifted their position of argument depending on the changes of ideas through persuasion by others. From the changes of their position, we can identify following types of students in Matsuura Group from the viewpoint of conceptual and procedural knowledge (Hibert 1986; Isoda 1992, 1996):

Type A Students who changed immediately at Scene 1
Although they knew both the procedure of division and the meaning of the quantity on fractions, they did not used well or understand the meaning of the quantity fraction at the beginning. They applied the dividing procedure of divisional fraction at the original question and some of them immediately remembered the quantity

fraction from Minamiyama's answer and recognized the difference between 'from' and 'of' by themselves. Even if they moved 'either way is fine' but did not moved to Minamiyama Group at Scene 1, it means that they well understand both the procedure of dividing and the meaning of the quantity on fractions. They well understood that the confusion originated from the difference between 'of' and 'from' and that is why they could understand both position.

Type B Students who changed to Minamiyama Group until Scene 6
They knew the divisional procedure and did not well understand the quantity on fractions at the beginning. They applied the dividing procedure to the original question. They did not change their position when they met Minamiyama's answer. It means that their understanding of the quantity on fractions was not enough to distinguish the difference of 'from' and 'of' in this moment. From the contradiction of $\frac{3}{3}$ m = 2 m and additional contradiction of $\frac{2}{2}$ m = 2 m which deduced from their divisional procedure at original question, they gradually recognized the difference of 'from' and 'of' and until Scene 6, they understood the meaning of the quantity on fractions. Then, they moved to Minamiyama Group. We can say that their understandings of the meaning of the quantity on fractions and the quantity itself were different depending on when they had changed.

Type C Students who did not changed until Scene 6.
They knew the dividing procedure. They produced their arguments for explaining how their conclusion was correct and did not want to think of others' ideas which were based on the quantity on fractions, and finally did not understand the relationship among quantity. In their case, their diving procedure produced their belief which is strong enough to disregard the contradiction and change the meaning of the quantity. For their persuasion, they have to use their dividing procedure on original question and they must support it. Their confrontational dialectic position for trying to explain how their original idea was correct, enhanced their procedure as the hard core, which should be kept.

From the view point of the meaning and procedure, Type A students used well dividing procedure and understood the quantity on fraction at the beginning. Type B students used well dividing procedure but did not understand the quantity on fraction at the beginning and finally learned the quantity on fraction from the contradictions until Scene 6. Type C students used dividing procedure but they did not need to understand the contradiction, sympathetically, and failed to conceive the meaning of the quantity on fraction. Some of them rejected to conceive the quantity on fraction for asserting their conclusion to be true. In their case, their procedure is functioning like a meaning as the base to explain why their conclusion is true. They had to choose their necessary assumption for deducing conclusion to be true and never share the assumption which can be generalize. Those categorization from the viewpoint of the procedural knowledge and the conceptual knowledge which were observed in the process of argumentation is summarized in following Table 5.

Table 5 Students understanding of fraction from the conceptual and procedural knowledge

In the process, they applied	Type A	Type B	Type C
The appropriate procedure of division	Kept	Kept	Kept
The appropriate meaning of the quantity	Kept	non → having	non → non

Getting Others' Perspectives; Ways of Persuasion and Moments of Conviction

The four principles of hermeneutic effort are "Understanding," "Getting others' perspectives (the assumption of the position of others)," "Instruction from experience (self-understanding)," and "The hermeneutic circle." Here, I illustrate the difficulty of getting other's perspective for knowing the ways of persuasion and the moments of conviction.

We usually say the "logic of persuasion" states that "it is not possible to easily persuade people unless one does so with an understanding of their perspective first". This is a kind of ancients' dialectic which begins the reason: "if what you say is true…." Suzuki knew the idea of both groups and repeated the same explanation. But depending on students, understandings were different because they do not share the same reasoning.

In Fig. 3, even though the reasoning and understanding of each student are not the same we could categorize Matsuura Group into Type A, Type B and Type C. Suzuki's explanations did not change through the discussion but depending on their understanding their decisions were different. Type B students could not agree with Suzuki from the beginning because they do not have a sharable ground. As Hegel, G. described, the antithesis works positively and negatively. In the case of Type B, counter examples given by teachers supported them to develop (or remember) the meaning of the quantity. In the case of Type C, counter examples influenced them to develop the hard core for reasoning based on their conclusion. Both developments are different instructions from experience in this lesson. It means that counter examples given by teachers functioned positively for developing ground of discussion for Type B students but works negatively for Type C students.

However, in case, students can share the ground of discussion, they can share the ideas. In this lesson, the teacher prepared several strategies for developing the ground of discussion for the conviction. First, he gave students the opportunity to exchange the different answers. Second, he gave the opportunity to exchange their reasoning. Third, he gave counter examples. Fourth, he fixed the ground of reasoning by supporting Suzuki's explanation, especially, he posed different counter examples: $\frac{3}{3}$ m, 0.5 m and $\frac{3}{2}$ m in the next lesson. $\frac{3}{3}$ m is limited within fractions, 0.5 m is related with decimal notations and $\frac{3}{2}$ m is the extension of divisional fraction to the quantity. Each counter example has their different roles for developing the concept of quantitative fractions. He posed them in the sequence from specific to general: students could not represent $\frac{3}{3}$ by decimal notations. Divisional fraction divide whole and is not larger than 1.

Through these teaching strategies, students engaged in the hermeneutic cycle and were able to develop others perspective as the ground for sharable discussion, and developed appropriate understanding.

Significance of Hermeneutic Effort on Lesson Study for Humanizing Mathematics Education

Lesson study will succeed just in case the objective for teaching and learning among teachers, students and researchers in classroom are well shared. On the open class and the post-class discussion on lesson study, the necessary point at the post-class discussion is knowing the objective of the teacher's teaching activities and students' learning activities. Even if participants of open class give their alternative comments against the teacher's teaching activities, they have to make clear the difference of objectives for sharing their ideas reasonably in the lesson study community. To show the difference of objectives, they also respect the originality of teacher's activities. The students' behaviours are usually referred for showing evidence to make clear explanation of the effect of teaching practice on its hermeneutic cycle.

Hermeneutic efforts are usually done for making clear the human activity. Main component of hermeneutic efforts is based on the activity for getting others perspective. Even if it is purely subjective activities, it can produce sharable interpretation and deeper understanding of others.

In mathematics education, Piaget's epistemology and Constructivism had been functioning as the major theory for learning beyond contradiction and reorganizing organisms to be more viable. Vygotskiian epistemology and Social Constructivism had been functioning as the major theory for learning from intersubjective to subjective. The former perspective enhances solipsism in the environment and explains one's understanding, but not easy to explain the difference of environments and relationship between each individual. The later perspective enhances materialism, and existence of intersubjective knowledge such as instrument and norm, and explains learning as instrumentalization but not easy to explain difference of every mind. As Glassersfeld (1995) mentioned, there is no contradiction between Constructivism and Social Constructivism. However the weakness of both epistemologies has existed in their adaptations. Both epistemologies are independently adapted for describing the phenomena of learning metaphorically. On this context, each epistemology functioned as the model to explain learning phenomena in different manner and never connected. On the adaptations, researchers usually did not treat the objectives or aims of subjects such as students. For example, Freudenthal (1973) criticized Piaget's and others description of experiment as the inappropriate interpretations of students' activity beyond their willingness to think.

Traditional perspectives on Hermeneutics, which mentioned in this paper by the four principles on hermeneutic efforts provides our convictions of understanding others and appropriateness of interpretation. Hermeneutic efforts provide solipsism the possibility of understanding others and existence of others. Hermeneutic efforts also provide materialism the possibility of subjective interpretation of the object, such as humanities of mathematics. For example, students can say that I am thinking likely Pythagoras for using Pythagorean theorem even if we are not sure that he found the theorem or not.

As Yrjö Engeström problematized, the theories of education failed to provide the platform for treatment of objective or aims of teaching practice. Hermeneutic efforts provide major theory for lesson study through getting other's perspectives.

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History, Application, and Philosophy of Mathematics in Mathematics Education: Accessing and Assessing Students' Overview and Judgment

Uffe Thomas Jankvist

Abstract The Regular Lecture addresses the three dimensions of history, application, and philosophy of mathematics in the teaching and learning of mathematics. It is discussed how students' overview and judgment—interpreted as 'sets of views' and beliefs about mathematics as a discipline—may be developed and/or changed through teaching activities embracing all three dimensions of history, application, and philosophy. More precisely, an example of such a teaching activity for upper secondary school is described along with a method for both accessing and assessing students' overview and judgment. Examples of data analysis are given based on a concrete implementation of the teaching activity.

Keywords History, applications, and philosophy of mathematics · Overview and judgment · Students' beliefs, views, and images of mathematics as a discipline

Introduction

Recalling Imre Lakatos' introductory statement to his *History of Science and Its Rational Reconstructions* from 1970, "philosophy of science without history of science is empty; history of science without philosophy of science is blind", I intend in this Regular Lecture to address interrelations between the two dimensions of history of mathematics and philosophy of mathematics in the teaching and learning of mathematics, and further relate this to the dimension of applications of mathematics in mathematics education.

Besides my personal interest in history, application, and philosophy of mathematics in mathematics education, my academic motivation for wanting to address interrelations between history, applications, and philosophy in mathematics education is twofold; one from the international scene and one from the national.

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Research on History, Application, and Philosophy in Mathematics Education

Internationally speaking, with the creation of the ICMI affiliated *International Study Group on the relations between History, Pedagogy, and Mathematics*, known as HPM (http://www.clab.edc.uoc.gr/hpm/), the past four decades have offered a vast amount of literature on the inclusion of a historical dimension in the teaching and learning of mathematics (e.g. Jankvist 2009a). Although the first couple of decades mainly provided literature of an advocating or descriptive nature, often drawing on teachers' own positive experiences, a shift towards actual research studies, including empirical studies, has occurred in the past decade or so (Jankvist 2012a), and this in particular since the publishing of the ICMI Study on *History in Mathematics Education* (Fauvel and van Maanen 2000). One initiative which has been taken is that of setting up a working group at the *Congress of the European society for Research in Mathematics Education* (CERME) specifically devoted to research on history in mathematics education.

Regarding application—and modeling—equally much, if not more, has been done, in particular by the *International Community of Teachers of Mathematical Modelling and Applications* (ICTMA), which has as its declared purpose "to promote Applications and Modeling (A&M) in all areas of mathematics education—primary and secondary schools, colleges and universities" (http://www.ictma.net/). ICTMA has had biennial meetings since 1983 and in 2003 it became an affiliated study group under ICMI. As HPM, ICTMA is also present at general mathematics education conferences with topic study groups, discussion groups, working groups, etc. and an ICMI Study has been devoted to Modelling and Applications in Mathematics Education (Blum et al. 2007).

In comparison to the number of studies on history and applications/modeling of mathematics found in international journals of mathematics education, including special issues, etc., studies on a philosophical dimension in the teaching and learning of mathematics are close to non-existing. When one performs a search on 'philosophy' and 'mathematics education' the hits found almost all concern the use of philosophy in developing mathematics education theory, not its inclusion in the classrooms, etc. (For a list of the few exceptions I am aware of, see Jankvist 2013). Now, why is this so? Is it the case that a philosophical dimension has nothing to offer mathematics education? To the best of my knowledge, the answer to this question is 'no'. To illustrate it, I turn to the second part of the motivation; the national one.

"Overview and Judgment"

In Denmark in 2002 a report was published as the final product of a government funded project on *Competencies and Mathematical Learning*, edited by Niss and Højgaard (2011, English edition). Besides listing and discussing eight (1st order) mathematical competencies (mathematical thinking; problem tackling; modeling; reasoning; representation; symbols and formalism; communication; and the tools and aids competency), the KOM-project also lists three (2nd order) competencies, known as three types of *overview and judgment* (OJ). These are:

- OJ1: the actual application of mathematics in other subject and practice areas;
- OJ2: the historical evolution of mathematics, both internally and from a social point of view; and
- OJ3: the nature of mathematics as a subject.

Where mathematical (1st order) competencies comprise "having knowledge of, understanding, doing, using and having an opinion about mathematics and mathematical activity in a variety of contexts where mathematics plays or can play a role", or in other words a kind of "well-informed readiness to act appropriately in situations involving a certain type of mathematical challenge", the three types of overview and judgment are "'active insights' into the nature and role of mathematics in the world" (pp. 49, 73). Niss and Højgaard state that "these insights enable the person mastering them to have a set of views allowing him or her overview and judgement of the relations between mathematics and in conditions and chances in nature, society and culture" (p. 73). As we shall see below, the three types of overview and judgment largely resemble the dimensions of history ($\sim$OJ2), applications ($\sim$OJ1), and philosophy ($\sim$OJ3) in mathematics education.

The first type of overview and judgment (OJ1) concerns actual applications of mathematics to extra-mathematical purposes within areas of everyday life, society, or other scientific disciplines. The application is brought about through the creation and utilization of mathematical models of some kind. As examples of questions to be considered in relation to this, Niss and Højgaard (p. 74) mention: "Who, outside mathematics itself, actually uses it for anything? What for? Why? How? By what means? On what conditions? With what consequences? What is required to be able to use it? Etc."

The second type (OJ2) should not be confused with knowledge of the history of mathematics viewed as an independent topic (as taught per se). The focus is on the actual fact that mathematics has developed in culturally and socially determined environments, and the motivations and mechanisms responsible for this development. On the other hand, the KOM-report says, it is obvious that if overview and judgment regarding this development is to have any weight or solidness, it must rest on concrete examples from the history of mathematics. Examples of OJ2 questions are (p. 75): "How has mathematics developed through the ages? What were the internal and external forces and motives for development? What types of actors were involved in the development? In which social situations did it take place? What has the interplay with other fields been like? Etc."

The third type (OJ3) concerns the fact that mathematics as a subject area has its own characteristics, as well as the characteristics themselves. Some of these, mathematics has in common with other subject areas, while others of them are unique. As examples of OJ3 questions Niss and Højgaard (pp. 75–76) mention: "What is characteristic of mathematical problem formulation, thought, and methods? What types of results are produced and what are they used for? What science-philosophical status does its concepts and results have? How is mathematics constructed? What is its connection to other disciplines? In what ways does it distinguish itself scientifically from other disciplines? Etc."

Narrowing Down the Problématique

As can be seen from the set of example questions above, the third type of overview and judgment (OJ3) embraces quite a few elements related to aspects of philosophy of mathematics. Clearly, due to the lack of studies discussing a philosophical dimension in the teaching and learning of mathematics, not much has been said about the interrelation between such a dimension and the dimensions of history and application, respectively.

One observation which may be made in regard to the use of history, application, and philosophy in mathematics education is that these dimensions may play either the role of a *means* for improving the usual mathematical instruction in one way or another, or they may play the role of an *end*. Regarding application (and modeling), Niss (2009) argues that, on the one hand, such a dimension may serve as a means to support the learning of mathematics, either by providing interpretation and meaning to mathematical ideas, constructs, argumentation, and proof, or by motivating students to study mathematics. On the other hand, it may be seen as an end in itself that students become acquainted with the use of mathematics in extra-mathematical contexts (and in relation to modeling, also become able, themselves, to actively put mathematics to use in such contexts). As discussed in Jankvist (2009b), the same situation applies to the historical dimension: history may be used as a *tool* for teaching and learning mathematical ideas, concepts, theories, methods, algorithms, ways of argumentation and proof; and history may be used as a *goal*, meaning that it is considered a goal to teach students how mathematics has come into being, the historical development of it as well as both human and cultural aspects of this development, etc. Not surprisingly, the role of a philosophical dimension subordinates to a similar categorization (Jankvist 2013): where philosophy as a means/tool would embrace arguments stating that philosophy may assist students in their sense-making of e.g. mathematical argumentation and the notion of mathematical proof, including also how and why we argue and prove, mathematical ideas and constructs, etc.; philosophy as a goal would include arguments stating that it serves a purpose in its own right for students to know something about e.g. the epistemology and/or ontology of mathematics and its concepts and constructs, the philosophical foundations of mathematics as a discipline as well as questions of why mathematics is constructed the way it is, its science-philosophical status, etc.

Looking at the above description of the three types of overview and judgment, it is clear that these concern history, applications, and philosophy in the role of ends/goals. However, what is not clear, nor from the KOM-report's normative description of overview and judgment, is:

1. How teaching activities may be designed in order to assist students in their development of the three types of overview and judgment?; and
2. How students' possession and/or development of the three types of overview and judgment may be both accessed and assessed?

In respect to these questions—which shall make up the research questions of this Regular Lecture—it is worth noticing that the KOM-report talks about a person who is able to master his or her active insights in relation to the three types of overview and judgment as being equipped with a *set of views* regarding mathematics and its role in relation to nature, society, culture, and the world in general. Without entering into the long discussion of the difference between knowledge and beliefs, it seems fair to say that such a set of views is related to a student's beliefs about mathematics as a (scientific) discipline as well as his or her knowledge. Thus, I shall address these issues next.

Beliefs About Mathematics (as a Discipline)

One recent definition of beliefs, although given in the context of teacher education, is that of Philipp, who in the *Second Handbook of Research on Mathematics Teaching and Learning*, describes beliefs as "lenses through which one looks when interpreting the world" and:

> Psychologically held understandings, premises, or propositions about the world that are thought to be true. […] Beliefs might be thought of as lenses that affect one's view of some aspect of the world or as dispositions toward action. Beliefs, unlike knowledge, may be held with varying degrees of conviction and are not consensual. (Philipp 2007, p. 259)

Thus, what is knowledge for one person may be belief for another. Regarding beliefs, people are generally aware of the fact that others may believe differently and even that their stances may be disproved. Concerning knowledge, on the other hand, people find "general agreement about procedures for evaluating and judging its validity" (Thompson 1992, p. 130).

Various attempts have been made to try and organize people's beliefs about mathematics. One often cited categorization of students' mathematics-related beliefs is that of Op't Eynde et al. (2002). They review four earlier categorizations of students' beliefs (due to Underhill 1988; McLeod 1992; Pehkonen 1995—also to be found in Pehkonen and Törner 1996—and Kloosterman 1996), and provide a new more comprehensive framework of their own, structured under three different topics:

1. *Beliefs about mathematics education*: (a) beliefs about mathematics as a subject; (b) beliefs about mathematical learning and problem solving; (c) beliefs about mathematics teaching in general
2. *Beliefs about the self*: (a) self-efficacy beliefs; (b) control beliefs; (c) task-value beliefs; (d) goal-orientation beliefs

3. *Beliefs about the social context*: (a) beliefs about the norms in their own class (a1. the role and the functioning of the teacher; a2. the role and the functioning of the students); (b) beliefs about the socio-mathematical norms in their own class (Op't Eynde et al. 2002, p. 28)

I shall not go into a detailed discussion of the components of these three categories of beliefs, only mention that in the context of this framework of students' mathematics-related beliefs it is point 1a, students' beliefs about mathematics as a subject, which is closest related to the KOM-report's 'set of views' in relation to overview and judgment. However, due to the embeddedness of point 1a in an educational context, it is fair to argue that there is a dimension missing in the above categorization—one we may call *beliefs about mathematics as a discipline*. In fact, this dimension is dealt with more independently by Underhill (1988) and Pehkonen (1995), but Op't Eynde et al. (2002) play it down significantly. Nevertheless, the shortage may be remedied by adding the extra dimension—as in Fig. 1 (right). (For further discussion of this, see Jankvist 2009a, 2015).

Such a dimension about *mathematics as a discipline* of course *embraces a student's 'set of views' in relation to the three types of overview and judgment*. The reason for placing the dimension outside the triangle in Fig. 1, i.e. turning this into a tetrahedron instead of a square, has to do with the fact that mathematics as a *discipline* is rather different than mathematics as a *subject*, as included under beliefs about mathematics education. However, if students are to obtain an image of and develop beliefs about mathematics as a discipline through their teaching and learning of mathematics, then this can only happen in the interplay between their social context (class), their mathematics education, and their self, which is to say the triangle making up the base of the tetrahedron.

Thus, one way of trying to create a setting in which students' beliefs/set of views about mathematics as a discipline may be developed is to design an activity to be

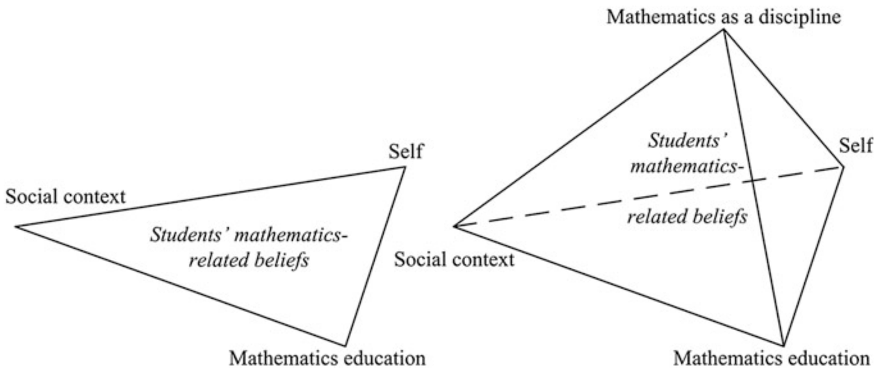

Fig. 1 *Left* "Constitutive dimensions of students' mathematics-related belief systems" illustrated by a triangle with the corners: *mathematics education*, *social context* (the class), and the *self* (Op't Eynde et al. 2002, p. 27, Fig. 2). *Right* An expansion of the *left hand side triangle* to a tetrahedron, the dimension above the triangle illustrating *mathematics as a discipline*

carried out in class (social context) as part of the students' regular mathematics program (mathematics education), during which the students also are exposed to both individual questionnaires and interviews (self) in order to access and assess their overview and judgment. I shall explain this in detail in the following sections.

Designing a History, Application, and Philosophy (HAPh) Module

As stated by Niss and Højgaard (2011), it is clear that if overview and judgment regarding the historical evolution of mathematics (OJ2) is to have any *weight* or *solidness*, it must rest on *concrete examples* from the history of mathematics. Although the KOM-report only states this in relation to OJ2, it is equally clear that something similar applies to OJ1 and OJ3 as well: to know about actual applications of mathematics in other subject and practice areas one must be exposed to concrete examples; and to know about the nature (and philosophy) of mathematics as a subject or discipline, one also needs concrete examples in order to hold one's beliefs/set of views more *evidentially* (Green 1971) or more *knowledge-based*.

Mathematical Problems: Euler Paths, Shortest Path, and Minimum Spanning Trees

The main idea of the design to be described is to have the students read and work with one original source for each of the three types of overview and judgment, all of them adhering to a common mathematical theme and/or topic. I shall illustrate this by describing a concrete teaching module, which was implemented in a mathematics class in first year of Danish upper secondary school in 2010. The class consisted of 27 students of age 16–17 years.

The three texts (in Danish translation) included in the teaching material for this module were:

- Leonhard Euler 1736: *Solutio problematis ad geometriam situs pertinentis*
- Edsger W. Dijkstra 1959: *A Note on Two Problems in Connexion with Graphs*
- David Hilbert 1900: *Mathematische Probleme—Vortrag, gehalten auf dem internationalen Mathematiker-Kongreß zu Paris 1900* (the introduction only).

The overall theme was *mathematical problems*, which was what Hilbert addressed in general terms in the introduction of his lecture from 1900. To make Hilbert's general observations a bit more concrete, the students were first to read the two other texts, each of which addresses a mathematical problem. Euler's paper from 1736 is on the *Königsberg bridge problem*: how to take a stroll through Königsberg crossing each of its seven bridges once and only once—and today the paper is considered the beginning of mathematical graph theory. Two centuries later, with the dawn of the computer era, graph theory (and discrete mathematics in general) found new applications. *Dijkstra's algorithm* from 1959 solves the problem of finding shortest path in a connected and weighted graph, and today it finds its use in almost every Internet application that has to do with shortest distance, fastest distance or lowest cost. Furthermore Dijkstra also discussed a method for finding *minimum spanning trees*, a problem relevant for the building of computers at the time, but also highly relevant today.

Because original sources often are difficult to access, the presentation of these were supplied with explanatory comments and tasks along the way—a so-called 'guided reading' of the sources, inspired by the format developed by Barnett, Lodder, Pengelley, Pivkina and Ranjan (2011) and others related to the group at New Mexico State University which considers the use of original historical sources in the classroom. Practically no mathematical requirements were needed beforehand on the students' behalf to study the text of Euler—a major reason for choosing this text initially—and many of those needed for the Dijkstra text were introduced in commentaries along with the Euler text, thereby also bringing the students somewhat up to date with modern notation, etc.

The students' way into the first original text was by looking at Euler's diagram of landmasses and rivers in Königsberg (Fig. 2 middle) and then verify that this is in fact an accurate representation (or model) of the Königsberg bridge problem by comparing with an illustration of the town (Fig. 2 left). Afterwards the students were told that in modern graph theory, landmasses are represented by vertices (or nodes) and links between them by edges. Students were asked to transform Euler's diagram into such a modern graph individually and then compare their own representation to their fellow classmates, this illustrating that graph representations can look different. The idea was to have the students adapt more and more schematic representations of the Königsberg bridge problem until arriving at something looking like Fig. 2 (right), gradually increasing the level of abstractness.

Once being familiar with the modern representation of a graph, the students were introduced to the problem of representing *multiple edges*, such as for example the two edges between vertices *A* and *B* in the Königsberg graph. These cannot be represented by only their pair, (*A*, *B*), since this causes ambiguity (which is why Euler also named them *a* and *b*, respectively). To illustrate a formal and general way of dealing with this to the students, they were provided with the following modern definition:

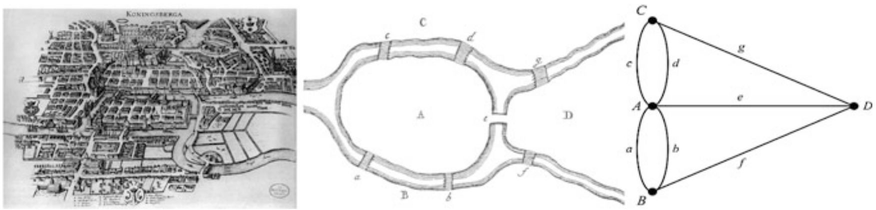

Fig. 2 *Left* An illustration of Königsberg with its 7 bridges from 1652. *Middle* Euler's 1736 simplification of Königsberg's bridges. *Right* A modern graph representation

A graph $\boldsymbol{G}$ is a set of vertices $V(\boldsymbol{G})$ and a set of edges $E(\boldsymbol{G})$ together with a function ψ, which for every edge $e \in E(\boldsymbol{G})$ assigns a pair, called $\psi(e)$, of vertices from $V(\boldsymbol{G})$.

The students were then asked to write up the sets $V(\boldsymbol{G})$ and $E(\boldsymbol{G})$ for the Königsberg graph and the seven function values of $\psi(e)$. On the one hand, the idea of this was to enable them to perceive the definition of a graph as a triplet $\boldsymbol{G} = \{V(\boldsymbol{G}), E(\boldsymbol{G}), \psi_{\boldsymbol{G}}\}$, and on the other hand to have them realize how the above definition in a general fashion resolves the problem of ambiguity, when two vertices in a graph have multiple edges.

As Euler, in his text, introduces various constructs, the students were introduced to the somewhat equivalent modern terminology in the intermediate commentaries, e.g. *route*, *path*, *Euler path* (open and closed), *subgraph*, *degree* of a vertex as well as a few small theorems which Euler explicitly or implicitly uses, such as for example the so-called *handshake theorem*. At the end of his paper, Euler states his three main results (Euler 1736, p. 139 in Fleischner 1990, p. II.19, numbering is mine):

(i) If there are more than two regions with an odd number of bridges leading to them, it can be declared with certainty that such a walk is impossible.
(ii) If, however, there are only two regions with an odd number of bridges leading to them, a walk is possible provided the walk starts in one of these two regions.
(iii) If, finally, there is no region at all with an odd number of bridges leading to it, a walk in the desired manner is possible and can begin in any region.

The students were first asked to formulate these three results using the modern terminology and notation they had been introduced to. Next, they were provided with a modern definition of a *connected graph*, i.e. that there exists a route between every pair of vertices, a property Euler does not state explicitly. Using this property, the three results may be reformulated as:

If a connected graph $\boldsymbol{G}$ has more than two vertices of uneven degree, then it does not contain an Euler path.
Let $\boldsymbol{G}$ be a connected graph, then $\boldsymbol{G}$ contains an (open) Euler path if and only if $\boldsymbol{G}$ contains exactly two vertices of uneven degree.
Let $\boldsymbol{G}$ be a connected graph, then $\boldsymbol{G}$ contains a (closed) Euler path if and only if all vertices of $\boldsymbol{G}$ have even degree.

Most of Euler's efforts goes into proving his first result (i), and regarding the third (iii), which today is considered the main theorem of the paper, he only proves it in one direction. To introduce the students to the notion of if-and-only-if theorems, they were to consider result i as being of the form $\mathbf{P}$: $\mathbf{A} \Rightarrow \mathbf{B}$, and then identify $\mathbf{P}$, $\mathbf{A}$, and $\mathbf{B}$. After having the students prove that $\mathbf{A} \Rightarrow \mathbf{B} \equiv \neg\mathbf{A} \Leftarrow \neg\mathbf{B}$ (by means of a truth table), they were asked to write up $\neg\mathbf{B} \Rightarrow \neg\mathbf{A}$ for result i, i.e. formulating the contrapositive of this theorem, which states that

If $\boldsymbol{G}$ is connected and has an Euler path (open or closed), then $\boldsymbol{G}$ has two or less vertices of uneven degree.

Since Euler has shown, in his own context of course, that a graph will always contain an even number of vertices with uneven degree, we may distinguish between two different cases: when ***G*** has exactly two vertices of uneven degree and when it has none, i.e. when all vertices have even degree. These cases correspond to the ⇒–direction in results ii and iii, respectively. Thus, by looking at Euler's original text again, the students would be able to deduce that the missing parts of the proofs are the ⇐–directions for results ii and iii. For result iii this is ascribed to Carl Hierholzer (published posthumous in 1873), and the students were shown this proof. Then they were asked to prove the ⇒–direction for iii and both ways for result ii using modern terminology.

While employed at Mathematical Centrum in Amsterdam in 1956, Dijkstra was asked to demonstrate how powerful the center's computer, the so-called ARMAC, was. He did so by devising an algorithm for finding shortest path between two nodes in a connected, weighted graph—today known simply as *Dijkstra's algorithm*. Dijkstra's description of his algorithm appeared in 1959 in a paper which also describes an algorithm for finding minimum spanning trees in connected, weighted graphs. Unlike Euler's text the text by Dijkstra is short and builds on a large apparatus of existing graph theory. In fact, the text is only a few pages long. Also, Dijkstra only provides the description of his algorithms and he gives no examples of running these and no proofs of their correctness either, only a few remarks about running time. Thus, this text needed some 'unpacking' for the students in the form of explanatory comments, additional examples, tasks, etc. For example, the students were provided with definitions of a *weighted graph*, a *tree*, and a *spanning tree*:

> A connected graph ***T*** without any subgraphs that are circuits is called a tree, and a tree that for some graph ***G*** contains all vertices of *V*(***G***) is called a spanning tree.

To illustrate that finding a least spanning tree is not trivial, the students were asked to look at the Königsberg graph (Fig. 2 right) and find the number of different spanning trees that can be constructed from this, and then explain their method for finding the answer. (The answer, which is 21, may be calculated using the so-called (Kirchhoff-Trent) *Matrix-Gerüst-Satz*. Deleting the *i*th row and column of this matrix and taking the determinant of the one dimension smaller matrix reveals it. But the students had to do it by systematic inspection.)

In fact, Dijkstra's motivation for devising an algorithm for finding minimum spanning tree had to do with a very specific problem related to the construction of the ARMAC computer. The massive size computers at the time required vast amounts of expensive copper wire to connect their components. Finding a minimum spanning tree corresponds to leading electricity to all electric circuits while using the least amount of expensive copper wire. (A few comments were of course made to the students about the earlier discoveries of the algorithms by Jarník, Borůkva, Kruskal and Prim, respectively.)

Having worked through the Dijkstra text, the commentaries and examples to this, and a modern proof of the shortest path algorithm's correctness, the students got to the third text by Hilbert; the introduction of his 1900-lecture in which he discusses 'mathematical problems'. Paraphrasing Hilbert roughly, he states that often some mathematical development is spurred on by a problem in the extra-mathematical world. Then it is drawn into mathematics and rephrased so that it is hardly recognizable anymore and embedded in a much more general context. Years later, when this has grown into a mathematical discipline, what often happens is that it may then again be used to solve some new extra-mathematical problem:

> Surely the first and oldest problems in every branch of mathematics spring from experience and are suggested by the world of external phenomena. […]
>
> But, in the further development of a branch of mathematics, the human mind, encouraged by the success of its solutions, becomes conscious of its independence. It evolves from itself alone, often without appreciable influence from without, by means of logical combination, generalization, specialization, by separating and collecting ideas in fortunate ways, new and fruitful problems, and appears then itself as the real questioner. […]
> In the meantime, while the creative power of pure reason is at work, the outer world again comes into play, forces upon us new questions from actual experience, opens up new branches of mathematics, and while we seek to conquer these new fields of knowledge for the realm of pure thought, we often find the answers to old unsolved problems and thus at the same time advance most successfully the old theories. (Hilbert 1902, quoted from the 2000-reprint, p. 409)

In a certain sense, the case of graph theory illustrates this: first, spurred on by the Königsberg bridge problem, which Euler generalized so that the answer to the original problem falls out as a small corollary to his more general results; and next, two centuries later when we have a much clearer idea about the discipline of graph theory, Dijkstra solves the extra-mathematical problem of shortest path (and also considers minimum spanning trees) in this graph theoretical context.

Three Student Essay-Assignments

For the students to realize this connection between the three original texts, and thus the three dimensions of history, application, and philosophy, they were asked to identify the criteria that Hilbert proposes for a good mathematical problem (e.g. that it must be explainable to laymen and that it must be challenging but not inaccessible, etc.) and see to what degree the problems treated by Euler and Dijkstra fulfill these, and then relate these cases to Hilbert's comments on the development of mathematics in general. The context in which they were asked to do so was as part of a so-called *essay assignment*. In a previous study, I found that having groups of students prepare small essays was a good way of bringing them to work with the history of mathematics (Jankvist 2009b, 2010, 2011). So the same approach was taken to bring in the two other dimensions of application and philosophy. The module included three essay assignments, each addressing different aspects in relation to overview and judgment. The first essay was on the just discussed topic of *mathematical problems*, linking the three texts by Euler, Dijkstra, and Hilbert together.

The second essay was on *mathematical proofs* and first dealt with different kinds of proofs and proof techniques as well as the use and need for new signs and symbols (both arithmetical and graphical) in the development of new mathematics (concepts, definitions, etc.), something that Hilbert also addresses. The students were asked to discuss this with relation to Hilbert's text and try to draw connections to the two cases, in particular the advantages Dijkstra had in 1959 with a fully developed graph theoretical and conceptual apparatus at his disposal, as compared to Euler who had to start from scratch in 1736. In the end, this essay moved into Hilbert's actual discussion of proofs and their role in solving mathematical problems as well as the role of rigor in mathematical proofs. On the overall, the idea of this was to spur some reflections on the students' behalf regarding the epistemological development of the notion of proof.

The third essay was about *mathematics' status as a (scientific) discipline*, in its own right and in comparison to other disciplines, e.g. physics. Based on their readings of Hilbert, and the two texts by Euler and Dijkstra, the students were asked to try to point out some characteristics of mathematical problems, methods, and ways of thinking as well as to say something about the types of results mathematics delivers and what they may possibly be used for. They were invited to discuss this by comparing mathematics to other academic disciplines. Then they were asked to identify what Hilbert says about the differences and connections between mathematics and other disciplines, and then discuss to what extent they agree or disagree.

Accessing Students' Overview and Judgment

In order to access the upper secondary students' overview and judgment, and the development of this, they were given an 'overview and judgment questionnaire' and a selection of the students (half the class) was interviewed about their answers. This questionnaire included three sets of questions, each set connected to a type of overview and judgment. The first set, which was connected to OJ1, asked about application and sociologically oriented aspects of mathematics as a discipline:

- (a1) Do you believe it to be important for you to learn mathematics? If 'yes', why? If 'no', why not?
- (a2) Do you believe it to be important for people in general to learn mathematics? If 'yes', for whom is it then most important and why? If 'no', why not?
- (a3) From time to time you hear that mathematics is used in many different contexts. Can you mention any places from your everyday life or elsewhere in society where mathematics is being applied, either directly or indirectly?
- (a4) Not counting the ordinary types of calculation (the four basic arithmetical operations, calculation of percentages, etc.) where do you then find mathematics applied in your everyday life and society in general?
- (a5) Do you think mathematics has a greater or lesser influence in society today than 100 years ago?
- (a6) What do you understand by a mathematical model or that of carrying out mathematical modeling?

The second set, related to OJ2, asked about historical and developmental aspects of mathematics as a discipline:

(b1) How do you think that the mathematics in your textbooks came into being?
(b2) Why do you think it came into being?
(b3) When do you think it came into being?
(b4) From when does the coordinate system, as we know it, originate do you think?
(b5) When did Euclid live (approximately)?
(b6) What do you think a researcher in mathematics (at universities and the like) does? What does the research consist in?
(b7) When do you think the negative numbers were (formerly) defined in mathematics (in the Western world): approx. 2000 BC; approx. 300 BC; 15th century; or 19th century?
(b8) When do you think mathematicians (and others) began using negative numbers (in the Western world): approx. 2000 BC; approx. 300 BC; 15th century; or 19th century?

And the third set, for OJ3, asked more philosophically oriented questions of mathematics as a discipline:

(c1) Do you think that parts of mathematics can become obsolete? If yes, in what way?
(c2) Do geometrical figures, e.g. triangles, exist independently of us humans or are they human constructions?
(c3) Does the number 'square root of 2' exist independently of us humans or is it a human construction?
(c4) Are the negative numbers discovered or invented? Why?
(c5) Do you believe that mathematics in general is something you discover or invent?
(c6) What is a mathematical problem?
(c7) Can you give a short description of how an area of mathematics is built?
(c8) Why do we prove mathematical theorems?
(c9) Is mathematics a science? If 'yes, about what? If 'no', what is it then?
(c10) If you believe mathematics to be a science, is it then a natural science? Why or why not?

The word *science* in c9 and c10 is to be understood in the more broad sense of Scandinavian *videnskab* and German *Wissenschaft*, including natural science, social science and the humanities (see Jankvist 2009a for further explanation and elaboration).

During a two-year period the students were asked to answer the above questions three times, intervened by two HAPh-modules; the one on mathematical problems, graph theory and shortest paths described above and another on the unreasonable effectiveness of mathematics, Boolean algebra and Shannon circuits (see Jankvist 2012b, 2013). HAPh-module 1 ran over twelve 90-min lessons and module 2 over

Table 1 Timeline for complete empirical study

Year	Dates	Activity
2010	February 8th	1st O&J questionnaire
	February	Follow-up interviews (round 1)
	April–May	Implementation of HAPh-module 1
	May 8th	1st test questionnaire
	May	Follow-up interviews (round 2)
2011	May 4th	2nd O&J questionnaire
	September–October	Implementation of HAPh-module 2
	October 12th	2nd test questionnaire
	November	Follow-up interviews (round 3)
2012	March 8th	3rd O&J questionnaire
	March–April	Follow-up interviews (round 4)

seven 90-min lessons (due to practical constraints). After each module, the students were also given a test on the content of the module including its related tasks and essays. For a timetable of the study, see Table 1.

Assessing Students' Overview and Judgment

When assessing the students' possession and development of overview and judgment, I shall regard this as a kind of *reflected image of mathematics as a discipline*. This requires an explanation—or two to be more precise.

First, with reference to the literature describing beliefs (see earlier), beliefs are considered as something rather persistent. Therefore, to state that any developments and/or changes in beliefs are in fact permanent is difficult, if not impossible, based on a two-year study as the one described here. Of course, following a class of students for two years should make it more possible to verify observed developments and/or changes than if only following them for one year—but still. However, what we may say is that observed developments and changes occur in the students' views (and 'sets of views'), if we think of (and define) *views to be something less persistent than beliefs, but with the potential to develop into beliefs at a later point in time*. In this respect, we may also consider (and define) *students' images of mathematics (as a discipline) to be made up of their beliefs as well as their views* (Jankvist 2009a, 2015), including of course also their *knowledge*, e.g. in the form of evidentially or knowledge-based beliefs/views.

Second, what then are we to consider as a student's *reflected* image of mathematics as a discipline? The definition of this will be based on empirical findings from a previous study, which concerned the use of history in mathematics education (as a 'goal') and students' development of their image of mathematics as a discipline, in particular in relation to OJ2. In this study it was found that upper

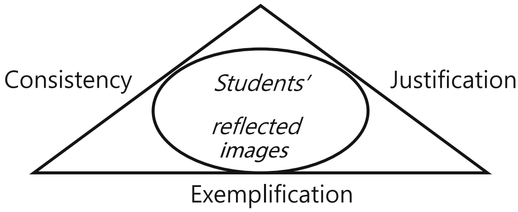

Fig. 3 Students' reflected images of mathematics as a discipline as consisting of three dimensions on a basis of explicitness: consistency, justification, and exemplification (evidence) (Jankvist 2009a)

secondary student' beliefs/views of mathematics as a discipline developed in (at least) three ways, namely in terms of (Jankvist 2009a):

- a growth in *consistency* between a student's related beliefs/views;
- the extent to which a student sought to *justify* his or her beliefs and views; and
- the amount of provided *exemplifications* in support of the beliefs and views a student held, i.e. that the beliefs appeared to be held more evidentially or knowledge-based.

Of course, a prerequisite condition for this is that the students are explicit about their beliefs, i.e. that they are able to express them, but the continuous rounds of questionnaires and interviews provided a setting for this—in the previous as well as the present study.

On the one hand, the three observed dimensions of consistency, justification, and exemplification may be used define what is to be understood by students' *reflected images of mathematics as a discipline* (held on a basis of explicitness)—a definition which also provides some depth to the previously introduced fourth dimension of "students' mathematics-related beliefs" in Fig. 1. The situation is illustrated in Fig. 3.

On the other hand, we may consider Fig. 3 as a model of students' reflected images which tells us that in order to understand the degree of reflection in their images we need to consider each of these three dimensions. That is to say, the model in Fig. 3 provides us with a way of assessing the students' development of overview and judgment—and, so to speak, operationalize the KOM-report's normative description of overview and judgment.

Examples of Data Analysis

To illustrate how the three O&J questionnaires and the follow-up interviews, from the study presented above, can be used to access students' beliefs and/or views about mathematics as a discipline, and thereby assess their possession/development of overview and judgment, I shall make some illustrative 'downstrokes' in the data and questionnaire questions, taking an approach of tracking changes on an individual student basis.

Accessing and Assessing: Consistency

One of the places where consistency, and a growth of such, often shows is in the philosophy oriented questions c2 through c5. As illustration of how to search for consistency in the answers to these questions we shall look at two students, Samuel and Larry.

In the 1st questionnaire, Samuel answered the following to the question about geometrical figures and square root of 2, c2 & c3: “I guess it always existed, but we have defined it.” To the question of the negative numbers, c4: “I don’t think they are discovered. Should they have been discovered on a rock? Neither were they invented, we had to think to find them.” And regarding mathematics in general, c5: “Neither. You can’t discover mathematics in the ground, like fossils, for example. You can’t invent mathematics, because then it’d be concrete—like a pencil, for example. It has always been there, but we had to think it, not discover or invent. In a way it is quite philosophical.” What we may notice here is that Samuel’s answers are both contradictory and somewhat inconsistent, because on the one hand mathematics has always been there, but on the other hand humans had to think it and define it. The statement that mathematics is neither a discovery, nor an invention, may be explained by the student not counting immaterial things as something which is subject to discovery or invention. One year later, in the 2nd questionnaire, Samuel provides the following answer to c2: “Human construction—nothing is coincidental” and refers to this answer in the related questions. Two years later, in the 3rd questionnaire, he answered to c2: “It is a human construction. Let us say that triangles can be found in nature, but humans have defined them, i.e. a human construction.” In c3 he referred to this answer also, and in c4 and c5 he answered: “It is something which has been invented.” Now, even though Samuel states that maybe triangles occur in nature (many students said so with reference to the shape of mountains, etc.), his answers in both the 2nd and the 3rd questionnaires are very consistent; in all answers he believes ‘invented’. Thus, there is a growth in consistency from the 1st to the 2nd and 3rd.

Larry provided the following answers to c2 in the 1st questionnaire: “Can’t think of places in nature where ‘perfect’ geometrical figures exist, so (even though there may be places where they do) it is human made.” And for the square root of 2 in c3: “Can’t give a qualified answer.” Regarding negative numbers in c4, he wrote: “In the beginning invented. But physics have now proven the energy of the universe to equal 0, because of negative forces and quantum mechanics oscillation (which was believed to create everything), so now it is also proven.” And for mathematics in general, in c5: “First invent (including providing arguments) and then proven by means of physics, etc.” As the majority of students, Larry had never given thought to the question of invention versus discovery of mathematics before. But once the question had been raised, it kick started some thought processes. Already in round 1 of the follow-up interviews it was clear that he had thought about this, and perhaps altered his view a bit:

> Well, it depends on how you define 'invention'. I mean, if it is something where we say it's like that, and it is something that humans invented themselves. I mean, we know that there are some connections between things. […] It is just the way we describe nature, by formulas. […] It is our way of writing it. On the one hand, it is discovered, but at the same time it's ourselves who have invented it… […] All things considered, it may be discovery.

In the 2nd questionnaire, Larry answered to c2: "They exist, we just defined them." To c3 he provided a conditioned answer: "Does it describe a relationship which exists in nature? If not, it is a human construction." To c4 he said: "Discovered. Things can have negative energy (electrons), i.e. something exists on the other side of the spectrum." And finally regarding mathematics in general in c5: "Discover through understanding of earlier mathematics." Clearly, there is a shift in Larry's answers from the 1st to the 2nd questionnaire, somehow reflecting his considerations from round 1 of the interviews, although there still appears to be some ambiguity present, e.g. in his answers to c2 and c5. By the time of the 3rd questionnaire, however, Larry's answers appear more consistent. c2: "Yes, a lot of math exists independently of us, the figures mentioned here." c3: "If the constant is used as for example π, then independently." c4: "Discovered." c5: "Discover, most often."

Accessing and Assessing: Exemplification

In order to illustrate how exemplification can play a role in the development of a student's overview and judgment, let us stay with Larry. In round 3 of the interviews, after HAPh-module 2, Larry said the following about discovery and invention:

Larry: Well… There can be connections in mathematics which we discover. For example the equation with Euler's number in the power of π times i minus or plus 1 equals 0 [$e^{i\pi} + 1 = 0$]. These are some interrelations which we have not made ourselves. It is a lot of independent things which we have found and which then fits together and reveals a beautiful connection. […] I think it is a good example of something which we just discover. As far as I know these [π, e, and i] were not that associated. But that they fit together in this way, it kind of shows… that there must be a… that no matter what, we did something right

Interviewer: Yes?

Larry: So regarding invention or discovery in mathematics, I think… I think that some things are invented and some discovered. I will risk claiming that

Interviewer: Alright. Can you give some examples?

Larry: Well… for example our way… in graph theory, to translate bridges into numbers and the way of writing it all up. That is something we've made. While things as… what is a good example? Things as π is something we discovered. […]

Interviewer: Okay. Is it possible to say if one precedes the other? Does discovery precede invention or does invention precede discovery?

Larry: In most cases it must… well, not necessarily… With π, for instance, I guess that discovery was before invention, because… If we say that we invented, that we set a circle to 360 degrees. But when we calculate π […] then we don't use the 360 degrees, as far as I recall. […] It is different within different areas of mathematics, but with π I think we discovered that there was a connection first, and then we built on that. But it's quite related; when we choose something we quickly arrive at some further discoveries

Worth noticing here is that Larry, in his process of going from believing mainly in invention (1st questionnaire) to mainly in discovery (3rd questionnaire), is able to provide several examples of one or the other. To illustrate discovery he first mentions Euler's identity—on the interrelation between the additive identity 0, the multiplicative identity 1, the base of natural logarithms *e*, the imaginary unit *i*, and the number π—as something which he finds it unlikely to have been invented. And he then carries on to elaborate on the number π in relation to discovery (see also answer to c3 in 3rd questionnaire). As an example of mathematics which is invented he refers to the cases of the HAPh-modules, as seen above when mentioning graph theory and Euler's way of solving the Königsberg bridge problem, but also elsewhere in the interview when he refers to Boolean algebra as a human construction and invention. Such exemplification helps Larry to hold his beliefs and views more evidentially. And it also assists him in justifying his beliefs and views as well as the development and changes in these. However, as we shall see below, justification does not always include exemplification.

Accessing and Assessing: Justification

In the 1st and 2nd questionnaires the student Jean justified his answers to question a5, on the influence of mathematics in society today as compared to earlier, as follows: "I think it has greater influence today, since mathematics is used to more and more and many things are about numbers." "I think it has greater influence since it is used for so many things." As compared to the students who would just answer either 'greater' or 'lesser', Jean clearly tried to justify his answers. But he did so without any exemplification whatsoever, which may be illustrated by his answer to a5 in the 3rd questionnaire: "Greater. Everything today is based on numbers and models. In particular computers and cell phones, which we couldn't live without, are based on numbers and do thousands of calculations per second." The mentioning of computers and cell phones is an exemplification made to support the justification that mathematics has greater influence today. Whether there is a connection between Jean's mentioning of computers and the HAPh-modules is not clear, although it was emphasized that Dijkstra's algorithms made up the basis of software used practically everywhere and that Shannon's circuit design ideas among other places were used in computer hardware.

Another illustration of the development of overview and judgment and the role of justification is that of the student Salma and her answers to questions c9 and c10 on mathematics being a scientific discipline and if it belongs to the natural sciences. In the 1st questionnaire she answered to c9: "I don't know if I'd call it so. But it is a tool for the other scientific disciplines within the natural sciences." And to c10: "A means for describing and understanding natural science." A year later in the 2nd questionnaire she said for c9: "It is a scientific discipline as well as a tool for understanding other sciences." And for c10: "Yes, it is a natural science since it is primarily used within the natural sciences. But at the same time it is also used within the social sciences, and for that reason it is difficult to 'classify'." And yet a year later in the 3rd questionnaire, she answered to c9: "Yes, I would say it is. In mathematics you show a theorem's validity through proofs, so yes." And for c10: "It is its own scientific discipline, which may be used within the faculty of natural science as well as that of social science." Salma's answers to the questions c9 and c10 illustrate well how her image of mathematics as a (scientific) discipline gradually becomes more and more reflected over the two year period—a finding which is supported both by the four rounds of interviews and by the essay assignment hand-ins from HAPh-module 2, where one essay concerned the five faces of mathematics: a pure science; an applied science; an educational subject (both taught and studied); a system of tools for societal practice; and a certain kind of platform for gaining aesthetic experiences (Niss 1994).

Recapitualtion and Final Remarks

As the reader will have noticed, this Regular Lecture is mainly concerned with method:

- method for designing teaching modules bringing out aspects of all of the KOM-report's three types overview and judgment; and
- method for accessing and assessing students' possession and development of overview and judgment (through considerations of their beliefs and/or views of mathematics as a discipline)

But let us briefly recapitulate these methods as presented so far in order to provide possible answers for the two research questions.

The way of addressing the first question on how to design activities which can assist the development of overview and judgment has been one of 'answer by example'. By this I am of course referring to the presentation of the HAPh-module on early graph theory in the form of Euler's solution to the Königsberg bridge problem, Dijkstra's algorithms, and Hilbert's discussion of mathematical problems. Through the description of this module and the essay-assignments that the students were to work with it was illustrated how a setting can be created in order for students to develop their overview and judgment regarding mathematics by challenging their existing 'set of views'. As pointed to earlier, in order for students' overview and judgment to have weight and solidity they must be provided with concrete examples. And that was what the module sought to do; provide examples in regard to the three dimensions of history, applications and philosophy of mathematics.

Regarding the second question on how to access and assess students' overview and judgment, the above may be rephrased to state that the HAPh-modules sought to provide the students with 'evidence' in order for them to make and hold their beliefs and views more evidentially and knowledge-based. As we know from Green (1971), we cannot expect students' to change or alter their beliefs and views—and consequently their overview and judgment—are they not provided with concrete evidence to 'measure' these against, the reason being that

> Not until students have access to evidence – or counter-evidence – are they likely to criticize rationally, reason about, and reflect upon their beliefs, and possibly accommodate and change them, should they find it necessary. (Jankvist 2009a, p. 257)

This was illustrated in particular by the student Larry, when he used the mathematical cases from the HAPh-modules as evidence for examples of invention and Euler's identify and the number π as evidence of discovery of mathematics. Hence, by means of examples Larry was able to justify by exemplification his gradually more consistent views regarding the question of invention versus discovery; illustrating the more reflected image of mathematics as a discipline, which he ended up with. (Something similar could be argued for the student Salma by a further display of data.) Of course, the HAPh-modules are not the only things which may cause the students to alter and accommodate their view of mathematics as a discipline—Larry's example with Euler's identity was not part of the modules and neither was Salma's example with mathematics being used within the social sciences. But whether the students' possible changes in beliefs/views can be linked to the modules through the choice of examples or not, the study illustrates that an approach through students' beliefs with repetitive questionnaires and follow-up interviews does appear to be one sensible way of accessing students' overview and judgment about mathematics and the development of such. Furthermore, basing the assessment on the presence and growth of consistency, exemplification, and justification is an approach which reveals some insight regarding the students' beliefs and views—and knowledge—of mathematics as a discipline. Due to the connectedness of the definition of students' mathematics related beliefs regarding mathematics as a discipline (cf. Fig. 1) and the KOM-report's three types of overview and judgment, it is clear that the students for whom it may be concluded that they have come to possess more reflected images of mathematics as a discipline, also are the students who have developed their overview and judgment about mathematics. Again, Larry and Salma served as examples of such students.

Having now described and discussed the proposed methods for accessing and assessing students' overview and judgment as well as a design method for including the three dimensions of history, application, and philosophy in the upper secondary school mathematics program, let us return to our point of origin; the Lakatos quote. Thus, with apologies to Lakatos for abusing the quote, I end this Regular Lecture by stating that:

> history and/or applications of mathematics (as well as other concrete, clarifying cases) can assist in making a use of philosophy of mathematics in mathematics education less empty;
>
> applications of mathematics can assist in making a use of history of mathematics in mathematics education less blind; and
>
> philosophy (and/or philosophizing) can assist in making uses of history and/or applications of mathematics in mathematics education less blind.

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Implications from Polya and Krutetskii

Wan Kang

Abstract Enhancing mathematical problem solving abilities, George Polya gave tremendous contribution to mathematics educators. He identified 4 steps in the problem solving process; (1) understand the problem, (2) devise a plan, (3) carry out the plan, and (4) look back and check. For each step, Polya revealed many useful habits of thinking in forms of questions and suggestions. V.A. Krutetskii analysed mathematical abilities of school children, which suggest valuable implementation to many trying to develop effective ways of expanding mathematical problem solving abilities. Krutetskii's research was inspecting mathematical behaviour in 3 stages of information gathering, processing, and retention. He concluded that mathematically able students show strong trends to gather information in more synthetic way, to process information in more effective, economic, and flexible way and to retain indispensable information more than inessential.

Keywords Mathematical heuristics · Mathematical abilities · Elementary teacher education · G. Polya · V.A. Krutetskii

Introduction

Many mathematics teachers have been trying to enhance their mathematics teaching to be more meaningful and powerful in various ways. To teach mathematical concepts meaningfully, some teachers may provide activities to their students of constructing meaningful concepts through students' real life situations. In this case the Freudenthal's idea of mathematization should be very helpful. Some other teachers who want to enhance their students mathematical capabilities may provide activities to their students of drill and practicing to solve mathematics problems. On the other hand, many think it is important to enhance mathematical capabilities of teachers themselves.

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M.A. Clements and N.F. Ellerton in Australia tried to reflect both ideas of Polya and Krutetskii on problem solving in school mathematics. Their efforts give much substantial help to mathematics educators (Clements and Ellerton 1991). Now, such kind of efforts may be applied to pre-service teacher training courses. Let's look at how such attempts might give shape to mathematics teacher education.

Today I will talk about my ideas about ways of enhancing mathematical capabilities of students and/or mathematics teachers of elementary school level. About ways of teaching mathematical concepts meaningfully, I will talk at another adequate opportunity.

Enhancing Teachers' Capabilities to Teach Mathematical Problem Solving

Polya (1957) gave tremendous contribution to mathematics educators. He identified 4 steps in the process of problem solving; (1) understand the problem, (2) devise a plan, (3) carry out the plan, and (4) look back and check. For each step, Polya revealed many useful habits of thinking in forms of questions and suggestions.

According to Polya, for the first step of understanding the problem, such questions and suggestions as "What is the unknown?", "What are the data?", "What is the condition?", "Is it possible to satisfy the condition?", "Is the condition sufficient to determine the unknown?", "Draw a figure.", "Introduce suitable notation.", and so on, are useful.

For the second step of devising a plan, such questions and suggestions as "Have you seen it before?", "Do you know a related problem?", "Could you restate the problem?", "If you cannot solve the proposed problem try to solve first some related problem.", "Did you use all the data?", and so on, are useful.

For the third step of carrying out the plan, "Carrying out your plan of the solution, check each step.", "Can you see clearly that the step is correct?", and "Can you prove that it is correct?" are useful.

For the fourth step of looking back, "Can you check the result?", "Can you check the argument?", "Can you drive the result differently?", "Can you use the result, or the method, for some other problem?" are useful.

These useful habits of thinking are very recommendable for every mathematical problem solver. It can be said that especially mathematics teacher should build up these habits for their students as well as even more for themselves. The right way of building up good habits is to act out.

As a professor of an institute of elementary teacher education, I have been giving a special course to students who are going to be elementary school teachers. In this course, my students should solve a mathematical problem of elementary school level a day during the semester. Of course, they are recommended to think in forms of Polya's questions and suggestions to solve the problems. In addition to solving a problem, they have to note the mathematical solution with didactic analysis on the ways of helping elementary school children solving the same problem. The following Figs. 1 and 2 are examples of course notes of the students.

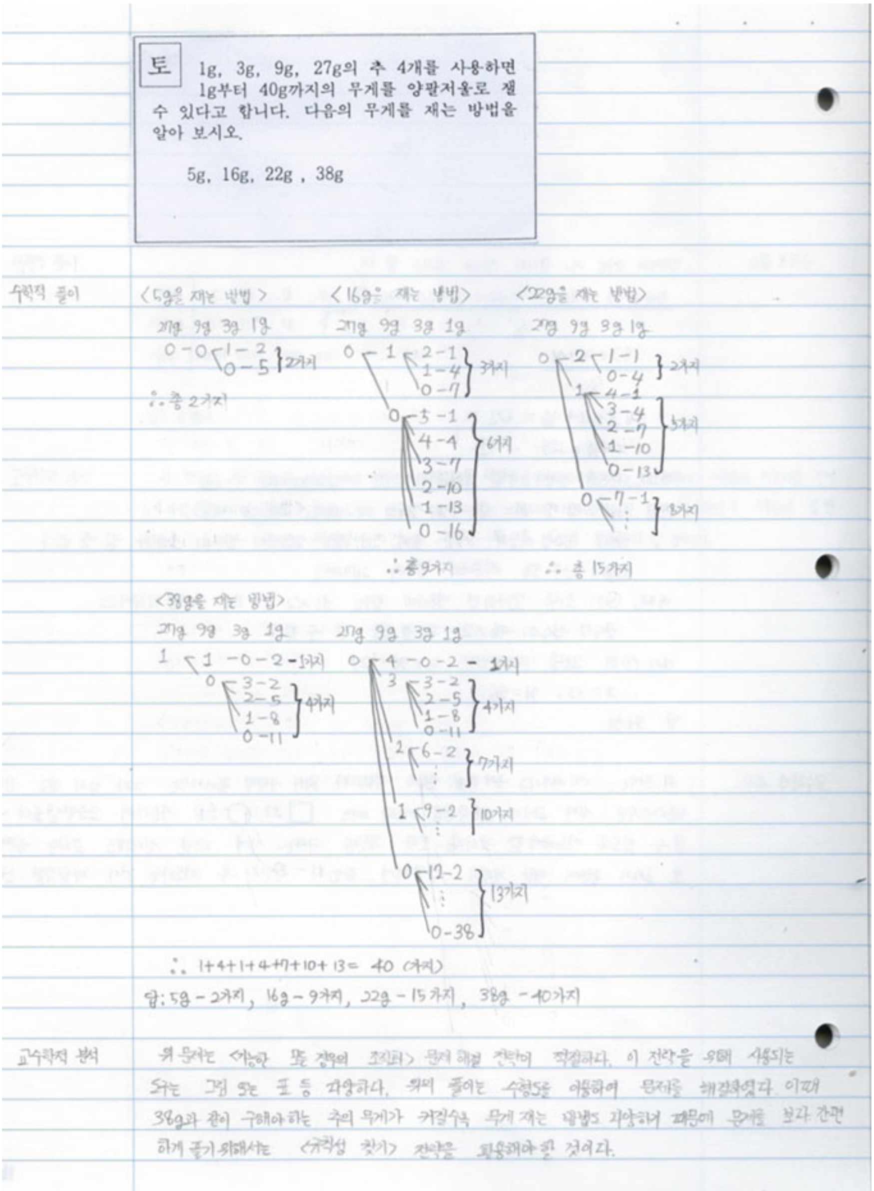


Fig. 1 A student's note (problem of saturday)

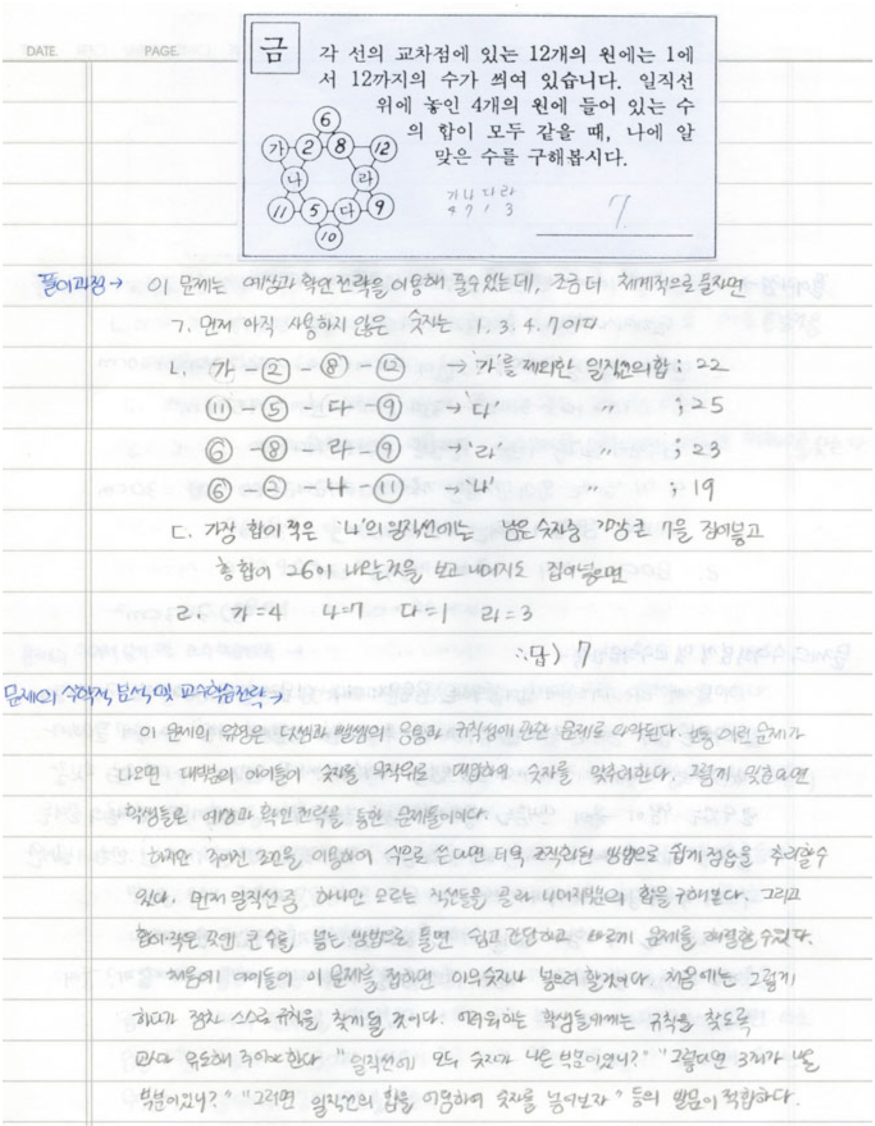


Fig. 2 A student's note (problem of friday)

- In English -

[Problem of Saturday]

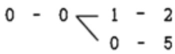
Using 4 poises of 1g, 3g, 9g, 27g, we can measure all the weights from 1g to 40g with a balance. Explain the ways of measuring following weights:

5g, 16g, 22g, 38g

[Solution]

<Ways of measuring 5g>

27g 9g 3g 1g

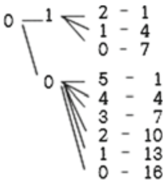


} 2 ways

∴ 2 ways in total

<Ways of measuring 16g>

27g 9g 3g 1g



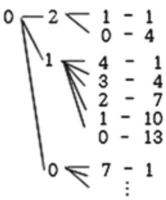
} 3 ways

} 6 ways

∴ 9 ways in total

<Ways of measuring 22g>

27g 9g 3g 1g



} 2 ways

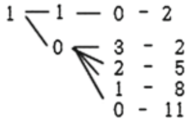
} 5 ways

} 8 ways

∴ 15 ways in total

<Ways of measuring 38g>

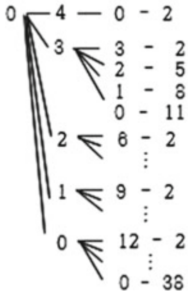
27g 9g 3g 1g



} 1 way

} 4 ways

27g 9g 3g 1g



} 1 way

} 4 ways

} 7 ways

} 10 ways

} 13 ways

∴ 1+4+1+4+7+10+13=40 (ways)

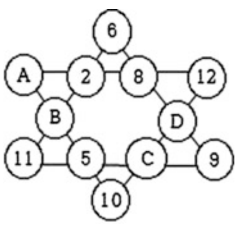
Answer: 5g – 2 ways, 16g – 9 ways, 22g – 15 ways, 38g – 40 ways.

<Didactic analysis>

In this problem a problem solving strategy of <organizing all possible cases> is suitable. Devices for this strategy are various such as diagram or table. In the above solution the tree diagram is used. Because the heavier the poise is like 38 g, the ways of measuring are more various, strategy of <searching patterns> should be used to find the solution easier.

[Problem of Friday]

Numbers from 1 to 12 are written in 12 circles which lie on the intersections of the lines. The sums of numbers in 4 circles on a same line are all equal. Find the number in the circle B.



[Solution]

This problem can be solved by using a strategy of guess and check. To solve systematically,

a. Numbers which are not used yet are 1, 3, 4, and 7.

b. Ⓐ-②-⑧-⑫ → sum of numbers except A : 22

⑪-⑤-Ⓒ-⑨ → sum of numbers except C : 25

⑥-⑧-Ⓓ-⑨ → sum of numbers except D : 23

⑥-②-Ⓑ-⑪ → sum of numbers except B : 19

c. On the line of B which has smallest sum, among the remaining digits, put the largest one 7 in B. Then the sum of numbers of a line becomes 26. So, we can put suitable digits in the remaining circles.

d. A=4, B=7, C=1, D=3

∴ Answer 7.

<Didactic analysis>

I grasped the type of this problem as <an application of addition and subtraction> and <a problem about pattern>. Usually for this kind of problem, most children try to match numbers by substitute numbers randomly. If they matched numbers in that way, students solved the problem using the strategy of <guess and check>.

However if you write expressions according to the given conditions, you can derive easily the right answer in more systematic way. First among lines choose a line which include the unknown [circle] and get the sum of the rest part [numbers in the other three circles]. And if you use the way of putting a large number in a place where the sum is small, you can solve the problem easily, simply, and rapidly.

At first children who come in contact with this problem would put an arbitrary number [in the blank circle]. Then gradually they would begin to find patterns. For the students who feel difficulty, teacher should guide to find patterns. "Is there a part on a line, where all numbers are shown?", "If not, is there a part where 3 numbers are shown?", "If so, put a number [in the blank circle] by estimating the sum of [the other three numbers on] the line" are useful questions and suggestions.

Enhancing Students' Mathematical Capabilities

Krutetskii's studies of mathematical abilities of school children gave me motivation to develop the ways of enhancing students' mathematical capabilities in elementary school level.

Krutetskii summarized factors of mathematical ability as below:

Obtaining Mathematical Information

- The ability for formalized perception of mathematical material, for grasping the formal structures of a problems.

Processing Mathematical Information

- The ability for logical thought in the sphere of quantitative and spatial relationships, number and letter symbols; the ability to think in mathematical symbols.
- The ability for rapid and broad generalization of mathematical objects, relations, and operations. (Generalization)
- The ability to curtail the process of mathematical reasoning and the system of corresponding operations; the ability to think in curtailed structures. (Condensation)
- Flexibility of mental processes in mathematical activity. (Flexibility)
- Striving for clarity, simplicity, economy, and rationality of solutions. (Being economical)
- The ability for rapid and free reconstruction of the direction of a mental process, switching from a direct to a reverse train of thought (reversibility of the mental process in mathematical reasoning). (Reversibility)

Retaining Mathematical Information

- Mathematical memory (generalized memory for mathematical relationships, type characteristics, schemes of arguments and proofs, methods of problem-solving, and principles of approach). (Structural memory)

General Synthetic Component

- Mathematical cast of mind.

These components are closely interrelated, influencing one another and forming in their aggregate a single integral system, a distinctive syndrome of mathematical giftedness, the mathematical cast of mind (Krutetskii 1976, pp. 350–351).

Krutetskii's experiment clearly implies that the process of mathematical reasoning does not always coincide with the psychological process of children. In fact, mathematical abilities of gifted children, including ability of understanding generalized data, prompt generalization, reversibility, flexibility, condensation can be also developed by average students later. Considering differences of mathematical abilities, of course, does not mean ignoring logical aspects of mathematical contents in schools.

For example, in the aspect of reversibility, teachers sometimes present solutions only in one direction and ask students to solve problems that require reversibility of thinking process, assuming students can reverse without difficulties. Students can find $\sin 75^\circ$ after they learned $\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$. However, when $\sin 15^\circ + \cos 15^\circ$ is given, most students are puzzled because this requires reversibility of thinking process.

It may be suggested that students need thinking process disciplines considering their psychological characteristics before they fully develop their mathematical abilities. In other words, you may use a metaphor from baseball that a hitter needs to be trained in stands, swings, and concentrations in order to achieve higher hit potentiality and so do children in mathematics.

In this sense, types of learning activities enhancing students' mathematical capabilities can be listed as follows:

(1) Making problems with given materials
(2) Modifying problems with incomplete information
(3) Modifying problems by removing surplus information
(4) Developing insights of figures
(5) Classifying problem patterns
(6) Making problems by using given types
(7) Crypt-arithmetic
(8) Problems with several solutions
(9) Problems changing contents
(10) Recomposition of thinking
(11) Direct and reverse problems
(12) Logical inferences

Among these activities, <(1) making problems with given materials>, <(2) modifying problems with incomplete information>, and <(3) modifying problems by removing surplus information> aim to develop ability to recognize given facts in problems and their relationships. The activity <(4) Developing insight of figures> trains ability to derive geometrical factors from given figures and backgrounds. Activities <(5) classifying problems patterns> and <(7) crypt-arithmetic> help students to condense their deduction processes. The activity <(6) making problems by using given types> targets to sharpen generalized recognition of problems. Activities <(8) problems with several solutions>, <(9) problems changing contents>, and <(10) recomposition of thinking> pursue to develop flexibility of thinking process and elegant solutions. Activities <(11) direct and reverse problems> and <(12) logical inferences> are for improving reversibility and deducting ability, respectively.

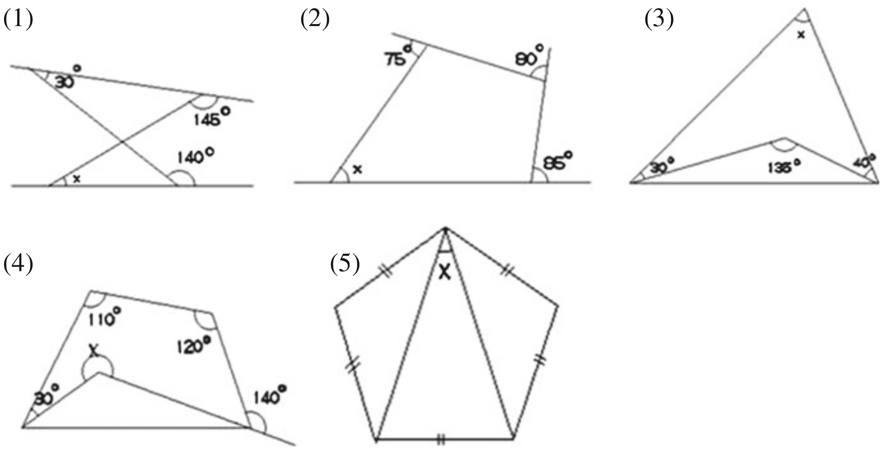
Examples of activities <(10) recomposition of thinking> and <(11) direct and reverse problems> are presented as below.

Examples of Activity (10) Recomposition of Thinking

In case of repeating same types of thinking process, a habitual and fixed thinking process can be settled down, which is referred as “mental set.” Even if mental set may raise efficiency of thinking process, but it can jeopardize flexibility of thinking process when new patterns of problems are given. In general, most mathematics materials are focusing only efficiency of thinking process. In order to broaden thinking ability, disciplines of recomposing thinking process based on appropriate conditions are required.

In the following examples of recomposing, five questions form a cluster of problems, representing categories (1)–(5). In order to solve question number (5), students need to produce different thinking process from question number (1) to (4). Therefore, after letting student solve five questions in order, the examiner are required to closely observe changes of student’s thinking process when he/she start to solve question number (5). And also, students by themselves need to find out difference between the solution of question (5) and solutions of the other previous questions.

[Example 1] Find angle x.



All questions from (1) to (5) require to find angle x using features of interior and exterior angles of polygons. However, the major difference is that no concrete measure of angles is given in question (5) while given in the other questions.

[Example 2] Formulate an equation between x and y.

(1) A bicycle with 38 cm wheels moves y cm after x revolution of wheels.
(2) A rectangular with area 200 has length x m and width y m.
(3) A sector with radius 5 cm has central angle x and arc y cm.

(4) X (the number of notebooks) notebooks can be bought with 2000 won (2$) when the price of each notebook is y won.
(5) Y L of seawater is required to gain 420 g of salt when X g of salt can be obtained if 2L of seawater is evaporated.

From question (1) to (4), students directly find whether the relationship between variable x and y is direct proportional or inverse proportional with one concrete number is given. However, in question (5), students need to set up a proportional expression to find out the relationship between x and y with two concrete numbers are given.

Examples of (12) Direct and Reverse Problems

In many cases, mathematical principles and formula are formed by two-way thinking processes such as symmetric relations, operations and inverse operations, theorems and inverse theorems. However, most students tend to use principles and formula only in one way just as the way they learned. For advanced mathematical thinking, prompt reversibility of thinking process is required.

Examples of direct and reverse problems below are presented in pairs of direct problem A and reverse problem B. While reverse problems use similar materials as in direct problems, given data of problem A become the unknown of problem B, and vice versa. The examiner let student solve direct problem A first and right after turn to reverse problem B without any explanation on solutions. It is helpful to observe how and how fast the direction of student's flow of thinking changes. At this point, the fact that problem B is actually reverse version of problem A does not need to be notice to student.

[Problem A] Worker A and B laid 500 and 700 bricks, respectively. The wage is proportional to the number of bricks they stacked. If the total wage of two workers is 180,000 won, what is the wage of A and B each?

[Problem B] The total wage of worker A and B is 48,000 won after they stacked bricks. The wage is proportional to the number of bricks they laid. If worker A stacked 200 bricks and received 30,000 won, how many bricks were stacked by worker B?

Problem A can be solved by dividing total wage (180,000 won) in the ratio of the number of bricks (5:7), while problem B requires to find ratio of wages of worker A and B first, and multiply this ratio by the number of bricks that A stacked.

[Problem A] Anne, Tim, and Rachel divided 24 marbles in ratio of 3:5:4. How many marbles does each of them have?

[Problem B] Beth, Daniel, and Joshua have marbles in ratio of 5:7:3, respectively. If Beth has 15 marbles, how many marbles do Daniel and Joshua have?

Problem A demands ratio distribution while problem B is involved with the ratio of marbles that Beth has.

[Problem A] If a person was loaned 500,000 won with monthly interest rate 1.3 % for four months, what is the total amount that he should pay back?

[Problem B] A person had deposited 150,000 won with annual interest rate 10 %. When he withdrew, the amount became 195,000 won. How long did he deposit the money?

Problem A requires the amount with interest when original amount, interest rate, and period were given, and problem B demands the period when original amount, interest rate, and the amount with interest were given.

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Constructionism: Theory of Learning or Theory of Design?

Chronis Kynigos

Abstract Constructionism has established itself as an epistemological paradigm, a learning theory and a design framework, harnessing digital technologies as expressive media for students' generation of mathematical meanings individually and collaboratively. It was firstly elaborated in conjunction with the advent of digital media designed to be used for engagement with mathematics. Constructionist theory has since then been continually evolving dynamically and has extended its functionality from a structural set of lens to explanation and guidance for action. As a learning theory, the constructionist paradigm is unique in its attention to the ways in which meanings are generated during individual and collective bricolage with digital artefacts, influenced by negotiated changes students make to these artefacts and giving emphasis to ownership and production. The artefacts themselves constitute expressions of mathematical meanings and at the same time students continually express meanings by modulating them. As a design theory it has lent itself to a range of contexts such as the design of constructionist-minded interventions in schooling, the design of new constructionist media involving different kinds of expertise and the design of artifacts and activity plans by teachers as a means of professional development individually and in collective reflection contexts. It has also been used as a lens to study learning as a process of design. This paper will discuss some of the constructs which have or are emerging from the evolution of the theory and others which were seen as particularly useful in this process. Amongst them are the constructs of meaning generation through situated abstractions, re-structurations, half-baked microworlds, and the design and use of artifacts as boundary objects designed to facilitate crossings across community norms. It will provide examples from research in which I have been involved where the operationalization of these constructs enabled design and analysis of the data. It will further attempt to forge some connections with constructs which emerged from other theoretical frameworks in mathematics education and have not been used extensively in constructionist research, such as didactical

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design and guidance as seen through the lens of Anthropological Theory from the French school and the Theory of Instrumental Genesis.

Keywords Constructionism · Design · Meaning generation · Theory networking · Digital media

Introduction

Constructionism was a term coined by Seymour Papert almost half a century ago with a two-legged agenda based on the context and time of the emergence of constructivist theory as a more sensible approach to study and understand learning than the prevailing behaviorist frame at the time, which had been established a few decades before that. Papert's agenda was on the one hand to challenge Piaget's constructivist theory by drawing our attention to the meanings actually generated by learners rather than to describe their shortcomings in understanding taken-as-ontological meanings at different stages in life. It was also to change the perception that concrete thinking was a 'lesser' kind of cognitive process in relation to abstract thinking by pointing out that proper and rich exposure to the former was pivotal in ever hoping to reach the latter (Ackermann 1985). On the other, it was to claim that meanings are naturally generated in our social, intellectual and physical environment and that digital technology makes it possible for us to enrich this environment so that learners would enjoy more opportunities for the formation of meanings. Papert's work has been an ode to kids mathematical thinking, he has been provocative in arguing that we have not paid enough attention to how children think mathematically and to the nature of meanings they form given the language and tools they use, the activities they engage in and the communicative situations they find themselves in. He also argues that for children, a key to learning is the process of engagement in activity, the ownership of ideas and style of learning and the exposure, i.e. expressing their ideas to others, for reasons of exploration and communication.

So, what happened to constructionist theory since its first articulations? What has the mathematics education community learned so far and how has it been put to use? Has the theory been developing all these years or is it a well recognized but now rather blunt instrument associated with outdated technologies and ideas of how they can be used for learning? We live at a time of growing connectivity and resource availability, at a time where 'watch and practice' technologies and administrational infrastructures are popular and politically publicized and immersion in collective virtual worlds where mathematical representations are given very low priority by media designers. We also live at a time where several theoretical frameworks and constructs in mathematics education are in danger of lying in fragmentation each to be used by a community of researchers close to the context from within which it emerged. So, is constructionism relevant and useful today and in what capacity? Can it be meaningfully connected to other theories and is there some mutual benefit from trying to forge such connections? How can it be useful in the age of jings, blogs, portals and LMS? Is there scope for further development of constructionism theory in an era of ever-changing technologies and a wealth of theoretical frameworks and constructs and how can this be justified? In this paper I attempt to contribute to the argument that constructionism is essentially an epistemology creating continual need for an evolving theory of learning in collectives and individually and at the same time a theory of design of new digital media, new kinds of activities facilitating the generation of meanings and techniques and processes for systemic interventions at various levels such as school cultures, resource systems and educational systems.

Epistemology

Constructionism started out as an epistemology of mathematics as a discipline and of doing mathematics. Mathematics is portrayed as a human construct under continual development rather than an ontological reality to be explained or as the production of unquestionable truths. Papert perceives mathematics as essentially fallible (in the sense of Ernest 1980), i.e. that each mathematical definition, lemma, theorem, proof, has the status of a proposition for others to try to refute. Even in cases where theorems are proven and problems are solved, they lie in wait for other mathematicians to question the process, the context, the point of view of articulating the question, the assumptions. This process is essentially part of doing mathematics and it does not matter in the end whether and to what extent an axiomatic system remains robust or a theorem is proven to have shaky foundation. Engaging in mathematical activity necessarily involves the process of refutation together with logical thought, deduction, generalisation and proof. Lakatos was at the time equally provoking in his book 'proofs and refutations' (Lakatos 1976) where he analysed mathematicians' activity as a process of conjecture as a public expression of thought and subsequent engagement with a cycle of refutation, redrafting and new proofs.

> …deductivist style tears the proof generated definitions off their 'proof-ancestors' presents them out of the blue in an artificial and authoritarian way. It hides the global counterexamples which led to their discovery. Heuristic style on the contrary highlights these factors. it emphasises the problem-situation: it emphasises the 'logic' which gave birth to the new concept (Lakatos 1976, p. 144).

Papert (1972) argues that this kind of mathematical activity cultivates learning not only in established mathematicians but for all, even for young children. He goes further to say that this activity is natural and that traditional schooling somehow denies students the opportunity and encouragement to engage in the logic which gives birth to concepts as Lakatos put it, imposing an artificial picture of mathematics to be the practice of trying to understand the abstract and irrelevant products of mathematical activity rather than the activity itself. Papert discusses mathematical activity which is based on the expression of mathematical meanings through the use of and tinkering with representations in the form of digital artifacts. His argument was that this technology could play an important role in generating learning environments which would be more authentic, rich and dense in opportunities for the type of mathematical activity which was termed as 'heuristic style' by Lakatos. Papert was inspired by the ways in which computer scientists engaged in solving difficult algorithmic problems by using LISP programming as an expressive tool for this kind of heuristic process and thought that there was no reason not to develop a LISP-like tool for children to do mathematics with. Logo was the tool he built together with Feurzig (Feurzig and Papert 1971) maintaining the LISP philosophy for programming as an emergent problem solving process (Sinclair and Moon 1991) but changing the syntax and most importantly the object and the objective of programming to Turtle Geometry. The choice of Geometry was not accidental. At the time, Geometry was perceived by educationalists as a problematic domain for learning mathematics. Although it carried a unique potential to connect children's experience with space to mathematical meaning making, educational approaches had steered it away towards a problematic attempt to teach deduction in an abstract explanatory way. At the time, Freudenthal (1973) had a similar approach to learning mathematics in his discussion about the educational potential of geometry:

> the deductive structure of traditional geometry has just not been a didactical success. people today believe geometry failed because it was not deductive enough. In my opinion the reason was rather that this deductivity was not taught as reinvention, as Socrates did, but that it was imposed on the learner. If geometry as a logical system is to be imposed upon the student it would better be abolished (73, p.402-406). Geometry can only be meaningful if it exploits the relation of geometry to the experienced space. If the educator shirks this duty he throws away an irretrievable chance. Geometry is one of the best opportunities that exists to learn how to mathematize reality (p. 407).

The essence of Turtle geometry as a context for mathematical programming was not only its differential kernel (the state of the turtle entity is defined by its difference to its immediately previous state) but also connectivity with children's experienced space though the embodied metaphor of the turtle and its state changes, something which Papert called 'body syntonicity' (Healy and Kynigos 2010). So the idea was to embody this digital medium with a coherent mathematical kernel, a mathematical language connecting formalism to graphical representations and a naturalistic metaphor to enable even young children to engage in heuristic activity. The latter issue reflected an epistemology towards learning itself which was constructivist in nature.

Papert called Turtle Geometry a mathematical microworld, a term which became central to constructionism and was originally defined as:

> self-contained worlds where students can "learn to transfer habits of exploration from their personal lives to the formal domain of scientific construction" (Papert 1980, p. 177).

In a recent review, Healy and Kynigos (2010) reflect on the ways in which the microworld idea was developed, used and modified within the framework of constructionist theory. They argue that although initial focus was on microworld as a digital object it quickly became apparent that it made much more sense to discuss the term in association with the kinds of activity emerging from the use of microworlds and the scope of each microworld with respect to the conceptual field it was designed to embody. Healy and Kynigos point out that these issues are apparent in a more recent definition of microworlds as computational environments embedding a coherent set of scientific concepts and relations designed so that with an appropriate set of tasks and pedagogy, students can engage in exploration and construction activity rich in the generation of meaning (Sarama and Clements 2002). In a reflective paper discussing dynamic geometry systems and Turtle Geometry Balacheff and Sutherland (1993) use the term 'epistemological validity' to denote that the design of expressive media for mathematical learning inevitably incorporates an epistemological approach to mathematics and to the learning of mathematics.

So, even though research on student activity based on the use of microworlds focused on contributing to learning theory, to the design of digital media and educational interventions and to the processes of learning and teaching as design activities, what has been pertinent in constructionism is epistemology. Constructionist epistemology has shown to be quite challenging at many levels including institutional levels, using the work in the sense used in the Anthropological Theory of the Didactic (Artigue et al. 2011). It has caused both enthusiasm and reaction in many cases and has certainly given inspiration for the term used by Laborde in another context 'perturbation' of norms and mindsets (Laborde 2001). What is interesting is that after so many years it continues to inspire and perturb at the same time and find itself relevant in many societal and technological issues emergent today. Take for instance the instructionist 'watch and practice' paradigm which has recently been enabled technologically and has gained such kudos by the media, the educational administrators and policy makers. The argumentation behind it is that since the instructional part of mathematical education is now covered without the need to spend classroom face to face time, there is space for better supporting experiential learning of mathematics be it constructionist or inquiry learning or problem solving. How sound is this argument and how can it be approached from an epistemological basis?

Learning Theory

The visionary nature of applying this kind of epistemology to mathematics education for young children coupled with the new technology just made available motivated researchers to study both the process of mathematical meaning generation and the nature and use of meanings. The early eighties was good timing for researchers to investigate mathematical learning processes since a concurrent problem solving movement emerged at least in the US and was applied widely in schools. Process oriented studies identified types of student activity such as (U.D.G. S.) Using, Discriminating, Generalising and Synthesising (Hoyles and Noss 1987) and characterised students' abstractions as situated (Noss and Hoyles 1996), i.e. as emerging from the specific situations including the problem, the notation (such as for instance the notation for variable values), the computer feedback, the types of argumentation around the figural models built with the turtle.

The emergent meanings were inevitably seen as tightly connected with the activity of working with Turtle Geometry. The first characteristic of such activity was the style envisioned by Papert and pedagogically encouraged by researchers in the role of teachers which he characterised as 'bricolage', borrowing the term from the Bourbaki school of thought in France at the time. The idea was that students used the computer to construct models of figures by means of programming the turtle. These models, both their figural and formal descriptions were seen as externalized expressions of ideas and thoughts. The act of constructing a model constituted making a sequence of ideas public for discussion and change. The models or artefacts thus had the status of being malleable, of being questionable or improvable propositions for an on-going discussion around subsequent changes to the artefacts. They had the same role as Lakatos' mathematical propositions expressed by mathematicians with formal notation in order for a refutation and proof dialectic process to start. Only here, the representations albeit mathematical were connected and designed for youngsters to use and give meaning to, the expressive tool used provided feedback, and interactivity and the constructions were extensible (Papert 1980). Mathematical meanings we generated at different levels, at the level of using the notation to express an idea, for instance a variable to express generalized number, at the level of a concept underlying a model and its behaviour or at the level of the model itself, i.e. the model of a variable pentagon and the patterns which can be constructed with this as a building brick.

But of course this kind of learning does not happen just by providing students with the media. Research showed that just like giving the opportunity for constructivist learning by no means sufficed for students to go on and become advanced mathematical thinkers, poorly designed pedagogically unsupported constructionist activity with Turtle Geometry quickly resulted in a-mathematical constructions and plateaus with respect to mathematical development. Furthermore, it showed that the meanings themselves were idiosyncratic and bound to the situations from which they emerged, students found it difficult to synthesise them with more general applications of the same concepts or with traditional mathematical notations. Kaput et al. (2002) describe this mathematics as learnable but also as a different kind of mathematics which is shaped by the users of the medium which in turn shape the medium itself. Edwards talks about functional and structural views of microworlds (Edwards 1988) and more recently Rabardel's ergonomy theory of Instrumental genesis has been elaborated for the use of digital media for mathematics education to highlight the ideas of instrumentation and instrumentalisation (Verillon and Rabardel 1995; Guin and Trouche 1999; Artigue 2002). Both constructs are elaborations of the idea that expressive media are shaped by the people who used them who in the process form evolving schemes regarding the meanings underlying the use of the media.

There has of course been extensive research on the meanings generated by students around specific mathematical concepts which could be mapped onto any standard curriculum. Some such studies have been on symmetry (Healy and Hoyles 1999), geometrical systems (Kynigos 1992, 1993), properties of geometrical figures and functions (Hoyles and Noss 1987), generalised number (Sutherland 1987), fractions and proportional thinking (Psycharis and Kynigos 2009), trigonometric functions (Keisoglou and Kynigos 2006; Kynigos and Gavrilis 2006), navigational mathematics Yiannoutsou and Kynigos 2004), angle and spatial geometry (Kynigos 1997; Clements and Batista 1992; Latsi and Kynigos 2011). There has also been research addressing constructionist learning through the design of digital artifacts (Kafai et al. 1998; Kafai 2006). This research was mainly done in the context of students designing games and playing them, focusing on the mathematical rules of the games. In this paper we will not extensively cover this research, more can be found in Kafai and Resnick (1996). This research has moved on to consider design in teacher professional development contexts where teachers design artifacts and scenarios for student activity creating thus a context for pedagogical reflection (Kynigos 2007a).

Constructionist activity has been addressed from early on as a social activity where typically students discuss and argue over working with a microworld (Hoyles and Sutherland 1989; Hoyles et al. 1992). Meaning generation is afforded by a conjunction of noticing computer feedback, using the representations and the semantics of the microworld and generating a language to express arguments in the context of this kind of activity. It has taken some time for the wider education research community to notice that constructionism was not perceived as an individualistic learning theory and at the same time for constructionists to develop more explicit language and frameworks for the social discursive aspect of this kind of learning (Artigue 2009a, b; Kynigos and Theodossopoulou 2001).

Apart from the early research which perceived constructionist learning as essentially discursive (Hoyles et al. 2002; Simpson et al. 2007), special attention was given to the design of digital support for collective constructionist learning. The first was that of distributed constructionism, the second involved literal artifact exchanges across the net and the third study which is introduced to the mathematics education community and was on-going at the time this paper was written addressed the idea of socio-metacognition, i.e. students learning to learn in collectives engaged in constructionist activity.

Distributed constructionism was the first attempt to focus on collaboration and discussion around and about constructions (Resnick 1996). It was based on the idea of combining constructionism with distributed cognition, i.e. that is addressing learning 'not as a property of a person but as the process of interaction with others and the environment' (p. 281). Computer networks mediated three main types of constructionist activities enabled by the available networking infrastructure at the time. The first type involved discussing constructions through email, newsgroups and bulletin boards, the second type of activity involved sharing constructions which made use of a web based version of Logo which allowed file storing in LogoWeb, a network created to support this activity due to technological restrictions that did not allow the use of world wide web, which offered easy access to the files among users. The third type of activity involved collaboration on constructions where students were expected to collaborate in real time in one project. The aim of the project as described by Resnick (1996), was the creation of an ocean ecosystem where each student was expected to program the behavior of an artificial fish and place it to function in the ecosystem. Student collaboration was facilitated by chat tools and focused on the programming and functioning of the ecosystem through discussions emerging from the observed interactions of the programmed fish.

The concept of sharing discussing and collaborating constructions through the internet was revisited in 2002 in a multi-organizational European project titled 'WebLabs' which was based on an explicit attempt to develop technologies supporting the social aspect of constructionism. Web Labs built a system that allowed early teenage students to construct models of their emerging mathematical ideas, to share the models and to pay attention to the process-based descriptions of the models (Simpson et al. 2007). Collaboration in weblabs was mediated by "webreports" which included working models—not just descriptions of models—along with multi-media descriptions, interpretations and reflections. According to the project' design, the complex behavior of generating meaning by tinkering with the rules with which digital models operate, questioning their behaviors and thinking about the phenomena they simulate requires the students to have shared models, argumentation skills, and practice with reflection on their learning activities.

Another approach to collective constructionism has been taken in an on-going multi-organizational project in the European TEL setting titled 'METAFORA'.[1] The project is about socio-metacognition, i.e. enhancing students' awareness of group learning processes and their roles in learning collectives. The phrase used in the project is 'learning to learn together'—L2L2 (Dragon et al. 2013). The project is based on a wider 'challenge based learning' paradigm where attention is given to learning process in students addressing challenges consisting of fuzzy broad realistic problematic situations. A challenge typically refers to a complex open problem situation relating to a real or realistic phenomenon within some kind of socially relevant situation. In a constructionist challenge the pedagogical intervention strategy and the design of the challenge is such that students will try to understand the rules, properties and relations underlying a model in order to change or re-create a model of their own, generating meanings from mathematics as they work. In METAFORA students are encouraged to focus on how they learn with and from each other as they work in groups to solve challenging problems in science and math. L2L2 is perceived as a complex enterprise not easily decomposed into a set of underlying skills. However some key skills were identified as necessary to any process in which students are learning together, and on a higher level, learning

[1]The Metafora project is co-funded by the European Union under the Information and Communication Technologies (ICT) theme of the 7th Framework Programme for R&D (FP7), Contract No. 257872.

how to learn together. The students must gain skills at working collectively to be more successful group learners. These competencies included Distributed leadership, Mutual engagement, Help seeking/giving and Reflection on the group learning process. This process of group meaning generation requires skills from across the different aspects of L2L2, organizing, discussing, seeking and offering help from peers when needed. A digital L2L2 built within the METAFORA project integrated tools for group planning and discussion with microworlds for students to collectively engage in constructionist activities. Within this framework, the "social" in constructionism involves not only the process (discussion, sharing and collaboration) but also the output that is, the actual product of construction. Constructions have the special status as public entities which means that they are open for others to inspect how they work, to comment on the underlying mechanism of the construction, to change, customize and re-use parts of this mechanism. These activities which focus on "how things work" and which are central and rather unique in constructionist environments can shape collaborative activity and differentiate it from other learning environments (such as inquiry based learning). So, the question raised here and that is going to be further investigated in "METAFORA" is how the placing of constructions at the centre of collaboration shapes L2L2 activity. Microworlds in this case are perceived as boundary objects which offers an analysis of how meaning is generated through discussion and negotiation about constructions among participants that belong to different communities (see also Hoyles et al. 2004; Kynigos 2007b).

Artifact Design

The design of constructionist media quickly expanded beyond mathematics education. Even from the days of the MIT-initiated LCSI Logo the idea of programming as a means of expression for all, a new kind of literacy, grew quite rapidly making the idea of constructionist mathematics look like a special case of something outside the mathematics education community. The idea of constructionism without programming in the sense of using a formal language also gave birth to digital applications, such as the use of macros, building with ready-made construction kits. Furthermore, programming got fused in icon-driven interfaces and semantics making the use of a formal language not necessary.

Important ideas however emerged for constructionism not necessarily addressing the learning of mathematics per se, such as that of principled design to enhance deep structural access to the functionalities of digital tools (diSessa 2000), the idea of interplay between private and public expression, of transparency (Resnick et al. 2000; Eisenberg 1995) and of black-and-white box designs (Kynigos 2004).

There were however issues particularly pertinent to mathematics education. Firstly, the issue of programming as a mathematical activity. Logo was intentionally designed to be as mathematical as possible in syntax, rules and semantics maintaining however in full its property of being a full functional, list processing, recursive and structured programming language (Harvey 1993). However, the very role of a mathematical formalism to be the notation for programming involved differences to traditional formalism in mathematics (Abelson and diSessa 1981). In the old days, this was a relative detail since there was no other way to approach mathematical expression with a computer. However, in the past decade with the advent of mathematical text editors and Computer Algebra Systems a programming language is a programming language and the attempts to call it mathematical formalism look feeble. There is a deeper issue at hand however than wether the connection between mathematical formalism and programming is technically avoidable: it is that now we can reflect on how conducive is formal representation to learning mathematical thinking, especially constructionist mathematics. This is an open question: there are those who perceive formalization as an obstacle for students to access mathematical meaning and thus a design objective for digital media is to find ways to by-pass formalism by means of alternative representations (Laborde et al. 2006; Kaput et al. 1992). There are those who argue that digital media has made it possible for formalisation to become meaningful to students since they can use it to construct models and other mathematical representations such as graphs (Kynigos and Psycharis 2009; Dubinsky 2000; Artigue 2002). The proponents of the latter perceive formalization as one of the representation registers which lie interconnected in digital media and that meaning is generated through expression with connected representations precisely because students gradually dissociate meaning from a single representation.

Another issue is why program in order to do mathematics now that it's not necessary? Dynamic Manipulation Systems, Computer Algebra Systems, Data Handling systems and even simulation authoring systems allow for constructionist learning often with mathematical representations but only with an indirect notion of programming and certainly not by means of a formal programming language.

Furthermore, technical advances in e-book technologies are allowing for the consideration of narratives containing constructionist 'widgets'.

This is now been attempted in the 'M C Squared' project where 'c-books' (c for creativity) invite the reader to also tinker with a variety of widgets and the author of such c-books to re-think the role of the reader and the this particular resource.

Intervention Design

The previous section discussed some issues of constructionist media design pertaining to mathematics education. As more technological choices become available, it is a continual task to re-appraise what is a rich constructionist tool for doing mathematics with respect to the representations, to the functionalities and semantics of the medium, the ways representations can be manipulated and to the relationship of constructionism to programming and the exact definition of programming itself. In the section before that, the notion of design as a learning process was discussed within the framework of students designing games and other digital artefacts.

In this section we look at a different aspect of design, that of designing and implementing an intervention in an institutionalized educational practice. This issue has been termed 'design research' and has been elaborated in the past 10 years or so as a research method (Cobb et al. 2003). Researchers working within the frame of constructionism however, always had a special agenda for intervention and challenge in some established educational practice. Artigue uses the term 'concerns' to describe researchers' agendas which are not fully elaborated or expressed even in academic publications yet shape the method and the produced knowledge to an important extent (Artigue 2009a, b). Constructionist epistemology was from the beginning articulated as something which was real and natural but which the institutionalization mechanisms and processes had somehow corrupted or diverted mindsets, values and practices to more artificial views of knowledge teaching and learning (Papert 1972). The growth of very large communities outside educational institutions such as the Scratch or the NETLOGO communities are an indication of this (Brennan et al. 2011). An important part of constructionist research was carried out to challenge institutionalized education through a design research paradigm. There are different ways of synthesising such research of course. In this section we look at constructionist interventions at the level of teacher professional development, the school as an educational institution and the education system at a wide scale (Childs et al. 2006; Healy and Kynigos 2010).

With respect to teachers, an early study—the 'microworlds' project (Hoyles et al. 1991)—centred on allowing a design role for teachers by supporting them in the design and construction of microworlds. This focus has seen the development of new methodologies, tools and experiences of teachers as designers has been progressing (Funglestad et al. 2010). What does design for constructionist activity have to offer, not only to planning for and assessing learning but also to professionals engaging in such designs (Kynigos 2007a). In the framework of exploring such issues, the idea of "half-baked" microworlds as digital artefacts designed so that learners would directly engage with, question and change their structure was introduced (Kynigos 2007b). These are artifacts explicitly designed so that their users would want to build on them, change them or de-compose parts of them in order to construct an improved artifact for themselves or one designed to be changed by others. They are meant to operate as starting points, as idea generators and as resources for building or decomposing. The essence of such microworlds is not only that they are built to allow changes but also that they are mediated as malleable, questionable and improvable objects. Half-baked microworlds can be designed for teachers to engage in pedagogical or epistemological reflection as they de-construct or re-construct the microworlds in a context of designing tools for students. Half-baked microworlds operate like diSessa's toolsets (diSessa 1997) in that they are not built and presented as ready-made environments to be understood by the teachers and then used by students. Instead, the point is to change and customize them and thus to gain ownership of the techniques and the ideas behind microworld construction. In this way the operate s boundary objects for teachers and researchers to discuss pedagogy and intervention agendas (Star and Griesemer 1989; Kynigos 2007a).

Constructionism has naturally not only been perceived as a theory for the design of tasks and digital artifacts for students. That would mean that to get students to work in a contructionist way would be unproblematic in traditional schools and schooling systems. On the contrary, both constructionist epistemology and practice have been portrayed by Papert himself as constituting a challenge involving the need for re-thinking the educational paradigm of schooling and the epistemological approach of mathematics (Papert 1993). This rationale came about in the historical context of some bold attempts for curricular changes in the United States based on the problem-solving movement which focused on problem-solving methods. Constructionism was seen as the ideal boost to the movement through the use of technology. Inevitably, the attention of the wider community was on the technology that accompanied the constructionist vision at the time which became a victim of the educational innovation pendulum (Agalianos 1997; Noss 1992). Constructionism was seen as a technological ticket to generate mathematical thinkers on a wide scale. When this not surprisingly proved to be a much lengthier and complex issue than the advent of a specific technological support, constructionism was in many ways attached both to a specific ageing technology and to a sense of an unfulfilled promise at the cultural level (Papert 2002). Technologies were very hard to access, machines were slow and non-dependable, the internet had not arrived and people were not widely using digital technologies in their daily activities. Many of these disappointing features were attributed to the original perspectives and theories of how digital media could be used for mathematical learning.

Constructionist interventions at the level of school originated as far back as the early eighties either as a school wide implementation of an innovation (Noss 1985) or at the level of a longitudinal mathematics curriculum for a particular group of students (Hoyles and Sutherland 1989). They also addressed wider issues of implementing an innovation as a means to challenge epistemology and school culture (Blikstein and Cavallo 2002; Kynigos 2002). Both at the school and at the institutional level however, these interventions did not generate any kind of self-growing culture change. On the contrary, the system diluted the essence of constructionist learning in a number of ways insightfully analyzed by Hoyles (1993).

Up till now, however, constructionist interventions in educational institutions have been perceived by the community at large as designed by researchers for radical innovation to be implemented in small scale situations. The era when this approach was valid and seductive to the research community seems to be ending. There is now demand for large scale initiatives and accreditation of new efforts before they have had the chance to become infused in educational curricula. The ideas behind the constructionist culture are proving hard to grasp and accept not only by school systems but also by other stakeholders in education such as new computer science and telecommunication communities. A mathematics designed with a constructionist agenda in mind can only become part of school mathematics if the associated practices are given legitimacy by the various stakeholders involved. This poses new challenges to the community of finding methods and avenues for communicating these ideas to a wider audience in a language which is widely understood (Papert 1996, 2002).

This kind of language can only emerge from experience of communicating and negotiating ideas and reform processes with such communities, it is not something which can be defined at a theoretical level. The process will require being explicit about what happens when collaborative design and implementation is taking place. An example of such an effort in a broader context of digital media in mathematics education including microworlds is that of a European project titled 'ReMath, Representing Mathematics with Digital Media' where six research teams engaged in cross-experimentation in order to develop a more specific language to communicate theories, contexts and use of representations (Bottino and Kynigos 2009).

Connectivity and Networking

Within the constructionist community, there is recent discussion on whether constructionism is better addressed as an epistemology or a learning and design theory. At the crux of the discussion is not only what is more prevalent in the articulation of constructionism and in the ways it has been put to use in designing and studying educational processes but also, if it is to be addressed as a learning theory what kind of theory has it turned out to be. There are several possible reasons why reflections on the essence and the status of contructionism as a theory for mathematical learning have been intensified in the past few years. Some may be seen as a result of challenges put to the community by other developing theories in mathematics education and also by pendulum-like trends in educational systems, such as the different versions of back-to-basics- reforms. Others, as a result of perceiving constructionism as a theory prescribing a method through which deeper mathematical understanding will be achieved, attaching, that is, an element of predictability and controllability to the theory (see for instance the Pea—Papert debate in the eighties, Papert 1987).

A different kind of challenge comes from the changing trends in what is considered as added value in the uses of digital media in education and in society at large. A first wave was the dynamic manipulation and mathematical text editing technologies which questioned programming as an effective mathematical meaning-making activity. Programming and formal code were seen as a kind of unnecessary noise to doing mathematics. A second wave was the advent of social media, portals, LMS and the recently widely advertised 'watch-and-practice' video portals considered as an infrastructure relieving teachers of the need for frontal lecturing. This means that attention is currently given to the use of technology for mathematics education which supports traditional curriculum delivery so that human time and focus can be given to discussing questions and supporting the generation of meaning. So is constructionism going to be considered as an unnecessary noise to content delivery?

Reflections on the place and role of contructionism in amongst mathematics education theories however have also emerged as a result of a wider initiative to consider the landscape of theories in the field, to better identify their nature, status and functionalities and to develop strategies for integrations amongst them so that there is a better understanding and communicability of the progress of mathematics education as a field to stakeholders outside academia and educational reformers. These initiatives began without reference to theories in the use of digital media dating from the CERME conference in 2005 and have already output important literature on the ideas of integration and networking between theories (for example, Prediger et al. 2008; Niss 2007).

Significant work on bringing constructionism and other theories developed or shaped to study the uses of digital media for mathematical learning into this game was done through the work of six European research teams for a period of 6 years [2004–2009, the TELMA European research Team in the Kaleidoscope Network of Excellence and the European Information Society Technologies programme (FP6) titled 'Representing Mathematics with Digital Media' (ReMath)]. Theories such as the Anthropological Theory of the Didactique, The Theory of Didactical Situations, Social Semiotics, Semiotic Mediation, Activity Theory, Instrumental Genesis were considered together with Constructionism to be part of the same phenomenon happening more widely in mathematics education, i.e. a fragmentation and polysemy slowing down and diluting the production of knowledge in the field. The teams worked under the initiatives of Michele Artigue (2009a, b) to elaborate a process of networking amongst these frameworks initially at the level of conceptualizing and proposing a networking process and subsequently at the level of operationalizing the process to actively articulate connectivities between frameworks through joint research. The initial framing of the networking process involved an articulation of these theories through the lens of their didactical functionality and the language of concerns. Special attention was give to the aspect of representations of mathematical concepts through digital media (Artigue 2009a, b) and the formative influences of the context of the educational system and the processes of design and development of both media and research interventions (Kynigos and Psycharis 2009). Pilot methods for implementing research enabling this kind of networking were originated in the TELMA ERT involving the process of cross experimentation, i.e. a research team designing and carrying out research based on the use of a digital medium designed and developed by another and vice versa. In the ReMath project, networking involved the whole cycle of designing and developing six original state of the art digital media for learning mathematics, the design of interventions and classroom experiments and the implementations of these analyzing students' meanings in realistic classroom situations. Several networking tools were developed for cross experimentation which operated as boundary objects to identify and articulate connectivities between frames. A key element of the project was the cross-case analysis of these studies, i.e. an integrated meta-analysis or two research studies carried out by two different teams in respectively different contexts involving the use of the same digital artifact. This section discusses a set of three examples of the formation of an elaborated language connecting theoretical frames through the ReMath work and in particular as a result of the cross-case analysis method. The examples are draw from the ways in which one particular theory, constructionism, which was developed specially as a tool to think about learning mathematics with digital technologies, was connected to three different theories in respective cross-case analyses. These were two-way connectivities articulated between constructionism and (a) instrumental theory (Kynigos and Psycharis 2013), (b) social semiotics (Morgan and Kynigos 2014) and (c) the anthropological theory of the didactique (Artigue and Mariotti, in preparation). What is particular about the enterprise of connecting constructionism with other theories is that as perhaps the oldest theory on this particular issue, it has had enough time to become fragmented largely due to its interpretation as a static theory and in parallel, enough time has passed for it to evolve and develop from a theory focusing on the individual to addressing social and distributed cognition, many types of technologies and representations, new ventures such as for instance the design of activities and interventions and most importantly interventions challenging institutions. This developmental nature has not really been recognized or noticed much outside the constructionist community and yet connectivities with at least some other theories could provide mutual benefit and reveal complementarities useful to elaborate in the future.

Take for instance the theory of instrumental genesis. With respect to connectivity, it was originally seen as a tool to explain the instrumentation of CAS-based techniques as discussed earlier within an anthropological framework. There have also been some perceptions of IG providing a more elaborated tool to describe the process of mediation within the framework of Activity Theory (e.g. Lagrange and Vandenbrouk, in preparation). IG has given a lot of attention to instrumentation as a notion to describe what happens when digital artifacts are put to use by denoting the formation of a conceptual schema which users develop about the functionality of the artifact in question, the underlying concepts, the kinds of things is can be used for, the meaning of its representations etc. The process of instrumentation has been seen as incorporating changes made to the medium itself and this aspect has been termed intrumentalization. Instrumentalization was coined to show that the artifact itself is shaped by each individual through its use and that there is a reciprocal relationship between these two processes, i.e. that instrumentation is affected by instrumentalization and vice versa. Little attention however has been given to instrumentalization itself. Activity theory was not articulated at a time when the medium was susceptible to functional and operational changes as is the case with digital media and therefore gave no detail into the process by which schemes of artifact use were formed through the mediation of artifacts. Instrumental theory identifies instrumentalization and situates this process within the context of mediation and schemes of use but does not elaborate on its definition. What is meant by changes to the artifact? What constitutes a change? What constitutes a change which is relevant to instrumentation and are there changes which are less relevant or irrelevant? Is instrumentalization a process which inevitably happens during instrumentation or does it depend on the design and the nature of the activity and on the nature of the artifact. Are there artifacts which invite instrumentalization more than others? What are the issues involving the design for instrumentalization (Kynigos and Psycharis 2013). These ideas are coherent with the notions articulated about a decade earlier by Noss and Hoyles (1996) that a medium shapes the mathematical meanings generated through its use and at the same time is itself shaped by use reciprocally. What is interesting however is that the design element of constructionist theory offers a more elaborate articulation of the process of designing media so that they afford useful and rich kinds of instrumentalization. A relevant notion here is that of 'half-baked microworlds' developed by Kynigos (see e.g. 2007a, 2009), i.e. digital artifacts intentionally designed and given to students as malleable and improvable asking of them to engage in discovering faults and shortcomings and changing them. This process is at the heart of fallibility and bricolage activity and discusses instrumentalization processes through a language of concerns pertaining to design and meaning generation.

With respect to social semiotics the focus is on the use of external representations and their connectivity and interdependence in digital media. Each medium carries one or more interconnected representational registers which are used together with traditional representations such as language, tangible manipulatives, gestures and written language. Digital media have the particular property that representational registers are connected and therefore expressing on manipulating one representation has immediate reciprocation on the other. They also have the unique property that representations can be dynamically manipulated and the manipulation becomes part of the representation itself. The social semiotics perspective takes a pragmatic view of the use of these registers alongside with others outside the medium such as language and pencil-paper notations (Halliday 1978). What constructionism seems to bear on such a perspective is an educational weight, i.e. on addressing a representation as a facilitator or an obstacle to the generation of meanings, of designing representations for the former and of giving primary importance to one representation over others in cases where there are more than one in connection. For example, mathematical formalism is placed in a driving role of creating graphical representations with the didactical intention to find ways in which formalism may become meaningful to students. In that sense, it can be placed in digital media with educational goals even though it may be possible from a technical or ergonomic point of view to avoid it by an icon driven interface for instance. Constructionism also gives importance to the idea of artifacts or models as representations and thus the activity of making changes to representations by making changes to the artifacts. It encompasses the idea of levels of representations. Functions represent the properties and behaviors of models of newtonian objects and at the same time the objects themselves and their behaviors are representations of science and mathematical concepts.

Finally, a comparison between the Anthropological Theory of the Didactique (Chevallard 1992) and Constructionism may allow for socio-constructivism to play the role of a common basis. A key issue where these two theories are complementary however, is the role and status of control of the didactical process. This may well be attributed to epistemology or simple to the notion of concern. Constructionism takes on board the notion that meanings are in anyway generated to some extent outside the control of a teacher or the sequencing of an activity. In designing educational activities therefore didactical intervention can at most aim to help create an environment rich or dense in opportunities and challenges for meaning generation. There is an element of randomness and uncontrollability in that process which needs to be appreciated if there is learning to be done. Otherwise, intense attempts to control the learners activities may result in disengagement and trying to guess what's in the teacher's head rather than ownership of knowledge. This does not mean that design is 'looser' with respect to activity sequencing, the designed tools to be used or the interactions between teacher and student collectives. It means however that the kinds of interactions are more strategic from the teacher's side, more participatory in a joint enterprise and more allowing for the unexpected. The teacher elicits meanings in formation and mathematics in use and helps students elaborate emergent ideas and generalizations. Also they allow and recognize fallibility, i.e. the status of suggestions, student created artifacts, student solutions etc. to be in evolution or in flux rather than that of an expression of thought awaiting a final verdict. In this wake the construct of half-baked microworlds was developed to describe artifacts especially designed to invite changes and improvements and given to the students in that capacity, rendering them engineers (Kynigos 2007a). ATD on the other hand elaborates controlled scenarios and designs where didactical interventions are pre-designed, expectations of activities and understandings are precise and stepwise and teaching sequences are defined in terms of responses to specific pre-defined questions and tasks.

> From the identification of fundamental situations expressing the epistemological characteristics of a mathematical concept or theme to the determination of the didactical variables which condition the efficiency of solving strategies or condition students' adidactical interaction with the milieu, the design of situations reflect an ambition of control and optimization. The importance attached to a priori analysis and to its anticipative dimension also attests this ambition, deeply rooted in the role of *phenomenotechnique,* with the meaning given to this term by Bachelard, devoted to didactical engineering (Artigue, in preparation and 1989).

These are three kinds of connectivity elaborations between constructionism and other theoretical frameworks in mathematics education. The process of networking is perceived as essential for the de-contextualization of the theories and a better sense of the richness of theory building in the field. Constructionism as a theory which studies meaning generation through activities of collective and individual bricolage with expressive artefacts (mostly but not exclusively digital) where meaning is drawn through the use of representations, engagement with discussion and reflections on how to make changes to them and on their behaviors as they change.

Discussion

In this paper I argue that Constructionism can be addressed as an epistemology of learning associated with a theory of learning and design. There is also an implicit suggestion that it may be useful to think of Constructionism not only from a scientific perspective but also from a strategic perspective with respect to intervening and pushing for change in institutionalized educational practices. Can constructionism transcend time and be considered as a theory in continual flux and relevance as society and expressive media change? can it be taken seriously in the networking process of theories for mathematics education? what niche does it cover in today's landscape and how can it be used today—for explanation, for guidance for action, as a structural set of lens? Today's society is torn between (a) the digital natives, complex and changing society, the need for flexibility, agency, identification of problems lying in fuzzy realities, integrated domains with respect to traditional schooling and (b) school mathematics, the need for engagement with mathematical processes such as generalization, proof, rigour, analytic-synthetic ability and also the need to understand and use traditional concepts such as number sense, algebra, analysis, geometry, space, navigation, statistics, probability, digital models. Constructionism is relevant since digital society is full of objects to be tinkered with and tools for collective mathematical activity and communication. Traditional mathematics can be given meaning since representations can be used to create models, can be manipulated, connected and visualized. Mathematics concepts and representations in use. Didactic explanatory approaches can become an infrastructure rather than the object of education. This may leave new institutionalized space for constructionist practices to develop. So in the paper I argue that constructionist epistemology is transcendentally relevant and useful to drive strategy and educational knowledge in a society where expression, bricolage, collectivity change with media.

Selected Projects and Products

Microworlds Pro, Imagine, NETLOGO, TNG—StarLogo, Scratch, E-slate Turtleworlds, ToonTalk, FMS-Berkeley Logo, Elica Logo, Scheme, BYOB-Scratch, MachineLab Turtleworlds.

Weblabs, http://www.lkl.ac.uk/kscope/weblabs/theory.htm.

TELMA, ReMath, http://telma.noe-kaleidoscope.org, http://remath.cti.gr.

METAFORA http://www.metafora-project.org/.

Mathematical Creativity Squared, http://mc2-project.eu.

The 'Constructionism' conferences, 2010, 2012, 2014, http://constructionism2014.ifs.tuwien.ac.at/.

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Mobile Linear Algebra with Sage

Sang-Gu Lee

Abstract Over the last 20 years, our learning environment for linear algebra has changed dramatically mathematical tools take an important role in our classes. Sage is popular mathematical software which was released in 2005. This software has efficient features to adapt the internet environment and it can cover most of mathematical problems, for example, algebra, combinatorics, numerical mathematics, calculus and linear algebra. Nowadays there are more mobile/smartphones than the number of personal computers in the world. Furthermore, the most sophisticated smartphones have almost the same processing power as personal computer and it can be connected to the internet. For example, we can connect from mobile phone to any Sage server through the internet. We have developed over the years on Mobile mathematics with Smartphone for teaching linear algebra (Ko et al. 2009; Lee and Kim 2009; Lee et al. 2001). In this article, we introduce Sage and how we can use it in our linear algebra classes. We aim to show the mobile infrastructure of the Sage and the mobile-learning environment. We shall also introduce mobile contents for linear algebra using Sage. In fact, almost all the concepts of linear algebra can be easily covered.

Keywords Linear algebra · Sage · Mobile math · Matrix calculator · Smartphone · Technology

ZDM Classification: D30, H60, N80, U6.

2000 Mathematics Subject Classification: 15A04, 97B40, 97U50, 97U70.

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Introduction

After the 'e-Campus Vision 2007 (2003–2007)' (MOE-Korea 2002; Lee and Ham 2005) for we were able to have our classroom where we can explain, write, read, go, find, see, do, hear, discuss. Furthermore students can review lectures not only text but also real lectures with sound and movie clips, even they can enjoy real time class activity from a remote place. Furthermore all of our lectures are saved and can be reviewed right after the lecture over and over again. Also it can be done without anyone's help. By-products of this task were all lecture clips and web contents with CAS tools. We realized all of these outputs can be used in mobile environment.

The information and communications technology (ICT) has been the leading source of innovation in education. After we found some possibility, we started to develop CAS tools, lecture clips and to design our lectures in mobile environment from 2009. Korean Minister of Education announced "Strategies for Advancement of Mathematics Education" in January, 2012. Ministry of Education and Korea Science Foundation in Korea had a Declaration ceremony of 'The Year of Mathematics Education: 2012' in April 17th, 2012. According to that, CAS tools use in mathematics education at all level in Korea will be visible for the first time.

For over 20 years, the issue of using an adequate CAS tools in teaching and learning of linear algebra has been raised regularly (Shiskowski and Frinkle 2010; Butt 2011). A variety of CAS tools were introduced in several linear algebra textbooks. However, due to several problems in adopting those tools none of these has been adopted in the class and the theoretical aspect of linear algebra has been focused in teaching and learning. We have tried to find or make a simple and comprehensive method for using ICT in our classes.

We now introduce what we have developed over the years on Web and Mobile mathematics with Smartphone for teaching Linear Algebra.

Due to the dramatic changes in educational environment, we are aware that CAS tools and internet resources can be incorporated into classes sooner or later. Most of countries having advance information technology have been interested in the applications of the ICT in their educational environment. Korea is no exception to this tendency. Most of teachers have wanted a comprehensive and uncomplicated learning process in their classes using CAS and ICT with low cost. We had to keep those issues in mind. Korean students are not well trained with hand calculators or graphing tools, but they are now prepared for internet-based computational tools. Linear Algebra is the first abstract mathematics subject for most of new college students. Hence, most of the students face some difficulties to deal with various novel mathematical concepts in linear algebra.

The internet service has lot of information whenever our students want to get, and most of instructors now understand this situation and make a constant effort to provide good educational information to their students. We have tried to have contents and tools to show a lot of visualization in our classes.

However, there were no reasonable CAS tools made in Korea. So most of secondary students have depended on the foreign software, such as Mathematica, MATLAB, Maple, GSP and so on. They also faced the language barrier. The cost of this software was also an obstacle.

Korean Version of Sage

Korea now has a decent internet infrastructure, but there have been virtually no reasonable CAS tools for our mathematics classrooms in general. Sage (Free open source software to do mathematical computations) was developed by William Stein at the University of Washington (Stein and Joyner 2005). It has a familiar grammar system, like the commercial software, and it follows the GNU Product License (GPL) and is therefore free. Sage has been revised by many developers after the National Science Foundation grants for William Stein. The latest version of Sage is 5.4 and it will be updated constantly. Sage is released on the website, http://www.sagemath.org. Furthermore, Sage has a client-server model which is well-adapted to the internet.

Sage showed its potential over other softwares in our need. Furthermore, Sage also has compatibility with Mathematica, Matlab, Maple commands.[1] The reform of the educational environment and the development of educational tools and contents in this reform has been a hot issue. In the reform of our educational environment, we had to develop good tools which most students can use when they have a job after their degrees.

We found that Sage could be an excellent solution for our needs, so a lot of efforts have been made at our 'Brain Korea 21 Mathematical Modelling Human Resources Development Division'. We found that Sage can be used for students' better understanding in linear algebra teaching.

Now instructor can show and compute whatever he likes to do in his class. Many students can see and handle visualized outputs. It also helps students to understand the complicate mathematical concepts. Students may install Sage on their laptop if they have a limited access to internet. A snapshot of the website is provided in Fig. 1. In order to use full feature of Sage, Google Chrome or Firefox browsers are recommended. The main hosting site[2] in Korea is the following.

Sage is programmed in the Python programming language, and it can be modified freely. We can utilize many pre-published Sage worksheets for linear algebra in http://math1.skku.ac.kr/pub.

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[1] http://www.sagemath.org/doc/reference/sage/interfaces/mathematica.html.

[2] Sage Server 1 (http://math1.skku.ac.kr), Sage Server 2 (http://math2.skku.ac.kr), Mobile Sage Server (http://math3.skku.ac.kr), SKKU Sage Cell Server (http://sage.skku.edu).

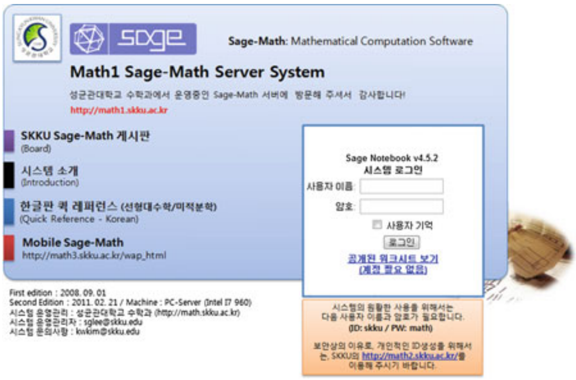

Fig. 1 Front page of one author's Sage server. http://math1.skku.ac.kr (*ID* skku, *Password* math)

As mentioned earlier, language barrier has been an obstacle for software to be popular in Korea. Therefore, we had to develop a Korean edition of the Sage software. Since source codes of Sage can be modified, we could change the source for the use of Korean language in it. Because of many touches of developers, the sage source code is complicated and there were some difficulties to apply our local information into this software. But we were able to make Korean localized edition which can make many modules to treat Korean 2-byte characters (Kim et al. 2010). Now Korean students can use the Sage in Korean language.

Mobile Sage

Sage can be applied on the various platforms having internet connectivity, for examples, tablet PC, and cellular phones. This function is based on the XML-RPC technique. If we can use this function on the Sage, then it means that all clients which are able to connect to the internet service can get the mathematical computation results (including the graphic visualizations) using the Sage server.

The XML-RPC can be applied to cellular phones, so, we can show mathematical results freely using sage wherever we want and its XML-RPC functions. Therefore cellular phones can be changed to the scientific calculator with the Sage. Especially, the Sage and the XML-RPC based on the network environment, students do not need to spend their time to install specific software in their cellular phones. Just connection with the internet, students can show their problem's answer without too many hassles.

We have made a mobile service module which is suitable for the common cellular phones. To use the mobile service on the Sage, connect the following URL[3] in the mobile internet services.

[3]http://math3.skku.ac.kr/wap_html.

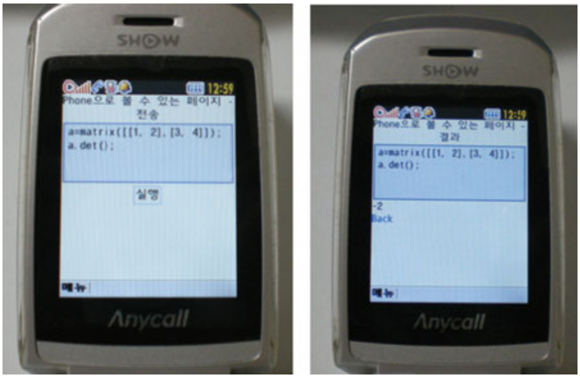

Fig. 2 Sage in a normal phone

When we connect the mobile service, we can see the following Fig. 2 with the input function. We can modify some commands or input new commands in the input windows. After that, click the “Execution” then we can see those results of inputted codes. If we want to go back after results, click the “Back” button and we can see the first page. In the Fig. 2, we can show the determinant of the matrix $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$.

Actually, code inputs on the cellular phone are difficult. Since, the keypad on the cellular phone is smaller than the keyboard in a personal computer, we have prepared some examples for convenient use of these functions.

We have developed most of Sage commands for teaching of elementary linear algebra class as a web contents. A snapshot of the website[4] is provided in Fig. 3.

The followings (Table 1) are developed as Sage contents for linear algebra.

Students can save time to learn or memorize most of Sage commands and programming languages. They only have to find in the organized web page or published pages, and use them at http://sage.skku.edu. They may revise whenever it is necessary.

Linear Algebra with Sage and Mobile Sage

Smartphone has been popular since 2009 as we see in the following Fig. 4.

Now our smartphone has lot of functions for this mobile linear algebra. One example is a QR-code that we made. We can do similar computations just by changing a few numbers or commands after scanning those QR-codes.[5]

[4] http://matrix.skku.ac.kr/2012-Sage/sage-la.

[5] http://matrix.skku.ac.kr/CLA12/Sage-2-2-2.html.

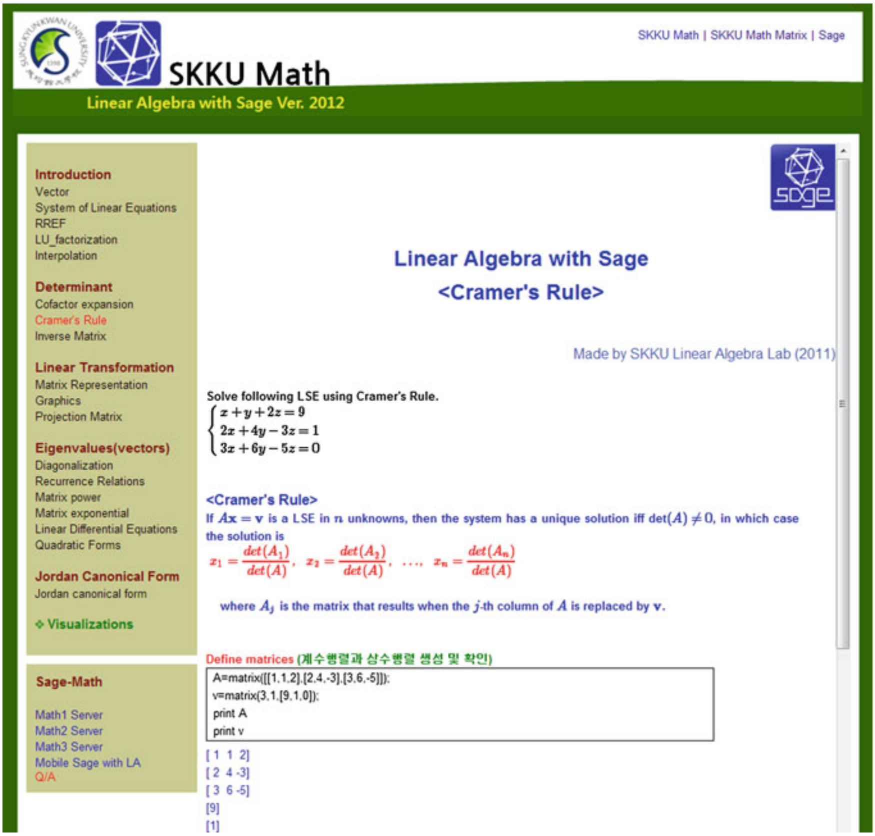

Fig. 3 Web: Sage commands for linear algebra class

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 2 \\ 0 & 0 & 1 & 1 & 3 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 1 & 3 \end{bmatrix}$$

Our Android App for linear algebra 'Sage Calculator for Matrix[6]' can be found in Google Play store with 'SKKU Matrix'.

It has features of lecture notes of a course, video lectures, and solutions of most of problems in a textbook and Sage tools with following contents (Figs. 5, 6, 7, 8, 9, 10, 11 and 12).

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[6]https://play.google.com/store/apps/details?id=skku.la.sol.

Table 1 Developed Sage tools for linear algebra

Chapter	Contents
Introduction	Vector
	System of linear equations
	RREF
	LU factorization
	Interpolation
Determinant	Cofactor expansion
	Cramer's rule
	Inverse matrix
Linear transformation	Matrix representation
	Graphics
	Projection matrix
Eigenvalues	Diagonalization
(-vectors)	Linear algebra with Sage
	Matrix power
	Matrix exponential
	Linear differential equations
	Quadratic forms
Jordan canonical form	Jordan canonical form

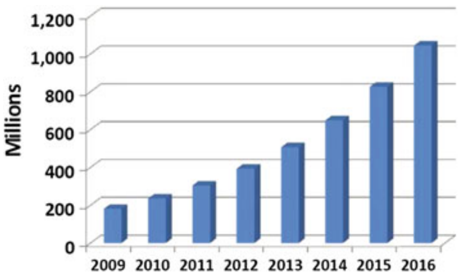

Fig. 4 Changes in global smartphone sales. *Source* Telecom Trends International, Inc.

Students' Feedback

In our linear algebra class of the spring semester in 2012, more than 83.5 % students out of 20 students gave a very positive answer (Table 2, above 4/5) for the effectiveness of such a tool in their learning process of linear algebra (Franklin and Peng 2008).

Fig. 5 QR code and cover page of the App

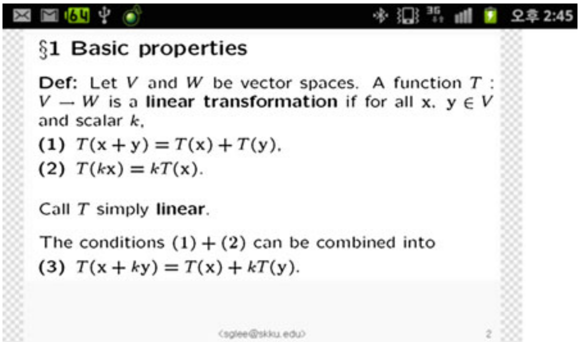

Fig. 6 Mobile lecture notes

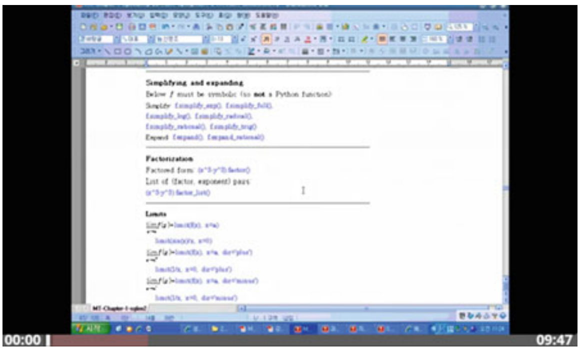

Fig. 7 Video lectures

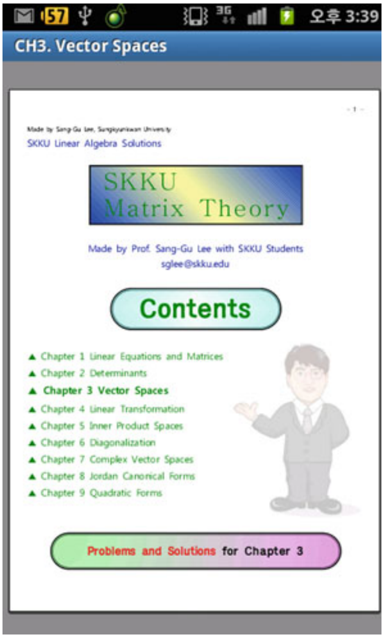

Fig. 8 Problems and solutions

Fig. 9 Sage calculator for matrix

Fig. 10 Usage of the Mobile Sage App (http://math3.skku.ac.kr/spla/LT_Dilation)

Fig. 11 SNS feature of the Mobile Sage App

Determinant | Characteristic polynomial | Eigen system | LU-decomposition

Fig. 12 Contents of Mobile Sage App (Determinant http://matrix.skku.ac.kr/CLA12/sage-4-1-4.html, Characteristic polynomial http://matrix.skku.ac.kr/CLA12/sage-4-5-2.html, Eigen system http://matrix.skku.ac.kr/CLA12/sage-4-5-6.html, LU-decomposition http://matrix.skku.ac.kr/CLA12/sage-4-R-1.html)

Table 2 Feedback of students in a linear algebra course with Sage

Questions/ students	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	AVR
1. Positiveness for learning	4	5	3	4	4	5	3	5	5	5	3	4	4	4	4	4	2	2	3	3.84
2. Quality of teaching	3	5	4	5	4	5	5	5	5	5	4	5	5	5	5	5	5	4	5	4.68
3. Improvement of knowledge	4	5	5	4	4	5	5	5	5	5	4	5	5	5	4	5	4	3	4	4.53
4. … of reasoning skills	2	5	5	4	4	5	4	3	5	4	4	4	4	5	4	3	4	3	3	3.95
5. … of self-directed learning skills	4	5	3	4	4	5	4	4	5	5	4	5	5	5	4	5	3	3	4	4.26
6. … of problem solving skills	4	5	5	5	4	5	4	4	5	5	4	5	5	5	4	5	3	3	4	4.42
7. … of expertise	3	4	4	5	4	5	4	4	5	5	3	4	4	5	3	4	3	3	4	4
8. Fairness of assessment method	4	5	5	5	4	5	5	4	5	4	4	4	5	4	3	5	5	3	4	4.37

Conclusion

In this article, we have mentioned what made us to use Sage and Mobile Sage in our linear algebra, and what we have done for our mobile infrastructure of the Sage in mobile-learning environment.

We have developed over the years on Mobile mathematics with Smartphone in teaching of linear algebra. We have also included some of Sage mobile contents for linear algebra. Taking advantage of these, almost all basic concepts of linear algebra can be easily covered. Even the size of matrices can be expanded without any difficulty.

More and more students in Korea, who never used a graphing calculator or personal computer with commercial mathematics software, now have a smartphone in their hand. We can make them to be equipped with several free linear algebra smartphone applications that can handle linear algebra problems. Such a small change has made a lot of impact in learning linear algebra in Korea.

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Discernment and Reasoning in Dynamic Geometry Environments

Allen Leung

Abstract Dynamic Geometry Environments (DGE) give rise to a phenomenological domain where movement and variation together with visual and sensory-motor feedback can guide discernment of geometrical properties of figures. In particular, the drag-mode in DGE has been studied in pedagogical settings and gradually understood as a pedagogical tool that is conducive to mathematical reasoning, especially in the process of conjecture formation in geometry. The epistemic potential of the drag-mode in DGE lies in its relationship with the discernment of invariants. In this lecture, I will discuss means of discernment and reasoning for DGE based on a combined perspective that puts together elements from the Theory of Variation and the Maintaining Dragging Scheme. My focus is on an idea of invariant as the fundamental object of discernment. Furthermore, an idea of instrumented abduction is proposed to frame how such reasoning can be developed. Exploring by dragging is a powerful tool supporting geometrical reasoning. At the end, I will introduce a Dragging Exploration Principle that might help to cognitively connect the realm of DGE and the world of Euclidean Geometry.

Keywords Dynamic geometry environments · Dragging · Variation · Abduction

Introduction

Dynamic Geometry Environments (DGE) give rise to a phenomenological domain where leaner manipulated movement and variation together with visual and sensory-motor feedback can guide discernment of geometrical knowledge. DGE are modelled after theoretical systems like Euclidean Geometry and the dynamism that characterizes DGE phenomena gives a new perspective for geometry and geometry

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education (see for example, Laborde 2000; Strässer 2001; Laborde et al. 2006; Lopez-Real and Leung 2006). In particular, the drag-mode in DGE has been studied in pedagogical settings and gradually understood as a pedagogical tool that is conducive to mathematical reasoning, especially in the process of conjecture formation in geometry (see for example, Arzarello et al. 2002; Baccaglini-Frank 2010; Baccagalini-Frank and Mariotti 2010). The epistemic potential of the drag-mode in DGE lies in its relationship with the discernment of invariants. In this lecture, I will discuss means of discernment and reasoning for DGE based on a combined perspective that puts together elements from the lens of variation in DGE (Leung 2008), the Maintaining Dragging Scheme (Baccaglini-Frank 2010) and abduction in DGE (Baccaglini-Frank 2011), and how they contribute to reasoning in DGE.

Invariant, Discernment and DGE

Resnick (1997) advocated that mathematics is a "science of patterns". I interpret mathematical pattern is an emerging invariant structure when a phenomenon that concerns number and/or shape is undergoing changes or variation and mathematical experience as "the discernment of invariant pattern concerning numbers and/or shapes and the re-production or representation of that pattern." (Leung 2010) Variation is about what changes, what stays constant and what the underlying rule is. In phenomenology, one of the hermeneutic rules is to "seek out structural or invariant features of the phenomena" (Ihde 1986, pp. 39–40). Dienes (1963) attributed the abstraction and the generalization processes in mathematical thinking by what he called the perceptual variability principle and the mathematical variability principle:

> The perceptual variability principle stated that to abstract a mathematical structure effectively, one must meet it in a number of different situations to perceive its purely structural properties. The mathematical variability principle stated that as every mathematical concept involved essential variables, all these mathematical variables need to be varied if the full generality of the mathematical concept is to be achieved." (Dienes 1963, p. 158)

Discernment, variation and simultaneity are central concepts in Marton's Theory of Variation (Marton and Booth 1997; Marton et al. 2004). In particular, discernment of critical features occurs under systematic interaction between learners and the thing to be learnt, and variation is an agent that generates such interaction. Variation in different aspects of a phenomenon unveils the invariant structure of the whole phenomenon. Thus invariants are critical features that define or generalize a phenomenon. This matches what doing mathematics is about, for a major aim of mathematical activity is to separate out invariant patterns while different mathematical entities are varying, and subsequently to generalize, classify, categorize, symbolize, axiomatize and operationalize these patterns (see for example, Mason et al. 2009).

A key feature of DGE is its ability to visually represent geometrical invariants amidst simultaneous variations induced by dragging activities. The variations of the moving image are perceived in contrast to what simultaneously remains invariant. The movement and the identification of invariants are what lies at the heart of activities that aim at exploiting the epistemic potential of DGE (for example, Laborde and Laborde 1995; Hölzl 1996; Healy 2000; Arzarello et al. 2002; Olivero 2002; Leung 2008; Baccaglini-Frank and Mariotti 2010). Hölzl (2001) considered dragging as "a tool to find different representations of one and the same figure in continuous transition. Because dragging acts on a drawing with the effect being determined by the figure, a mediating function emerges." (Hölzl 2001), and Lopez-Real and Leung (2006) suggested if dragging in DGE is accepted as a tool that can bring about invariant structures and patterns, then "…we have new 'rules of the game', or even a new game, for experiencing geometry." (op cit, p. 676) Furthermore, interaction with dynamic figures through dragging in DGE

> could induce a special type of reasoning (or explaining) in DGE in which a signified object in DGE could have a diachronic nature. That is, one has to conceptualize a draggable object in DGE as it varies (over time) under dragging. Hence, a whole object in DGE should be understood as a (continuous) sequence of the "same" object under variation. (Leung and Or, 2007)

"Sameness" of a sequence of figures recognized as one is given by the perception of *invariants* that characterize each figure of the sequence. In general, invariants are determined both by the geometrical relations defined by the commands used to construct the dynamic-figure, and by the relationship of dependence between the original relations of the construction and those that are derived as a consequence within the theory of Euclidean Geometry (Baccaglini-Frank et al. 2009; Laborde and Strässer 1990). To illustrate, let us consider the following construction (see Mariotti and Baccaglini-Frank 2011):

> *ABCD is a quadrilateral in which D is chosen on the parallel line to AB through C, and the perpendicular bisectors of AB and CD, r and s respectively, are constructed.*

Figure 1 shows a dynamic figure derived from such construction in Cabri (a DGE). It has a set of constructed invariants (the parallelism between AB and DC, the perpendicularity of s to DC and of r to AB, and the passing of s and r through the midpoints of DC and AB respectively), and consequently all the invariants derived from these (for example the parallelism between *r* and *s*) which are all conserved simultaneously during dragging. In particular, in this Cabri construction, A, B, C are free base-points which can be moved by dragging them directly, D is restricted to move along a line parallel to AB through C, *r* and *s* are dependent objects which can't be moved by dragging them directly but can be moved indirectly as a consequence of the movement of other base-points.[1] Hence as they appear simultaneously during dragging, different invariants may have different status according to the type of control allowed by the DGE (in this Cabri case,

[1]We remark that in other DGE like Geometer Sketchpad, for the same construction, *r* and *s* can be moved by directly dragging on them.

direct or indirect). In a constructed figure there is a relationship of dependency between the constructed invariants and any that are derived from these according to Euclidean Geometry. It may happen that a relationship between invariants is an invariant itself, and such a relationship can be discerned through dragging. In our example there is an invariant relationship between invariants that can be expressed by the following statement.

If two lines are respectively perpendicular to two parallel lines, then the first two lines are parallel.

Discerning invariants and discerning invariant properties between invariants are cognitively quite tasks. I introduce the following terminology:

Level-1 invariants: aspects of a dynamic figure, potentially corresponding to geometrical properties that are perceived as constant during variation of the figure through dragging. For example, "AB parallel to CD" and "s parallel to r" are level-1 invariants of the dynamic figure constructed in Fig. 1.
Level-2 invariants: invariant relationships between level-1 invariants. For example, "AB parallel to CD causes (or implies) *s* parallel to *r*".

Different dragging modalities can be employed to formulate strategies to unveil these two levels of invariant. Making sense of the sensory-motor feedback under the drag-mode is entirely up to the learner who will need to interpret the "construction steps" as invariant geometrical properties, relate them to other invariants, discover new ones, and eventually through reasoning, logically link the perceived geometrical properties and relationships to one another. In the literature, there are two broad categories of dragging modality (and examples of them) that are in line with Hölzl's (2001) description of two principle drag modes.

Dragging for testing consists in dragging to check the presence of desired (known) properties in a dynamic figure. It "presupposes an expectation as to the

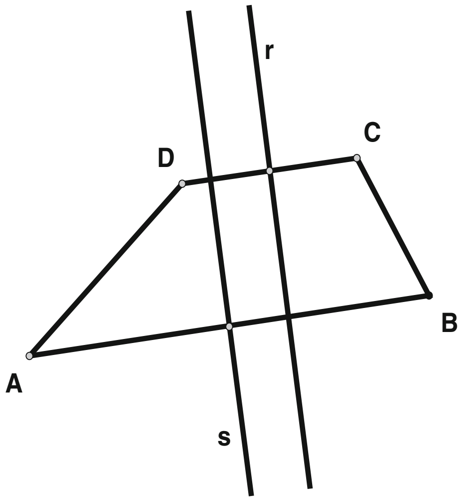


Fig. 1 A dynamic trapezium with free vertices A, B, C and perpendicular bisectors r, s. D is on the line parallel to AB that goes through C

reaction of the construction when it is being dragged" (ibid, p. 83). The movements on the screen can meet that expectation or not. The learner movement will be used to induce on the figure to find the invariants that are being sought after. Examples are: the dragging test (Arzarello et al. 2002, p. 67); the soft dragging (conjecture) test (Baccaglini-Frank and Mariotti 2010); the robust dragging test (or an adaptation of linked dragging or bound dragging (Arzarello et al. 2002; Olivero 2002).

Dragging for searching/discovering consists in dragging to look for new properties of the figure. In other words, "the changing appearance of the drawing must be evaluated under aspects which are still unknown" (ibid, p. 83). These may be possible configurations it might assume, invariants, and/or relationships between them. If, for example, the task is to formulate conjectures on the figure, this type of dragging will be used to discover new properties through the perception of invariants and relationships between them. Examples are: wandering dragging (Arzarello et al. 2002; Olivero 2002); guided dragging (Arzarello et al. 2002; Olivero 2002) and dragging to fit (Lopez-Real and Leung, 2006); lieu muet dragging (Arzarello et al. 2002; Olivero 2002) or maintaining dragging (Baccaglini-Frank and Mariotti 2010; Baccaglini-Frank 2010).

Discernment Through Variation Interactions

As mentioned above, discernment, variation and simultaneity are the central concepts in Marton's Theory of Variation (Marton et al. 2004; Marton and Booth 2007). Discernment of critical features occurs under systematic interaction between a learner and the thing to be learnt, and variation is the agent that generates such interaction. Local variation in different aspects of a phenomenon unveils the invariant structure of the whole phenomenon. Invariants are critical features that define or generalize a phenomenon. Four basic patterns of variation were proposed by Marton: contrast, generalization, separation and fusion. They form the kernel for discernment under variation in the Theory of Variation. These patterns of variation have the potential to be used to organize a variation experience and generate interactions between learners and the "thing" to be learnt.

In the context of DGE, I consider them as possible types of *variation interaction under the drag-mode* while a learner is exploring for geometrical invariants.

> A variation interaction in DGE is a strategic use of variation to interact with DGE objects in order to bring about discernment of geometrical invariant.

Drag to contrast is the strategy to discern whether a DGE object satisfies a certain condition or not, that is, it seeks to differentiate different DGE phenomena.

Drag to separate is the strategy to bring about awareness of critical geometrical features that may become invariants. It is an awareness of part-whole relationship realized by constraint dragging strategy that purposely varies or not varies certain aspects aiming to separate out invariant features in the whole.

Drag to generalize is the strategy to explore whether after contrast and separation, an observed geometrical feature can occur in a varied situation. It is a conjecture-making activity checking the general validity of a geometrical feature.

Drag to fuse is the strategy to integrate geometrical critical features under simultaneous co-variation. By fusing the separated-out critical geometrical features together, a whole invariant concept may appear. By contrasting critical features, fusion reveals how parts of a whole vary in interconnected ways.

These four types of variation interaction under the drag mode act together in a concerted way to bring about discernment of geometrical properties. I will use the following DGE exploration as an illustrative example (cf. Leung 2003):

(**E**): Explore how many circles can be constructed through any two given points and explain the construction.

In Fig. 2, a circle is constructed that passes through a free base-point (hence arbitrary) A with a draggable base-point C as its centre, B is another draggable base-point. C is dragged (while not moving A and B) to make the circle visually passing or not passing through B with direct control over C.

Thus drag to contrast based on simultaneous focus on the varying position of C (hence a sequence of circles) and B allows the learner to discern that many circles can pass through two given points.

In Fig. 3, Trace (a DGE function) is activated for C and C is dragged while keeping point B visually on the circle (A, B are not dragged, hence are the aspects that are being kept constant). It results in a visual *path* appearing on the screen. This path can be interpreted as a visualization of simultaneous focus because it is a unique simultaneous representation of a time sequence (different appearances in time) of positions that *visually satisfied* the desired condition. That is, as long as point C varies along this path with A and B kept fixed, the corresponding circles *seem to* pass through points A and B.

The path separated out in Fig. 3 can be "re-generated" for different positions of A and B (Fig. 4). Focussing on the re-appearance of different paths for different positions of A and B through contrast and simultaneous focus allows the discernment of an invariant property of all these paths leading to the generality of such

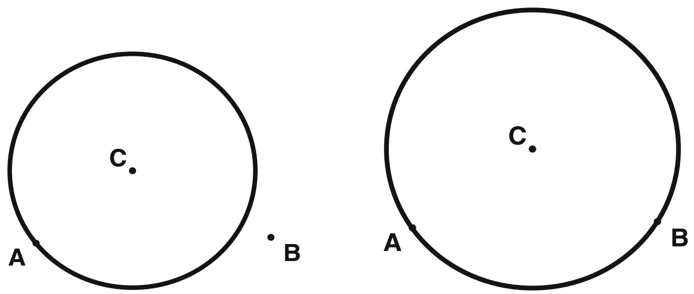


Fig. 2 Contrast via simultaneous focus: circle centred at C through B or not through B

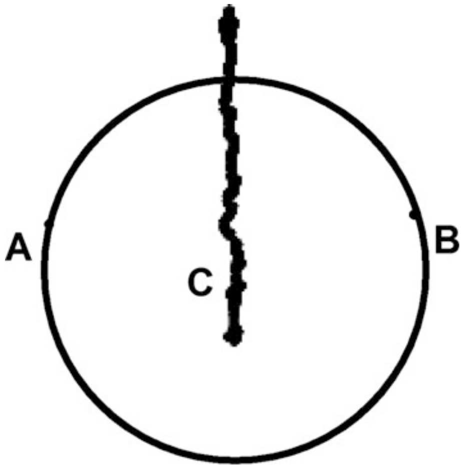


Fig. 3 Separation via awareness of critical features: a *path* is traced while maintaining a circle that passes through A and B

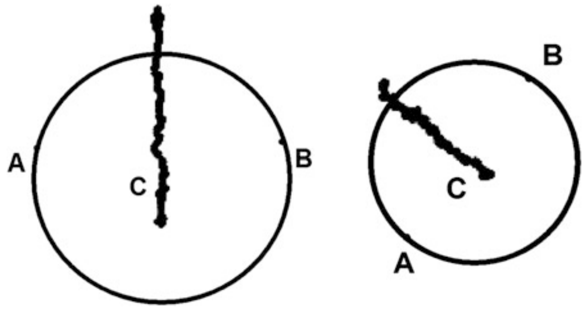


Fig. 4 Generalization: the repeated appearance of an invariant (path) for different positions of A and B

invariant, which may be described geometrically: C seems to lie on a path that is the perpendicular bisector of AB. In this way, a level-1 invariant is being discerned.

The above variation interactions under the drag-mode bring about fusion (a simultaneous awareness) of two inter-relating critical features: (1) a family of circles where A lies on with C as centre passing through B, and (2) C belongs to a path. Under generalization, the invariant property of all the paths discerned can be interpreted geometrically as "the perpendicular bisector of AB". This integration and interrelation discerned through fusion can be expressed through the following conjecture:

> *Given two free points A and B, if a point C moves along the perpendicular bisector of AB, then the circle with C as a centre passes through A and B* (Fig. 5).

Thus in this case fusion brings about the awareness of an invariant relationship between two level-1 invariants (the two critical features); that is, a level-2 invariant. The transition from a traced path to a geometrical path (a perpendicular bisector) holds the key to the formation of a conjecture.

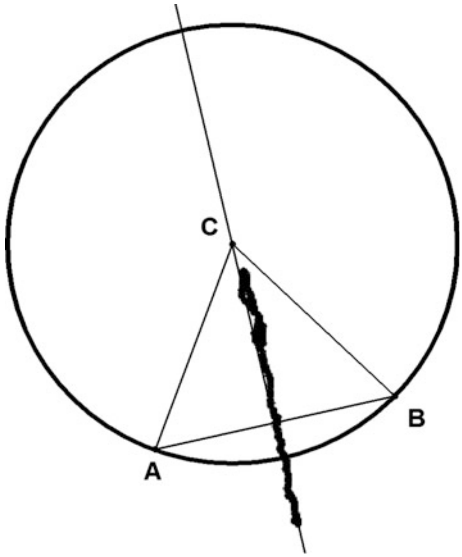


Fig. 5 Fusion: the perpendicular bisector of AB as an invariant relationship between two critical features

In this discernment process, drag to contrast and drag to generalize interact in a mutually enhancing way (to differentiate, to test validity) driven by a drag to separate strategy in order to pin down an invariant property. The drag to separate strategy is a maintaining dragging modality (see a full discussion in the next section) with the Trace turns on. When different critical features (for example, the traced path, perpendicular bisector, isosceles triangle) are discerned, a co-varying drag to fuse strategy can be used to test their correlation. For example, robust or soft dragging tests are types of drag to fuse strategy that focus on simultaneous variation of different geometrical properties testing the validity of a geometrical conjecture. Furthermore, drag to fuse is an underlying variation interaction that permeates all dragging interactions as simultaneous attention on different varying and invariant aspects are fundamental in discernment. Different dragging modalities (for example, those mentioned in the previous section) can be employed to furnish the four variation interactions under the drag-mode.

Discernment Through the Maintaining Dragging Scheme

I discussed how the development of potential awareness of direct and indirect control over level-1 invariants can occur through sensory-motor experience *accompanied by reasoning* and how variation interactions can be used as a means to discern invariants. The form this reasoning might take on is dependent on the context it is developed in, and in particular, on the learner's goal (or task) and therefore on the chosen dragging modality. The "reasoning" can take on the form of what has been referred to as *scheme* or *utilization scheme* within the instrumental approach (Artigue 2002; Rabardel 1995). This approach puts an *artefact* in relation to a task and a learner. The learner develops a utilization scheme in order to accomplish the task using the artefact. The combination of the artefact and the developed utilization scheme is the *instrument*. If *dragging in a DGE* is considered from an *instrumental* perspective, particular dragging modalities can be seen as artefacts supporting the task of conjecture-generation. Together with a utilization scheme developed by a learner during a process of *instrumental genesis*, a particular way of dragging may become an *instrument*. These utilization schemes developed by the learner are referred to as "dragging schemes", and they constitute the "reasoning" that accompanies particular uses of dragging (see for example, Leung et al. 2006).

The complexity in a dragging scheme is constituted by becoming aware of the *hierarchy induced* not only on the elements of a DGE figure but *on their properties* (that is relationships between elements) by the steps of the construction, and of the fact that such a hierarchy corresponds to logical relationships between the properties of the "geometric figure." Thus not only can the learner experience different types of control over elements, *direct and indirect control, depending on the type of DGE used* (*cf. footnote* 1)*, may also be purposely exercised over invariants*. A fundamental means of discernment is awareness of these types of control, in particular, in the context of inducing new invariants on a dynamic figure. For the example depicted in Fig. 1, it is possible to try to induce a new invariant like "coinciding perpendicular bisectors". This is often called a *soft* property (Healy 2000; Laborde 2005). A learner can try to maintain such interesting property by dragging a base point. This type of dragging modality is referred to as *maintaining dragging*. To illustrate, let's choose point B as the base point. Figure 6 shows how such property can become a *soft invariant* as a base point is dragged.

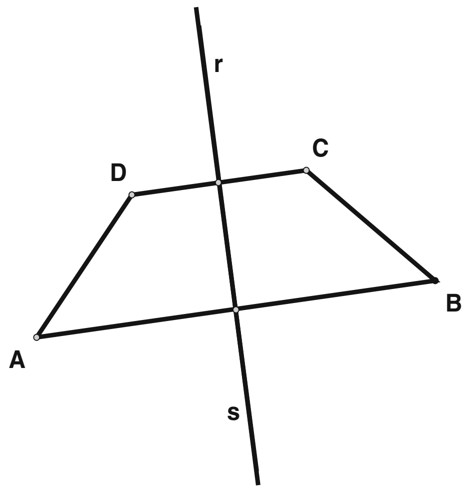


Fig. 6 The figure shows a learner trying to maintain the coincidence between the perpendicular bisectors, by dragging point B

The movement of B is controlled by the learner and is not random. This *control* exercised over the movement of B is intentionally direct while that over the invariance of the desired property (*r* and *s* coincide) is consequentially indirect. Such awareness developed through sensory-motor experience accompanied by reasoning becomes fundamental when maintaining dragging is used as a means of exploration (Baccaglini-Frank 2010; Mariotti and Baccaglini-Frank2011). Inducing soft invariants on a dynamic figure can be done through the maintaining dragging modality. When this modality is used to search for new conjectures, that is relationships between possible invariants (level-2 invariants), it is referred to as the *MD scheme* (Baccaglini-Frank 2010; Baccaglini-Frank and Mariotti 2010). This scheme is described through a set of (possibly implicit) tasks the learner addresses in the quest to formulate conjecture:

Task 1: Conceive a configuration to be explored by dragging intentionally to maintain the appearance of the desired configuration, thus inducing it as a soft invariant. This conceived configuration is called an *intentionally induced invariant*.
Task 2: Look for a condition that makes the intentionally induced invariant visually verified through maintaining dragging. This can occur either through

- a geometric interpretation of the movement of the dragged base-point or,
- a geometric interpretation of the trace mark (path) of the dragged base-point.

Propose a geometric description of the movement or the path observed.
Task 3: Verify the *conditional link* through the dragging test. A conditional link is a relationship of logical dependency between geometrical properties. This requires the accomplishment of at least some of the following subtasks:

- representing the *invariant observed during dragging* through a construction of the proposed geometric description of the path;
- performing soft dragging test by dragging the base-point along the constructed geometric description of the path;
- performing a robust dragging test by providing (and constructing) a geometric description of the path that is not dependent upon the dragged-base-point and redefine the base-point on it in order to have a robust invariant, then perform the dragging test.

A key requirement for conceiving a conditional link is the learner's experience of simultaneity together with direct control; that is, control over the direct movement induced by dragging a base point along the path. In fact, after discovering the invariant observed during dragging, the learner can directly act on the base point to maintain the invariant observed during dragging, and as a consequence simultaneously feel and observe the maintaining of the intentionally induced invariant indirectly. Thus it suggests that a bridge between the experiential field (in dynamic geometry) and the formal world of Euclidean geometry is provided by the following interpretation:

As learners induce invariants in a DGE, the types of control that learners experience over these invariants enable discernment of a-symmetric status among them in spite of the fact that they appear simultaneously. This may lead the learners to interpret the dynamic relationship between invariants as a *conditional relationship (If ... then…) between geometric properties*.

This discernment hypothesis is summarized as follows:

Simultaneity + Direct control via Maintaining Dragging over a soft invariant
→ Premise of a possible conjecture (IF)
Simultaneity + Indirect control via Maintaining Dragging over a soft invariant
→ Conclusion of a possible conjecture (THEN)

When developing such scheme the role of the path is crucial. The notion of *path* was introduced by Baccaglini-Frank and Mariotti (2009) and is consistent with Leung and Lopez-Real's (2002) notion of *locus of validity*. A path can be conceived as a visual record of a controlled variation, but at the same time it may be a record of simultaneous invariance: of the maintained invariant and of a new invariant causing it. A path is a trajectory such that when a base point of a configuration is being dragged along it, the configuration will satisfy a certain prescribed condition. In particular, in the discernment of level-2 invariants a critical role can be played by the conception of a path. As the exploration proceeds the representation of the path (both mental and within DGE) passes through a sequence of steps that capture an evolution process that is representative of the means of discernment that were discussed above. Firstly the path is envisaged (envisaged path), then a path is roughly traced (traced path), then a path is constructed and along such path a base point can be dragged (drag-along path), finally, if possible, a generalized robust path is constructed (generalized robust path).

Discernment Through Path

Continue with the dynamic figure depicted in Fig. 1, I now use it to illustrate how the MD (Maintaining Dragging) Scheme can be applied together with variation interactions utilizing a sequence of paths to formulate a possible conjecture. The exploration problem is:

Explore the possible positions of B such that the perpendicular bisectors r and s coincide

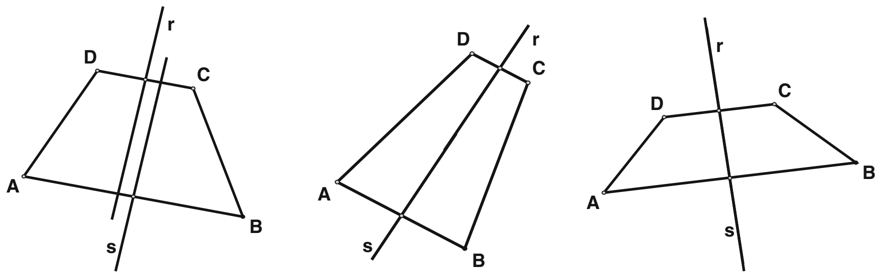


Fig. 7 Envisaged path

Envisaged Path (Task 1 in MD Scheme)

Wandering dragging B to *drag for contrast*, for different fixed positions of A, C and D find the possible positions of B that makes the perpendicular bisections r and s coincide (intentionally induced invariant), and the perception of a possible path is forming in the dragging process (Fig. 7).

Traced Path (I) (Task 2 in MD Scheme)

Use the *maintaining dragging modality* on B with the Trace turned on (*lieu muet dragging, drag-to-fit*) to *drag for separation*. A path is traced out while keeping *r* and *s* coincide. The path roughly takes the form of a circular arc and the perception of a soft invariant (invariant observed during dragging) is forming in the dragging process (Fig. 8).

Traced Path (II) (Task 2 in MD Scheme)

Perceptually asserting that the traced path is part of a circle, the next task is to locate the centre of the circle. As the coincided r and s appears to "lie in the middle" of ABCD and ABCD looks like a symmetric trapezium, a reasonable guess would be that the centre may be the intersection of the two diagonals AC and DB. Indeed, *simultaneous focus* on AC, DB and the coincided r, s with direct control on B via maintaining dragging over a soft invariant (*drag to fuse*) reveals that the centre of the circle that causes the *r*, *s* coincidence (geometric interpretation of the traced path) does seem to lie on the intersection of AC, DB and the coincided r, s (Fig. 9).

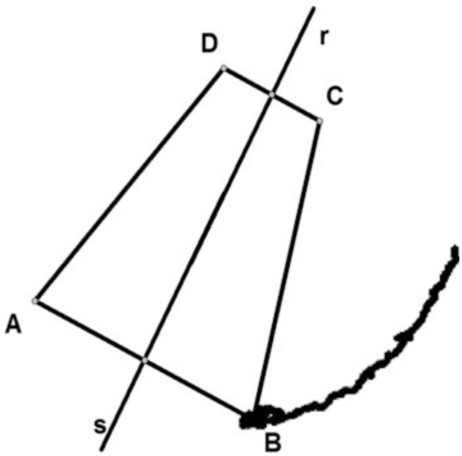


Fig. 8 Traced path

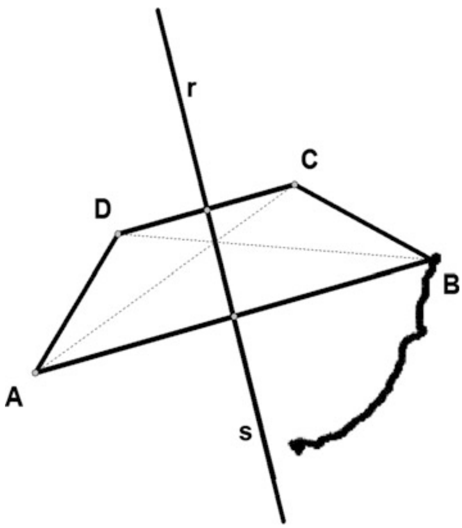


Fig. 9 Exploring the geometrical interpretation of the traced path

Drag-Along Path (Task 3 in MD Scheme)

Construct a circle with the intersection of AC and BD as centre using A as the terminus of a radius. Drag B along the constructed circle (*soft dragging*) to test the validity of the perceived geometric interpretation of the traced path (*drag to generalize*). That is, direct control on an invariant observed during dragging causes an indirect control on an intentionally induced invariant. This drag-along path will become the soft invariant that forms the premise of a possible conjecture (Fig. 10).

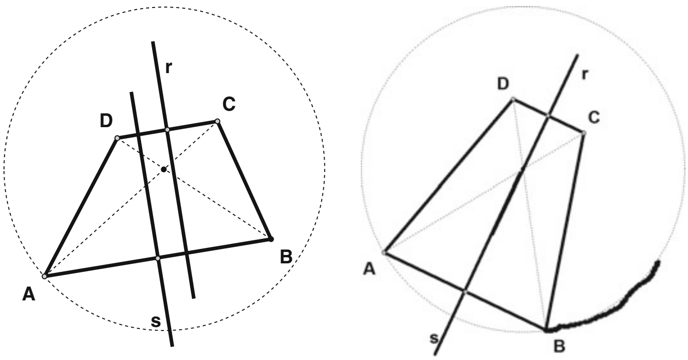


Fig. 10 Soft dragging on the geometrically interpreted path

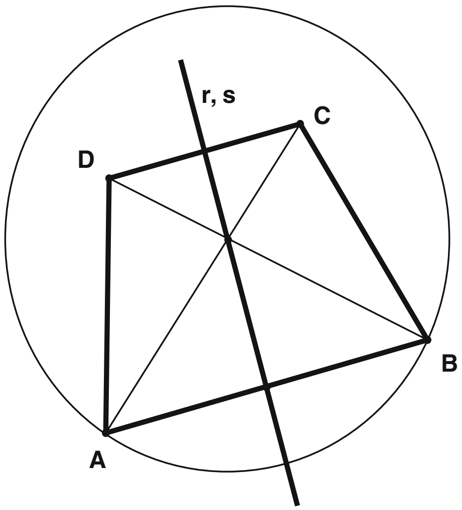
Generalized Robust Path (Task 3 in MD Scheme)

Construct a robust circle using the critical features discerned in Trace Path (II) to confirm the conclusion of a possible conjecture and to perform a *robust dragging test* (*drag to generalize and drag to fuse*) on the final construction (Fig. 11).

A Possible Discerned Conjecture

Let O be the intersection of the diagonals AC and BD.
IF B lies on the circle centred at O with OA as radius
THEN the perpendicular bisectors of AB and CD coincide.

Fig. 11 Generalized robust path



This dragging exploration illustration shows how discernment through variation interactions and discernment through the MD Scheme can be merged together under the perception of different types of path resulting in a possible conjecture. Notice that this conjecture is stated in a DGE context where dragging, variation and invariants are implicit.

Reasoning Through Instrumented Abduction

Abduction describes a particular form of reasoning that leads to discernment of experiential phenomena through the generation of *explanatory hypotheses*. During explorations in DGE and the learner focuses on relating invariants, that is discerning level-2 invariants, the generation of explanatory hypotheses can be a key. In Magnani's terms (2001) abduction is:

> the process of inferring certain facts and/or laws and hypotheses that render some sentences plausible, that explain or discover some (eventually new) phenomenon or observation; it is the process of reasoning in which explanatory hypotheses are formed and evaluated." (pp. 17–18)

Research studies suggest that when certain utilization schemes have been appropriated during exploration in DGE, associated to dragging modalities, such explanatory hypotheses may be generated through a particular form of abduction. Baccaglini-Frank and Mariotti have described such form of abduction supported by an instrument (in their case the instrument of maintaining dragging), as *instrumented abduction* (see Baccaglini-Frank 2011). Two characteristics used to define instrumented abduction are: (1) that it occurs at a meta-level through a *meta-rule* developed during a process of instrumental genesis, and (2) that it is *encapsulated* in the use of the artefact in the task it is used for.

When using MD to elaborate a conjecture, that is to define a level-2 invariant, the learner perceives invariants and tries to logically link two (or more) of such invariants in a conditional statement. Such a statement (a level-2 invariant) constitutes an "explanatory hypothesis" for the observed phenomenon. With respect to the illustration in the previous section, dragging B to induce the property "*r* and *s* coinciding", the two invariants can be described as: *r* and *s* coinciding (III: intentionally induced invariant), and B lies on the circle centred at O with OA as radius (IOD: invariant observed during dragging). The instrumented abduction performed by the learner can be illustrated as

Fact Observed and Experienced *DGE Phenomenon*	Perceptual experiences: III (*r* and *s* coinciding) and IOD (B lies on the circle centred at O with OA as radius) are observed simultaneously under variation interactions + direct control over IOD + indirect control over III
MD Discernment Meta-rule *Instrumental Genesis*	Meta-cognitive experiences: "if IOD then III" + Simultaneity of soft invariants IOD and III+ direct control over IOD + indirect control over III
Explanatory Hypothesis or Conditional Link *Logic*	Conjecture statement: If IOD (B lies on the circle centred at O with OA as radius), then III (*r* and *s* coinciding)

The learner observed and experienced simultaneous phenomenal facts in a DGE while exercising direct and indirect MD controls over them under variation interactions. This learner-centered experience transcends to a meta-level where the learner elaborates a MD discernment meta-rule through a process of turning MD into a conjecture-generation instrument. Then this meta-rule *becomes* an explanatory hypothesis (in Magnani's interpretation of abduction) or a conditional link. Thus the rule in an instrumented abduction evolves from the learner's meta-level reflection on her/his MD experience of simultaneity and variation.

The data analyzed in the study by Baccaglini-Frank and Mariotti (Baccaglini-Frank 2010) suggested that once the MD scheme has been appropriated, the process of conjecture-generation as described by the MD-conjecturing model seems to become "automatic", and the learner proceeds through steps that lead smoothly to the discovery of invariants and to the generation of a conjecture, with no apparent abductive ruptures in the process. The abduction seems to be concealed within the MD-instrument. This can now be explained through the meta-rule developed during the process of instrumental genesis of the tool (in this case MD) that allows the learner to interpret the phenomenon in her/his experience in dynamic geometry in terms of logical dependence between invariants, and therefore to produce a conditional statement of the type: "if property A (the second invariant perceived) then property B (the first invariant induced)". Such conditional statement is the product of the learner's exploration, an explanatory hypothesis, but that becomes "automatic" once the process of instrumental genesis is complete (i.e. the MD scheme has been appropriated). In this sense the abduction seems to be of the type: it occurs at a meta-level through a *meta-rule* developed during a process of instrumental genesis, and is *encapsulated* in the use of the MD-artefact in the task of conjecture-generation.

Geometry with a Dragging Principle

DGE are software environments designed to embody Euclidean Geometry in a dynamic and interactive way. Basically they are computer programs that can induce all the properties that are Euclidean consequences of the properties of construction, but not as consequences of the Euclidean axioms. Lopez-Real and Leung (2006) suggested that if dragging in DGE is accepted as a tool that can bring about structures and patterns, then "…we have new 'rules of the game', or even a new game, for experiencing geometry." (op cit, p. 676)

A discernment and reasoning model has been developed in the above discussion where explorations within DGE can lead a learner to transform acting and perceiving to conceptual counterparts and to the fundamental theoretical aspects of the exploration, thus gives us a means to access Euclidean Geometry through the drag-mode. Hence exploring by dragging is a powerful tool supporting geometrical reasoning. This can be expressed as the following *Dragging Exploration Principle*:

> During dragging, a figure maintains all the properties according to which it was constructed and all the consequences that the construction entails in Euclidean Geometry.

This principle implicitly embraces variation, invariants (soft and robust) and sensory-motor perception. Furthermore, it is embedded with a time factor that can be associated to a figure since the possibility of dragging implies variation, which is a temporal phenomenon. A learner has control over a DGE figure through dragging its base points, and can add and take away properties, for example, by freely inducing or relaxing soft invariants. In this way a learner travels back and forth thus imposing a *controllable time frame* on the dynamic figure, *time of a dynamic figure* (*tf*), with respect to a real time sequence *tp* of a dragging experience. *tf* can be thought of as reversible and stoppable (with respect to *tp*) time travelling back and forth with respect to the III and the IOD in which it is possible to modify the figure. These two time-related aspects are direct consequence of the Dragging Exploration Principle. The principle makes possible the appearance of a *tp* phenomenon and serves as a rule to allow further *tf* exploration to look for a cause for (or explanation of) the *tp* phenomenon. In particular, the principle is behind the maintaining dragging modality that drives the path sequence discussed in the previous section which converges to a conditional link (a conjecture) between an III and an IOD.

The Dragging Exploration Principle imbues DGE with an epistemic quality that is process-oriented and user-centred. In particular, the idea of time of a figure opens up a type of geometrical reasoning that could be distinct from deduction and induction, and possibly suggests a different type of pedagogical process. Equipped with this principle, learners can search, via dragging, for reasonable explanations that are consistent with the Euclidean axioms. Fishbien (1993) proposed the notion of *figural concept* which stated that in geometrical reasoning, geometric figures are mental entities that simultaneously possess both conceptual properties (consistent with Euclidean theory) and figural properties (consistent with sensorial perception such as shape, position and magnitude). This notion is particularly fitting with the

Dragging Exploration Principle as a DGE figure is a dynamic geometric entity that is simultaneously Euclidean and sensorial. Under dragging, this simultaneity (with respect to both time frames *tp* and *tf*) between abstraction and perception fosters a type of figural reasoning which I call *DGE figural reasoning*. It is roughly a type of abductive reasoning instrumented by the drag-mode seeking to harmonize between perceived dynamic DGE phenomena and corresponding Euclidean concepts. Variation interactions and the MD Scheme are discernment means that could instrument the drag-mode to bring about such reasoning process. This lecture serves as a starting point for an investigation to pin down the logic behind DGE figural reasoning.

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Riding the Third Wave: Negotiating Teacher and Students' Value Preferences Relating to Effective Mathematics Lesson

Chap Sam Lim

Abstract The "Third Wave" is an ongoing international collaborative mathematics education research project, involving 10 countries conducted over the years 2009–2011. Adopting the theoretical framework of social cultural perspective, the project aimed to explore the contextually-bound understanding and meaning of what counts as effective mathematics lesson from both the teachers and pupils' perspectives. This paper will begin with a brief description of the Third Wave Study Project, the research framework and the general methodology used. Thereafter, it will concentrate on the main focus of the paper featuring a detailed discussion of the related findings from the Malaysian data. The data involved six mathematics teachers and 36 pupils from three types of primary schools. Multiple data sources were collected through classroom observations, photo-elicited focus group interviews with pupils and in-depth interviews with teachers. During each class lesson observation, the six selected pupils (as predetermined by their teacher) were given a digital camera to capture the moments or situations in the observed lesson that they perceived as effective. Pupils were then asked to elaborate what they meant by effective mathematics lesson based on the photographs that they have taken. Teachers were also interviewed individually immediately after each lesson observation and pupil's focus group interview. Findings of the study show that both teachers and pupils shared two co-values and two negotiated values in what they valued as an effective mathematics lesson. The two co-values are "board work" and "drill and practices" while the two negotiated values are "learning through mistakes" and "active student involvement". However, there are minor differences in teachers' and pupils' value preferences, for instance, pupils valued more of "clear explanation" from their teachers and active participation in classroom activities whereas teachers put emphasis on using different approaches to accommodate different types of pupils. More importantly, it was observed that an effective mathematics lesson is very much shaped by the continuous negotiation between

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teachers' and pupils' values and valuing. This paper will end with reflections on some possible implications and significant contributions of the study in mathematics education.

Keywords Effective lesson · Expert/excellent teacher · Mathematics education · Primary school · Photo-elicited interview · Third Wave · Values

Introduction

This paper aims to discuss the value negotiation between teachers and pupils about what constitutes an effective mathematics lesson. More specifically, what are the characteristics of a mathematics lesson that are valued as effective by the teacher and his/her pupils in a mathematics classroom? Do they co-value the same characteristics? How do the teacher and his/her pupils negotiate to shape an effective mathematics lesson?

To set the context, this discussion will begin with a brief introduction of the Third Wave Project, in which the data of discussion were based upon, followed by the research framework and methodology used.

The Third Wave Project and Its Research Framework

The Third Wave Project was a multinational research project coordinated by Seah Wee Tiong from Monash University, Australia (see Seah and Wong 2012 for more details). The project was conducted by 12 research teams in 11 economies. These 11 economies included Australia, the Chinese mainland (2 teams from 2 different provinces), Hong Kong, Japan, Macau, Malaysia, Singapore, Sweden, Taiwan, Thailand, and USA. Conducted over the years 2009–2011, the project aimed "to investigate the harnessing of relevant values to optimize school mathematics teaching and learning" (Seah and Wong 2012, online first version, no page number). Considering the cognitive dimension as the first wave and the affective dimension as the second wave, the project adopted the value perspective as the third wave dimension for examining effective mathematics teaching and learning.

The study was framed by three main knowledge domains, namely effectiveness in mathematics learning, values (in mathematics education) and the role of interaction in education. First, the project argued that the notion of effectiveness in mathematics teaching could be cultural related and thus value laden. This cultural context is not restricted to only the national cultures, but also the ethnic, occupational, religious and gender culture. For example, it was found by Cai and Wang (2010) that mathematics teachers from the East (Chinese mainland, in particular) and their Western counterparts (such as the USA) differed in their view of effective mathematics teaching, and this difference was closely related to their values about the nature of mathematics.

Moreover, results of a few international comparative studies such as PISA 2003 (OECD 2004) have also shown that effective teaching is more about responding to the socio-cultural aspects of the learning environment rather than adopting a particular pedagogy. Therefore, unpacking what each culture values and considered as important will help to identify the characteristics of effective mathematics teaching that are really suited to that particular cultural context.

Second, recognizing that values play a central role in any culture, values (in mathematics education) is thus regarded not as an affective factor, but more a socio-cultural product, drawing their form and meaning from discourses, practices and norms of participants and the interaction among themselves (Seah and Wong 2012). Consequently, mathematics classroom is considered an important venue where teachers and students negotiate their values, and where complex interaction and discourse exchanges are being carried out. It is, therefore suggested that classroom observation becomes the major means of data collection in this project.

Next, the interaction between the teacher and pupils, particularly in mathematics classroom activities is proposed as the place where social mediation and negotiation of what these teacher and pupils considered as important is taking place. What the teacher values as important for his/her pupils will be enacted and emphasised in teaching. Likewise, pupils also bring in their individual values about what an effective mathematics lesson should be based on their own prior knowledge and experiences. However, the teacher's values might sometimes be in agreement but at other times in conflict with his/her pupils' values. It is through active interaction and negotiation between the teacher and pupils that what needs to be valued by the class as a whole will be perceived as being co-valued by both parties.

Purposes of the Malaysian Study

In concomitant with the Third Wave Project, there were three main purposes in the Malaysian study: (i) to identify the elements of mathematics lessons considered as effective primary mathematics lessons by the teachers and their pupils; (ii) to identify any common value(s) underlying effective primary mathematics lessons across different cultural groups; and (iii) to explore the negotiation and co-emphasis of values in effective mathematics lessons between mathematics teachers and students. However, the discussion of this paper will focus mainly on the third purpose mentioned.

Methodology

The research framework of the study as discussed in the earlier section suggested a socio-cultural approach to explore teacher and pupils' preferred values in an effective mathematics lesson. The following is a brief discussion of the participants and methods of data collection.

Participants

This study involved six primary school mathematics teachers selected from the latest published list of "Excellent Teacher" released by the Ministry of Education in 2009. In Malaysia, 'Excellent Teacher' (or "Guru Cemerlang" in the Malay language) is a promotion scheme introduced by the Malaysian Ministry of Education since 1993 (Ministry of Education, 2011). Under this scheme, the award is conferred on a teacher who exhibits these four main characteristics: (a) possesses high expertise, knowledge and skills in his/her subject area; (b) dedicated and responsible; (c) highly motivated especially in teaching and learning in the classroom; and (d) had at least five years of teaching experience in school. This study deliberately chose the excellent teachers as the participants based on the assumption that these teachers were more likely to produce quality or effective teaching than the normal teachers in schools. Nevertheless the study acknowledged that not every lesson taught by these "Excellent Teacher" ought to be an effective lesson.

Table 1 provides brief background information of the six teachers. Both teachers G and K were Indian male teachers who were teaching in two different Tamil Primary schools whereas teachers C and L were Chinese female teachers teaching in two different Chinese primary schools. Teacher Z was a male Malay teacher while teacher R was a female Malay teacher, both were teaching in two different national primary schools. Among these teachers, Z, C and L were very experienced teachers with more than 30 years of teaching experience while R, G and K were younger teachers and newly accredited as "Excellent Teacher".

Besides the teacher participants, there were also 36 pupils who participated in this study. Each teacher was asked to select 6 pupils (two high performing; two averages and two low performing) from the class that was observed for teaching. These pupils were given a camera each to capture the critical moments of the lesson and to elaborate in the focus group interview after the observed lessons. More detailed discussion about this method of data collection or better known as photo-voice will be elaborated in the next section.

Table 1 Respondents' profile

Teacher	Z	R	C	L	G	K
Gender	Male	Female	Female	Female	Male	Male
Race	Malay	Malay	Chinese	Chinese	Indian	Indian
No. of years of teaching mathematics (years)	30	10	30	31	11	9
Year awarded with ET	1999	2009	2001	2008	2008	2008
Types of primary school	Malay	Malay	Chinese	Chinese	Tamil	Tamil

ET = Excellent Teacher

Methods of Data Collection

This study employed three methods of data collection as briefly described below:

1. **Observations of classroom teaching**
   Two video cameras were used to record the classroom teaching so as to provide a holistic view and the teaching process of the lesson. One video camera focused on the teacher while the other focused on the pupils. Each teacher was observed for three mathematics lessons. These lessons were prepared by the Excellent Teachers on a particular mathematics topic that they choose and would like the researchers to observe. The duration of each lesson ranged from 50 min to one hour. Table 2 displays the grade level and topic of each lesson taught by each teacher.
2. **Photo voice or photo-elicited interview with pupils**
   In view of the limitation of vocabulary, language competency and expressive ability of young children, this study adopted the concept and method of photo voice which was first introduced by Wang and Burris (1997). They defined photo voice as a participatory research methodology that uses photographic technique that allows the participants to "identify, represent and enhance their community" (p. 369). Photo voice might be new as a data collection tool in mathematics education research, nevertheless it has been widely used in community research studies (e.g. Wang 1999; Young and Barrett 2001; Wang et al. 2000). In fact, Darbyshire et al. (2005) had adapted photo voice as one of their data collection methods to explore the Australian children's perceptions and experiences of place, space and physical activities. The children were provided with a disposable camera and asked to take photographs on any physical activities. They would then write a brief comment and reasons that they took each photograph. The children were found to express themselves better during the interview based on the photographs taken.
   In this study, during each lesson observation, the six selected pupils were given a camera each to capture any moment that they deemed the teacher was teaching effectively. At this junction, my research team faced a methodological challenge. On one hand, we expect the pupils to give as verbatim as possible their meaning of "effective mathematics lesson". Thus, we did not provide them a

Table 2 Grade level and topics taught on each lesson observation by each teacher

Teacher	Grade of class taught	Lesson 1	Lesson 2	Lesson 3
G	Grade 6	Mixed operation	Area	Pie chart
K	Grade 5	Percentage	Mass	Perimeter
C	Grade 5	Multiplication	Percentage	Mass
L	Grade 3	Volume	Volume of liquid	3-D shapes
Z	Grade 4	Fraction	Division	Money
R	Grade 4	Length	Perimeter	Volume of liquid

definition or an example. However, on the other hand, most pupils would ask for an example of an effective lesson or the meaning of "effective". To resolve this problem, we opted to give the pupil an analogy of medicine. We told the pupils that when they have stomach pain, the doctor prescribe them a kind of medication. They take the medicine and they get cured. So they would say that the medicine is effective. Similarly, what kind of teaching you would like to have or you think it is important to be an effective teaching. We acknowledged that this ambiguity could be a limitation for this study. Therefore, after each lesson observation, we asked each pupil to explain based on the photographs that they have taken. The aim of the focus group pupil interview was to explore the characteristics that the pupils valued as an effective mathematics lesson.

3. **Teacher Interview after each lesson observation**
   Immediately after each lesson observation, the pupils were interviewed then followed by the teacher. During the teacher interview, the teacher was first asked to reflect whether the objectives of his/her lesson were achieved. The teacher concerned was also asked to indicate the moments of teaching that he/she perceived as effective. Later, the teacher was shown the photographs taken by their pupils to check if they agreed with the pupils' view about the elements of an effective mathematics lesson.

Findings and Discussion

In total, 18 lessons were observed, 18 individual teacher interviews as well as 18 focus group pupil interviews were video recorded and transcribed for analysis. Each of the 36 selected pupils took photographs of effective moments during the three observed lessons for each school.

As shown in Table 3, each pupil took photographs ranging from as few as four photographs to 85 photographs. Altogether 1057 photographs were taken. There was no significant observed difference in the number of photograph taken by the pupils in terms of their ability levels or type of school. Comparatively, pupils from SJKC T appears to take the least number of photographs, perhaps these pupils came

Table 3 Number of photographs taken by each pupil from the six schools

Pupil	SK A	SK B	SJKT L	SJKT P	SJKC T	SJKC M	Total
A1	15	22	11	85	26	18	177
A2	15	19	26	48	8	17	133
B1	44	40	26	64	9	18	201
B2	27	28	34	41	42	13	185
C1	19	32	56	38	4	16	165
C2	14	51	7	81	14	29	196
Total	134	192	160	357	103	111	1057

Table 4 Themes representing espoused and observed classroom practice of teachers across different schools

Theme	Type of school					
	SK		SJKC		SJKT	
School	A	B	M	T	L	P
Teacher	Z	R	C	L	G	K
1. Board work	✓○	✓○	✓○	✓○	✓○	✓○
2. Drill and practice	✓○	✓○	✓○	✓○	✓○	✓○
3. Different approaches for different types of pupils		✓	✓		✓	✓
4. Use of real or concrete objects	✓○	✓○	✓○	✓○	✓○	✓○
5. Engaging ICT or courseware	✓○	✓○		✓○	✓○	✓○

✓ = espoused in interview ○ = observed in classroom teaching

from the younger grade (they were Grade 3 pupils). Meanwhile, pupils from the SJKT P seem to capture the most number of photographs, but many of the pictures taken were repetitive.

The data were then analysed separately based on the video recorded classroom teaching, individual interview for teachers, and photo-elicited focus group interviews and photographs taken for pupils. A list of themes emerged from the analysis of the data on the classroom observation (a total of 59 themes); teacher interviews (a total of 40 themes) and student interviews (a total of 83 themes). For the purpose of this paper, only the five most common themes from each data source were compared. Tables 4 and 5 display the distribution of themes for teachers and pupils respectively.

As displayed in Table 4, all the six participating teachers espoused (in the interviews) as well as enacted (as observed in classroom teaching) five common themes: namely board work, drill and practice, different approaches for different pupils, use of concrete objects and integration of courseware. Similarly, the 36 pupils (see Table 5) co-valued board work, exercise or practice, learning through mistakes, explanation and students' involvement as the five common elements of an effective mathematics lesson. Hence, two out of the five common themes were co-valued by both teachers and pupils, that are, board work and exercises or practices.

Table 5 Themes valued by pupils from all schools

Theme	School					
	SK A	SK B	SJKT L	SJKT P	SJKC T	SJKC M
Board work	✓	✓	✓	✓	✓	✓
Exercise or practice	✓	✓	✓	✓	✓	✓
Learning through mistakes	✓	✓	✓	✓	✓	✓
Explanation	✓	✓	✓	✓	✓	✓
Students Involvement	✓	✓	✓	✓	✓	✓

Co-value 1: Board work

In this study, board work refers to asking pupils to come in front of the class to demonstrate their working or writing out their solutions on the board. There were 85 photographs taken by the pupils exhibiting board work. During the focus group pupil interviews, the following reasons were given:

> "when he (the pupil) works in front, everyone in the class can see what he is doing]" (translated from Malay language, SK B_B1).
>
> "The mistakes made can be discussed together" (translated from Malay language, SK B_B2).
>
> "Sir will call the students who know how to do it to do so that whoever don't know, they also will learn from it" (translated from Tamil, SJKT P_A1).
>
> "Maybe she doesn't understand it well…to make her clearer, sir called her out to do it" (translated from Tamil, SJKT P_C1).
>
> "In this photo, teacher asked a pupil to come out and do,…Teacher called the boy who can do to come out and do, or else we may not understand later" (translated from Mandarin, SJKC M_B1).
>
> "Teacher asked a pupil to come out and do, if he did wrongly, teacher can then correct him" (translated from Mandarin, SJKC M_A1).

Analyzing these pupils' reasons indicates that pupils valued board work as a platform for them to learn from their peers. This can happen in two ways. If the better ability pupils were asked to demonstrate their solutions in front, the weaker peers can learn from them the correct methods of solving the problem. Conversely, if the weaker ability pupils were called out, the other peers can learn from the mistakes made.

Meanwhile, engaging pupils in board work was also commonly mentioned as one of the elements of an effective mathematics lesson during the teacher interviews. For instance, Teacher R from SK B espoused in the interview that she liked to call pupils to show their work in front of the class, particularly the academically weaker pupils. Teacher R mentioned in her interview that, "*the children come to show in front, their friends can spot where they have made mistakes*" (translated from Malay; *SK, Teacher R interview after 1st lesson).* Teacher K from SJKT P also echoed that asking pupils to present their solutions in front is a good strategy as "*when one group had learned, if got some mistakes, they can share…*" (SJKT, Teacher K interview after 3rd lesson). Hence, it is observed that what the teachers have espoused was coherent with what were mentioned earlier by their pupils. In brief, engaging pupils in board work allows pupils to learn from each others, particularly from the peers' mistakes. Therefore teachers would encourage pupils to show their workings through board work while their pupils also support this kind of activity. Thus, both teacher and pupils co-valued the importance of board work as an important element of an effective mathematics lesson.

Co-value 2: Exercises and Practices

Giving exercises for drill and practice was observed to be a common norm for most mathematics classes in all the three types of primary schools in Malaysia. Thus, it was not surprising to notice that many of the pupils took photographs on worksheet or textbook exercises that were assigned by their teachers (see some sampled photos): (Figs. 1, 2 and 3).

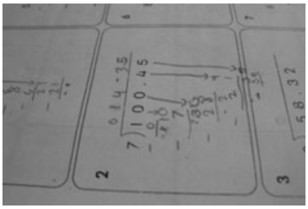

Fig. 1 *Photograph* taken by SK_B1

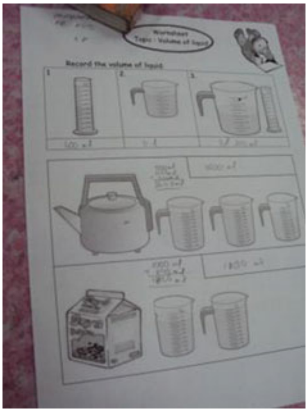

Fig. 2 *Photograph* taken by SK B_C1

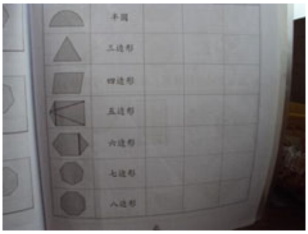

Fig. 3 *Photograph* taken by SJKC T_A2

During interview, pupils gave practical reasons such as:

> "Sir Vanthu, (Sir) give us exercises to [do and to see if] students understand or not" (SJKT L_B2).
> "He [the teacher] will give them [the pupils] more exercises and allocate time for them and explain to them" (translated from Tamil, SJKT P_B1).
> "Teacher gave us worksheets to practice. If we do, during examination, there are a lot of questions. If those that the teacher photocopied are similar to the one in examination, then we will know how to do" (translated from Mandarin, SJKC T_C2).

The reasons given indicate that pupils valued exercises and practice for enhancing their understanding, and more importantly, so that they can score well in examinations. The examination-oriented culture, particularly in Chinese primary schools, is clearly rooted as early as in Grade 3 (illustrated from the interview with the Grade 3 pupils of the selected school—SJKC T as displayed above).

The value of drill and practice was also clearly reflected in teachers' interview data. For instance, Teacher R highlighted in one of the interviews that, "In my mindset, learning mathematics must do a lot of exercises… if lack of exercise, we will be weak." (Translated from Malay, SK, Teacher R interview after 2nd lesson). Her opinion was supported by Teacher K that, "that is my secret for me, I give more activities, more exercises…I apply [this] for the Year 1 [pupils] also, I think that is a better way, they can do well." (SJKT, Teacher K interview after 3rd lesson). Teacher K further added that, "Before the exam, I have given them 3 or 4 books, the exercise books, the small squares, so many activities, so many exam papers, I prepared them to do well." In brief, these teachers appeared to hold a strong belief of "practice make perfect" and drill and practice is one of the effective ways of ensuring their pupils do well in examination.

Clearly, this co-sharing of the value of drill and practice has allowed both the teacher and their pupils to negotiate and to support each other's effort. The teachers gave a lot of drill and practice to their pupils in the forms of individual exercises, group exercises, memorization of multiplication tables and home work. The pupils were willing to carry out these practices in school as well as at home as they hold the same belief of practice makes perfect, and practice as the way to ensure better performance in examination.

Besides the above two clear-cut co-values: board work and exercises or practices, further analysis of the data show that there were a few negotiated values as discussed below:

Negotiated value 1: Learning through mistakes

"Learning through mistakes" is classified as a negotiated value because this value was explicitly mentioned by the pupils but only implicitly reflected in teachers' interviews. For example, a pupil (A1) from one of the national primary schools (SK A) took a photograph that depicts a comparison of four pupils' solutions written on a small board (see Fig. 4). During the interview, the pupil (A1) explained that, "Can correct what is wrong or not?" (Translated from Malay language). His explanation

Fig. 4 A *photo* taken by SK A_A1 showing a comparison of different solutions on the blackboard

was supported by his peer (B1) that, "Teacher wants us to see and to show which is right which is wrong" (Translated from Malay language, SKA_B1).

Similar photograph content and similar argument was also given by a group of the Chinese primary school (SJKC M) pupils as illustrated in their interview conversation below: (The following conversations were translated from Mandarin):

R: Why does the teacher want to compare one right and one wrong?
C2: So that we can know…
A1: Which is wrong, which is right
B1: Because (the teacher) wants us to know where the mistake is
A2: Can get us remember, because after teacher discussed about it, we can remember it better

The above pupils' reflections show that pupils valued their teachers correcting their mistakes immediately in class or in their exercise books, so that they can learn through their own or their peers' mistakes.

However, no teacher mentioned explicitly about the value of learning through mistakes. Instead, they negotiated this value through the effective use of board work. Teachers liked to call pupils, particularly the academic weak pupils to display their working or solution steps in front of the class. Their purpose is to assess to what extent these pupils have achieved the teaching and learning objectives. Three of the teachers explicitly mentioned that,

"I choose mostly weak pupils" (translated from Malay; SK, Teacher R interview after 2nd lesson); "Just now I chose all are weak or slow learners." *(*SJKT, Teacher K interview after 2nd lesson); *and "*those who are lagged behind, I will call them especially" (translated from Chinese, SJKC, Teacher L interview after 3rd lesson). Nevertheless, their intentions were well received by their pupils who took them as opportunities for learning.

Negotiated value 2: Active student involvement

Student involvement refers to the moments or situations whereby the pupils mentioned explicitly in the interviews or in the photographs taken that they were asked by their teachers to go out to the front of the class to be involved in teaching and learning activities such as demonstration, role play, testing memorising of multiplication table or playing games. In total, there were 85 photographs taken which depicted pupils' active involvement in some activities. Figures 5, 6 and 7 illustrate some of the scenes taken.

Fig. 5 A pupil was asked to fill in the table on the board, regarding the number of sides and angle of different polygons. (*Photograph* taken by SJKC T_A2)

Fig. 6 Two pupils were asked to role play as seller and buyer in the lesson on 'mixed operation'. (*Photograph* taken by SK A_C2)

Fig. 7 Pupils were asked to explain their solution in front of the class. (*Photograph* taken by SJKT P_A2)

The finding that pupils like to be involved in activity did not surprise the teachers. Teacher C mentioned that, "They (her pupils) are very active. They like to come in front (of the class) and show" (translated from Mandarin, SJKC_M, interview after 1st lesson). Her response was echoed by another male teacher from SJKT_P that, "Because this class, they every…every day they ask me to call (them) in front. Students they want to do in front" (SJKT, Teacher K interview after 2nd lesson). Likewise, Teacher L also found some of her pupils liked to show their work in front of the class. She recalled, "They like to involve themselves in learning, so they like to come out. I gave them chances to answer, come out to the front, they like to write it out at the front. *(Translated from Chinese, SJKC, Teacher L interview after 3rd lesson).*

Thus, the above teachers' reflection indicated that the teachers have tried to fulfil their pupils' value of student involvement through the negotiation of giving board work and group activities. However, most teachers were constrained by the shortage of time and the overloaded syllabus that they could only provide selective activities for their pupils.

Implications and Conclusion

The above analysis implies that both the teachers and their pupils shared two co-values and two negotiated values. The two co-values are "board work" and "drill and practices" while the two negotiated values are "learning through mistakes" and "active student involvement".

In fact, board work provides a platform for the pupils to display their solution or working steps in front of the class. In this way, the teacher can assess the progress or achievement of the pupils as well as diagnose the mistakes of his/her pupils. At the same time, board work allows pupils to learn through their peers' mistakes, as well as provide the pupils with opportunities to engage themselves actively in the teaching and learning process. Indeed, board work was commonly observed in many Asian mathematics classrooms. For examples in Japanese classroom (Shimizu 2009) and Australian classroom (Seah 2007), students are commonly asked to come in front to present their work too. Likewise, Lim (2007) observed that Shanghai mathematics teachers also like to "call individual students to demonstrate in front of class, answer or explain orally" (p. 81 in Table 1) as a way to engage the pupils.

More importantly, to make board work as a platform for effective learning, the teacher needs to carefully pose appropriate questions and choose pupils' solution steps to display on the board for discussion. As highlighted by Takahashi (2011), in a typical Japanese mathematics lesson, the teacher usually presents a problem to his/her pupils without demonstrating a procedure or method. The aim is to encourage pupils to bring in different approaches and different possible solutions. The experienced Japanese teacher will then go round the class to select appropriate exemplar solution methods. These anticipated solution methods should include both the most efficient methods and also those caused by misconceptions. In this way, the teacher can then facilitate the pupils to compare and justify which is the best method to use, as well as correcting some misunderstandings of pupils.

The next implication is that board work could be particularly effective for mathematics teaching and learning as compared to other subjects. Solving mathematical problems requires showing the working or solution steps precisely. Pupils are expected to derive step by step logically and to clearly demonstrate how a solution is arrived at. This is unlike other subjects such as language (e.g. English) or even arts or science subjects (e.g. history, biology) which might be too content laden or need too much space and time to demonstrate their solutions on the board. Therefore, this finding implies that using board work could be an effective teaching approach that provides a platform to engage pupils actively as well as providing a two way communication between teacher and pupils, particularly suitable for mathematics teaching and learning.

The second characteristic that was co-valued by both teachers and pupils as effective mathematics lesson was the beliefs about the role of practice and exercises in mathematics learning. Mathematics composes of both concepts and skills. It is generally believed that drill and practice is an effective way to sharpen mathematical skills. Particularly, drilling of basic facts can help pupils to memorise the multiplication tables and formulae which consequently resulted in automatic response, which could be useful for recalling of facts and formulae during problem solving. Indeed, this belief of "practice makes perfect" is also a common norm in many of the examination oriented Asian countries (such as China and Singapore). Holding strongly to this belief, therefore the teacher tends to give a lot of exercises and problems for his/her pupils to practice and solve. As the pupils also hold strongly to the same belief of practice makes perfect, they do not complain about the amount of exercises or home work that they were given. Thus the teacher strives hard to provide more exercises for the pupils, while the pupils work hard to complete them, so as to achieve excellence in the mathematics examinations.

However, one major implication to be cautious about is that drilling without conceptual understanding (or some called it as procedural learning) might not lead to effective mathematics learning in the long run. The amount of time spent and the quantity of exercises given do not necessarily equate to the quality of learning. Therefore careful selection of exercises and variation of exercises given are crucial in applying the belief of practice makes perfect.

Finally, the findings of this study imply that there were minor differences in teachers' and pupils' value preference in relating to an effective mathematics lesson. Pupils valued more of "clear explanation" from their teachers and active participation in classroom activities whereas teachers place emphasis on using different approaches to accommodate individual differences of pupils. Nevertheless, to ensure an effective mathematics lesson, both teachers and pupils will have to negotiate continuously in order to suit the pupils' and the teachers' value preferences. As in this study, the teacher negotiated pupils' value preference of active participation through board work and group activities occasionally. Pupils valued clear explanation from their teacher which is negotiated with their teacher's value of "using concrete objects". Through the use of concrete objects, the teachers believed that they would give a more concrete and clearer explanation to their pupils that would further enhance their understanding of abstract mathematical concepts.

In brief, for any mathematics lesson to be effective, ideally, both teachers and pupils should share and co-value collectively. Or else, they will have to negotiate continuously between teachers' and pupils' value preferences.

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Learning Mathematics by Creative or Imitative Reasoning

Johan Lithner

Abstract This paper presents (1a) a research framework for analysing learning difficulties related to rote learning and imitative reasoning, (1b) research insights based on that framework, (2a) a framework for research and design of more efficient learning opportunities through creative reasoning and (2b) some related ongoing research.

Keywords Learning difficulties · Rote learning · Creative reasoning · Problem solving

Introduction

A central problem in mathematics education is that we want students to understand mathematics and to become efficient problem solvers, but even after 30 years of research and reform many students still do inefficient rote thinking (Hiebert 2003; Lithner 2008). This is one of the main reasons behind learning difficulties in mathematics. Even the learning of routine procedures does not function well by rote learning, since students largely following the rules "like robots with poor memories" (Hiebert 2003, p. 12).

There are probably several reasons behind this problem such as social, cultural, political, etc. This paper will focus on some of the reasons that are directly related to how the subject mathematics is handled in teaching and learning situations. Even with respect to this aspect, the reason that the rote learning problem is (largely) unsolved in many countries is probably a combination of several factors related to the immense complexity of mathematics learning (Niss 1999) and to the lack of research insights concerning the effectiveness of different teaching designs (Niss 2007). In addition, there seems to be many choices made in ordinary teaching that lead to rote

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learning as an unintended by-product, mainly connected to attempts to help students by reducing the cognitive complexity (Doyle 1988; Schoenfeld 1985, 1991).

The purpose of this paper is to present research frameworks for (1) analysing existing learning difficulties related to rote learning and for (2) a design research approach to constructing more efficient learning opportunities for mathematics students. A research framework is in this paper seen as "a basic structure of the ideas (i.e., abstractions and relationships) that serve as the basis for a phenomenon that is to be investigated" (Lester 2005, p. 458). These two parts respond to the questions "what are the causes and characteristics of mathematical learning difficulties?" and "what can be done to improve the situation?" respectively. Of course, this paper does not fully respond to these questions but aim to treat some central aspects of them.

Part One: Insights in Shortcomings of Imitative Learning

Part one of the framework contains the following components:

- Goals for mathematics teaching and learning, as a basis for analysing what to aim for and where we fail. Problem solving, reasoning and conceptual understanding will be in focus.
- Rote learning, the unintended but common way to try to learn mathematics through superficial imitation, which is a main cause behind learning difficulties since the goals mentioned above are not attained.
- Imitative and creative mathematical reasoning, a characterisation of the thinking processes that students activate in learning situations. A basic assumption in this paper is that students opportunities to learn mathematics are largely determined by the thinking processes they activate in learning situations, and their explicit reasoning is seen as traces of their thinking processes. The empirically based reasoning framework will be used to specify how students reason by imitation in rote learning and what is missing with respect to more efficient creative reasoning.

Learning Goals in Mathematics: Problem Solving, Reasoning and Conceptual Understanding

In order to understand the learning difficulties that students encounter and in order to suggest measures to take, it is not sufficient to describe learning goals merely in terms of mathematical content since such descriptions do not capture what students are supposed to be able to do with the content. The NCTM Principles and Standards (NCTM 2000) complement five content standards (number and operations, algebra, geometry, measurement, and data analysis and probability) with five process standards (problem solving, reasoning and proof, connections, communication, and representations). Judging by the impact and large number of references to the NCTM Principles and Standards (and its earlier versions) it seems fair to say that this type of learning goal description is largely accepted by the mathematics education research community. A framework presented by Niss and Jensen (2002) and Niss (2003) contains factors similar to the NCTM process standards but denotes them competencies. Competence is the ability to understand, judge, do, and use mathematics in a variety of mathematical contexts and situations. Three competencies are particularly relevant for this framework: problem solving competence, reasoning competency and conceptual understanding. The latter is not formulated as a competency in the mentioned frameworks, but is in this paper seen as a somewhat intuitive term including the representation and connection processes of the NCTM (2000) framework, see below for definitions.

In this paper *problem solving* is defined as "engaging in a task for which the solution method is not known in advance" (NCTM 2000, p. 51). This definition implies that in this perspective there are only two types of tasks: problems and non-problems (often denoted 'routine tasks'). Note that some aspects often included in similar definitions of problem solving are not included in the definition above, for example that the task is necessarily a challenge (Schoenfeld 1985) or that the task requires exploration (Niss and Jensen 2002). The main difference between solving a problem and a routine task is that in the former the solver has to (at least partially) construct the solution method by herself, while in a routine task the method is already known by the solver or provided by an external source such as the book or the teacher. One may also note that in determining if a task is a problem or not it is insufficient to consider properties of the task alone, instead the relation between the task and the solver has to be considered (Schoenfeld 1985). For example, to find the number of combinations when having 3 pants and 4 shirts available to choose from may be a problem for a grade 3 student but a routine task for a grade 7 student. And if the grade 3 student (that master elementary addition and/or multiplication) has solved several similar problems and realised that the number of combinations can be found by multiplication or repeated addition, or if the teacher describes this method to the student, then every task of this particular type becomes a routine task to this student.

The problem solving competency includes identifying, posing, and specifying different kinds of problems and solving them, if appropriate, in different ways (Niss 2003). Schoenfeld (1985) formed through a series of empirical studies the probably most cited problem solving framework based on four key competencies (p. 15): Resources (basic knowledge), Heuristics (rules of thumb for non-standard problems), Control (metacognition: monitoring and decision-making), and Belief Systems (one's mathematical world view). He found that novices often had sufficient resources but were lacking in the other three competencies.

The NCTM Principles and Standards (2000) recognize *reasoning and proof* as fundamental aspects of mathematics. "People who reason and think analytically tend to note patterns, structure, or regularities in both real-world situations and symbolic objects; they ask if those patterns are accidental or if they occur for a reason; and they conjecture and prove" (p. 56). The reasoning competency goes beyond constructing reasoning, and includes abilities like following and assessing chains of arguments, knowing what a proof is and how it differs from other kinds of reasoning, uncovering the basic ideas in a given line of argument, and devising formal and informal arguments (Niss 2003). Creative and imitative reasoning are examples of such different forms of reasoning, which will be described in more details below, and used in the subsequent analyses of students reasoning.

The concept of *understanding* is very complex (Sierpinska 1996), and will not be pursued here beyond noting that several of the theoretical constructs concern relations between rote learning and deeper understanding. Skemp (1978) distinguishes between 'instrumental understanding' and 'relational understanding' of mathematical procedures. The former can be apprehended as 'true' (relational) understanding, but is only the mastering of a rule or procedure without any insight in the reasons that make it work. A similar distinction, between 'action' and 'process' is made by Asiala et al. (1996), and Hiebert and Lefevre (1986) distinguish between conceptual and procedural understanding. It seems difficult to find a precise definition of mathematical understanding, but the NCTM (2000) representation and connection standards can be seen as more well-defined sub-components of understanding. Inspired by (but not using the notions in exactly the same way) Brousseau (1997, p. 12) and NCTM (2000), the notion 'conceptual understanding' (or just 'understanding') will here be used in a relatively intuitive way, referring to insights in the origin, motivation, meaning and use of a mathematical fact, method or other idea, where insights include suitable representations and connections.

Seeing problem solving ability, reasoning ability and conceptual understanding as key learning goals, the next section will describe how learning environments that promote rote learning deny students of proper opportunities to reach such goals.

Rote Learning

Rote learning is "the process of learning something by repeating it until you remember it rather than by understanding the meaning of it" (Oxford Advanced Learner's Dictionary). Rote learning in mathematics mainly includes facts and procedures, and can vary from simple, e.g. the fact that a + b = b + a or the procedure of one-digit addition by finger-counting, to complex such as a long proof or a set of calculus techniques for integration of composite functions. The characteristics, causes, and consequences of rote learning in mathematics can to a large extent be connected to an unwarranted and far-reaching reduction of complexity in terms of an algorithmic focus (Skemp 1978; Hiebert and Carpenter 1992; Tall 1996; Vinner 1997; Hiebert 2003; Lithner 2008).

> "After several years of executing procedures they do not understand, students' behaviour is so rule-governed and so little affected by conceptual understanding that one can model their behaviour and predict the errors they will make by looking only at the symbol manipulation rules they have been taught and pretending that they are following these rules like robots with poor memories" (Hiebert 2003, p. 12).

Rote learning alone is not likely to be a goal of any mathematics curricula. Schoenfeld (1985) argues that many of the counterproductive behaviours that we see in students are unintended by-products of their mathematics instruction that result from a strong classroom emphasis on performance, memorising, and practising, which ultimately causes students to lose sight of rational reasons. Referring to "massive amounts of converging data" in studies from USA, Hiebert suggests that the baseline conclusion is that students are learning best the kinds of mathematics that they are having the most opportunities to learn, which is simple calculation procedures, terms and definitions through memorization (Hiebert 2003). Similar opportunities to learn mainly how to handle procedures were found in a Swedish large-scale study including observations of 200 mathematics classrooms (Boesen et al. 2012).

Rote learning is in itself not problematic. On the contrary, memorising facts and procedures, even without understanding, is a central aspect of mathematics learning. It is not reasonable to expect that students should be able to understand or (re) construct every mathematical idea. At least some of these ideas are too difficult to be fully understood (for a specific educational level) and may have to be learnt by rote or with limited understanding. The problem is when rote learning becomes dominating since it is not possible to develop other central competencies like problem solving ability and conceptual understanding by rote learning alone. For example, it is well known from the extensive research on problem solving from the eighties (e.g. Schoenfeld 1985) that there is no transfer from rote learning of basic facts and procedures to the ability to solve non-routine mathematical problems. From literature reviews (e.g. Hiebert 2003) and from the empirical studies exemplified below, it is reasonable to draw the conclusion that rote learning is one of the main causes behind the learning difficulties that large groups of mathematics students of all age levels encounter. As will be discussed below, the avoidance of meaning is a key to clarifying both advantages and disadvantages of rote learning.

So far, the main ideas in the presentation of rote learning and problem solving have been quite general. However, one reason that impact of rote learning may be particularly strong in mathematics is that the historical progress of the subject itself is to such a large extent based on the inventions of powerful concepts (such as the zero and the decimal position system) and powerful procedures (such as algorithms for arithmetic calculations), of which many can be learnt by rote. As an extreme example, it is fairly easy to teach seven-year old kids to differentiate simple polynomials, which would yield them some points on an upper secondary mathematics exam. These kids have of course no insight whatsoever into the underlying concepts of polynomials, functions or differentiation and have learnt the procedure only by rote. At the same time, due to the clear and simple structure of mathematics and the possibility to (partially) detach mathematical tasks from the complexity of the real world and hereby choose a suitable task difficulty level, mathematics is particularly suited to learn problem solving and reasoning (Pólya 1954).

Creative Reasoning

This and the next section contain a modified summary of selected parts of a research framework (Lithner 2008) that is based on the outcomes of a series of empirical studies on the relationship between reasoning and learning difficulties in mathematics.

"Mathematical reasoning is no less than a basic skill" (Ball and Bass 2003, p. 28). Despite this pronouncement, the term 'reasoning' is often used by mathematics educators without being defined under the implicit assumption that there is universal agreement on its meaning (Yackel and Hanna 2003). The purpose of this section is to provide three things: (1) a broad definition of reasoning that allows the inclusion (and comparison) of both low- and high-quality arguments; (2) the underlying notions that make it possible to define creative, mathematically founded reasoning; (3) a characterization of imitative reasoning as the opposite of creative reasoning.

Reasoning is defined in this paper as the line of thought that is adopted to produce assertions and reach conclusions when solving tasks. Reasoning is not necessarily based on formal logic and is therefore not restricted to proof; it may even be incorrect as long as there are some sensible (to the reasoner) reasons supporting it. This example illustrates that "reasoning" is used in a broad sense in this framework to denote both high- and low-quality argumentation; the quality of the argument is characterized separately. Reasoning can be seen as thinking processes, as the product of these processes, or as both. The data for the investigations discussed here are behavioural; thus, we have to at least partially speculate about the underlying thought processes (Vinner 1997). Because one purpose of this framework is to characterize data, the choice here is to see reasoning as a product that (primarily) appears in the form of written and oral data in the form of a sequence of reasoning that starts in a task and ends in an answer.

In a task-solving situation (including sub-tasks) two types of argumentation are central.

1. *Predictive argumentation* (why will the strategy solve the task?) can support the strategy choice. The 'strategy' can vary from local procedures to general approaches, and 'choice' is defined in a broad sense (choose, recall, construct, discover, guess, and so forth).
2. *Verificative argumentation* (why did the strategy solve the task?) can support the strategy implementation.

School tasks normally differ from the tasks addressed by professionals such as mathematicians, engineers and economists. Within the didactic contract (Brousseau 1997) in the school context, it is allowed, and sometimes encouraged, to guess, to take chances, and to use reasoning without any strict requirements on the logical value of the reasoning. Even in examinations, it can be acceptable, as in Sweden for example, to have only 50 % of the answers correct, while it would be absurd if mathematicians, engineers, or economists were satisfied in being correct in only 50 % of their conclusions. This framework proposes a wider conception of logical value that is inspired by Pólya (1954): "In strict reasoning the principal thing is to distinguish a proof from a guess, […] In plausible reasoning the principal thing is to distinguish a guess from a guess, a more reasonable guess from a less reasonable guess." Thus, a *plausible argument* can be constructive without being logically valid (in contrast to, for example, a proof which must be logically true).

What does it mean for an argument to be based on mathematics? Schoenfeld (1985) found that novices used naive empiricism and judged that geometrical constructions were correct if they 'looked good,' whereas experts used more relevant properties (for example, congruence). Thus, the reference to the mathematical content is important: what are the arguments about? To address this question, the notion of *anchoring* is introduced (Lithner 2008). Anchoring does not refer to the logical value of the argument but refers to its fastening the relevant mathematical properties of the components one is reasoning about—objects, transformations, and concepts—to data. The *object* is the fundamental entity; it is the 'thing' that one is doing something with, for example, numbers, variables, functions, and diagrams. A *transformation* is what is being done to the object, and the outcome of the transformation is another object. A sequence of transformations, finding polynomial maxima for example, is a *procedure*. A *concept* is a central mathematical idea built on a set of objects, transformations, and their properties, such as the concept of a function or of infinity. The status of a component depends on the situation. $f(x) = x^3$ can be seen as a transformation of the input object 2 into the output object 8. If f is differentiated, then the differentiation is the transformation; $f(x)$ is encapsulated (Tall 1991) into an input object, and $f'(x)$ is the output object.

Arguments can be anchored in either *surface* or *intrinsic properties*, and the relevance of a mathematical property can depend on context. In deciding if 9/15 or 2/3 is largest, the size of the numbers (9, 15, 2, 3) is a surface property that is insufficient to resolve the problem (a conclusion based on this property alone is that 9/15 > 2/3 since 9 and 15 are larger than 2 and 3), while the quotient captures the intrinsic property. The intrinsic/surface distinction was introduced because one of the reasons behind students' difficulties was found to be the anchoring of arguments in surface properties (Lithner 2003).

Because this framework addresses ordinary students' thinking, imputing creativity only to experts is not sufficient.

> "Although creativity is often viewed as being associated with the notion of 'genius' or exceptional ability, it can be productive for mathematics educators to view creativity instead as an orientation or disposition toward mathematical activity that can be fostered broadly in the general school population" (Silver 1997, p. 75).

The aspect of creativity that is emphasized in this framework is not 'genius' or 'exceptional novelty,' but the creation of mathematical task solutions that can be modest but that are original to the individual who creates them. Thus, creative is the opposite of imitative.

The discussion above leads to a definition of *Creative Mathematically Founded Reasoning* (CMR) that fulfils all of the following criteria (See Lithner 2008, for references to underlying empirical data).

(i) Creativity. A new (to the reasoner) reasoning sequence is created, or a forgotten one is re-created, in a way that is sufficiently fluent and flexible to avoid restraining fixations.
(ii) Plausibility. There are arguments supporting the strategy choice and/or strategy implementation explaining why the conclusions are true or plausible.
(iii) Anchoring. The arguments are anchored in the intrinsic mathematical properties of the components that are involved in the reasoning.

Imitative Reasoning

The empirical studies behind this framework have identified two main types of imitative reasoning: memorized and algorithmic. In *Memorized Reasoning* (MR), the strategy choice is founded on recalling an answer by memory, and the strategy implementation only consists of writing it down. This type of reasoning is useful as a complete solution method in only a relatively small proportion of tasks (Lithner 2008), such as recalling every step of a proof or the fact that one litre equals 1000 cm^3. When school tasks ask for calculations, it is normally more appropriate to use *Algorithmic Reasoning* (AR) (Lithner 2008), where the strategy choice is to recall an algorithm and the strategy implementation is to apply the algorithm to the task data.

The term 'algorithm' includes all pre-specified procedures (not only calculations), such as finding the zeros of a function by zooming in on its intersections with the x-axis with a graphing calculator. "An algorithm is a finite sequence of executable instructions which allows one to find a definite result for a given class of problems" (Brousseau 1997, p. 129). The importance of an algorithm is that it can be determined in advance. The nth transition does not depend on any circumstance that was unforeseen in the $(n - 1)$st transition—not on finding new information, any new decision, any interpretation, or thus on any meaning that one could attribute to the transitions. Therefore, the execution of an algorithm has high reliability and speed (Brousseau 1997), which is the strength of using an algorithm when the purpose is only to solve a task.

However, if the purpose is to learn something from solving the task, the fact that an algorithm is independent of new decisions, interpretations or meaning implies that all of the conceptually difficult parts are taken care of by the algorithm, and thus only the easy parts are left to the student. This segmentation may lead to rote learning. In particular, the resultant argumentation is normally superficial and very limited, as seen in the main AR types that are found in studies. *Familiar AR/MR* includes a strategy choice that can be characterized by (perhaps superficial) attempts to identify a task as being of a familiar type with a corresponding known solution algorithm or a complete answer. Justifying a successful solution by simply describing the algorithm is an accepted socio-mathematical norm (Yackel and Cobb 1996) in most practise and test situations studied (Lithner 2008). In *Delimiting AR*, the algorithm is chosen from a set of algorithms that are available to the reasoner, and the set is delimited by the reasoner through the included algorithms' surface property relationships with the task. For example, if the task contains a second-degree polynomial $p(x)$, the reasoner may wrongly choose to solve the corresponding equation as $p(x) = 0$ even if the task asks for the maximum of the polynomial (Bergqvist et al. 2008). In *Guided AR*, the reasoning is mainly guided by two types of sources that are external to the task. In person-guided AR, a teacher or a peer pilots the student's solution. In text-guided AR, the strategy choice is founded on identifying, in the task to be solved, similar surface properties to those in a text source (e.g., a textbook). Argumentation may be present, but it is not necessary because the authority of the guide ensures that the strategy choice and the implementation are correct.

In students' attempts to resolve problematic task solving situations, the CMR criteria i–iii (see section "Creative Reasoning") were found to capture the main differences seen in reasoning characteristics between MR/AR (where i–iii are absent) and constructive CMR (Lithner 2008). A task solution in MR is immediate through recollection, in AR, it follows a known algorithm and in CMR, it is created (although CMR normally includes elements of MR/AR). Furthermore, in CMR the epistemic value (degree of trust, see Duval 2002) lies in the plausibility and in the logical value of the reasoning. In MR and AR, it is determined by the authority of the source of the imitated information.

Students often use superficial imitative reasoning of the types presented above in laboratory tests and when working with tasks (e.g. textbooks or assessment) in regular classroom contexts, which is a major hurdle both when it comes to learn and to use mathematics (e.g. Lithner 2000, 2003, 2008, 2011; Bergqvist et al. 2008; Boesen et al. 2010). In addition, teaching, textbooks and assessments mainly promote rote learning in the sense that Guided AR is provided by teachers and textbooks, and that most practise and test tasks can be solved by AR (e.g. Bergqvist 2007; Palm et al. 2011; Bergqvist and Lithner2012; Boesen et al. 2012). Judging from the quote by Hiebert in the introduction this may be the case also outside Sweden, for example as found in common American calculus textbooks (Lithner 2004).

Summary Part One

Rote learning is sometimes necessary in mathematics and in general efficient in a limited and short-sighted perspective, but when dominating it does not provide students with opportunities to develop central mathematical competencies such as problem solving competency, reasoning competency and conceptual understanding. The message from large parts of the mathematics education research community (e.g. NCTM 2000) is quite clear: students need to engage in activities including problem solving and reasoning in order to develop mathematical competence. Under the assumption that learning mathematics is affected by the reasoning that the student actually activates when solving practise tasks, it is hypothesised that in order to learn mathematics better students need to engage in problem solving and CMR to a larger extent than what seems to be the case in for example in USA and Sweden. The next section will present ongoing research on the design of such learning opportunities.

Part Two: Designing Teaching Through Creative Reasoning

Part two of the framework contains the following components: (1) A summary of the principles of design research, the approach chosen to study alternatives to rote learning; (2) Brousseau's Theory of Didactical Situations, that clarifies in what ways and why learning through problem solving can be more efficient than rote learning; and (3) An on-going teaching experiment comparing learning by Algorithmic and Creative reasoning.

Design Research

Concerning constructive teaching there are some insights that certain approaches can better enhance learning, but in general we lack deeper knowledge regarding *how* and *why* different teaching approaches affect different aspects of learning (Niss 2007). The on-going research described below can be characterised as *design research* which in this paper refers to the use of scientific methods to develop theories, frameworks and principles of existing or envisioned educational designs. An *educational design* can be seen as a plan produced to show the function (including purpose and means) of an educational artefact or practise. Such plans can be of different grain size and character, for example from local informal to global formal. The plan can refer to various components of the educational system, e.g. the classroom, teacher education, textbook production and large scale assessment.

The meaning of design experiments have not been settled in the literature (Schoenfeld 2007). Plomp (2009) argue that authors may vary in the details of how they picture design research, but they all agree that design research comprises of a number of stages or phases:

- preliminary research: needs and content analysis, review of literature, development of a conceptual or theoretical framework for the study
- prototyping phase: iterative design phase consisting of iterations, each being a micro- cycle of research with formative evaluation as the most important research activity aimed at improving and refining the intervention
- assessment phase: (semi-) summative evaluation to conclude whether the solution or intervention meets the pre-determined specifications. As also this phase often results in recommendations for improvement of the intervention, we call this phase semi-summative.

A key characteristic of design research is thus that it is strongly aligned with effective models linking research and practise, which, according to Burkhardt and Schoenfeld (2003), "the traditions of educational research are not". This is also emphasised by Cobb et al. (2003): "Design experiments have both a pragmatic bent—'engineering' particular forms of learning—and a theoretical orientation—developing domain-specific theories by systematically studying those forms of learning and the means of supporting them." This is also in line with (Gravemeijer and Cobb 2006): "the purpose of the design experiment is both to test and improve the conjectured local instruction theory that was developed in the preliminary phase, and to develop an understanding of how it works."

All empirical design research do not have to take place in classrooms, but may for example be in the format of laboratory pre-stages to classroom design in form of the clinical trials described by Schoenfeld (2007). Research questions in design research are typically in the form "what are the characteristics of an intervention X for the purpose/outcome Y in context Z?" and the research results in interventions (programs, products, processes), design principles or intervention theory and professional development of the participants involved in the research (Plomp 2009). In contrast to most research methodologies, the theoretical products of design experiments have the potential for rapid pay-off because they are filtered in advance for instrumental effect. They also speak directly to the types of problems that practitioners address in the course of their work (Cobb et al. 2003).

The Theory of Didactical Situations

The theoretical foundation for the attempts presented in this paper to design better learning opportunities for mathematics students is Brousseau's Theory of Didactical Situations (1997), which is a theory of how mathematics can be learnt through non-routine problem solving. It emphasises "the social and cultural activities which condition the creation, the practise and the communication of knowledge" (p. 23).

In the theory, the milieu is "everything that acts on the student or that she acts on" in a learning situation (p. 9). Didactique studies the communication of knowledge and one central aspect of Brousseau's didactical situations is the devolution of problems. The student has to take responsibility for a part of the problem solving process, but she cannot in general learn in isolation. The teacher's task is to arrange a suitable didactic situation in the form of a problem. Between when the student accepts the problem as her own and the moment when she produces her answer, the teacher refrains from interfering and suggesting the knowledge that she wants to see appear. This part of the didactic situation is called an adidactical situation. The student must construct the piece of new knowledge and the teacher must therefore arrange not the communication of knowledge, but the devolution of a good problem. If the student avoids or does not solve the problem, the teacher has the obligation to help. Then a relationship is formed that (mainly implicitly) determines what each party will be responsible for: the didactic contract that ensures the functioning of the process. Wedege and Skott (2006) note that the term 'didactic contract' is used outside France as a metaphor for the set of implicit and explicit rules of social and mathematical interaction in a particular classroom, which is an extension outside Brousseau's didactical situations and more in line with a definition by Balacheff (1990).

Temporarily incomplete or faulty conceptions in the form of obstacles are in Brousseau's theory not in general seen as failures but are often inevitable and constitutive of knowledge. An obstacle produces correct responses within a particular, frequently experienced context but not outside it and may withstand both occasional contradictions and the establishment of a better piece of knowledge. Clarifying obstacles helps the student see the necessity for learning, not by explaining what the obstacle is but to help her discover it. Good problems will permit her to overcome the obstacles. The teacher may (e.g. to reduce complexity) try to overcome the obstacle and force learning by devolving less of the problem to the student. Brousseau exemplifies this by the Topaze effect (p. 25) when the teacher lets the teaching act collapse by taking responsibility for the student's work and letting the target knowledge disappear (as in Guided AR). Telling the student that an automatic method exists relieves her of the responsibility for her intellectual work, thus blocking the devolution of a problem. If this is the normal didactic situation the student meets then the didactical contract is formed accordingly, which may not be the teacher's intention. The teacher expects the student to learn problem solving reasoning, while the student expects that an algorithm should be provided that relieves her of the responsibility of engaging in the adidactical situation. This avoids dealing with the obstacle that can therefore become insurmountable.

So the key issue with respect to this paper is to find a suitable devolution of problem, with the aim of providing learning opportunities through CMR instead of AR. It is in general easy to design imitative (MR or AR) tasks, since the structure of the task is based on repeating the fact or algorithmic procedure and follows therefore directly from the fact or procedure. For example, after the procedure to solve linear equations ($ax + b = cx + d$) is described then a large number of AR tasks are obtained trivially by just formulating different equations. If the purpose is just to design any mathematical problem (recall that a problem is a non-routine task) suitable for a particular student group, then the situation is a bit trickier but the literature and the internet is full of good mathematics problems. However, if the purpose is to design a problem that can help the student to construct (by devolution) a particular target knowledge then the design becomes much more complicated. In addition, the central target knowledge within mathematics curricula is often such that a set of problems (and adidactical situations) rather than a singular task is required. For example, if the goal is that the student shall herself construct a general method for solving linear equations it is unrealistic that this can be done in a single adidactical situation. It probably requires the solution of a series of equations of increasing complexity, and unpublished pilot studies indicate that students can get quite far this way.

A Teaching Experiment Comparing Learning by Algorithmic and Creative Reasoning

This pre-clinical (Schoenfeld 2007) design experiment is a part of a larger project that studies teaching designs that give students different opportunities to learn with respect to imitation or creative construction of knowledge. In this experiment two ways of teaching are compared:

(I) An algorithmic method for solving a type of tasks is presented, and students apply this method on a set of practise tasks. The structure is founded in the framework for AR and in the empirical studies of common teaching mentioned above.

(II) Guiding the individual into by herself constructing a solution method for the same type of tasks as in I. This structure is founded in the Theory of Didactical Situations, in research on mathematical problem solving and in the framework for CMR.

In order to be able to compare these two ways of teaching, it is prioritised (a) that similar target knowledge of the teaching experiments can be reached by both ways and (b) that the target knowledge may be learnt both by rote and by other types of learning leading to higher understanding. A suitable form of target knowledge is task solving methods that can be economised as algorithmic mathematical procedures. This is a central aspect of mathematical knowledge (Kilpatrick et al. 2001) and the teaching of such procedures seems to constitute some 50–100 % of mathematics teaching (Lithner 2008; Boesen et al. 2012) at least in Sweden but maybe also in other countries (Hiebert 2003). Other types of knowledge, e.g. understanding of concepts, the heuristic strategies or metacognitive control ability, is impossible (or at least unlikely) to be learnt by imitative reasoning and therefore not suitable as target knowledge when comparing these two different teaching modes.

One consequence of this choice of target knowledge is that this study does not primarily address the question of how to better learn non-routine problem solving, which is another central aspect of the rote learning problem. However, this has been extensively researched with uniform (at a general level) results: In order to become proficient solvers of non-routine problems students must practise non-routine problem solving, there is no automatic transfer from extensive drills of routine algorithms alone to this competence (Schoenfeld 1985). Thus the overall background question posed is: "how to best learn mathematical task solving methods that can be formulated as algorithms"? Is it to practise standard algorithms by large amounts of drill exercises, or by the students' own construction of the algorithms? Concerning this issue the discrepancies between research and practise, and between different research perspectives seem large (Arbaugh et al. 2010). In addition, there seem to be little empirical evidence backing the rather few theoretical claims made.

In the imitative teaching mode I a set of algorithms (e.g. rules for solving equations or rules for two-digit multiplication) is described and a set of practise tasks (e.g. equations that can be solved by the given rule) is given to the subject. This teaching mode is hypothesised to lead the subject into rote learning of algorithms by AR without understanding the foundations of the algorithm. It is relatively easy (Lithner 2008) in mathematics, which is essential for the experiments in this project, to design teaching situations where students are likely to learn a task solving algorithm without understanding it.

In the creative teaching mode II the subject is not given a method that can be directly applied to solve a set of practise tasks. Instead, a sequence of exploratory tasks is given. The tasks are denoted problems, meaning that the solver does not from the start have access to a complete solution scheme and CMR is required if the tasks are to be solved successfully. This devolution of problem is intended to make pure rote learning impossible and the subject has to understand the method in order to solve the task.

Compared to rote learning, the individual's construction of knowledge is to a larger extent the ideal among educational researchers and in particular within the constructivist paradigm. However, as far as I can see it is still not sufficiently clarified empirically if, why and in what sense this approach should be better. For example, Brousseau's motivation why devolution of problem is necessary is somewhat vague: "The student must construct the piece of new knowledge since she can only truly acquire this knowledge when she is able to put it to use by herself in situations outside the teaching context." (Brousseau, p. 30). One argument behind the hypothesis that task that require CMR will lead to a constructive adidactical situation with a real devolution of problem is related to the three defining criteria of CMR: (i) Novelty, that the task cannot be solved by familiar imitative reasoning, ensures the devolution of some kind of reasoning that the student has to be responsible for. (ii) The presence of arguments, supporting the plausibility of the conclusions, is necessary to guide and verify the construction of new insights. (iii)

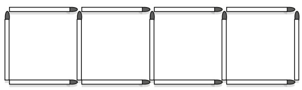
The necessity to anchor the reasoning ensures that the mathematical obstacles are addressed and that the resolutions are based on properties of relevant mathematical facts and concepts (in contrast to, for example only superficial clues and imitative reasoning).

The research question of this experiment is: What are the characteristics of an adidactical situation that leads to devolution of problem where learning through CMR is more efficient than learning through AR (in a context where AR is in the format common in school)? The present pre-clinical experiment is carried out in a laboratory context with no peer-peer or peer-teacher interaction, and serves to clarify basic phenomena as a preparation to pose the same question in a real classroom context.

Several iterations and revisions of task designs have been carried out. In one of the designs two groups of students, matched by basic cognitive tests, learn task solving methods in the form of algebraic formulas by AR and CMR respectively. An example of an AR practise task is given in Fig. 1 and a corresponding CMR practise task is given in Fig. 2. 14 different task sets on mathematical patterns, with 3–5 sub-tasks each, were given to the practice groups. One week after the practise session both groups take the same post-test, and among the data registered are the number of correct responses and corresponding response times.

One may note that compared to the CMR group, the AR group has an advantage since they are provided with more information. This implies that the AR group could solve the task in exactly the same way or a better way than the CMR group. And by the implicit dominating teaching ideal "help the students by providing solution methods" the AR group should learn better. However, the empirical studies mentioned above show that if students are given an algorithmic solution method to a task, they will probably mainly apply AR to solve the task without considering the underlying meaning of the concepts, representations or connections. Thus they will probably not even try to understand the meaning of the algebraic formula, which in this example is the relation between the figure of matches and the formula $y = 3x + 1$. If this actually is the case in the experiment and if the devolution of problem is successful, then the Theory of Didactical Situations implies that the CMR group may

When squares are put in a row it looks like the figure to the right. 13 matches are needed for four squares:



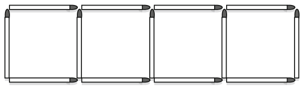
If x is the number of squares then the number of matches y can be calculated by the function $y=3x+1$

Example: If 4 squares are put in a row then $y=3x+1=3\cdot4+1=13$ matches are needed.

How many matches are needed to get 6 squares in a row?

Fig. 1 Example of an AR practise task

When squares are put in a row it looks like the figure to the right. 13 matches are needed for four squares:



How many matches are needed to get 6 squares in a row?

Fig. 2 Example of a CMR practise task

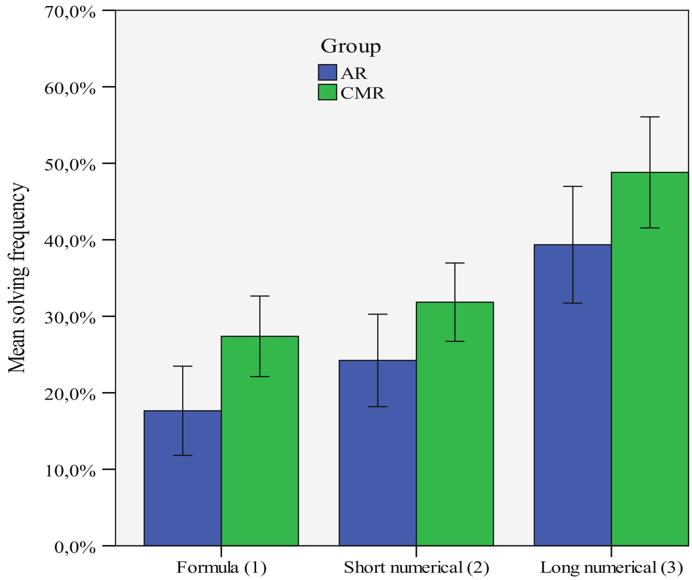


Fig. 3 Post-test results for the whole sample (n = 99), summarised over the 14 task sets. The test sub-tasks “Formula” ask for recalling the formula, “Short numerical” for recalling and applying a solution method and “Long numerical” for (re)constructing a solution method

learn better in some ways. The preliminary analyses of data indicate that this is the case, in the sense that the CMR group on average has significantly higher post-test results (Fig. 3) in spite of shorter response times.

In addition, the common belief that only the best pupils can benefit from learning through their own construction of solution methods is not supported by this experiment. On the contrary, in the 30 % with lowest cognitive index (a composite grade and pre-test score) the difference to the advantage of the CMR group is even larger than for the whole sample (Fig. 4).

Parallel to the experiment above, other complementary studies are carried out within the research project. One example is an on-going study using functional Magnetic Resonance Imaging (fMRI) to compare brain activity for students from AR and CMR training groups. This study is exploratory with the aim to analyse non-behavioural information about students’ thinking processes. One question

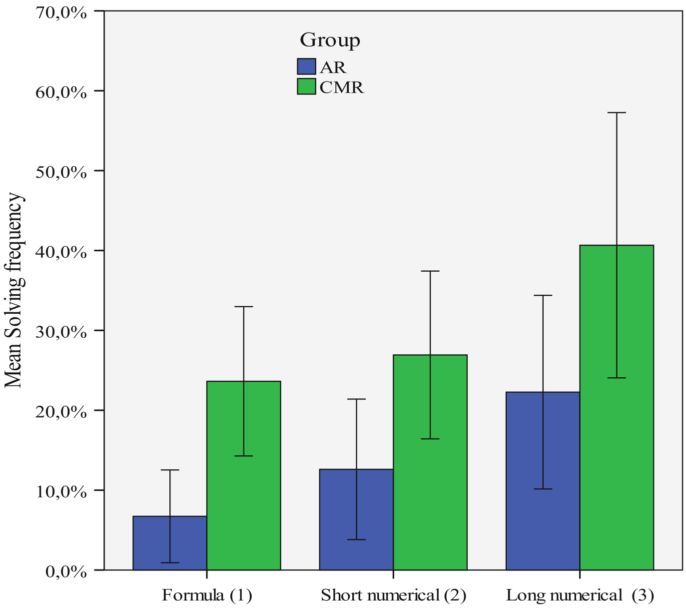


Fig. 4 Post-test results for the 30 % lowest cognitive index sample

asked is if students from the two groups activate different neural networks, and how this relates to earlier research findings about the brain and mathematics. Another question is if students from one group show higher brain activity (in some regions), and what the cause may be. For example, brain activity in the CMR group could be higher if they have created some kind of richer neural networks or lower if they have developed more rational solution methods. Another example of ongoing research uses eye-tracking methods to compare the strategies used by the AR and CMR student groups.

Further Research

Schoenfeld (2007) provides a structure that is useful in design research, with four phases of evidence-based educational research and development: Pre-clinical studies, design experiments, contextual studies and large scale validation studies. The ongoing research presented above reside in the pre-clinical stage and concerns the design of mathematical tasks that are suitable for devolution of problems where students may solve the tasks by CMR. One aim is to form a basis for clinical (classroom) studies in phase two. However, it is not just to take tasks designed and evaluated in the pre-clinical phase into the classroom. Stein et al. (2008) argue that teachers who attempt to use inquiry-based, student-centred instructional tasks face challenges that go beyond identifying well-designed tasks and setting them up appropriately in the classroom:

> "Because solution paths are usually not specified for these kinds of tasks, students tend to approach them in unique and sometimes unanticipated ways. Teachers must not only strive to understand how students are making sense of the task but also begin to align students' disparate ideas and approaches with canonical understandings about the nature of mathematics" (p. 314).

Thus one major challenge for the further research is how the design research can incorporate peer—peer, teacher—peer and teacher—class interaction that enhances suitable devolutions of problems through CMR. A second challenge is to design tasks that are more open to the students' own initiatives, and a third to design adidactical situations that encompass wider and deeper target knowledge than the algebraic formulas in the design experiment above.

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Features of Exemplary Lessons Under the Curriculum Reform in China: A Case Study on Thirteen Elementary Mathematics Lessons

Yunpeng Ma and Dongchen Zhao

Abstract Dramatic changes in mathematics education in China have taken place since the new mathematics curriculum standard was implemented in 2001. What do new features of exemplary lessons appear under the context of the curriculum reform? This paper will answer this question by presenting a case study on 13 elementary mathematics lessons that were evaluated as excellent exemplary lessons by mathematics educators in China. This study found that, consistent with the ideas advocated by the new curriculum, the selected lessons demonstrated the features of emphasizing on student's overall development, connecting mathematics to real-life, providing students the opportunities for inquiring and collaborating, and teachers' exploiting various resources for teaching. Meanwhile, the selected lessons also shared other common features in the lesson structure, interaction between the teacher and students, classroom discourse. The results reveal that the exemplary lessons have practiced the advocated ideas of the current reform, while they also embodied some elements that might be the stable characteristics of Chinese mathematics education.

Keywords Chinese mathematics classroom · Teaching practice reform · Exemplary lesson · Elementary mathematics

Introduction

In the past decades, investigating and understanding Chinese mathematics education, especially the mathematics classroom in China, has been of interest to many educators and researchers (e.g., Gu et al. 2004; Huang et al. 2006; Huang and

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Leung 2004; Leung 1995; Ma et al. 2004; Stevenson and Stigler 1992). Recently, efforts to improve the quality of classroom instruction have led to ever-increased interests in research on excellent lessons (e.g., Li and Shimizu 2009; Huang et al. 2009; Li and Yang 2003; Zhao and Ma in press). From different perspectives, the existent studies have deepened the understanding of Chinese mathematics classroom. Yet, the picture of Chinese mathematics classroom is not clear enough. More investigations and studies on Chinese mathematics classroom are needed.

In pursuit of knowing and understanding the characteristics of Chinese mathematics classroom, it should be aware of that changes might be taking place in Chinese mathematics classroom with the global change and the development of Chinese society. At the turn of the 21st century, with the aim of preparing younger generations for an age in which the economy is globalized, and the society is information-rich and "knowledge-based", mathematics curriculum in many education systems around the world have undergone dramatic changes (Wong et al. 2004). In such a situation, mathematics curriculum in China is no exception. In September, 2001, China initiated and implemented the new round of curriculum reform of compulsory education (Ministry of Education of China 2001). According to some reports, changes have taken place in the classroom as a result of the current curriculum reform (Song 2003; Li 2002). If the reform could be implemented deeply and continuously, we may expect that the practice of China's mathematics classroom will demonstrate many differences from that of the classroom in past decades. While as a cultural activity, teaching has its relative stability. In some comparative studies, both differences and similarities were found in the exemplary mathematics lessons in different decades (Huang et al. 2009; Li and Yang 2003; Zhao and Ma in press).

What characteristics do the "excellent" lessons have in the current curriculum reform? Or, how mathematics is taught and learned in the exemplary lessons in the context of reform? How are the ideas advocated by the new curriculum embodied in these exemplary lessons? Are there any other common features shown in these lessons? Much remains unknown about these questions. This paper will attempt to answer these questions by presenting a study, in which 13 elementary mathematics lessons valued as the excellent exemplary lessons under the new curriculum reform, were analysed.

The paper is structured with four parts: firstly, the background and main changes of the mathematics curriculum will be briefly introduced; secondly, the background of the lessons analysed in this study and the analysis method will be described; thirdly, we will present the results of this study in two aspects: one is the lessons' features that were consistent with the ideas advocated by the curriculum reform; the other one is other common features embodied in the lessons; and at last, we will give a short summary and discussion on the results. Based on the discussion, we will share our implications drawn from this study with you.

Background: Current Curriculum Reform in China

Mathematics curriculum in China has experienced several waves of changes since the founding of People's Republic of China in 1949 (Su and Xie 2007). The current new round of mathematics curriculum reform of compulsory education[1] was initiated and implemented under the guidance of *Mathematics Curriculum Standard for Full-time Compulsory Education* (*Manuscript for consultation*) (hereafter *Standards*) in September, 2001 (Ministry of Education of China 2001). Since September, 2005, all the students in the first academic year of primary school and junior high school have used the new curriculum. Now, the new curriculum has spread out nationwide.

Before the implementation of the new curriculum, the latest mathematics curriculum was developed under the guidance of Mathematics syllabus for elementary school of nine-year compulsory education and Mathematics syllabus for junior high school of nine-year compulsory education issued in 1992. The mathematics curriculum guided by the above-mentioned two syllabi was suitable for social development at that time, but there are still some problems left to be solved. For example, the syllabi issued in 1992 over-emphasizes "two basics" (basic knowledge and basic skill) and did not take into account students' development of affection and attitude, and this resulted in student's unbalanced development. Some contents of the curriculum were too difficult and narrow, and was not related to the students' real life (Zhang 2002; Ma 2001). Furthermore, the teaching method was monotonous; teachers useed textbooks as the only reference for their teaching, and perceived teaching as transferring knowledge from textbook to students (Ma 2001). In order to solve these problems and make the mathematics curriculum more responsive to the need of the development of both students and society, the Ministry of Education (MOE) of China initiated the mathematics curriculum reform.

The new mathematics curriculum at the stage of compulsory education aims at providing a solid foundation for students' full, sustainable and harmonious development, and to provide mathematics education for all students (MOE of China 2001, p. 1). Students' over-all development has been the most important goal of China's education especially because quality-oriented education was advocated by Chinese government since the 1990s (CCCPC and the State Council 1999). The *Standards* takes student's affection and attitude as one important dimension of their over-all development, and takes students' learning "process" as important as "outcome". For example, the *Standards* emphasizes students' full development by focusing curriculum objectives on four aspects: knowledge and skill, mathematical thinking, problem solving, as well as affection and attitude. The curriculum contents consist of four dimensions: Number & Algebra, Space & Graph, Statistics & Probability, and Integration & Practice. Nine-year Compulsory education is divided

[1]Compulsory education at the present time in China includes 9-year schooling from elementary school to junior high school.

into three phases: the first is for Grade 1 to 3; the second is for Grade 4 to 6; and the third is for Grade 7 to 9. For each phase, objectives for knowledge and skills, mathematical thinking, problem solving, and affect & attitude are elaborated in the *Standards*. Some contents in the former mathematics curriculum was trimmed down, meanwhile, some new contents was added to the new curriculum. Calculating and solving problem in multiple ways and strategies are encouraged.

The new curriculum also proposes some new ideas for improving mathematics classroom practice. It suggests that teaching should be closely related to students' daily life so that students can connect mathematics with real world (MOE 2001, p. 51). It emphasizes that mathematics teaching and learning should "begin from student's primary experience of real life, and encourage student to experience the process of abstracting mathematics model from real-life problem, and the process of interpreting and applying." (MOE of China 2001, p. 1) "Contents of mathematics learning for school children ought to be realistic, meaningful and challenging. These contents should facilitate school children to engage actively in mathematical activities, such as observation, experimentation, guessing, hypothesis testing, inference making, and communication." (MOE of China 2001, p. 2) It is also claimed that "effective mathematics learning activities cannot simply rely on imitation and memorization. Instead, hands-on practical work, autonomous investigation and cooperative exchanges are important modes of mathematics learning." (MOE of China 2001, p. 2) Besides, the *Standards* also encourages teachers to design and enact their lessons creatively rather than to perceive teaching as transferring knowledge from textbook to students mechanically (MOE 2001, p. 51).

In a word, dramatic changes have taken place in the mathematics curriculum in China since 2001. The ideas advocated by the new curriculum bring both opportunities and challenges for mathematics teachers. How to implement the new ideas in mathematics classroom? And what should an excellent mathematics lesson be like? Mathematics educators have been thinking about these questions and putting their understanding into their classroom practice. It is also of interest to researchers to identify and examine the features of the excellent lessons in this reform context.

Methodology

Research Questions

The analysed lessons in this study were the prized exemplary lessons at the national level in the current context of curriculum reform. This study aims at answering two questions as following: (1) how the ideas advocated by the *Standards* are implemented in the exemplary lessons? And (2) what other common features could be found in the exemplary lessons?

The Selected Lessons

In China, the institutions responsible for administrating educational research at the national or provincial levels often organize teaching contests and teaching exhibitions (see Li and Li 2009). In 2008, the NCCT (National Centre for School Curriculum and Textbook Development) of the Ministry of Education organized the 1st National Contest in Exemplary Lessons of Elementary Mathematics in the new curriculum reform context. Elementary mathematics teachers were encouraged to design and implement mathematics lessons to show how the new curriculum was taught and learned in their classrooms. The teachers had many choices in the teaching topic, grades, mathematics content fields, and lesson types. They had their lessons video-taped and submitted the lessons to the NCCT. At last, about 820 video-taped lessons were called up from each province (municipality and autonomous region) in China. These lessons were evaluated by an Expert Evaluating Group which was constituted of Mathematics educators and researchers. Finally, 55 lessons were selected and honoured as the First Prize. These 55 lessons covered grade 1–6, four fields of content (*Number and Algebra, Space and Graph, Statistics and Probability,* and *Integration and Practice*), and 3 types of lesson (*XinShou Ke*–Teaching and learning new content, *FuXi Ke*–Reviewing the previously learned content, and *ZongHe ShiJian Ke*–Integrated using knowledge to solve problems).

We focused on the lessons in type of "*Number and Algebra*" and "*XinShou Ke*", and selected lessons only from those in grade 3 or 4. Finally, thirteen lessons in total were selected for analysis. Their topics of teaching and learning, grades of students', and their codes in this study are shown in Table 1.

In addition, the textbooks used or referred by these lessons and the lesson plans of eleven lessons were also collected for analysis.

Table 1 General background information about the selected lessons

Lessons taught in grade 3		Lessons taught in grade 4	
Code	Topic of teaching and learning	Code	Topic of teaching and learning
A	Knowing and understanding second (time unit)	H	Countermeasure
B	Knowing and understanding fractions	I	Multiplication: 3-digit by 2-digit
C	Division with remainder	J	Using letters to present numbers
D	Year, month, and day	K	Multiples and factors
E	Year, month, and day	L	Solving the problems of planting trees
F	Year, month, and day	M	Solving the problems of planting trees
		N	Solving the problems of planting trees

Note Lesson D, E, and F focused on the same topic, and lesson L, M, and N focused on another same topic. This is coincidental

Method of Analysing

The framework and the method of analysing are decided according to the research questions. For the first research question, we firstly extracted the advocated ideas relevant to mathematics teaching and learning from the *Standards* and then examined where these ideas could be found in the lesson videos or lesson plans and how they were implemented.

As introduced in the part of "Background" in this paper, many ideas relevant to mathematics teaching and learning were proposed in the *Standards*. However, some of them are difficult to be examined and identified in a lesson by video analysis. For example, in the "Suggestion for teaching" in the *Standards*, it is suggested that teachers should create contexts and guide students learning in the contexts (MOE of China 2001, p. 51, 64). What is context? The *Standards* does not give a definition. Instead, it gives some suggestions for creating contexts for the phase one (Grade 1–3) and phase two (Grade 4–6) as following.

> Design the lively, interesting, and visual mathematical activities, such as the use of storytelling, games, visual demonstration, and scenario performance, to stimulate students' interest in learning, so that it can help the students know and understand the mathematics knowledge in a vivid and specific context (p. 51). (Suggestions for the phase one)
>
> Create the contexts relevant to students' living situation and knowledge background in which the students are interested (p. 64). (Suggestions for the phase one)

From these suggestions we can infer that the purpose of creating contexts is to make the mathematics lively, interesting, and relevant to real-life, so that it can provide a motivation and an experience foundation for students' learning. Even so, only from researcher's perspective without consulting the concerning students' opinions, is it difficult to judge whether teaching and learning are lively and interesting. By contrast, it is practicable to judge whether the teaching and learning are related to real-life.

Due to the limitation of the method used in this study, finally, only four ideas about teaching were identified for examining the selected lessons. They are: (1) taking students' over-all development into consideration; (2) connecting mathematics to real-life; (3) providing opportunity for student to inquire and collaborate; and (4) exploiting resources for teaching rather than just obeying to the textbook. The first idea was examined by analysing the objectives listed in the lesson plans. The second and the third were examined by analysing the lesson videos. And the fourth was examined by contrasting the actual taught content and the content in the textbook.

For the second research question, we adopted an open method for analysing rather than determined any analytical framework. The "constant comparison method" (Glaser and Strauss 1967) was used for analysing the selected lessons. We watched the lesson videos and read the transcripts of the lessons several times until some themes came to our attention. Then these themes were further examined until they were found to represent the common features of all the 11 selected lessons. In other words, common features were gradually summarized. Finally, six common features were found in the lessons. More details will be reported in the next section.

All the selected lessons were analysed by two researchers. The results of their analysis were tested and discussed to get them be consistent between the two researchers.

Results: Features of Exemplary Lessons

The Features that Were Consistent with the Ideas Advocated by Curriculum Reform

Students' Over-All Development Was Concerned

By analysing the instructional objectives in the lesson plans, it was found that students' over-all development was concerned by all of the lessons. Both objectives about the results of learning and the process of learning were all shown in the lesson plans. Students' mathematical development and the non-mathematical-relevant development all could be found in the lesson plans. The objectives of two lessons were shown as following:

> Help students (1) estimate the range of the product of 2-digit by 3 digit multiplication in a specific context, and calculate the 2-digit by 3 digit multiplication by listing vertical formula; (2) explore the methods of 2-digit by 3 digit multiplication, compute correctly, and be willing to exchange the methods with others; (3) develop the interests in calculating and a good habit, and improve the ability to use multiplication to solve practical problems; and stimulate students enthusiasm to love science by introducing current events. (Extracted from the plan of lesson I "2-digit by 3 digit multiplication")
>
> Help students (1) construct the preliminary concept of fractions based on their exploring and discussing the things in their real-life and geometric figures, correctly read and write simple fractions, and explain the meaning of a fraction by using geometric figures; (2) compare two fractions whose numerators are 1 by using geometric figures; (3) develop students' awareness of collaboration with others, and their ability of observation and analysis, hands-on skills and language skills, and develop students' mathematics thinking. (Extracted from the plan of lesson B "Knowing and understanding fractions")

The traditional mathematics teaching has been criticized for its over-emphasis on the results of learning (mathematical knowledge and skills) and neglecting the learning process. According to the objectives listed in the lesson plans, we found the learning results as long as the learning process was taken into consideration by teachers. Furthermore, some non-mathematics-relevant skills, such as the awareness of cooperation, communication, and interests, also were covered in the instruction objectives. The broader scope of the objectives indicated that students' over-all development was considered by these lessons.

Mathematics Was Connected with Real-Life

By analysing the lesson videos, it was found that all the lessons contained the real-life contexts during which mathematics was taught and learned. Three strategies were found in these lessons to create such a context. One is to begin a lesson with a real-life event or problem. All the lessons used this kind of strategy. The contexts created in these lessons were summarized as shown in Table 2.

The second strategy is to use real-life tasks or problems during teaching and learning the new content. The third strategy is to provide opportunity for students to apply the learned new content to the real-life.

Inquiry Learning and Collaborative Learning Occurred During Lessons

It was found that inquiry learning and collaborative learning existed in all of the lessons. The students had the opportunities of exploring knowledge and methods by themselves and communicating or discussing their opinions or findings with deskmates or group members. The inquiry learning and collaborative learning in these lessons were summarized as shown in Table 3.

Teacher Adapted the Textbook and Exploited Other Resources for Teaching

By contrasting the curriculum resources used in the lessons with the resources listed in the corresponding textbook, it was found that none of the 13 lessons completely conform to the textbook. In these lessons, the teachers selected some resources, such as the pictures, examples and exercises, from the textbook for their teaching and also exploited various resources by themselves. These results reveal that the teachers have made their adaption and creation while they designed and implemented their lessons. This is consistent with the ideas advocated by the new curriculum that the teachers should actively utilize various teaching resources and creatively use the textbook. However, a further analysis showed that, although the adaptations on the textbooks and the development of new resources were made in all of the lessons, the content of teaching and learning in the lessons do not show

Table 2 The context of teaching and learning at the beginning of each lesson

Lesson	The context
A	Watched video: opening ceremony of Olympic Games. Felt the scene of countdown. Led to the time unit “Second”. Then students gave examples that they used “Second” in daily life
B	Students allotted several types of learning tools equally with their deskmates, and recorded the numbers of each type of learning tool that each student received. They finally found that a half could not expressed by any whole number. So 1/2 was introduced
C	Students played the game of splicing flowers with 12 petals. Two results emerged: one is all the petals were used; the other is one or several petals was/were left. These lead to the “divisible division” and the “division with divisor”
D	Watched video: Opening ceremony of Olympic Games. Felt the scene at the time, recalled the date of the Olympic Games, lead to the topic of “Year, Month, and Day”
E	Watched the pictures of history events and holidays, students answered the dates of these events and holidays, and then the topic of “Year, Month, and Day” was introduced
F	Students interchanged the memories about the Olympic Games, introduced the topic of “Year, Month, and Day” from the date of Olympic Games
H	The teacher played cards with the class. The teacher always won the game by using countermeasure. Students felt curious. Then the topic of “countermeasure” was introduced
I	Students watched a simulative animation in which a satellite was running around the Earth. After having known the circumference of the orbit, students were asked to raise mathematical problems from this event
J	Students sorted 13 pieces of playing cards (2 to 10, and J to A). Students looked J, Q, K, and A as the number of 11, 12, 13, and 14. Then the topic “Using letters to present numbers” was introduced
K	Students made up a big rectangle with 12 small squares, and then expressed the length and width of the rectangle with a multiplication formula. They found, with the different splicing method, the multiplication formula was different. This led to multiples and factors
L	Students observed their finger spacing, gave examples of spacing in daily life, then raised the problem of planting trees
M	Appreciated the picture of the urban landscape, led to the topic of urban greening, and then raised the problem of planting trees
N	Began from a riddle: “Two trees have 10 branches, but they have not leaves and do not flower” (The answer is two hands). Students observed finger spacing, and then gave examples of spacing in daily life, which led to the problem of planting trees

differences from the content in the textbooks regarding of the coverage on mathematical knowledge and skills. Therefore, from the ways in which teachers used textbooks, we can see the teachers in the selected lessons did not depend on the resources in the textbook, but intended to follow the mathematical objectives embodied in the textbook.

Table 3 Overview of the inquiry learning and collaborative learning in selected lessons

Lesson	Summary of the inquiry learning or collaborative learning
A	Groups studied how to prove 1 min was equal to 60 s
B	Students communicated how they got the fractions that they wanted to learn by folding square papers
	Students divided a square paper into 8 equal parts with different methods. Then they were asked to discuss in pairs whether two parts in different shapes were in the same size
C	Groups played the game of splicing 5-petal flowers to investigate the relationship between the remainder and the dividend
	Students discussed the problem of planning a schedule for cleaning
D	Students observed the calendar independently, and then shared their findings with deskmates
E	Students observed calendar in groups and collected the data about year, month, and day. Then the whole class compared and analysed the data to investigate the relationship between year, month, and day
	Students discussed the methods of calculating the days of a common year
	Students communicated the methods of calculating the days of a leap year
F	Students observed the calendar independently. Then they found there were 12 months in a year, and the number of days varied in the 12 months
H	Groups designed the program of horse racing with the method of countermeasure. Pairs played a game to apply the method of countermeasure
I	Pairs communicated the methods of estimating
	Groups compared two different methods of calculation
J	Groups discussed how to present the relationship of two persons' ages that were increasing simultaneously
K	Groups communicated the methods of enumerating the factors of a number
L	Groups studied the different methods of planting trees and recorded the number of trees and corresponding spacings. Then students analysed the data to find the relationship between the numbers of trees and spacings
M	Groups discussed why the number of planted trees was one more than the number of spacings
N	Groups studied the relationship between the number of spacings and the number of trees
	Groups discussed the relationship between the number of spacings and the number of trees in two different situations (planting trees from one end to another end of a line, and planting trees in a line without planting at the two ends

Other Common Features

Except the features reported above, some other common features were also found existing in the selected lessons.

Table 4 Seven types of teaching activities and their purposes

Type	Purpose
Introducing topic	To arouse students' interests or to activate students' previous experience relevant to the topic of current lesson (including reviewing previous lesson), and then introduce the topic
Teaching and learning new content	To acquire knowledge, concepts, skills, or procedures that have not been learned in earlier lessons
Practicing the new content	To consolidate the new content or to apply it in a new situation, including solving routine exercise and non-routine problems
Summary	To help students get an overall view on the previously learned new content or previous teaching activity in the current lesson
Homework assignment	To give students assignment for them to accomplish at home
Extended learning on non-mathematical content	To have a relaxation or celebration, or to introduce a current social event. The content or activity is irrelevant to mathematics
Proposing problems for future study	To invite students raise questions or problems for studying in future lessons

Features of Lesson Structure

Seven types of teaching activities with different purposes for student's learning were found in the 13 lessons. They are defined in Table 4.

The "Extended learning on non-mathematical content" only existed in three lessons. For example, in the lesson N "solving the problem of planting trees", students sung a "*Tree-planting Song*" to celebrate their accomplishment of previous learning. Taking another example, at the end of the lesson I (Multiplication: 3-digit by 2-digit), the teacher introduced a current affair of the manned space rocket. Only lesson L had the "Proposing problems for future study". Lesson D, E, and J had "Homework assignment". By contrast with these three types of activities, the other four types of activities were very popular in all of the 13 lessons.

Figure 1 shows a picture of the lesson structure, in which each type of activity was presented according to its location in the teaching process and the percentage of its duration to the whole lesson. As it is shown in this figure, in all of the lessons, it started with introducing the topic of current lesson, during which the students' interests were aroused and their previous experience was activated. After introducing the topic, the new content was taught and learned. The new content was divided into several parts and each part was taught and learned gradually. Practices were set following some parts (not all) of the new content or were set after all of the new contents were finished. One or more summary was/were given during the lesson or at the end of the lesson. Overall, all the lessons showed three features as following: (1) introducing topic; (2) teaching and learning new content accompanied by practicing; (3) summarizing during the lesson or near to the end of the lesson.

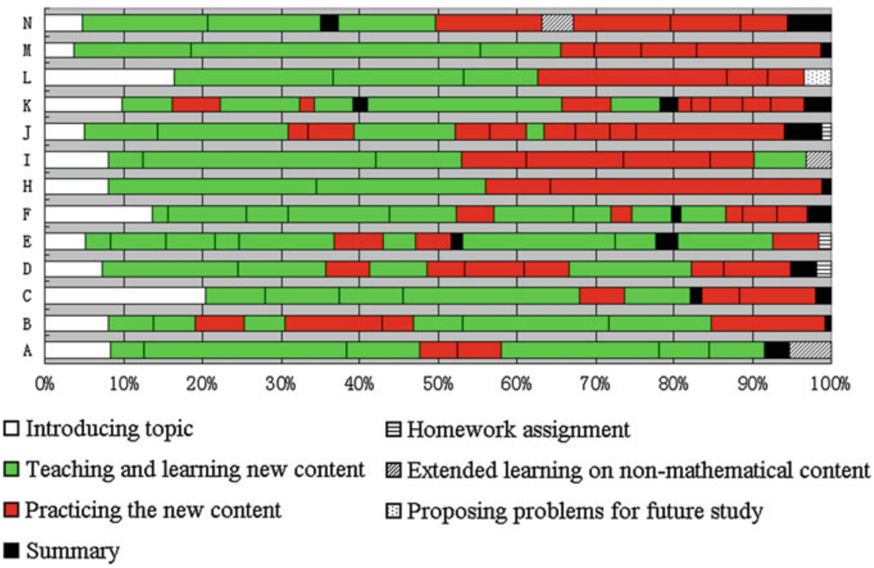

Fig. 1 Lesson structure

Teaching and Learning New Content Accompanied by Practicing

Practices were found in all of the lessons. The type of these lessons is "Xinshou Ke", which means that teaching and learning the new content is the main purpose of these lessons. However, most of the lesson time was not only spent on teaching and learning new content but on both teaching and learning new content and practicing the new content. The percentages of time spent on teaching and learning new content and practicing in each lesson are shown in Fig. 2. As far as the percentages of time spent on "practicing" is concerned, the highest one is lesson J (48.3 %),

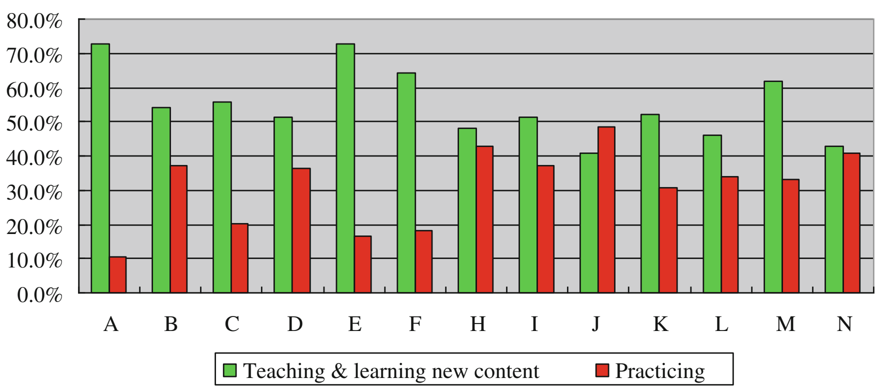

Fig. 2 The percentages of time spent on teaching and learning new content and practicing

and the lowest one is lesson A (10.5 %). Nine lessons spent more than 30 % of their lesson time on practicing. In some lessons (H, J, and N), the total time spent on practicing was nearly as much as the time spent on teaching and learning new content. Therefore, the selected lessons showed an obvious feature that teaching and learning new content was accompanied by practicing.

A further analysis found that two strategies were used for accompanying practicing with teaching and learning new content. One is to arrange the practice after all of the new content was taught and learned (e.g. lesson H, L, M, and N, see Fig. 1). Another one is to place one or more practice between the sections of teaching and learning new content (e.g. lesson B, D, K, et al. See Fig. 1).

Most Lessons Included Summary, and Some of Which Were Made by Students

Except lesson I and lesson L, the other 11 lessons all contained at least one section of summary. There were two types of summary: (1) it took place during the process of teaching and learning and intended to in review the key points of the just learned content or the just accomplished activity; (2) it occurred near to the end of a lesson and intended to review the whole lessons. From Fig. 1, we can conclude that the lesson A, B, D, H, J, and M used the first type of summary, while the lesson C, E, F, K, and N used both types of summary. A further analysis found that the teachers always invited students to give summary during the second type of summary. In this occasion, several students reflect what they had acquired in the lesson, including the knowledge and skills, as well as their experience and affection.

Public Interaction Dominated the Lessons

It was found that, although the students had opportunities to explore mathematics knowledge by themselves or to discuss and cooperate with their classmates, most of lesson time was spent on the whole-class work. By referring to the TIMSS 1999 Video Study (Hiebert et al. 2003, pp. 53–54), two categories of classroom work patterns were used in this study. One is public interaction, in which the teacher and students interact publicly, with the intent that all students give their attention to the presentation by the teacher or one or more students. Another category is private interaction, in which students complete assignments either individually, in pairs, or in small groups. An analysis of the different types of interaction showed that more than 70 % of lesson time in the selected lessons was spent in public interaction. The percentage of private interaction in most of lessons was not more than 20 %. It was under 10 % in seven lessons. This indicates that all these lessons were dominated by public interaction (more details see Fig. 3).

The private interaction in the 13 lessons included the students discussing or communicating in pairs or groups, working with tasks individually or collaboratively, doing exercise at seat, and reading textbook. The public interaction included

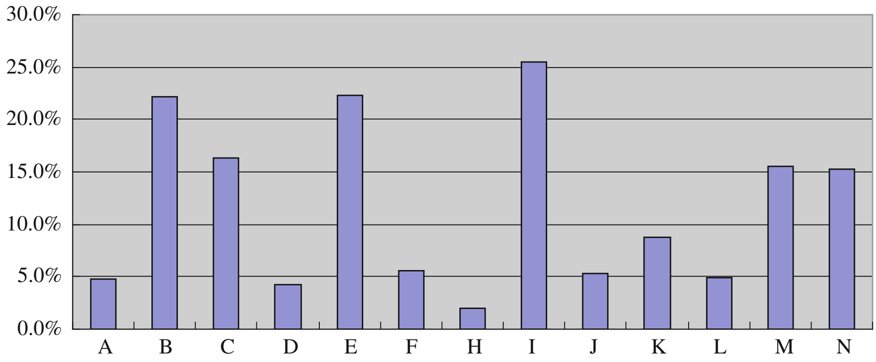

Fig. 3 Percentage of time duration of private interaction

presenting information by teachers (such as explaining, questioning, and blackboard writing) and by students (such as answering questions, reporting findings, demonstrating personal or group work). Most of the public interaction took place in the form of dialogue between teacher and students.

Teacher Had More Opportunities to Talk, and Students' Talking in Chorus Was Obvious

It has found that most of the public interactions were in form of dialogue between teacher and students. The discourse of teacher's and students' was analysed for further examining the dialogue between teacher and students. By watching lesson videos, we found there were four types of talking during the public interaction. They were: (1) the teacher talking individually; (2) single student talking; (3) students talking in chorus without teacher's participation, in which two or more students talked together; and (4) teacher and students talking in chorus, in which the talk made by two or more students accompanied their teacher. In this study, the teacher talking includes both (1) and (4), while the student talking refers to all of the (2), (3), and (4).

All of the discourse during the public interaction were transcribed verbatim, based on the frequency of teacher talking (FT), frequency of student talking (FS), ratio of FT to FS, number of teacher's words (TW), number of students' words (SW), and the ratio of TW to SW were analysed quantitatively. The results are shown in Table 5.

As shown in Table 3, the ratios of FT to FS ranged from 1.0 to 1.3, which indicates the FT is not much more than the FS. In another words, the opportunity for students talking was near to that for teachers. In addition, the frequencies of student talking in the 13 lessons are all more than 100. This reveals that the students in these lessons were not the passive, quiet listeners.

Table 5 The results of quantitative analysis in dialogue during public interaction

	A	B	C	D	E	F	H	I	J	K	L	M	N
FT	214	215	239	184	197	245	147	158	259	396	149	137	154
FS	223	164	217	155	181	210	149	135	205	325	139	118	142
FT:FS	1.0	1.3	1.1	1.2	1.1	1.2	1.0	1.2	1.3	1.2	1.1	1.2	1.1
TW	3336	3921	5007	3596	3915	5739	3525	3456	4030	5238	4080	3218	3985
SW	1753	2015	1629	1842	2130	1621	1451	2125	1834	2114	1587	1656	1433
TW:SW	1.9	1.9	3.1	2.0	1.8	3.5	2.4	1.6	2.2	2.5	2.6	1.9	2.8

Regarding the spoken words during the public interaction, it could be seen in Table 3 that the ratios of TW to SW in the 13 lessons are different to some extent, among which the highest one is 3.5:1 and the lowest one is 1.6:1. However, the feature that the teachers talked more than their students was shared by all of the lessons. The ratio of average TW to average SW is 2.3:1.

A further analysis in the amount of student talking found that the student talking in chorus (including the chorus accompanied teacher's participation) was frequent. More than 25 % of student talking was in chorus. The highest frequency was found in lesson E (66 %).

Questioning-Responding Occurred Frequently, but Students Rarely Asked Questions

We found that many dialogues between the teacher and students appeared in the way of teacher's asking questions and the students' responding. The frequency of mathematical questioning (not including the questioning for lesson management) during each lesson is all more than 40. The questioning and responding took place not only during teaching and learning new content, but also occurred in other sections of a lesson.

However, nearly all mathematical questions were raised by the teachers. Students' asking question on their own initiative (not including the questioning motivated by teacher's invitation) was found only in three lessons (once in lesson B, once in lesson J, 5 times in lesson A). No student presented any questions in any of the other ten lessons.

Discussion and Implication

Discussion

By analysing 13 elementary lessons, we found some features of the exemplary lessons under the curriculum reform in China. On one hand, some of the features were consistent with the ideas advocated by the new curriculum, such as emphasizing on student's overall development, connecting mathematics to real-life, providing students opportunities of inquiring and collaborating, and exploiting various of resources for teaching. These features have demonstrated the possibility of the further improvement of mathematics teaching. If the current curriculum reform could be implemented efficiently and continuously, we may expect that China's mathematics classroom will show many differences from that of the classroom in past decades.

On the other hand, the selected lessons in this study also shared many other common features in the lesson structure, interaction between the teacher and students, classroom discourse. Some of these features are consistent with the findings of other studies on Chinese mathematics classroom. For example, regarding the lesson structure, all the selected lessons began with introducing topic, accompanied teaching and learning new content by practicing, and summarized during the lesson or near to the end of the lesson. This is consistent with the findings of Zhao and Ma (in press)'s comparative analysis of four exemplary lessons in different decades in China. Chen and Li (2010)'s study on a Chinese competent teacher's four consecutive lessons also found that the teacher tended to structure the lesson into reviewing previous lesson, teaching and learning new content, and summary, which resulted in an instructional coherence. Other studies reveal the instructional coherence seems be a characteristics of Asian mathematics classroom. For example, Shimizu (2007) found the summary also played an important role in Japanese mathematics classroom.

The selected lessons in this study also showed a feature that teaching and learning new content was accompanied by practicing. This feature also was found in Zhao and Ma (in press)'s study. It is well known that the Chinese mathematics classroom in the last half of 20th century, had been predominated by the belief that "students should have sufficient exercises in order to consolidate the learned knowledge" (Zhang et al. 2004) and that "practice makes perfect" (Li 2006). We could not conclude whether the practice in the lessons in this study was emphasized as much as it was in traditional classrooms, but it is evident that none of the lessons in this study neglected the role of practice.

It was also found that all the lessons in this study were dominated by public interaction, and teachers talked more than their students. These two features are consistent with the findings of TIMSS 1999 Video Study (Hiebert et al. 2003) on eight-grade mathematics lessons in seven countries. However, it should be noted that the ratio of teacher's words to students' words in this study is 2.3:1, which is much less than the ratio found in TIMSS 1999 Video Study[2]. Based on the data of Learner's Perspective Study (LPS) (Clarke et al. 2006), Cao et al. (2008) analysed the discourse in Chinese competent mathematics teachers' lessons. It was found the ratio of average teacher words to average students' words is 6.6:1. Taking these

[2]In TIMSS 1999 Video Study, the average number of teacher words to every on student word per eighth-grade mathematics lesson in six countries/regions was reported. They were as following: Australia, 9:1; Czech Republic, 9:1; Hong Kong SAR, 16:1; Japan, 13:1; Netherlands, 10:1; and United States, 8:1. (Hiebert et al. 2003, pp. 109–110).

findings as a whole, we may hypothesize that the differences might exist in the discourse between classrooms in different stage of schooling.

In addition, the phenomenon of students' talking in chorus found in this study is similar to the findings of Wang (2010)'s study on two elementary mathematics classrooms in China. And the feature of frequent questioning-responding in the lessons in this study also was found in other studies on Chinese exemplary mathematics lessons in different decades (Zhao and Ma in press; Huang et al. 2009).

By reviewing the existing studies and comparing their findings with the features found in this study, we may find the lessons in this study embodied some elements that might be the stable characteristics of Chinese mathematics education. Stigler and Hiebert (1999, p. 86) pointed out that the teaching is a cultural activity. As a cultural activity, teaching has its relative stability. Therefore, it is understandable that both differences and similarities exist in mathematics lessons of different periods of time.

Implication

In this paper, we have reported the features of 13 exemplary lessons under the curriculum reform in China. We hope our findings could help you understand the current changes in elementary mathematics classroom. In addition, as noted above, the classroom under the reform not only reveals the new ideas advocated by the reform, it also contains some stable elements that might be inherited from the traditional classroom. This reminds us that the classroom under the reform and traditional classroom are not completely conflicting and exclusive. We should not ignore reflecting upon the tradition while implementing the new ideas. The traditional mathematics classroom may contain the asset that is worth preserving and carrying forward, and may also hide the drawbacks to be discovered. From this point, the teaching reform is a successive and gradual changing process, during which the reflection on present and history is always needed. This is the implication drawn from our study. Perhaps it also could be a reference for mathematics educators in a reform.

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Teachers, Students and Resources in Mathematics Laboratory

Michela Maschietto

Abstract This paper deals with the methodology of mathematics laboratory from two points of view: the first one concerns teacher education, the second one concerns teaching experiments in the classes. Mathematics laboratory (described in the Italian national standards for mathematics for primary and secondary schools) can be considered as a productive "place" where constructing mathematics meanings, more a methodology than a physical place. It can be associated to inquiry based learning for students. An example of mathematics laboratory with cultural artefacts such as the mathematical machines (www.mmlab.unimore.it) is discussed.

Keywords Mathematics laboratory · Instrumental genesis · Semiotic mediation · Mathematical machine · Teacher education

Introduction

The idea of mathematics laboratory is rooted in the pedagogical studies fostering active methods at the end of XIX century. It spread in different forms and times in several countries in the world. This paper refers to the case of Italy where this idea is related to methodological aspects based on various and structured activities, aimed to the construction of mathematical meanings. Italian mathematics laboratory provides for the use of tools (digital but also classical tools as compass) in mathematics teaching and learning. In this paper, we deal with mathematical machines,[1] which are mechanical tools for geometry and arithmetic drawn from historical treatises or strongly related to the history of mathematics (a geometrical mathematical machine is "a tool that forces a point to follow a trajectory or to be transformed according to a

[1]http://www.mmlab.unimore.it (Accessed December 2012).

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given law", Bartolini Bussi and Maschietto 2008, p. 183). Mathematics laboratory with mathematical machines is considered here as the object of a particular teacher education programme, whose aim is the diffusion and consolidation of laboratory methodology in school at each level. Concerning teachers' professional development, the topic of resources becomes relevant. Following Adler (2000, p. 207), "It is possible to think about resource as the verb *re-source*, to source again or differently". Adler considers not only material resources, but also cultural ones. In a wide sense, a resource can be thought as that is considered in the teaching of mathematics. From this point of view, basing on the several teaching experiments carried out in different classrooms we have put forward the hypothesis that the mathematical machines could be considered as resources for teachers. So, research questions concern the processes of appropriation of these resources for teachers (not only for teacher-researchers) in the context of mathematics laboratory. At the same time, another question is if and how a teacher education programme on mathematics laboratory with mathematical machines can 're-source' the professional practice of its participants. On the other hand, this work aims to contribute to the discussion on mathematics teacher education, in which interest in research in mathematics education is growing (Ball et al. 2008).

This paper presents five parts. The first part places the idea of mathematics laboratory in the history of mathematics teaching. Essential elements of a theoretical background for mathematics laboratory and teacher professional development are hinted in the second part. The third part contains the description of the education programme, which is analysed in the fourth part. A discussion with some elements from the development of the education programme ends this paper.

Mathematics Laboratory in Mathematics Education

Roots and Development of the Idea of Mathematics Laboratory

Between the end of the nineteenth century and the beginning of twentieth century, different European and North American mathematicians discussed their reflections on the ways to teach mathematics, often in opposition to the traditional lesson (Giacardi 2012). In this trend, in England John Perry (1850–1920) proposed a new didactic method, called *Practical Mathematics*. In France, Émile Borel (1871–1956) wished the creation of *atelier mathématique*.[2] In Germany, the main advocate for the use of concrete models and dynamic instruments was Felix Klein (1849–1925) (Bartolini Bussi et al. 2010). In Italy, Giuseppe Vailati (1863–1909) contributed with his innovative idea of *school-laboratory*, with a methodology based on problem solving.

[2]http://smf4.emath.fr/Publications/Gazette/2002/93/smf_gazette_93_47-64.pdf (Accessed December 2012).

Those educational questions were also present in many discussions in *L'enseignement mathématique* (the official journal of ICMI from its creation in 1908). They were summarized in some contribution to the Working Group 4 at ICMI Symposium 2008,[3] where there emerged the diffusion of a sort of mathematics laboratory in different countries. Over the last ten years, the mathematics laboratory urges a renewed interest at institutional and international levels. Even if the name 'laboratory' is not always used, there are several institutional positions that are consistent with the principles of mathematics laboratory (an experimental approach to mathematics, practical tasks for students, use of tools, …); for instance, the Inquiry Based Science Education fostered by the European Commission (Rocard report[4]), or the *démarche d'investigation*[5] in French mathematics curriculum. Furthermore, the expression 'mathematics laboratory' is related to several forms of laboratory (e.g., a room for students after school time, Kahane 2006), with different time, places and organization (Maschietto and Martignone 2008). In this paper, we will consider the idea of mathematics laboratory developed in Italy.

Mathematics Laboratory in Italy

In Italy, in the Sixties, a new relaunching is given by the work of Castelnuovo (1963), that has inspired mathematicians and teachers. Some of them formed the core of the Laboratory of Mathematical Machines (MMLab) at the University of Modena e Reggio Emilia (Maschietto 2005). Mathematics laboratory was a kind of methodology shared by certain groups of Italian teachers, even if the institutional acknowledgement happened when the Italian Mathematical Union (UMI) proposed mathematics standards from 5 to 18 years old students, in 2001–2004, on the behalf of the Ministry of Education (Bartolini Bussi and Martignone 2013). The document for secondary school (AA.VV. 2004)[6] emphasizes the importance of the following aspects in a mathematics laboratory:

- Relationships among people: peer interactions by group work, group discussions as well as interaction between students and the teacher as an expert;
- Presence of a question to understand, a problem to solve, an object to discover;
- Students' engagement: every student can contribute, even and especially students that do not "make mathematics" in traditional classes;

[3]http://www.unige.ch/math/EnsMath/Rome2008/WG4/WG4.html (Accessed December 2012).

[4]http://ec.europa.eu/research/science-society/document_library/pdf_06/report-rocard-on-science-education_en.pdf (Accessed December 2012).

[5]http://ife.ens-lyon.fr/editions/editions-electroniques/dies2010/ (Accessed December 2012).

[6]http://umi.dm.unibo.it/old/italiano/Matematica2003/prima/premessa2.pdf (Accessed December 2012).

- Construction of mathematical meanings: putting together manipulative aspects, gestural and procedural intertwine with theoretical aspects;
- Role of teacher (in the sense of cognitive apprenticeship);
- Presence of tools (digital or classical ones), not only as technical instruments (Vygotsky 1978).

According to Chiappini and Reggiani (2004), a laboratory is a phenomenological space of students' conceptualisation and reflexive thinking. It is "finalised to the construction of the experiential base which is necessary for the appropriation of the mathematical meanings" (p. 3). The authors emphasizes that a process of transposition of mathematical knowledge is present from the viewpoint of the didactic implementation of a mathematics laboratory.

In September 2012 (as in 2007), the Italian Ministry of Education published guidelines[7] for the compulsory school curriculum, in which the methodology of laboratory is a key component (not only for mathematics and sciences teaching and learning). The laboratory has also became an important element of prospective primary school teacher training programs (Bartolini Bussi and Maschietto 2008).

Before and after UMI standards appeared, several research works have been carried out in terms of mathematics laboratory. In a certain sense, they contribute to frame the institutional description. At the same time, a theoretical framework drawn on Vygotsky's work has been defined for mathematics laboratory. The next section presents it, together with other references on teachers' professional practices.

Theoretical References for Mathematics Laboratory

In all the research studies (for some example, see Bartolini Bussi and Maschietto 2006) carried out by the team of the MMLab about mathematics laboratory with classical technologies (e.g., the mathematical machines), at least three analytical components are present (Arzarello and Bartolini Bussi 1998):

- an epistemological component, with attention to mathematical meaning;
- a didactic component, with attention to the classroom processes;
- a cognitive component, with attention to processes of learning.

Those studies have been framed by the Theory of Semiotic Mediation (Bartolini Bussi and Mariotti 2008) and the instrumental approach (Vérillon and Rabardel 1995). They are briefly resumed below.

[7] http://hubmiur.pubblica.istruzione.it/web/istruzione/prot7734_12 (Accessed December 2012).

Instrumental Approach

The instrumental approach (Fig. 1a) contributes to the cognitive component. It identifies and studies the processes of instrumental genesis for a subject who has to accomplish a task by the means of an artefact (a material or an abstract object, already produced by human activity). In doing that, the subject develops utilisation schemes and construct own instrument (a psychological construction, made up artefact and utilisation schemes). The instrumental genesis is composed of two kinds of processes: instrumentation and instrumentalisation. The former is relative to the emergence and evolution of utilisation schemes; the latter concerns the emergence (as a first level) and evolution of artefact components of the instrument. As stressed by Trouche and Drijvers (2010), the notion of scheme can be related to Vergnaud's definition, i.e., a scheme is an invariant organisation of behaviour for a given class of situation. A scheme is composed of action and anticipation rules, but also operational invariants and inferences. From an educational point of view, Trouche (2004) highlights the need to take into account and support students' instrumental genesis when a new artefact to perform a task is introduced in the classroom. In a technological rich environment, the teacher has to face to several instruments (instrumental orchestration).

Theory of Semiotic Mediation

The Theory of Semiotic Mediation has been elaborated and applied to mathematics education by Bartolini Bussi and Mariotti (2008) within a post-Vygotskian perspective. It is founded on the relevance of the use of artefacts in human activities. The process of semiotic mediation is described schematically by means of the

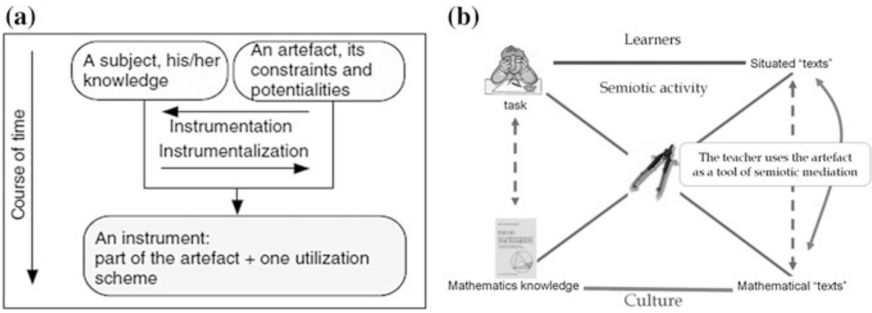

Fig. 1 **a** Instrumental approach (from Maschietto and Trouche 2010, p. 37). **b** Theory of semiotic mediation (from Bartolini Bussi and Maschietto 2008, p. 192)

following drawing (Fig. 1b). It involves the teacher, students, contents, space and methodology. The essential elements of this framework are:

- When an artefact is introduced in the process of solving a given task, a double semiotic link (named semiotic potential of an artefact) is recognizable: the first link is between the artefact and the task, and the second link is between the artefact and a piece of knowledge;
- The activity with a specific artefact fosters the production of signs (see Fig. 1b, cognitive component: the higher triangle 'task—artefact—situated texts');
- The teacher guides the evolution of students' signs produced using the artefact into mathematical 'texts' (see Fig. 1b, didactic component: the right triangle 'task, situated texts, mathematical texts');
- The importance of teacher's role as a cultural mediator with respect to mathematical contents (see Fig. 1b, right and left arrow bottom-up).
- The choice of an artefact from the mathematical culture (see Fig. 1b, epistemological and cultural components: the triangle 'mathematical knowledge, artefact, mathematical texts');

When the teacher uses an artefact to mediate mathematical meanings, according to the elements above, he/she uses it as a tool of semiotic mediation.

The analysis of the semiotic potential of an artefact includes an epistemological component concerning the knowledge evoked by the artefact and a cognitive component by the identification of utilisation schemes. For the latter, the instrumental approach is taken into account (Mariotti 2012). From a didactic point of view, the process of semiotic mediation is grounded in a specific structure of activities (called the "didactical cycle"): activities with artefacts usually in small groups that promote the emergence of signs (words, sketches, gestures, …) in relation to the use of the chosen artefact; individual written production of signs (drawings, writing, ….) and collective moments leading to social production of signs. In the latter, the mathematical discussion is the fundamental didactic strategy. The teacher acts in the definition of tasks and the orchestration of mathematical discussions. The content of the mediation can be a specific mathematical topic but also a fundamental process (or cultural component) of mathematics, like argumentation and proof (Mariotti 2010).

This framework takes into account a part of teacher practise before the introduction of the artefact into the classroom with specific tasks (see Fig. 1, triangle 'mathematical knowledge—task—artefact'). If a perspective of teacher education is considered, that kind of work should be seen in a wider perspective, characterising the professional practice of mathematics laboratory with tools. Such a perspective could comprise mathematical machines as resources. Some research questions arise: how can a teacher education programme be organised? Which kinds of proposition for teachers? How can professional practise be taken into account?

Documentational Approach

Gueudet and Trouche (2009) recall the principles of the instrumental approach and propose a an expansion concerning the specific work of mathematics teachers. Following the distinction between artefact and instruments, they introduce a distinction between resources (in place of artefact) and document (in place of instrument), with the notion of documentational genesis. Documents are developed throughout documentational geneses involving resources (or systems of resources). The authors retain the formula: Document = Resources + Scheme of Utilization. Schemes are related to operational invariants, which can be inferred from the observation of invariant behaviours of the teacher for the same class of situations across different contexts. As a result of the genesis, "a document is saturated with the teachers' experience" (p. 205). Behind utilisation schemes, there are professional knowledge (Trgalova 2010). In addition, documentational genesis process pays attention to instrumentalisation process because it shows how the teacher does transform them and construct a document for the class. In these geneses, some process of conception 'in use' and conception 'for uses' are recognisable (Folcher 2007). Gueudet and Trouche (2009) also focus on collaborative work among teachers.

Research studies on the integration of technologies in classrooms and teachers practices highlight the complex processes that occur when a new tool is proposed/introduced into the classroom. In particular, in her research on spreadsheet, Haspekian (2011) identifies processes of double genesis for teachers: a first genesis to construct the instruments 'spreadsheet' in order to do mathematics, a second genesis in order to have a didactical instrument for teaching (or as 'vehicle of learning', following Wisløw 2003). In addition, she shows how teacher's geneses affect students' geneses.

Some questions arise for mathematics laboratory. With respect to mathematical machines, does the double genesis occur? What kinds of genesis for mathematics laboratory methodology? What is the relationship, if any, between documentational genesis and double genesis? How does a teacher education programme effectuate and support these geneses, if any? In such a way are the cultural aspects taken into account in a teacher education programme? How could a teacher education programme be included in teacher professional practice? How could collaborative work be fostered?

Teacher Education in and for Mathematics Laboratory

Maschietto and Bartolini Bussi (2011) discuss a paradigmatic example of mathematical laboratory activities with the pantograph for reflection. Different activities are proposed to prospective teachers in order to: be experienced in a mathematical laboratory session; provide a model that might serve for future class activity; make they think over the relationships between manipulative and theoretical aspects in doing mathematics, since manipulation alone is not enough to construct mathematical knowledge. Their analysis seems to show that laboratory methodology is a great challenge for the teachers that can not be taken for granted.

But there is another important element to take into account. As quoted in the first section, mathematics laboratory is strictly related to cultural aspects (Boero and Guala 2008). They concern the view on mathematics and doing mathematics on one hand, educational systems and teachers' professional development on the other hand.

On the basis of the experiences of the MMLab research group, an education programme on mathematics laboratory for practising teachers, with mathematical machines, has been considered in the MMLab-ER project (Laboratories of Mathematical Machines for Emilia-Romagna[8]). Since 2008, this project has been funded by the Region Emilia-Romagna (with the collaboration of several policy makers). It aims at constructing a network of mathematical laboratories in the provinces of the region and a network of practising teachers working within laboratory methodology in their classes (Bartolini Bussi and Maschietto 2010, Bartolini Bussi and Martignone 2013). The first part of the project was carried out in two years (2009–2010), while the second part of the project is ongoing.

The teacher education programme is composed of two phases: the first phase considers six presential laboratory sessions, in the second phase the teachers have to experiment mathematics laboratory sessions in their classes. The participants teach at different school levels (primary and secondary school levels, 6–16 years old pupils). In the first phase, each session concerns a specific mathematical content, with different mathematical machines (starting with the compass, then pantographs for geometrical transformations, curve drawers for conics section, and finally the arithmetical machine Zero + 1 (Maschietto 2015)). It is structured following a laboratory methodology: [1] group work for the exploration of a mathematical machine with a working sheet or geometrical constructions with ruler and compass; [2] collective discussion on the exploration of the chosen machine and analysis of tasks and resolution processes (Bartolini Bussi et al. 2011). Ideas for teaching experiments are also discussed. In the second phase, teachers carry out teaching experiments in their classes. At the end, they have to hand in a logbook with their analyses of the progress of the activities. The teacher educators are researchers in mathematics education affiliating to MMLab team.

In the second year of the first part of the project (for the base of Modena and Bologna, school year 2009–2010), the teachers had an e-learning platform (Moodle) to support training and experimentation phases (inspired by Guin and Trouche, 2006). They could not only download material worked during the laboratory sessions, or concerning mathematical machines and teaching experiments, but also upload their own files. One goal of that implementation in our project is to provide a tool to accompany and support teachers and to foster the development of

[8]http://www.mmlab.unimore.it/site/home/progetto-regionale-emilia-romagna.html (Accessed Dcember 2012).

a collaborative work (Maschietto 2010). By means of the Wiki tool, teachers (split into groups) were asked to write a report for each meeting (each group was in charge of only one report). Each report had to contain: [1] an introduction of the topic of the session, in which teachers are invited to see the training sessions from their personal points of view (this can call 'situated analysis'); [2] remarks on what and how they made in those sessions, as well as on processes activated during their activities ('analysis of shared and distributed knowledge').

The aim of this paper is analysed this programme by the lens of instrumental and documentational genesis, in which mathematical machines are considered as resources for teachers.

Analysis of the Teacher Education Programme

This programme can be analysed at different levels of granularity: at the institutional level (the institutional contexts is described in Bartolini Bussi and Martignone 2013); at the macro level, where we could see the articulation between the two phases of the programme; at a meso level, where the two phases are analysed; and at a micro level, where the individual teacher and the impact of this education in his/her professional development can be studied.

At the macro level, the project realizes a cycle between the phase of presential sessions on mathematics laboratory and teaching practice, due to the fact that it is addressed to practising teachers and it requires an experimental phase. At the end of the education programme, a last session is organised in which the teachers share their experiences and reports, as logbooks and papers for the book of the project.[9]

The analysis of the MMLab project aims to investigate if and how the double genesis is present for those teachers, what the relationships with the documentational genesis are, what kind of support for teachers. So, the education programme will be analysed through the lens of appropriation at meso and micro levels:

- Appropriation of the methodology of mathematics laboratory;
- Appropriation of certain mathematical machines as resources, in two levels: as instruments to make mathematics and as resources to make students do mathematics;
- Appropriation of historical elements and analysis of students' activities.

These terms correspond to those components that characterise mathematics laboratory as a cultural and professional challenge for teachers. The two phases of the education programme will be analysed by connecting and combining instrumental and documentational approaches and the Theory of Semiotic Mediation.

[9] http://www.mmlab.unimore.it/site/home/progetto-regionale-emilia-romagna/risultati-del-progetto/libro-progetto-regionale.html (Accessed December 2012).

Working Sessions in Mathematics Laboratory

In our analysis of the laboratory sessions, we will refer to the reports written by the group of teachers in the Wiki tool, at the end of each session.

In the first step of each working session, the teachers are in a position of students, with a mathematical machine to explore by means of a worksheet. The mathematical machines, apart from the compass, appear as new objects for teachers. With respect to the Theory of Semiotic Mediation, this phase corresponds to the higher triangle "task—artefact—situated texts" in Fig. 1b. It represents the first component of a didactic cycle.

The questions asked in the worksheets (Bartolini Bussi et al. 2011) support the instrumental genesis of mathematical machines, as a tool for doing mathematics. They promote instrumentalisation (by questions as how the artefact is made, which components and their characteristics) and instrumentation (how using that mathematical machine to do something) processes. This first step corresponds to an instrumental genesis, in which the teachers construct an instrument to deal with certain mathematical objects. In the worksheets, there are also questions concerning the proof about the 'product of the machine', corresponding to identify the mathematical principle at the base of its functioning. The educator wants to bring out the invariant components of action (that is the processes allowing a certain result to be obtained), on which the construction of utilisation schemes is based. Some conclusive questions concern the change of some elements of the machine (instrumentalisation process), in order to study the relationships between certain changes and mathematical meanings evoked by the machine. Those questions are classified as problem solving in (Bartolini Bussi et al. 2011). Following Chiappini and Reggiani (2004), the idea of laboratory as an experiential space where mathematical knowledge is restructured through the introduction of the artefact seem to be performed and, above all, experienced by the teachers.

With respect to the framework of semiotic mediation, we can say that in this first step, the mathematical machines are not chosen to mediate specific mathematical meanings by the educator. Instead, the content to be mediated is represented by some elements of the laboratory methodology, the analysis of processes that can be promoted in the laboratory and the analysis of exploration schemes of mathematical machines. The analyses were carried out in the second step of the working sessions, usually consisting in collective discussions. The teacher educator uses all the elements appeared in group works, starting with the answers to worksheets, in order to pay attention to processes of exploration of a mathematical machine (instrumental genesis), to the analysis of answers and argumentation and proof performed by the teachers themselves. In this way, the teachers experimented a form of mathematical discussion, in the position of student. The presence of teachers from different grade levels enriched this discussion. For instance, the primary school teachers asked to specify some mathematical elements (e.g., certain steps in a proof) that seem transparent to secondary school teachers:

Exc1. Session 1 We start working, finding and comparing the possible solutions among ourselves, split by groups, we have "animated" the group conversation, reflecting on the various educational implications and any insights/ideas, which are derived from our geometric constructions

In their situated analysis, the teachers comment their instrumental genesis, in which schemes of exploration of mathematical machines appeared from a session to the next one:

Exc2. Session 2 Thus, the methodology of work seems to be established; we discuss the object as an artefact and as an instrument without being tied to the proposed worksheet

On the other hand, the teachers are expert, that connect the mathematical machine to a precise curricula content, in a mix of mathematical knowledge and pedagogical knowledge. Different connections are observed depending on the grade school where teachers work. For example, the theory of conic sections is not a mathematical content taught at primary school, so those teachers' mathematical knowledge appeared weaker than for secondary school teachers. These differences represent richness for group work and collective discussion, even if it could become dangerous if discriminating on what to invest energy during the education programme.

The teachers worked on mathematical knowledge and cultural aspects, presented by the questions and for the use of mathematical machines as tools grounded in the history of mathematics (this is an important component of the training, because those elements are not very considered in teacher education, as stressed by Adler, 2008). In their analysis written in Wiki tool, the teachers have taken into account all these aspects (mathematical, cultural and cognitive):

Exc3. Session 2 [we] highlight the different processes, but above all the various mathematical knowledge(s) latent to the different [geometrical] constructions, detecting, among other things, a certain difficulty in transmitting orally the procedure [of construction]

Exc4. Session 4 In our opinion, this experience can make us reflect both on the aesthetic value as well as formal [value] of a proof, and on the mental scheme of each of us, who is induced to more easily follow a type of argumentation rather than another one, and then to consider it [the first one] more good

Based on this kind of comparison, a second genesis seems to start. We could call it instrumental-documentational genesis, in which a mathematical machine begins to be seen as a tool to make students do mathematics, a resource for the teacher as an expert. Collective discussions, comparisons among different exploration strategies, argumentations but especially teachers' experiences with machines (movements, constrictions…) allow to detect didactic uses of the mathematical machines and their semiotic potential, which is an essential element for the use of an artefact as tool of semiotic mediation (Mariotti 2012). At the same time, other resources were considered, as teaching experiments previously carried out (discussed in Bartolini Bussi and Maschietto 2006). In addition, teachers share their constructions and their proofs after the sessions, uploading files, in general dynamic geometry files, to the platform. This material represents another kind of resources, i.e., the result of the collective discussions (we can see a germ of collective work).

The analysis of written report on Wiki shows that collaborative writing is carried out in different ways by the groups, with a different participation of their members. These reports highlight several voices[10]: teacher-as-student voices in the situated analysis (when they work with artefacts), teacher-as-professional voices, in the analysis of knowledge and cognitive processes. A third voice is represented by the educator voice, which drags teachers toward didactic, cognitive and cultural analysis. But other voices are presents: students' voices (students from teachers' classes and proposed teaching experiments), and the voice of constructors of mathematical machines.

Exc5. Session 2	We are students and teachers at the same time; these activities solicit us to make a meta-reflexion, toward a deep reflexion that reinforces the meta-cognitive teacher (aware of what he knows and what he wants to do)

During the first phase of the education programme, the teachers were supported to enlarge or their systems of resources for teaching (or to construct a new system). The resources were not only constituted by the artefacts mathematical machines, but by the instruments (according to instrumental approach) with elements for the analysis of their didactic uses. In this sense, they are complex resources, as 'secondary resources' for teachers (Maschietto and Trouche 2010, p. 45). So, this first phase seems to support the appropriation of the mathematical machines as resources for the mathematics laboratory. With respect to the double genesis (Haspekian 2011), the analysis of this phase through the lens of documentational approach seems to pay attention to an intermediate genesis, aimed at the appropriation of the didactic potentialities of the mathematical machines in mathematics laboratory.

Design and Teaching Experiments

Together with the laboratory sessions, teachers initiated the design of experiments to be carried out in their classrooms, according to a training contract (it can be analysed at the institutional level). This second phase demands the conception and

[10]The term 'voice' is used according to Bachtin, to mean a form of speaking and thinking that represents the perspective of an individual.

implementation of didactic projects of mathematics laboratory using the mathematical machines. This request forces the teachers to activate a process of documentational genesis (Gueudet and Trouche 2009), which is supported by the tutors (an intermediate figure between teachers and educator) and educator, as well as by the platform, whose use allows collaborative work. From the viewpoint of the documentational genesis, the teacher constructs a document for his work in the classroom, starting with the new resources (in the sense previously assumed) and existing systems of resources. He performs instrumentalisation processes, as he fits and constructs worksheets for his students, and instrumentation processes, performed by a priori analysis, strategies for anticipating students' strategies, classroom management by didactical cycles. These strategies can become parts of utilisation schemes for the methodology of mathematics laboratory. Concerning resources, the use of personal resources, such as dynamic geometry software, pays attention to the question of instrumental orchestration (Trouche 2004). Following Chiappini and Reggiani (2004), in this design phase, the teacher performs a re-configuration of content to teach, in order for making that content an object of investigation for students and encouraging the construction of mathematical meanings.

In the training in Modena, the teachers have decided the machines and the mathematical contents to propose to their students. They were split into four groups, each of them had a reserved space on the platform (Maschietto 2010). For this design, a grid is proposed, with the aim of supporting a priori analysis of the experiment to bring out (components of laboratory methodology, kinds of tasks for students and ways to manage students work and collective discussions). This grid also asks to anticipate students' difficulties and crucial points of the teaching path. The group work allows to appropriate the new resources, contextualise the use of the machines (as fostering argumentation and proof, or construct specific meanings, that are related to their didactic functionalities) and, last, support documentational genesis.

We consider now an example to show the elements mentioned above. It concerns teaching experiments on conic sections[11] (standard topic in Italian secondary school), carried out by three teachers working in three different schools (two senior high schools specialized in classical studies and one vocational school). The teachers met twice to discuss about their projects and to define worksheets for students. During those meetings, they explored the curve drawers with tightened thread (see Fig. 3, in the top left-hand corner) before writing tasks for students. In so doing, they deepened their analysis of didactic functionalities in terms of semiotic potential (related to utilisation schemes and potential personal meanings of their students) and didactic aims (to introduce conic sections by the mediation of the mathematical machines).

[11] http://www.didatticaer.it/cerca_didattica/macchine_matematiche_classe.aspx (Accessed December 2012).

On the basis of this collaborative work, each teacher has constructed his/her own document for the class, taking into account knowledge of students and content, knowledge of the curriculum and time constraints. So, part of the documentational genesis is highlighted by the design grid, which shows different documentational geneses for the three teachers, even if there are common elements. For the three experimentations, the first session was carried out in the MMLab at the Department of Mathematics, where students were introduced to the topic through a historical perspective on conic sections in Greek mathematics. The ppt file used for this introduction is another resource of the training. It was instrumentalised by each teacher, that deleted some slides and added other ones. Another example of different documentational geneses concerns the content: the teacher (GB) working in a vocational school did not propose hyperbola and preferred working first on ellipse and then on parabola as an example of function. At the same time, the implementation of this session required an instrumental orchestration for teachers (orchestration of video projector, large models of cones, and mathematical machines for students, paper and pencil). Concerning didactical components and the implementation of didactic cycles, in general group works had a more important place with respect to collective discussions.

In order to deepen the analysis of teachers' genesis, we have to carry out now a micro level analysis of the work of one teacher (GB). The teacher wanted to introduce the parabola by the mathematical machine with tightened thread in order to study its geometrical definition and properties, before an analytical approach, which she considered masking those properties. She assumed that the laboratory methodology allowed student work within a synthetic geometrical approach. For her, in the mathematical machines environment, students could link synthetic and analytical geometry. For the teaching experiment, the teacher chose certain material resources, as dynamic geometry files (Fig. 3). With respect to resources offered by the education program (Fig. 3, in the top and at the bottom left-hand corner), she also considered other resources (Fig. 3, in the centre at the bottom).

The PowerPoint file for historical introduction was modified. A new resource was conceived for her teaching: in order to support the historical presentation about the genesis of conic surfaces in the Menaechmus-Euclide approach, she constructed three models of triangles to be rotated (Fig. 2, at the bottom right-hand corner).

The construction and sequence of worksheets for her students takes into account students' instrumental genesis. The unique worksheet of the training (Fig. 2, at the bottom left-hand corner) is instrumentalised: the teacher prepares four worksheets. Each of them considers a question, decomposed into several tasks; questions to control are presented in the back of the paper (Fig. 3).

This case is paradigmatic of the second phase of the training programme, in which individual and collective documentational geneses seem to take place.

At the end of class work, teachers have to present a logbook, which requires the a posteriori analysis of the experimented laboratory sessions. The goal is to bring out the role of mathematical machines used in the classroom, the students' cognitive processes, critical points and difficulties, but also positive aspects. In general, teachers paid attention to students' motivation and engagement in the proposed

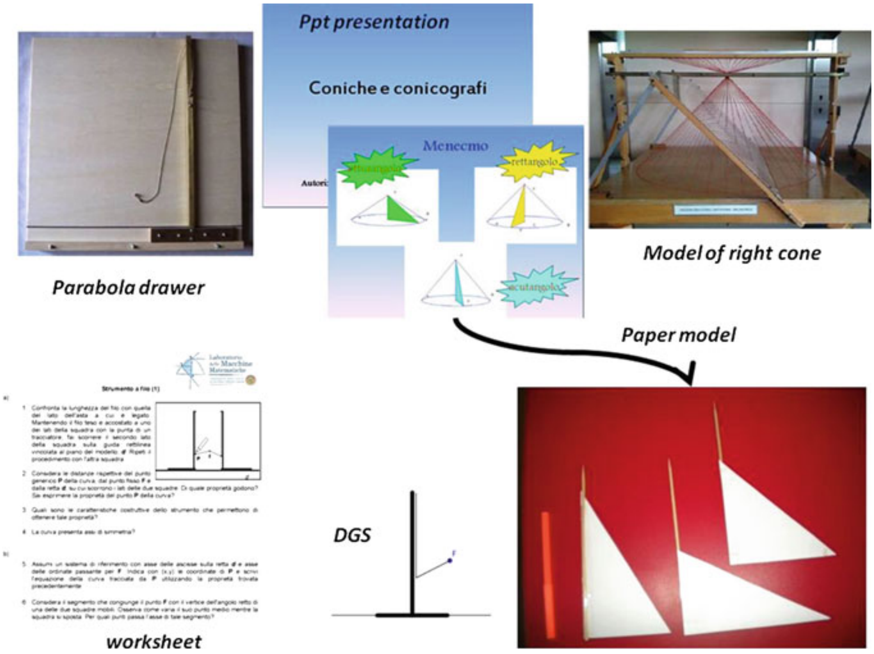

Fig. 2 Teacher GB's resources for conic sections

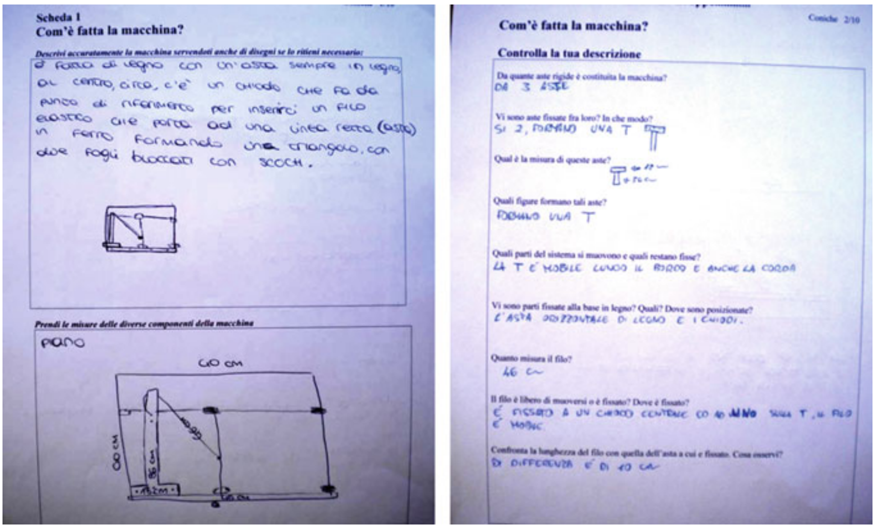

Fig. 3 The first worksheet (front and back) concerning the parabola drawer

activities. In most logbooks, a great difficulty for students lied in language and writing or expressing their explorations, formulating conjectures and argumentations. Teachers were aware of their role during a mathematics laboratory, very different from a classical lesson, and the management of didactic time. In their analysis, teachers also reported the differences between the design and the experimentation, due to time and changes depending on the answers of their students (some part to expand, new questions arose). For instance, teacher GB wrote:

Exc6. Each exploration proposed by the worksheets can be the beginning of an unexpected development proposed by the students: [the teacher] needs to be aware during the session and change her plan in order to seize that opportunity

Discussion

In the previous section, we have analysed a teacher education programme concerning the methodology of mathematics laboratory through the lens of instrumental and documentational genesis. We have carried out the analysis of the two phases of the education programme and the work of a teacher for the teaching experiment in her classroom. According to Haspekian (2011), a double genesis is recognised. It is supported by the two phases of the education programme. But the documentational approach allows an intermediate genesis to be detected, focusing on the appropriation of didactic potentialities of the mathematical machines for mathematics laboratory (or, in other terms, the appropriation of mathematical machines as vehicle of learning).

The appropriation of mathematics laboratory methodology seems to result from teachers' reflections on their works during the first phase, as the reports on Wiki tool show. The Moodle platform supports it.

Teachers' reports seem to highlight that the structure of the education programme following a laboratory methodology allows teachers to play a role on two levels: as-a-student and as-a-professional. In the latter role, the teacher can have a reflexive regard on mathematics laboratory and processes happening during each sessions. This kind of regard seems to foster the appropriation of artefact and methodology. Collaborative work seems to begin to act.

During this training, the teachers tried laboratory sessions and constructed resources for it. At a meso level, we can observe a cycle of resources-documents: from the document of the educator to resources for teacher. This is a complex cycle from a macro level of analysis, in which several elements come into play: primary and secondary resources as the mathematical machines, teacher's personal resources, knowledge of content and pupils, analysis of content and cognitive processes, etc. These geneses are supported by the structure of the training and by the use of a platform, in order to develop collaborative work.

In the documentational approach, the documentational genesis is characterised as an ongoing process rather than a process with a final step. During the education programme, the teachers have constructed documents for their own use. But those documents, as well as final reports and logbooks, can be considered as resources for others teachers. When a teacher considers those as resources, he starts a documentational genesis.

In the year following the training programme, some teachers in Modena have continued to meet and work together: some primary and secondary school teachers on compass, some secondary school teachers (different school level) on pantographs for reflection, a secondary teacher on conic sections. The platform is always available for teachers, in particular resources produced during the training. Afterwards, the case of compass is discussed.

Two primary school teachers have considered the logbook written by a secondary school teacher as a new resources. Even if they have participated in all the sessions the year before, they needed a certain time for appropriating this new resource. Their aim was to use the compass to mediate the meaning of circle (5th grade), but at the same time to work within a secondary school perspective for their pupils. The work together with the secondary school teacher helped them in the instrumentation of his logbook. The analysis of the teaching experiment carried out the year before paid attention to the question of time and tasks for students, in order to support students' instrumental genesis and the construction of the mathematical meaning. On the other hand, working within the laboratory methodology contributed to consider compass from another point of view:

Exc7. In standard teaching, a compass is usually presented as a technical tool with certain functions that are explained to students and become a support for some aspects of geometry. On this occasion, we have used a different perspective: the tool, bringing into play the mathematical knowledge intrinsic to its structure, was placed at the centre of the mathematics activities, through its discovery by pupils it has proved to be an invaluable vehicle for the acquisition of other skills, not only mathematical. (primary school teacher's logbook)

In the design of the teaching experiment, as in teachers' logbooks, it is possible to identify some elements that can be considered in terms of invariant of teacher behaviour for mathematics laboratory with mathematical machines. This is relevant from the point of view of teachers' documentational genesis on mathematics laboratory and their appropriation of mathematical machines as a vehicle of learning.

We try to develop the idea expressed in Maschietto and Trouche (2010), seeing a resource as an artefact, becoming, in a complex appropriation process, an instrument for a given teacher. This model of thinking resources design can be seen as a tertiary artefact for modelling teachers' development.

In addition, in order to realize teaching experiment in their classes, the teachers also have to take into account conditions and constraints (Bosch 2010). Following Bosch, a great challenge for the diffusion of teaching methodology is to study in a systematic way the circumstances allowing a certain kind of activity to live in a class, in order to change blocking components into developing ones.

Systems of resources also affect educators. Indeed, at the same time, teachers' documents can become resources for the educator, as papers on previous teaching experiments are. For instance, the continuation of the training programme starts by reading certain logbooks, in order to deepen the analysis of experiments. So, teacher training changes too. But a question arises: what is the relationship between the resources for educator and resources for teachers? When a lot of resources are available, which of them are important for the training if any, in order to support teachers' geneses?

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The Common Core State Standards in Mathematics

William McCallum

Abstract The US Common Core State Standards in Mathematics were released in 2009 and have been adopted by 45 states. We describe the background, process, and design principles of the standards.

Keywords Standards · United States · Common Core

Background

In 2009, 48 out of the 50 states in the U.S. came together under the leadership of the National Governors Association (NGA) and the Council of Chief State School Officers (CCSSO) to write school standards for Mathematics and English Language Arts that would ensure students leaving high school are ready for college and career. The Common Core State Standards (NGA and CCSSO 2010) were released on 2 June 2010, and they have been adopted by 45 states.[1]

Twenty years earlier, in 1989, the National Council of Teachers of Mathematics made the first move in the modern era towards a common understanding of school mathematics (National Council of Teachers of Mathematics [NCTM] 1989). Before that time curriculum varied among the nation's numerous school districts. The NCTM standards were not themselves an act of government, but in response to them the governments of the 50 states started developing their own standards, bringing a measure of consistency to the mathematics curriculum within states. However, consistency between states proved elusive and in the years after 1989 state standards diverged greatly. For example, Table 1 shows the 2006 distribution

[1]A 46th state, Minnesota has adopted the English Language Arts Standards.

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Table 1 Distribution of grade levels where state standards introduce addition and subtraction of fractions

Grade	1	2	3	4	5	6	7	8
Number of states	2	0	7	22	9	1	1	0

of grade levels at which 42 state standards introduced addition and subtraction of fractions.

Schmidt et al. (1997) described the situation:

> There is no one at the helm of mathematics and science education in the U.S. … No single coherent vision of how to educate today's children dominates U.S. educational practice …. These splintered visions produce unfocused curricula and textbooks …[that] emphasize familiarity with many topics rather than concentrated attention to a few …[and] likely lower the academic performance of students who spend years in such a learning environment. *Our curricula, textbooks, and teaching all are "a mile wide and an inch deep*." [emphasis added]

The mile wide inch deep curriculum is partly a natural result of the U.S. system of local control, in which there is no central Ministry of Education, authority over education is delegated to the states and, on many questions of policy, to the 16,000 school districts. Another cause was the math wars, an ideological conflict about almost every question related to mathematics education: curriculum, assessment, methods of teaching, the nature of mathematics itself. The fragmented system of local control allowed this debate to rage unchecked, drawing school boards, parent groups, curriculum reformers, policy makers, and university faculty into a draining conflict about curriculum materials at the expense of work on other important problems in mathematics education.

Various efforts in the last decade have had a primary or secondary goal of improving this situation by bringing the different sides together and aligning state standards with each other and with international standards: the American Diploma Project (Achieve 2004), Finding Common Ground in K-12 Mathematics Education (Mathematical Association of America 2005), Adding it Up (Kilpatrick et al. 2001), the NCTM publications *Curriculum Focal Points for Prekindergarten through Grade 8 Mathematics* (NCTM 2006), and *Focus in High School Mathematics* (NCTM 2009), reports by ACT (ACT) and College Board (College Board), and the report of the National Mathematics Advisory Panel (U.S. Department of Education 2008).

These efforts came to a head with the surprisingly rapid consensus around developing common standards in 2009. The initiative was spearheaded by NGA and CCSSO, two new actors in the world of standards writing, who showed agility in bypassing old stalemates. Key to the success of the endeavour in political terms was that unlike previous efforts, the Common Core effort was led by state policymakers, not the federal government. The standards drew on many sources, including the reports and publications mentioned earlier, and also including standards of U.S. states and high achieving countries, particularly in East Asia. A recent analysis has shown that the standards are closely aligned with the standards of the A+ countries, a group of countries that formed a statistically significant group of top achievers on TIMSS 1995, and that state achievement in mathematics on the

National Assessment of Education Progress is correlated with the closeness of previous state standards to the Common Core (Schmidt and Houang 2012).

With the great majority of states adopting CCSS, the U.S has entered an unprecedented time of opportunity, with long-standing arrangements in mathematics education now open to renegotiation. For over 20 years discourse in mathematics education has been dominated by the math wars, in which apparently divergent views of mathematics competed for dominance; it was difficult, for example, to advocate both that students acquire fluency with algorithms for addition, subtraction, multiplication, and division in elementary school, and that they engage in serious work with statistics in high school, without being viewed askance by both camps. But CCSS incorporates both stances, viewing the two as integral parts of a coherent progression in skill and understanding.

What are the opportunities? First, the opportunity for curriculum developers to produce more focused and coherent materials, without having to attend to diverse demands for topic placement made by different state standards; second, the opportunity for teacher preparation and professional development to become less generic and more focused on the mathematics taught at a given grade level; and finally, the opportunity for teachers from across the country to share tools for implementation based on common standards.

What Should Standards Look like?

In countries with a fully functioning education system, they can look like Fig. 1. I was one of the lead writers of the Common Core; we sometimes dreamed of the ability to make simple bulleted lists like this. How does Singapore get away with

O Level Mathematics Syllabus Secondary One

Secondary One	
Topic/Sub-topics	**Content**

Algebraic representation and formulae	Include:
	• using letters to represent numbers
	• interpreting notations:
	  • ab as $a \times b$
	  • $\frac{a}{b}$ as $a \div b$
	  • a^2 as $a \times a$, a^3 as $a \times a \times a$, a^2b as $a \times a \times b$, ...
	  • $3y$ as $y + y + y$ or $3 \times y$
	  • $\frac{3 \pm y}{5}$ as $(3 \pm y) \div 5$ or $\frac{1}{5} \times (3 \pm y)$
	• evaluation of algebraic expressions and formulae
	• translation of simple real-world situations into algebraic expressions
	• recognising and representing number patterns (including finding an algebraic expression for the nth term)

Fig. 1 A page from the Singapore secondary standards

this? we asked ourselves. The answer is that Singapore has a Ministry of Education that produces curriculum and exams; their standards document is a description of that system, not a prescription for it.

Figures 2 and 3 show two images of standards in the U.S. The NCTM standards (NCTM 2000) have 14 "standards" (bulleted items) for Number and Operations, Grades 6–8, followed by 7 pages of narrative. To a certain extent the NCTM had an education system available as well, or rather two systems: the system of commercial and NSF-funded textbook projects for producing curriculum, and the system of 50 state departments of education for producing exams. The system was divergent, chaotic, and voluntary.

The document illustrated in Fig. 2 (which is representative of state standards at the time) has 82 standards for Number and Operations in Grade 6 alone, and no pages of narrative. This is much more detailed and performance-based than the NCTM standards. Unlike the NCTM standards, state standards have direct policy and legal consequences, and are used as a basis for writing assessments. They are flat lists of performance objectives of even grain size, designed to be delivered into the hands of assessment writers without the need for too much discussion or interpretation.

MATHEMATICS STANDARD ARTICULATED BY GRADE LEVEL

GRADE 6

Strand 1: Number Sense and Operations

Every student should understand and use all concepts and skills from the previous grade levels. The standards are designed so that new learning builds on preceding skills and are needed to learn new skills. Communication, Problem-solving, Reasoning & Proof, Connections, and Representation are the process standards that are embedded throughout the teaching and learning of mathematical strands.

Concept 1: Number Sense

Understand and apply numbers, ways of representing numbers, the relationships among numbers and different number systems.

PO 1. Express fractions as ratios, comparing two whole numbers (e.g., ¾ is equivalent to 3:4 and 3 to 4).

PO 2. Compare two proper fractions, improper fractions, or mixed numbers.

PO 3. Order three or more proper fractions, improper fractions, or mixed numbers.

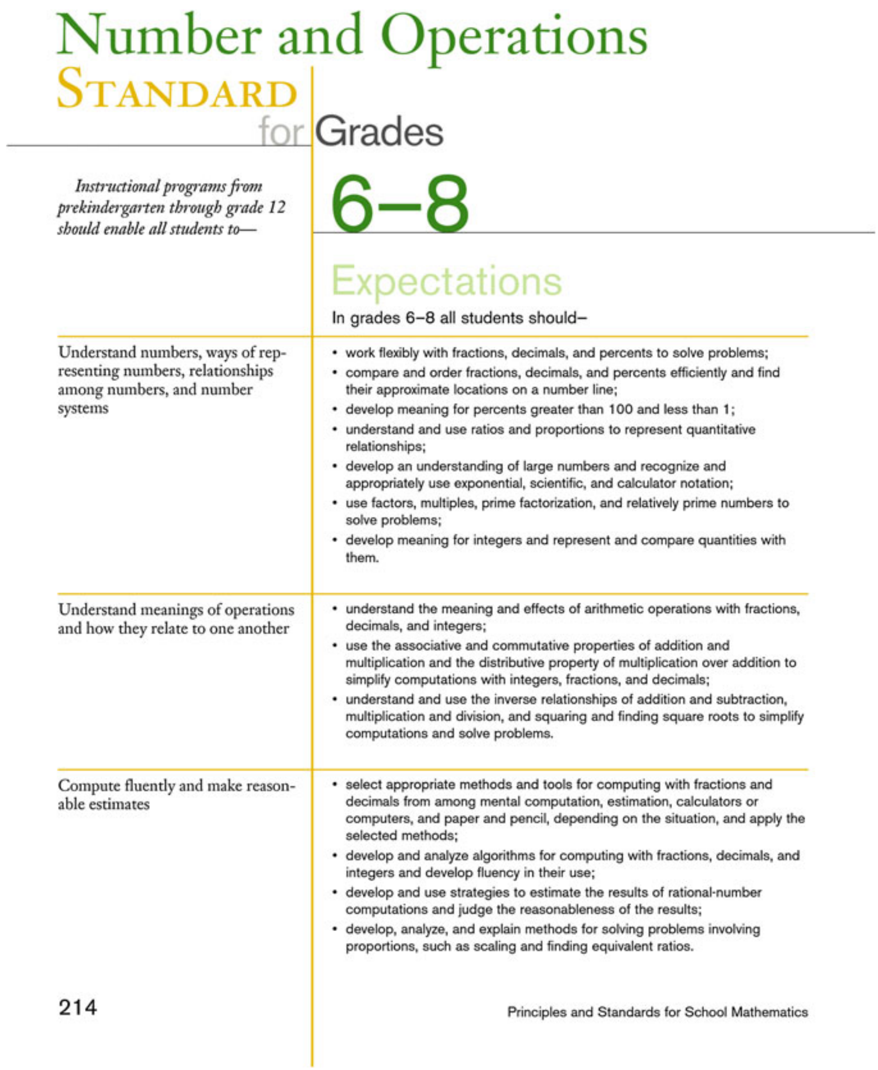
PO 4. Determine the equivalency between and among fractions, decimals, and percents in contextual situations.

PO 5. Identify the greatest common factor for two whole numbers.

PO 6. Determine the least common multiple for two whole numbers.

PO 7. Express a whole number as a product of its prime factors, using exponents when appropriate.

Fig. 2 A page from a typical set of state standards, 2008



Number and Operations

Standard for Grades 6–8

Instructional programs from prekindergarten through grade 12 should enable all students to—

Expectations

In grades 6–8 all students should–

Understand numbers, ways of representing numbers, relationships among numbers, and number systems	• work flexibly with fractions, decimals, and percents to solve problems; • compare and order fractions, decimals, and percents efficiently and find their approximate locations on a number line; • develop meaning for percents greater than 100 and less than 1; • understand and use ratios and proportions to represent quantitative relationships; • develop an understanding of large numbers and recognize and appropriately use exponential, scientific, and calculator notation; • use factors, multiples, prime factorization, and relatively prime numbers to solve problems; • develop meaning for integers and represent and compare quantities with them.
Understand meanings of operations and how they relate to one another	• understand the meaning and effects of arithmetic operations with fractions, decimals, and integers; • use the associative and commutative properties of addition and multiplication and the distributive property of multiplication over addition to simplify computations with integers, fractions, and decimals; • understand and use the inverse relationships of addition and subtraction, multiplication and division, and squaring and finding square roots to simplify computations and solve problems.
Compute fluently and make reasonable estimates	• select appropriate methods and tools for computing with fractions and decimals from among mental computation, estimation, calculators or computers, and paper and pencil, depending on the situation, and apply the selected methods; • develop and analyze algorithms for computing with fractions, decimals, and integers and develop fluency in their use; • develop and use strategies to estimate the results of rational-number computations and judge the reasonableness of the results; • develop, analyze, and explain methods for solving problems involving proportions, such as scaling and finding equivalent ratios.

214 Principles and Standards for School Mathematics

Fig. 3 A page from the 2000 NCTM principles and standards for school mathematics

It was against this background that the Common Core State Standards were written. On the one hand they were commissioned by the states and therefore had to be the type of document states were used to: detailed bulleted lists describing what we want students to know and be able to do. On the other hand, we were being asked to do something new, to break out of the system that produced the mile wide inch deep curriculum.

Design of the Standards

The fundamental design principles for the Standards are focus, coherence, and rigor.

Focus means attending to fewer topics in greater depth at any given grade level, giving teachers and students time to complete that grade's learning.

Coherence means attending to the structure of mathematics and the natural pathways through that structure, where "natural" means taking into account both the imperatives of logic and the imperatives of cognitive development in designing the sequence of ideas. Since these two imperatives are sometimes in conflict, attaining coherence is a complex exercise in judgement, requiring a certain amount of professional craft and wisdom of practice not easily obtained from any one source.

Rigor means balancing conceptual understanding, procedural fluency, and meaningful applications of mathematics. Here the word rigor is used not in the way that mathematicians use it, to indicate a correct and complete chain of logical reasoning, but in the sense of a rigorous preparation for a sport or profession: one that exercises all the necessary proficiencies in a balanced way.

Organization of the Standards

The Standards are divided into Standards for Mathematical Content and Standards for Mathematical Practice. The content standards are further subdivided into K-8 standards and high school standards. The K-8 standards are specified by grade level and organized into *domains*, topics which follow a coherent progression over a certain grade span (see Fig. 4).

The organization by domains is different in an important way from the organization by strands typical of previous state standards. Under the latter scheme, four or five strands (e.g., Number and Operations, Algebra and Functions, Data and Measurement, Geometry) would extend from Kindergarten to Grade 12. The homogeneity of this scheme with respect to time is at odds with the progressive

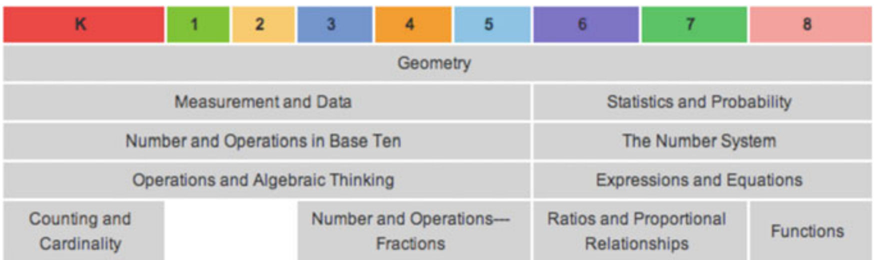


Fig. 4 Domains in the Common Core, Grades K-8

nature of mathematics, and resulted in a tendency to fill in every cell of the grade-by-strand matrix, one of the causes of the mile wide inch deep curriculum.

By contrast, domains operate at a finer level (there are 12 domains in the K-8 standards), and have a beginning and an end, each preparing for and eventually giving way to higher domains that both build on and encapsulate previous work. Domains allow for convergence and consolidation of ideas, as when the K-5 number work in the domains Operations and Algebraic Thinking, Number and Operations in Base Ten, and Fractions, is consolidated into a unified understanding of The Number System in Grades 6–8. The abbreviated life time of a domain also allows for the delineation of foundational domains that support more than one future domain: the work on Fractions in Grades 3–5 is a basis for The Number System, but also for the work on Ratios and Proportional Relationships in Grades 6–7, leading to Functions in Grade 8.

The high school standards are arranged into the broad *conceptual categories*: Number and Quantity, Algebra, Functions, Geometry, Modelling, and Statistics and Probability. These are further divided into domains as in the K-8 standards. However, the high school standards are not arranged into grade levels, and so the domains do not always exhibit a temporal progression. Some of the domains are conventional topics (e.g. Congruence is a domain in Geometry); others describe ways of thinking that help students bind their mathematical knowledge into coherent packages rather than trying to remember innumerable different formulas and techniques. For example, the Algebra category has a domain Seeing Structure in Expressions, which undergirds a student's work during the entire high school experience, from linear functions to logarithms.

Just as the content standards attempt to describe the complex structure of mathematical knowledge, the Standards for Mathematical Practice (see Fig. 5) describe the contours of mathematical practice; the various ways in which proficient practitioners of mathematics carry out their work. These are not intended to be free-floating proficiencies, unconnected with content, nor are they uniformly applied over all the work that students do. Just as a rock climber's various skills are called on differently during different parts of a climb, so specific aspects of practice become salient in specific pieces of mathematical work. For example, students learning how to complete the square in a quadratic expression benefit from

Fig. 5 Standards for mathematical practice

1. Make sense of problems and persevere in solving them
2. Reason abstractly and quantitatively
3. Construct viable arguments and critique the reasoning of others
4. Model with mathematics
5. Use appropriate tools strategically
6. Attend to precision
7. Look for and make use of structure
8. Look for and express regularity in repeated reasoning

consciously looking for structure and seeing the regularity in reasoning with a sequence of well-chosen examples (SMP 7 and 8); students constructing geometric proofs will learn to critique arguments and use precise language (SMP 3 and 6); students designing a study to see if there is a connection between athletic and academic proficiency will construct a statistical model and choose appropriate methods and technologies (SMP 4 and 5).

Taking Focus Seriously

Four out of the six domains in K–5 deal with number and operations (see Fig. 4): Counting and Cardinality (Kindergarten), Number and Operations in Base Ten (K–5), Operations and Algebraic Thinking (K–5), and Fractions (3–5).

The focus on number and operations in elementary school is even stronger than this count would suggest, because many standards in the other domains are designed to support the focus on number and operations. For example, the Grade 2 data standard in Fig. 6 supports the principal work with addition and subtraction of whole numbers.

As another example, many geometry standards in elementary school deal with composing and decomposing figures, supporting the unit fraction approach to fractions starting in Grade 3.

In order to make room for the focus on number and operations, some topics are given much less time in elementary school than was the case with previous state standards. This was a necessary step to make good on the promise of repairing the mile wide inch deep curriculum. For example, standards on data, patterns, and symmetry are reduced to a trickle in elementary school. This was one of the more controversial shifts in the Common Core, and it is worth looking at in a little more detail. Debate about curriculum in the United States has suffered from an all-or-nothing quality, and nowhere is this seen more clearly than in the debate about data and statistics in elementary school: it has seemed that the only choices were embracing a rich stream on data work in elementary school, as advocated by the GAISE report (Franklin et al. 2005) or drying it up to nothing. In contrast, the Common Core is based on progressions that start with a trickle before they grow into the full flow of a domain. Thus the data standards in elementary school are neither to be ignored nor to be given undue prominence. In due time, in high school, statistics and probability becomes a major topic.

2.MD.10. Draw a picture graph and a bar graph (with single-unit scale) to represent a data set with up to four categories. Solve simple put-together, take-apart, and compare problems using information presented in a bar graph.

Fig. 6 A Grade 2 measurement and data standard

The function concept is another topic that is delayed compared to previous state standards. There is a trickle of pattern standards in elementary school, carefully worded to support the emergence of an incipient notion of function (see Fig. 7).

In middle school, further preparation for functions is provided in the domains Ratios and Proportional Relationships and Expressions and Equations. The function concept finally makes its appearance in its own domain in Grade 8, and becomes a major conceptual category in high school.

As the examples of statistics and functions illustrate, taking focus seriously means delaying favored topics until their time, which will be a difficult shift for the educational system in the U.S.

The payoff for this approach occurs in high school, where the subject matter focus broadens as the foundations developed in K-8 allow for a variety of work in number and quantity, algebra, functions, modelling, geometry, statistics, and probability. Focus in high school means not so much a small number of topics as a concentration of skills and practice into a small number of underlying principles.

Preserving Coherence

The act of writing standards for a subject is inherently in conflict with the goal of showing the structure of the subject. Daro et al. (2012) liken this to shattering an intricately decorated Grecian urn into pieces and expecting the shape and decorative details to be visible in the pieces (see Fig. 8).

In order to avoid this problem and preserve a coherent view of the subject, both in the broad contours and in the small details, the Common Core breaks with a long-standing tradition that each individual standard should have the same "grain-size". Mathematics itself does not come in pieces of equal grain-size, and neither should a description of it. Consider, for example the Grade 2 cluster of standards shown in Fig. 9.

The first standard is large and fundamental, figures in much of the work of elementary school, and will show up again and again in a curricular implementation

3.OA.9. Identify arithmetic patterns (including patterns in the addition table or multiplication table), and explain them using properties of operations.

4.OA.5. Generate a number or shape pattern that follows a given rule. Identify apparent features of the pattern that were not explicit in the rule itself.

5.OA.3. Generate two numerical patterns using two given rules. Identify apparent relationships between corresponding terms. Form ordered pairs consisting of corresponding terms from the two patterns, and graph the ordered pairs on a coordinate plane.

Fig. 7 The trickle of pattern standards in elementary school

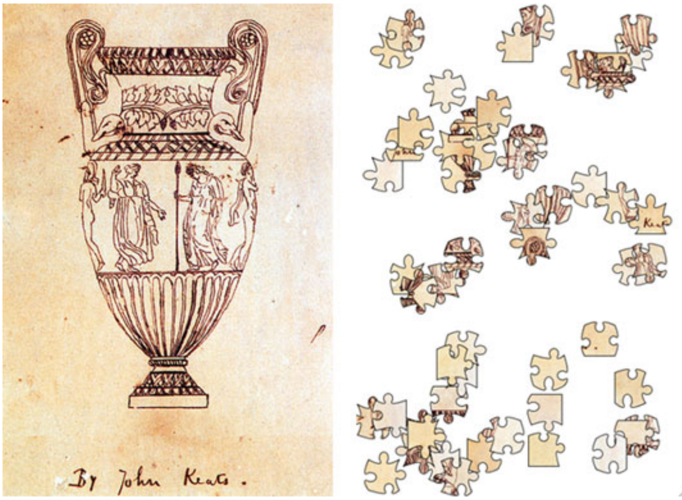


Fig. 8 Standards for a Grecian urn

Understand place value

2.NBT.1. Understand that the three digits of a three-digit number represent amounts of hundreds, tens, and ones; e.g., 706 equals 7 hundreds, 0 tens, and 6 ones. Understand the following as special cases:

a. 100 can be thought of as a bundle of ten tens—called a "hundred."
b. The numbers 100, 200, 300, 400, 500, 600, 700, 800, 900 refer to one, two, three, four, five, six, seven, eight, or nine hundreds (and 0 tens and 0 ones).

2.NBT.2. Count within 1000; skip-count by 5s, 10s, and 100s.

2.NBT.3. Read and write numbers to 1000 using base-ten numerals, number names, and expanded form.

2.NBT.4. Compare two three-digit numbers based on meanings of the hundreds, tens, and ones digits, using >, =, and < symbols to record the results of comparisons.

Fig. 9 Grade 2 cluster of standards on place value

of the standards, reinforced and deepened by work in later grades. The second standard is a discrete performance objective that, once secured, recedes from importance.

This example illustrates another feature of the standards designed to provide coherence: the clusters and cluster headings. In the Common Core, the individual statements of what students are expected to understand and be able to do (the "standards") are embedded within cluster headings, which are in turn embedded in domains. "The Standards" refers to all elements of the document's design, including the wording of domain headings, cluster headings, and individual

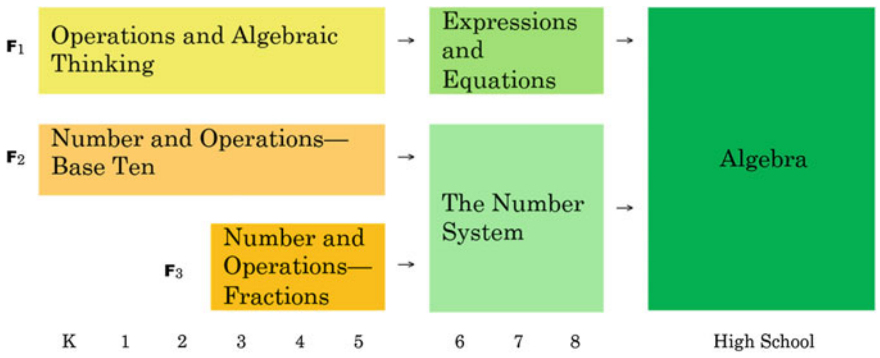


Fig. 10 Flows leading to algebra

statements. In this case, the cluster heading "Understand place value" says clearly the fundamental purpose of this cluster, in a way that is not completely captured by any individual standard within the cluster.

Another aspect of coherence is the flow of domains across grade levels, described earlier. As a further example of this, Fig. 10 shows the flow of domains to high school algebra. Building a viable ramp to algebra was a design requirement implied by the mandate to write standards that prepared students for college and career.

Balancing Understanding, Fluency, and Applications

State standards before the Common Core were often formulated in terms of concrete observable performances, following a hierarchy of verbs, in which some verbs describe higher levels of performance than others (e.g., memorize, interpret, formulate, analyze). One verb in particular was often avoided, however, the verb "understand." Many times during the writing of the standards we were told we could not use this verb because standards had to be measurable, and understanding was ill-defined and either impossible or very difficult to measure.

Nonetheless, if the goal of standards is to express our desires for our children's achievements, it is hard to argue that understanding is not among them. The Common Core calls explicitly for understanding in a number of standards and cluster headings (see Fig. 11).

Other standards explicitly call for fluency with addition and multiplication facts and with standard algorithms for addition, subtraction, multiplication, and division. These are capstone standards, occurring after adequate groundwork in earlier grades on strategies and algorithms based on place value and the properties of operations.

Yet other standards call for students to apply the mathematics they have learned. Modelling with mathematics is one of the Standards for Mathematical Practice

- Understand and apply properties of operations and the relationship between addition and subtraction (Grade 2)
- Understand concepts of area and relate area to multiplication and to addition (Grade 3)
- Understand ratio concepts and use ratio reasoning to solve problems (Grade 6)
- Understand congruence and similarity using physical models, transparencies, or geometry software (Grade 8)
- Understand solving equations as a process of reasoning and explain the reasoning (High School)
- Understand and evaluate random processes underlying statistical experiments (High School)

Fig. 11 Selected cluster headings using the word "understand"

7.EE.3. Solve multi-step real-life and mathematical problems posed with positive and negative rational numbers in any form (whole numbers, fractions, and decimals), using tools strategically. Apply properties of operations to calculate with numbers in any form; convert between forms as appropriate; and assess the reasonableness of answers using mental computation and estimation strategies.

Fig. 12 Linchpin standard in Grade 7

(see Fig. 5) and many of the high school standards are flagged as particularly important venues for modelling with a special symbol. The elementary and middle school standards build towards this with a progression of standards from simple word problems involving addition of whole numbers in Grade 2 to the culminating Grade 7 standard shown in Fig. 12.

Concluding Thoughts

Let me conclude by mentioning two projects designed to round out the description of the standards:

- The Progressions Project, ime.math.arizona.edu/progressions/.
- The Illustrative Mathematics Project, illustrativemathematics.org.

The standards were founded on progressions, narrative descriptions of domains provided by experts on the working team. The original progressions documents did not keep up with the rapid revision process, and the Progressions Project aims to produce final versions of them. The progressions provide another view of the standards, useful for curriculum designers, teacher educators, and could support research aimed at making recommendations for revisions to the standards.

The Illustrative Mathematics project is collecting sample tasks to illustrate the standards. It uses a community based approach, in which tasks are submitted, reviewed, edited and finally published by a growing community of experts who gain discernment and craft by participating in the process. It aims to become a permanent virtual destination that is both a repository of materials and a place where an expert community works.

I have tried to give some idea of what the standards look like, but ultimately a close reading of the standards is necessary to gain a complete picture. The standards are not designed to be easy reading, but they are designed to be read. The promise of the Common Core is having a shared text that, whatever its virtues and flaws, provides the basis of disciplined innovation in curriculum and shared tools for teaching.

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From Practical Geometry to the Laboratory Method: The Search for an Alternative to Euclid in the History of Teaching Geometry

Marta Menghini

Abstract This paper wants to show how practical geometry, created to give a concrete help to people involved in trade, in land-surveying and even in astronomy, underwent a transformation that underlined its didactical value and turned it first into a way of teaching via problem solving, and then into an experimental-intuitive teaching that could be an alternative to the deductive-rational teaching of geometry. This evolution will be highlighted using textbooks that proposed alternative presentations of geometry.

Keywords Practical geometry · History of mathematics education · Textbooks

Introduction

As is well known, *practical geometry* was present from the very beginning of the history of geometry. Different populations, such as the Egyptians, the Babylonians, the Chinese and the Indians, developed practical geometric skills. A practical geometry was present in the Greek world too; however, "basing their work on the practical geometry of the Egyptians", the Ancient Greeks developed "a system of logic which culminated with the great work of Euclid" (Stamper 1909, p. 4). As we know, the *logical-deductive* aspect of Euclid's *Elements* greatly influenced the teaching of geometry.

The Romans were mainly interested in practical geometry (land-surveying and the engineering of warfare), and in its *teaching in their schools*. In this environment, we find one of the first *didactical appreciations* of practical geometry: "Leaving its use in warfare, many maintain that this science, different from others, is not useful when it is part of the knowledge, but *at the moment in which it is learned*" (Quintilianus, *Istitutio Oratoria*, I, 34 etc.). With these words Quintilianus

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(1st century AD) attributed to practical geometry, through the attention to the *process* of learning, the same educational role that *proving* has within rational geometry.

Many attempts have been made to teach geometry following a path different from that proposed in the *Elements* of Euclid (Barbin and Menghini 2013). The most diffused way was to change the order of the theorems and of the problems, while accepting hypothetical constructions and a limited use of arithmetic. In this paper we will look at those attempts whose main purpose was not in a (new) logical order for geometric deduction, but rather in an "exploration" of geometry that kept a strong link with its practical origin, and whose contents and methods can be seen as an evolution of the medieval practical geometry.

De Practica Geometriae

We will begin our excursus among the *textbooks* of practical geometry with Fibonacci's *De practica geometriae* (1223). This work by Leonardo of Pisa (Fibonacci) gave rise to that stream of practical geometry which characterised the geometry of the Late Middle Ages and presented not only the very practical rules and methods for calculating distances and areas for land-surveyors, but also problems that, towards the end of the Middle Ages, turned into "*mathematical games*" (as in Leon Battista Alberti's *Ludi matematici*, written in the mid-15th century).

Fibonacci's text was largely used in the last grades of the Italian *Scuole d'abaco*, which were parish-schools for pupils who wanted to learn a trade. In these schools pupils had to memorize only the rules, but in the text of Fibonacci there was something that went beyond this idea.

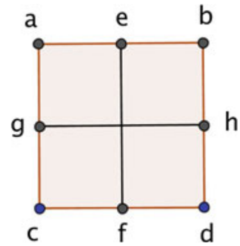
Fibonacci didn't write his book in contrast to the Euclidean one. On the contrary, he often refers to Euclid's propositions. His geometry is simply a different thing. Fibonacci never speaks of axioms or theorems; at the beginning he lists definitions and "principles", which broadly correspond to axioms or to constructions that can be done. He uses *numbers*, arithmetic, and practical examples; proofs are often only verifications with numbers. In the case of the theorems from the 2nd book of Euclid, concerning geometric algebra, he reports the proofs "translated" in an algebraic language.

Fibonacci doesn't present drawing or measuring instruments. He doesn't even refer to angles. His geometry has the scope of "measuring all kinds of fields" and "dividing fields among partners". In fact one of the first rules shown by Fibonacci concerns the *calculation of the area of a square*. As he always does, the rule is given with an example (p. 14).[1]

[1]For the English translation, as well as for the pages for the quotes, we refer to the edition of Hughes (2008).

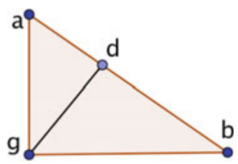
> Given a quadrilateral, equilateral and equiangular field having 2 rods on each side, I say that its area is to be found by multiplying side *ac* by its adjacent side *ab*, namely 2 rods by 2 rods.

Let lines *ab* and *cd* be divided into two equal parts [this is one of the allowed constructions listed at the beginning] at points *e* and *f*, and draw the line *ef*. Likewise […] draw line *gf*. Thus quadrilateral *abcd* has been divided into four perpendicular squares, each of which is measured by one rod on a side. Thus there are 4 plane rods in the entire square quadrilateral *abcd*.



This explanation will be sufficient for all later practical geometry. But Fibonacci wants to show that in fact the four quadrilaterals are squares. The "proof" consists in an *observation*: *ef* being equal to and equidistant from *ac* and *bd*, and *gh* being equal to and equidistant from *ab* and *cd,* then *ae* is also equidistant and equal to *cf*. Therefore the angle *aef* is equal to the right angle *acf*, etc.

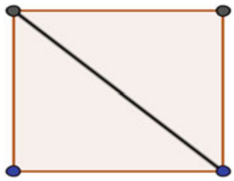
Fibonacci uses the theorem of Pythagoras to measure areas, and he proves it by referring to Euclid's theorems concerning the similarity of the involved triangles; in the triangle *agb*, we have (p. 69)



$$db{:}bg = gb{:}ab, \text{ hence } db \cdot ba = bg^2. \text{ Similarly } ad \cdot ab = ag^2.$$

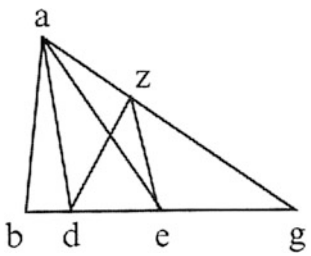
> So $db{\cdot}ba + ad{\cdot}ab = bg^2 + ag^2$. But the first part equals the square on *ab*, as proved by Euclid.

Then Fibonacci gives the rule for the area of a triangle: "To find the square measure, namely the areas of all triangles, multiply half the cathete [the height] by the whole base or half the base by the whole cathete" (p. 66) Fibonacci shows, only in the case of a rectangular triangle, that a triangle can be seen as half a rectangle. For the other triangles, he tries to come back to right-angled triangles by dividing the figure by using the height. He gives many examples in order to consider various positions of the height.



The many problems concerning “dividing fields” into equivalent parts are more abstract, although they originate from concrete problems. Here we start to find “puzzles”, challenges. For instance “To divide a triangle in two equal parts by a line drawn from a given point on a side” (p. 188):

> In the triangle *abg*, consider the point *d*. I will divide side *bg* in two equal parts at point *e*, and I will join lines *ad* and *ae*. And through point *e* I will draw line *ez* equidistant from line *ad*, and I will draw line *dz*. I say therefore that triangle *abg* is divided in two by line *dz*. The proof follows. Two equal triangles *ade* and *adz* [are] sitting on base *ad* with sides *ad* and *ez* equidistant from one another. To both triangles add triangle *abd*. The two triangles *abd* and *adz* become quadrilateral *abdz* equal to two triangles *abd* and *ade*, that is triangle *abe*. But triangle *abe* is half of triangle *abg*. Whence quadrilateral *abdz* is half of triangle *abg*. What is left, namely triangle *zdg*, is the other half of triangle *abg*. Therefore triangle *abg* is divided into equal parts at point *d* by line *dz*.



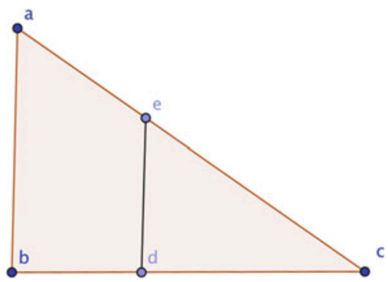
> Numerically: $ed \cdot ga/gd = az$, for instance considering the triangle with sides $ab = 13$, $bg = 14$, $ga = 15$, and choosing d such that $de = 2$, then $gz = 11$ and 2/3.[2]

These kinds of problems are not found in Euclid. In the following centuries, they became typical problems in texts of practical geometry.

In the chapter about “measuring heights, depths, and longitudes of planets”, we find practical problems such as the following (p. 346):

> If you wish to measure a height [*ab*], fix a staff [*ed*] perpendicular to the ground. Step away from the staff and the object you wish to measure. Stoop down the ground level from where you can see the top of the object across the top of the staff, and mark the place from where you looked [*c*].

[2]The triangle 13–14–15 was often used in practical problems, as it has a “good” area (42).



Considering the involved similar triangles, Fibonacci explains how to find the height. He gives different examples, always repeating how the rules are used ("This can be shown with numbers. Let *ed* be 5 palms […]", p. 347).

With Fibonacci's text the students' learning is facilitated because the rules are introduced through numerical examples, and theorems are "proved" by substituting numbers to the considered lengths, and, sometimes, with the help of algebra (as in the theorem of Pythagoras). On the other hand, the student is challenged with problems, such as the division of fields.

Books similar to Fibonacci's, maybe with a little more Euclidean proofs, lasted for over 300 years and influenced the teaching not only in the old universities, but also in the first secondary schools of the 16th century. Examples are the texts of Luca Pacioli in Italy (1494) and of Orontius Finaeus in France (1556).

"Geometriae Libri": The Way to Geometry[3]

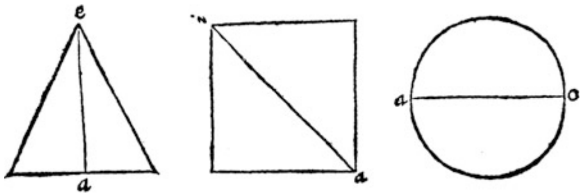
In Renaissance, Italy lost its mathematical "supremacy" to France. In fact, in the 16th century we find the interesting work of Pierre de la Ramée (Petrus Ramus). The text by Ramus (1569—we refer below to the edition of 1636) should be about practical geometry. But, even though Ramus presents more drawing and work instruments than Fibonacci, he is also a philosopher and an educator and explicitly criticizes the presentation of Euclid.

Ramus writes in the preface that "geometry is the art of measuring well" (*geometria est ars bene metiendi*). To measure well it is necessary to consider the nature and "affections" of everything that is to be measured: to compare such things one with another, to understand their reason, proportion and similarity… This aim of Geometry appears much more beautiful when one observes "astronomers, geographers, land-meters, sea-men, engineers, architects, carpenters, painters, and carvers" in their description and measurement of the "starres, countries, lands, engines, seas, buildings pictures, and statues or images".

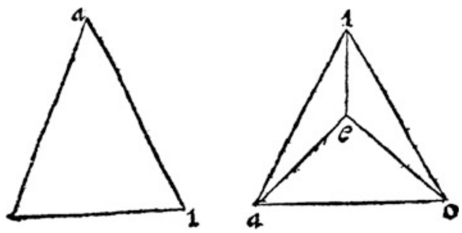
[3]*The way to geometry* is the title of the English translation of Bedwell (1636) that we use here for the English quotes.

Ramus starts with an extended *presentation* of geometric entities and figures through drawings, measures and simple constructions. We could call this an *observational* geometry, through which the reader becomes acquainted with the figures and their properties. We can say that in Ramus we find the concept of the different *levels* of geometric understanding. To reach his aim, Ramus also presents *unusual* objects and relations. Some examples (Book 4):

> 6. [The Diameter is a right line inscribed within the figure by its center]. The diameters in the same figure are infinite.

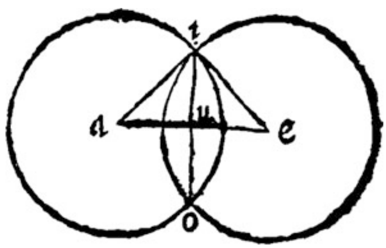


> If a figure has all equal diameters it is a circle.
> 11. A prime or first figure, is a figure which cannot be divided into any other figures more simple then it selfe.



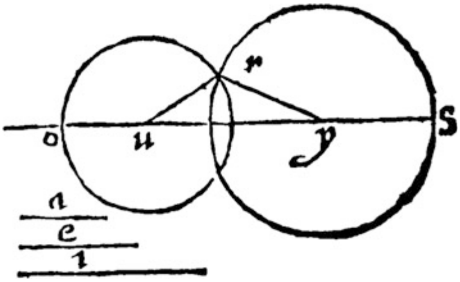
Ramus introduces drawing instruments (ruler and compass), then, also referring to previously mentioned properties, he states:

> 12. If two equal peripheries from the ends of a right line given, doe meete on each side of the same, a right line drawne from those meetings, shall divide the right line given into two equall parts (Book 5).



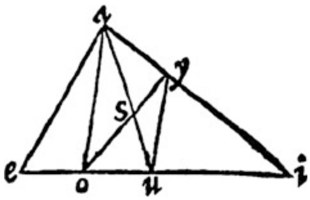
Ramus explains that the segment *ae* of the figure is divided into two equal parts by *io*, using the equality of triangles. Then he shows how to construct a triangle given three sides.

10. If of three right lines given, any two of them be greater than the other, and peripheries described upon the ends of the one, at the distances of the other two, shall meete, the rayes from that meeting unto the said ends, shall make a triangle of the lines given (Book 6).



In all cases Ramus explains his constructions, he proves that they work, linking the theoretical to the practical aspect. The text is full of drawings and of properties of the figures that we normally don't find in textbooks. He brings the reader to observe in an active way; and he supplies the proofs. We also find problems about the partition of fields, as in Fibonacci:

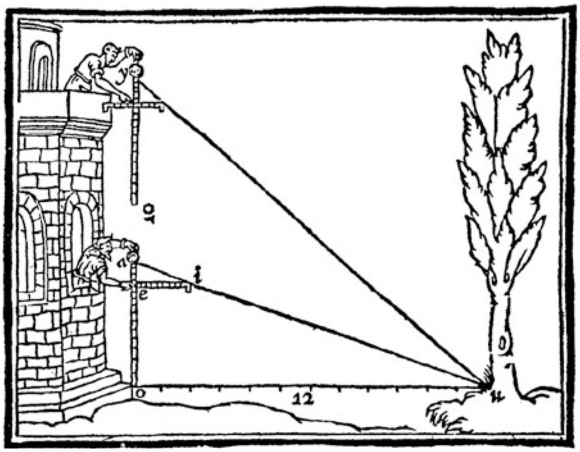
10. If a right line be drawn from the top of a triangle, unto a point given in the base (so it be not in the middest of it) and a parallel be drawn from the middest of the base unto the side, a right line drawn from the toppe of the said parallel unto the sayd point, shall cut the triangle into two equall parts (Book 7).



We omit the proof, which is the same as in Fibonacci.

Then Ramus arrives to the "classical" problems of practical geometry, giving rules (and instruments) for the measuring of segments (Book 9, the quotation refers to the lower part of the figure):

7. If the sight be from the beginning of the Index right or plumbe unto the length, and unto the farther end of the same, as the segment of the Index is, unto the segment of the transome, so is the height of the measurer unto the length.



> Let therefore the segment of the Index, from the top, I mean, unto the transome be 6. parts. The segment of the transom, to wit, from the Index unto the optique line be 18. The Index, which here is the height of the measurer, 4. foot: The length, by the rule of three, shall be 12. foot. The figure is thus, for as *ae*, is to *ei*, so is *ao*, unto *ou*, […] for they are like triangles.

Up to this point, Ramus' text is concerned with the *measuring of lengths*. The way to measure the *area of a rectangle* by dividing it in unit squares is shown only after a long chapter on quadrilaterals and their properties.

In Ramus' text, measurement is still the main aim of geometry, but the activity of measuring is preceded by a very interesting and original part of observation, a "play" with figures. The theorems that are proved mainly belong to the classical tradition, but their proofs are strongly supported by geometric constructions, and the use of drawing is added to the practical activity of measuring by means of proper instruments.

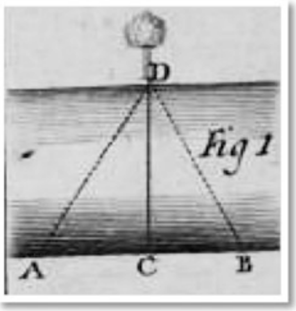
"Éléments de Géométrie": Geometric Constructions

In 1741, again in France, Alexis Clairaut wrote his *Éléments de Géométrie*. His first chapter is about the *measurement of fields*; nevertheless Clairaut was *not* interested in teaching a practical geometry. With Clairaut we see a shift from measurement as a goal to measurement as *a means* to teach geometry via problems. This is seen by the fact that the part about measurement *doesn't contain numbers*; there is only a hint at the necessity for a comparison with a known measure (See Clairaut, 1743).

The aim of Clairaut is to solve a problem "constructing" the elements he wants to measure. The focus is on the *process* of constructing and in a *narrative* method.

> A person placed on *D*, on the bank of a river, wishes to ascertain how far it is from the other bank *AB*. It is clear, in this case, that to have this measure, it is necessary to take the shortest of all the right lines *DA*, *DB*, etc., which may be drawn from the point *D* to the right line

> *AB*. Now it is easy to observe that the line wanted is *DC*, which must not lean to the line *AB* more in the direction of the right hand than of the left. On this therefore, which is called a perpendicular, we must move our known measure […] We are therefore under the necessity of finding a method of drawing perpendiculars (p. 2).



Clairaut shows other cases in which a perpendicular is needed, for instance, to draw a rectangle. And then he goes on with the construction:

> The point C might be found by repeated trials, but this method would leave the mind unsatisfied […] Take a common measure, a rope line, or a pair of compass with a certain opening, according as you may going to operate either on the ground or on paper…

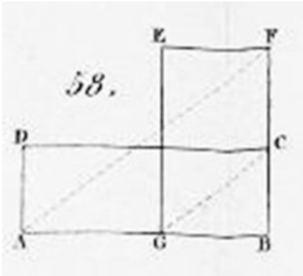
Even if a more detailed part has been omitted, this quotation shows the precision of the description. There is no definition of perpendicular, but the term is clear when it appears. We can note here the importance of the "moment" of learning, mentioned by Quintilianus.

Clairaut's text doesn't contain proofs, but constructions and arguments. The third part of the text also presents concrete models for space geometry.

Numbers appear rarely in Clairaut's treatise: mainly to show that the area of a rectangle is given by the product of its base by the height (dividing the rectangle in horizontal stripes whose height is one, and then each stripe into unit squares). The area of a triangle is then half the product of its base and height, because "we may easily perceive that this figure [the rectangle] transversely divided by the line *AC*, which is called a diagonal, resolves itself into two equal triangles" (p. 12). Clairaut seems not to be worried, as Fibonacci was, of showing that the property holds for all triangles.

The second part of the text is more *rigorous*, as Clairaut himself asserts, not because there are proofs, but because the allowed instruments are only a ruler and a compass. Clairaut continues via his problem-solving method to transform, for example, a rectangle into another rectangle with equal area.

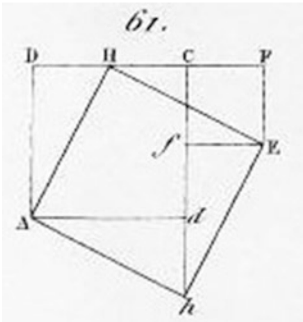
> Let it now be proposed to change the rectangle *ABCD* into another, which shall have the same surface but with the altitude *BF*. […] the new rectangle (the altitude of which is to be greater than *BC*) must necessarily have a base less than *AB*; […] All therefore we have to do is to divide the line *AB* in such manner that *AB* shall be to *GB*, as *BF* to *BC*. This (according to Part I, XLI) will be done by drawing the line *FA*, and, from the point *C*, the line *GC* parallel to *FA* (p. 68).



Clairaut introduces proportions and gives numerical examples.

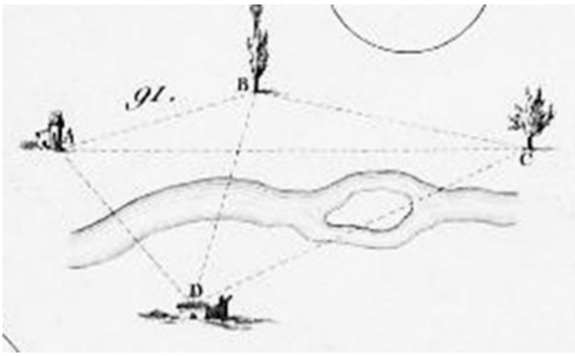
To sum two squares "we can transform the smallest square into a rectangle whose height is the side of the other square, so obtaining a new rectangle".

To sum two squares in order to obtain another square, we can work on the decomposition of the two squares, as in following figure, where *H* is taken so that *DH* = *CF* (and consequently *HF* = *fh* = *DC*) (p.76).



So the theorem of Pythagoras comes to mind when speaking of equivalent figures and of comparison of figures (though Clairaut doesn't explicitly mention Pythagoras).

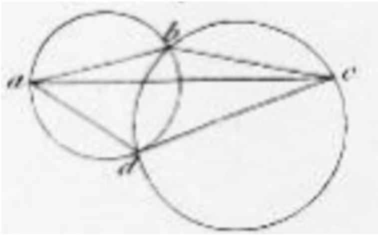
Clairaut doesn't make use of the classical theorems of Euclid, but he does prove the theorems concerning the angles in a circumference. He then goes back to problems of land-surveying; this has the purpose of using properly the protractor so as to construct a similar figure on the paper: "Knowing the distances between the given points *ABC*, we want to find their distances to a fourth point *D* from which the three are seen" (Part 3, n. 91).



Then two circumferences can be constructed, which have as a periphery angle the angle *adb* and the angle *bdc*…

This application of the theorem about the periphery angles is one of the few challenging problems of Clairaut's text.

The challenges of Clairaut are not in solving difficult problems, but rather in the request for the learner to follow him in the process of the solution to a problem, by paying attention to the needed construction.



Clairaut's success came a century after his text was written. In 1836 it was translated for use in the Irish national schools (Clairaut 1836)[4] and was reprinted in French in 1852 and officially adopted. It was also used in Italian Technical Schools (which correspond to the first three years of technical instruction) till the beginning of the 20th century.

"Anfangsgründe der Geometrie": Drawing, Constructions, Proofs

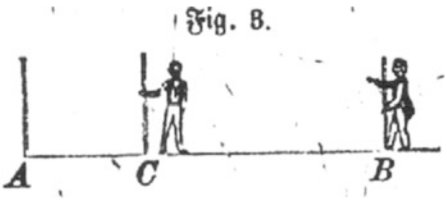
In 1846 Franz (Ritter von) Mocnik wrote a text that had many reprints and adaptations up to the beginning of the 20th century. It was used in the Austro-Hungarian territories, including the Northern part of Italy before the Italian unification, and was mainly intended for technical and professional schools (see Mocnik 1873). It is another text for the practical application of geometry, but it is now a textbook aimed at an extended school-system; and as a result pedagogical theories start to emerge.

Mocnik divides his text into a first part called *Formenlehre*, which includes free-hand drawing, some hints about projective geometry and practical measurements, and a second part called *Grundlehre*, which includes constructions with ruler and compass (as in Clairaut) and also proofs. *Formenlehre* reminds us of the educator Johann Heinrich Pestalozzi, who was particularly successful in Middle- and North-European countries, and who supported an initial approach to teaching based on intuition. Projective geometry is included in *Formenlehre* because it describes the way we see.

[4]This is the translation that we used for the quotes of the first two parts.

Mocnik starts with practical indications on how to represent points, on how to draw a line by free-hand or with a ruler. In fact, he gives many exercises that require free-hand drawing. There are also problems applied to land-surveying, such as:

> If we want to plant a pole, in a field, that must be aligned with two other poles *A* and *B*, we need to be behind pole *B* while an assistant goes with pole *C* to where the pole should be planted. Then we can indicate to him with our hand where he should move until we can see that pole *C* is aligned with *A* and *B* (p. 9, translated by the author).



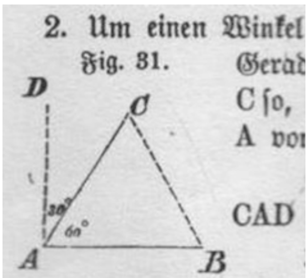
The concept of distance is primitive, and there is an exercise that asks, for instance, to draw five horizontal lines at equal distances; then Mocnik suggests to compare approximately the length of two segments and to draw their sum or difference (there is no reference to numbers). Also approximate multiples or sub-multiples of a segment have to be found by drawing them.

Some information is given about new and old unit measures, and about measuring instruments. The angle is introduced as the difference between the inclination of two straight lines, and the perpendicular [to a line *AB*] is defined, as in Clairaut, as a line *which must not lean to the line AB more in the direction of the right hand than of the left.* But Mocnik also adds that it will form two equal angles with the line *AB*.

This geometric description and others (such as the diagonals of a square) help pupils to understand the concept and thus be able to draw perpendiculars free-hand. In a field a perpendicular has to be drawn using a special instrument.

Measuring instruments are only introduced after the concept has been explored through free-hand drawing. But the use of instruments is not an aim, their use is not often suggested. For instance, after the introduction of the protractor, Mocnik shows how to draw a 60° angle (p. 30):

> We take a segment *AB*, and a point *C* whose distance from *A* and *B* should be as the distance from *A* to *B*. Then we draw *AC*. *BAC* = 60°. If we then draw the perpendicular *AD* to *AB*, then *CAD* = 30° (translated by the author).

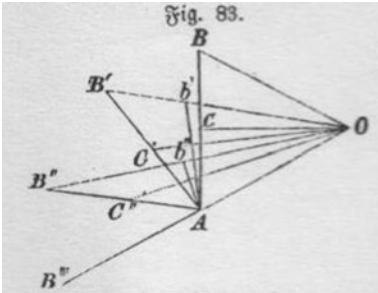


Mocnik also gives a method to "construct" our own protractor for angles less than 16° (approximating the values of the tangent), but he also suggests to guess the value of an angle.

Mocnik doesn't explain all the rules and methods that he gives. Nevertheless we cannot say that his practical geometry follows the "medieval" meaning of the term. The suggested work aims at giving the pupil a familiarity with the topics. In a certain sense it has something in common with Ramus. And in fact Mocnik slowly increases the level of rigor.

He describes solids in space and gives the first elements of perspective, for instance:

> If a line [a segment] is rotated from the position *AB*… around *A* to the positions *AB*', *AB*'', *AB*''', it can be seen under different angles and with different lengths. The apparent length of the segments is obtained when, from point *A*, we draw—inside the corresponding angle of view—a perpendicular to the central ray (p. 64, translated by the author).[5]

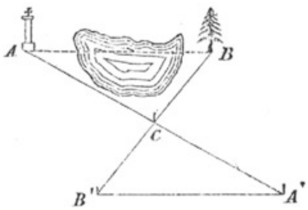


Congruent triangles are, in this first part of the text, simply triangles that have three equal sides and three equal angles. Mocnik suggests drawing them using *motions*. In the part concerning *Grundlehre*, the first proof is about the equality of two triangles that have three equal sides. Indications are given on how to construct, with ruler and compass, a triangle that is equal to another one, and then how to construct a triangle given three elements.

Theorems are always followed by exercises that require constructions. We find practical applications of the congruence of triangles (p. 93):

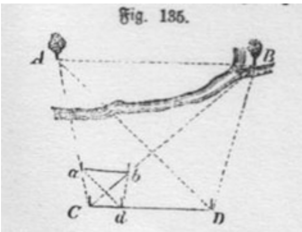
> To determine the distance of two points in a field, if this cannot be measured directly due to an obstacle between them, but it is possible to measure the distance from a third point to both…

Fig. 119.



[5]The drawing doesn't seem very precise, but the description is correct.

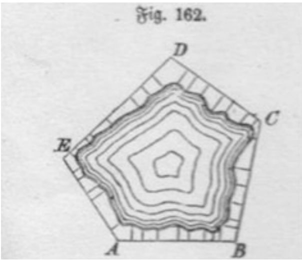
As well as practical applications of similarity (p. 104): “To determine the distance between two un-reachable points” you have to fix two points C and D and measure the angles CDB…



Only at this point does Mocnik introduce the measurement of areas. The area of a square is determined as in Clairaut; Mocnik first sees the figure as subdivided in rectangles, and then in unit squares. He proposes many numerical examples for this rule.

The theorem of Pythagoras too is proved in the way of Clairaut, but Mocnik also proposes to draw the figure used for the proof (see Sect. “Éléments de Géométrie”) on paper and then to cut its various parts in order to re-obtain the considered squares.

Again we find practical problems, such as to represent parts of a land on paper. Here Mocnik also describes instruments used by surveyors.



And again we come back to theorems. Mocnik proves those concerning the circumference and also conic sections and geometry in space.

“Lehrbuch der Elementar Geometrie”: Geometric Transformations

In the second half of the 19th century, many texts tried to modify the introduction to geometry by also including “new geometries”. *Geometric transformations*, in particular translations, were used to introduce the concepts of straight line and parallelism, and rotations were used to introduce the concept of angle and perpendicularity (as in Méray 1874). In this period, new topics such as conic sections and analytical geometry made their first appearance in the school curriculum.

Among those new texts we find a text by Henrici and Treutlein (1881–1883). It is not a book about practical geometry, even though it contains a part explicitly devoted to practical geometry. The extended use of *geometric transformations* is linked to movement and to geometric constructions: the authors state in the preface that it is important that the pupils concretely perform the transformations in the classroom using models.

In fact the importance of practical geometry is not so much in its use for applications, but rather in its educational value; in particular the interpretation of the term *practical* as "to be performed concretely" is the meaning that it mainly acquired in the 1900s.

The text of Henrici and Treutlein is a text of synthetic geometry for the Gymnasium (for the Tertia, that is for pupils older than 13), with axioms and theorems. But it presents relevant differences from the Euclidean text. The authors state that they pursue "a logical order of the concepts rather than of the theorems". In particular the subject is developed according to the kind of transformations that are necessary for the proofs.

Here the use of geometric transformations is more relevant than in Méray. At first, figures with a centre are introduced, and the central symmetry is used to define parallel lines (p. 23). To *draw* parallel lines the use of a set square is also suggested (p. 21).

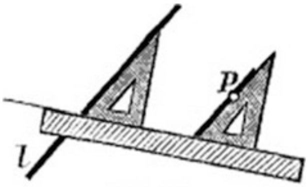


Fig. 39.

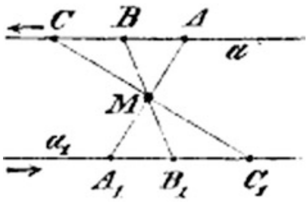


Fig. 43.

A *new kind of proof* can be performed using geometric transformations. Let's see how central symmetry is used to prove that the medians of a triangle are concurrent (p. 36):

If two of them, AA_1 and BB_1, meet in S, we can rotate the segment CS around B_1 in AB_2 and around A_1 in BA_2, so that CS is parallel and equal to AB_2 and BA_2. Also AB_2 and BA_2 can be superimposed and the centre of the rotation will be S. The three parallel lines CS, AB_2, BA_2 will cut equal segments on AB, on AA_2 and BB_2 [so CS is a median].

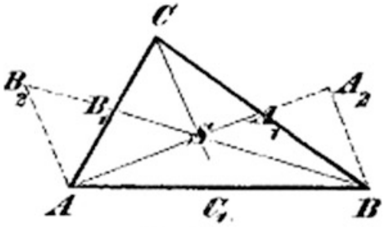


Fig. 77.

A *new kind of theorems* is also possible.

For instance, after the introduction of translations and rotations, the authors show how to find the transformations that bring two equal triangles to coincide (p. 40).

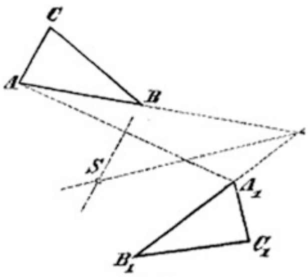


Fig. 84.

After the circle is introduced, a drawing with ruler and compass can be performed, and it is a relevant part. A *method* is given on how to face problems of constructions. At first, we find the usual problems that require the construction of perpendicular or parallel lines, as we can find in Clairaut, but we soon arrive at more complicated problems, such as drawing tangents to a circle under certain conditions or drawing the common tangent to two given circles (p. 64).

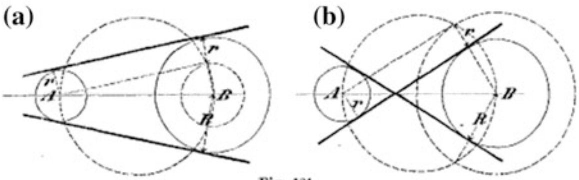


Fig. 121.

Areas are introduced with reference to Euclid's Book II (geometric algebra), to the proportions of the area of rectangles of equal bases, to equi-decomposition (seen also as equivalent transformation, p. 93).

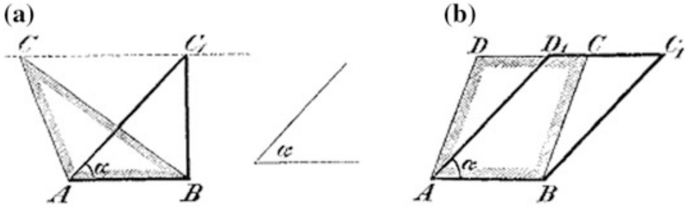


Fig. 178.

The theorems of Euclid and Pythagoras are thus proved with reference to equivalent transformations, and not to proportions. This chapter is about the "transformations of figures" and contains exercises similar to those of Clairaut in his analogous chapter about "comparison of figures". But there are also problems of "divisions of figures" that are exactly like those of Fibonacci in his "division of fields", even if they have now lost their practical origin. For instance:

"To divide a triangle in *n* [3] parts from a point on a side" (p. 96).

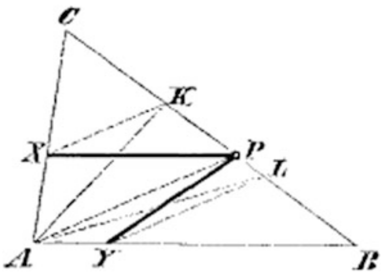


Fig. 184.

The solution is suggested by the figure.

In the appendix of the text by Henrici and Treutlein, the measurement of areas is introduced by giving the rule to calculate the area of a rectangle; this procedure is always that of Clairaut or Mocnik. We now find the *first numerical examples* of this text.

In the second volume similarity is introduced, and also elements of projective geometry. Besides elements of prospective representation, as we find in Mocnik, there is a more extended part containing topics such as harmonic points, pole and polar, projective correspondences, conic sections. Problems of measurement are taken up again and solved by algebraic means.

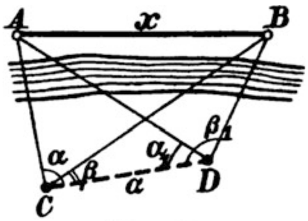


Fig. 162.

Problems of *practical geometry* are treated as an application of trigonometry. For instance, the problem that Mocnik solves with the help of similar representation and concrete measurement is solved by means of trigonometric formulas that lead to the value of x (p. 157):

$$x^2 = \left(\frac{a\sin\alpha_1}{\sin(\alpha+\alpha_1)}\right)^2 + \left(\frac{a\sin\beta_1}{\sin(\beta+\beta_1)}\right)^2 - 2\cdot\frac{a\sin\alpha_1}{\sin(\alpha+\alpha_1)}\cdot\frac{a\sin\beta_1}{\sin(\beta+\beta_1)}\cdot\cos(\alpha-\beta)$$

Measuring is no longer the aim of teaching geometry, as it was for Fibonacci or Ramus; nor an "excuse", as it was for Clairaut (with the only exception of the applications of trigonometry). Numbers have a very small role. The aim of Henrici and Treulein is the comparison of figures by means of transformations. But the concrete references to the operations of transforming and the geometric constructions place this book in a new stream of synthetic "practical" geometry that evolved in the succeeding decades.

Experimental Geometry

At the beginning of the 20th century an international reform movement started, which aimed at introducing modern topics, such as analytical geometry and calculus, in secondary school, and which also proposed new methodologies in mathematics teaching. As to geometry, a practical-intuitive approach was suggested. In many countries new syllabi were established. Borel (1905) wrote a text in accordance with the new French syllabi of 1905 (see Nabonnand 2007). The various reprints of Henrici and Treutlein's text seemed to be suitable for the German syllabi of 1907 (Becker 1994).

The *experimental geometry* of John Perry had a major influence at an international level. Perry (1900) emphasized the educational value of experimental procedures in the first approach to Euclidean geometry. He thought that a major part of elementary geometry had to be assumed as primitive and that the subject matter should be taught with reference to its utility as well as being interesting to the pupils (Howson 1982; Barbin and Menghini 2013).

The text of *practical geometry* by Harrison (1903) was written according to the proposals of Perry and even presented an appreciation of Perry himself in the preface.

Harrison states in the preface that many of the British schools and colleges are equipped with *laboratories* in which "experimental work involving quantitative measurements can be carried on as part of the ordinary school course", and that it is "coming to be recognized that elementary mathematics should be taught in relation to such work".

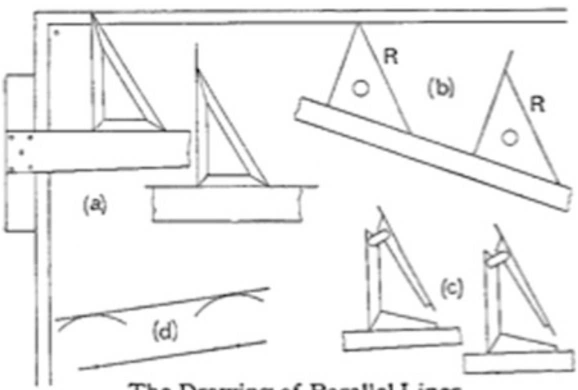
The syllabus that the author cites at the end of the book (Science Subject I, Board of Education, South Kensington) states that

> This subject [practical plane and solid geometry] comprises the graphical representation of position and form and the graphical solution of problems. […] In the elementary stage the main object of the instruction will be to familiarize the student with the fundamental properties of geometric figures and their applications […] It is not intended that the student shall follow the Euclid's sequence […]. In the Advanced stage […] the subject will be developed in its applications [Engineering and physical group…].

We can see that the advanced stage doesn't propose Euclidean Geometry.

Harrison starts with a sort of definition of points, lines and surfaces which is a mixture of Euclid's definitions, Ramus' drawings and Mocnik's representation of points. The aim of drawing points is to explain how a point can be seen in a drawing.

He then gives a detailed description of drawing instruments and their uses, and explains how to draw parallel and perpendicular lines using various kinds of instruments (ruler and set square, tee square and clinograph, p. 29).

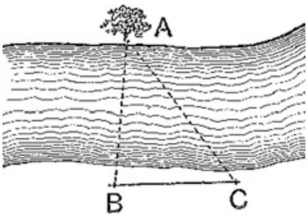


The Drawing of Parallel Lines.

Harrison gives a long explanation of how to measure angles, and he also defines the basic trigonometric functions and explains the trigonometric tables (we can thus understand that the considered level is not a primary or beginning secondary school level).

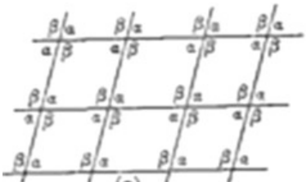
One of the problems is, as was in Clairaut, that of determining the width of a river. The suggested method is different (please note that Harrison doesn't suggest to work in scale, so the solution is not very practical…)

The figure shows how the width of a river could be ascertained by a person on one bank. He might select and measure a base *BC*. Then note some conspicuous object *A* on the remote bank, and by a sextant or other instrument measure the two angles *ABC* and *ACB*. The triangle *ABC* could then be plotted, and the width measured (p. 61).



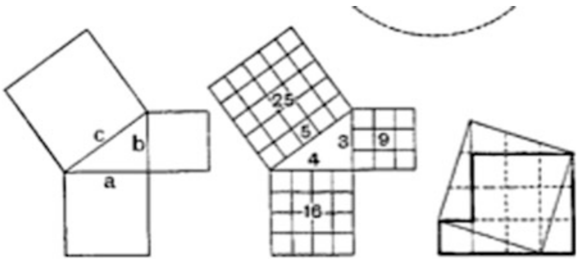
Defining a parallelogram Harrison states:

A parallelogram is a quadrilateral in which opposite sides are parallel. Thus in fig. *c* the two systems of parallel sides give rise to a series of parallelograms. From the figure it is evident that opposite angles of a parallelogram are equal. And by measurement the student will easily satisfy himself that opposite sides are also equal (p. 42)



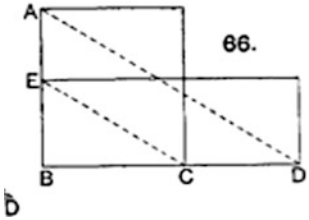
No further explanation is given.

Again through measurement or by paper cutting the student can verify the *Pythagoras theorem* (p. 47).



Harrison uses the same drawings as Fibonacci to show that the area of any plane figure may be measured by finding how many unit squares would be required to cover it (pp. 77–79).

Another problem that we have already found in Clairaut is “To reduce a given rectangle *AC* to an equivalent rectangle on the base *BD*”.



And its solution, different from Clairaut, is without explanations: "Join *DA*. Draw *CE* parallel to *DA*. Draw the rectangle ED, which is the one required" (p. 83).

The same kind of "proof" is given for other theorems, such as "The angle in a semicircle is a right angle":

> Verification: Draw any semicircle, diameter *AA*. Set out several angles *ABA* in the semicircle. Verify that in all cases the angle *ABA* is 90°.

Harrison introduces also vectors, their representation, sum and difference. Practical geometry in space is about methods of representing solids, mainly by projections on the coordinate planes. These exercises are definitely not elementary technical drawing exercises. They require, for instance, the use of trigonometry to calculate angles and measures.

Not all English books on practical or experimental geometry are of the same kind; in the text by Godfrey and Siddons (1903) the part devoted to measurement is shorter than in Harrison, and moreover it presents exercises which require an idea or a form of explanation (Fujita and Jones 2004). Anyway, the main focus of practical geometry in that period is on measuring or calculating measures, with some technical drawing; it is quite obvious that objections would eventually be raised about this methodology:

> Most recent text-books of geometry contain an introduction on practical geometry. While presupposing a short preliminary course of this nature, we have preferred to leave it to the teacher to devise himself. In this direction we think that the recent reforms have gone too far, and we feel sure that, as regards secondary schools, it will be necessary to retrace our steps. Too much time spent on experimental and graphical work is wearisome and of little value to intelligent pupils. They can appreciate the logical training of theoretical geometry, while experiments of far greater interest can be made in the physical and chemical laboratories (Preface, Davison 1907).

Intuitive-Introductory Geometry

Notwithstanding these criticisms, experimental geometry has influenced, from the beginning of the 1900s, texts of *intuitive* or *introductory geometry* devoted to the lower school grades. These texts were influenced not only by experimental geometry, but also by the whole stream of texts of practical geometry (Menghini 2010). An earlier intuitive-experimental approach was considered a good aid for students to overcome the difficulties caused by the logical deduction of Euclid's textbook.

So we find *free-hand drawing* (Veronese 1901) and *drawing instruments* (Borel 1905), and parallel lines introduced by means of central symmetry or translation. *Geometric transformations* are considered not only useful to introduce simple concepts, but also to compare segments and figures, as they were carried out practically. The equality of triangles is established through the possibility of constructing them (as in Clairaut), and *measurement* and the use of *numbers* become widespread.

We also find the use of *paper folding* and *concrete materials.* For instance: two successive foldings of a piece of paper are used to introduce the concept of perpendicularity (Frattini 1901; Borel 1905); in order to prove that *the diagonals of a parallelogram bisect each other*, a parallelogram is cut out from a piece of paper, filling again the resulting empty space after a rotation of 180° (Frattini 1901); to know the sum of the angles of a triangle, the corners of a triangle drawn on paper are cut and placed next to each other to check that they form a straight angle (Amaldi 1941).

Emma Castelnuovo, who was also largely inspired by the approach via problems of Clairaut, uses simple tools, such as a folding meter, to show how to modify a quadrilateral into a different one and to analyze the limit situations (Castelnuovo 1948).

Developments of Practical Geometry in the 20th Century

Axioms for Measure

The textbook *Basic Geometry*, by George David Birkhoff and Ralph Beatley, was published in an experimental edition in 1933, and officially edited in 1941 (Birkhoff and Beatley, 1941). The text starts with many exercises that require the practical use of a scale and a protractor: measuring of lengths and angles, and verifying rules about their addition and subtraction. Numbers are very important from the beginning. It would seem to be a text about practical geometry, but *measuring is not an aim*. The aim is to help pupils to accept the *Principles*. The first is about:

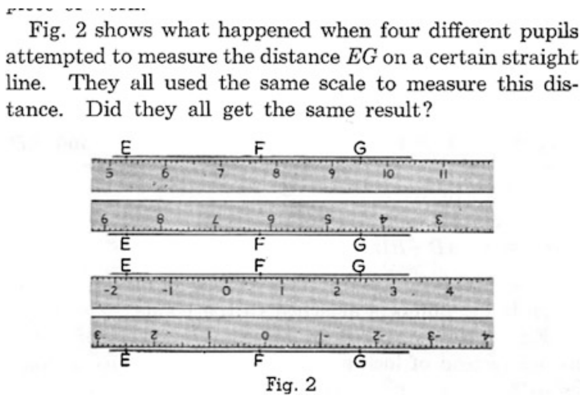


Fig. 2 shows what happened when four different pupils attempted to measure the distance EG on a certain straight line. They all used the same scale to measure this distance. Did they all get the same result?

Fig. 2

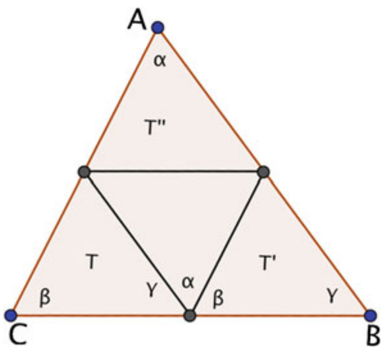
Line measure: the points of any straight line can be numbered so that number differences measure distances (p. 40).

And the third about:

Angle measure: All half lines having the same end point can be numbered so that number differences measure angles (p. 47).

Two more axioms state that "there is one and only one straight line through two given points" and that "all straight angles have the same measure", while the fifth axiom corresponds to a criterion of *similarity of triangles*. An example of the application of the axioms is the proof of the theorem about the *sum of the angles of a triangle*:

Principle 1 guarantees that the sides of the triangle *ABC* can be divided by two, while Principle 5 guarantees the similarity of *T*, *T*', *T*" to *ABC*, and thus the similarity of *ABC* and the internal triangle. So… (see figure).

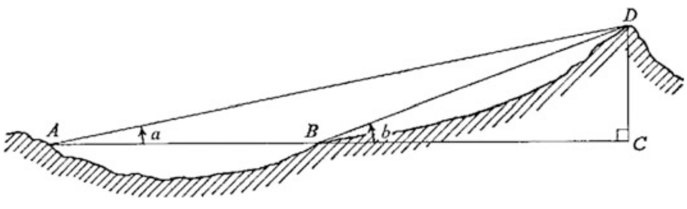


The ideas of Birkhoff slowly found their way into American teaching, and inspired many other textbooks on geometry, such as those produced by the School Mathematics Study Group.

Old Applications and New Theorems

In the period of the New Math reforms in the 60s, the British School Mathematics Project (1970) still presented elements of practical geometry. A chapter was devoted to the measuring of an angle, to the use of the protractor, and to applications such as:

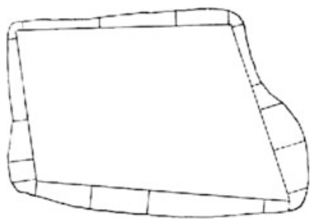
> To measure the height of a hill, land-surveyors measure the angles *a* and *b* from two different points *A* and *B* which are at the same height, and whose height and distances are known. If *AB* = 1000 m, *a* = 19° and *b* = 36°, draw a diagram in a scale 1 cm–200 m and find *DC*. If *A* and *B* are at 200 m height, what is the height of the hill? (p. 82)



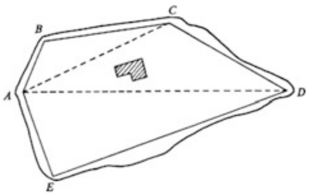
Pupils are supposed to find the answer by measuring, as was seen in exercises of Clairaut or Mocnik.

After chapters about polygons and polyhedra, approximations of areas by tessellation, observation of the symmetries of a figure, and paper folding, a chapter is again devoted explicitly to land-surveying. The aim is not only to apply what has been studied in the previous chapters, but also to introduce new concepts. Here the old applications of practical geometry are the basis for exercises in the new practical geometry. There is also the suggestion to do real exercises in the open air.

These exercises mainly require to draw on paper a piece of land by measuring (or taking as given) certain distances and angles using similarity (p. 298); *irregular* forms (p. 300) can be drawn by using the way proposed by Mocnik.



So we find many of the topics that we have met from the beginning of practical geometry.



The book also presents *geometric transformations* and *new theorems* based on their *composition* (as we have seen in Henrici and Treutlein) and on their *invariants*.

Conclusion

For Fibonacci, instruments and their use to measure distances were taken for granted. With the exception of a short part about the unit measures used at that time, his text is mainly concerned with rules for calculating areas and with theorems that can help in comparing the areas. The rules arise mostly from a generalization of numerical examples, and numerical values are given to the considered segments in order to verify theorems. These simplifications are an aid for the reader, who at the same time is challenged with concrete problems that require finding a distance (using similarity), or with more abstract problems that require the division of a field, given certain conditions. For the solution to these problems, only an initial hint to allowed constructions is given.

With Petrus Ramus we find on the one hand an increase in the *activity* of measuring, through the explicit presentation of measuring and drawing *instruments*; but on the other hand this activity is postponed in order to *present* geometric figures, their properties, and to "play" with them.

We can see that the "puzzles" of the practical geometry of the Middle Ages and the "plays" of Ramus are characteristic of a *freedom* that is not in the Euclidean tradition. Euclid is absolutely not "playful".

Clairaut focuses again on activity, which is now aimed at *constructing* the geometric elements to be measured. As in Fibonacci, not only is measuring per se not important, but also the starting problems are not as important as the process of solving them. The role of the process of constructing is principally an *educational* one.

The educational role of constructions, of concrete activities, and of observation can also be found in the text of Mocnik, whose aim is again practical. Finding distances and areas is still a goal, but the use of measuring instruments is in the background, while theorems acquire importance.

The text of Henrici and Treutlein cannot be considered exactly along the same line of thinking, but it takes from practical geometry examples, problems and an idea of concrete activity, which is then applied to new objects such as the geometric transformations. Here we can see how practical geometry suggested a methodology that also influenced the more theoretical study at the Gymnasium.

In the *practical geometry* of the 1900s, we find something similar to Fibonacci. If Fibonacci verified the theorems by substituting lengths with numbers, Harrison verifies them by measuring. The use of instruments and the activity of measuring are central to Harrison; but we find nothing that can be considered an intellectual challenge to the pupil.

The *intuitive geometry* for the lower grades can be considered as a middle ground: those activities of measuring, of concretely performing transformations, of using instruments, can be seen as an introductory activity *before* starting with Euclidean deduction.

Another way of combining practical and deductive geometry is proposed in the book of Birkhoff and Beatley, where measuring must help pupils to understand the role and the meaning of axioms for line and angle measure. This book shows that a *deductive* geometry based on measure needs new axioms.

Avoiding axioms, the SMP proposes a curriculum that encompasses many of the topics that we have seen in the development of practical geometry: from the concrete problems of land-surveying, to drawing, to the *new theorems and proofs* based on geometric transformations. The proposals of SMP in the 60s show that practical geometry can have a role that goes beyond an introductory or intuitive presentation.

As we understand from the presented development, a practical-intuitive teaching of geometry requires an active role of the pupil, where reasoning is supported by action and by construction. We have also seen examples of rational teaching of geometry supported by practical elements. Even if the challenging problems proposed by the practical geometry of the late Middle Ages entered only occasionally in the successive geometry textbooks (with the exception of the *division of fields*), we can recognize the heritage of this geometry in a certain freedom in choosing problems and methods (including *intuition*). However the main methodological heritage lies in the shift of the meaning of *practical* from "useful for applications" to "to be performed concretely". This shift happened towards the end of the Middle Ages, even in texts whose aim was very concrete, through the *description* of the activity of measuring or of constructing, and the use of instruments.

Practical geometry can thus suggest alternative methodologies, not only for an intuitive introduction *before* rational geometry, not only for examples of useful applications, but also for a different way of teaching based on a conception of knowledge which includes the *practical process of knowing*.

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Research on Mathematics Classroom Practice: An International Perspective

Ida Ah Chee Mok

Abstract Research on Mathematics Classroom Practice encompasses very comprehensive themes and issues that may include any studies and scientific experiments happening inside the classroom, including consideration of the key agents in the classroom (the teachers and the students), undertaken with diversified research objectives and theoretical backgrounds. To a certain extent, seeking an international perspective provides some delineation of the topic. Studies will then focus on those issues already prioritised as of interest by existing international comparative studies and those issues seen as significant within an educational system. This lecture will draw upon the work of an international project, the Learner's Perspective Study (LPS), an international collaboration of 16 countries with the aim of examining in an integrated and comprehensive fashion the patterns of participation in competently taught eighth grade mathematics classrooms.

Keywords Mathematics classroom practice · Cross-cultural practice · Teaching strategies · Learning tasks · Student perspective

Introduction

Research on mathematics classroom practice encompasses very comprehensive themes and issues. Reviewing the abstracts of papers and research reports in key journals with the key words "classroom practice", a comprehensive range of presentations and discussions emerges, and the broad topic of "classroom practice" can be seen to include any studies and scientific experiments happening inside the classroom, including the investigation of the key agents in the classroom (the teachers and the students), and addressing diversified research objectives and drawing on a wide range of theoretical backgrounds. To a certain extent, seeking an

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international perspective provides some delineation of the topic. International research offers insight into possible explanatory frameworks within which differences and similarities between cultures can be seen as offering complementary features to supplement the understanding of the nature of mathematics teaching and learning practices in one's own culture. Studies will then focus on those established as being of legitimate interest, as informed by existing international comparative studies and by the perspectives of the educational system within which each researcher is working. This paper will draw upon the work of an international project, the Learner's Perspective Study (LPS), an international collaboration of 16 countries with the aim of examining the patterns of participation in competently taught eighth grade mathematics in an integrated and comprehensive fashion (Clarke et al. 2006a).

The promising design and emerging findings of the project have been well received in international conferences and by the research community in education, and the LPS research community has continued to attract additional international membership. Despite their progress being at different stages, the LPS researchers from different countries greatly complement each other in terms of findings and the development of new research agenda. Some of these are reported in the publications of the LPS book series (Clarke et al. 2006a, b; Shimizu et al. 2010; Kaur et al. 2013). The large body of complex data supports both the characterisation of practice in the classrooms of competent teachers and the development of theory. Participation in the LPS community provides valuable insights for understanding research on mathematics classroom practices as well as providing an opportunity for the sharing of international perspectives through the various project-specific international collaborations that form within the LPS community.

In this paper, I attempt to demonstrate the diversity of international perspectives available within the LPS research community by sharing some results taken from the first three books in the LPS series. However, I must admit that the picture will be limited to a certain extent by my personal experience and by the nature of my engagement in the project. The presentation will be divided into the following sections:

The Learner's Perspective Study,
Stories from the insiders,
Comparing lessons in different cultures,
Learning tasks, and
Conclusion

The Learner's Perspective Study (LPS)

Being in a familiar environment, one usually takes things for granted and may fail to see the characteristics of that environment as special or different (Runesson and Mok 2005). A fundamental belief in the Learner's Perspective Study (LPS) was that "international comparative studies are likely to reveal patterns of practice less evident in studies limited to a single country or community" (Shimizu et al. 2010, p. 11). A significant stimulus for comparative classroom studies was the seminal work undertaken in the TIMSS Video Study. The initial stage of TIMSS Video Study involved only three countries and was reported in the book *The Teaching Gap* (Stigler and Hiebert 1999) and later extended to seven countries in the follow-up TIMSS 1999 Video Study (Hiebert et al. 2003). The originators of the LPS project, Clarke, Keitel and Shimizu, were motivated to a significant extent by a perceived need to complement the survey-style approach characteristic of the work of Stigler and his co-workers with an in-depth approach that captured more of the perspective of the learner. The positioning of the research methodology for the Learner's Perspective Study (LPS) was that the design must be "sufficiently sophisticated to accommodate and represent the multiple perspectives of the many participants in complex social settings such as classrooms. Only by seeing classroom situations from the perspectives of all participants can we come to an understanding of the motivations and meanings that underlie their participation. Our capacity to improve classroom learning depends on such understanding." (Clarke 2006, p. 15) The design of the project centred on three key requirements: (i) the recording of interpersonal conversations between focus students during the lesson; (ii) the documentation of sequences of lessons, ideally of an entire mathematics topic; and, (iii) the use of post-lesson interviews with the teacher and selected students to identify the intentions and interpretations underlying the participants' statements and actions during the lesson.

Essential features of the LPS research design are (i) the on-site mixing of the images from two video cameras to provide a split-screen record of both teacher and student actions and (ii) the use of video-stimulated recall in interviews conducted immediately after the lesson to obtain participants' reconstructions of the lesson and the meanings which particular events held for them personally. The participants were invited to play back the lesson video and comment on lesson episodes which they see as important. A third camera recorded "corporate" student practices—that is, the practices common to the whole class group. Two students were interviewed after each lesson. Each teacher participated in three video-stimulated interviews and completed two substantial questionnaires before and after videotaping, as well as a shorter questionnaire after each videotaped lesson. Copies were also made of student written materials, textbook pages, and worksheets used in class. With regard to the teachers, classes, and classrooms being sampled: eighth grade lessons were recorded in three classrooms for each country, for a minimum of ten consecutive lessons for each class/teacher. This produced a data set of at least 30 lessons for each country. The particular teachers whose classrooms were studied had been identified by the local members of the research team, in consultation with colleagues and members of the local mathematics education community, as engaging in "quality teaching practice" consistent with local curricular emphases.

The research teams in the LPS community are now based in universities in Australia, China, the Czech Republic, Germany, Israel, Japan, Korea, New Zealand, Norway, The Philippines, Portugal, Singapore, South Africa, Sweden, the United

Kingdom and the USA (www.lps.iccr.edu.au/). The LPS database documents the teaching of sequences of lessons and provides a rich integrated data set generated from very diverse education systems, values and cultures. The dataset allows researchers to reconstruct accounts of lesson events by combining the video-records with the teachers' and the students' perspectives as expressed in post-lesson interviews. A continual generation of new findings and insight is accompanied by the addition of new international partners leading to the steady expansion of the dataset and even greater potential for cross-cultural analysis. While the international partnerships afforded by the LPS provides opportunities for contextualized understanding of the systems, issues and tensions from the perspectives of both insiders and outsiders, the international collaboration has also contributed to the creation of a new agenda for parallel and comparative analysis of classroom practice in mathematics classrooms internationally.

Stories from the Insiders

In the first book in the LPS series, *Mathematics Classrooms in Twelve Countries: The Insider's Perspective* (Clarke et al. 2006a), researchers from 12 countries were invited to write chapters based on the data within their own countries. The meaning of "the insider's perspective" was conceived as two fold in the preparation of the book. Firstly, the writers of each chapter are considered insiders as their analysis and writing are based on the data generated from their own cultures and school systems, and they voiced their analyses from that position. Secondly, the other "insiders" were the participants in the classrooms, i.e., the teachers and the students, and the LPS analyses have taken into account these "insider" voices in the dataset.

How teaching and learning take place inside the classroom is a complex system comprising different interacting components that include the teacher, the students, the curriculum, the learning materials and the interactions in the activities in the lesson. It is difficult to describe all details in full, but, in fact, the details in each aspect matter significantly. When each researcher constructed their insider's chapter, they looked into the LPS data collected from their home country and reported the analysis of an important aspect according to their own perspectives. The resulting reports in the book, taken as a whole, therefore do not represent a consensus on how classrooms should be viewed. More importantly, the differences represent what researchers from each community saw as relevant and significant. The complementary conglomerate vision created by the efforts of the international team gives a rich and culturally nuanced picture of classroom practice. Here, in this section, I summarize a few examples.

Keitel (2006) analysed "setting a task" in German schools. Keitel and her team analysed classroom practice in three LPS German schools to find out what kind of tasks were set, what were the differences in how the teachers set tasks, and how these differences (if there were any) might affect the students' learning of mathematics as a school subject and an important scientific and social enterprise. The team's basic assumptions were that tasks had a double function: they represented the means for teaching and assessing, and also provided a mechanism for securing objective standards of evaluation. In the words of Keitel (2006), "a strange amalgam of setting up various tasks for conflicting purposes developed. The teaching and assessing on one hand means that tasks are designed to develop performance abilities, to prepare for assessment, and to serve as means of assessment that evaluates these abilities" (p. 42). Keitel compared the lessons of two teachers G1 and G2. The teacher G1 regularly started with a series of tasks for the purpose of practicing and memorizing specific algebraic rules, and he told the students repeatedly that it helped them prepare for the regular assessment tests. Therefore, the students followed him and were accustomed to this kind of routine task. The students mostly worked individually with little collaboration. In one lesson the teacher introduced a non-routine question aiming to link algebra with geometry through a problem involving a proof of the algebraic expansion of the square of (a + b) via the display of the areas of squares and rectangles. However, analysis showed "the overwhelming emphasis on mastering routines and algorithms seemed to be extended to all parts of mathematics classroom practice in G1 and overshadowed even those events that actually broke with routines and tried to offer a new kind of insight and enlightenment" (p. 47). In contrast, the teacher G2 structured the lessons less formally. Although he also used formal routine tasks, his lesson plans allowed more improvisation, and he was more supportive of students' self-initiated activities and collaboration. Students in this class had more time for discussion in groups and for the presentation of their ideas and methods. In the students' interviews, they reported that they liked working together, and they felt that the collaborative activities were more demanding, as they had to find new ways on their own to solve the problem, and they appreciated this as "thinking mathematically" (p. 52).

"Seatwork" is an organizational segment in the lesson in the analysis of work in the Third International segment and is examined in terms of its frequency, length and pattern and process (Stigler and Heibert 1999). Using the definition of Stigler and co-workers (1999), seatwork activity refers to the period in the lesson when students work on assigned tasks independently or in small groups. Hino (2006) examined the teacher's support for individual students during seatwork, the role of seatwork in three Japanese classrooms (J1, J2, and J3) in the development of the lesson and its relationship to students' learning. In Japan, seatwork is called "Kikan-Shido" meaning instruction at students' desks that includes the teacher's purposeful scanning of the individual student's problem solving process (Shimizu 1999). The analysis by Hino (2006) showed that seatwork was important in Japanese lessons because the teacher used the students' work for a variety of purposes including: sharing and making use of their work, eliciting their mistakes, eliciting their puzzlement, eliciting opposing solutions, pointing out different solutions and giving explanation, pointing out difficulties and giving explanations, taking up students' questions and making their way of thinking visible to the group. For the Japanese lessons in the three classrooms, seatwork occurred before presenting the main content. This sequencing let the teacher support individual students by orchestrating the development of the lesson, drawing on the teacher's knowledge of the students' understanding obtained from observation during the seatwork activity. From the analysis of the students' interviews, seatwork activity was viewed as important for two reasons. First, it provided time to exchange information and opinions with their neighbours and to think about the problem together. Second, it provided a foundation for students' thinking and understanding in the later part of the lesson.

Wood et al. (2006) attempted to find out how the objectives of mathematics education reforms were realized in three US classrooms. The analysis of the Third International Mathematics and Science Study (TIMSS) 1995 and 1999 Video Study data (e.g., Jacobs et al. 2006 and Leung 2005) reported that the US lessons in the study remained "traditional." Despite these findings, Wood and co-authors (2006) argued that the analysis of the LPS data of the US classroom for the evidence of curriculum elements was essential for two reasons: (1) the three US classes (US1, US2, US3) in the LPS data contained features that were important in the realisation of the current curriculum standards in mathematics in the US, and (2) the LPS data consisted of a sequence of consecutive lessons that could provide greater detail about the nature of teaching in the US classrooms than could be obtained from the TIMSS Video Study. Moreover, an essential feature was that the data provided insights into both teachers' and learners' practices. The researchers analysed the data collected from three LPS US schools, to examine how the goals of the school mathematics curriculum were realized in these classrooms. In the analysis, the team applied a conceptual framework with two dimensions, student participation and student thinking, aiming to describe the differences among the three classrooms in terms of the relationship between these two dimensions. Close examination revealed that the students did work in groups and were given autonomy to check their homework and to clarify what they did not understand. However, the quality of mathematical experiences was generally focused on procedural solutions for the problems. The teacher spent the majority of the lesson in routine activities such as taking attendance and checking homework. Consequently, listening to students was not an essential activity, because a majority of the teacher's actions did not require knowledge of students' thinking. Illustrating with a class episode, Wood and co-authors described, "In this class, the teacher rarely provided mathematical reasons for the steps in the procedures she presented. Instead, when students encountered difficulties executing the procedures, she relied on the authority of the text (often explicitly as "they said") for definitions and created rules, such as "you can't do that", to justify the steps in a procedure" (p. 82). Based on their analysis, Wood and her co-workers further argued that "the gap between research on mathematics teaching and learning practices and current school instruction still exists after nearly two decades; it is clear from the analysis of this data that one reason is that the degree of change is more far-reaching than initially perceived" (p. 83).

Mok (2006) addressed the issue of teacher dominance by exploring the teacher's and the students' perspectives on their mathematics lessons as revealed in the Chinese data. Mok aligned the teaching in the lessons of a Shanghai classroom (SH2) with the teacher's self-evaluation of the lesson and the students' perspectives of the same lesson. The findings showed that the teaching had features consistent with the prevailing image found in the studies of Asian classrooms (e.g., Mok and Morris 2001; Park and Leung 2006); the teacher has a very influential role. The teacher showed a deep understanding of the subject and attempted to guide his students to understand the same level of detail. The teacher explicitly said that his teaching was not what usually done in traditional Chinese classrooms and that a central thesis was to provide students the opportunity to comprehend and to think. From the analysis of the lessons, it was clear there were opportunities for discussion, but the teacher exercised much control through the deliberate design of the classroom tasks. The teacher encouraged students to express their ideas in their own words, but at the same time he emphasized the use of standardized mathematical language. From the analysis of the students' interviews, the students showed a deep appreciation and respect for their teacher and the learning of mathematical knowledge, demonstrating a culture that the students would not challenge what the teacher did. The results showed a correspondence between the teacher's expectations and the students' expectations for the lessons, while there was a mismatch between the teacher's espoused belief and his pragmatic practice based on his interpretation and synthesis of different teaching models.

Kaur et al. (2006) studied the role of the textbook and homework in two Singapore classrooms. Textbooks in Singapore are produced commercially and compete for official adoption by the Ministry of Education (MOE). Therefore, the content of textbooks very closely matched the mathematics syllabus reflecting the expectations of the national standards. Ulep (2006) studied how a Philippines teacher used a motivational strategy called "Ganas", in which the teacher gave additional points to the students who first finished the exercises with all answers correct. The case study demonstrated how improvisation in a teacher's strategies made an impact on the classroom practice. Williams (2006) studied how selected Australian lessons might have supported creative mathematical thinking. Park and Leung (2006) studied the characteristics of the Korean mathematics classroom and delineated the dimensions of variation in the learning experiences (Marton et al. 2003). Furthermore, interpreted the results in terms of the underlying cultural values that shared with other East Asian countries.

Comparing Lessons in Different Cultures

Comparison of cases among different cultures and backgrounds is the principal approach in the analyses reported in the second published LPS book. *Making Connections: Comparing Mathematics Classrooms Around the World* (Clarke et al. 2006b). Aspects of a familiar education system can easily be taken for granted losing sight of the underlying value system. When comparisons of education systems at different levels are made, some characteristics are often seen as less or more prominent because of the comparison, bringing about a better understanding of the affordance and limitation within a culture and among cultures. Therefore, one of the objectives for international comparison is to learn from other systems about teaching and learning to enable reflection on one's own system (Runesson and Mok 2005).

The fundamental questions addressed are what to compare and how to compare. Clarke et al. (2006b) argues that the complementary approach employed in the LPS research design can be used to study lesson structure at three levels: whole lesson, topic and lesson events. In the first half of the book, "Lesson Events" (regularity in the form and function of types of key lesson activities/events from which lessons are constituted) are used as an entry point for data analysis in relation to a variety of foci provided by different countries. The lesson events included: Beginning the lesson; Between desks instruction ("Kikan-Shido" in Japanese); Students at the front; Summary of the lesson ("Matome" in Japanese); and 'Learning task' lesson events.

Beginning of the lesson: Mesiti and Clarke (2006) examined classroom practices related to the beginning of the lesson over sequences of ten lessons from the LPS classrooms in the USA, Australia, Japan and Sweden. Purposeful selection was made to include data from culturally-different settings. This particular lesson event was defined as beginning from the moment the teacher first communicated with the whole class followed by the next ten minutes. The patterns of practice identified by the analyses represented coherent sets of actions regularly and/or effectively used specifically for the beginning of a lesson, possibly, but not necessarily, including the whole of the first ten minutes. Mesiti and Clarke identified a variety of components with different purposes for beginning of the lesson. They developed a coding scheme that include: the pre-education component (administrative, organizational, pastoral care); the review component (focusing or warm-up, recap or run-through); the instruction component; the student practice component; the student assessment component (diagnostic, assessment); and the correction component (whole class, independent).

Between desks instruction ("*Kikan-Shido*"): O'Keefe et al. (2006) investigated "Kikan-Shido" or "between desks instruction" in eighteen classrooms located in Berlin, Hong Kong, Melbourne, San Diego, Shanghai and Tokyo. In the Japanese lessons, the "between desk instruction" is given a special name in Japanese "Kikan-Shido" representing a significant component of the activities supporting learning. The analysis applied very detailed coding of the principal functions taken place in kikan-shido and showed a range of variation between the practices of different teachers in different classrooms in different cultures. For example, the Japanese teacher walks around the classroom, predominantly monitoring or guiding student activity, and may or may not speak or otherwise interact with the students. The teachers in the LPS Australian classrooms did a lot of scaffolding and the practice of kikan-shido was the implicit devolution of the responsibility for knowledge generation from the teacher to the student. Some US classrooms also did a lot of between desks instruction in the body of the lesson as part of seatwork.

Students at the front: Jablonka (2006) studied the students' activities when they were called to the front of the classroom, i.e., the side of the room on which the teacher's desk, the board, an overhead projector (OHP), a flip chart, or a screen was located. The selection of the six classrooms for analysis was based on the principle of maximizing contrast but at the same time keeping a reasonable basis for comparison. The classrooms included two from Berlin, two from Hong Kong and two from San Diego. The forms taken by this lesson event might include one or several students working at the front at the same time. Variations on the "student at the front" lesson event included: an extra chance to get the teacher's comments, solving a new task in public, publicizing work, explaining work, providing a division of labour between teacher and students. The students might be writing solutions on the board, presenting an account of completed work, showing products of group work, or assisting the teacher in a demonstration.

Summing-up ("*Matome*" *in Japanese*): Shimizu (2006) discussed the form and functions of the particular lesson event "Summing-up" (Matome) in the LPS classrooms in Australia (Melbourne), Germany (Berlin), China (Hong Kong, Shanghai), Japan (Tokyo), and the USA (San Diego). He first analysed the form and purpose of the event within the local contexts of the Japanese classrooms, followed by the comparison with corresponding events from the other countries. Matome in Japanese classrooms had the main features of teacher public talk, effective use of chalkboard and reference to the textbook. Matome was seen by teachers (and the researcher) to be indispensble in traditional Japanese classrooms for sharing and pulling together the students' solutions in the light of the goals of the lesson of the day. Students also perceived Matome as important in the Japanese classrooms. The analysis showed that the Australian teachers did not give a specific summary at the end of each lesson, but rather they tended to wait until the end of the topic before delivering a summary. In the German classrooms, the teacher did give some summary or provided some general comments on students' procedure, but it did not seem to be common for the German teachers to conclude the lesson by discussing or summarising retrospectively what students had learned during the whole lesson. The US teacher made summaries like the Japanese examples, but the summary often appeared at the end of each activity, instead of at the end of a lesson. In contrast, the Asian classrooms showed both commonalities and differences. Japan and Shanghai showed similar engagement of the teacher and students in Matome-like events. They made summary to the whole class regularly in each lesson, whereas the teachers in Hong Kong only gave summary of the lesson only occasionally.

'*Learning task*' *lesson events*: Mok and Kaur (2006) developed a definition for learning task lesson events (LT event). A learning task lesson event was defined as comprising not only the description of the task itself but also the actual lesson episode in which the teacher and the students were engaged in the task (that is, both the stated or written task and the subsequent social activity). The class organization might be in any format such as whole class discussion or group work. In their definition, Mok and Kaur differentiated between a learning task and a practice item. A learning task was intended to teach the students something new, and the sequence of learning tasks showed a coherent development of the object of learning, whereas, a practice item was mostly repetition of a previously taught skill. They compared 18 learning tasks from Australia, Germany, Hong Kong, Japan, Shanghai, Singapore and United States. The data were chosen by the researchers in each country from their LPS home data according to the developed definitions. Three common dimensions emerged in the analysis: differentiation of the mathematical process, building a realistic context and building connections.

Learning Tasks

Mathematics tasks are the major vehicles in mathematics lessons (Hiebert and Wearne 1993; Shimizu et al. 2010). The content of teaching and student activities in mathematics classrooms are organized around mathematical tasks. Therefore, the tasks are influential determinants of the implemented curriculum and the content of the lessons. However, how the students are engaged in the tasks is important. Depending on how the teacher sets the task and engages students in the task, the learning experience can be very different. For example, a simple task of expanding an algebraic formula ($a^2 - b^2 = (a + b)(a - b)$) might be carried out in the form of exploratory pair work, where the students shared their reasons and argument in private in a German classroom, while the same task might be a teacher's example for demonstration in which the teacher gives strong guidance with limited whole class interaction in a Singapore classroom. Both can be effective ways for helping students' learn, but the students may learn different things, such as skills of making argument and communicating mathematics ideas, from these very different experiences. Due to the restricted number of lessons sampled, there was no intention to claim any of the methods as being typical of the region or school system. Nonetheless, it is possible that one event is more likely to happen in a classroom than the other as a result of the values embedded in the teacher's beliefs and the long history of education practice in that particular school system or locality (Mok and Kaur 2006).

The third LPS book *Mathematical Tasks in Classrooms Around the World* (Shimizu et al. 2010) is devoted entirely to research into the role of mathematical tasks. It enhances the understanding of the nature of tasks and provides alternative analytical frameworks for tasks and their role in mathematics lessons. The book carries on the LPS theme of examining established practice via international comparative research. In this book, focusing on the nature, role and implementation of mathematical tasks, the authors offer the reader a variety of images of classrooms from the countries participating in the Learner's Perspective Study.

In this section, the analysis reported in the chapter by Mok (2010) shows some images of Australian lessons and of Shanghai lessons, aiming to show how the enactment of mathematical tasks in different countries can be revealed through the lens of a learning model. Mok (2010) used the learning task lesson events with purposeful sampling to compare the lessons from classrooms with very different demographic backgrounds. Two consecutive Australian lessons (A1-L04 and A1-L05) on the topic of circumference of a circle and two Shanghai lessons (SH2-L02 and SH2-L03) on the topic of linear equations in two unknowns were compared. At the first level of analysis, all the tasks used in the lessons were classified, and task events were classified as learning task lesson events (LT events) or non-learning task lesson events (non-LT events). The LT events included demonstration of new skills, explanation of new concept, investigation and solution of new problems. The non-LT events included review, repetition and practice. The first level of analysis gave a brief overall picture of the lesson, an illustrative example comparing A1-L04 and SH2-L02 is shown in Table 1. Although the two topics were different, both teachers began their lessons with review and followed with one focus LT event representing the major development of the topic of the day. At the same time, the comparison also showed that the Shanghai students had to do more tasks than the Australian students in one lesson—the Shanghai students had to do four practice items classified as non-LT event. The content of the items required the students to scrutinize and apply what was taught in the LT-event, even if the purpose was practice, and the task provided a good context for consolidation of new knowledge. In terms of similarity and differences in the routines of the lessons, the author wrote:

> It is very obvious that the pace in the Shanghai lessons was much faster than that in the Australian lessons. A Shanghai lesson always covered more tasks (including LT and non-LT) than an Australian lesson. Despite this difference, both teachers had a similar major routine of giving instruction about the task, followed by students carrying out the task and ending with the whole class paying attention to teacher-led interactive feedback. (Mok 2010, p. 124)

The next stage of analysis is to identify a platform for investigating the nature of learning taking place. The first question is how to compare the teaching of different topics taken place in different cultural system and curriculum. Mok (2010) resolved this by making the cognitive process the focus for comparison. The chosen framework was a learning model based on the constructivist and social constructivist perspectives. In the framework, the elements of concrete preparation, cognitive conflict, metacognition and bridging were essential features for effective learning in the lessons.

Table 1 A brief overall comparison between lessons A1-L04 and SH2-L02

A1-L04
• 2 tasks: 1 review and 1 LT event • The LT event with 2 sub-tasks: (1) measurement of radius (r), diameter (d) and Circumference (C) of circle either in the board or in the textbook; (2) Find the ratio of C/d
SH2-L02
• 6 tasks • Review (non-LT event): recall the meaning of a linear equation in one unknown • LT event: chicken and rabbit problem: 4 sub-tasks • 4 students' practice (non-LT event): • Non-LT1: rewrite an equation and write down the solution. • Non-LT2: identify a linear equation in two unknowns from some give pairs • Non-LT3 and non-LT4: determine whether a given pair of numbers is a solution for a given equation or not.

> Concrete preparation covers a range of activities which include the provision of relevant technical vocabulary which will be useful for students' subsequent activities.
>
> Cognitive conflict appears if children find a problem or task in which their methods and strategies appear not to work or to yield a contradiction; then the resulting mental conflict may challenge them to produce a higher-level strategy which does work.
>
> Metacognition means reflecting on one's own thinking.
>
> Bridging involves divergent thinking, where a person uses their imagination to invent other uses in contexts quite different from those in which they have learnt an idea or skill. (Mok 2010, pp. 121–122)

In the analysis of the selected lesson segments, the features of concrete preparation were explicit in the interaction elicited in the review activities. Often the teacher reviewed the concepts that would be needed for the activities students would do, and the students answered readily. For example,

Lesson L05 in Australian school A1 [A1-L05]:

T	[to all] What is perimeter?
	[to Michael] Michael.
Michael	It's the distance around the outside of a circle.
T	[to Michael] Wonderful.
	[to all] What do we call the perimeter of a circle?
	[to Sam] Sam.
Sam	The circumference
T	Good girl.
	[to all] What other parts have we looked at, what other distances of a circle have we looked at?
S1	The radius.

Lesson L03 in Shanghai school SH2 [SH2-L03]:

T	Guess it: how many chickens and rabbits are there? There are x rabbits and y chickens in a cage. There are altogether twelve heads, and forty legs. How many rabbits and chickens are there in the cage?
Cobot	I'll set up two equations.
T	Oh, he said that he would set up two equations, tell me the first one.
Cobot	x plus y equals twelve.
T	[Writing on the board] x plus y equals twelve.
Cobot	Four x plus two y equals forty.

Nonetheless, evidence of essential cognitive features did not suggest these were necessarily exclusive of each other in interactions. For example, very often features such as cognitive conflict and metacognition were found in the class interaction when the teacher helped the students recognize conflicting mathematical phenomenon and invited students to a deeper reflection.

For example, in lesson A1-L05, the teacher led the students to reflect upon the mathematics when inspecting two different ways of representing a formula. This might happen spontaneously, when the teacher asked a why-question concerning the meaning of the variables. In this example, the teacher gave the students plenty of time to try their ideas and encouraged them to test their unconfirmed answers.

[A1-L05]

T	So I'm just going to substitute, instead of 'D', I'm just going to write 'Two R'. So that now says the circumference is equal to two pi two R. How does that match up with what I wrote just above? Are they the same?
Ben	[not audible]
T	Who said 'No'?
Ben	[not audible]
T	[to Ben] Ben, why not?
Ben	[not audible]
T	So they don't look identical, do they? Um, what does two pi R mean in expanded form?
Ben	Two times Pi times R
…	
T	π by two, then by R? Two by pi, then by R. So if I were to substitute a radius of five centimetres into this one, and into that one, what would I get? Would I get… You don't know? [to Stephen] Stephen?
Stephen	The first one is bigger than the top one
T	This one would be bigger than the top one? Why don't we try it, then? Michael, I'm going to need your help with the calculator, O.K.

On the other hand, the example found in the lesson by the Shanghai teacher SH2 showed a different nature. The teacher used the chicken and rabbit as a familiar context to guide the students through scaffolding the mathematical ideas between a realistic context and abstract context. One main difference between this Shanghai example and the earlier Australian example is that the teacher moved to different levels of abstraction of the mathematical objects by employing a very compact and dense instructional mode, via his strategic scaffolding questions. Conflict and metacognition were experienced when the teacher guided the students to inspect a mathematical phenomenon from an alternative perspective. The teacher invited the students to give the equations, suggest the different answers to the same equations, look back on their pairs of answers and reflect upon the context of generalization of equations.

[SH2-L02]

T	Please think about, think about the first question. The first question, … in the cage, there are chickens, and rabbits as a sum of twelve, we have the equation x plus y equals to twelve, think about how many chickens are there, how many rabbits are there?
…	
T	Tell me, what does the solution of linear equation in two unknowns mean? There are only three students willing to put up their hands! Okay, this student, tell us?

Catty Among the numbers of solutions of the linear equation in two unknowns, each of the solution is the solution of the equation

…

T Okay, for this question, it is said that there are eleven pairs of solutions, if x and y indicates not real situation, that doesn't stand for rabbits, or chickens, so how many solutions are there for such an equation?

Class Infinite.

The essential point is that consideration of the contribution of a task to classroom practice and to the learning of students must look beyond its simple written or spoken form and take into account the social interaction when the task is implemented in each classroom setting. Only by attending to the social enactment of the task can we come to understand its function in the mathematics classroom, which is so much more important than its simple mathematical form. This distinction between form and function was a pervasive comparative tool throughout LPS analyses in a wide range of investigative contexts.

Conclusion

What does the study of mathematics classroom practice encompass? Mathematics classroom practice refers to the teaching and learning of mathematics that take place inside the classroom. It concerns the mathematical content, the organization, both the perceptions and activities of the teacher and the students, and the nature and content of the interaction in which the participants engage themselves (Laborde 2006).

What are international perspectives? International studies often involve comparisons of learning achievement and outcomes. In the results of international studies such as the Program for International Student Assessment (PISA) or the Trends in International Mathematics and Science Study (TIMSS), the students in some Asian countries consistently outperform the other countries. Therefore, it is tempting to suggest the less-successful non-Asian countries would do well to adapt for their use the instructional practices of Asian classrooms. Nonetheless, such a claim may be problematic for there may be a "pseudo-consensus" imposed across systems (Keitel and Kilpatrick 1999) that predetermines the criteria for comparative evaluation. While interpreting these results of comparison and Clarke and co-authors (2006) also point out that such assumptions have not given adequate attention to issues of cultural heterogeneity and differences in the students' experiences. Classrooms around the world differ in their underlying cultures, systems and beliefs. It is not possible to identify one model or system that may fit the purposes of all places. Nor is it right to say a certain kind pedagogical approach is better than the other in all instances and all settings. However, comparative studies help identifying pedagogical values, realizing the affordance and limitations on one's own system, that may not be recognized in a uniform system. Therefore, the search for differences and similarity is well justified.

This paper discusses international perspectives about mathematics classroom practice based on the work or results of the Learner's Perspective Study LPS. The analyses research reports generated by the LPS series demonstrate a variety of research foci and agenda about classroom practices, carried out by researchers from different education systems. Multiple facets and paradigms of classroom practice are unfolded in various analytical frameworks. The nature of research is very different from that of comparative studies of student assessment which attempt to show one is better than the other. What can be learned about research on mathematical classroom practice from LPS?

A key concept in LPS is complementarity.

> Complementarity is fundamental to the approach adopted in the Learner's Perspective Study. This applies to complementarity of participants' accounts, where both the students and the teacher are offered the opportunity to provide retrospective reconstructive accounts of classroom events, through video stimulated post-lesson interviews. It also applies to the complementarity of the accounts provided by members of the research team, where different researchers analyse a common body of data using different theoretical frameworks. (Clarke et al. 2006a, pp. 4–5)

LPS demonstrates other facets and perspectives for international comparative studies of classroom practice. The idea of complementarity embedded in the project design and methodology is an essential feature. Also embedded in the design and methodology is the accommodation of diversity. The project recorded a sequence of consecutive lessons by the same teacher to capture the variation in the development of the topic or the diversity of learning opportunities created by the same teacher. The use of purposeful selection of competent teachers ensured capturing the practices by teachers who were locally recognized as representing a conception of accomplished practice. Inside the classroom, both the teacher and the learners are the main agents, and LPS has recorded the voices of both parties in the data collection. The design of the project has allowed researchers to construct complementary accounts of what happens in the lessons from a variety of foci and theoretical perspectives. Similar to many international comparative studies, whenever the results of LPS are discussed in conferences and seminars, it is always of interest for the audience as to whether the reports observed in the classrooms in the study showed consistency of form and purpose that distinguished the classrooms situated within one school system from other classrooms in different school systems, such as to suggest a culturally-specific character for a region. However, it has never been the intention of the LPS project to make claims about national typification of practice. Complementary and collaborative international efforts are the essential elements for activating the international perspectives. To that end, I conclude with the following quotes from the three books in the LPS series:

> The Learner's Perspective Study is guided by a belief that we need to learn from each other. The resulting chapters offer deeply situated insights into the practices of mathematics classrooms … an insider's perspective. (Clarke et al. 2010)

> The comparisons made possible by international research facilitate our identification and interrogation of these assumptions. Such interrogation opens up possibilities for innovation that might not otherwise be identified, expanding the repertoire of mathematics teachers internationally, and providing the basis for theory development. (Clarke et al. 2006b)

> By making comparison possible between the classroom use of mathematical tasks in different classrooms around the world, the analyses reported in this book reveal the profound differences in how each teacher utilizes mathematical tasks, in partnership with their students, to create a distinctive form of mathematical activity. (Shimizu et al. 2010)

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Mathematical Literacy for Living in the Highly Information-and-Technology-Oriented in the 21st Century: Mathematics Education from the Perspective of Human Life in Society

Eizo Nagasaki

Abstract This paper discusses mathematical literacy for living in our highly information-and- technology-oriented society in the 21st century. First, it inquires into the significance of thinking about mathematical literacy in terms of how it benefits modern individuals, as well as modern society. A summary of the past trends of mathematical literacy in Japan is given. This is followed by a consideration of a framework for thinking about mathematical literacy in the future. Here, the focus is on mathematical methods and the need to re-visit the meaning of studying mathematics. This is followed by a discussion of the design of school mathematics curricula that aim to nurture mathematical literacy. The discussion includes an examination of the general structure of school mathematics as it pertains to mathematical literacy, and the framework of school mathematics that addresses diversity. Concrete examples of the designs of school mathematics curricula based on research on mathematics education in Japan to date are given. Lastly, the maintenance and development of mathematical literacy outside school is touched upon.

Introduction

It is thought that Japanese mathematics education has produced definite results. In fact, Japan has been ranked high in the international mathematics studies in which Japan has been a participant since 1960. On the other hand, IEA and PISA found that Japanese students in secondary school tend to see mathematics as fixed knowledge (in the First International Mathematics Study in 1964), and their motivation to study mathematics is low (in the Second International Mathematics Study in 1980). More recently it was found that Japanese students' recognition of the significance of the study of mathematics is low (PISA 2003). Furthermore, it has

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been found that Japanese adults are low in terms of awareness of the cultural value of mathematics. These factors in Japan now contribute to a phenomenon referred to as "the drift away from science and technology". Of course, behind these issues, there is a long tradition of attaching the utmost importance to testing centered around entrance examinations (Amano 2007).

These tendencies bring to mind a discussion on mathematization and demathematization (Jablonka and Gellert 2007). Japanese students get high marks for performance in school, at least in mathematics examinations. However, seeing the previous tendencies, I cannot help thinking that demathematization in Japan has its roots in school education.

As we entered into the 21st century, the crisis of the earth's sustainability became the subject of public outcry. We are called on as a society to grapple with many such issues in a democratic way. Especially in modern society, where the role of science and technology is of vital importance, scientific and technological literacy is expected of all adults (Project "Science for All" 2008a).

In this paper, the concept of mathematical literacy for lifelong enrichment of all adults as an objective of mathematics education will be investigated and clarified. By reviewing many papers authored by researchers in Japan and overseas (Abe 2010, etc.), we found that this concept of mathematical literacy and its importance for all adults is regarded as a common idea (AAAS 1989; Project "Science for All" 2008a).

The Significance of Mathematical Literacy

Mathematical literacy is aimed at all adults who are already functioning in a society. Mathematical literacy seems to benefit people who are trying to better themselves. Also, mathematical literacy may benefit people who are trying to better society in a certain age. In the former, mathematical literacy depends more on humans, even though society in each era has an effect upon them. In this sense, mathematical literacy in a way transcends generations. In the latter, since mathematical literacy depends more on society in the present and near future, mathematical literacy may be different according to various images drawn of a society. In this section, the significance to think mathematical literacy will be considered from these two viewpoints keeping all adults in mind.

Addressing the Individual

If we examine human history, it is clear that people have always had the will and potential to improve themselves. In this section, we will investigate that drive by looking into two kinds of people who strive to improve themselves; those who broaden their horizons to see themselves, and those who broaden their horizons to see the world.

People Who Broaden Their Horizons to See Themselves

Learning mathematics makes us aware of human potential (Hirabayashi 1987). In other words, learning mathematics broadens our view of ourselves. For example, when we learn mathematics, we engage ourselves in finding mathematical rules, solving and drawing up mathematical problems, and thinking of problems in the real world mathematically. By using mathematical expressions to present problems and by brainstorming together to come up with solutions, we become aware of our individual potential. We also come to realize that we are subjects working in our own ways to construct mathematics (Earnest 1991).

Furthermore, there may be situations where people think mathematically about various problems they encounter in daily life. People are capable of broadening their ideas of themselves through recognizing themselves as individuals who can think mathematically.

People Who Broaden Their Horizons to See the World

Learning mathematics also broadens people's perspectives on the world (Steen 1990). Looking back to elementary school, as children's lives and worlds broaden, mathematical content and methods also expand. At the end of compulsory education, students proceed from the world of concrete numbers to the world of abstract mathematical expressions. They enter into the world of two dimensions as they are introduced to problems that are solved with quadratic equations, Pythagorean theorem and quadratic functions. Furthermore, students proceed from the realm of informal demonstration to the realm of formal demonstration mainly through learning geometry.

People encounter more complex problems when they are out there in the real world. However, basic contents and methods to deal with the problems are included in what they already learned in school. Therefore, students' view of the world is broadened by learning mathematics in school. Also, from time to time, they learn more mathematics out in the world. We can expect this kind of cycle to develop and continue, with people broadening their views of themselves and their views of the world.

Addressing Present Society

It is said that our present society is highly information and technology-oriented and will continue to be so in the future. Development of science and technology, especially development of information technology such as the internet, has gone beyond our imagination. Also, the decline of the birth-rate and increase of the aging population continue in Japan. In this section, such issues as sustainable society, democratic society, and a society of lifelong learning, both important for Japan and profoundly related to learning mathematics, will be considered.

Sustainable Society

Realization of a sustainable society is a global challenge in the 21st century. To protect the global environment and construct a sustainable society, each and every adult is called upon to use their judgement when faced with various challenging situations. In order to think and judge, scientific and technological literacy is needed. Think of what happened on March 11, 2011 in Japan. North-Eastern Japan was struck by a catastrophic earthquake and tsunami. These disasters caused nuclear power plants in Fukushima to break down. It shocked not only Japanese people, but the whole world.

The disaster at the nuclear power plants made us realize how important it is to make information available to the public, especially in a crisis. It made us reflect upon ourselves—how stunned we were, and how important it is to keep thinking for ourselves (Fromm 1976).

Democratic Society

Further development of a democratic society has been a continuing challenge since the beginning of the 20th century for Japan, as well as the world. Though Japan has nearly 70 years of experience as a democratic state, it seems to lack a sense of democracy. Moreover, in times of economic uncertainty, there seems to be a welcoming atmosphere for dictatorial leaders.

To further construct this democratic society, it is imperative that people consider diverse perspectives and engage in logical dialogue about current problems. Critical thinking must be emphasized in these arguments (Freire 1970).

Lifelong Learning in Society

Responding to a rapid change in society, and demand for lifelong education began world-wide in the 1960s. Changes that took place after that period, particularly the development of information technology, are remarkable. It is causing massive information overload, as well as big changes in how we process information.

The need for lifelong learning and the lifelong education that supports lifelong learning is more and more increasing. Even today, learning *how to learn,* which was advocated in lifelong education remains important (Lengrand 1970).

Trends in Mathematical Literacy in Japan to Date

It was in the latter half of the 20th century that discussion of mathematical literacy began in Japan (Abe 2010). Here I will give you a summary of trends in mathematical literacy in Japan with a focus on activities involved in and resulting from the Project “Science for All”.

The Latter Half of the 20th Century

In Japan, the term "mathematical culture" was used in a national course of study of mathematics education in upper secondary school published in 1955. In the course of study, ways of mathematical thinking were shown as specific examples of mathematical culture. Since then, there has been almost no discussion of mathematical culture or mathematical literacy in mathematics education. In the 1980s, discussions about computer literacy, information literacy, and media literacy arose in response to changes in upper secondary school education, as well as in society. These discussions were continued until the mid 1990s. During the latter half of 1980s, there were some discussions on mathematical literacy, but not many. From the mid 1990s, there were various discussions on mathematical literacy (including NCTM's views), matheracy, numeracy, mathemacy, quantitative literacy, and statistical literacy. Since the beginning of 2000 until now, mathematical literacy as defined by PISA has been the core of various discussions.

Among these, there are three types of literacy that seem to be unique to Japan. They are: (1) mathematical literacy for fostering intelligence in the majority of upper secondary school students, (2) matheracy, and (3) mathematical literacy for ordinary citizens in a highly information-oriented society.

Mathematical literacy for fostering intelligence in the majority of upper secondary school students was advocated by a mathematician, Professor H. Fujita, in 1983. According to Professor Fujita, it is important to foster mathematical intelligence in upper secondary school students. He believed that mathematical intelligence was composed of mathematical thinking and mathematical literacy. He felt that mathematical thinking was necessary for those students who wanted to become specialists in the future, whereas mathematical literacy was necessary for the remaining majority of students. He assumed that mathematical literacy included mathematical laws, mathematical logic, and the ability to apply mathematical concepts for talking about science at the ordinary citizens' level.

Matheracy, which was derived from mathematical literacy, was advocated by then-president of Japan Society for Mathematical Education (JSME), Professor T. Kawaguchi, in 1983. Matheracy was an idea for building curricula based on mathematical cognitive activities. It also split mathematical content into four categories: objects, activities, inferences, and problem solving activities. Problem solving activities integrated the other three elements. Here, the aims and objectives of the curriculum were explained based on the existing premises, and activities were described as the core of educational content and methods.

A report on mathematical literacy for ordinary citizens in a highly information-oriented society was made by an ad hoc committee of the JSME in 1987. In the report, discussions of mathematical literacy were expanded into discussions of computer literacy in response to the growing importance of computers and technology.

As for discussions on mathematical literacy in Japan, although some attempts were made to pursue something unique to our country, the discussions became influenced by NCTM's and PISA's definitions of mathematical literacy. After all, in those days, ideas on mathematical literacy in Japan were understood in terms of school education, and there was no consensus that the mathematical literacy was for "each and every" student. Also, the discussions did not include societal context. Thus, the discussions were unable to truly embody mathematical literacy.

Entering into the 21st Century: Project "Science for All"

In 2005, a body of scientists and educators used government funds to start Project "Science for All" to outline scientific and technological literacy. The term of this project was divided into three periods and is still going on now (Nagasaki 2010a).

From 2005 to 2006, the study "Research on Building Scientific and Technological Literacy" laid the groundwork for scientific and technological literacy. About seventy scientists and educators took part in this study and examined the current states of scientific, mathematical, and technological literacy in Japan, several foreign countries, and international institutions. In addition, they delved into the significance and necessity of literacy, as well as the organizational systems for developing literacy (Kitahara 2006; Nagasaki 2006). The foreign countries included the USA, Canada, England, and China. World organizations such as UNESCO and OECD were also examined. In particular, Project 2061 by the American Association for the Advancement of Science (AAAS) and its great fruits, "Science for All Americans" (AAAS 1989) were investigated in detail, as well as research on mathematical literacy (Jablonka 2003). Findings were published in special issue of the Journal for the JSME (Vol.89, No.9, 2009) and in other publications.

In 2006–2008 the Project "Science for All" defined scientific and technological literacy as "the knowledge, skills, and ways of viewing science, mathematics, and technology that all adults are expected to acquire" (Project "Science for All" 2008a). About 150 members, including scientists, engineers, educators, science museum personnel, members of the media, and various NPO's took part in the project. The seven special subcommittees to define literacy were composed; Mathematical Science, Material Science, Life Science, Informatics, Earth, Space and Environmental Science, Human and Social Science, and Technology.

The Project "Science for All" drew an image of what we wanted Japan to be like in the future. According to this image of the future Japan, (1) every individual would be recognized as an irreplaceable member of our society, (2) every member of our society would care for planet Earth, share wisdom, and take action toward the realization of sustainable society, and (3) our society would be equipped with an effective system in which young people can entertain hope for the future and inherit a culture. We decided that we must make scientific and technological literacy a reality for the enrichment of the Japanese people and for a sustainable and democratic society.

We believe that the significance and the need to define scientific and technological literacy lie in the following four objectives: (a) To make judgments concerning science and technology; (b) To transmit scientific knowledge over generations; (c) To provide a coherent, long-term perspective for primary and secondary school education in science, mathematics, and technology; (d) To convey the sense that learning, including the learning of science, mathematics, and technology, is a life-long activity. Recent advancements in science and technology, as well as characteristics of Japanese culture, were incorporated into defining literacy. In preparing the framework for the reports, we made an effort to build its framework with human society at the core, and to include perspectives from the present and the future.

In 2008, the Project published one volume of the "*Integrated Report on Scientific and Technological Literacy*" and seven volumes of "*Summary Reports*" on scientific and technological literacy (Project "Science for All" 2008a, b).

The Integrated Report on Scientific and Technological Literacy is composed of six sections as follows: Section 1. Towards Science for All and Lifetime Enrichment in the 21st Century; Section 2. The Essence of Science and Technology; Section 3. Science for All—Seven Doors; Section 4. Perspectives on Science for All; Section 5. The Application of Science for All; and Section 6. The Future: Succession and Sharing of Science for All.

The Summary Report of the Mathematical Science Subcommittee is composed of five sections as follows: Section 1. The Essence of Mathematics; Section 2. The World of Mathematics (A): Objects and Important Concepts of Mathematics; Section 3. The World of Mathematics (B): Methods of Mathematics; Section 4. Several Topics; and Section 5. The Relationship Between Mathematics and Humanity. This report was mainly developed by mathematicians and centred around mathematics as language (Namikawa 2009).

The results of the Project have been published in various magazines, such as the Journal for the Japan Society for Science Education (JSSE; Vol.32, No.4, 2008), which includes discussions on mathematical literacy.

Since 2009, the Project has shifted its emphasis to publicity and revision of scientific and technological literacy. In 2011, meetings for the revision were held in which the project members explained scientific and technological literacy to elementary school teachers who in turn critically investigated the concepts. Based on their exchange, the members and the teachers held a dialogue and worked together to prepare a report for the revision.

Update on Mathematical Literacy

Researchers on mathematics education involved in the Project "Science for All" have continued to engage in ongoing research focused on mathematical literacy (Iwasaki 2010; Nagasaki 2011a, b). Among their research, there are several on mathematical literacy, such as research on the construction of a framework for mathematical literacy that includes "the nature of mathematics," "mathematical concepts," "mathematical abilities" and "mathematical application" (Nagasaki 2009); curriculum research focused on fostering critical thinking in regard to mathematization and demathematization as advocated by Jablonka et al. (Iwasaki and Hattori 2010) and PISA's key-competencies for school mathematics (Shimizu 2010). The framework also drew heavily on research on the significance of learning mathematics in mathematical literacy (Nagasaki 2010b), and mathematical modelling as one of mathematical methods (Abe 2010). Also, the researchers discussed the collaboration of science education, technology education, and mathematics education for scientific and technological literacy (Nagasaki 2011a, b).

Framework for Considering Mathematical Literacy in the Future

The previous sections outlined the context and significance of mathematical literacy in Japan. This section deals with the mathematical literacy that all adults will be expected to acquire in the future.

Addressing Individuals in Contemporary Society

Mathematical literacy, as mentioned above, must address both individuals and contemporary society as a whole. Mathematical literacy must be meaningful for people and society (Senuma 2002). That is to say, mathematical literacy must be relevant to daily life. Moreover, the connection must continue over the course of a lifetime.

So it is important how people, contemporary society and mathematics will be considered for mathematical literacy. People autonomously live in contemporary society and individuals are diverse and equal. As contemporary society, sustainable and democratic society is aimed. And mathematics is a creature by human and has the application in the society. Individuals, contemporary society and mathematics are closely interconnected.

Focusing on Mathematical Methods

As compared to mathematical contents, mathematical methods are applicable to a wider range of information. For example, the inductive method can be used for finding rules or laws in various situations and the deductive method is needed to verify rules or laws that were found. In this sense, it is natural to focus on mathematical methods as mathematical literacy. Even if information multiplies immensely as times change, mathematical methods do not change so much. Mathematical methods are always used, intentionally or unintentionally, beginning with learning various mathematical contents in elementary school.

In connection with mathematical methods, it is important to duly appreciate the role played by computers. As computers were indispensable for solving the four-colour problem, it must not be forgotten that computers demonstrate their great power in numerical calculation and processing of geometrical figures.

Mathematical methods can be divided into two categories. One is the category of general methods in science that can be obtained through not only learning mathematics, but also learning other subjects such as science, and the other category is that of intrinsic methods to mathematics.

General Methods in Science

The most typical of general methods in mathematics is logical thinking. From early times, logical thinking has been considered related to mathematics. But logical thinking is not intrinsic to mathematics. Rather, it is a method of thought for science in general (Todayama 2011). However, needless to say, mathematics education involves a great deal of logical thinking.

A more general method that involves logical thinking is critical thinking. In critical thinking, we reflect upon various matters through interactions with others. Since mathematics is a tool for communication, communication is also a general method. When an individual's subjective knowledge becomes objective knowledge, collaboration is needed as a general method. In mathematics, there are several general methods that are useful in society.

Methods Intrinsic to Mathematics

There are some methods that are intrinsic to mathematics. Typical of such methods is a method of mathematization (Freudenthal 1983) that transforms phenomena to mathematics or transforms one form of mathematics to another for solving problems. In Japan, mathematization was emphasized in 1940s. With respect to abstract mathematics, there are methods of expressing concepts as symbols and methods of generalization and extension of concepts. On the other hand, with respect to the connection between mathematics and the real world, there is the method of mathematical modelling.

Further, there are several methods of thought that are more deeply dependent on mathematics. They include analytic and geometrical methods that relate mathematical expressions and geometrical figures, methods to find rules or laws in mathematics, methods to think functionally, methods to think statistically, and methods to think based on a limit and others.

Re-Examination of the Meaning of Learning Mathematics

When we consider mathematical literacy, we need to focus on mathematical methods. At the same time, in Japan, we need to place special focus on the meaning of learning mathematics. The goals of mathematics education are usually discussed from three points of view, namely: disciplinary, practical, and cultural (Nagasaki and Takii 2007). Japanese students, however, as already mentioned, seem to have lost their motivation to learn mathematics, as they focus on nothing but entrance examinations when they study mathematics. This means that Japanese society as a whole seems to have lost the real meaning of learning mathematics.

If we understand as a society why we learn mathematics as part of mathematical literacy, it will encourage students to study not only what mathematics is, but how it applies to the real world. In order for Japanese mathematics education to break away from thinking "mathematics as a sieve" and in order for adults and students to enjoy learning mathematics, we need to build a society that shares a true understanding of learning mathematics, even though we have a long way to go to achieve such society.

Furthermore, these are related to necessity of knowledge on mathematical literacy, namely meta cognitive knowledge. Process of meta cognition that consists of monitor, self-evaluation and control (Shigematsu 1990) will suggest for mathematical literacy.

Designing School Mathematics for Mathematical Literacy

As I mentioned in the previous chapter, mathematical literacy is mainly concerned with mathematical methods and the importance of learning mathematics. Keeping in mind that mathematical literacy is for all adults who have been through school, school education must change. One big reason for this is that adults do not retain the results they learn after they finish school, as I mentioned in my introduction. Now, let us review Japanese mathematics education up to this point and consider how to design school mathematics so that adults can retain and even further develop mathematical literacy.

General Structure of School Mathematics in Regard to Mathematical Literacy

Mathematics education in elementary, lower secondary, and upper secondary school gives a span in both time and space, if we think in terms of mathematical literacy. Here, the span in time and space is considered as general structure of school mathematics.

Life-Long Learning

When we think about mathematical literacy, we need to formulate a plan that looks at learning as a life-long activity. This clearly differs from the "mathematics for entrance examination" or "mathematics as sieve" mindset. We must break the vicious circle of thinking that "the harder we work on mathematics education, the less people like mathematics." Since mathematical literacy takes life-long learning into account, it is our hope that children will learn mathematics of their own free will. When we teach them, we must help them connect with the real world and continue to guide them as they grow.

Life-long learning means that we should not think of learning as finished at school. People continue to learn even after they go out into society. Educational systems must be changed in accordance with this fact, and individual awareness must be raised.

Integration of Various Academic Cultures

Mathematical literacy integrates general mathematics education for all students and mathematics education for students who go on to pursue science and technology or advanced mathematics. The mathematical methods and the meaning of learning mathematics, which are important elements of mathematical literacy, are useful not only for all students, but also for students who go on to pursue science and technology because it gives them an overview of their future career.

Thus, the cultures of science and technology and the humanities (Snow 1964) can be integrated by mathematical literacy. Furthermore, since the target of mathematical literacy is all adults in a democratic society, collaborative learning among heterogeneous groups of students is imperative. This idea is connected "Interacting in Heterogeneous Groups" in OECD's concept of key-competencies (Rychen and Salganik 2003). Mathematical literacy will integrate these diverse cultures.

Enjoyment of Learning and Preparation for Future Endeavours

With an awareness of mathematical literacy, we can see that there are two aspects of the curriculum for general education. One is enjoyment of learning in the present and the other is preparation for future endeavours.

Enjoyment of learning means that all students can collaboratively engage in appropriate mathematical activities with common mathematical content for the greatest possible development. Students think and express themselves freely in their groups.

Preparation for future endeavours is composed of two types of learning. One is for all students to jointly learn common mathematical methods and the meanings of learning mathematics. The other is for different students to learn different mathematical content.

Addressing Diversity in the Design Framework of School Mathematics

If we keep mathematical literacy in mind when revising the general structure of school mathematics, we see that many practices and studies conducted in Japan should not be changed altogether. The clarification and renewal of concepts in some practices and studies yield a design framework relevant to improving the general structure. Here the central concept for the framework is diversity. I have been saying that although mathematical literacy is for all adults, it includes diversity. In Japanese mathematics education, we have a long tradition of giving consideration to students' diverse ideas. In Japan, mixed-abilities classes have been conducted for a long time. Though consideration for individual instruction is occasionally lacking, in a sense, our mathematics education has been addressing students with diverse backgrounds. In this part of my lecture, I would like to consider the design framework of school mathematics in relation to this diversity and give some concrete examples.

Collaborative Research by Elementary, Secondary School, and University Teachers

In Japanese mathematics education, results of a collaborative research project by a team of elementary, secondary school, and university teachers has ongoing for some time. The team began to discuss the framework based on a certain idea. While many teachers make plans and conduct experimental classroom lessons based on the framework, other participants observe the lessons. The team discusses findings and issues based on lesson records. The subjects of the research projects are diverse in their inclusion of elementary, lower and upper secondary students. The researchers are also diverse, including teachers from elementary and secondary schools, as well as those from universities.

Such collaborative research among various teachers made it possible for us to put mathematics education in long-term perspective. The findings of practical research in each school level become more objective through researchers' viewpoints of different school levels. And this type of collaborative research leads participating teachers to common understanding of not only mathematical development throughout elementary to university levels, but also students' ways of mathematical thinking. The collaborative research is appropriate for taking people's lives into account, which I elaborated on when I spoke about the general structure of school mathematics with viewpoints of mathematical literacy. In addition, as a matter of course, these research results will become relevant in society beyond school.

Integration of Mathematical Content and Methods in Educational Curriculum

The mathematics curriculum for each grade in Japan is known internationally as a large unit curriculum. In a given mathematical unit, the following processes are aimed. Students start with a problem prepared by a teacher. The problem should be relevant to society or daily life. The students must determine how the new mathematics can solve the problem, examine the properties of newly introduced mathematics, summarize the procedures and properties, and tackle new problems using the new mathematics (Shimada 1970). This type of integration of mathematical content and methods not only allows us to see mathematical culture integrated, but also to put both enjoyment of learning and preparation for future endeavours in perspective.

Problem solving plays a vital role here. Students understand the content and acquire the methods through problem solving. This idea became clear in elementary school mathematics in the 1930s, and in upper secondary school mathematics curriculum in 1950s. But the idea evolved in elementary school mathematics, just as in other educational reforms in Japan.

Setting of Learning Situation Based on Students' Mathematical Activities

Though mathematics learning situations are different depending on teaching and learning content, it is thought in Japan that students' mathematical activities are central to understanding concepts and acquiring methods. Also it is thought that in order to better evaluate mathematics education, planning situations in which students can freely display their own abilities is essential.

Mathematical activities are conducted based on situational mathematical problems. The mathematical contents include pure mathematical problems, as well as problems in the real world. The mathematical activities include solving mathematical problems, finding mathematical rules, and making new mathematical problems. In these kinds of learning situations, the following processes are commonly found: (a) thinking individually, (b) presenting various ideas, (c) whole class discussion. The last process (c) is called "*neriage*," which means to "polish up". In this process, students and teachers talk together about differences and similarities or relations among various ideas so that they can understand concepts more deeply as a whole class.

Design of School Mathematics

Based on the design framework of school mathematics in regard to diversity, let us consider several findings from research on mathematics education practices, as well as future possibilities in Japan. Two recent publications are helpful. "*Handbook of Research in Mathematics Education*" (JSME 2010) is a comprehensive summary of research in mathematics education over 50 years in Japan, and "Toward the Problem-Centered Classroom: Trends in Mathematical Problem Solving in Japan" (Hino 2007) summarizes research in problem solving since 1980 in Japan. In this part of my lecture, I will mainly discuss the results of collaborative research by elementary, secondary school, and university teachers that are still valuable in terms of mathematical literacy.

Classroom Lessons that Make Students Think

Mathematics education in elementary school in which children think mathematically based on concrete problems was explicit in elementary mathematics textbooks in 1930s in Japan. Before these textbooks were published, there were two books on reforms for secondary mathematics education published in 1924. They are "*Fundamental Problems for Mathematics Education*" (Ogura, K., Idea-shoin) and "*Fundamental Investigation on Elementary Mathematics Education*" (Sato, R., Meguro-shoten).

The idea that children should use mathematics by themselves to solve problems given by their teachers remains at the core of teaching and learning. For example, there was a study that investigated various aspects of children's mathematical thinking in elementary and secondary schools (Matsubara 1971). "Classroom lessons that make students think" was also conducted in elementary and secondary schools (Matsubara 1987; Handa 1995). The authors claim that "The essence of the classroom lesson is not to teach. It is to have children be active. In order to realize that, the classroom must become a forum for thinking." They also say, "The most important thing that teachers should do to hold good lessons is to know children well." They discuss how the lesson should be introduced, how teachers should bring up questions, and how to work out ways to make children think for themselves. In this study, mathematical thinking was thought as functional thinking. And it seems for me that it is mathematization.

This type of teaching and learning that explicitly accept diversity of children's ideas was developed into "Teaching that appreciates students' various ideas" (Kotoh and Niigata Study Group on Elementary School Mathematics Education 1992). In the lesson, mathematics is thought to offer multiple ways to solve a problem with only one correct answer. The ideas, which reflect the students' various perspectives, are summarized at the conclusion of the lesson. This concept showed us a course of action to treat all children with diversity important in whole class teaching of a mixed-ability class.

Open-End Approach

In 1960, the IEA First International Mathematics Survey found that Japanese students thought of mathematics as something to memorize. In response to this finding, development of mathematical thinking became an issue to be tackled. So ways of thinking mathematically were emphasized as an objective of the official mathematics curriculum of elementary and secondary school, and "integrated and developmental" became a slogan in 1960s–70s (Nakajima 1981). The slogan means development of mathematics with awareness of integration. It is similar to the idea that subjective knowledge becomes objective knowledge in society (Ernest 1991).

Under these circumstances, elementary, secondary school, and university teachers developed a project to evaluate ways of thinking mathematically. For this purpose, they thought the priority should be to prepare a situation in which children could conduct comprehensive mathematical activities. Thus, the open-end approach was advocated: ways of teaching and learning that present an open-ended problem with multiple correct solutions (Shimada 1977; Becker and Shimada 1997). It was the first book on mathematics education in Japan that explicitly advocated mathematical modelling to explain mathematical activities.

The open-end approach is a form of instruction that "sets an open-ended problem as a task, develops the classroom lesson positively using a variety of correct solutions of the problem, and gives students experiences in which they combine learned knowledge, skills, and ways of thinking in various ways to discover something new in the process." This instruction had a big impact on mathematics education, since the prevailing idea in those days was that mathematics had only one correct solution. The project developed open-ended problems that required three types of mathematical activities. They are: how to measure, how to classify, and how to find rules. Later on, the project included problems with excessive information or insufficient information in open-ended problems.

Developmental Approach to Mathematical Problems

The creation of mathematical problems was focused on expanding the open-end approach, or as one type of mathematical activities, or as an extension of activities to think in various ways. Another team of elementary, secondary school, and university teachers started up a project. The team came up with the idea of teaching and learning focused on a learning situation in which students first solve a problem and then formulate new problems based on the one they have solved. They proposed teaching and learning through a developmental approach to mathematical problems (Takeuchi and Sawada 1984).

Instruction through the developmental approach to mathematical problems involves "engaging students in spontaneous learning activities such as starting from a given problem and having students make up new problems which they solve by themselves by changing elements of the problem for similar or more general ones, or by thinking the converse etc." This is "instruction through problem creating."

In the beginning of the 1900s in Japan, other type of instruction through creating problems was conducted. Then, elementary school children created mathematical problems in class by looking at their surroundings. The developmental approach to mathematical problems is based on mathematical problems. It has real difference from problem solving method. Students who were thought to be unable to participate in problem solving were now involved in mathematical activities.

Connection Between Society, Culture, and Mathematics

The IEA Second International Mathematics Study conducted in 1980, as well as subsequent international surveys, found that Japanese students had little awareness of a connection between mathematics and society. Based on the findings of these studies, another elementary, secondary school, and university teachers began to study ways to put mathematics instruction into societal context. They started thinking about learning situations that focused on solving problems relating to children's lives, society, and culture. In other words, they began to think about "connecting mathematics to society and culture" (Nagasaki 2001).

Connecting mathematics to society and culture means "teachers in the classroom focus on mathematical activities and interactions that help to prepare children to tackle problems in society or problems related to mathematical culture. It means helping those children acquire competencies and attitudes to improve society and culture."

The ability to connect mathematics with society includes four domains, namely, understanding of quantities and shapes in society, ability to solve social problems mathematically, ability to communicate using mathematics in society, and ability to use approximation. There, mathematization of phenomena was the necessary method first and mathematical modelling was the central method. Though this kind of mathematics education was carried out during the 1950s in Japan, it failed due to the weakness of the mathematical aspect. Here our instruction was conducted having objectives as mathematics appropriately against the failure. It has the same idea as mathematical enculturation (Bishop 1988).

Future Possibilities

Research on the design of school mathematics in Japan has gradually been introduced into Japanese mathematics textbooks and spread widely. Recently, research has expanded further.

Let me give you some examples of these research works. Concerning open-end approaches, there is a study on fostering the ability to make decisions by actively treating societal problems as open-ended problems (Shimada 2011). On the subject of developmental approaches to mathematical problems, they conducted a study on problem creating in which university students used computers (Shimomura and Imaoka 2009). As for connecting mathematics with society, there are study on development of teaching materials and instruction (Nishimura et al. 2010), study on the ability to perform mathematical modelling (Nishimura 2010), and study on the ability to think statistically (Matsumoto 2010).

Also, there are several studies on mathematical activities, including a research project focused on ways to handle mathematical methods in elementary and lower secondary level textbooks (Japan Textbook Research Centre 2006), studies on systematizing mathematical competencies (Nagasaki et al. 2008), and a study on teacher education focusing on types of classroom teaching such as problem solving (Kubo 2012), and a study on teacher education focusing processes of thinking mathematically (Ohta 2012).

The design of school mathematics was based on practice and research in mathematics education. "Mathematics for all" was implicitly understood. Above all, we asked, "can there really be mathematics for all?" Based on the findings of cognitive science (Inagaki and Hatano 1989), mathematics education for all children which employs the concept of collaborative and shared learning is starting to become the subject of research (Matsushima 2012) and will be theoretical by design research that demonstratively verifies the design of learning environment (Masukawa 2012).

Furthermore, it seems that the main goal of mathematics education so far has been to build a system of mathematical knowledge by proving true propositions deductively. However, it seems to be equally important to find false propositions in a society flooded with information that is interwoven with both truths and falsehoods. That is to say, it is necessary to actively and intentionally introduce problems that enable refute by counter-example (Lakatos 1976). Concerning to these, socio-mathematical activity theory is proposed on the basis of theory of mathematics by Lakatos and theory of development by Vygotsky (Ohtani 2002).

Maintenance and Extension of Mathematical Literacy Outside of School

Most of our lives are spent outside of school. People must maintain and extend what they learned in school. Ways of doing this include conversations at home, traditional media such as books, newspapers, radio, TV, magazines etc., and internet. However, in addition to these informal means, some formal activities are needed. There are some sorts of formal activities related to scientific literacy.

Lessons from Diffusion of Scientific Literacy

In the beginning, scientists diffused scientific literacy by giving lectures on it. But it became clear that a mere outpouring of knowledge was not sufficient. Later on, concepts such as science communication, science interpretation, and consensus conferences for all adults have been implemented. Here, I will mention four activities for the diffusion of scientific literacy.

First, there are activities in science museums or science centres that families can visit (Ogawa 2011). Second, there are activities in science cafes and science pubs run by NPOs, universities, and other organizations. Third, there are consensus conferences (Kobayashi 2004) at which scientists and ordinary citizens exchange opinions about important policies related to science and technology and prepare reports. Fourth, there is a science communication movement that scientists do not communicate scientific literacy to citizens unilaterally, but use two-way communication to share scientific literacy.

Reconsideration of Mathematical Literacy

Looking at the diffusion of scientific literacy from the standpoint of mathematical literacy, science museums, science centres, and science cafes could also be used for mathematical literacy. And consensus conferences offer one answer to the question of who develops 'mathematical literacy for all adult'. Consensus conferences show us that mathematical literacy should not be developed by specialists alone, but by the collaborative efforts of specialists and non-specialists. And science communication reminds us that mathematics is a "powerful means of communication" (Cockcroft 1982).

Mathematical literacy must be geared toward all adults and take people's lives into account. Every day, adults face family, job and social events that compel them to think and judge. Adults also need thoughtful behaviours to cope with many matters such as marriage, child caring, jobs, building one's home, retirement and old age according to individuals' growth. Mathematical literacy is necessary for these thinking, judging and acting. Sudden real problems are even more important. Today, for example, adults are compelled to make judgements about global warming and nuclear power plants, mathematical literacy is vital.

Ultimately, mathematical literacy must be organically connected and integrated within each adult. It is not enumeration of mathematical content and methods, but a representation in language and figures that allow us an overhead view of the whole of the connection; connection between mathematics and the problems of individuals, society, and culture, and connections among various areas of mathematics (Project 'Science for All' 2008a).

Conclusion

Mathematical literacy varies from time to time and from society to society. In this paper, mathematical literacy has been discussed keeping Japan's present and near future in view. I have attached importance to taking people's lives into account and making connections with society when mathematical literacy is considered.

These ideas about mathematical literacy still leave us with challenging issues. For example, what happens to curriculum development (Howson et al. 1981) for fostering mathematical literacy in school education? Though mathematical methods are focused on as part of mathematical literacy, how would mathematical methods be structured by age group? What other types of teaching and learning could be developed by practice and research of elementary, secondary, and university teachers? How can mathematical literacy be maintained and developed in society? How can mathematical literacy be evaluated?

Also, ideas about mathematical literacy seem to raise several issues for mathematics education. For example, why is mathematics taught as a core subject in school? Are screening tests compatible with ideas of mathematical literacy? How should mathematics and mathematics education be connected with society? And mathematics education so far has aimed at 'written mathematics', is it necessary to re-examine the mathematics education from the viewpoint of language as the human nature (Sakai 2002), specially oral language?

By raising such serious issues, ideas of mathematical literacy re-examine mathematics education at its foundations.

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Exploring the Nature of the Transition to Geometric Proof Through Design Experiments From the Holistic Perspective

Masakazu Okazaki

Abstract The gap between empirical and deductive reasoning is a global problem that has produced many students who have difficulties learning proofs. In this paper, we explore the conditions that aid students in entering into proof learning and how they can increase their ability before learning proofs through design experiments. First I discuss the theoretical backgrounds of the holistic perspective and didactical situation theory, and set a research framework as the transition from empirical to theoretical recognition consisting of the three aspects of inference, figure, and social influence. Next, I report my design experiments in plane geometry redesigned for the seventh grade, and examine how students may enter the world of proof by learning geometric transformation and construction as summarized in the three aspects of the framework. Finally, I suggest key ideas for designing lessons that promote transition.

Keywords Transition to geometric proof · Holistic perspective · Empirical and deductive reasoning

Problems with Prerequisites and the Necessity of Learning Proof

Proving is an essential activity in mathematics that has occupied an important content area in school mathematics. However, difficulties with proof learning have continued to be a global problem. Research has indicated that the gaps between empirical and deductive reasoning cause a large number of secondary students to fail to learn proof (Hirabayashi 1986; Harel and Sowder 2007). Nevertheless, it remains unclear how students can increase their abilities to bridge this gap, which is the issue we explore in this paper.

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The gap between empirical and deductive reasoning as identified by the van Hiele model (1986) can be considered as a difference between the second and third levels of thinking. In the second level of thinking, students "explicitly attend to, conceptualize, and specify shapes by describing their parts and spatial relationships between the parts" while in the third level they "explicitly interrelate and make inferences about geometric properties of shapes" and "logically organize sets of properties" (Battista 2007). Several researchers have indicated that even after students learned formal proof many of them continue to use empirical arguments from the second level of thinking (Chazan 1993; Koseki 1987), such as basing their responses on appearances in drawings or proving statements by providing specific examples, and are not able to distinguish between inductive and deductive arguments (Harel and Sowder 2007). However, our intention is not to compare between the empirical-inductive and deductive forms of argument, but rather to reexamine them from the perspective of transition from the former to the latter.

Moreover, several researchers with teaching experience have indicated a problematic situation in proof teaching whereby students do not feel why a proof is needed (Ohta 1998; Souma 1998; Kunimune 2003). As reasons for this they suggested that students already know the properties and thus few new properties need to be explored, so it is a determined geometric system that is given to students. For example, the equality of base angles in an isosceles triangle has already been confirmed by folding a piece of paper in elementary school, and therefore the students find it hard to understand the significance of proving the properties of isosceles triangles and parallelograms. Instead the researchers proposed the idea of "proving as inquiry" (Ohta 1998; Sekiguchi 2002). Ohta insists that we should "situate proofs in the learning tasks to solve problems from a real life or to explore the interesting properties of geometric figure, and to encourage students to see the significance and structure of proof through gradually systemizing them". Stylianides (2007) also indicated that the sudden introduction of proof causes difficulties. We think that these factors cause students to miss the point of why a proof is needed.

I thus consider it important to emphasize the functions of proof (de Villiers 1990), in particular the function of explanation because formal proof that only focuses on verification causes students to remove much of the significance (Hanna and Jahnke 1996). Furthermore, as Hanna (1991) shows, mathematicians highly regard the usefulness of a theorem and it may be necessary to consider what interesting situations the theorem can treat and solve. Tall (2008) also indicated that "written formal proof is the final stage of mathematical thinking; it builds on experiences of what theorems might be worth proving and how the proof might be carried out, often building implicitly on embodied and symbolic experience". Without considering this, students may not direct their attention towards formal proof.

The present paper examines what conditions enable students to learn proof and how they can develop their reasoning abilities before learning proof so that they overcome the gap between empirical and deductive reasoning. I think that this is consistent with recent attempts that continually foster proving of statements from the elementary stage under a wider concept of proof (NCTM 2000; Stylianides 2007). For this purpose, a large number of practical studies will be necessary. However, very little research has so far looked at proving practices in the classroom (Knipping 2008). I later examine how students get their ability for proof learning through our classroom design experiments.

Below, we begin by considering a perspective that permits proving as inquiry.

The Holistic Perspective and Didactical Situation Theory

I now introduce the philosophical stances of the holistic and systemic perspectives (Miller 2007; Wittmann 2001), not distinguishing between the two terms but showing the central ideas.

The holistic perspective is in a sense an antithesis of the mechanistic and atomistic view that has dominated our way of thinking and living in various respects since the industrial revolution (Hirabayashi 1987; Sato 1996; Wittmann 1995, 2001). In the atomistic view, the whole consists of the separate parts and so the whole is acquired by learning the parts separately. This view may induce teachers to break up the content, arrange it sequentially and teach it piece by piece. Meanwhile, holism is the view that the whole is not the sum of its parts but is the whole itself in principle. When students, even those who have good mathematical competence, often say, "I don't know what we are learning, where we are going, what mathematics is good for", these comments can be interpreted as anxiety at not seeing the whole. In this regard, I may note that holism does not approve of totalism which consolidates all things into the fixed total, but emphasizes the relations between the whole and the parts, or that the whole can come to be a part of the larger whole.

Holistic education is also a result of reflection on the history of education in which policy has gone back and forth between emphases on the individual child' knowledge and skills and on the application to real life contexts, and tries to synthesize both to open up new horizons for education (Yoshida 1999). I think that the definition of holistic education given by Miller (2007) is important in mathematics education:

> The focus of holistic education is on relationships: the relationship between linear thinking and intuition, the relationship between mind and body, the relationships among various domains of knowledge, the relationship between the individual and community, the relationship to the earth, and our relationship to our souls. In the holistic curriculum the student examines these relationships so that he or she gains both an awareness of them and the skills necessary to transform the relationships where it is appropriate.

Holistic education has several keywords such as "relationships", "balance", "inclusion", and "connection". I consider at least three aspects of this even if we limit our scope to mathematics. One aspect is knowledge formation where emphasis is placed on connections between intuition, logic, prior or everyday knowledge, and the ideas of others. The second aspect is human relationships, which may include fellowships, norms, or attitudes in the classroom. The third aspect is the innermost self. Education aims to give students encounters of experience so that the self is moved, where there is a connection between the external ego and the internal self. Here "the student is not reduced to a set of learning competencies or thinking skills but is seen as a whole being" that includes aesthetic, ethical, physical and spiritual aspects.

Holistic education integrates two strands. One is humanistic education that concerns the growth of humanity in each student, and the other concerns social change towards an equal and cooperative society. In mathematics education, the former has been integrated into mathematics as a human activity by several founders, such as Gattegno, Wheeler, Brown and Hirabayashi (Hirabayashi 1987, 2001; Koyama 2007). The latter strand has been recognized by excellent teachers. Some excellent teachers share their thoughts: "Only after the feelings of 'everybody is different and everybody is nice' come into being in child's mind the rich mathematics lesson in creativity is possible" (Tsubota 2001), and "I specialize elementary mathematics teaching. Through teaching mathematics, I am guiding students' life, deepening child's compassion, and fostering classroom camaraderie" (Takii 2001). I think that the holistic view may reflect more or less an East Asian way of thinking. The theme of PME 31 in Korea was "School Mathematics for Humanity Education". The holistic view sympathizes with the Chinese philosophy of Laozi and Zhuangzi (Hirabayashi 2001; Wittmann 2001). Wittmann expresses it as "leaders should not interface with the natural powers and inclinations of their clients, but should instead build upon self-organization and offer help for self-help". Here, Wittmann considers mathematics education as design science, which sympathizes with our research stance.

We may then consider how we can realize the holistic view in the classroom. We think that it may be a basic didactical device for constructing rich situations in which students can identify various relationships (Hirabayashi 2001; Wittmann 2001; Yoshida 1999). In this sense, we turn out attention to Brousseau's (1997) didactical situation theory for that realization. Brousseau conceives of knowledge (knowing) as characterized by a (or some) "adidactical situation" and describes the learning process by which knowledge and situation develop reciprocally. He distinguishes three statuses of knowledge in the history of mathematics: protomathematical, paramathematical, and mathematical. An example he gives of the protomathematical stage is that al-Khowarizmi had constructed many ideas with rational numbers but not really with real numbers. An example of the paramathematical stage is function in the nineteenth century, where "in the absence of recognized mathematical status, their terms used are tools which respond to the needs of identification, formulation and communication and that their use is based on a semantic control". In the final mathematical stage, a concept is put "under the control of a mathematical theory" and has "its exact definition in terms of structures in which it intervenes and of the properties that it satisfies".

The three types of knowledge are sustained by situations for action, formulation, and validation, respectively. Students first construct their (informal) ideas from their

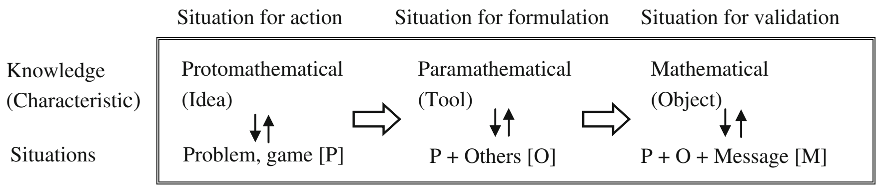


Fig. 1 Learning process in didactical situation theory

interaction with a fundamental situation like a game or a problem that is the situation for action. In situations for formulation they use their ideas practically or explain them to others, and the ideas may therefore take on a character of usability (tool) and the ideas of others enter the situation. Furthermore, in situations for validation, students objectify the messages that have been exchanged with others and develop their knowledge of mathematical and theoretical ideas. Okazaki (2003) schematized the process by which the three situations develop as follows (Fig. 1).

As a holistic characteristics, the process shows that students develop their ideas among various relationships through interacting with the situations and with other people. If we see the theory in terms of proving activity, it places us in situations for validation. The theory suggests that the preceding stages are necessary to reach proof learning. Brousseau (1997) also indicates that the paramathematical stage has continued for a long period in the history of mathematics. If we wish to invite all students into a world of proving, we should clarify what activities are needed in situations for action and formulation to prepare for effective proof learning, and how these proving activities can emerge and develop.

Next, we examine three components that constitute our analytical framework of students' development from empirical to deductive reasoning.

Transition from Empirical to Deductive Recognition

Harel and Sowder (2007) indicate several typical views of proof that students have: (A) external conviction proof schemes (authoritarian, ritual, and non-referential symbolic), (B) empirical proof schemes (inductive and perceptual), and (C) deductive proof schemes (transformational and axiomatic). (A) is a heteronomous case that depends on the authority of the teacher and textbook, or the superficial appearance of a written proof. (B) is a conception of proof by giving examples or experimenting and measuring. Kunimune (2003) reported that 92 % of eight graders and 77 % of ninth graders thought it acceptable to use experimentation and measurement. Chazan (1993) also reports that students have the belief that evidence is proof and that proof is just evidence. (C) is divided into deductive reasoning and the understanding of the meaning and roles of axioms.

(B) and (C) may be historically likened to pre-Greek and Greek mathematics, respectively. Harel and Sowder see not just that the Greeks pushed mathematics from a practical tool to the study of abstract entities and produced a proof method, but also that the consideration of the nature of existence and inference applied to existence progressed in parallel, namely that the object and the method are epistemologically dependent. If so, even if we teach students just the method of proof without changing the existence of a geometric figure, an inconsistency between the method and the object may arise. Moreover, they indicated that Greeks wished to create a consistent system that avoided paradoxes. The study of consistency presupposes the existence of a substantial number of properties and relations as was the actual situation in Greece. However, it does not seem that current geometry students have as many properties and relations as they feel necessary for organizing them. Hirabayashi (1991) indicated that "an introduction of proof may be impossible until students come to view geometric figure as a set of properties and relations, but not as the intuitive shape".

They also point to the continuity between the empirical and the deductive proof schemes. They state that "the construction of new knowledge does not take place in a vacuum but is shaped by existing knowledge," and "the empirical proof schemes are inevitable because natural, everyday thinking utilized examples so much. Moreover, these schemes have value in the doing and the creating of mathematics… The question is how to help students utilize their existing proof schemes, largely empirical and external, to help develop deductive proof schemes?" Our research question addresses this very point. We consider the teaching and curriculum for developing empirical proof schemes towards deductive schemes below.

The Research Framework for Analyzing the Transition from Empirical to Deductive Recognitions

Here, we consider the research framework for analyzing the transition process students follow towards geometric proof, which consists of inference, figure, and social influence.

Learning geometric construction is crucial for extending students' empirical to deductive reasoning. Mariotti (2000) states that "geometrical constructions have a theoretical meaning. The tools and rules of their use have a counterpart in the axioms and theorems of a theoretical system, so that any construction corresponds to a specific theorem," where she emphasized a need to shift from the construction procedure to a justification of the procedure itself. Tall (2008) also refers to the shift of the focus of attention from the steps of a procedure to the effect of the procedure in the compression from procedure to process.

Okazaki and Iwasaki (2003) identified several functions of geometric constructions in the teaching experiments: evoking shapes and putting their properties into play, constructing propositions, facilitating the recognition of hypothesis-conclusion, and enhancing the recognition of definition. However, we do not think this is possible with the current teaching where the procedures of drawing

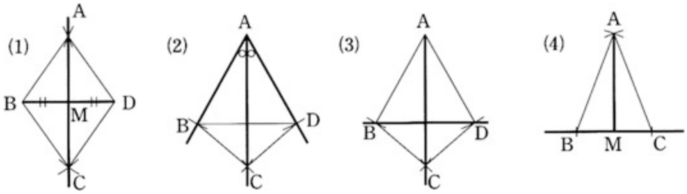


Fig. 2 Constructions of perpendicular bisector, angle bisector, and perpendicular based on a kite

perpendicular bisector, perpendicular, and angle bisector are taught in a manner such as: 'Draw a circle with center O and any radius and let the intersections with side OA and OB be C and D respectively. Draw circles with centers C and D of equal radius and let the intersection be P. Draw the ray OP'. Such knowledge may not provide a good basis for students in their proof learning.

Instead, we should consider that perpendicular bisector, perpendicular, and angle bisector are integrated in a kite or a rhombus (van Hiele 1986). Thus, if we imagine a kite, then we can construct each of these (Fig. 2). Because of the definition for constructing a kite ("two pairs of adjacent sides are equal"), perpendicular and angle bisector are then deduced as properties of the diagonals. If we symbolize this as if $AB = AD$ and $BC = DC$ then $AC \perp BD$, then it may be changed into a proposition. Thus, geometric construction is not just the procedure but also the tool for exploring the relationships between the properties of geometric figure. Therefore, it is important to adequately situate geometric construction in the transition stage.

Next, we must also consider that proving can be essentially characterized as the interactions between an individual's discursive inferences and visualizations (Koseki 1987; Duval 2002; Battista 2007). Murakami (1994) states that the figures in proof have characteristics of the proof model which shows structure, tool and variability. Also, Duval (1998) sees the figure in proof as configurations of several constituent gestalts in 1D, 2D, and 3D, and states that the relationships between several constituents need to be recognized by discursive language for each constituent figure. Thus it is necessary to recognize that the figure can represent geometric relations and contain the data, and if we can clarify the nature of reasoning using shape then proof teaching will be improved. However, the opportunities for reasoning by seeing a figure as a set of constituents are rarely encountered before the introduction of proof in the current curriculum.

Moreover, social influence is another factor that we regard as essential in the proof learning. Fawcett's (1938) study remains fresh today. He emphasizes the importance of re-examining the hypothesis of orienting human behavior behind belief or of clarifying the significance of definition and premise, with the purpose of cultivating critical and reflective thinking in accepting or rejecting conclusions. We think it necessary to clarify the characteristics of the social influences when students construct proof in classroom discussions.

Elementary stage: To comprehend the properties of figure inductively and empirically

↓ Transitional stage:

- Aspects of inference
- Aspects of figure
- Aspects of social influence

Secondary stage: To prove the statements deductively and using symbols and languages

Fig. 3 Triad of aspects for analyzing the transition towards geometric proof

The present study investigates the three aspects of the transitional stage in a bottom up manner through the design experiments of geometric constructions and transformations. Then, we regard the three aspects not as independent but as interrelated with each other (Fig. 3).

Design Experiments on the Transition Towards Proof

Participants, the Design of Teaching Units, and Methods

The research group conducted design experiments for the seventh grade unit on plane geometry. The participants were two classrooms at a public junior high school. We had 21 lessons of 40 min each for each classroom. The classrooms were typical in Japan; the students' abilities were judged as average for public schools according to the usual tests. They had already learned the properties of geometric figures empirically in elementary school.

We intended to redesign the lessons on geometric transformations and constructions to bridge the gap to formal proof in the eighth grade. We chose the hemp leaf figure (Fig. 4) from traditional Japanese design as a fundamental reference figure for the whole unit, and constructed the teaching based on figure. The teaching unit was divided into four subunits: (1) the discovery game for figures, (2) the

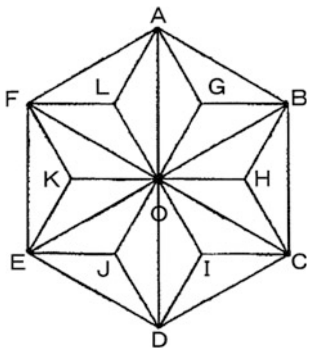


Fig. 4 The hemp leaf figure from traditional Japanese design

jintori game (transformations), (3) the discovery game for constructions, and (4) the construction and proof of the center of rotation. In designing the teaching sequence for each subunit, we referred to the ideas of situations for action, formulation and validation from Brousseau's (1997) didactical situation theory. Moreover, for teaching and learning, the classroom was often divided into two groups where the students were encouraged to propose their discoveries and refute them with each other. Brief lesson overviews are given in the following table.

Subunit	Nth lesson	Overview of the lesson
1	1	Making the hemp leaf using Origami
	2	The construction of a hexagon using compass and ruler
	3–7	The discovery game for figures. The students found the figures included in the hemp leaf (e.g. rhombus, kite, cube) and the properties (e.g. parallel, perpendicular, symmetry), and justified their findings to one another
	8–9	The properties of the figures of line symmetry and point symmetry
2	1	Introduction of three transformations and jintori game (described below)
	2	Games played within small groups and reflections
	3	Multiple transformations from a base triangle to target triangle
	4	Whole class discussion and summary
3	1	Introduction and discovery games for geometric constructions
	2–3	Construction of a kite and its relationship with the constructions of perpendicular, perpendicular bisector, and angle bisector
	4	Symbolizing the procedures of the three constructions
4	1–4	Construction and proof of the center of rotation (described below)

Our design experiment was conducted according to the methodologies of Cobb et al. (2003). All lessons were recorded on three video cameras and using field notes. We then made transcripts of the video data and conducted two types of data analysis; Ongoing analysis after each lesson and retrospective analysis after all the classroom activities had finished. In a latter analysis, we divided the transcript into the meaningful minimal units of conversation, examined the students' reasoning in each unit and how the reasoning was related to the three aspects of inference, figure and social influence. We also examined whether one activity related to more than one of the above aspects, rather than relating to just one. Next, we tried to interpret and understand the processes the students developed their reasoning by comparing the individual analyses of conversation units with each other and checking their consistency. Thus, we analyzed both the students' reasoning in each conversation unit and the whole process of the students' development. Finally, we tried to clarify the components of each aspect in the framework by integrating and summarizing our analysis.

Jintori (Position Taking) Game by Transformations in the Hemp Leaf

The second subunit dealt with geometric transformations in the Jintori (position taking) game. Our aim was to clarify how students may enhance their views of geometric figures and reasoning towards geometric proof. The rules of the game are that a pair of students first each decide upon a base position from the 18 isosceles triangles in the hemp leaf, then obtain the remaining positions alternately by translation, symmetry, and rotation from the base position. The person with the most positions wins. We also added the rule that they obtain a position if their explanation of how to transform from the base to the target position is accepted by the partner, since the intention is to improve recognition of relationships between figures and to have rich experiences of justification. We also prepared a tool for checking the transformations of each student.

(1) Introduction of game and imagining transformations

The teacher explained translation, line symmetry, and rotation with demonstration, and then introduced the game with a help of a student (Noza) seated in the front row. Noza (base position Δ*JDE*) and the teacher (Δ*HOC*) alternately got the positions shown in Fig. 5 (No.: turn, triangle: teacher, star: Noza, T, S, and R: translation, symmetry, and rotation). For example, Noza first rotated his base triangle 60° and obtained triangle FEK, the teacher next translated his base triangle and obtained triangle FLO, Naza then reflected his base triangle to obtain triangle EJO, and so on. After the demonstration, all students played the game in pairs without clear understaning of the rules. The teacher instructed them to articulate the direction of translation, the axis of symmetry, and the center and the angle of rotation.

We noted several characteristics among the students. First, they often envisioned their partner's move by pointing to the target place just after his or her initial utterance like "I chose *BE* as the axis". Second, they tried to get to positions that the

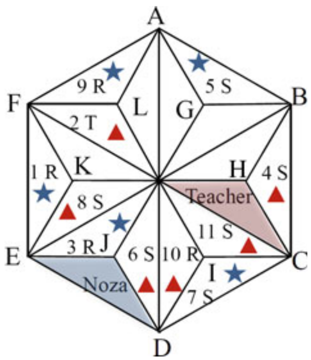
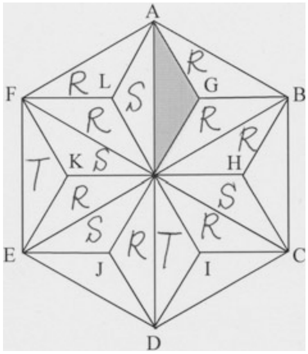


Fig. 5 Noza versus teacher

Fig. 6 Noza's note



partner may target. That is, they imagined how their partner may act. We also observed that they discussed with each other how the remaining positions could be reached beyond winning or losing.

(2) Utilizing transformations as tools

In the second lesson, we found that they used several strategies such as trying the same moves with their partner and changing the home position. Here, it seems that their interests were shifting from just enjoying the game to exploring the strategies and structure of the game. When the teacher asked what strategies they used, Noza stated that "I summarized all the positions I can get" (Fig. 6: base position: ΔAGO). We think that Noza's summary diagram is essentially his strategy for winning the game and more importantly that it played a role in orienting the learning of the structure of the hemp leaf by transformations. The teacher asked whether there are other transformations different from Noza, and many students showed their interest by stating that there were the multiple transformations. Moreover, although he also asked whether moves to the three empty places in Noza's note were possible, nobody was able to respond.

The teacher proposed that all students create Noza's diagram for the base triangle position, ΔKEF, after confirming that there were two types of the base triangles, outer and inner. We found that they came to fully comprehend what places can be reached. We consider that the transformations gradually became their tools for exploring the structure of the hemp leaf.

(3) Objectifying transformations

In the third lesson, the teacher again asked them whether it is possible to move to the empty places. After some small group discussion, Miya explained "We use two moves. We first rotate it 60° clockwise with the center at O and next flip it with BH as a crease. Then we can move it to the empty ΔBCH". However, the other students argued against this because the rules did not allow two consecutive transformations. The teacher then negotiated with the students and they agreed on the new rule that two moves are permitted. Under this rule they were able to fill all the empty places.

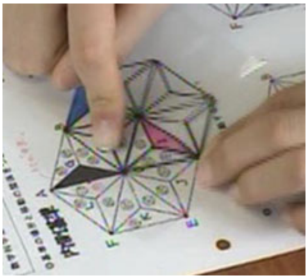


Fig. 7 Checking

The teacher again asked if these transformations were possible by a single move. Ura stated “if we use the midpoint of *BO* and rotate 180°, we can move to Δ*HCB*” and demonstrated it on the blackboard. The teacher gave each student a checking sheet and asked them to check it (Fig. 7). As many students had thought that one move was impossible, they were surprised and became inquisitive. Next, Noza proposed “we can maybe move to Δ*EJD* if we put *C* as the center and rotate 120° anticlockwise”, though the angle was wrong. The students checked the transformation using a sheet and marveled at being able to move to a place they had thought impossible. When the teacher asked Noza how he thought of the transformation, Noza stated “there is a rhombus *A*, *B*, *C*, and *O*, and *C*, *O*, *E*, and *D* also form a rhombus. Then, I displaced them.” Hence, he not only considered the move from Δ*AGO* to Δ*EJD*, but the correspondence in terms of the rhombuses *ABCO* and *EOCD*, which include the object figures (Fig. 8). This idea helped the students recognize the rotation more clearly.

At this point, the only remaining target was Δ*ICD*. The students were convinced that some move existed, saying “we can absolutely move it”. A little later, Hato discovered and stated that “point H is center and we rotate 120°” and all the students checked the transformation using a sheet. They were astonished about this, too. When we examined Noza’s notebook after the lesson, there was the picture of a kite in it (Fig. 9). While we assumed that the discoveries made by Ura, Noza, and Hato initially were exclusively for them, all students were able to confirm them, with feelings of surprise, imaginarily or using the checking sheet.

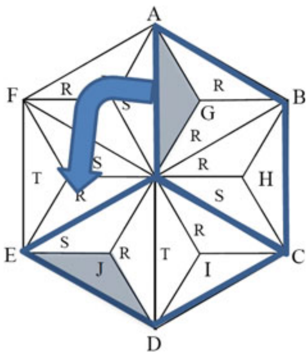


Fig. 8 The *rhombuses ABCO* and *EOCD*, which include the object figures

Fig. 9 Noza's notebook, there was the picture of a kite in it

(4) Discussion

We next discuss the three aspects of the transition in the framework.

In terms of inference, the idea of 2-fold correspondences as seen above can be described as a deductive reasoning; namely (a) the transformation from rhombuses *ABCO* to *EOCD* is possible (the major premise), (b) the triangle *AGO* in rhombus *ABCO* is same position as triangle *EJD* in *EOCD* (the minor premise), and therefore (c) the transformation from triangles *AGO* to *EJD* is possible (the conclusion). This reasoning echoes Duval's (1998) view of proof in which the relationships among the subfigures in the given figure are recognized through discursive language. It is noteworthy that this emerged from the students' own explanations.

The above is rooted in several conditions. First, the students used their abilities to imagine the transformations in their minds. They seemed to progress from predicting their partner's move from his or her words to trying to take positions that the partner would not wish them to take. Next, the students were able to examine the possibilities and limits of transformations. Here, Noza's note that organized the transformations beforehand was a breakthrough. It led to examining different moves and pursuing whether all positions can be reached. It is considered that the transformations became a tool for exploring the hemp leaf situation in the students' recognition.

Next, the aspects of figures are conceived with reference to the above ideas as seeing various correspondences or relationships among the figures and the umbrella figures. As a premise for this, it may be necessary to be able to variously combine the figures.

We should also pay attention to the above factors emerging with some social influences. It was by imagining the partner's moves and pursuing the others' thinking that the initial activity proceeded. The influence of Noza's note was crucial as stated above. Moreover, the idea of 2-fold correspondences emerged as a way of explaining concepts to others. Thus, it is considered that the students' learning progressed as a process that internalizes and objectifies others peoples' actions and thinking.

Geometric Construction of the Center of Rotation and Its Proof

The task in the fourth unit was the construction of the center of rotation and the corresponding proof, since the students had already realized that rotation played a large role in the Jintori game in the second subunit and had just learned geometric constructions in the third subunit. As the students had not been taught what formal proof was, we regarded it as important that their arguments were based on some given premises and that they had feelings of conviction and persuasion.

In the first lesson the teacher explained that in the jintori game, rotation made moves from one place to many other places possible. Then, after putting two triangles from the hemp leaf in arbitrary places on the blackboard, he asked "We want to superpose triangle ABC on triangle DEF. Where is the center of rotation?" (Fig. 10).

(1) Investigation by trial and error

The students first formed small groups to explore the problem. At the end of the lesson, each group presented its investigation to the whole class (Fig. 11).

Construction (a) is an exploration based on the figures done by connecting an intuitively assumed point with the vertices of the triangles. Construction (b) shows an equilateral triangle that has edges of the length of AD. In (c), the students find the points equidistant from the vertices B and E and connect them with the other vertices, where they try to find the center by trial and error. In (d), we found from the remarks that they considered the perpendicular bisectors of AF, CD and BE. In (e), they connect the corresponding points to each other. We think that all ideas are effective in solving the problem, since if these are integrated it would lead to the discovery of how to construct the center of rotation.

(2) Construction using the perpendicular bisector

In the second lesson, a student, Simi, connected the pairs of corresponding points A and D, B and E, and C and F, and drew the perpendicular bisectors of the three

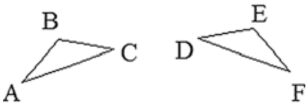


Fig. 10 How can we superpose *triangle ABC* on *triangle DEF*?

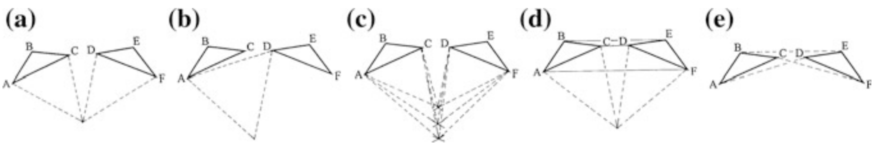


Fig. 11 Students' constructions (**a–e**)

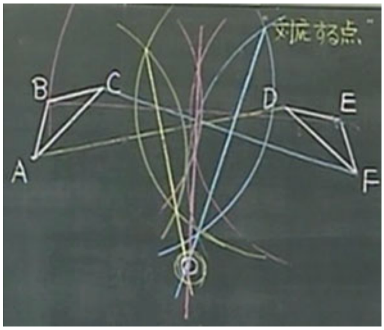


Fig. 12 Perpendicular bisectors of the tree segments and consciousness of concurrency by making a *small circle*

segments. Although the third perpendicular bisector deviated a little from the intersection of first two bisectors, he displayed these as a point by making a small circle (Fig. 12). He seemed to be conscious of concurrency. Next, Miya drew three circles centered at the point Simi found that passed through the points A, B and C, and stated that the points D, E and F were on each circle (Fig. 13). The classmates agreed. By the end of the second lesson, Simi's construction was accepted as a way of finding the center.

(3) Justification of the construction

In the third lesson, the teacher asked the students to justify why Simi's construction was correct. After the individual activities, Seki stated, "I used the hemp leaf" (Fig. 14) and explained:

Seki The transformation from triangle AGO to CIO is possible by a rotation with the center at O. And we connect the corresponding points of triangles AGO and CIO. Then the intersection of the perpendicular bisectors of segments AC and GI is point O. So it is correct.

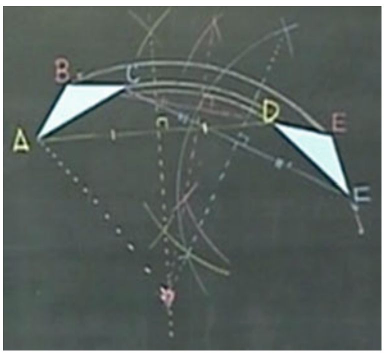


Fig. 13 Miya drew three *circles* centered at the point

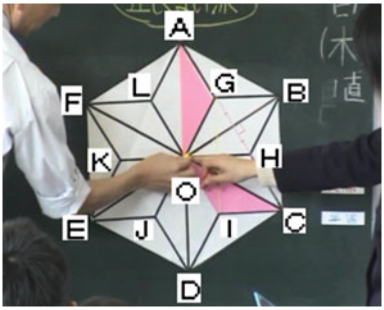


Fig. 14 Seki justified through the hemp leaf situation

They became more convinced by her idea that used an example from the hemp leaf and thus extended the application field inductively. We found that the activities involving the hemp leaf in the former subunits formed a foundation. However, Oda then tried to refute the idea.

Oda When they are lined up like triangles FLO and OHC, the perpendicular bisectors of the segments connecting the corresponding points do not intersect. So the explanation does not apply to the parallel case. Therefore, it can't be valid (Fig. 15).

Many students agreed with the counterexample Oda presented. However, at the beginning of the fourth lesson Oda himself stated his opinion that the explanation is valid if we exclude the parallel cases, and the classmates again agreed with it. Here, while the teacher reformulated it as a proposition, his argument seemed to be acknowledged as natural by the others, as they were able to immediately begin the proof construction activities in small groups. After the group session, some students explained:

Matu The center of A and D is this line, the center of C and F is that line. Also, the center of the circle of B and E is the perpendicular bisector. So, the intersection O of these lines is the center of all circles (Fig. 16).

Iori O is a point that satisfies all things.

Noza They are equal anywhere on perpendicular bisector CF.

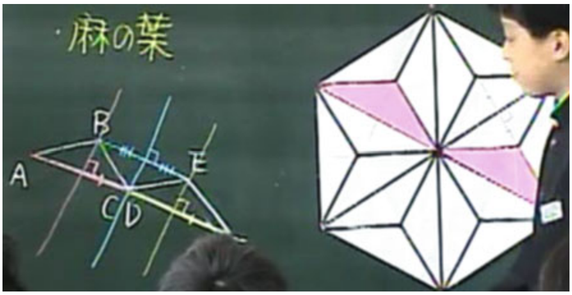


Fig. 15 Oda gave a counter-example

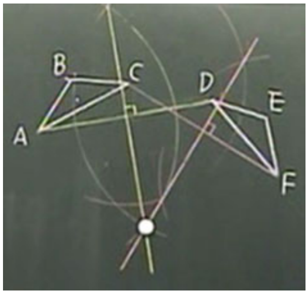


Fig. 16 Matu tried to explain that *O* is the point of intersection of the perpendicular bisectors

We think that Matu was explaining that if O is the point of intersection of the perpendicular bisectors of segments AD and CF, then O is also at the center of a circle passing through B and E, and is the center of rotation of the figure. The teacher then augmented the sketch (Fig. 17), which was followed by Noza's statement "Because O is equidistant from A and D as well as from C and F, it is a center". The other students agreed with this statement. We observe here that the teacher's indication of the relationship using the segments in Fig. 17 guided Noza's second statement.

To complete the proof, the students had only to show the congruence between △OAC and △DOF by adding the statement of AC = DF. However, because they had not learned the theorem of congruence of two triangles, such complete proof may be a next stage for them.

(4) Discussion

We now discuss the three aspects of transition in the framework.

We consider there to be two critical points in inference: examining the validity and extent of the proposition in inductive and empirical ways and by presenting a counterexample, and objectifying the construction procedure and using it for justification.

There are several important reasons for examining the validity and extent of the proposition. First, the students' justification began with a demonstration to confirm

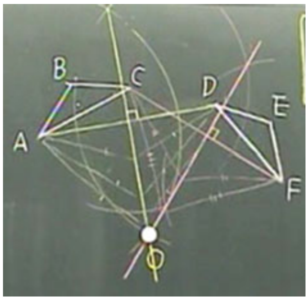


Fig. 17 The teacher drew the segments *AO*, *DO*, *CO*, and *FO*

whether the corresponding points were on the same circles, and they checked inductively to see if the construction method remained valid in the hemp leaf situation. It is considered that they then were able to enhance their conviction of the proposition through these approaches. Second, they reexamined the extent to which the construction method was sound after Oda proposed the counterexample of the parallel case, and the teacher confirmed the method as a proposition in the if-then form. As the students could start their proof construction after these activities, it is considered that these activities played a role in shifting empirical reasoning to deductive reasoning.

Furthermore, the students' proving was proceeded by objectifying the construction procedure and then using it to support the justification. The process is outlined as follows. First, Matsu reflected on the construction procedure and stated that the distances from two corresponding points to any point on the perpendicular bisector were equal to each other. Next, the teacher visualized these as segments. Finally, Noza made the logical step 'any point on the perpendicular bisector of segment AD is equidistant from A and D, and likewise any point on the bisector of segment CF is equidistant from C and F. The intersection O of the perpendicular bisectors satisfies OA = OD and OC = OF. Therefore, O is a center of rotation'. Note that the students reinterpreted the construction procedures as the conditions for justification and drew a conclusion based on them.

Next, we examine the aspects of figure so as to understand the development of their reasoning in terms of recognition of the figures. First, we focus on the students drawing the equilateral triangles in exploring the construction and trying to find the center of rotation by changing the radius. The figures here were not static for them but entailed images of equilateral relations and the continuous movement of points. It is considered here that these were the resources that they used to discover the construction method. Second, it important to note that, when a student found that the distances from two corresponding points to any point on the perpendicular bisector were equal to each other in the justification stage, the teacher visualized these as segments, because through that visualization the later reasoning proceeded to a successful conclusion. The segments here are variable and with equality relations, and thus consist of a part of the reasoning. The the figures implying movement, variability or relationship may induce the construction and proving.

Finally, regarding the aspect of social influences, we first found that the counterexample led to exploring the extent to which the construction method was valid and to making sense of the proposition. Also, through the proving process, the efforts to reformulate and improve each others' explanations helped to develop their reasoning. The social influences in this subunit can be listed as follows; the classroom lessons in the form of conjectures, refutations and agreements, absorbing criticism from others, giving the counterexample and reflecting on its meaning, and stating other people's ideas more clearly.

Designing and Clarifying the Transition from Empirical to Deductive Recognition

We here clarify our framework for transition by incorporating the findings of our analysis in design experiments and give some suggestions on the design of lessons and units for the transition.

On the Framework for the Transition from Empirical to Deductive Recognition

The experiments on the aspect of inference in the second unit (U2) identified the importance of finding the transformation by using and combining figures and properties, including examination of different or multiple transformations and working with the idea of 2-fold correspondences. In particular, we note that the idea of 2-fold correspondence takes on an aspect of deductive reasoning, though it may not be developed using only language. In the fourth unit (U4), we identified two main things. One is the enhancing recognition of the proposition through examining the validity and extent of the construction method, in the same way that students checked validity empirically in the special situation of hemp leaf and shifted their focus on the limit within which the construction method worked. The other is that it was necessary to objectify the construction procedure and using it for justification. We regard this as deductive, and also as the reflective abstraction that extracts ideas from actions of the construction and reorganizes them mathematically (Piaget 2000).

The aspects of figure, as emerging from inference, are the flip side of the same coin. For example, the idea of 2-fold correspondences in U2 was dependent on seeing the configurations and relations in the figure. Additionally, the reasoning in U4 was based on seeing the figure dynamically to actualize the meaning of construction. It is thus considered that it is crucial to be able to see the figure as implying variables and relations for the transition to proof.

Last, when we examine the aspects of social influence, we may see whether the first two aspects go hand in hand with this one. In U2, we observe the students' attitudes towards assuming or pursuing others' actions and thoughts, trying transformations different from others, and revising explanations in their justification. Further, in U4 we may consider the attitude of giving a counterexample and efforts to reformulate and revise the explanations of others as solid background factors for development from the game to proving.

We now summarize these three aspects of the framework in Fig. 18. We think that this framework will play a useful role in designing classroom lessons for transition and in evaluating students' activities. Moreover, we think that with refinement and further design experiments, this framework can be applied more widely, in particular to the upper grades in elementary school. It is our intention to further investigate this.

<u>Elementary stage</u>: To comprehend the properties of figure inductively and empirically

<u>Transitional stage</u>:

A. Aspects of inference

(1) to find methods of construction and transformation by using and combining figures and their properties.

(2) to confirm whether a proposition works empirically and the extent to which it works inductively and with counterexamples, and to enhance the understanding of the proposition.

(3) to reinterpret and use the construction procedures as conditions for proving.

B. Aspects of figure

(1) to select suitable figures and to use them in combination for constructions and inference.

(2) to see figures as variables which can be changed by dynamic transformations.

(3) to see figures as relations among the whole and partial figures through reasoning diagrammatically.

C. Aspects of social influence

(1) to base the learning environment on students' conjectures, refutations and consensus.

(2) to make and develop conjectures while accepting criticism from others.

(3) to interpret others' explanations and express them more precisely.

<u>Secondary stage</u>: To prove the statements deductively and using symbols and languages

Fig. 18 Revised triad of aspects for analyzing the transition towards geometric proof

Note that our discussion does not suggest that these three aspects work separately, but rather that they have an integrated and interdependent nature.

Moreover, if we think that a certain type of learning transformations and constructions produced the above aspects in students' recognition, it may be reasonable to see these aspects as in Fig. 19. Note that the students' reuse of the construction procedures in their justifications and seeing the figure as implying variables and relations, both of which aided the proving process, are reflectively abstracted from the dynamic actions when learning constructions and transformations. We may thus proceed to study the curriculum for how teaching the content of geometric construction, transformation and proof can be reorganized as teaching units for introducing secondary geometry to students.

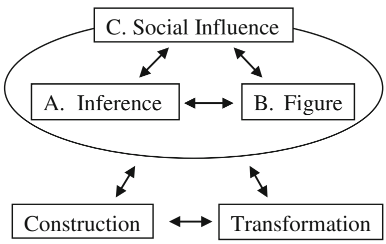


Fig. 19 A modality of three aspects

Some Suggestions for Designing Lessons for the Transition to Geometric Proof

This paper considers that, to accomplish the transition to geometric proof, it is essential not only to propose the theoretical framework but also to develop and show the classroom practices and situations for the transition, which was one of the purposes of this paper. We finish by commenting on the design of the lessons and units in terms of the continuity of context and necessity, the recursive growth of explanation, and the humanistic aspects.

Let us first consider that the continuity of context through the whole unit was crucial for the students' construction of the proof. This should be made clear by conceiving of the fourth unit as the situation for validation in terms of didactical situation theory. The first three units can then be regarded as situations for action and formulation, where the fundamental ideas and concepts, such as rotation and perpendicular bisector, the ways of reasoning using the figure, and the learning style of conjecture and refutation were fostered. Such stages seem to be inevitable when we consider how long the stage of formulation has lasted in the history of mathematics. Moreover, as the students explored proving by returning to the hemp leaf situation, we can conclude that the Jintori game provided a solid foundation for them as the situation for action. If this situation had not existed, they would not have explored the meaning of the proposition and the extent to which the construction method works. In other words, we consider that learning the three aspects of our transition framework in the same situation facilitated the students' development of proving.

In association with the above points, our experiments show that the students' activities in understanding the theorem itself occupied more time than of the actual proof. In the Jintori game, it was important to find whether a single rotation allowed the move from the base triangle to the separate target triangle. Additionally, in constructing the center of rotation the exploration of whether the three perpendicular bisectors intersect was important. If we regard the activity of understanding the theorem as a situation for formulation in didactical situation theory, it can be considered that more focus should be given to understanding theorems and how this understanding is intertwined with the proving process. This is important for solving the problem of necessity.

Second, let us consider that it is important to design the lessons for transition as a generalization process. In our experiments, the idea of 2-fold correspondences in the Jintori game has the same structure as the proof of the center of rotation. They are both deductive, while the former is action based and the latter is a language-symbol based exploration of the general case (Fig. 20). Thus the students can reciprocate between the particular and the general cases. We use the terminology of Pirie and Kieren (1994), concluding that students develop their understanding of the latter situation through folding back to the former situation. This implies that the students' proving ideas may grow transcendent-recursively. This process will be an important topic of research.

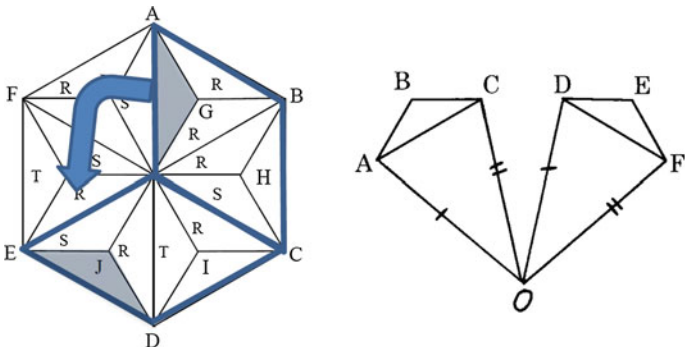


Fig. 20 Action-based reasoning (particular) and language-symbol based reasoning with the same structure

Finally, it is important to consider the aspects of holistic perspective when introducing proof to students. That is, students' justification and proving progress parallel with exploring and explaining curious phenomenon, in the classroom environments where there is collaboration, the acceptance of criticism from others, and refining of other students' explanations, as our design experiment demonstrated. Beyond this, it is important for students to have encounters of experience so that the self is moved, as the idea of two fold correspondence in the hemp leaf for instance lead to the students' feelings of surprise and good emotion. We may regard this as an indication of the relationship between humanity and mathematics.

Final Comments

In view of the gap between empirical and deductive reasoning in geometry, this paper explored connections existing between two forms of reasoning and how the connections can be developed. In order to this, design experiments were carried out from a holistic perspective. One reason for adopting the holistic perspective is the belief that students can better grasp mathematical concepts with affirmative emotions such as interest, surprise and beauty, whereas many students have negative experience related to their difficulty in understanding proof. Therefore, the study investigated not only the proving process, but also how the students enhance their power of reasoning from the three aspects of inference, figure, and social influence, and how they can develop proving as their inquiry.

It is hoped that the theoritical framework discussed in this paper will contribute to the researches clarifying the development from empirical to deductive reasoning. More importantly, I hope that the study make a substantial contribution to practical requirements for teachers and students in the mathematics classroom.

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Laying Foundations for Statistical Inference

Maxine Pfannkuch and Chris J. Wild

Abstract In this paper we give an overview of a five-year research project on the development of a conceptual pathway across the curriculum for learning inference. The rationale for why statistical inference should be part of students' learning experiences and some of our long deliberations on explicating the conceptual foundations necessary for a staged introduction to inference are described. Implementing such a pathway in classrooms required the development of new dynamic visualizations, verbalizations, ways of reasoning, learning trajectories and resource material, some of which will be elucidated. The trialing of the learning trajectories in many classrooms with students from age 13 to over 20, including some of the issues that arose, are briefly discussed. Questions arising from our approach to introducing students to inferential ideas are considered.

Keywords Secondary-university students · Sampling variability · Visualizations · Verbalizations

Introduction

Traditionally statistical inference is the focus of the final year of high school with previous learning experiences featuring constructing plots and describing them. Research over many years (e.g., Chance et al. 2004) has consistently demonstrated that for the majority of students formal statistical inferential reasoning eludes them. One conjectured reason is that inference is grounded in mathematics and is presented as a procedure to follow. The resultant mathematical manipulations and calculations then act as obstacles to understanding the thinking behind inference. The second

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conjectured reason is that concepts underlying inference such as sample, population, and sampling variability should be developed over time in the curriculum rather than presenting such a complex network of integrated ideas in the final school year.

In the last decade, with increased access to technology, some curricula have adopted an exploratory data analysis (EDA) approach allowing students to investigate real data to look for interesting patterns and trends. But as Konold and Kazak (2008, p. 1) explain:

> As researchers began studying settings in which students were introduced to EDA, an unsettling picture soon emerged. Students given considerable exposure to and instruction in data-analysis techniques nevertheless had difficulties performing one of the most basic tasks in analyzing data — judging whether two groups appeared different by comparing their averages or the approximate centers of their distributions.

Pfannkuch (2006) in her research also noted that students had difficulties in comparing two groups. One reason was students did not know what game they were playing: Game One involving reasoning only about the sample or Game Two involving reasoning about the population from the sample. If students were meant to be playing Game Two then another reason for their difficulties was that they had no learning experiences of sampling variability and other underpinning concepts such as the notion of a population distribution. In further research Pfannkuch (2008, 2011) noted that even when students experienced sampling variability for quantitative and qualitative data the basic problem of how they should judge whether one group tended to have larger values than another group still remained. These required understandings to make a judgment were conceived as *informal inference* by an International Forum of Statistical Reasoning, Thinking and Literacy led and founded by Joan Garfield and Dani Ben-Zvi (see special issue of Statistics Education Research Journal, Pratt and Ainley 2008). Biehler (2011), when reflecting on the research presented at the 2011 Forum on how students were making inferences, argued that the responses of students across the year levels from junior to middle to senior needed to provide some evidence of growth in conceptual understanding and he raised questions about how that growth might be characterized and the type of learning approaches that might contribute to such growth.

Noting that problems were occurring with statistical inference using both the traditional and EDA approaches and the fact that comparing two groups was a problem for New Zealand teachers with respect to national assessment of 15 year-olds, we embarked on a journey to find a pathway for realizing statistical inference ideas and in particular to place substantive conceptual foundations into the statistics curriculum.

Overview of Research Project

Our five-year project used a four-phase design research cycle incorporating identification of the problematic situation, promulgation of the conceptual foundations to inform the design process, designing the learning trajectories, and testing them in the classroom (Hjalmarson and Lesh 2008). It is an interactive cycle whereby, for example, the design of learning trajectories can raise conceptual issues, which need to be revisited. The cycle is continued as new problems are identified, hypotheses are generated and theories are conjectured. The instructional materials are designed in an attempt *to engineer and support a new type of learning and reasoning*. Design research engages researchers in improving education and provides results that can be readily used by practitioners (Bakker 2004a; Cobb and Gravemeijer 2008; Schwartz et al. 2008). The methodology employed in the project used a mixed methods approach of pre- and post-tests, interviews, observations, and reflections.

In the first year of the project in 2008 a statistics education researcher and three statisticians debated, argued, and mapped out a potential pathway for introducing inferential ideas from the beginning of secondary school to the first year of university. In the next two years a team of two statisticians, two statistics education researchers, and nine teachers collaboratively worked through two design research cycles. Four classes in both years (a total of about 200 students), whose teachers were in the project team, participated. These classes covered a range of school socio-economic levels, student abilities, ethnicities and ages (13–16). In the final two years, 2011 and 2012, a team of 33 people consisting of statisticians, education researchers, Year 13 (last year of high school) teachers, and university statistics lecturers collaborated on identifying conceptual underpinnings and developing learning trajectories for statistical inference using bootstrapping and randomization methods. About 3000 university students and 200 Year 13 students participated in the implementation. The main data collected were: pre- and post-tests from all students, pre- and post-interviews of a sample of these students, videos of some classes implementing the teaching unit, and teacher reflections.

Conceptual Foundations: Informal Statistical Inference

A staged introduction to the conceptual foundations of statistical inference in the Year 10 (aged about 14) to Year 13 (aged about 17) New Zealand secondary school curriculum was constrained by time (four weeks per year for Years 10–12), access to technology (limited to one computer and a data projector per classroom), resources (very limited school budgets), and national assessment at Years 11, 12, and 13 with its requirement to show progression of growth in skills, concepts, and thinking. With these constraints in mind we chose to start with teachers' current problem in national assessment on how to compare two groups using box plots. As Biehler (1997) had already noted, a rich conceptual repertoire underpinned these comparisons such as sample, population, sampling variability, distribution, and sample size effect. We deliberated on and debated many conceptual foundations in response to the literature, analyses of data from the research project, and our own reflections. In this section we briefly discuss three issues, namely, making a call, sample-population ideas, and sampling variability, give some examples from our research and highlight some pivotal moments in our thinking.

Making a Call

First, we deliberated on how students could make a call directly from the comparison of two box plots that had a statistically sound basis. Starting with current class practice and what was possible using hands-on simulations we determined that initially the sample size should be fixed at about 30 and through running simulations found a quick rule-of-thumb for making or not making a call in terms of shift of the boxes (the middle 50 %) and location of the medians (Fig. 1). To progress from these notions the next stage was to consider that spread and sample size matter when making a call, which is then taken further to consider an informal confidence interval based on the work of Tukey (see Wild et al. 2011 for more detail). At the next level, still using the idea of taking a random sample from the population the confidence interval is formalized using the bootstrap method. In addition, students are introduced to comparative experiments where random samples are not taken from populations rather convenient samples are used and units are randomly allocated to one of two treatment groups and hence the randomization method is used for making or not making a claim.

Introducing secondary students to making a claim or call through the direct comparison of box plots necessitated the development of new dynamic visualizations and verbalizations. In particular, a breakthrough in determining some of the verbalizations occurred when we were looking at the example in Fig. 2 from the GAISE K-12 Report (2007, p. 47).

> The median sodium content for poultry hot dogs is 430 mg, almost 50 mg more than the median sodium content for beef hot dogs. The medians indicate that a typical value for the sodium content of poultry hot dogs is greater than a typical value for beef hot dogs. The range for the beef hot dogs is 392 mg, versus 231 mg for the poultry hot dogs. The ranges indicate that, overall, there is more spread (variation) in the sodium content of beef hot dogs than poultry hot dogs.

We realized that the verbalizations in this example were mixing up descriptive and inferential thoughts. That is, describing what could be seen in the plots and inferring what might be happening back in the populations. We noticed the subtleness of the language with the use of the definite article when referring to the sample and non-use of the definite article when referring to the population. We wondered how students not exposed to sample and population ideas would know which statements were descriptive and which were inferential. We realized that for learners we would need to be careful in separating out the descriptive from the inferential but often the boundary between them was blurred. Another consideration was contextual thoughts that also needed to be invoked when making a claim (for the full discussion see Pfannkuch et al. 2010).

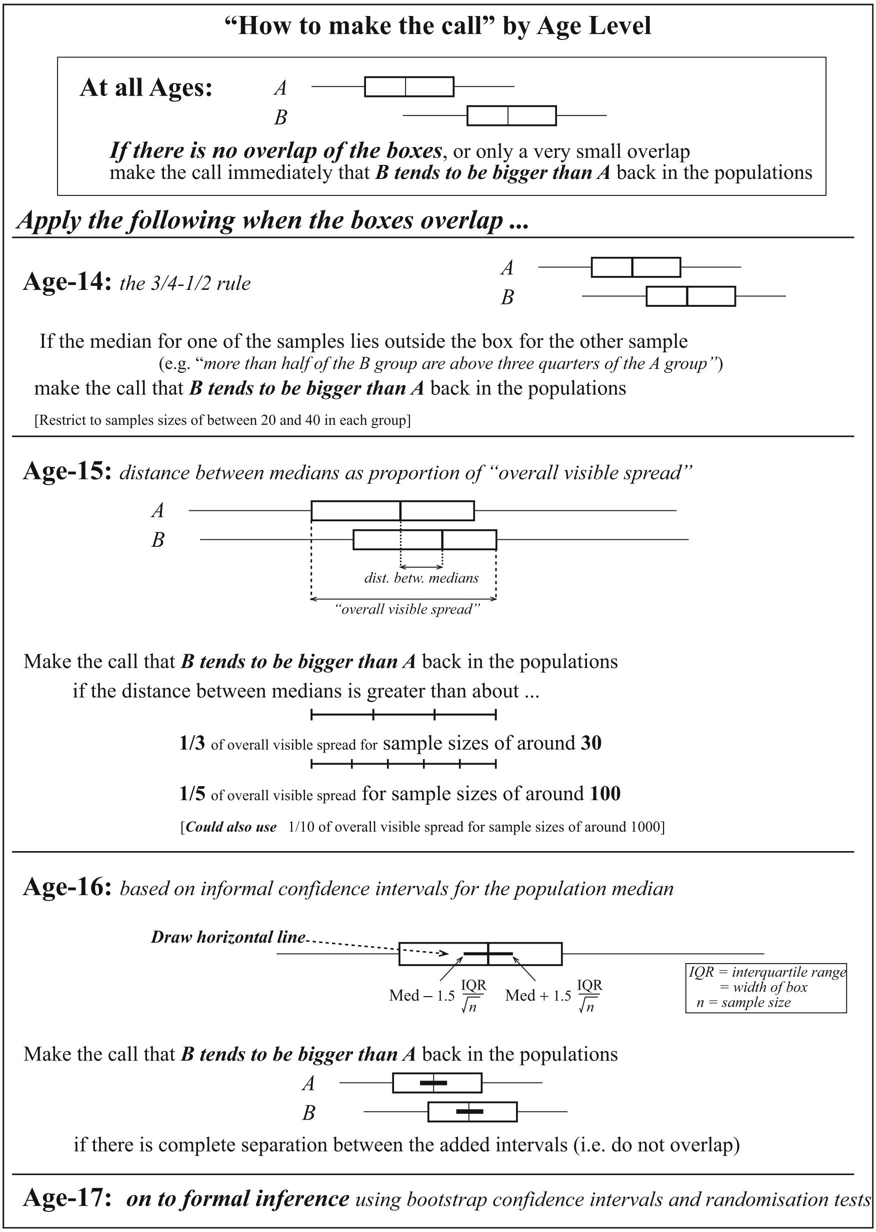


Fig. 1 Guidelines for making a claim

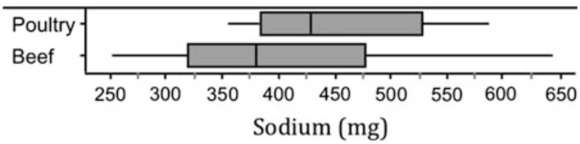


Fig. 2 Sodium content of hot dogs

Sample-Population Ideas

A second conceptual foundation we deliberated on was the notion of sample and population and the links between them. From previous research (Pfannkuch 2006), we knew that students thought they were reasoning about the sample not the population. To assist students to conceptualize that they were reasoning about the population from the sample, we gave students population bags of over 600 datacards from which they drew a sample. Each datacard had 13 pieces of information about a student such as height, year level and gender. The datacard information about each student was drawn from a CensusAtSchool online survey of over 30,000 New Zealand students. To reduce confusion about the New Zealand school student population, the subpopulation that answered the survey and the sub-subpopulation of the datacards, we chose, after much debate, not to refer to all these populations but simply present the bag of datacards as the population. Students were gradually introduced to the notion that they were reasoning about a population from a sample through individually physically sampling from the population bag, plotting the data, and comparing their plots to other students' plots to see whether the message in the data was similar. Teachers in the study reported that a powerful reminder to students that they were reasoning about the population was the population bag, to which they frequently drew students' attention.

The notion of a population in a bag and a sample as a collection of individuals on which measurements are taken, however, does not give students a sense of the population distribution of the variable of interest. A retrospective analysis of the 2009 student data alerted us to the fact that the students had an impoverished understanding of and language for distributional shape and underlying plausible shapes of population distributions (Pfannkuch et al. 2011a). We regarded the situation as problematic, since in practice contextual knowledge and statistical experience are used to conjecture an expected population distributional shape and if the sample distribution is at variance with the expected shape then further investigation of the data is warranted. Furthermore, we considered that students' statistical knowledge should include images of population distributions of everyday situations such as height and reaction times. The question was raised about how students could build the contextual knowledge necessary for thinking about population distributions and consequently sample distributions. The breakthrough occurred when we realized that *the story in the data was often in its distributional shape*; that shape is a key foundational concept. In a conversation with Cliff Konold in 2009 it emerged that there was little research in this area but Konold had realized

Fig. 3 Resting pulse rate of a sample of 40 Year 13 NZ students

from his own research, and Bakker (2004b) also, that students needed assistance in *seeing shape*. Closely allied with such a problem is the well-documented difficulty for students to conceptualize and reason from an aggregate perspective of data (Konold et al. 2004). Consequently we devised new learning trajectories to build students' conceptions, contextual and statistical knowledge, and the language for engaging with shape. One learning task involved students matching data plots to context, while in another students were given the context and asked to draw the shape of the plot. Our research findings on conceptualizing shape are currently being reported (see Arnold and Pfannkuch 2012).

To understand some of our research in this area, consider the plot in Fig. 3. Before our teaching intervention many students sketched the shape of the plot by drawing an outline or "skyline" and described in detail how the plot went up and down. After the intervention consider the responses from interviews of two students aged 14 who now sketched curves over the plot. One student thought the underlying population shape would be normal with perhaps a right skew. On being questioned further she said she expected this shape based on her general knowledge but this particular sample just happened to be bimodal. The other student believed the underlying population shape was bimodal because of fit and unfit people. Both these students seem to be beginning to realise that shape is closely aligned with unlocking the story in the data, the contextual part, with the first student beginning to consider also the notion of sampling variability, the statistical part.

Sampling Variability

A third conceptual foundation was building students' concepts about sampling variability. The breakthrough occurred when Chris Wild recognized the potential to develop dynamic animations to visualize sampling variability when he was listening to how Pfannkuch (2008) conducted a small-scale teaching intervention on developing students' notions of sampling variability. She explained how plots of random samples were projected on to a whiteboard and each time the sample median in the case of quantitative data or the sample percentage in the case of qualitative data was recorded manually with a horizontal line. In this way a sampling-variation band was built up and students started to become aware of sampling variability including the sample size effect. By taking this idea and realizing that the visualization of sampling variability was more effective when connected directly with the plot rather than increasing cognitive load by referring to a sampling distribution of a statistic, Wild developed a number of dynamic visualizations based on the research literature to assist learning. In order for students to fully conceptualize sampling variability related hands-on activities were developed with attention on developing verbalizations, and imagery, which included gestures (see Arnold et al. 2011).

With regard to developing students' sampling variability reasoning, we believe, that our learning trajectories are assisting students' awareness when they are confronted with static sample distribution plots. For example, we will describe one 14 year-old student's reasoning before and after the teaching intervention. In Fig. 4 is a question that was in the pre- and post-test.

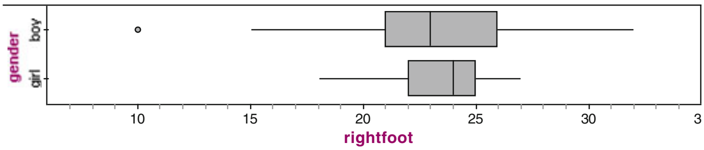
In the pre-test the student gave the following written response:

> No I would not make the same claim as Emma because the Year 8 NZ boys right foot lengths are spread out across the graph whereas the Year 8 NZ girls right foot lengths are found close together at the place where a normal bell curve would be found.

Note that she is not making the same call as Emma and makes her judgment on the spread of the data. Also some prior knowledge about how the data would be distributed in a normal curve is used, which she said she had learnt in science. In her interview she was asked about drawing another random sample, to sketch what the plots might look like, and whether she would make the same call. Intuitively she knew that another sample would give different plots. She sketched them with a focus on the spread and said:

> I think that, well the overall results would still be the same but I've just spread out the girls more across the graph and made the boys a bit more close to one point on the graph.

Emma is interested in comparing the right foot lengths (in cm) of Year 8 NZ boys and girls. Emma takes a *random sample* of 30 Year 8 NZ boys and a *random sample* of 30 Year 8 NZ girls. She can find out their right foot lengths without shoes on in cm. She plots her graph correctly.



Emma looks at her graph and claims that the right foot lengths of Year 8 NZ girls tend to be bigger than the right foot lengths of Year 8 NZ boys. Would you make the same claim as Emma? Why?

Fig. 4 Pre- and post-test question

In the post-test her written response was:

> No I would not make the same claim as Emma. I would not be prepared to make this claim because on Emma's box plots both the medians are in the overlap. This makes it hard to make an accurate claim because I know that another random sample could easily show the medians the other way round.

Note that her focus is now on the medians and she is aware of the variation in them due to sampling and that she cannot make a call. To check out her imagery of a sampling-variation band for the medians the interviewer asked her to use her hands as the box plots to show her how the plots might change if she took many random samples, which she demonstrated in a similar manner to the dynamic visualizations and said:

> Okay so maybe they would go like this and then maybe like this and the next one would be like this again and then maybe the next one would be like this. There's not much difference but …

Hence from our research data we believe that this student and many others are gaining a sense of sampling variability (see Pfannkuch et al. 2015 for a fuller discussion).

Conceptual Foundations: Formal Statistical Inference

Statistics students at Year 13 and first-year university levels are introduced to formal statistical inference. Since statistical practice is rapidly changing to simulation methods we developed learning trajectories using the bootstrap method for sample-to-population inference and the randomization method for experiment-to-causal inference. A key idea we used with these methods was to mimic the data production process (Hesterberg 2006). We again especially developed dynamic visual inference tools that can be used for learning and analysis (see VIT—visual inference tools—http://www.stat.auckland.ac.nz/~wild/VIT). In this section we will briefly highlight one conceptual issue among many issues that arose for each method based on data from a pilot study in 2011.

Bootstrap Method

From previous work on sampling variability, our learning trajectory incorporated the prior visual imagery of a sampling-variation band, which transformed into a re-sampling distribution of a statistic so that the confidence interval was quantified (Fig. 5). All the students in the pilot study seemed to be able to articulate how the bootstrap method worked to compute a confidence interval and to explain all the components of the dynamic visualizations for the bootstrap. When faced with a word-only question in the post-test about confidence intervals, however, the

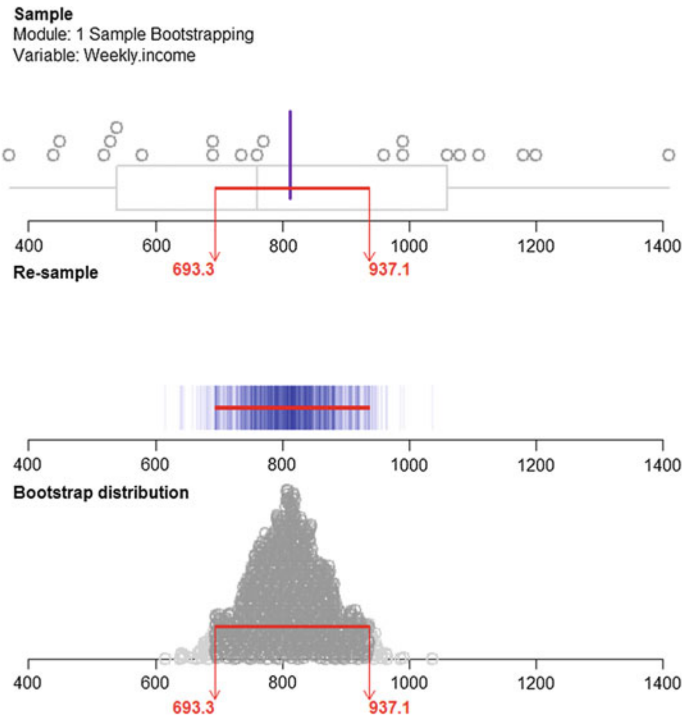


Fig. 5 Graphics panel of software for performing bootstrap process

concept and image of a re-sampling distribution of a statistic seemed to slip from their minds. We then appreciated that we did not pay enough attention to a key conceptual transition. In primary school students view data displays as distributions of data while at secondary school our conceptual pathway led students to view them as distributions of sample data from which they make inferences about populations. With the bootstrap method students needed a new conception of a familiar distribution as a distribution of a statistic. We realized our verbalizations, learning trajectories, and resources were remiss in addressing such a key conceptual transition (see Parsonage et al. 2012 for more detail).

Randomization Method

Since we worked on the principle for beginning formal inference that the method must mimic the data production process, students were introduced into a new world of experiments and the very new idea of *causal* inference via the randomization method. For example, in comparative experiments volunteer participants are randomly allocated to one of two groups, the treatment or the control. The experiment is then conducted and the data are plotted. One question for us was how to explain the observed difference in the two groups. We decided to use "the treatment is effective" or "chance is acting alone" as the two possible explanations for the observed difference (see Pfannkuch et al. 2011b, for full discussion on many other issues with respect to verbalizations and language). In the pilot study some students reported difficulty in understanding the concept of "chance alone" (see Budgett et al. 2012, for more detail on findings from the study). Furthermore, the concept that the "treatment is effective" comprises the two components of chance and treatment. The problem of how we might explicate such concepts started to be resolved when we developed dynamic visualizations to show, for example, the observed difference in weights, which could be obtained through random allocation to two groups. Such difficult concepts, however, need more thinking and debate.

Conclusion

Our research is concentrated on improving the *quality* of the content of the statistics that is taught to students. Overall, we believe, that our attempt at providing students with learning experiences to develop their inferential thinking seems to be improving their understanding of inference at a conceptual level (Arnold et al. 2011; Pfannkuch et al. 2012). In terms of curriculum-time constraints we think that by automating graphics students can spend more time on learning to interrogate and reason from data and less time on learning how to construct plots. Ideally we envisage students learning statistics within an EDA environment, building inferential concepts, and being enculturated into a statistical way of thinking.

In this paper we discussed some of the conceptual foundations necessary for statistical inference that we have wrestled with and debated. We also highlighted some pivotal moments in our research that led us to reconsider some conceptual foundations. Our journey into laying foundations for statistical inference involved us in exploring and thinking about our own conceptual understandings at a very deep level. Through explicating concepts, which were previously implicit, we have come to appreciate the complex nature of inferential thinking in statistics. It has also made us more aware of the many gaps in curricula and textbooks such as no attention to shape of distributions, apart from naming the shapes, the imprecise use of language, and the mixing up of descriptive and inferential thoughts. To reveal the complexity of inference to students requires careful attention to learning trajectories within and across year levels. Our learning trajectories incorporated a coherent set of experiences and ideas of concepts such as sample, population, and sampling variability through interconnecting hands-on simulations, dynamic visualizations, verbalizations, gestures, visual imagery, language, and discourse.

To build this conceptual pathway for laying foundations for inference we started with a desired learning goal in mind that students would be able to make formal statistical inferences for observational studies and experiments. Working from a problematic situation for New Zealand teachers with the comparison of box plots we theorized a pathway and devised learning activities and software. As our journey progressed with discussion, implementation, and modification through the period 2008–2012, we learnt a great deal. The question we now ask ourselves is whether the pathway would look the same with what we now know. The answer is probably not. We believe that there could be other pathways for building students' inferential reasoning such as developing paths for sampling variability using both quantitative and qualitative data and for inferences from observational studies and experiments. Or perhaps the randomisation method path could be used for all types of data and study (e.g., Holcomb et al. 2010) although this does violate the Hesterberg principle of mimicking the data production process.

At the moment, however, there is too big a gap between statistical practice and statistics education (Cobb 2007; Efron 2000). To not try to advance the quality of statistics learnt so that it is more closely aligned to practice through laying foundations for inference at secondary school and incorporating methods such as bootstrapping and randomization at the upper levels, is to leave it, paraphrasing Efron's (2000, p. 1295) words "stuck in the 1950s." We hope that our work will contribute to a paradigm shift in what is learnt in statistics. Presenting statistics at the secondary school level without attention to inferential conceptual foundations is no longer viable.

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Mathematics Education in Cambodia from 1980 to 2012: Challenges and Perspectives 2025

Chan Roath

Abstract The Kingdom of Cambodia was a world leader in technology and scientific understanding from the ninth to the fifteen century as the Khmer Empire. Unfortunately the Pol Pot regime destroyed the education system in Cambodia between 1975 and 1979. The process of rebuilding the educational system of Cambodia was started by collecting the surviving educated people and by adapting the slogan: "The one who knows more teaches the one who knows less and the latter transfer's knowledge to illiterates". Mathematics education in Cambodia currently faces many problems such as a lack of well qualified teachers, a lack of knowledge in curriculum development, text book writing, methodology of teaching and use of ICT. Currently no quality assurance mechanism is available to ensure Cambodia's mathematics curriculum is up to international standards. The relatively low salary of teachers in the Kingdom remains an impediment to our educations system as it provides little motivation for people to become teachers. The Cambodian Mathematical Society (CMS) was established on the 4th of March 2005 and recognized by the Royal Government of Cambodia to play a part in addressing the problems and improving the capacity of mathematical education in Cambodia. CMS is committed to promoting mathematics as a key "enabling" discipline that underlies other key disciplines and is at the heart of economic, environmental and social development in Cambodia. A successful outcome for mathematical education in Cambodia depends on the creation and implementation of developmental goals that are appropriate for Cambodia. The CMS has identified goals that will be made priorities in addressing the needs of mathematical education in Cambodia. These goals include improving the level of qualification of Cambodian mathematical teachers, upgrading the mathematical curriculum to a modern and internationally competitive level, improving the quality of teaching materials and textbooks available in the Khmer language, improving the pedagogical methods of teaching

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mathematics, promoting and supporting the use of information communication and technology (ICT) in mathematical instruction and encouraging participation in international mathematical programs and competitions as well as developing such competitions further in Cambodia.

Keywords Cambodia · Mathematics education · Teaching skills · Information technology · Human resources

Introduction

The Kingdom of Cambodia was a world leader in technology and scientific understanding from the ninth to the fifteenth century, during the Khmer Empire. The scientific understanding of our ancestors was at that time among the most developed in the world and their appreciation of science was profound. During that period our intellectuals were world leaders in their knowledge of mathematics, astronomy and other scientific pursuits, and their strong understanding of the natural world enabled them to create some of the most advanced and sophisticated architecture on the planet. The complex of temples and shrines in the Angkor region is recognized by international scholars as the world's largest preindustrial city.

Unfortunately, the progress did not last because of the unrest and war with neighbouring kingdoms. More recently the persecution of intellectuals under the Pol Pot regime, from 1975 to 1979, had a profound impact on the societal and intellectual development of Cambodia. During the Khmer Rouge regime, education was dealt a severe setback and the great strides made in literacy and in education during the two decades following independence (1953) were destroyed systematically by the regime. Schools were closed and educated people and teachers were subjected to particularly harsh treatment, often facing exile or execution. Soviet sources reported that 90 % of all teachers were killed under the Khmer Rouge regime. Only 50 of the 725 university instructors, 207 of the 2300 secondary school teachers, and 2717 of the 21,311 primary school teachers survived. During the Khmer revolution, young people were rigidly indoctrinated but literacy was neglected, so that an almost entire generation of Cambodian children grew up illiterate. After the Khmer Rouge was driven from power, the educational system in Cambodia had to be re-created from almost nothing, as illiteracy had climbed to more than 40 %, and most young people under the age of 14 lacked any basic education. This process of rebuilding was started by collecting the surviving educated people and by adopting the slogan: "The one who knows more teaches the one who knows less, and the latter transfers knowledge to illiterates".

The aftermath of this regime resulted in many difficulties, which we still face today. This paper offers a survey of the current situation of mathematics education in Cambodia, discusses some of the challenges that need to be met and presents some perspectives on actions now being planned or having been implemented so to address the developmental needs of mathematics education.

(Further general information about education in Cambodia can be found at the url http://en.wikipedia.org/wiki/Education_in_Cambodia)

- Land area: 181,035 km^2
- Provinces/municipalities: 24
- Population (2011): 14,521,275
- Household size: 4.7
- GDP per capita (2011): $912
- Occupation classification:
  Agriculture: 72.3%
  Industry: 8.5%
  Services: 19.2%
- Literacy rate: 77.6%
- Gen. educ. students: 3,123,082
- Higher educ. students: 222,016 (37,7% F)
- HE students per 100,000 (2012): 1430
- No. math teachers: 6082
- No. PhD in mathematics (2012): 6

Table 1 Net enrolment ratio per age group

	Kindergarten	Primary	Lower secondary	Upper secondary	Higher education
Age	<6	6–11	12–14	15–17	18+
% attending	52.7	96.4	35.1	19.6	1.53

Source MoEYS 2012

Current Situation of Mathematics Education

Resources

Cambodia allocated 15.9 % of its annual budget in 2012 to education. But the annual education budget is only 1.6 % of Cambodia's GDP (Gross Domestic Product). This is very low in comparison with other countries. However 83 % of the funds are allocated to remunerations and operational expenses, which leaves little funding for facilities maintenance and proper teaching materials such as computers and internet connections.[1] Many schools in Cambodia are currently understaffed, with 45,296 teachers teaching 2,562,010 primary school students, 27,067 teachers teaching 541,147 lower secondary students and 10,160 teachers teaching 318,165 upper secondary students. Among all these teachers, only 6082 are involved in mathematics education. These teachers will teach over the 38 weeks of the academic year 18 h per week at lower secondary school, 16 h per week at upper secondary school, 14 h in teacher training centres and 12 h in higher education. Mathematics is taught for 7 h/week per class in grades 1–3, and 6 h/week per class in grades 4–10. Students studying advanced mathematics in grades 11–12 receive 8 h/week per class, but students choosing to study basic mathematics only receive 4 h/week per class (see footnote 1).

The percentage of the population in each age group attending an educational institution is shown in Table 1. It should be noted that only 1.53 % of people in Cambodia in the age group 18+ can currently afford to pursue tertiary education.

Policy Implementation

Provincial/Municipal Offices of Education are responsible for supporting the Ministry of Education, Youth and Sport (MoEYS) in implementing educational policies, for preparing and submitting plans for further development of education, and for providing data and statistics about schools. However, there is currently a lack of congruence between research and policy making, linked possibly to the inadequacy of our budget and research facilities, which exemplifies the weakness of

[1] http://en.wikipedia.org/wiki/Education_in_Cambodia

our educational system with regard to analytical research and development. As a result, there exists a significant gap between policy formation, implementation and monitoring within the educational system that prevents from addressing the specific problems which both educators (mathematics teachers) and children face. Moreover currently no quality assurance mechanism is available to ensure that Cambodia's mathematics curriculum is up to international standards.

Gender Disparity

Although the literacy rate and number of girls graduating from primary school in Cambodia is increasing, a gender disparity remains. This can be partly attributed to the cost of sending girls to school, as there will be one person less at home to contribute to the family's income. The trade-off between school participation and economic activity increases as the child gets older—this trend is particularly prevailing among girls, whose place in the Khmer culture has traditionally been at home. Among the Cambodians who do obtain an education, there is still a significant gender disparity in relation to mathematical literacy. At Phnom Penh's Khemarak University, only 20 % of students who achieve a Bachelor's Degree in mathematics and 5 % of those who achieve a Master Degree in mathematics are women.

Tertiary Education

In 2009, Cambodia had a tertiary enrollment rate of 10 %, which appears to be low when compared with other nations (see footnote 1). Of the students who study at a tertiary level, only 10 % study science and of those students, only 1.5 % study mathematics. Students who study biology, chemistry, and social sciences only spend 45 h studying calculus or basic mathematics during their foundational year, while students studying business, economy, accounting and related fields study slightly more than this. Students completing a Bachelor degree in mathematics will need to acquire 120 credits and each credit will take approximately 15 h to complete. A Master degree will require an additional 45 credits and a successful thesis. Cambodia's higher education institutions still lack world recognition and are currently not acknowledged by *QS World University Rankings*.[2] There is also inadequate communication between schools and corporations. This hinders curriculum development in mathematics and in other subjects that is necessary to ensure that students have the skills to meet the demands of the current labour market (see footnote 1).

[2]http://en.wikipedia.org/wiki/QS_World_University_Rankings

Poverty Hindering Education

The poverty line in the rural areas of Cambodia is set at US$1 per person per day, but minimum daily food requirements will cost at least US$2.50 (see footnote 1). Due to poverty, children in Cambodia are forced to give up the chance of receiving an education and will be forced instead to work to supplement their family's income. Mathematics teachers in Cambodia currently earn US$65–US$150 a month. In Phnom Penh, monthly living expenses for a family of four average at around US$500 a month. This forces many teachers to find a second source of income, often charging students for after school classes. This additional workload distracts teachers from their primary vocation and hinders the educational system of Cambodia.

Mathematics Teachers

Current statistics indicate Cambodia has a fast growing and youthful population compared to many other countries (MoEYS 2012). As teachers are required to possess a certain minimal qualification and as Cambodia's capacity for teacher training is still in development, the current teacher shortage could become increasingly severe.

The obstacles to mathematics teachers in Cambodia include:

- a lack of opportunities and time for teachers to properly learn the curriculum, and develop their knowledge and skills;
- the absence of a mechanism to motivate teachers to update their knowledge and skills, as well as of a system to determine which teachers should be promoted;
- the fact that the salary of teachers is relatively low and their status in Cambodian society is not as high as it should be, given its importance to Cambodia's development. This aspect is a substantial obstacle to the development of our educational system, as it hinders the motivation for people to become teachers.

Many mathematics teachers in Cambodia currently face serious impediments to their effectiveness as instructors, such as:

- a lack of knowledge of mathematics and pedagogy due to the destruction of educational resources and authorities under the Khmer Rouge regime;
- a lack of motivation for teachers to improve and update their knowledge due to a absence of systems rewarding self-improvement;
- a low research capacity for higher education teachers, because of the lack of time caused by the fact that many teachers are working second jobs during what would otherwise be free time devoted to development.

Lack of Resources

Due to a lack of resources and government funding for schools in Cambodia, there is a shortage of teaching material and school facilities. According to official statistics from the MoEYS, as mentioned earlier, the education budget for 2012 is 15.9 % of the national budget. This amounts to 1.6 % of Cambodia's GDP being spent on education (see footnote 1). Even though the Cambodian government promises to provide $1.50–$1.75 per student per year to each primary school for teaching materials and school operating costs, this amount is often insufficient to even cover the basic operational cost of schools (see footnote 1). And so, almost no funds are available for instructional technology. Technologies such as graphing calculators, Maple, MathCad and Mathematica are not available to Cambodian students, teachers and researchers.

Curriculum and Mathematics Textbook

The curriculum of mathematics in Cambodia is currently upgraded every five years, compared to every ten years in the past. Cambodia's mathematics curriculum is written by experienced local teachers with international assistance, as Cambodia currently lacks specialists in mathematics curriculum development.

The textbooks used in Cambodian schools are written by experienced local teachers. As previously mentioned, these teachers do not have the same qualifications for this job that their international counterparts often have.

Cambodian mathematical textbooks often suffer from the following problems:

- the textbooks often contain mathematical errors, so that less experienced teachers do not recognize these mistakes and may teach incorrect mathematics to students;
- textbooks may not reflect modern educational psychology and pedagogy;
- teachers' guides are not widely available and there is a lack of reference books for teachers in Khmer language, as most teachers can only speak, read and write in Khmer;
- text is printed only in black and white—there are no high quality colour graphics as typically found in international textbooks;
- there is little correlation between the mathematical content taught and its application to real life situations.

School Administration and Mathematics Management

School administration and mathematics management suffer from the following problems:

- a lack of vision, educational leadership and management that is caused by a deficiency in specialist knowledge in these fields;
- the low contributions to education from communities and the private sector in Cambodia;
- a lack of school networking to support development, experience sharing and best practices.

Future Perspectives to 2025

The Cambodian Mathematics Society was established in March 2005 and is recognized by the Royal Government of Cambodia as playing a part in the developmental needs of mathematical education in Cambodia. Success in improving the Cambodian situation depends on the creation and implementation of developmental goals that are appropriate for our country. The CMS has identified goals that will be made priorities in addressing the needs of mathematical education in Cambodia.

These goals include:

- to gather mathematicians, lecturers and teachers of mathematics to discuss current issues in mathematics education and discuss how to combat these problems;
- to advance the quality of mathematical knowledge in both applied and educational capacities, to update curricula and textbooks and to promote innovative teaching methodology;
- to communicate with the international scientific and mathematical community society in order to support mathematics in Cambodia.

Goals for the development of mathematics in Cambodia from 2013 to 2025 include:

- developing and improving the level of qualification of mathematics teachers at all levels by:
  - building the capacity of mathematics teachers in Cambodia through the creation and implementation of training programs;
  - increasing the number of people holding Ph.D.'s in mathematics from 4 to 50 in 10 years, so to provide the necessary intellectual leadership and management;

  - increasing the number of M.Sc. in mathematics, both pure and applied, from 60 to 500 in 10 years, so to provide technical assistance, curriculum development, textbook creation, supplementary reading material and a variety of other important roles within the educational system;
  - increasing the number of B.Sc. in mathematics, both pure and applied, from 2000 to 5000 in 10 years, so to provide an adequate number of qualified teachers for the growing number of mathematics students in Cambodia;
  - improving the quality of mathematics education at all levels;

- preparing and publishing educational materials for students and teachers of mathematics in Cambodia. This process will involve identifying and engaging authors, writers and publishers, and assembling experienced mathematics teachers who have the skills required to record high-quality instructional VCD-DVD materials;
- improving the pedagogical methods of teaching mathematics at all levels, in particular by promoting and supporting the use of information communication and technology (ICT) in mathematics education;
- upgrading the mathematics curriculum at all levels to modern and internationally competitive standards through the development of quality curriculum materials suitable for all grades and by holding workshops to promote the exchange of ideas. This activity will be concentrated at the six Regional Teacher Training Centres and the eighteen Provincial Teacher Training Centres through one-week training courses and longer training programs of up to three months in order to build mathematical knowledge and teaching capacity.
- encouraging participation in international mathematical competitions as well as developing such competitions further in Cambodia. In that connection a committee will be formed to establish a national mathematics competition. The national committee will be composed of members from both provincial and urban locations and relevant institutions, and will prepare entry requirements for capable students. This committee will also identify, support and motivate the most outstanding students and teachers and will distribute awards to promising students and teachers annually.

Conclusion

The Cambodian Mathematical Society is committed to the promoting mathematics as a key “enabling” discipline underlying many other key disciplines and at the heart of economic, environmental and social development in Cambodia.

Although progress has been made through the support of societal, national and international agencies, there still remain significant challenges in ensuring quality mathematics education for all Cambodian students. Consequently the CMS will continue to be a passionate advocate for international support and partnerships in this important mission. Desired outcomes of the actions of the CMS include:

- enhanced knowledge and expertise in mathematics education for mathematics teachers at all levels within the system;
- access to higher quality learning materials, including textbooks in Khmer language for all students, teachers and researchers;
- stronger relevance of the mathematics curriculum to the needs of students and communities, in alignment with the most modern and internationally accepted standards in science and technology, and thus enjoying attention similar to other curriculum areas;
- enhanced interest in, and commitment to, mathematics education by students, researchers and potential mathematics teachers.

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The Challenges of Preparing a Mathematical Lecture for the Public

Yvan Saint-Aubin

Abstract As public curiosity and interest for science grow, mathematicians are invited more often to address a public that is not a classroom audience. Such a public talk should certainly convey "mathematical ideas", but it obviously differs from the classroom lesson. Preparing for such a talk offers therefore new challenges. I give examples from recent public lectures given by prominent mathematicians and by myself that try to tackle these challenges. I also reflect about how these efforts have changed my behavior in the classroom.

Keywords Mathematical lecture · Public awareness · Public interest for mathematics · Science awareness · Mathematics communication

Introduction

Over the last centuries, some learned societies have felt the responsibility to foster public awareness of their field. In Canada, the Royal Canadian Institute in Toronto has held public lectures for more than a century. Their lectures touch upon all sciences and mathematics. With the creation of many mathematics research institutes around the world, the last ten years have seen the launch of a few lecture series for the public specifically on mathematics. Still public lectures on mathematics remain a novelty and not many mathematicians or mathematics educators have had the opportunity to explore this way of communicating mathematics.

The advantages of mathematical lecture series are numerous. They may present mathematics as a living discipline very much in development and share with the public the intellectual adventure of research. By the range of problems covered, pure to concrete, they can show to students and their parents that scientific activity

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may translate into career opportunities. They may also show that mathematics and science are useful to attack complex issues, but they do not necessarily provide definitive and clear-cut answers. Public lectures are therefore a fruitful addition to “teaching mathematics” in its broadest sense, one that has an impact on citizens, governments and, of course, scientists.

It may seem that the question of how to prepare such public talks is faced by only a limited community, that of professional mathematicians. But a question, just slightly modified, touches a much wider audience than this limited group: How do we tell people around us what we do? Students in science and math teachers from high schools to universities are asked on a regular basis to explain the purpose of their work. Their successfulness might have an impact as diverse as fostering interest in science among teenagers or stopping the erosion of the number of hours devoted to mathematical activities in curricula.

In this short paper I would like to share my experience of public talks on mathematics. This experience is limited, but yet covers the two distinct aspects of listening to and giving a talk. I have attended to several around the world and I have the privilege to be the organizer with my colleague Christiane Rousseau of the *Grandes Conférences* series held by the Centre de recherches mathématiques based in Montréal. I have also prepared several and given them, some numerous times, to diverse audiences. The point of view expressed here is resolutely personal. It is not based on any scientific assessment of the impact of these talks. I will not even attempt to define a set of goals at which such series should aim. Still I think this personal point of view might be of interest. The public of the *Grandes Conférences* has changed significantly over the six years of the series existence. The audience, constituted at the beginning largely by the community of the departments of mathematics of the four universities in Montreal, has now evolved to a point where at least half comes from outside academia. Slowly this series is reaching its target. And the public is becoming more discerning: After the lectures, the participants will not hesitate to express their view to my co-organizer or to me and their comments almost always coincide with my personal view. It seems that what is a good public talk for the layman is also a good public talk for me. As an organizer, I find this is reassuring.

The next section identifies the difficulties of preparing such talks and proposes ways to tackle them. The next two sections are devoted to examples, first to the overall structure of the talk, second to specific examples of what I think might be efficient sharing of mathematical ideas. Concluding remarks follow.

Pitfalls and Challenges

There are several challenges facing someone about to design a public lecture. Most come from the practice that every good teacher has honed over the years in her or his classes. Indeed many skills developed for the classroom interfere with what is called for to design a good public talk. Here are some of the pitfalls.

Who Are They?

A good teacher is always aware of the inhomogeneity of the students in the classroom: They have different background, talents, interests and dedication to the topic taught. But, compared to the audience of a public talk, these differences are minor. After all, most of the students in a trigonometry class will have mastered the basic rules of algebraic manipulations. Their levels of virtuosity will definitely vary, but it should be easy to remember them that such or such manipulation can be used at a given point. In a public lecture, the audience will be in general *very* inhomogeneous. One could imagine that the background could spread from high school mathematics to that of a researcher in mathematics. Moreover even those with a university degree in mathematics might not have used this knowledge since a long time and have forgotten most of it. Such a spread in backgrounds is indeed common. A lecture of general interest, say about mathematical modeling of biological phenomenon, will pique the curiosity of the widest crowd. Often these public lectures are given within a professional mathematical meeting; it is then aimed, at least in the official discourse, to the participants' accompanying family and the public of the city where the meeting takes place. But a substantial part of the audience will be the mathematicians attending the event. To whom should the lecturer speak?

The large spectrum of backgrounds is not the only difficulty stemming from the inhomogeneity of the audience. The varying interest level is another one, the practice with long scientific explanations still another. People with a keen interest in science, but without a mathematical training, will find it hard to follow long mathematical arguments. Who has the ability to concentrate on an hour-long difficult reasoning?

Once exposed these pitfalls are obvious. Still I feel they need to be stated. It makes the challenge ahead clearer. When about to prepare a public lecture, the speaker should (i) decide to whom the conference will be targeted, (ii) evaluate honestly the background of this audience and (iii) break down the lecture into pieces of various difficulties making sure that most if not all of them are accessible to the chosen public.

A Story Is Being Told, a Question Being Answered

For each level of teaching, for each course in a program, a syllabus is imposed or agreed upon. The validity of an education or a diploma is seen in the educators' mind as tied to the goals of this syllabus being reached. In a classroom, the flow of ideas is slow, each argument being substantiated and explained. A logical argument may overlap two classes and the properties of triangles in Euclidean geometry will be covered during several weeks. It is therefore normal that a student at the end of, say, high school Euclidean geometry might fail to see that she or he was introduced to the basic geometric tools and their use and, more importantly, to the foundations of logical reasoning. These overall goals are often obscured by the details of the syllabus.

Here are a few titles that would raise my interests: *Leonardo's architectural works, Global warming and Kyoto protocol, The international community and the Arab-Israeli conflict, Popular music in the 18th and 19th centuries*. They circumscribe a topic well, but they could be titles for a public talk *and* for a one-term course. For the latter, an exhaustive point-of-view will be taken and many detailed aspects will be studied. For the former, a few outstanding points will be chosen for their significance and they will be tied together in a way that informs, makes a point and, if possible, entertains. Often a story will be told or a question will be raised at the beginning so that the lecture has a sense of direction, a goal, a unity. This is surely not the only way to construct an interesting lecture. But there is a lesson for mathematicians here. Mathematics often grows out from concrete questions, it is useful to our society, it is taught and developed by human beings, and its breakthroughs often come after years of conjectures and fruitless efforts. Even though mathematics is indeed a deeply human enterprise, our teaching rarely shows it that way. There is no reason to hide these human aspects, but for the lack of time in our classrooms. Why couldn't a mathematical public lecture tell a breathtaking story? Or solve a question so intriguing that it enthralls the public for 1 h? The pitfall here is clear. The planning of a public lecture should not proceed as that of a lecture for the classroom. The challenge is to find a sense of direction that will carry the mathematical ideas and make them inescapable.

The first thing that comes to mind while preparing a public lecture is obviously the mathematical ideas that one would like to communicate. But, from then on, the path is different from that of preparing for the classroom. I believe that the next steps should be (i) to seek elements complementing the mathematical concepts that relate to the genesis of the latter and show their importance within and without mathematics and (ii) prepare a plan, or more precisely, a *script* or a *storyboard* of the whole talk orchestrating all the parts. The complementing elements can be related to human, historical, sociological and even political aspects; they can reveal hidden connections with other parts of science and unexpected applications; they can raise open questions and, ideally, they should be entertaining. As for the script, should it be a thriller (like the efforts over almost two thousand years to prove the fifth Euclidean axiom from the first four), a drama (like the birth of logic in the 20th century with its disillusionments, the suicides of some main characters, and Gödel's final coup de théâtre), or a historical epic (like the history of estimating the age of the earth, as seen in the next section)?

The Lure of Rigor

Rigor is one of the outstanding characteristics of mathematics. Among all human intellectual endeavours it is the only one that can claim such an excruciating level of precision and rigor. Many mathematicians feel that an argument has been presented only when it is made rigorous beyond the slightest doubt. A serious pitfall is the temptation of bringing this level of exactness to a public hall. This can manifest itself is several ways, either by giving too much details to make sure all bases are covered or by organizing one's talk in a long chain of arguments where *A* implies *B* and then *B* implies *C* and so forth. It is unlikely that a general public will have the will or interest to go through such a grueling exercise. But, if proofs should be downplayed, will there be any mathematics left?

The challenge here is to construct a talk that both keeps mathematics as the central focus of the event *and* captures the attention of the audience from beginning to end. I feel that the following guidelines might be of use: (i) choose a handful of mathematical ideas, three or four maximum, that can be linked logically and whose union will solve satisfyingly the key question raised by the talk, (ii) think of the best way to explain each of these mathematical ideas in a convincing way (graphics, animation, even metaphors could be used), (iii) make sure that each explanation is accessible to the targeted audience and that, if one is missed, the following ones can still be understood and (iv) organize the whole so that the logic between these ideas stand out.

The audience is likely to have done science courses before and experienced the thrill of the "ah!" moment, this magical moment when a scientific idea suddenly makes sense, is adopted and tied to one's existing scientific knowledge and everyday life experience. If this audience comes to such an event outside the classroom, it is probably looking for living such an experience again. The care put in point (ii) above is therefore intimately tied to the success of the event.

The Script of a Mathematical Lecture

The remaining part of this article is devoted to examples: In this section, of the overall structure of mathematical public talks and, in the next, of ways to communicate a well-circumscribed mathematical idea.

The public interested in intellectual quests is likely to have been exposed to various conferences, documentaries and movies with a significant scientific content. She or he will have seen Al Gore's *An inconvenient truth* (2006), or NOVA television programs like the adaptation of Brian Green's *The elegant universe* (1999), or *The Proof* about Andrew Wiles's proof of Fermat last theorem. These large-budget popular successes have been written by scientists together with professionals of movie scripting and editing. The single mathematician does not have these means. But she or he might want to learn from them. As argued in the previous section, a good script mingling purely mathematical ideas with elements with a broader scope but still related to them seems a sound way to a successful public talk. I chose to discuss two scripts of public lectures that I think reach this balance.

When the Earth Was to Young for Darwin, a Public Lecture by Cédric Villani

Cédric Villani is a French mathematician who won one of the four 2010 Fields Medals, the single-most prestigious award given to mathematicians under forty years of age. He was invited to give a public lecture in the series *Grandes Conférences* of the Centre de recherches mathématiques based in Montréal, Canada. His lecture was entitled *Quand la terre était trop jeune pour Darwin* and told the story of the clash between two titans of British science of the 19th century. Charles Darwin, the first titan, in his *On the origin of the species*, argues that species transform, appear and disappear, and that what is observed at one time is the consequence of a long evolution process that still continues. One of his assumptions, largely based on geological knowledge of his period, is that the age of the earth is larger than 10^8 years and allows for such an evolution. Lord Kelvin, the second titan, is the physicist who set out to use the recent developments in mathematics launched by Jean-Baptiste Fourier to compute the age of the earth. His best estimates placed the age of the earth between 20 and 40×10^6 years, far too short for Darwin. Both scientists were aware of the other's prediction and a heated debate ensued, one that the educated society of the time followed with great interest. The topic cannot be better for a public lecture: It offers drama, suspense and humour, a vivid look at science in progress, a surprising resolution and even a sociological commentary. A good script is the last element required to make the story a great mathematical lecture. Villani's was essentially faultless. The main parts of his lecture are the following.

- Presentation of William Thompson, Lord Kelvin: short bio, major works, e.g. involvement in the construction of the first transatlantic cable;
- the estimates of the age of the earth, from the beginning of the written word to the 19th century: James Usher and the old testament, then George-Louis Leclerc de Buffon, his estimate (one of the first based on a scientific argument) and his prudent stance toward the Church;
- Joseph Fourier and his treatise *Théorie analytique de la chaleur* that creates a new chapter of mathematics; a short description of ideas behind the new tools;
- the equation describing heat propagation, Kelvin's hypotheses and computation of the age of the earth using the temperature within the earth but close to the surface;
- Charles Darwin and the geologists estimates for the age of the earth and the debate that follows the two conflicting predictions;
- Ernest Rutherford observes that the recently discovered radioactivity violates one of Kelvin's hypothesis (that the earth does not internally generate heat);
- John Perry suggests that the center of the earth is liquid which increases convection;
- recent results (post-1960) and conclusion.

The two first points above are mostly historical; they both inform and engage the audience. The discussion of Buffon's efforts also shows the power of the Church and how scientists of the 18th century were still constrained by religious and political forces of their time. Villani spends some time explaining Buffon's needle, a problem tied to probability theory that offers a mechanical way to compute the number π. This mathematical curiosity is not important for the following, but the simple argument acts as an easy mathematical warm-up and probably provides a sense that the level of the lecture is really for the participants.

The real science starts with the next two points on Fourier's new theory and Kelvin's computation. Here Villani relies mostly on drawings to remind his public of trigonometric functions and to show how they can be used to approximate other functions. Of course no formal definitions are given, nor theorems or general statements. The relationship between trigonometry and Fourier analysis is explained elementarily, but again its necessity for understanding the next step is minimal. The heat equation requires mathematically advanced concepts usually covered during the last two years of a B.Sc. This is clearly beyond such a lecture. To capture the meaning of this complex equation, Villani has recourse to everyday experience: A hot object whose temperature is uniform throughout is brought in a cool room. As it cools down, the external layers of the object will have a lower temperature than the inner ones. This is why one stirs a bowl of soup or a cup of tea. The temperature of the various layers of the object being cooled can actually revealed how long it has been loosing heat. This is one of the uses of the heat equation and this is how Kelvin estimated the age of the earth. The description of these ideas takes about 10 min of Villani's talk. The trigonometric functions, their role in Fourier analysis, the notion of variation of the temperature of a cooling mass like the earth, and the general meaning of the heat equation are the main mathematical points discussed by Villani. To a great extent, they are independent and, if a participant misses one of them, she or he may hope to catch the next.

In the next point, Villani introduces the second main character, Charles Darwin, and discusses the disagreement between Darwin and Kelvin's estimates. It is interesting to see how the personalities of both characters are revealed by their respective reactions to this clash. A few contemporaries' comments are also quoted; they show awe for the scientific debate in progress.

The remaining parts of Villani's talk are devoted to the resolution of the clash. The discovery of radioactivity and Rutherford's observation that it weakens one of Kelvin's crucial hypotheses are presented. They rest on the understanding of Kelvin's hypotheses presented earlier, but are otherwise independent of the previous mathematical parts and surely easier to understand. But Rutherford's observation cannot yet bring together Darwin and Kelvin's windows for the age of the earth. The crucial proposal, that the inner core of the earth behaves for practical purposes as a liquid, was made by John Perry. But it took more than half a century for this idea to be accepted and for models to include it properly. The ideas in science do not progress in a straightforward way and their acceptance is often slowed by the prejudices or the inertia of the scientific community.

By telling the history of the ideas and of the human efforts leading to the accepted age of the earth, Villani's script provides a breathtaking drama whose main characters are probably science and mathematics, more than Darwin and Kelvin.

The Archaeology of the Tabla, a Lecture by the Author

The *tabla* is a traditional Indian percussion instrument. It is constituted of two drums, a small one called *dayan* played by the dominant (right) hand and a larger one called *bayan*. The small one, also referred to simply as tabla, has a remarkable property: It has a recognizable pitch, one that a musician can easily identify and sing. Drums used in occidental orchestras, kettle drums, timpani and others, produce a sound whose pitch is difficult to identify and even though composers of the classical tradition write in their scores the note the timpani should play, many musicians find it hard to sing the notes played. The remarkable harmonic character of the tabla was first noticed by two Indian physicists, Raman and Kumar (1920). The former is the first Indian (and first non-European- or American-trained scientist) to ever win the Nobel Prize for Physics. Raman and Kumar suggested that its remarkable sound was due to the black patch in the middle of the membrane, known as the *gab*, that is constituted of flour, water, iron powder and other ingredients and added by the artisans. The density and the radius of this patch are very uniform over all known tablas. But, how the Indian artisans could possibly devise this musical instrument and find the proper density and radius? These questions are very similar to those asked by the subfield of archaeology known as *experimental archaeology*. How civilizations responsible for an implement, a monument, a technique could achieve these results? In the case of the tabla, an "experimental archaeological" explanation was proposed recently: Gaudet, Gauthier and Léger's argument (2006) is based on a mathematical analysis of the natural vibration modes of the membrane with the added patch.

This tentative archaeological explanation provides the backdrop of a mathematical lecture that I developed for students registered in Quebec cegeps. (In Quebec, the cegeps are education institutions offering a two-year program to high-school graduates. Their diploma is necessary to register to Quebec universities.) The topic does not have the drama of Villani's script, nor does it offers personalities like those of Darwin and Kelvin. (The role of Raman is limited to the observation that the sound of the tabla is unlike any occidental drum.) Even though it might lack in intrigue, the topic leads to a scientific understanding of what a musical sound is and how mechanical objects produce it. It is therefore intimately related to an everyday life experience of these young adults. Moreover the homogeneity of the targeted audience allowed me to be a little more ambitious for the mathematical concepts presented. As for Villani's talk, here is the breakdown into approximately equal-length parts of the script.

- What is experimental archaeology? Some examples: the mystery of Inca quarrying and stonecutting and Protzen's work (1985) and the mystery of the steel of Damas with Juleff's discovery (1996) and the computational work of Tabor et al. (2003) on Sri Lanka furnaces;
- Raman and Kumar's observation: The sound of the tabla is remarkable. An example of traditional music played on the tabla;
- Experiment 1: what is a musical sound? What distinguishes it from noise?
- Experiment 2: how physical objects "choose" certain frequencies?
- Experiment 3: can one write any function in terms of a sum of trigonometric functions? The analysis of the result of Experiment 1 using this mathematical tool;
- Experiment 4: what are the natural frequencies of the vibrating string? of the drum? Chladni patterns;
- consonant and dissonant tabla as a function of the density and radius of the patch. An archaeological hypothesis to explain the mystery of the tabla.

Like Villani, I prefer, almost instinctively, to start with a story. To define what experimental archaeology, I give first the example of the incredible work of Inca masonry and Protzen's efforts to reproduce the stonecutting techniques. This example does not bring in any mathematics, but my second example does. The quality of the steel of Damas was observed by Europeans as soon as during the Crusades. Only recently furnaces in Sri Lanka were discovered that might explain the mystery. Indeed numerical computation showed that these furnaces could use wind to bring temperatures high enough to produce such a high quality steel. This introduction is, I believe, entertaining and uses minimal scientific knowledge. Most students at this level will know that airplane wings are designed through numerical computation and will understand that the internal working of a furnace might be understood by similar methods. The observation of Raman and Kumar and the description of the tabla is at the same level.

The scientific part starts next with the understanding of what is so special about a musical sound. The four next parts are described above as "experiments" and I actually ask a member of the audience to join me in front to do each of them. For the first, I call for a musician. I ask the volunteer to listen to musical extracts and to sing the pitch of one note in particular. The extracts chosen are of string instruments, of timpani and of the tabla. Most volunteer will succeed in singing correctly the notes of the string and of the tabla and fail (or even refuse) to sing that of the timpani. This demonstrates the particular nature of the tabla. (I also record the volunteer's voice for ulterior use.) The second volunteer is asked to produce a repetitive pattern with a heavy chain. After some exercises, the volunteer is usually able to produce the three first natural modes of this chain, showing that physical objects vibrate at natural frequencies. The third volunteer, chosen for his or her ability at video games, is asked to match two curves on my laptop. This experiment will be described in more details in the next section. Its goal is to have the audience realize that the graph of a function can be approximated by a sum of well-chosen trigonometric functions. This is again a consequence of Fourier analysis, the same tool Villani described in his lecture. Very often the volunteer, after playing with the small simulation, will guess correctly that this approximation can be as good as desired by adding more terms to the sum and that the choice of the terms is in fact unique. At this point, I stop the experiments and explain that this mathematical tool, known as Fourier analysis, can be used for many types of functions, including the graph drawn by musical waves. I then proceed to make the Fourier analysis of the volunteer's voice recorded earlier. The result is shown to be similar to that produced by the third experiment. But the result contains a surprise: The human voice always sings many frequencies at the same time, as would any string instrument. This means that if the singer wants to sing an A at 440 Hz, she or he does simultaneously sings tones at 880, 1320, 1760 Hz,…, on top of the desired one. These tones have frequencies that are integer multiples of the base note. This is a well-known fact among musicians, but a surprising one for many young scientists. Each experiment teaches a particular observation about the vibration of a medium and its mathematical description. The relationship between the three experiments is stated, but the participants do not need to understand them all to have a clear idea of the whole argument.

The last experiment aims at obtaining the natural frequencies of the vibrating string and of the square drum. Even though the students know the concept of derivative (the only one really necessary), this is a more difficult exercise. I usually invite one of the professors in the audience. The result shows that the natural frequencies of a string and those of the volunteer's vocal cords are integer multiples of the lowest frequency, the *fundamental one*. For the square drum, the frequencies do not follow an easily recognizable pattern.

Finally the last part reveals the possible discovery by the Indian artisans of the best tabla possible. It follows easily from the previous experiments and a simple drawing giving a quantitative measurement of the consonance of tablas as a function of their density and radius.

Two Scripts, Common Characteristics

Clearly the topic of the last lecture is more modest than Villani's. After all, the clash between Darwin and Kelvin is a determining moment in the history of science as a whole, a statement that can be hardly made of the history of the tabla. The lecture on the tabla has the advantage of calling for students' participation, an opportunity that is offered by the homogeneity of the audience. This participation slows down the flow—this is a good thing—and, each time a volunteer is in front of the hall, the attention is increased dramatically.

Despite their differences, the two scripts have many things in common. They both start by an introduction that wants to be engaging and bring confidence among the audience. The scientific explanations of each part are based on simple elements, but the understanding of one part is not crucial for the following. And more importantly, the whole lecture is constructed as to answer *one* question, *one* mystery. The solution, revealed at the end, gives a feeling of completeness and satisfaction, as does the script of a well-plotted thriller.

Examples of "Experiencing Mathematics"

The previous section was devoted to examples of the overall structure of public talks. The present one focuses on parts of public talks where a single mathematical idea is captured with minimal notations and definitions. They deliver to the public the "experience of mathematics" at its best.

Ziegler's Checkboard

Günther Ziegler of the Freie Universität Berlin was invited to deliver one of the public talks at the last International Congress of Mathematicians (ICM), held in 2010 in Hyderabad, India. His talk was entitled *Proofs from THE BOOK*. Proofs are at the heart of mathematics and it seems natural to make an effort to describe to the public what they are and what they achieve. In fact, Ziegler wrote a book, with Aigner (2003), about some proofs that are so clear, direct and insightful, that they seem to have been written by God's hand. His book and lecture share the same title. The "public" for the public lectures at the ICM of Hyderabad comprised a large part of high-school students, brought in by the bus load (Casselman 2010). It is clear that this fact is in Ziegler's mind.

The first few slides of his presentation are devoted to explaining the role of proofs in mathematics and for the mathematical community. This is done by formulating some simple questions that are very difficult to prove and also reproducing some quotes from famous mathematicians about proofs. Some are deep, others simply witty. Then Ziegler introduces THE BOOK, the one containing "definitive" proofs. And then he gives an example. It is his first real mathematical moment and it is likely to have been magical for many of the high-school students. Here it is.

A slide appears, totally blank, but for one sentence and one drawing. The sentence is: "Theorem. The 'chessboard without corners' cannot be covered by dominos." And the drawing is the one on the left in Fig. 1. One can guess that Ziegler explains the statement that, even though it is possible to cover a whole chessboard with dominos, each covering two neighboring boxes, once two opposite corners of the chessboard are deleted, this task of filling the new chessboard by dominos becomes impossible. A student who has not encountered this problem before might feel the urge to play with dominos to see why this is impossible.

But a very simple argument proves the Theorem, without any such attempt! It is given in Ziegler's next slide that contains only the right drawing of Fig. 1 where the usual pattern of alternating black and white boxes has been added. There, in this

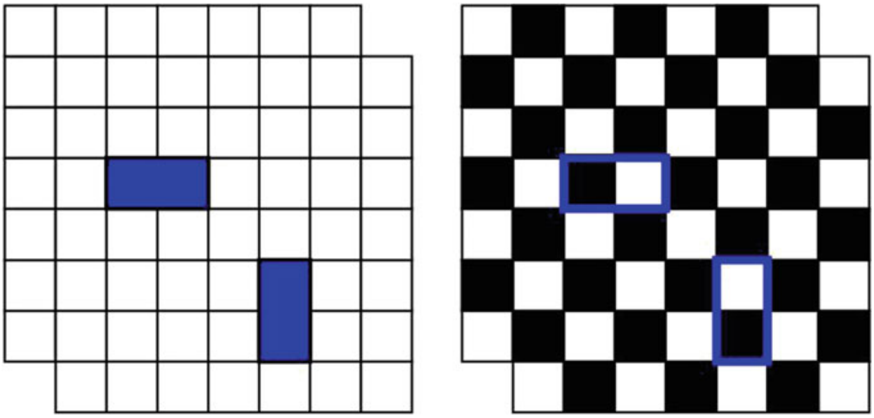

Fig. 1 A chessboard *without corners* cannot be covered by dominos

single drawing, one is reminded that each domino will always cover precisely one black and one white boxes, wherever they are placed. But one can also see at once that the removal of opposite corners of the chessboard has deleted two boxes of the same color. The number of black boxes is now two less than the number of white ones and it is therefore impossible to accomplish the covering by dominos.

The argument, by its simplicity and elegance, can be understood by most people and creates one of these magical mathematical moments in the mind of any scientifically inclined person. It is beautiful and, being placed at the beginning of the talk, captures all listeners in the experience. And it certainly belongs to THE BOOK!

The Original Algorithm that Launched Google

Whoever is old enough to have discovered the beginning of the world wide web will remember how Google imposed itself, over all other search engines, by the efficiency of its algorithm. In fact many of us wondered how a computer algorithm could possibly guess what was important for each of us!

One of my public lectures is aimed at explaining to a large audience how this algorithm works. The lecture supposes little mathematical knowledge and has been given to high-school graduates. It covers first the basic working of classical search engines, like library catalogues or police registries, and second the new problems raised by cataloguing the pages of the world wide web: The lack of uniformity among webpages, the wide range of information quality of webpages, the exponential increase of their number with time, the lack of uniformity of users, and the absence of consensus about what should be searchable.

After, the real mathematical idea behind Google's algorithm is explained. Figure 2 depicts a very small "world wide web". Each of the five circles marked

A, B, C, D and E represents a webpage. For example C could be the home page of the ICME-12, A its scientific program, E the home page of EXPO 2012 Yeosu Korea, and so on. The arrows between the circles represent links between the pages. The arrow starting from C and leading to A means that the webpage C (the page of ICME-12) has a link which, once clicked, will bring the browser to webpage A (the scientific programme of the Congress). Of course, the world wide web has more pages. (On July 25 of 2008, Google passed the threshold of 10^{12} indexed pages.) But the small graph allows us to understand how an order of importance among these five pages can be computed.

Suppose that an impartial web surfer is put at time $t = 0$ on page C. After each period of time, say of one second, the impartial surfer is asked to click on one of the arrows that leave the page where it stands. If there are more than one, it will choose with equal probability between all the arrows available. It is hard to determine where this surfer will be after a minute or an hour. Or more precisely, it is hard to determine what is the probability that it will stand, say, at page A. But, before explaining more in details the mathematics, I ask the audience on which page the impartial surfer is most likely to be after many clicks. A vote among the participants leads usually to the right answer. I guess most think as follows: The surfer can get to pages D and E only through a single arrow. It is therefore less likely to be there than on pages A, B and C. These three pages received exactly three arrows from other pages. But B is favoured as each time the surfer visits A (and D and E), the surfer will visit B one step after (and D after two and E after three). Since there are other ways to get to B, it is more likely to be on B than on A. For that reason, it is not surprising that the audience votes mostly for B. And the participants are right. The mathematical treatment developed after explains why, for most graphs like the one in Fig. 2, the behaviour of the surfer after a long period (the asymptotic behaviour) exists and is unique and it can be calculated using methods of linear algebra. [The asymptotic behaviour is the eigenvector associated to the largest eigenvalue of the matrix representing the Markov chain of the surfer. See for example Rousseau and Saint-Aubin (2008).] In the case of the example, page B will be visited $16/41 \sim 39\ \%$ of the time, more than any other.

The mathematics is not too difficult for the aimed audience. What is great with this simple example is that the audience can understand why the original algorithm used by Google is so remarkable. Once the probabilities of the visit by the impartial surfer are known, they are used to order the pages found after a user's request. The

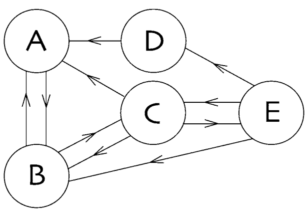

Fig. 2 A small graph representing the world wide web

first pages presented by Google in reply to the request are those that would have gotten visited the largest number of times by the surfer. Each time a webpage points to another page, the latter gets a "vote" and the algorithm is therefore poling the users by studying the arrows (the links) between the pages of the world wide web. And it is not the creator of a webpage who decides if this page is important. It is the other web users that do by linking to it.

I ask questions to the audience once the algorithm has been digested. For example, how should it be changed to avoid dead ends (a page that does not have any exiting arrow)? How would you do to get a better ranking of your personal webpage? Etc. It is clear that understanding the basics of Google algorithm has given the participants an awareness of the web community and its hierarchy.

The Basic Theorem of Fourier Analysis

Many technologies rely crucially on the mathematical theory founded by Fourier, especially those around signal transmission and analysis. Kelvin's estimate of the age of the earth was based on this theory, so was the analysis of the sound of the tabla. It is tempting to convey, in such talks, the basic ideas behind this theory. Unfortunately it requires fairly advanced mathematical concepts and, in an undergraduate program, it is usually introduced in the last two years. Describing even the simplest idea of this theory is a challenge.

The simplest (and central) result of Fourier analysis is somewhat intimidating when one sees it for the first time: Any periodic function of period L can be written as a series of sine and cosine functions whose periods are all of the form L/n for integer n. There exists a simple formula for the amplitudes of these functions and these amplitudes are uniquely determined by the function. (There are some minor hypotheses of smoothness on the function, but they can be ignored for the present discussion.) Another way to state it is the following: Draw the graph of a function over the interval $[0, 2\pi]$. Then one can always reproduce this graph by adding functions $\sin x$, $\cos x$, $\sin 2x$, $\cos 2x$, $\sin 3x$, $\cos 3x$,…, and maybe a constant. The coefficients in front of these functions are determined by Fourier analysis. In other words, such a series always exist and it is unique. Can one convince a non-mathematical audience of this fact or even give it an intuitive idea of what it means?

I chose to do it using a small interactive program. I invite a member of the audience to come play with it. The participant's experimentation will proceed in two steps. For the first, this participant is presented with the left image of Fig. 3, and then asked to move the cursor using the mouse so that the two curves meet. (When the cursor moves, the amplitude of the pale curve (green) is changed.) The participant rapidly finds the correct position of the cursor. I then ask her or him whether any other position of the cursor would lead to the same match between the curves. The answer is obviously "no" and the answer comes without hesitation. (Note that the dark graph (in red) is one period of the sine function and the pale graph (in green) is $a \sin x$ where the amplitude a is controlled by the cursor.)

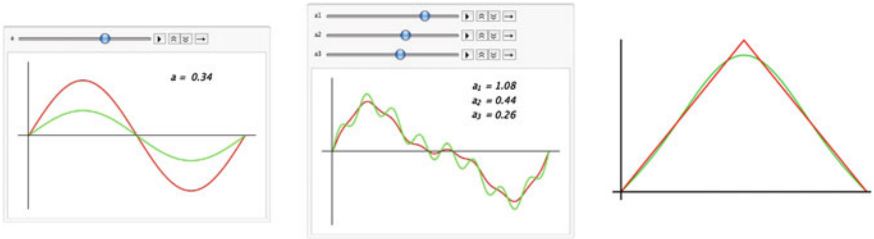

Fig. 3 Approximating a graph with trigonometric functions

The second step is the real challenge. There are now three cursors (middle image of Fig. 3) and the graph to be reproduced by moving these cursors is the dark (red) curve. This graph was chosen so that only three functions are necessary, namely sin x, sin $2x$, and sin $8x$. For Fig. 3, the cursors have been put away from their "correct" positions, those that would reproduce correctly the graph. The participant finds these correct positions with some trial and error. I then ask the same question: Are there any other positions of the cursors that would match the two curves? Usually, but not always, the participant claim, somewhat prudently, that no, there can be no others. This is the uniqueness stated above.

I keep the participant with me on the stage for one more question. I show her or him an animation where the triangle graph is approximated by the recursive addition of sine functions (right image of Fig. 3). The match is still not perfect at the end of the animation as I programmed it to stop after about ten non-zero terms and an infinity would be necessary. Still the approximation is very good. (On the Figure, only two non-zero terms are used.) I finally ask the participant whether a perfect match seems possible by adding more terms and whether such approximations could be done for any functions I could come with. Most think that it is plausible that, by adding more terms, the match will become as good as desired but, for the second part of the question, most hesitate. Some are bold enough to risk a positive answer (which is Fourier's result), others refuse to take position. I then thank the participant who, often, gets cheered by the rest of the audience. I conclude this experiment by saying that, yes, it is possible to approximate graphs by adding sine and cosine functions and that there is a unique way to do it.

The time spent for the whole experiment is between 5 and 10 min. I believe that the small interactive program gives a concrete visual picture of what an approximation of a graph through trigonometric functions is and that such approximation might exist for a large class of functions. These might seem modest achievements, but they are a reasonable success considering the time invested. The main disadvantage of the experiment is that, contrarily to other graphical arguments, this one does not give any idea how to prove the statement. But that might be beyond the goal of a public talk.

Concluding Remarks

As an organizer of a public lecture series, I seek leaders in their mathematical research field that are superb communicators. This is a very small group. However, when I visit Quebecer cegeps, the institutions whose level is just before the one where I teach, I consider that I am giving indeed a public talk. At least, I follow the templates put forward in the previous sections. In that extended sense, every mathematics educator could give a "public talk", that is, a talk to a group of interested students in a free setting. In fact, every mathematics educator should be given the opportunity to live such an experience. For once, no syllabus to abide by, no rush to make sure the material is covered, no need to manage students' stress, no exams to grade. Only the pleasure of communicating something one likes. This is a pregnant experience.

Since my first public talks years ago, I have come to distinguish between elements of my teaching experience that are applicable to public talks and those that are actually in conflict with them. Over the years, this analysis has hopefully improved the way I conceive a public talk. But this *is not* the most important point.

Public talks are only a small part of my professional life. I earn my life doing research and teaching in a university. And the conception of public talks had a definite influence on my teaching, particularly at the undergraduate level. Graphical arguments or even metaphors are useful to capture the essence of a mathematical concept. Most, I concede, would agree that this is not how a mathematical proof is done. But, when there are useful steps toward a mathematical formalization, I am definitely willing to use them.

Unfortunately, historical evolution, cultural impact, and real life applications are never part of syllabi of courses like differential geometry, group theory, real analysis or even linear algebra. Now, in all my classes, I try to keep fifteen, twenty minutes every two or three weeks to tell a story, as I would do in a public talk. These stories are intimately related to the course material, but definitely not in the syllabus. They try to paint mathematics as tightly woven into the fabric of human history. For example, in my linear algebra class, I tell the story of cuneiform tablet VAR 8389 of the Vorderasiatisches Museum in Berlin. This tablet, dating from 2000 BC, asks a simple question about the areas of two fields, given the rent the owner must pay on them (Grcar 2011). The problem leads to a system of two equations in two variables. This tablet is a clear indication that recipes to solve such systems were taught four thousand years ago. Clearly this linear algebra course has a fairly long history. Another example tied this time to my differential geometry class has, as main actor, the great architect Frank Gehry. One of his most famous buildings, the Guggenheim Museum in Bilbao, Spain, is made of bent and curved surfaces. These surfaces are covered with small tiles of metal. Gehry realized that, depending on the shape of the curved surfaces, the bending could be done manually or needed to be molded in a workshop. The first ones did not cost more than if the surface had been a plane, the second needed to be limited as they increased the budget. The difference between the two surfaces is that the Gaussian curvature, a central concept in the course, is zero in the first case and non-zero in the second.

Hence the important point *is* that public talks have had a direct influence on my classroom teaching. This is why I wish every teacher should get an opportunity to give a public talk.

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Computer Aided Assessment of Mathematics Using STACK

Christopher Sangwin

Abstract Assessment is a key component of all teaching and learning, and for many students is a key driver of their activity. This paper considers automatic computer aided assessment (CAA) of mathematics. With the rise of communications technology this is a rapidly expanding field. Publishers are increasingly providing online support for textbooks with automated versions of exercises linked to the work in the book. There are an expanding range of purely online resources for students to use independently of formal instruction. There are a range of commercial and open source systems with varying levels of mathematical and pedagogic sophistication.

History and Background

Assessment is a key component of all teaching and learning, and for many students is a key driver of their activity. Computer aided assessment (CAA) has a history going back over half a century, for example (Hollingsworth 1960) reports a "grader" programme which automatically checked some aspects of students' computer programmes. "*We are still doing considerable hand filing of punched cards at this stage. This large deck of cards which includes the grader program is then run on the computer. Our largest single run has been 106 student programs covering 9 different exercises.*" By the mid-1960s computers were being used to teach arithmetic, e.g. (Suppes 1967), and by the 1980s there were a number of separate strands of research in this area including the artificial intelligence (AI) community, e.g. (Sleeman and Brown 1982). The ambitious goals of such artificial intelligence-led systems have only been achieved in confined and specialized subject areas. These difficulties were acknowledged early, for example in their preface to (Sleeman and Brown 1982).

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> Early CAI (Computer Aided Instruction) workers set themselves the task of producing teaching systems which could adapt to the needs of individual students. It is now generally agreed that this is a very difficult task, and one that will only be accomplished as a result of extensive research in both AI and Cognitive Science.

For example, in the case of multi-digit subtraction, sixty separate skills were identified by Burton (1982), including "*27. Borrow once in a problem*", and "*46. Subtract numbers of the same length*". The majority of these skills involve borrowing, and they are linked in a hierarchy. Assuming a student gives incorrect final answers, their goal was to identify which parts of the algorithm are not being performed correctly. Clearly this is necessary to provide specific feedback to the student. They defined a "bug" as a "*discrete modification to the correct skills which effectively duplicate the student's behaviour*", such as $0 - n = n$ when subtracting individual digits.

> […] we were able to design a test capable of distinguishing among 1200 compound bugs with only 12 problems! A second important property of a test is that it cause each bug to be involved often enough to determine that it is consistent. The current tests that we are using are designed to cause each primitive bug to generate at least three errors. To accomplish this it was necessary to have 20 problems on the test (Burton 1982, p. 172).

The validity of such tests can be confirmed, they are reliable and (subject to the availability of computing resources) practical. A student's consistency in making particular errors can be checked in an online *adaptive system* by choosing the next question based on the responses received so far.

> The skills were organized into a hierarchy in order that an expert system could make inferences on student knowledge from the answers previously given and select the most appropriate question to ask next. This reduces the number of questions asked for a variable ability group (Appleby et al. 1997, p. 115).

The central difficulty is designing a good network of questions.

Alternative approaches have focused more on formative assessment. Rather than generating reliable diagnoses of a student's difficulties, they provide feedback which is designed to help students learn particular skills. For example, the group at Heriot-Watt University in the United Kingdom have more than a quarter of a century of experience in this area (see Beevers et al. 1991; Ashton et al. 2006). From the outset, one goal of this project was to assess students' steps in working.

> By giving the correct answer at each stage the student is encouraged to carry on but still has to work at each part before the answer is revealed. In a traditional text book example either the only answer given will be the final one or, if the question is answered in stages, it is difficult to avoid seeing all the answers at once (Beevers et al. 1991, p. 112).

In the CALM system of (Beevers et al. 1991), as in some contemporary systems, the extent to which the *mathematical properties* of students' individual mathematical expressions can be established was limited. We shall expand on this issue below. The problem of assessing a *complete mathematical argument*, without providing a template of teacher-designed answer boxes, remains an elusive goal.

However, by the early 1980s there was a backlash from some about the use, indeed as they saw it abuse, of computers for testing in this way. E.g.

> In most contemporary educational situations where children come into contact with computers the computer is used to put children through their paces, to provide exercises of an appropriate level of difficulty, to provide feedback, and to dispense information. The computer is programming the child (Papert 1980).

One result of this dissatisfaction was LOGO. Other exploratory computer environments, sometimes called microworlds, were another, see for example Noss and Hoyles (1996).

The difference between a focus on *skills and their acquisition* and *conceptual understanding* is a classic dichotomy in mathematics education. The classical algorithms of elementary mathematics, including place-value addition, subtraction, multiplication and division of numbers, together with algebra are sophisticated. The lattice of skills in basic subtraction of (Burton 1982) provides graphic evidence of this complexity. To achieve basic competence requires focus of attention and diligent practice. Methods in calculus, linear algebra and progressively more advanced topics rely on these foundations. Practice of such material has provided, and remains, a significant and important use of computer aided assessment systems. The popularity of freely available resources such as the Khan Academy (see http://www.khanacademy.org) is testament to this assertion. Publishers are increasingly providing online support for textbooks with automated versions of exercises linked to the work in the book. There are a range of commercial and open source systems with varying levels of mathematical and pedagogic sophistication. Therefore teachers are likely to increasingly mix CAA with traditional assessments. Furthermore, CAA is increasingly being used to automate high stakes final examinations. Currently, this is not widespread for school or university exams, but CAA forms a component of professional exams and in other areas such as the UK hazard perception test, an automated component of the UK driving test. This use is likely to increase significantly over the next decade. A survey of existing CAA can be found in Chaps. 8 and 9 of Sangwin (2013).

The focus in this paper is on CAA where the teacher may, at least in principle, author their own questions. However, this requires a new level of sophistication in using technical software packages, in particular of using CAS. In this regard, (Sangwin and Grove 2006) commented that "*teachers are often neglected learners*" themselves. It is necessary to have a detailed understanding in order to automate both the mathematics itself and details of the assessment process. Therefore CAA provides us with an opportunity to reconsider assessment, both its purposes and practices. All forms of assessment, from multiple choice questions, written answers under exam conditions, project work and oral examinations provide opportunities within certain constraints. CAA is no exception and while CAA does not claim to enable us to "assess everything" current CAA systems have a mathematical sophistication which enable valid assessment of significantly more than shallow procedural skills. Charles Babbage is reputed to have said the following:

> Propose to an Englishman any principle, or any instrument, however admirable, and you will observe that the whole effort of the English mind is directed to find a difficulty, a defect, or an impossibility in it. If you speak to him of a machine for peeling a potato, he will pronounce it impossible: if you peel a potato with it before his eyes, he will declare it useless, because it will not slice a pineapple.

CAA is just a *tool*, and it has some uses while not claiming to achieve everything.

It is interesting to recall how few tools were available to help the CAA system designer during the mid-1980s, and the power and sophistication of machines available to students. Even code to evaluate an expression at a point, crucial to establishing algebraic equivalence, had to be written from scratch by Beevers et al. (1991). In the last ten years software development increasingly takes advantage of code libraries for specific purposes, for example MathJax (http://www.mathjax.org) enables most web browsers to display mathematical expressions. Furthermore, teams of like-minded individuals collaborate on software projects and freely share the results. These include content management systems, such as Moodle (http://moodle.org/). For many students, at least in the more economically developed counties, the last ten years have enabled wide access to wireless networks through laptop computers and hand-held mobile devices.

STACK

In this section we describe STACK, an advanced general CAA system for mathematics, with an emphasis on formative assessment. The primary design goal was to enable a student to enter a mathematical answer in the form of an algebraic expression. While multiple choice questions (MCQ) have a place, the student selects an option from a list provided by the teacher. The purpose of many questions is grotesquely distorted by using a MCQ for mathematics, and hence the assessment is invalid. For example, solving an equation from scratch is significantly different than checking whether each potential response is indeed a solution. We preferred a system which evaluates student provided answers.

An example question, with student's response and feedback, is shown in Fig. 1, (see Sangwin 2010). The student's answer is a mathematical expression which they must enter into a form through a web browser. STACK then establishes the mathematical properties of the answer. For many questions the teacher will seek to establish that (i) the answer is algebraically *equivalent* to the correct answer and (ii) the student's answer is in the appropriate *form*, (e.g. factored). However, the answer need not be unique and STACK establishes properties of expressions, but remains an objective testing system. In the case of Fig. 1 the teacher does not simply establish that the student's answer is "the same" as their answer. A list of separate properties is needed, and many different answers may satisfy these. Notice that the feedback in Fig. 1 is specific to the answer and directly related to possible improvement on the task. STACK may include and display results of computer

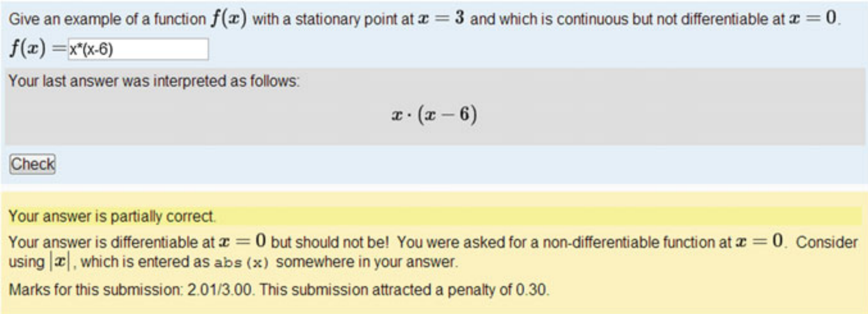


Fig. 1 An example STACK question

algebra calculations within such feedback which can be as detailed as appropriate to the question. This is a particular distinguishing feature of STACK.

In particular STACK uses the computer algebra system Maxima to

- randomly generate problems in a structured mathematical way;
- establish the mathematical properties of expressions entered by the student;
- generate feedback, as necessary, which may include mathematical computations of the student's answer;
- help the teacher analyse the attempts at one question, or by one student.

Version 1 of STACK was a stand-alone quiz system. Version 2 was partially integrated into the Moodle content management system while version 3 provides a question type for the Moodle quiz. STACK was designed and developed by the author, with substantial code for version 3 being written by Tim Hunt of the Open University in the UK. Numerous other colleagues have contributed to the design, code, testing and documentation. Version 2 of STACK has been translated into Finnish, Portuguese, German, Dutch and Japanese and is in regular use by large groups of students for formative and summative assessments. See (Sangwin 2010).

Establishing Properties

The key issue for teachers is to articulate the properties sought in an expression, and then encode an algorithm which establishes each of these. Where only some of the properties are satisfied the system may provide feedback to the student. The most common property a teacher will wish to establish is that two expressions are equivalent, for example the student's and teacher's respective answers. In many systems, e.g. (Beevers et al. 1991), the algebraic equivalence of two expressions is established by choosing random numbers and evaluating each expression at these points as floating point approximations. Numerical analysis assures us that a reasonable match between the values for each expression gives a reasonable probability of the two expressions being identical. Since students' expressions are usually relatively simple the probability of a false result is very low indeed. This approach was also used by Appleby et al. (1997) and many others. An alternative approach is to assign the variable SA to represent the student's answer and TA to be the teacher's and use the computer algebra command simplify and evaluate the following pseudocode.

if simplify(SA − TA) = 0 then true else false.

If this test returns true then we have established the two expressions are equivalent. However, if the test returns false are we really sure they are not equivalent? What matters in this approach is the strength of the "simplify" command, and the ability to know with certainty that the zero expression really means zero, and a non-zero expression is really non-zero. Different CAS implement functions such as "simplify" in a surprising variety of ways, and (Wester 1999) provides an interesting, but dated, comparison of CAS capabilities. From a theoretical perspective implementing this test is not possible in all cases! (Richardson 1966) showed that there is no algorithm which can establish that a given expression is zero in a finite number of steps for a reasonable class of elementary expressions. These results have been refined, e.g. by Matiyasevich (1993) and sharpened, e.g. by Caviness (1970) who showed that for the sub-class of polynomials over the complex rational expressions together with unnested exponential functions then zero really means zero, i.e. we do have a canonical form. Moses (1971) comments

> In fact, the unsolvability problem may lie in Richardson's use of the absolute value function. When one adds the absolute value function to a class of functions which forms a field (e.g. the rational functions), then one introduced zero divisors. For example, $(x + |x|)(x - |x|) = 0$, although neither factor is 0.

While Richardson's result is a theoretical restriction on the effectiveness of our test for algebraic equivalence, in practice for learning and teaching such tests work very well indeed on the limited range of expressions which arise as answers to typical assessments. As (Fenichel 1966) comments "*recursive undecidability can be a remote and unthreatening form of hopelessness*".

Many systems, including STACK, make use of computer algebra in this way. What then, does "simplify" mean? This is a phrase which is in common currency throughout elementary teaching we now argue that this is ambiguous and is often used for the opposite mathematical operations. For example, 1 is simpler than 7^0 but $7^{7^\wedge(10)}$ is probably simpler than writing the integer it represents. There are many algebraic examples, e.g. compare the expanded and factored forms of $x^{12} - 1$ and $(x - 1)^{12}$. Similar examples can be found for many other pairs of forms, e.g. single fractions and partial fractions, or various trigonometrical forms. It is not difficult to find examples in textbooks. The word "simplify" may mean little more than *transform* an expression into an equivalent, but perhaps unspecified form.

Fitch (1973) gave three reasons for simplifying expressions, the last of which was deciding if an expression is identically zero. The first is what he calls compactness of expressions, to make the expression smaller and this idea can be found in older writers, for example (Babbage 1827, p. 339) comments

> whenever in the course of any reasoning the actual execution of operations would add to the length of the formula, it is preferable to merely indicate them.

Or further back (Euler 1990, x50), "*[...] the simplicity of the equation expressing the curve, in the sense of the number of terms.*" Designers of contemporary CAA have also reached this conclusion, e.g. (Beevers et al. 1991, p. 113)

> It has long been accepted in science that "the simplest answer is the right one". We have translated this premise into "the shortest answer is the right one".

For these CAA designers the length of the representation was a key property of an expression. Of course, compactness is strongly related to the way in which information is represented, so this measure only makes sense in a particular context. Simplicity can also be interpreted as the ease with which calculations can be carried out. This view was developed by (Moses 1971):

> Of course the prevalence in algebraic manipulation systems of simplification transformations which produce smaller expressions is due mostly to the fact that small expressions are generally easier to manipulate than larger ones.

The second reason (Fitch 1973) gives for simplifying expressions, also discussed by Fenichel (1966), is intelligibility. That is making it easier for users to understand. It is not immediately clear that compactness and intelligibility are different. As one example, consider replacing trigonometric functions by complex exponentials. Using these we remove the need for any trigonometric identities. The formal rules $e^x e^y = e^{x+y}$, $(e^x)^y = e^{xy}$ and $e^0 = 1$ suffice. In this process we also remove the redundancy in using tan, cosec etc. and a plethora of separate rules. Hence, these transformations render expressions much easier for the machine to manipulate, with fewer rules and fewer operations. A user, on the other hand, may expect their answer in terms of these traditional trigonometric forms rather than as complex exponentials. *Ease of computation* and *intelligibility* are different issues.

Notice here the first issue we have to address, i.e. whether the teacher's expression is equivalent to the student's, immediately raises very interesting theoretical issues in computer science, and implicitly raises pedagogic issues. What is a student to make of the instruction to "simplify"? Is it any wonder some of our students remain perpetually confused? Appreciation of this potential ambiguity of 'simplify' suggests we develop a much more sophisticated vocabulary with which to talk about algebraic operations and the senses in which two expressions can be compared. Others, e.g. (Kirshner 1989) agree with this need.

> This analysis, we believe, points the way to a new pedagogical approach for elementary algebra, an approach that requires syntactic and transformational processes to be articulated declaratively, enabling more, rather than fewer, students to escape from the notational seductions of nonreflective visual pattern matching (Kirshner 1989, p. 248).

The approach in STACK is to enable the teacher to specify many senses in which two expressions might be the same or different, and separately to enable the teacher to test whether an expression is written a number of forms. To do this, STACK provides the user with a number of answer tests. Testing for properties is significantly different than performing calculations, and so requires specific computer algebra functionality to enable this. Inequalities, equations, and particularly systems of polynomial equations (see e.g. Badger and Sangwin 2011) all have interesting elementary mathematical issues of this type which teachers, and students would benefit from appreciating more deeply.

Question Models

In many situations the teacher will seek to establish more than one property and so need a mechanism by which a number of tests can be applied, perhaps in a specific order, and outcomes assembled on the basis of the results. An example STACK question is shown in Fig. 2 which was originally written by Vesa Vuojamo at Aalto University, Finland [see (Rasila et al. 2010) for details of their use of STACK] This question asks students to find the polynomial $p(x)$ which makes the function $f(x)$ continuously differentiable.

In this case the correct answer is the cubic spline which is unique up to algebraic equivalence. However, in practice STACK actually establishes five separate properties and provides separate feedback in each case. These properties establish that the student's answer is a cubic in x; that the polynomial passes through the points

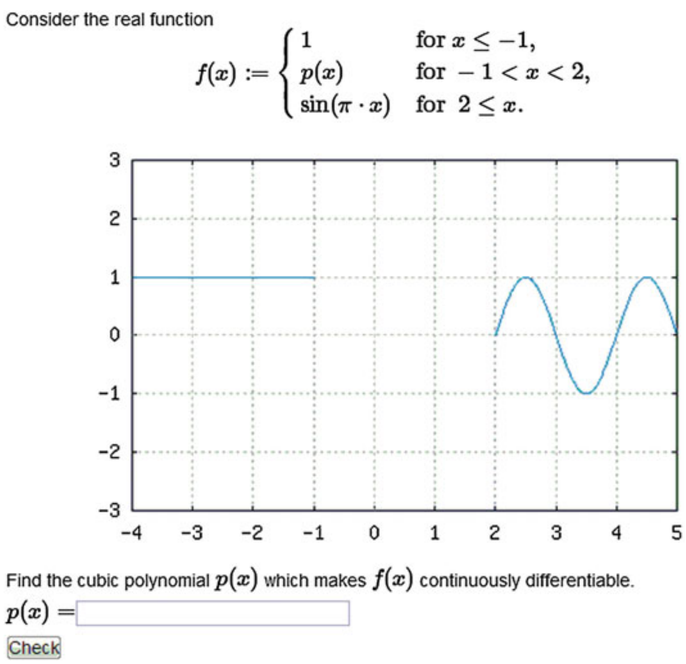


Fig. 2 Testing for individual properties in STACK

$p(-1) = 1$ and $p(1) = 0$, and the derivative of the student's answer matches $f(x)$ at $x = -1$ and $x = 1$. For brevity we have omitted a screen shot of this automatically generated feedback. Notice the significant shift here between "comparing the student's answer with the right answer" and articulating the properties needed separately. This is the central issue for the teacher.

An important goal of the STACK project was to enable teachers to write their own questions. Previous systems, in particular the AiM system of (Strickland 2002), forced the question author to become a computer programmer, in effect writing substantial pieces of computer algebra code to generate the required response processing commands for each question. While AiM does have an authoring system, many other CAA implementations do not: each question is essentially a bespoke computer programme making it impossible for anyone other than an expert developer to write questions. By providing answer tests which enable specific properties to be established we both reduce the amount of code and articulate what is intended in a way which is much clearer than expressions such as simplify (SA – TA). There is also an important conceptual shift needed here by the teacher, who must specify properties explicitly, and not use proxies for those properties. Experience has demonstrated, however, that writing reliable, valid questions remains a difficult task, requiring expertise.

While early CAA pioneers had to write everything from scratch, this did provide great freedom with respect to the underlying *interaction model* which the students are forced to use. Essentially the "model" is the flow-chart through which user's interactions change the internal state of the system until an end point is reached. For example, how many "attempts" can a user make? What form do these attempts take? What response does the system make and to what extent can a particular teacher make choices? How are questions sequenced, e.g. does the student see a fixed quiz at the outset, or are questions sequenced in an adaptive way as in the DIAGNOSYS system of Appleby et al. (1997)?

As we said before, the attempt to encode something forces you to be very specific about the details of what you are trying to do. We have discussed one example, "simplify", at the level of computer algebra. Next we consider one example at the level of the question model.

In moving the development of STACK from version 1 to version 2 we (somewhat naively) assumed that there would be a clean separation of the "question" from the "quiz". That is, it would be possible to "insert questions" into a more general quiz structure in a flexible way. This turns out to be exceedingly difficult to do, which was a significant and unexpected surprise. For example, many teachers using CAA for formative assessments will ask a group of students to complete a "quiz" of a predetermined number of questions each week. Formative feedback is available, and where necessary multiple attempts are encouraged to help students ultimately succeed in these tasks. However, there is a strict time limit, after which further attempts are prevented and the teacher's model solutions become available. This model of interaction essentially replicates traditional teaching in an online manner, which may or may not be efficacious. Notice that the concept of "due date" is a property of the "quiz", but availability of the "worked solution" is a property of the "question". It is simply impossible to divorce the two cleanly.

In STACK version 3, we have opted to provide a question type for the quiz system in Moodle. In these recent developments the designers of Moodle have provided much greater flexibility in the separation of "question" and "quiz" by the use of "behaviours", which enable the better integration of the model used by STACK which we have designed for use with a wide variety of types of mathematical questions.

When CAA "question types" are relatively simple, e.g. multiple response or numeric input and even when they are confined to single answers, a variety of models for interactions are available which are also relatively straightforward to understand. However, in mathematics, especially when we aim to accept (i) mathematical expressions as answers, or (ii) multi-part questions, then the situation becomes much more complex. Any model must provide interactions which the student clearly understands at each stage. There should be no doubt as to the consequences of each action with the system. In this way it should not raise concerns which distract from the actual mathematics. Teachers must also be able to author questions confident that forms of use are available which are sensible.

STACK implements multi-part mathematical questions. Figure 3 shows an example of a relatively elementary calculus question. Notice that all three parts refer to a single randomly generated mathematical object. Hence, we cannot really claim this is three separate questions. Furthermore, recall that unlike early CAA systems which were application software, students interact with STACK through a web browser. There is already an implicit interaction model here, which requires the student to submit a whole page at once. I.e. asking for individual parts to be marked separately is difficult. Indeed, if a student changes all parts, but only asks for one to be marked, then the model becomes quite intricate. What should the system do with data which has changed or input fields which are empty? Notice that we have used the word "part" without a definition. To the student, there are three inputs and so they might perceive this as having three parts. To the teacher, the first and second parts are linked, so are they separate parts or one?

In STACK a key design feature is a total separation of the inputs, into which students enter their answers, and potential response trees, which are the algorithms through which the mathematical properties are established. Response trees may rely on inputs in an arbitrary fashion, e.g. one-one or many-one.

Tied to the inputs is a concept of *validity*. The prototype input is an algebraic expression, and we expect the student to enter their answer using a traditional linear syntax into a web form box. Clearly the student needs to match brackets, indeed they need to enter a syntactically valid expression. STACK also enables teachers to permit a less strict syntax, e.g. omitting explicit * symbols for multiplication, where this is unambiguous. However, it is not clear in the twenty first century that this is helpful to students. As (Beevers et al. 1991) commented:

> We would like to make input simpler but have also recognised that restrictions can be advantageous. Most students will be using computers in other areas of their work and will need to learn to adapt to the rigours of computer input. The student is also forced to think much more carefully about the format of their answer since any ambiguity is punished mercilessly. This may be frustrating at first but can lead to a better understanding whereas a written answer may contain an ambiguity which is not recognised by the student and can lead to a misunderstanding later (Beevers et al. 1991).

The design issues associated with syntax were addressed in detail by (Sangwin and Ramsden 2007) with further discussion in (Sangwin 2013). Essentially, an unambiguous informal syntax is impossible, and when combined with international differences in notational conventions the situation becomes hopeless. Some conventions, particularly those for inverse trigonometric functions, are particularly problematic. Hence STACK provides a number of options for the input, which enables a teacher to tailor the meaning to the question and their group of students. Furthermore, the concept of syntactic correctness is only one part of the validation process. In some situations the teacher may wish to reject any floating point numbers, or rational coefficients not in lowest terms, as "invalid" and not "wrong". Not only is there a subtle pedagogic difference (e.g. "Since you have floats I'm not going to think about your answer!") but the way scores are calculated may depend on the number of valid attempts. Hence rejecting answers as invalid avoids penalising students on technicalities while reinforcing issues important to that teacher in a particular situation. All these decisions are at the control of the teacher of course. There are other reasons for invalidating an expression. If a student types an *expression* in place of an *equation* then the system can reject this as invalid, with explicit feedback of course. In practice students do need educating on how to enter their answer, and they take some time to become used to the interface. However, ultimately the majority of our students cease to find the interface especially problematic for the majority of questions. Entering a particularly complex expression is always going to be difficult. Notice however, that validity is significantly more involved that a syntactic check, and that validity is a concept tied to the input. It is separate from the notion of an answer being *correct*.

The algebraic expression is the prototype input, but the separation of inputs enables a variety of other interactions to be implemented. For example, HTML elements such as drop-down lists, "radio buttons" and checkboxes for MCQs have been implemented in a relatively straightforward way. MCQs do have a place, and often these can be combined as multi-part questions with algebraic inputs. Even when used alone, the support of CAS in randomly generating questions enables STACK to provide mathematical MCQs. The HTML text area enables multiple lines to be submitted by a student, although the primary use so far has been to provide more space when entering systems of equations. Systems of equations arise naturally when answering *algebra story problems*, see for example (Badger and Sangwin 2011). Asking a student to transform an algebra story into a system of equations, and then solve these, is a basic and classical mathematical task.

Normally, the student sees some validation feedback tied to the input, as shown in Fig. 2. The first time they submit their answer it is validated, and displayed in a

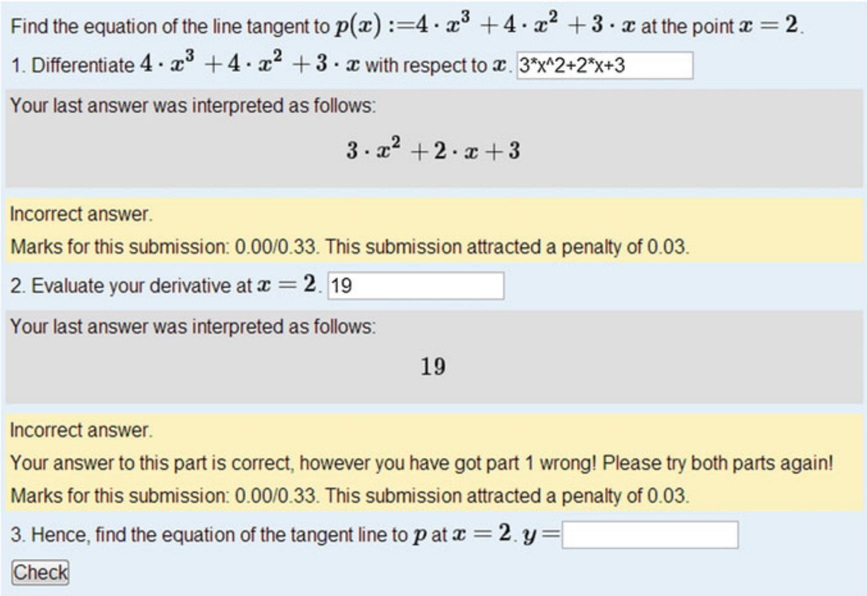


Fig. 3 Follow through marking in STACK

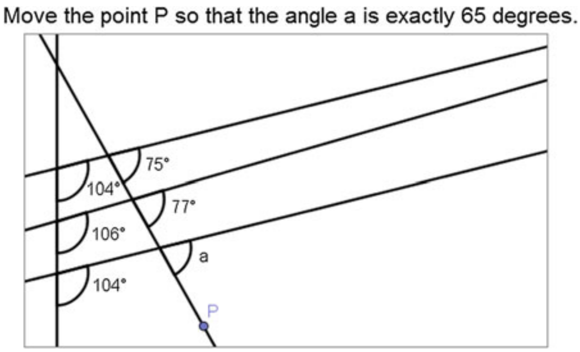
two dimensional format. This *double submission* is actually an artefact of the interaction model imposed by web page forms—there are other interaction models. For example, a student might see their expression build up in a two dimensional format as they type it, with brackets automatically closed, or mismatched brackets highlighted. In Fig. 3 feedback from the inputs showing expressions in two-dimensional traditional notation is separated out from feedback from the potential response trees which have established properties of the answers.

For some inputs, such as drop-down lists or multiple choice interactions, this double submission is irritating to users and the teacher can over-ride it. Another option is to make use of an equation editor to build up the expression in a two dimensional traditional way. Such equations editors are relatively standard in many CAA systems. STACK version 2 makes use of the DragMath editor, written by Alex Billingsly at the University of Birmingham.

The computer environment also enables other kinds of interactions which are not possible in a paper and pencil environment. Dynamic mathematics environments, such as GeoGebra, enable exploration and *mathematical experiments* to be undertaken. As exploration tools for learning, these are rather well established. These kinds of interactions can also be incorporated into assessments. The geometrical configuration of the diagram constitutes the mathematical answer.

One example of a STACK question with this type of interaction is shown in Fig. 4. Here, the student must move the point P by dragging on screen. As this is done, the values of the angles shown update dynamically as the dragging takes

Fig. 4 GeoGebra input interactions in a STACK question



place giving a particular form of immediate feedback. Once the student is satisfied they have the correct answer they can submit the page, and the value of the angle at *a* is returned as part of the answer. Such inputs potentially accompany other interactions in a multi-part question. Extensions to STACK to accommodate interactions such as this were first made at the University of Aalto in Finland, see (Rasila et al. 2010).

Now we turn to establishing the properties of an answer. Each potential response tree is an algorithm which relies on some (at least one) of the inputs. The fundamental requirement is that each input should provide a valid CAS expression. Once all the inputs upon which a potential response tree relies are valid, then the tree can be evaluated.

The potential response tree, technically an acyclic directed graph, consists of nodes with branches. Each node takes two expressions, e.g. the student's input and the teacher's answer, and compares them using a specified *answer test.* As discussed above the answer test might establish algebraic equivalence or one of a range of other properties. On the basis of this test, either the true or false branch is executed. The branches update numerical scores, generate feedback for students, create a "note" for later statistical analysis and then link to the next node or halt. Notice the first three of these actions generate outcomes which correspond broadly to the summative, formative and evaluative functions of assessment. Which outcomes are available to the student is an option which the teacher must choose. In Fig. 2 we have included only textual feedback and numerical scores have not been shown.

This separation between inputs and the algorithms which establish the properties of answers is not an obvious design advantage, however it actually enables many useful situations to be immediately implemented without having to have separate models for each "type" of interaction. The possibility of arbitrary mappings between inputs and outputs, together with an ability to place feedback in any position, enables a richer set of questions to be implemented.

The student in Fig. 3 has chosen to answer the first two parts. Notice that follow-through marking has been implemented, i.e. the student's expression in the first part has been used when marking the second. The student has correctly evaluated their expression, but actually they need to correct their original error and re-evaluate to correctly answer the whole question.

Immediately, a whole host of questions arise about how the student will interact with this system. Currently in STACK, all the "parts" of the question are visible to the student immediately. The question as phrased in Fig. 3 presupposes a particular method. Of course, other methods for finding tangent lines are perfectly valid. Does the teacher expect a particular method to be used, or are they simply interested in whether the student can find a correct answer using a valid method? Unless steps are laid out, to what extent can CAA assess which method is used?

From the outset, CALM system of Beevers et al. (1991) made a serious attempt to automate the assessment of steps in students' working. The interaction model they developed still had pre-defined templates through which students answered a question. However, in their system steps were revealed in a number of possible ways. For example, a student could ask for steps (possibly sacrificing some implied method marks as a result), or this could be triggered automatically if a student was unable to complete the whole question unaided. This kind of interaction model has been successful, and is widely used, e.g. see (Ashton et al. 2006). The STACK system has been modified and extended at Aalto University, Finland and is in regular use by large groups of engineering students. Their extensions include "question blocks" which can be revealed by correct or incorrect responses, providing very similar mechanisms to CALM. It is likely that a combination of the interaction model used by CALM, the mathematical sophistication underlying STACK, and the depth of adaptive design of DIAGNOSYS (see Appleby et al. 1997) will ultimately combine into a single CAA system.

Students' Reactions to STACK

The most important issue is students' reactions to tools such as STACK. Their reaction includes a number of aspects, such as their level of engagement, their reported affective reaction and ultimately their achievement. CAA tools do not operate in isolation: they are part of the whole experience. Clearly tools could be used in inappropriate, or even harmful ways, or they could simply replicate traditional teaching with little or no examination of the strengths and weaknesses relative to traditional methods. The difficulty in evaluating the effectiveness of such tools is isolating a specific effect from the general teaching. We also need to take account of the potential for innovator/novelty effects which might exaggerate the actual long term benefits. Of course, if a genuine benefit exists, even temporarily, some students will have taken advantage of this.

A common use of CAA at the university level is to automate routine practice exercises alongside traditional paper based tasks. This enables students to receive immediate feedback on tasks which assess competencies with manipulative skills. Teachers no longer have to undertake repetitive marking of such work by hand. This was our motivation for introducing CAA to the University of Birmingham mathematics degree programme in 2001. In the majority of situations where this kind of activity takes place, colleagues report a strong correlation between engagement with, and success on, STACK-based formative exercise questions and final marks in traditional exams. For typical data, see (Rasila et al. 2010). This is not surprising, and corresponds with the author's experience at the University of Birmingham. However, evidence and appreciation of this reinforces the ongoing need for regular monitoring of student activity to identify and support students who are not engaging with the online activities. These students have a high probability of ultimately failing the course. Many university courses have very large student groups, and with inevitable delays in marking paper-based formative work and collating such marks it is otherwise difficult to monitor such students individually.

To try to evaluate STACK we have undertaken focus groups with students to ask specifically about their experiences and reactions. The author has undertaken such focus groups, both at the University of Birmingham and with students at other institutions using STACK. A semi-structured interview provides a freedom to follow up themes or concerns in a way a paper-based questionnaire does not. From these interviews, some consistent themes emerge. Capital letters refer to individuals, although the quotations below are representatives from more than one focus group session.

Syntax is initially a problem

> A: I agree, but when you get used to STACK all that goes away [B: "yes"] but when you start that is a problem. It is very annoying when you try to type something… well you have the "check syntax", but if the check syntax is always incorrect you are like !!!

This is particularly problematic if a diagnostic test is the student's first experience of a university course. It is relatively common for students to sit such a test at the start of their course, Lawson (2003). However, the novelty of the university setting and unfamiliarity with the syntax combine to make this an unhappy experience.

> C: in the beginning, September or something like that, […] I think it was really annoying to use it then because you didn't know how to write it down, the syntax, […] You had a time to do the problems and it was very annoying that it said "wrong, wrong" all the time when it was the syntax. […] I didn't know how to write it down, so I got the wrong answer.

It was clear that it was syntax which was a barrier here. For one student prior knowledge removed this problem.

> B: Yes, I took this test but because I have a little bit of background of computer programming so I […] knew the syntax a bit. I was more frustrated because I didn't know the answers myself! [C: laughs] So I guess I have time to deal with the real mathematical problems, so I guess my frustration is based on my own lack of mathematical knowledge. So, I think the test worked quite well for me. But there you have it, I had some background with things like this.

Ultimately the syntax is learned in a relatively short space of time by the majority of our students. While there are differences between mathematical input notations, many systems share a common core of notational conventions. This in itself constitutes a valuable skill, as identified by Beevers et al. (1991), quoted above. However, it needs to be addressed specifically. I.e. we need to teach students how to express themselves unambiguously using this syntax. The feature most appreciated by students is the immediate feedback.

> C: Yes, I think it is good. Because of the feedback. […] with the paper you have to wait and then when you see the right answers you can look through those with the teacher probably too quickly, and you can't take your time to understand, but with STACK you can take your own time with those exercises. So that is the good thing with them.

Of course, a student could simply use a textbook for this. When specifically asked about this students responded

> A: yeah, yes but then you look at the answer before you have solved the problem. STACK won't tell you!
> B: it is a bit of cheating.
> A:You don't learn if you just go ahead and look at the back. And usually when we have homework during the course from the book they are usually problems you don't have the answers to so you can't find out if you are wrong or right.

When combined with the random questions, particularly when the teacher encourages repeated practice, this gives an interesting environment for self-motivated repeated practice. Students can try questions, respond to the feedback, perhaps look at complete worked solutions and then work on new problems from the same template. The use of random problems for each student provides a behaviour which does not occur when using fixed and static books.

> D: The questions are of the same style and want the same things but they are subtly different which means you can talk to a friend about a certain question but they cannot do it for you. You have to work it all out for yourself which is good.

This view was appreciated by other students on another occasion.

> B: I think one of the best things about STACK was the way it created the values, or the problems, for, like, meant for you. But they are still the same as your friend has so you can, like, collaborate on them and do some team work, and work on the difficulty with your friends, but you still have to do the exercise for yourself [A: yeah!] you have values and
> A: so you can't just copy!
> B: it won't help if you just copy the answer from your friend.

Notice here the appreciation of the surface variation in the context of an underling structure. In many situations, particularly in mastery of technique, the purpose of routine exercises is precisely to enable students to reach a point where they can recognise and successfully complete any problem from a particular class. Students who use STACK exercises at universities for mathematics are often amongst the highest achieving in their generation. It is not surprising to find recognition and appreciation of the mathematical sophistication.

> E: Recognising the turning points of the functions produced in question 2 was impressive, as there are a lot of functions with stationary points at $x = 1$ and it would be difficult to simply input all possibilities to be recognised as answers.

This was in response to a question such as show in Fig. 1 where the student is asked to find examples. In many situations the properties required by a correct answer can be established automatically, although it would be time consuming and somewhat tedious for a teacher to do so by hand. These kinds of questions have been widely discussed, e.g. (Watson and Mason 2002), but they reinforce the fundamental issue for CAA: the teacher must articulate the specific mathematical properties which an answer should satisfy.

Conclusions

Notice, however, the fundamental challenge remains here. We, as yet, have very few effective tools to encode a complete elementary mathematical argument on a machine. This document was originally prepared in LaTeX, which despite the steep learning curve still sets the standard for quality of mathematical *typesetting*. It was then converted to MSWord, which has very poor support for mathematics. What we cannot do is easily encode the meaning of an expression, and combine this with simple logic and automatic CAS calculations. Until we can achieve this simple interface, marking student's extended work automatically will be impossible. Given the theoretical difficulties of establishing equivalence of two expressions, establishing the validity of whole arguments automatically appears totally hopeless. For example, in finding the tangent line to answer the question posed in Fig. 3, the student could simply find the remainder when the polynomial is divided by $(x-2)^2$. The remainder after polynomial long division of $p(x)$ by $(x\text{-}a)^2$ always yields the tangent line at $x = a$ (see (Sangwin 2011) for details and other methods) without using calculus. Did the teacher want the answer using *a* valid method or using *the* method as taught? If we seek to automatically assess the working without providing a template this issue must be addressed.

Despite the fact that it is impossible to assess extended working, CAA is routinely used by thousands of students in many settings. These students and their teachers find many aspects of CAA very helpful. The ability to generate random questions, the immediacy of feedback and the detailed reporting are all cited as benefits. In particular, the use of randomly generated questions to enable discussion, and the ability to assess example generation tasks where the answers are difficult for a teacher to mark, are affordances which are unique to CAA. It is clear, at least to the author, that uptake of CAA will increase, in informal and self-directed situations and in formative settings. It is highly likely that CAA will become used in high-stakes examinations. In mathematics we do have objective notions of *correctness* and the progressive automation of mathematical knowledge provides our subject with an opportunity to implement valid assessment which are not apparent in essay or more subjective artistic disciplines. Hence, we have a responsibility to ensure the tools we use move beyond multiple choice questions or primitive *string match* to check expressions.

I end this paper with two comments. Firstly, for the arguments we encounter in many areas of elementary mathematics the theoretical difficulties do not arise: we can automatically decide if they are correct, or not (Beeson 2003). Secondly, automatic tools can be combined to establish the correctness, or otherwise, of parts of an argument. For example, a routine calculation within a longer proof can be checked automatically. It would be potentially very helpful to a student and teacher to have this confirmed automatically before the whole piece of work is submitted to an intelligent human marker. This semi-automatic approach, a pragmatic combination of human and automatic marking, seems to offer the most promising direction for future effort in computer aided assessment.

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Numerical Analysis as a Topic in School Mathematics

Shailesh A. Shirali

Abstract Concerns about the divide between school mathematics and the discipline of mathematics are known in math education circles. At the heart of the debate is the sense that imperatives in school mathematics differ from those in the discipline of mathematics. In the former case, the focus is on remembering mathematical facts, mastering algorithms, and so on. In the latter case, the focus is on exploring, conjecturing, proving or disproving conjectures, generalizing, and evolving concepts that unify. It is clearly of value to find ways to bridge the divide. Certain topics offer greater scope at the school level for doing significant mathematics; one such is the estimation of irrational quantities using rational operations. This problem is ideal for experimentation, forming conjectures, heuristic reasoning, and seeing the power of calculus. The underlying logic is easy to comprehend. It would therefore be very worthwhile if we could make such topics available to students in high school.

Keywords Numerical analysis · School mathematics · Discipline of mathematics · Estimation · Irrational quantity · Rational operation

School Mathematics and the Discipline of Mathematics

Concerns about the divide between school mathematics and the discipline of mathematics are well known in math education circles. At the heart of the debate is the sense that imperatives in school mathematics differ in a fundamental way from those in mathematics as 'done' by practicing mathematicians.

In the former case, the focus is on the concrete, measurable and reproducible: on remembering mathematical facts (formulas, theorems, etc.); mastering algorithms; reproducing proofs and derivations; answering questions in tests; and (typically)

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being answerable to an authority figure. In the latter case, the focus is on exploring; conjecturing; testing conjectures; proving or disproving them; establishing theorems; generalizing; creating; inventing; evolving concepts that unify; and so on.

Anne Watson writes in Watson (2008):

> In this paper I argue that school mathematics is not, and perhaps never can be, a subset of the recognized discipline of mathematics, because it has different warrants for truth, different forms of reasoning, different core activities, different purposes, and necessarily truncates mathematical activity. In its worst form, it is often a form of cognitive bullying which neither develops students' natural ways of thinking in advantageous ways, nor leads obviously towards competence in pure or applied mathematics as practiced by adult experts. For me, the starting point [in this debate] is what it means to do mathematics, and to be mathematically engaged. In the discipline of mathematics, mathematics is the mode of intellectual enquiry, and effective methods of enquiry become part of the discipline—so much so that mathematics theses do not have chapters explaining methodology and methods. 'Doing mathematics' is predominantly about empirical exploration, logical deduction, seeking variance and invariance, selecting or devising representations, exemplification, observing extreme cases, conjecturing, seeking relationships, verification, reification, formalization, locating isomorphisms, reflecting on answers as raw material for further conjecture, comparing argumentations for accuracy, validity, insight, efficiency and power. It is also about reworking to find errors in technical accuracy, and errors in argument, and looking actively for counterexamples and refutations. It can also be about creating methods of problem-presentation and solution for particular purposes, and it also involves, after all this, proving theorems.

Here is Ramanujam in Ramanujam (2010):

> There are several ways in which mathematics in school classrooms misses elements that are vital to mathematicians' practice. Here, we wish to emphasize processes such as selecting between or devising new representations, looking for invariances, observing extreme cases and typical ones to come up with conjectures, looking actively for counterexamples, estimating quantities, approximating terms, simplifying or generalizing problems to make them easier to address, building on answers to generate new questions for exploration, and so on. In terms of content area and the methodology of content creation, it may be hard to mirror the discipline of mathematics in the school classroom, but we suggest that bringing these processes into school classrooms is both feasible and desirable. This not only enriches school mathematics but can also help solve problems that are currently endemic to mathematics education: perceptions of fear and failure, and low participation.

And here is what the widely cited "National Curriculum Framework" document (National Council for Educational Research and Training [NCERT] 2005) states:

> What can be leveled as a major criticism against our extant curriculum and pedagogy is its failure with regard to mathematical processes. We mean a whole range of processes here: formal problem solving, use of heuristics, estimation and approximation, optimization, use of patterns, visualisation, representation, reasoning and proof, making connections, mathematical communication. Giving importance to these processes constitutes the difference between doing mathematics and swallowing mathematics, between mathematisation of thinking and memorizing formulas, between trivial mathematics and important mathematics, between working towards the narrow aims and addressing the higher aims.

Is There a Way Out?

The passages quoted above describe the problem eloquently. Is there a way out? Can we bridge this divide in any way?

Bridging the divide would imply helping students learn the ways that mathematicians think about problems, enabling them to experience the process of creating, conjecturing and exploring. If we do not attempt to do this, then we help perpetuate the divide.

Many writers have commented on the 'discontinuities' or 'transitions' that occur during a student's growth years, and while talking of the challenges that are faced by a mathematics teacher, Felix Klein has even talked of a 'double discontinuity' (Siu 2008). These singular points clearly need to be taken note of and factored into our teaching methodology. The most crucial of these transitions is perhaps in the area of problem solving: the fact that "the answer, if there is one, is not the end of the process" (Ramanujam 2010).

Is it possible to build bridges across these transition points? To help students learn how to mathematize alongside their learning of content? To let students in on the process of experimenting, creating and conjecturing? In this paper we suggest that this is indeed possible, and that numerical analysis is a convenient topic for enabling the passage through this transition. We report on a real classroom experience.

It is certainly true that in the ultimate analysis, it is the mode of transacting a class that is of greater importance than the subject matter itself: the manner in which the creative teacher opens up an issue for exploration or reflection; engages students as a group or as individuals; draws them into reflecting on a question or a problem; and turns even the tiniest of opportunities into avenues for learning. But there are some topics where this becomes easier to do, inasmuch as opportunities for asking open, accessible questions are greater, as also opportunities for using platforms such as computer software.

Two such topics which in our view offer succor and which do not find a place in the regular high school mathematics curriculum are *Elementary Number Theory* (ENT) and *Numerical Analysis*. The reasons for these choices are simple. In both cases it is possible to do many of the things described above—experimenting, looking for patterns, conjecturing, looking for counterexamples, and so on—in a way that is accessible at the school level. Both are well suited for computer based exploration. In Shirali (2010) the present author explored the possibilities offered by ENT. In this article we explore the possibilities offered by Numerical Analysis (viewed as a subtopic of mathematical modeling).

We list some desirable features of school level mathematical modeling activity: (i) The activity should use school level algebra, geometry, coordinate geometry and calculus, (ii) it should require working with computer software (e.g., *GeoGebra*, *Derive* and *Excel*, or any of their multi-platform equivalents), (iii) the methodology should be: exploration, data collection or generation, analysis of data, followed by theoretical investigation to understand the data.

A Topic for Exploration: Estimation of Irrational Quantities

The specific topic in numerical analysis that we take up is: *Estimation of irrational quantities using rational operations*. This simply stated problem contains a veritable wealth of opportunity. It is ideal for experimentation using software for computer algebra and dynamic geometry, and it provides a wonderful context in which we can form conjectures and reason heuristically, using elementary algebra and calculus. Finally, it provides a fertile ground for engaging with a historical perspective, because such kinds of reasoning go far back into human history.

Example: Estimating the Square Root of 2

We illustrate the theme by considering this simply stated problem: *Find good rational approximations to* $\sqrt{2}$. Numerous heuristic approaches can be envisaged, but here is one that is very straightforward.

We start with a known reasonably good approximation to $\sqrt{2}$, say $7/5$. We now make use of the fact that $\left|7-5\sqrt{2}\right|<0.1$. From this we deduce that $\left|7-5\sqrt{2}\right|^2<0.01$. Expanding the expression on the left side we get:

$$99-70\sqrt{2}<0.01, \quad \therefore \sqrt{2}\approx\frac{99}{70}. \tag{1}$$

This yields an estimate for $\sqrt{2}$ which is accurate to 4 decimal places. We can easily carry this further. Squaring the expression $99-70\sqrt{2}$ we get

$$\left(99-70\sqrt{2}\right)^2 = 19601-13860\sqrt{2}<0.0001. \tag{2}$$

From this we get the approximation

$$\sqrt{2}\approx\frac{19601}{13860}. \tag{3}$$

This is accurate to 8 decimal places.

Note the following important features of the underlying process:

- It needs an *initial good approximation*. If the starting estimate is not so close to the true answer, then the process will not yield good results.
- It is *iterative* in nature: starting with a close approximation, we get successively closer approximations.

- It may be applied to find a good rational approximation to the square root of any rational number, and if greater accuracy is required, the process allows for it; thus, it is *generalizable*.
- At the same time it is not infinitely generalizable: it can be applied to find good rational estimates for (say) $\sqrt{3}$ and $\sqrt{5}$, but not for π.
- The mathematics involved is simple; indeed, in this example, nothing more than tenth grade algebra was used. This is the case with many estimation problems: the underlying logic is fairly elementary in nature, yet the different elements combine together to give a result that is highly satisfactory.

Rational Approximations to Irrational Functions

Consider the problem of finding rational functions that yield close approximations to a given irrational function $h(x)$ in the neighbourhood of $x = 0$. Simple heuristic reasoning can lead us to find the first several terms of the Maclaurin series for this function. The reasoning proceeds thus: If we want two curves to stay close together in the neighbourhood of a point P where they meet, then surely we must ensure that they have equal slope at P; else the curves will quickly draw apart as we travel away from P. Next, if their second derivatives differ at P, then this will lead to a steadily widening gap between the first derivatives, i.e., the slopes, and hence to a widening gap between the curves; so we would do well to make the second derivatives coincide as well. Continuing, if their third derivatives differ at P, then this will lead to a widening gap between their second derivatives, thence to a widening gap between their first derivatives, and finally to a widening gap between the two curves themselves; and so on. So to achieve significant closeness of the graphs of two functions in the vicinity of a given value of x, we should try to ensure that the two functions coincide in their first several derivatives at that value—as many as possible.

Such 'kitchen' logic appeals readily to students, and the fact that we can actually test the resulting formulas using a hand-held calculator is most reassuring; a bonus in fact. As we show below, we can even reconstruct the famous Bakhshali square root formula by arguing this way.

Rational Approximations to the Square Root Function

Let $f(x) = \sqrt{1+x}$, defined for $x \geq -1$. The first few derivatives of $f(x)$, evaluated at $x = 0$ (and starting with the zeroth derivative, which is f itself), are:

$$1, \quad \frac{1}{2}, \quad -\frac{1}{4}, \quad \frac{3}{8}, \quad -\frac{15}{16}, \quad \frac{105}{32}, \ldots \tag{4}$$

Note that these numbers are rational. We wish to generate rational functions $g(x)$ of x that closely approximate $f(x)$ in the neighbourhood of $x = 0$; naturally, we want all their coefficients to be rational numbers (because all the computations must be in the domain of rational arithmetic). The simplest such functions are the *polynomials with rational coefficients*, and the polynomials of successively higher degrees which agree with $f(x)$ in its successive derivatives at $x = 0$ are simply the partial sums of the Maclaurin series of $f(x)$ about $x = 0$:

$$1, \quad 1+\frac{x}{2}, \quad 1+\frac{x}{2}-\frac{x^2}{8}, \quad 1+\frac{x}{2}-\frac{x^2}{8}+\frac{x^3}{16}, \ldots \tag{5}$$

The graphs of these functions, for $-0.5 \le x \le 2.5$ are shown in Fig. 1, together with the graph of $\sqrt{1+x}$. Their closeness in the vicinity of $x = 0$ is very visible, as also the growing error when x goes beyond 1.

Next in line are the *rational functions with rational coefficients*; they have the form p/q where p and q are polynomials with rational coefficients and low degree. Here we consider only the following three kinds of rational functions:

$$\frac{1+ax}{1+bx}, \quad \frac{1+ax}{1+bx+cx^2}, \quad \frac{1+ax+bx^2}{1+cx}. \tag{6}$$

By equating their successive derivatives at $x = 0$ with the respective derivatives of $f(x)$ and solving a few equations, we find the values of the coefficients. We get the following rational functions which we call, respectively, $r_{1,1}(x)$, $r_{1,2}(x)$ and $r_{2,1}(x)$:

$$r_{1,1}(x) = \frac{1+\frac{3x}{4}}{1+\frac{x}{4}}; \tag{7}$$

$$r_{1,2}(x) = \frac{1+\frac{5x}{6}}{1+\frac{x}{3}-\frac{x^2}{24}}, \tag{8}$$

$$r_{2,1}(x) = \frac{1+x+\frac{x^2}{8}}{1+\frac{x}{2}}. \tag{9}$$

The graphs of these functions over $-0.5 \le x \le 2.5$ are shown in Fig. 2. We see that the curves stay much closer to the curve $\sqrt{1+x}$ than do the polynomials obtained from the Maclaurin series. This comes as a surprise, as there is no obvious reason why such should be the case.

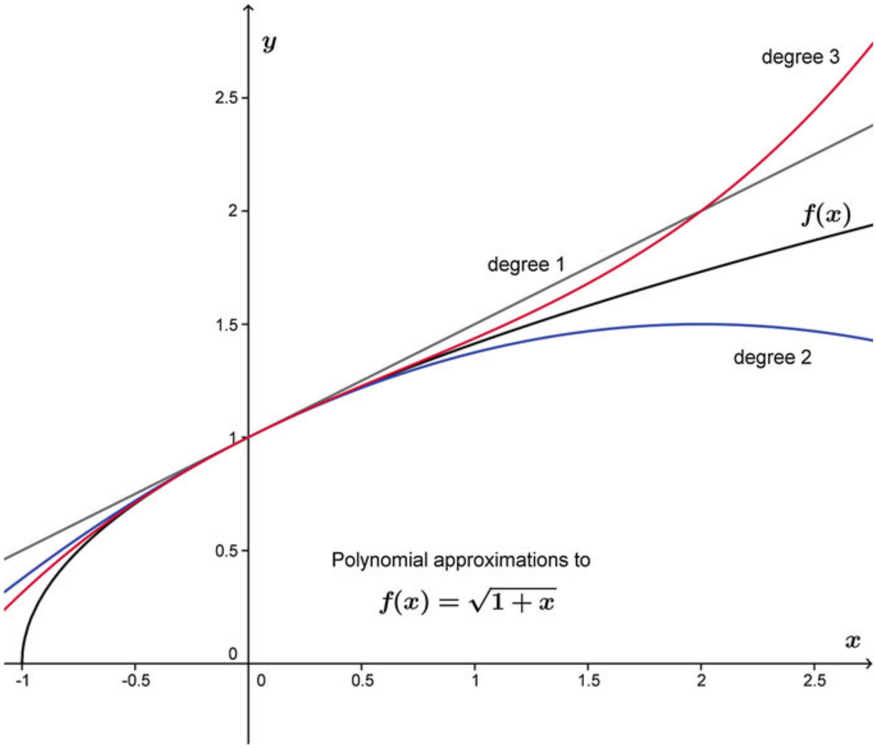

Fig. 1 Graphs of $\sqrt{1+x}$ and the first few partial sums of its Maclaurin series about $x = 0$

We also see that of the three functions considered, the one that best approximates $\sqrt{1+x}$ is $r_{2,1}(x)$. Now a straightforward manipulation yields the following:

$$\frac{1+x+\frac{x^2}{8}}{1+\frac{x}{2}} = 1+\frac{x}{2}-\frac{\left(\frac{x}{2}\right)^2}{2\left(1+\frac{x}{2}\right)}. \tag{10}$$

At this point we realize that something unexpected has happened: we have stumbled upon a very famous formula! For, the formula given on the right side is the *Bakhshali approximation*—a formula whose origins go back to the fourth century or earlier (the formula appears in a birch bark manuscript found in 1881 during an excavation in the village of Bakhshali, in North West Pakistan; see the image at http://treasures.bodleian.ox.ac.uk/The-Bakhshali-Manuscript). It is expressed in the following form:

> In the case of a non-square number, subtract the nearest square number, divide the remainder by twice this nearest square; half the square of this is divided by the sum of the approximate root and the fraction. This is subtracted and will give the corrected root.

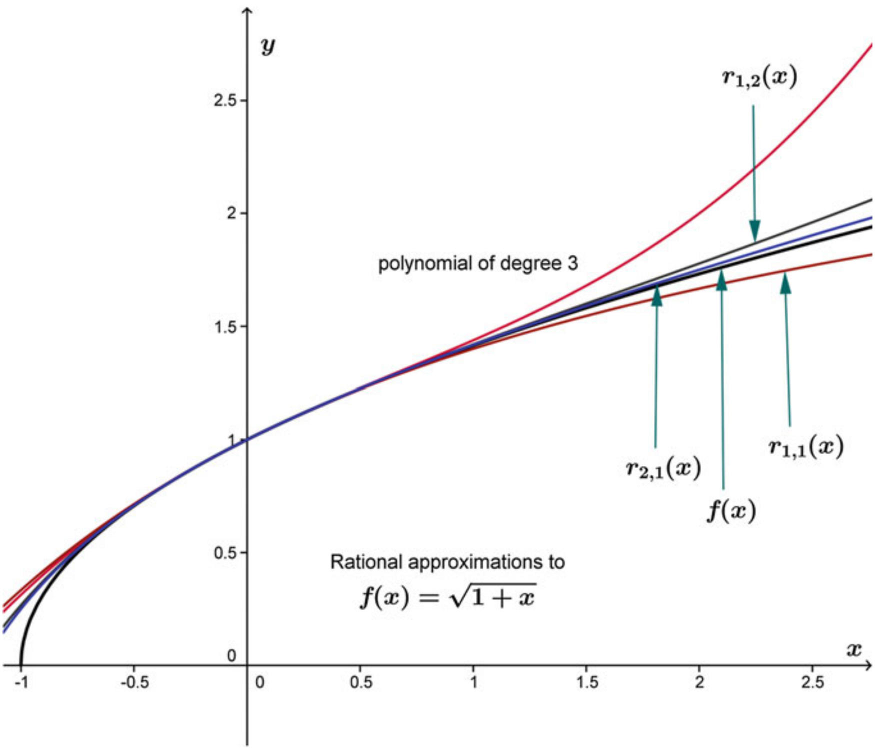

Fig. 2 Graphs of $\sqrt{1+x}$ and the functions $r_{1,1}(x), r_{1,2}(x)$ and $r_{2,1}(x)$

In other words:

$$\sqrt{A^2+b} \approx A + \frac{b}{2A} - \frac{\left(\frac{b}{2A}\right)^2}{2\left(A+\frac{b}{2A}\right)}. \tag{11}$$

To see the connection between this and the formula obtained above, put $x = b/A^2$ and do a few simplifications. Formula (11) readily emerges.

The Bakhshali formula yields extremely accurate estimates. For example, to estimate $\sqrt{11}$ we may take $A = 3, b = 2$ which yields $\sqrt{11} \approx 199/60$ or roughly 3.31666. This is accurate to five significant figures.

If instead we take $A = 3.3, b = 0.11$ (we are still in the realm of rational arithmetic), we get the estimate $\sqrt{11} \approx 79201/23880$ which is accurate to ten significant figures.

It is not very difficult to explain 'why' the Bakhshali formula yields such good results: one only has to compare the associated Maclaurin series with the corresponding series for $\sqrt{1+x}$ and note the matches between the successive terms.

It is a nice challenge to derive the formula. Using the Newton-Raphson formula, there is a very easy derivation (which incidentally showcases the great power of the Newton-Raphson formula). Less obvious is an algebraic derivation based on iteration. These matters are considered in detail in Shirali (2012). But the question that intrigues and at the same time frustrates us is this: *By what route might the original discoverers have found the formula?* Unfortunately we may never know the answer to this; there are no clues whatsoever. For more on the historical background of the formula and for why the dating of the manuscript has significance for historians, see Bakhshali manuscript (2012).

We see from this account how a fairly simple development of ideas has uncovered a formula that is not only extremely impressive but also connects with the distant past, allowing us to engage with questions about the people who lived in those regions, with questions about anthropology and history, and with questions about the history of mathematics itself.

Rational Approximations to the Tangent Function

We move now to conducting such an exercise for a few trigonometric functions. It is very easy to motivate the problem. To estimate square roots one can (in principle) get an answer without possessing any secret formula at all; for example, to estimate $\sqrt{11}$ one just tries many possibilities, one by one, and zeroes in on the answer; the only skill needed is the ability to do multiplication, and the only resources needed are computing power and time (a fair amount of this!). But for the trigonometric functions it is not clear how we might proceed. For example, there is no obvious way, intuitive or otherwise, by which one can find the value of $\sin 20°$ or of $\tan 40°$.

Here is an approach to finding a rational approximation to the tangent function, as described in Cheney (1945). Let $f(x) = \tan \pi x/4$. We seek a rational function $g(x)$ that closely approximates $f(x)$ over the interval from 0 to 1. To this end we impose the following conditions which are easy to justify:

- $g(0) = f(0)$ and $g(1) = f(1)$;
- $g'(0) - f'(0)$ and $g'(1) - f'(1)$ are very small.

We have the following values:

$$f(0) = 0, \quad f(1) = 1, \quad f'(0) = \frac{\pi}{4}, \quad f'(1) = \frac{\pi}{2}. \tag{12}$$

We search for a candidate $g(x)$ by using the approximation $\pi \approx 22/7$ and demanding that

$$g(0) = 0, \quad g(1) = 1, \quad g'(0) = \frac{11}{14}, \quad g'(1) = \frac{11}{7}. \tag{13}$$

Since

$$\tan\frac{\pi x}{4}=\frac{\sin \pi x/4}{\cos \pi x/4}\approx\frac{\pi x/4-\pi^3x^3/384}{1-\pi^2x^2/32}=\frac{\pi x}{4}\cdot\frac{1-\pi^2x^2/96}{1-\pi^2x^2/32}, \tag{14}$$

we look for functions $g(x)$ of the following form:

$$g(x)=\frac{ax(b-x^2)}{c-x^2}, \tag{15}$$

where a, b, c are real numbers. On setting up the equations and solving for a, b, c we get

$$g(x)=\frac{x}{7}\cdot\frac{(22-x^2)}{4-x^2}. \tag{16}$$

This very simply produced approximation yields accuracy to three significant figures for all values of x between 0 and 1. For example:

$$g\left(\frac{1}{2}\right)=\frac{29}{70}\approx 0.4143, \quad \tan\frac{\pi}{8}=\sqrt{2}-1\approx 0.4142; \tag{17}$$

and

$$g\left(\frac{1}{3}\right)=\frac{197}{735}\approx 0.26803, \quad \tan\frac{\pi}{12}=2-\sqrt{3}\approx 0.26795. \tag{18}$$

Now we are in a position to compute the value of $\tan 40^\circ$. The value of x we use is $8/9$. On substituting $8/9$ into (16) we get $3436/4095$ which yields $\tan 40^\circ \approx 0.83907$. This approximation is accurate to four significant figures.

Rational Approximations to the Sine and Cosine Functions

In much the same way, we may search for rational approximations to the sine and cosine functions. The exploration not only brings forth some striking results but also connects with an extraordinary formula dating from the seventh century. Unlike the Bakhshali formula this one is well documented; it occurs in the text *Mahabhaskariya*, written by Bhāskarā I, who belonged to the school founded by Āryabhatā. (However, here too we see no sign of a justification or rationale given for the formula.)

We consider the function $f(x) = \cos \pi x/2$ and look for rational approximations of low degree to $f(x)$, over the interval $-1 \leq x \leq 1$. Since $f(x) = f(-x)$ and $f(\pm 1) = 0$, we shall initially look for approximations of the following kind:

$$g(x) = \frac{a(1 - x^2)}{a + x^2}. \tag{19}$$

The form chosen ensures that: $g(0) = 1 = f(0), g(1) = 0 = f(1),\ g'(0) = 0 = f'(0)$. We find a using the condition $g'(1) = f'(1)$.

From $f'(1) = -\pi/2$ and $g'(1) = -2a/(a + 1)$, we solve for a and get $a = \pi/(4 - \pi)$. Using the approximation $\pi \approx 22/7$ this yields:

$$\cos \frac{\pi x}{2} \approx \frac{11(1 - x^2)}{11 + 3x^2}. \tag{20}$$

On testing this out we find that it yields only two decimal place accuracy, which is rather disappointing. Can we do better? It turns out we can. The analysis given below, from Shirali (2011), shows how.

Since the graph of $f(x)$ over $-1 \leq x \leq 1$ is a concave arch passing through the points $(\pm 1, 0)$ and $(0, 1)$, a first approximation to $f(x)$ over the same interval is the function $1 - x^2$, whose graph shows the same features. But this function consistently yields an overestimate (except, of course, at $x = 0, \pm 1$); see Fig. 3.

In order to fix the overestimate, we examine the following quotient more closely:

$$p(x) = \frac{1 - x^2}{\cos \frac{\pi x}{2}}. \tag{21}$$

Figure 4 shows the graph of $p(x)$ for $-1 \leq x \leq 1$. (At $x = \pm 1$ the indeterminate form $0/0$ is encountered, but if we require p to be continuous at $x = \pm 1$ and use L'Hospital's rule, we get $p(\pm 1) = 4/\pi \approx 1.27$.)

The shape is suggestive of a parabolic function, so we look for such a function to fit the data. To this end we mark three points on the graph: $(0, 1)$ and $(\pm 2/3, 10/9)$; rather conveniently for us, points with rational coordinates are available. For the parabola $y = d + ex^2$ to pass through them we must have $d = 1$ and $d + 4e/9 = 10/9$, giving $e = 1/4$. So the desired parabolic function is $y = 1 + x^2/4$, and we have the approximate relation

$$\frac{1 - x^2}{\cos \pi x/2} \approx 1 + \frac{x^2}{4}. \tag{22}$$

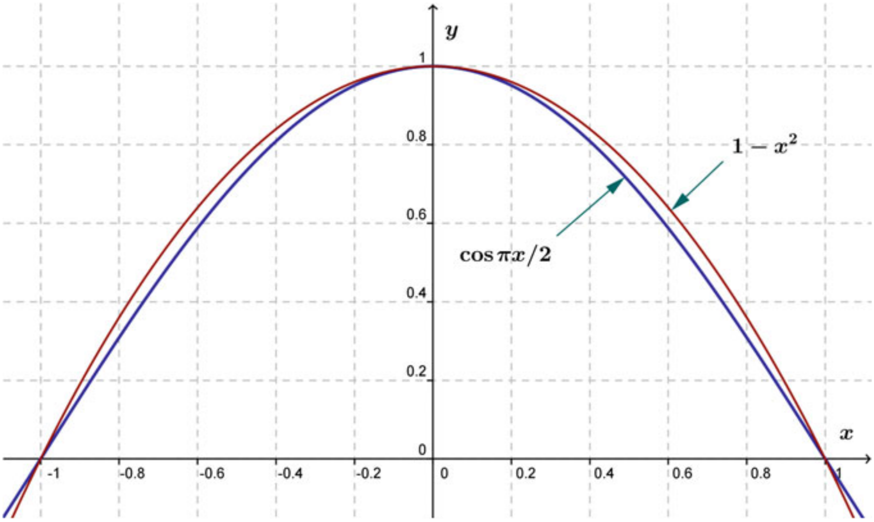

Fig. 3 Graphs of $1 - x^2$ and $\cos \pi x/2$

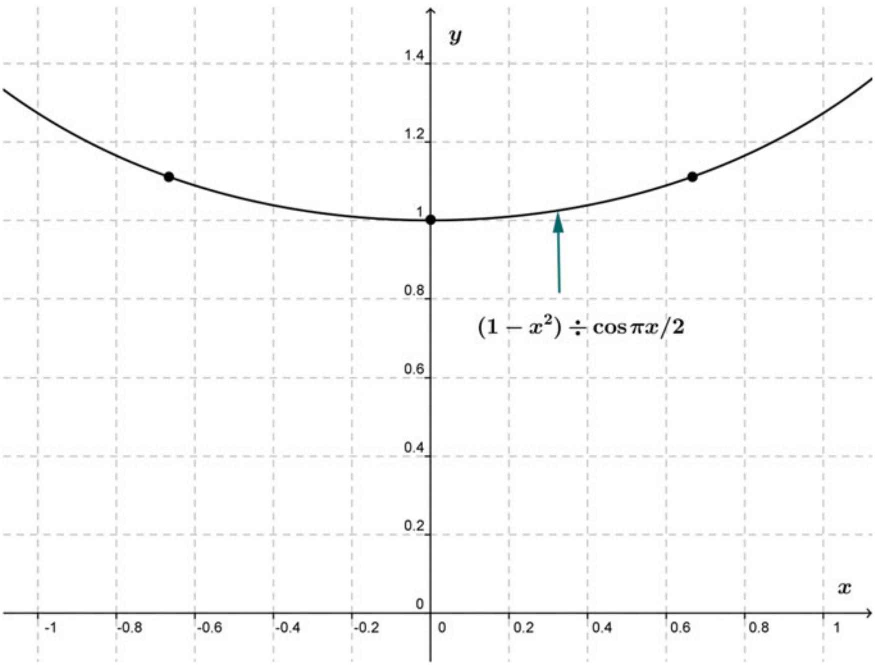

Fig. 4 Graph of $(1 - x^2) \div \cos \pi x/2$

Hence:

$$\cos\frac{\pi x}{2} \approx \frac{1-x^2}{1+x^2/4} = \frac{4(1-x^2)}{4+x^2}. \tag{23}$$

All the coefficients in this approximation are rational numbers. Note that it is of the type considered earlier, $a(1-x^2)/(a+x^2)$, with $a=4$; but earlier we had got $a=11/3$. The change of coefficient works wonders. The above approximation turns out to be extremely close.

Let us now change the unit of angle from radian to degree. Since $\pi x/2$ radians equals $90x^\circ$ the above relation may be written as

$$\cos 90x^\circ \approx \frac{4(1-x^2)}{4+x^2} \qquad (-1 \le x \le 1). \tag{24}$$

The replacement $x \mapsto x/90$ yields:

$$\cos x^\circ \approx \frac{4(8100-x^2)}{32400+x^2} \qquad (-90 \le x \le 90). \tag{25}$$

Finally, the replacement $x \mapsto 90-x$ yields:

$$\sin x^\circ \approx \frac{4x(180-x)}{40500-x(180-x)} \qquad (0 \le x \le 180). \tag{26}$$

This is the approximation given by Bhāskarā I. Call the function on the right side $B(x)$. Here is a comparison of the values of $\sin x^\circ$ and $B(x)$, given to three significant figures:

x	0	15	30	45	60	75	90
$\sin x^\circ$	0	0.259	0.5	0.707	0.866	0.966	1
$B(x)$	0	0.260	0.5	0.706	0.865	0.965	1

It is evident that $B(x)$ yields a very good approximation to the sine function over the interval from 0° to 180°.

See Plofker (2008) and Shirali (2011) for more on the Bhāskarā approximation. In the latter article the formula is examined from various points of view.

Classroom Study and Students' Responses

The author held three one-hour classes on these topics for a group of twelfth graders in his former school (Rishi Valley School in Andhra Pradesh, India), and asked the students to write about their experiences at the end. Here are some of the problems posed, which were used as a basis for class discussion:

1. A well known 'rule of thumb' in banking, relating to the number of years it takes for a fixed deposit to double in size, is the following: *If the interest rate is r % then deposits take 72/r years to double in size*. How do you justify this 'rule'? How accurate is it?
2. Examine the values given here of the tangents of some angles close to 90°, in each case to two decimal places, and explain the pattern you see in the values:

$$\tan 89^\circ \approx 57.29, \quad \tan 89.9^\circ \approx 572.96, \quad \tan 89.99^\circ \approx 5729.58,$$
$$\tan 89.999^\circ \approx 57295.78, \quad \tan 89.9999^\circ \approx 572957.80.$$

3. (This finding was reported to me two decades back by a student of the 11th standard. It led to a wonderful exploration. But we'll leave that story for another day.) We wish to find a good approximation to $\sqrt{2}$. We now use the following observed fact: *If* $x > 0$ *is close to* $\sqrt{2}$*, then* $(x+2)/(x+1)$ *is still closer to* $\sqrt{2}$. Using this iteratively, we get a sequence of steadily closer approximations. We stop when we feel we have come close enough to $\sqrt{2}$. How do you account for this strange 'rule'? For more about this episode, see Shirali (1997).

As all this was done in an informal way, in an interactive, problem solving mode, the students were relaxed; there was no 'exam pressure'. Here are a few of the comments they turned in (not categorized in any way):

> "Everything in maths is interconnected." "Calculus is powerful in answering questions whose solutions may not be intuitively apparent." "We now see how different areas in mathematics, learnt as separate chapters—calculus, binomial theorem, series and sequences—come together. It leaves in us a sense of wonderment about mathematics." "It shows the power of addition, subtraction, multiplication and division!" "We now see how by using just the basic functions $(+, -, \times, \div)$ we can get really good approximations to complicated functions." "I did not expect even for a moment that calculus could be used to find out the square root of a number." "It was nice to see things beyond our syllabus!" "The history of mathematics and mathematicians is itself an enticing topic."

We feel there is enough in this experience to suggest that dwelling on such topics in an enabling, problem solving environment will enhance students' mathematical maturity and their understanding of mathematics, and also their appreciation of the culture and history of mathematics; and to some measure help chip away at the Great Divide.

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Visualizing Mathematics at University? Examples from Theory and Practice of a Linear Algebra Course

Blanca Souto-Rubio

Abstract With this communication, I will try to promote a discussion on visualization adapted to university level: how I understand it, why may it be important to understand advanced mathematics and, mainly, how it is currently taught. With this aim, five examples—obtained by the observation and my reflective practice in a Linear Algebra course—will be presented. The analysis of these episodes will enable to go deeper into some issues of visualization, relevant in this particular context: characteristics of visualization, some obstacles and opportunities of teaching visualization and some actions needed to improve the teaching of visualization at university level.

Keywords Visualization · Linear algebra · Teaching at university level · Participant observation

Introduction and Motivation

There are several reasons to believe that visualization is helpful to understand advanced mathematics. In fact, so many authors have supported its use at university:

> Mathematical concepts, ideas, methods, have a great richness of visual relationships that are intuitively representable in a variety of ways. The use of them is clearly very beneficial from the point of view of their presentation to others, their manipulation when solving problems and doing research (De Guzmán 2002, p. 2).
>
> A variety of more cognitively appropriate approaches are postulated, some with empirical evidence of success. These include: […] the use of visualization […] to give the student an overall view of concepts and enabling more versatile methods of handling the information (Tall 1991, p. xiv).

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In my opinion, two words need of a better specification in order to argue the importance of visualization at university level: understanding and visualization. Inspired by Duval (1999) and the notion of concept image (see Vinner's chapter in Tall 1991) to *understand a concept* implies the construction of a network—a cognitive structure—in which the concept occupies a new node. This new node should be connected to previous nodes, which could be other elements such as: mathematical knowledge on this concept (definition, properties, theorems, proofs, etc.), mathematical experiences with this concept (examples, problems, conceptions, etc.), different representations of the concept (in the table, symbolic or geometric registers, in natural language, etc.), experiences with representations (creation, transformation (treatment and conversion), coordination with other representations, etc.), previous knowledge, intuitions, other concepts and so on (Fig. 13 could illustrate this idea). In order to achieve understanding, this network should be complete and adequately articulated in such a way that a rich mental image of the concept is formed on individual's mind. This image allows the learner to make sense of the concept and look at it from different points of view. Moreover, the construction of this rich mental image provides a new starting point, cognitively higher, that enables *Advanced Mathematical Thinking* (AMT) (Tall 1991); being this one of the main objectives of the teaching and learning at university.

On the other hand, I agree with Arcavi's definition of *visualization*:

> Visualization is the ability, the process and the product of creation, interpretation, use of and reflection upon pictures, images, diagrams, in our minds, on paper or with technological tools, with the purpose of depicting and communicating information, thinking about and developing previously unknown ideas and advancing understandings (Arcavi 2003, p. 217).

Eventually, the importance of visualization for the understanding in mathematics, underlined in the quotations above, could be illustrated by the following example.

> Imagine that I am in a foreign country where people speak an unknown language to me. I would like to walk from my hotel to another place. The only things I know about this place are its name and that it is close. Since I do not understand the person in the reception, I cannot ask. I decide to try by myself with Internet, as in similar situations in my country. First, I find out the address in the website of the destination. It is a sequence of words and numbers that do not make any sense. Then, I look for it in Google Maps. I introduce the addresses of the hotel and the destination, and voilà! I obtain my route's description in two different and complementary ways: one in written language (which I do not understand) and a map. Finally, I know that I can have the "street view" too. It is more accurate and could be helpful in order to determine the route in the real street. With all this information, I feel ready to go outside and walk to my destination by following the map.

When students arrive to some new subjects at university—such as Linear Algebra (LA)—and try to understand a concept, they may feel likewise in a foreign country, with an unknown language, in where they need to handle different representations (Duval 1999; Dorier 2000, p. 247–252 about Pavlopoulou's work), languages (Dorier 2000, chapter 6 by Hillel), modes of thinking (Dorier 2000, chapter 7 by Sierpinska) and points of view (Dorier 2000, p. 252–257 about Alves-Dias' work). In LA context, this is called "cognitive flexibility" and has been pointed out as a source of difficulties for students (Dorier 2000). I believe that, like in the example, visualization could be a useful tool to overcome such difficulties. Moreover, in this example, visualization has been essential in order to connect the two different points (origin and target). Similarly, visualization is essential to achieve deep understanding, that is, to obtain a rich mental image of concepts that enables AMT. However, is visualization available to students? In the example, I was supposed to have this ability at my disposal from previous experience. Likewise, I defend that teaching should facilitate it to students.

In order to find out more about how the teaching of visualization is at university courses, I participated as a teacher and observer in a LA course during the academic year 2010/2011. This course was taught at the School of Mathematics of the Universidad Complutense de Madrid (UCM), which is one of the most important universities in Spain. In this communication I will describe a story that took place during this period, which is composed of five episodes. With these episodes, I do not try to show an exhaustive analysis of the results of this participant observation, but to promote a discussion about some issues I found relevant in relation to the teaching of visualization at university level.

Episodes of Visualization in a Linear Algebra Course

About the Context

The LA course is part of the first year of a four year-long Degree of Mathematics. LA was one of the four annual subjects in this first year. It was divided into two semesters (November-February; February-June), each ended with a period of *partial exams*. There were two additional examination periods for *final exams* (one at the end of June and another in September). There were six hours per week for teaching LA: four hours of *lectures* dedicated to the theoretical contents of the subject and two hours of *seminars* dedicated to work on sheets of problems. There were fourteen sheets–one for each content unit– with 15 problems on average. Students were divided into three groups for seminars. I taught one of these seminar groups during the second semester. I also observed and videotaped lectures from January until the end of the course. In relation to the *materials* used, lectures followed a textbook (Fernando et al. 2010). For the seminars, in addition to the sheets of problems, other worksheets were given to students as voluntary homework. It counted for the continuous evaluation. Despite this, exams consist of the main method of *assessment*.

Episode 1: Spontaneous Use of a Diagram as an Aid for Solving a Problem in Class

This first episode is about the use of a diagram in a problem solving situation. It took place in a lecture as a consequence of a students' intervention. This episode

Obs. 1: If A is antisymmetric and n is odd $\Rightarrow$ $det\,A=0$, since

$$det(A) = det\,(-A^t) = (-1)^n\,det\,(A^t) = -\,det(A)$$

Obs. 2: Both, the rank and the antisymmetric character of a matrix, are preserved if I do this $\begin{cases} f_i \to f_j \\ and \\ c_i \to c_j \end{cases}$

Obs. 3: Let r be the rank of A. Let B the matrix obtained by interchanging rows and columns like in Obs. 2, so that the minor in B formed by the first r rows and columns is no null.

Obs. 4: $B = \left(\begin{array}{c|c} * & \\ \hline & \end{array}\right)$ antisymmetric $\Rightarrow$(*) antisymmetric

Conclusion: $\left.\begin{array}{r} \det(*) \neq 0 \\ (*)\ \text{antisymmetric} \end{array}\right\} \overset{Obs.1}{\Longrightarrow} r\ is\ even$

Fig. 1 Lecturer's solution to the problem as it was written on the blackboard

reminded me an anecdote told by Miguel de Guzmán in his works on visualization: Noerbert was giving a lecture and suddenly got stuck; he only was able to continue thanks to a figure that he drew (and erased quickly) in a corner of the blackboard (De Guzmán 2002). In the next episode, something similar happened. Fortunately, in this case, the diagram was loudly commented.

February's examination period was close and the lecturer left one day for students' queries. The day before, one student had asked the lecturer about one of the tasks in the voluntary worksheets: "*Let A be an antisymmetric matrix. Does A with odd rank exist*?" The lecturer began the class by explaining his solution to this problem on the blackboard (Fig. 1). Another student raised his hand and showed his disagreement with the "Observation 3". He said: "The third… How do you know that you can do this? […] If you have a 4 × 4 matrix and a 2 × 2 minor no null on the upper right corner, you cannot move it to the upper left corner by doing this" (20110202_23:02).[1] The lecturer wrote an example of a 4 × 4 antisymmetric matrix on the blackboard (Fig. 2, on the left). In this manner, a long discussion started,

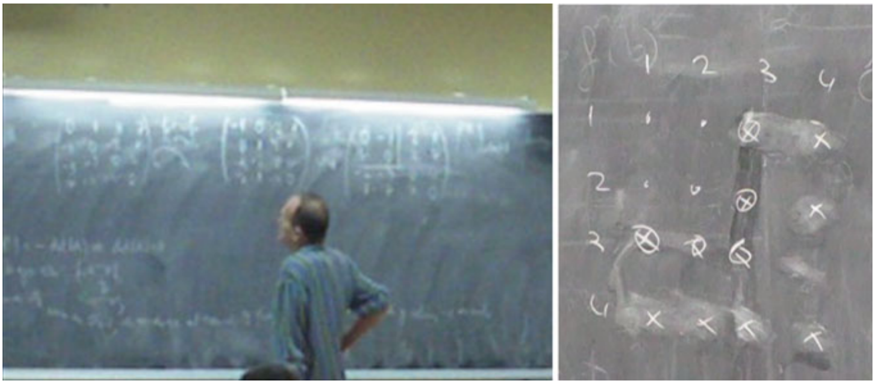


Fig. 2 The lecturer thinking about the problem with a concrete example (on the *left*) and with a diagrammatic representation (on the *right*)

[1]I refer the quotations from videos as follows: Date(YYYYMMDD)_Time(minute:second).

which lasted almost the whole class. The lecturer got stuck, trying to find out why his solution did not work and looking for a convincing explanation (with the tools available at this moment of the course). Then, the following diagram appeared (Fig. 2, on the right). The lecturer used it to think aloud, with generality, on the transformations that keep the antisymmetric character of any 4×4 matrix. The diagram was helpful to communicate these thoughts to the class.

Episode 2: The Paradox of Making Representations Explicit, the Transformation "Coordinates"

Students often confuse the vector and its representation in coordinates with respect to a basis; mostly if the vector space is IK^n. In this case, the original vector and its coordinates are both an n-tuple, looking like the same object (Dorier 2000, p. 201). This second episode concerns the transformation "coordinates", which I found useful to help students to overcome this difficulty. Moreover, this transformation helped me to realize about the existence of a paradox in relation to make explicit issues about representations.

The lecturer introduced the transformation "coordinates" ($coord_B$) during the explanation about the coordinates of a vector with respect to a basis B. This transformation maps an abstract vector space E onto IK^n and assigns each vector u to its coordinates in terms of B, that is, a n-tuple. At this time, it was described in symbolic language, though I found the same transformation represented in other registers (like the geometric) in other LA textbooks (Fig. 3). This transformation makes the change of representation explicit and allows the vector and its coordinates to be distinguished, which I believe facilitates the reasoning.

Nevertheless, the textbook followed a different approach. This transformation was not introduced at this point (unit 7 in the textbook). It appeared two units after, just to note that it was an isomorphism. Contrary to what happened in class, the isomorphism was not used to make any reasoning. It was just said: "The reader could recognize this isomorphism behind many arguments used in unit 7" (Fernando et al. 2010, p. 191). Why do authors in the textbook decided to hide this

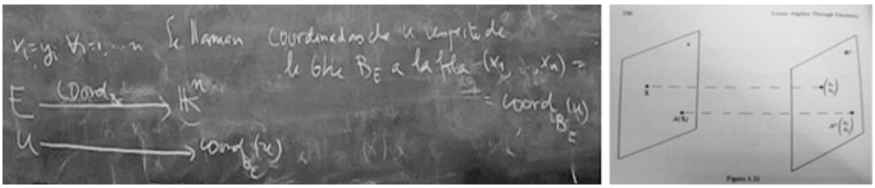


Fig. 3 Representations in symbolic register used on the blackboard (on the *left*) and geometric register found in other textbooks (on the *right* (The source of this image is: Banchoff and Wermer (1992)))

isomorphism while the lecturer decided to include it? I asked the lecturer about his decision to include this isomorphism in class:

> L: It is useful and we constantly use it. […] Thus, it is advisable to shape it, to give it a notation. I realized that notation in mathematics helps a lot. […] To transform things into equalities of symbols helps enormously to think. […] Many people have defended the opposite. Formulae: the less the better. I do not agree with this at all. Formulae: the more the better. Thus, the "general non sense" of mathematics can lead you.

Therefore, there is a difficult decision to be taken in relation to the communication of issues about representations such as: how to distinguish one representation from another or from the represented object; how a transformation is made; which representation is better for a particular aim and so on. The lack of clarity with such issues could be a source of students' difficulties, as it was pointed out above. In order to help students to avoid this kind of difficulties, a specific language—that allows the different kinds of representations and their transformations to be referred —is needed. In this case, mathematics provided this language[2] (the transformation "coordinates"). In other cases, different languages should be created and research, in Mathematics Education, could be a good source of inspiration (see macro/micro language explained in the Episode 3). However, the use of this language (initially introduced to help students) could be a new source of difficulties for students, as textbook's authors might have thought. Thus, the decision is: is it worthy, from students' point of view, to make explicit this kind of information about representations despite introducing a new language? This is what I have called the *paradox of making representations explicit*.

In this case, the lecturer had a clear answer to this question: YES, it was worthy since this transformation exhibits a common way of thinking in mathematics, in general, and in LA, in particular. It allows the work done with rows in the first part of the course to be translated to abstract vector spaces. This way of reasoning is well explained, in the corresponding extract of the class (Fig. 4, 20110124_41:42), through the previous symbolic representation of the transformation, gestures and natural language.

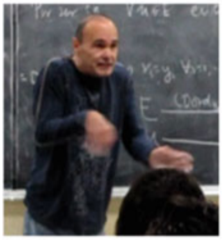
Episode 3: Making Sense of Difficult Concepts for Students Through Several Visualizations, the Concept of Quotient Vector Spaces

Most of the explanations in lectures used to be in the table and symbolic register and were exposed in a logical sequence. However, the next episode provides

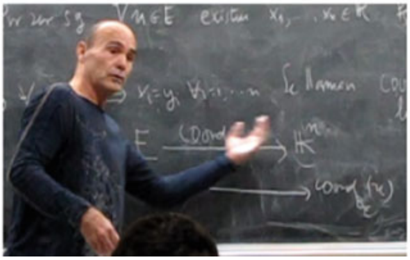
[2]In fact, there are many theorems and results in LA about changes of representations: matrix of a change of basis, matrix of a linear transformation, subspaces as intersection of hyperplanes, etc. This gives an idea of the degree of difficulty that this kind of processes could reach in advanced mathematics.

1. We will study this with more detail. But the idea is the following. Always that I am asked to do something in here,

2. I travel, via “coord”, pshium (he makes a sound while he passes the hand on the arrow) to IK^n.



3. I do there whatever, in IK^n [he shakes his hands], where I know well how to move–it is a space that I master, its elements are rows–



4. and I come back afterwards.

Fig. 4 Lecture’s explanation of a way of thinking via the transformation “coordinates”

evidences of certain variety in this routine. Explanations about quotient vector spaces (QVS) attracted my attention because different kind of representations, examples or metaphors—from now on, I will say “visualizations”—were used to introduce, motivate, favour intuition or, more generally, to make sense of the concept. Likewise, QVS were one of the few concepts geometrically represented in the textbook (Fernando et al. 2010, p. 176). In an informal conversation, the lecturer told me that QVS were one of the most difficult concepts for students to grasp in this LA course. This could have motivated the amount of visualizations (bigger than with other concepts). As a result, this episode serves to shed some light on how visualization could be used in order to help students making sense of difficult concepts, particularly on QVS: which different kinds of visualizations can be used, how the communication about them can take place, what are they useful for. Below, I will expose some of these episodes and visualizations that QVS gave rise. I describe before the contents that concern QVS in the chronological order in which they were explained during the course:

- The **formal definition** of *E/V* as quotient set—built from a vector space *E* and an equivalence relation dependent on a subspace *V*—provided with a vector space structure inherited from *E*.

- The **geometric representation** of *E/V* as a family of subspaces parallel to *V*. In lectures, it took place in between the formal definition: after the definition of the quotient set and before providing it of vector space structure.
- The **dimension and basis** of *E/V* and the calculation of the **coordinates** of a given equivalence class.
- The properties of the **canonical projection**, $\pi: E \rightarrow E/V$: it is linear, surjective (since $im\ \pi = E/V$) and no injective (since $ker\ \pi = V$).
- The Canonical Factorization (better-known as the **First Theorem of Isomorphism**) and its **application** (**factorization** of transformations via a QVS).

$\mathbf{Z_n}$ Example

This example concerns the group of integers modulo *n*. The lecturer used it to introduce the formal definition of QVS as an attempt to help students to connect it with previous knowledge acquired in other subjects of the Degree. Moreover, this example served to motivate the usefulness of quotients: "What for? Why? Why quotients were created? Well, you may have realized that the use of these Z_n is really useful to think about divisibility with simplicity" (20110217_13:43). Finally, Z_5 was used to introduce the notation of QVS as something familiar (Fig. 5) and Z_{17} was used to show why a transformation starting in a quotient has to be well-defined.

The Parameterization of the Circumference

This second example was introduced, with the following sentence, in the same lecture about the formal definition of QVS: "I think that quotients were created because of situations like this. Would you know how to parameterize the circumference S^1? (20110217_15:59)". He asked students to build such a parameterization. First, students proposed the parameterization obtained by isolating one of the variables in the equation of S^1. The lecturer started by considering this proposal just to show that it was not a good one, since two branches are needed to cover the whole circumference. He used a geometric representation to support this reasoning (Fig. 6). Second, students proposed a unique parameterization, with sine and cosine, that covered the whole circumference. The lecturer argued that it was not an

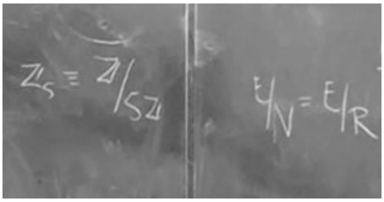


Fig. 5 Notations on the blackboard

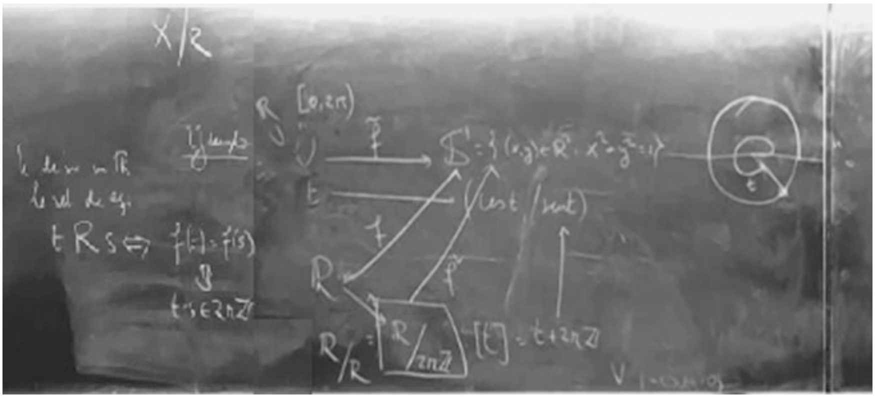


Fig. 6 Blackboard afterwards the explanation of the parameterization of the circumference

injective map. At this point, he said: "Whenever you have a map that is surjective but not injective, the reasonable thing to do in mathematics is what follows (20110217_21:45)". Thus, he introduced a relation of equivalence in *IR* and therefore, a quotient space. Eventually, this teaching sequence, provided by the problem of circumference's parameterization, served to motivate the introduction of the concept of QVS. Moreover, it was recalled—only with a few words this time—during the explanation of the First Isomorphism Theorem (the problem of having a non-injective map appears there again). This is an example of how a whole teaching sequence can be reified into a new unit of visualization, which could be easily referred to afterwards.

Geometric Representations in IR^2 (Micro and Macro)[3]

Two different episodes, concerning the geometric representation of QVS, will be described in this section. The first episode took place in relation to the formal definition too. The lecturer defined the relation of equivalence, proved that reflexive, symmetric and transitive properties held and described algebraically the classes of equivalence through several treatments of symbolic representations (Fig. 7, on the left). At this point, the geometric representation of QVS was presented as an example of how to visualize the equivalence classes. It was introduced with the following message:

> L: During my studies, some lecturers showed examples, though few. The examples they did were not useful, they were all trivial. And the examples I wanted them to do, lecturers did

[3] While the kind of representations and transformations referred in this section are possible in *IR*^{3} too (in fact, we also used them in seminars), I am going to refer here only representations in *IR*^{2} because they were more common

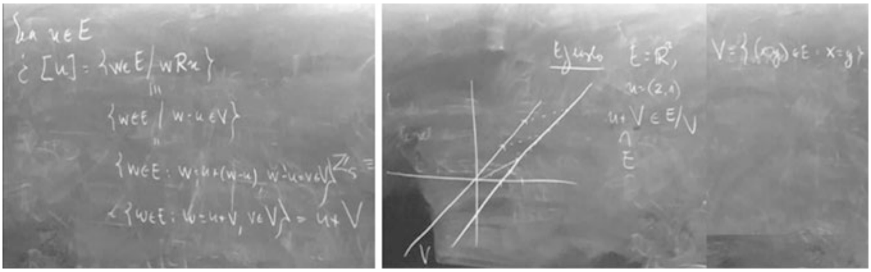


Fig. 7 Symbolic representations (on the *left*) and geometric representations (on the *right*) for classes of equivalence of QVS

> not. Thus … I have a bad experience with that. Of course, the thing I am going to do now [the example], it is not useful at all for a person who is in her right mind. But OK, there is always someone who is not in her right mind (20110227_36:38).

Additionally, the lecturer described the example as "antipedagogic". In his opinion, the main point about quotients was that their elements are subsets from the original set. "But even God cannot represent that! Thus, what am I going to do? To represent these subsets" (20110227_37:22). He turned back to the blackboard, wrote the equation of V (a subspace from IR^2) and represented geometrically V and a vector, $u = (2, 1)$. Step by step, he added u to vectors in V by the parallelogram rule and asked students: "Conclusion, what is $u + V$?" Some answered that it was a parallel line (Fig. 7, on the right). At this point, he insisted on the problem of the geometric representation of QVS:

> L: I have just represented, as a subset of *E*, this parallel line (he follows it with the hand). OK? Like a subset of E. Like a point of the quotient, there is no way! I do not know how to draw it. I cannot draw it as an element of the quotient. (20110227_36:48)

Finally, he recognized that the geometric representation was useful to see the properties of partitions: each element from the original space belongs to one and only one equivalence class; two any classes are either disjoint or the same; and the union of all the classes is the original space. He added a message to remark the particularity of the image: "It would be an unimaginable surprise that the opposite happened. This is a particular case (20110227_41:40)". I found this kind of message relevant to students.

The second episode, involving a geometric representation in IR^2, took place in one of my seminars. Several students came to me and claimed that they did not understand QVS. Despite I knew it could take some time, I decided to explain QVS again trying to use some different visualizations. I started the class with the *nails metaphor*, which will be explained below. It was followed by the *geometric representation in* IR^2. I asked one student to come to the blackboard and to find vectors related to a given u, which was represented geometrically. The idea was to use the definition of the relation of equivalence in order to find the class (instead of the $u + V$ description used in the lecture). However, I realized this approach may bring

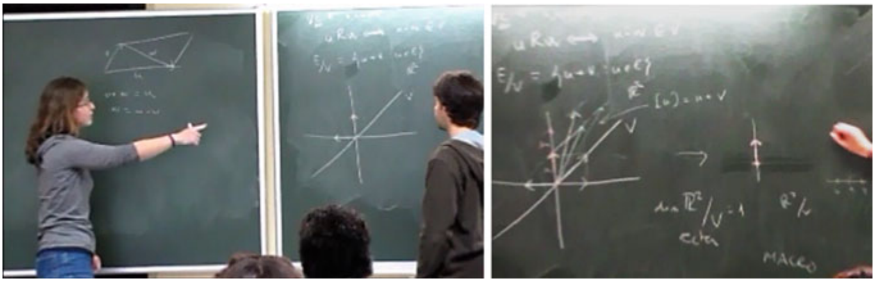


Fig. 8 Geometric representations for subtraction (on the *left*) and micro and macro representations of QVS (on the *right*)

more difficulties to students. It involves a subtraction, and the geometric interpretation of subtraction seemed to be less familiar to students. Thus, I felt the need of explaining it (Fig. 8, on the left part). Moreover, depending on how this subtraction is done, the result can be a vector which origin is not the point (0, 0). Therefore, this task was cognitively demanding and the degree of difficulty varied depending on the starting point chosen for the conversion from symbolic to geometric register.

Afterwards, a new geometric representation was introduced. I called it "macro", inspired by a previous work about the dual space (De Vleeschouwer and Gueudet 2011). The motivation was to solve the problem of representation for quotients pointed out by the lecturer. The name "macro" refers to the fact that this representation could be thought as the result of making "zoom out" from the previous geometric representation. Thus, in the example above, each parallel line to *V* becomes a vector and the quotient space obtained can be represented as a vector line. This representation enables to see QVS aspects' such as their vector space structure, a basis,[4] the dependence of classes, etc. However, it hides the subspace *V*, the properties of partitions and the fact that each element is actually formed by many vectors from the original vector space. I tried to solve this last issue by saying that the vectors in the macro representations were thick and I reproduced this idea in the geometric representation (Fig. 8, on the right). If all these properties of QVS want to be seen again, it is enough to undo the transformation by doing "zoom in". This recuperates the initial geometric representation that thus, I will call "micro". This micro/macro transformation of the geometric register enables to exhibit and to communicate a change in the point of view from which we look at QVS. Similarly to Episode 2, it involves the use of a new language and provides, as compensation, a good tool that could offer a deeper understanding of the concept.

[4]In fact, the transformation to the macro representation needs of the choice of a representative for each equivalence class, and therefore a basis of the quotient.

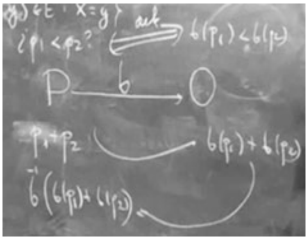


Fig. 9 Diagram to "transport the structure": (the bijection b allows traveling from P ("poor" and unknown space) to O ("organized" and well-known space))

Diagrammatic Representation

Once the lecturer had defined QVS and had represented them geometrically, he said: "You did not finish here when you were explained Group Theory. Likewise, we have not finished yet. The next step is to provide the quotient set with a vector space structure" (20110227_41:43). He continued saying: "And this is also done with the "general non sense" of Mathematics I told you about before. The hand writes by itself. You do not have to think anything" (20110217_41:43). He called this process "to transport the structure" and is very similar to the process described above, in Episode 2, for the "coordinates" transformation (Fig. 4). He started to explain this process by speaking but when he asked the students if they had understood, there was no answer. Thus, he decided to explain it with a diagram (Fig. 9).

Bags and Nails Metaphors

Metaphors were introduced in the course mainly to highlight that QVS are partitions, that is, QVS are the result of a process of ordination or classification into subsets. Moreover, these metaphors enable to talk about issues related to the choice of a representative of each class of equivalence; in particular, to the property of being something well-defined.

The *bags metaphor* consists of thinking that "the quotient space is a space that has bags as elements" and the "canonical projection is to move from one space to this space of bags". This metaphor was introduced during the explanation of the First Isomorphism Theorem, just after recalling the example of the *parameterization of the circumference*. It was also used to explain how quotients serve to make injective a non-injective transformation. The lecturer used a diagrammatic representation to support his explanation (Fig. 10, right part).

> L: How do I convert this [the transformation defined in the upper part of the diagram in Fig. 10] in injective? I pass from this space (he writes an arrow going down) to the space that has, as points, the bags. Let us see if I can do it… (He draws a square with three sets. They are similar to the upper representation but coloured instead of dotted). One, two, three. And I say: "look how this is bijective!" (He draws three arrows, one from each subset). Of course it is bijective, because this set has three elements: three bags. The first bag goes to the 1, the second to the 2 and the third to the 3. This is what it does, ok? It is obvious; I even do not know what to say! (20110228_26:00).

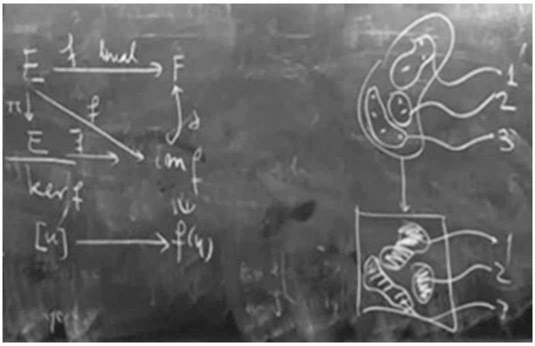
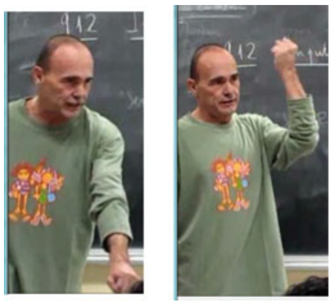


Fig. 10 Lecturer's explanation of the 1st isomorphism theorem with the diagrammatic (on the *right* part) and symbolic representations (on the *left* part)

Afterwards, this process was "translated" to the "less visual" symbolic register (Fig. 10, right part). To have both representations (one next to the other) made this process easier. Additionally, the lecturer used this metaphor in order to justify why it was necessary to prove that the built transformation was well-defined. In this case, gestures served to communicate it (Fig. 11). Finally, the lecturer used both, the diagram and gestures, to answer a student's question about the injectivity of the original transformation.

The *nails metaphor* was introduced in the seminar I explained QVS again as response to students' difficulties. This metaphor consists of thinking that "quotient is the desk of a hardware store", "classes are drawers of things such as nails" and "representatives are the labels on the drawers". I used this metaphor in order to highlight that the quotient is not a subspace of the original space, since the elements

L: All the transformations in the world that begin in a quotient except these here, with drawings, all use the following resource. We take the class of someone, the class of *u*. We introduce the hand into the bag. We take out someone, *u*. (He makes the gesture of introducing and taking out something from an imaginary bag). And we use it to define the transformed of the bag. Worry! What if other person introduces his hand, takes a different thing and, when applying *f* to it, the result is different? (He repeats the previous gestures). Thus, which is the image of the bag? I do not know! Let us prove that it does not matter who introduces the hand and what is taken from the bag. That is it, let us do it! This is called to be well-defined. (S6200110228_28:55)



"We introduce the hand into the bag" "We take out someone, *u*"

Fig. 11 Lecturer explanations with gestures

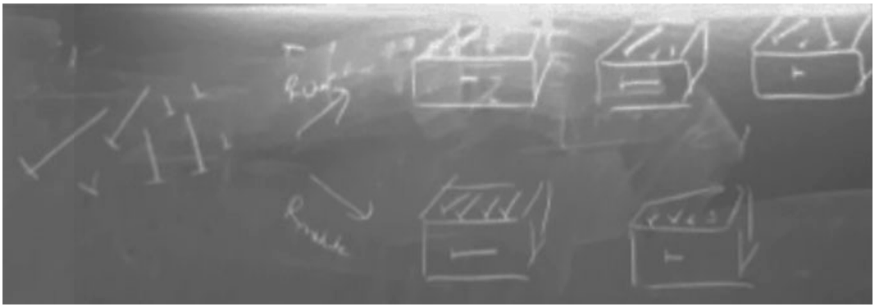


Fig. 12 Graphic representation used to support the explanation of the nails metaphor

in each space have different nature: nails in the original space and drawers in the quotient. Moreover, I used it to show the importance of notation for classes. I defined two different equivalence relations: to have the same colour and to have the same dimensions. Each relation led to different quotient sets (they had even different number of drawers). However, some drawers from different quotients can have the same label, the same representative, and it is not possible to distinguish them when using the bracket notation (*[u]*). In this metaphor's language: "brackets (*[u]*) do not allow us to see what is inside the drawers ($u + V$)". I supported the explanation with a graphic representation on the blackboard made with colour chalks (Fig. 12).

To sum up, in the following schema (Fig. 13) are represented the contents involving QVS explained in the course (light blue boxes in the centre), the different visualizations used to explain these contents (dark blue boxes with bold letters), the kind of language or representation used to communicate these visualizations (green boxes) and the aim for their introduction in the different episodes described above. The more visualizations to explain a content are used, the bigger the box for this content is.

Episode 4: Persistence of Students' Difficulties, the Gap Between Theory and Practice

In the previous episode, I pointed out how students' difficulties with a concept motivated different kind of visualizations. Now, I would like to reflect on the effect that these visualizations have in students' understanding of the concept. According to my view about how learning is produced, explained in the introduction, the presence of more representations, examples and different points of view should favour a better understanding. In a conversation with a student repeating this year,

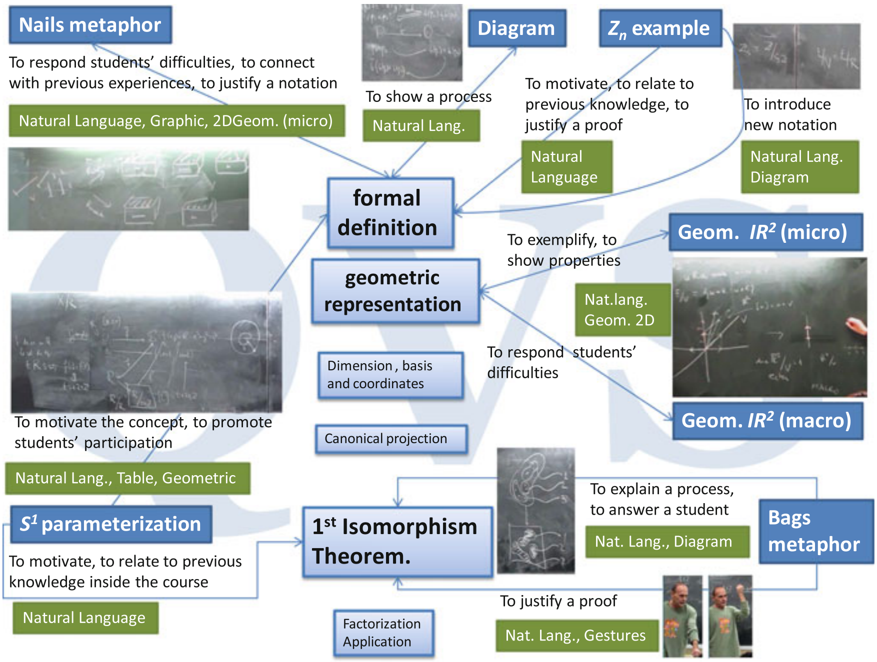


Fig. 13 Schema of the visualizations used to make sense of QVS

he recognized that QVS "were easier to understand in the way they had been explained this year". However, the fact is that, despite all these explanations, most of the students still have difficulties with the concept and they did not demonstrate a good understanding of it when solving problems.

The persistence of these difficulties in the understanding of QVS could be explained by saying that this is a difficult concept from both, the cognitive and epistemological viewpoints (for more details see Souto-Rubio and Gómez-Chacón 2012). Nevertheless, the effort made in the teaching of the concept was very valuable and it should have smoothed the path towards its understanding. At this point, I remembered a Chinese proverb: "*I hear and I forget; I see and I remember; I do and I understand*". Thus, I had a look at the fourteen problem sheets and at the two problems in each exam. In relation to QVS, I found nine problems in the sheets and one section in the first problem of each final exam (June and September). The schema in Fig. 14 summarizes the following information about these problems: the contents about QVS related to each problem (light blue boxes in the centre); the kind of the representation used (green boxes) and the aim of the problem. Problems are denoted with two numbers (dark blue boxes)—the first number refers to the content unit and the second to the position in the sheet—and some key words briefly describe them.

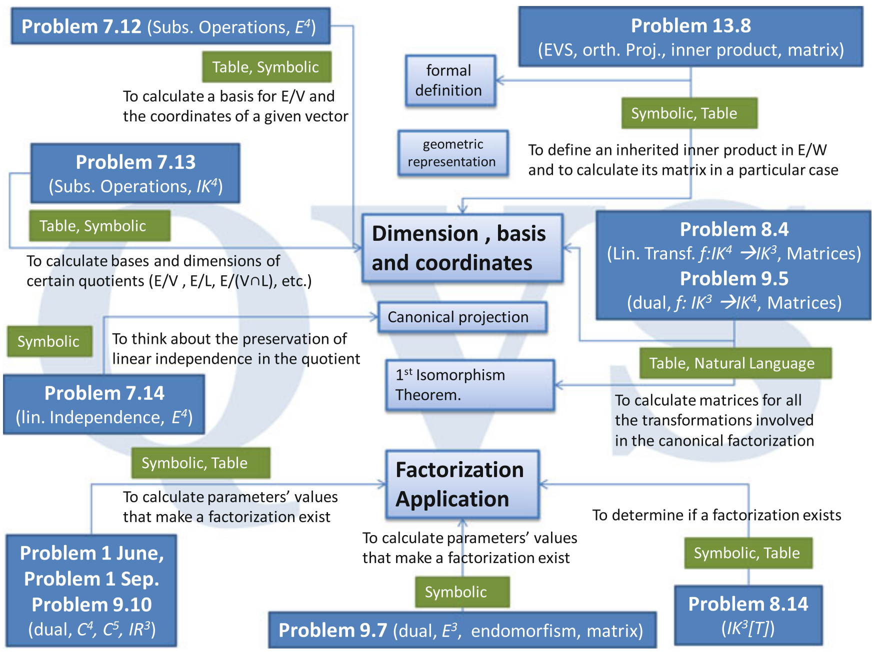


Fig. 14 Schema of the problems found in worksheets and exams that concerns QVS

The contents most referred in the problems are represented in bigger boxes: Dimension, basis and coordinates in QVS and the factorization of functions as an application of the 1st Isomorphism Theorem. This set of contents hardly intersects the set of contents explained in class with more visualizations (see Fig. 13). Moreover, all the problems are posed in a table or symbolic registers and most of them are routine and mechanic. I know from previous research (Souto-Rubio 2009) that problems with these characteristics normally need only instrumental understanding and table or symbolic registers to be solved. Thus, there is a gap between theory and practice: when solving these problems, students can leave aside all the images, metaphors and different kind of representations used in class. In other words and recalling the Chinese proverb: what students have heard and seen in explanations does not correspond to that they have to do. Therefore, the teaching effort done, in order to make sense of QVS, may actually not help students to understand this concept.

In order to avoid this gap, I believe that tasks of different nature should be included in the course: tasks that develop the instrumental character of visualization (it also exists); more constructive tasks which lead students to build rich mental images; more conceptual tasks, which need of these mental images, to be solved. How should these tasks be designed in order to help students to understand? How could these tasks be introduced in the LA course?

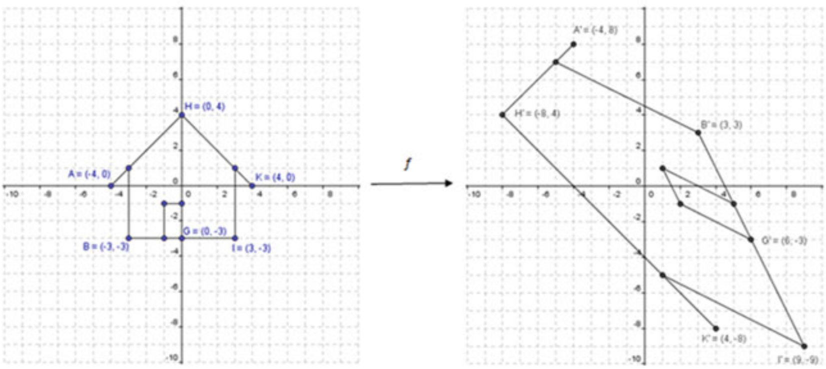
Episode 5: Visualization and Assessment, a Difficult Couple

This last question leads to reflect on the institutional point of view of visualization. It has been possible to witness—in some of the episodes above—facts, attitudes and messages that give an idea of the status of visualization in this LA course. In this last episode, I will narrate two experiences, around assessment, that make me specially reflect about this issue.

Last Year's Class, the Big Failure

The end of the course was coming. Only one week left for the second partial examination. I decided to give a special last class with the aim to review the most important contents seen in the course and to answer students' queries before the exam. This class was open to students from other seminars groups and some came. I had noticed that the closer the exam was, the more time I spent at the blackboard and the less visualization I included in my seminars. Thus, I decided to design a special material for this last class. The material started with a task of representing six endomorphisms in IR^2 either graphically (with a house, like in Fig. 15) or algebraically. Next, there were some questions to reflect on these endomorphisms and their representations. Finally, a conceptual map (about different points of view for a symmetric matrix) was included to be filled. I had planned to spend the first hour with this activity, since it could serve to review the main concepts in the

(6) (0.6 points) Let IR^2 be an euclidean space with the standard inner product and the following endomorphism $f: IR^2 \to IR^2$, which sends the first set of vectors into the second:



a) Diagonalize graphically the endomorphism, that is, find and draw a basis $B = \{u_1, u_2\}$ such as $M_f(B)$ is diagonal and write this matrix. Explain the procedure followed in order to find both, the basis and the matrix.

b) Write the matrix of f with respect to the standard basis $\mathcal{E}$ of IR^2 and find a matrix P invertible such as $P^{-1} \cdot M_f(\mathcal{E}) \cdot P$ is diagonal.

c) Is f a selfadjoint endomorphism? Why?

Fig. 15 Question included in the final exam of June course and to give a geometric interpretation of them. The second hour would be for students' queries.

However, this plan did not work. I was feeling nervous about the activity. I thought students may be only interested in solving their queries. Just one week left to the exam and such things were not going to be asked. Moreover, students were performing the activity much slower than I thought. At this point, I started to solve the problem on the blackboard. I found this harder than in the paper: I needed to place the exact points without a grid; I was too close to the blackboard to see the global picture, etc. The communication with students was also difficult, mainly with those who were not from my seminar group (fact that led me to think, once more, about the importance of the development of a language that enables to talk about visualization). Thus, after having made just four examples and motivated by a students' question, I decided to skip the activity and make a more traditional review. At the end, I recuperated the conceptual map's question from my activity. I thought it could be useful for summarizing what has been said. The result was that most of the students felt confused. They probably had never made such an activity and they did not know exactly what to do. After a brief comment about it, I felt defeated and left students to start asking their queries.

Attempting to Change, Visualization in the Final Exam

With this experience, I realized that assessment was a key factor to take into account in order to make changes in relation to the use of visualization in such LA course. It had strongly affected my way of teaching in seminars and my attitude to my own materials. Thus, when the lecturer wrote an email, asking for opinion about the second partial exam, I answered back encouraging him to introduce a more visual question. My proposal is shown in Fig. 15. It was based on one of the questions I skipped from the activity described in the section above.

The lecturer accepted to introduce it as an optative question, that allowed students to obtain 0.6 extra points over 10, and it would be as a part of the final exam (instead of the second partial). Twenty seven students took the final exam and only twelve tried to do something in this question. I note that some students, who passed the subject with partial exams, did not take the final exam and therefore they are not included in the following results (Fig. 16). The first thing to be noticed, in these results, is that students who tried this activity did not perform it very successfully. Thus, if it would have been higher considered in the final mark it would be detrimental for students. Second, the section with worse results is the more visual one, which is the section a) about what I have called "graphic diagonalization" (see Fig. 15). The section with better results is the more algebraic one, the section b), and it is remarkable that most of the students started with it instead of the a). Moreover, since both sections were asking the same in two different ways, this result is also saying that, in general, students are not able to connect the algebraic and the visual modes of thinking about diagonalization. From my point of view, this is worrying because it means that they do not completely understand this notion.

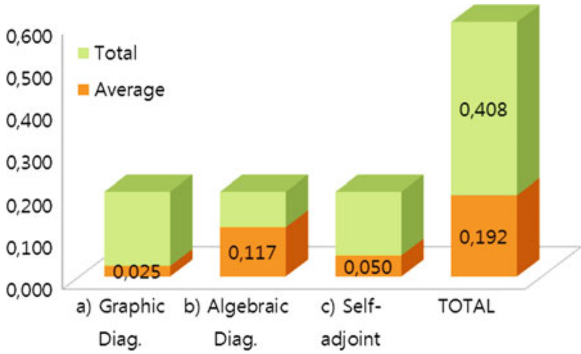


Fig. 16 Results of the question included in June's final exam

Finally, the third question admitted both a visual and an algebraic answer. Only three students answered it. All of them did it correctly by giving the algebraic one. This result can be interpreted in two very different ways: either students naturally use the algebraic mode of thinking (they may be more used to it); or students do not use intentionally the visual mode of thinking because either they think they are not expected to do it or they may even have been penalized for using it before (Souto-Rubio 2009).

Discussion and Conclusions

The story presented in this communication comes from a concrete case of a LA course, teachers and students. Many of the aspects of the teaching of visualization may be particular of this context. However, as I pointed out in the introduction, I think these episodes can serve to highlight important issues of visualization at university, or at least, at a LA course.

There is Visualization at University Level

Lectures with a proof-oriented focus, such as those observed in this LA course, are commonly criticized by its traditional style described as "definition- theorem-proof" (DTP) (Weber 2004). Nonetheless, I agree with Weber (2004) when he claims that the teaching styles included "under the umbrella of traditional DTP instruction may vary widely" and that more empirical studies on what happen during lectures are needed (and I would add seminars too). Episodes 1 and 3 give evidences of such variety. Moreover, such kind of episodes helps to give a better characterization of visualization in LA. This characterization is represented in Fig. 17, taking into account the three dimensions involved in Arcavi's definition: visualization as product (green and blue boxes), visualization as a process

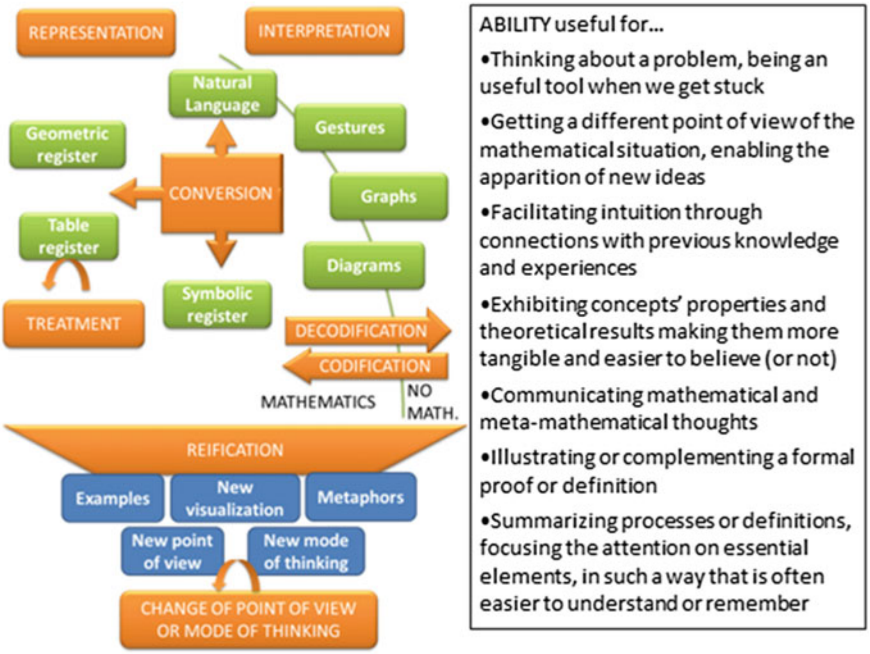


Fig. 17 Schema of the characterization of visualization in LA

(orange boxes with capital letters); and visualization as an ability useful for several aims (white box).

However, these episodes were exceptional, since most of the explanations and problems used table and symbolic registers and followed a logical sequence. What was special in these cases? In the first one, the lecturer got stuck and needed to think aloud in class. In the second one, students' difficulties with a concept could have motivated a considerable amount of visualizations. Therefore, visualization seems to be an aid in situations of cognitive difficulty. If this is the case, why is there no more visualization in the rest of the course?

There are Obstacles in the Teaching of Visualization

Authors like Eisenberg and Dreyfus (1991) and De Guzmán (2002) have pointed out some obstacles that make difficult the use of visualization in class. Some of these difficulties have been revisited through the episodes presented. I reformulate them as follows: visualization can lead to misinterpretations, visualization can be a limitation, visualization is not serious mathematics, visualization is not part of the curriculum, visualization is cognitively demanding, visualization needs of more knowledge and visualization is hard to communicate. I am going to comment the last two, since they may be less clear and I found them important.

Visualization needs of more knowledge—Visualizations in Episode 3 require more teacher knowledge (Ball et al. 2008) than a traditional class. The lecturer needed to know more than the common definitions, representations, properties and proofs around QVS (Common Content Knowledge). First, he also needed to know other different kind of representations, to explain the rules for their transformations, etc. (Specialized Content Knowledge). Second, he needed to know which visualizations are better in order to explain a particular property, or which example better motivates a concept, etc. (Knowledge for Content and Teaching). Similarly, students also need part of this knowledge in order to understand what the teacher is doing. For example, they cannot understand the micro geometric representation of QVS if they do not know how to add two vectors geometrically.

Visualization is hard to communicate—Different kinds of knowledge need of different levels of communication. First, it is difficult to convert, into a teaching sequence to motivate QVS, a non-linear and maybe personal relation between the parameterization of the circumference and this concept (Episode 3). Second, as happened in Episode 2, it could be important to make explicit some issues about representations. This needs of a specific language, which leads to a paradox in relation to students' difficulties. Regardless, the experience in the last year's class highlighted the importance of the development of such a language to talk about visualization.

At this point, it might be argued: if it implies so many difficulties, why to insist on teaching visualization?

In Defense of Visualization

First, as I explained in the introduction, visualization is essential to reach AMT. Second, as some of the episodes evidenced, visualization offers interesting opportunities for teaching and learning that should be exploited (see Fig. 13 and Fig. 17). However, the main reason to pay attention to visualization is that it is unavoidable, even in a subject such as LA. As the episodes showed, there is visualization, whether we want to or not. Geometry is one of the bases for the historical development of this subject (Dorier 2000) but it is also in the core of other subjects. Diagrams appear to represent transformations and relations among them and help us to think more abstractly. Metaphors or graphs could be a good aid to answer students—some could be visualizers (Presmeg 2006)—who ask for different explanations and so on. Thus, for me, the question is not if teaching or not teaching visualization, but how to handle visualization when teaching? And the answer is clear to me: it is important not to leave the whole responsibility only to students; on the contrary, the course should pay explicit attention to it. This is the only way to break the vicious circle that exists around visualization. I believe that the more you practice something, the easier it results. If students were used to the languages and characteristics of visualization before, it would take less time to explain a new one, its understanding and communication would be easier, possible misinterpretations of images would be more likely to be noticed by students, etc. Thus, visualization would become the useful and helpful tool for understanding that it could be.

In order to achieve this goal, I think three actions are needed. First, it is important to improve the teacher knowledge about visualization, thus more visualization and more conversations on visualizations could emerge in class. A systematization of this knowledge, either from a general approach (Fig. 17) or from a concept's perspective (Fig. 13), could benefit this improvement. Subsequently, the challenge will be how to transfer this knowledge to university teachers. Second, it is important, not only to expose students to visualization, but also to make them to practice and reflect on it. How to design such kind of tasks is still an open question. Third, none of the both previous actions will be effective if they are not accompanied by the legitimization of the visualization. This involves the institutional dimension, introducing changes in the curriculum and assessment, but also the personal dimension, promoting the individual reflection on this issue. For this reason, I consider very important to continue promoting this kind of debates among the community of mathematics educators. With this aim, and to show that there is still much to be done, I have presented here these episodes.

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On the Golden Ratio

Michel Spira

Abstract In this article we discuss some ideas associated with the Golden Ratio and its alleged appearances in art and biology.

Keywords Golden ratio · Golden number · Geometry

Introduction

The *Golden Ratio* is one of the most famous numbers. One gets used to "seeing" this number everywhere: in the Parthenon and the Pyramids, in the proportions of the human body, in the nautilus shell and so on. The aim of this article is to present a somewhat skeptical view on this.

Generalities About Φ

We begin our work with a quick look at the golden ratio.

In Euclid's theory of areas, one finds Proposition 11 in Book II of ***The Elements***: *To cut a given straight line so that the rectangle contained by the whole and one of the segments equals the square on the remaining segment*. In the notation of Fig. 1, one is asked, given AB, to find P such that the area of the rectangle on AB and BC (with $BC = PB$) is equal to the square on AP.

Setting $AP = x$ and $PB = y$, this is the same as

$$x^2 = y(x + y) \tag{1}$$

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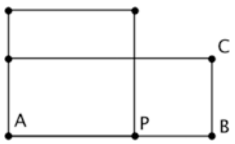


Fig. 1 Euclid's proposition 11, book II

One is not trying to find lengths here (since one does not know the length of *AB*), but rather the position of *P*; in other words, one wants to find the ratio $\frac{x}{y}$ in which *P* divides *AB*. We do so by rewriting (1) *as*

$$\left(\frac{x}{y}\right)^2 = \frac{x}{y} + 1. \tag{2}$$

This ratio, which following current usage we will denote it by Φ, was explicitly defined and called *extreme and mean ratio* by Euclid in Book VI of the *Elements*. There it is presented as the solution to the problem of finding a point *P* which divides a given segment *AB* in a larger part *AP* and a smaller part *BP* such that *the whole is to the larger part as the larger part is to the smaller part*. Still with the notation of Fig. 1, this amounts to

$$\frac{x+y}{x} = \frac{x}{y} \tag{3}$$

which gives us (2) again.

In passing, we note that the term *golden section* for Φ appears to have been used for the first time by Martin Ohm in 1835, in his textbook *Die Reine Elementar-Mathematik*; before him, Φ was called *divine proportion* by Luca Paccioli in his *De divina proportione*, in 1509.[1] One will also find *golden mean* and *golden number* as terms for Φ.

We now rewrite (2) as

$$\Phi^2 = \Phi + 1 \tag{4}$$

or, equivalently, as

$$\frac{1}{\Phi} = \Phi - 1 \tag{5}$$

and we get the well-known properties of Φ.

[1]Leonardo da Vinci illustrated this book, a fact that gave rise to the legend that da Vinci knew about Φ and used it in his works.

We remark that (4) and (5) follow from the very definition of Φ; in other words, they are not extraordinary or mystical properties of Φ, but only equivalent ways of saying that Φ is the positive root of $x^2 - x - 1$, which is the minimal polynomial of Φ over $\mathbb{Q}$. Hence Φ is an algebraic integer and, as is well known, all the algebraic properties of an algebraic number follow from its minimal polynomial; it is Φ's luck to have been defined by such a simple one. To belabor the point, Φ is defined as the unique positive number which satisfies (4); one should not be surprised it does so!

In this respect, it is quite sad to find (4) and (5) presented, in some texts (e.g., [1]) as follows: first the author somehow gets to $x^2 - x - 1$, finds

$$\Phi = \frac{1+\sqrt{5}}{2} = 1.618\ldots$$

and then "computes"

$$(1.618\ldots)^2 = 2.618\ldots = 1 + 1.618\ldots$$

with a similar computation for (5).

Let's go back to Φ. We first note that (4) gives rise, by repeated substitution, to the following interesting expression

$$\Phi = \sqrt{1+\sqrt{1+\sqrt{1+\cdots}}} \tag{6}$$

and, similarly, (5) gives rise to

$$\Phi = 1 + \cfrac{1}{1+\cfrac{1}{1+\cdots}} \tag{7}$$

Φ is also related to another geometrical problem. In Fig. 2, one asks for a rectangle $ABCD$ such that if one cuts off a rectangle $PBCQ$, similar to $ABCD$, the remaining rectangle $APQD$ is a square. It is easy to show that this reduces to

$$\frac{AB}{BC} = \Phi$$

and we get the so-called *golden rectangle*, i.e., a rectangle in which the ratio of the larger side to the smaller one is Φ.

One can also ask how to construct Φ with ruler and compass; the easiest construction (as far as this author knows) is given below (Fig. 3).

In this figure, $ABCD$ is a square of side 1 and M is the midpoint of AB; one draws $\mathcal{C}(M, C)$ and finds E. A quick computation, involving nothing more than Pythagoras's theorem, shows that $AE = \Phi$. More generally, if one starts with an

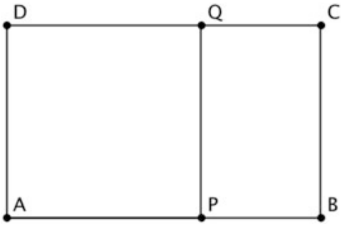


Fig. 2 Removing a similar rectangle from a rectangle

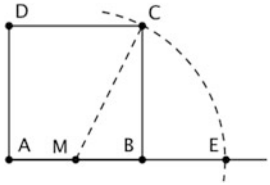


Fig. 3 Constructing Φ

arbitrary square, this construction gives $\frac{AE}{AD} = \Phi$ and, by completing the rectangle on A, E and D, gives us a golden rectangle.

We point out that the initial step this construction makes $\sqrt{5}$ enter the picture; in fact, we have $MC = \frac{\sqrt{5}}{2}$, which is all one needs to get Φ; the circle just contributes with the $\frac{1}{2}$ in $\Phi = \frac{1+\sqrt{5}}{2}$.

This utterly simple construction, relying only on a midpoint and an obvious circle, shows that every time an artist or an architect uses a square in his/her work, chances are that Φ will appear, either as a golden rectangle or as the ratio of some measures. In other words, finding Φ in a painting or a building is no indication that it was used intentionally, or even knowledgeably.

A common belief is that, among rectangles, the most "pleasing" or "beautiful" is the golden one. Presumably, this means that, given the choice between various rectangles of different proportions, people will favor the golden one or a close approximation. To check this, it is enough to make a survey and tabulate the results. This was done in [14]; the most popular rectangle has seems to be, not the golden one, but the one with sides in 1.83 proportion, with the longer side running horizontally. In [14] one can also find templates to run one's own experiment on which rectangle people prefer.

Another problem is how to divide a given segment in the golden ratio. To this end, there is a very simple ruler and compass construction, which can be done easily by paper folding[2], as shown in Fig. 4.

[2]Provided the reader has no difficulty in using corners of square paper as folding points.

Fig. 4 Dividing a segment in the golden ratio

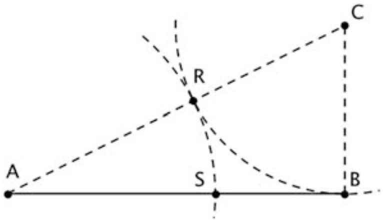
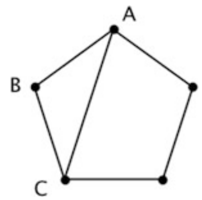


Fig. 5 Φ in a pentagon



In this figure, $BC = \frac{1}{2}AB$ is perpendicular to AB. One first draws $\mathcal{C}(C, B)$, determining R, and then $\mathcal{C}(A, R)$, determining S. An easy computation shows that S divides AB in the golden ratio.

A well-known appearance of Φ is in the pentagon. In the figure below, we have a pentagon and one of its diagonals (Fig. 5).

One can show easily that

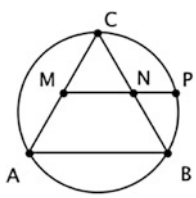
$$AC - AB = \frac{1}{AC}. \tag{8}$$

from which it follows that $\frac{AC}{AB} = \Phi$. In particular, if the pentagon has side 1, then its diagonals measure Φ.

Φ also makes unexpected appearances in many other contexts. Among those, we choose first to show that there is Φ in an equilateral triangle. In the following figure, ABC is an inscribed equilateral triangle and M and N are midpoints of the corresponding sides; the reader can easily show that N divides MP in the golden ratio (Fig. 6).

Now we look at the picture below, where $ABCD$ is an arbitrary rectangle, and ask what are the conditions on R and S for the three shaded triangles to have the same area (Fig. 7).

Fig. 6 Φ in an equilateral triangle



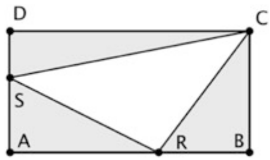


Fig. 7 Φ in an arbitrary rectangle

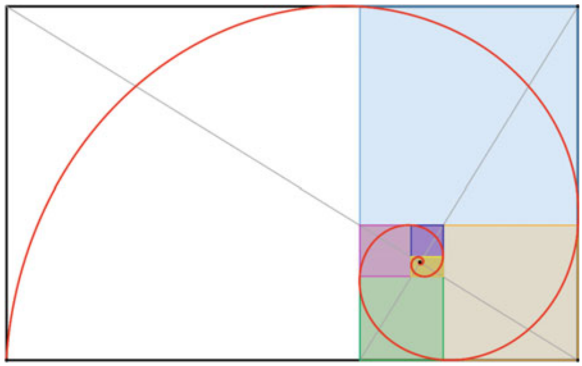


Fig. 8 The golden spiral

You guessed it: this happens if and only if *R* and *S* divide *AB* and *AD*, respectively, in the golden ratio.

Also well-known is the unexpected relation between the Fibonacci sequence 1, 1, 2, 3, 5, 8, 13, 21, … and the golden ratio: the successive quotients $\frac{1}{1}, \frac{2}{1}, \frac{3}{2}, \frac{5}{3}, \frac{8}{5}, \ldots$ give better and better approximations for Φ. Having in view that divisions in 5 and 8 parts are quite common in art and architecture, it is no surprise that measurements of proportions in paintings, sculptures and buildings show, quite often, that (good approximations for) Φ and/or Φ^{-1} are hidden there.

To finish this section, we talk briefly about the famous *golden spiral*, which we show in Fig. 8.

It starts with a golden rectangle, from which we extract a similar rectangle and then iterate this construction on each remaining rectangle. The golden spiral is the unique (up to similarity and rigid motions) logarithmic spiral through the division points as above. Seeing a (not necessarily) logarithmic spiral in nature, be it on nautilus shells, a sunflower or a galaxy is enough, for those so inclined, to say that Φ is hiding there; this is the origin of the idea that Φ is an important ingredient in the inner works of Earth and Cosmos.

Generalizing Φ

The reader has seen that Φ is, indeed, a wonderful number, full of beautiful and unique properties. We now try to put this in a wider perspective.

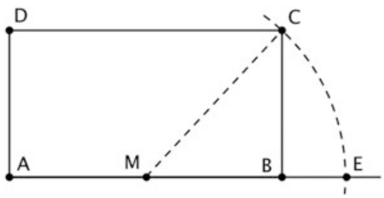


Fig. 9 The construction of the generalized golden ratio

First we go back to Fig. 3 and do the same construction, but now on a $p \times 1$ rectangle, where $p > 0$ is any given real number; Fig. 2 corresponds to the particular case $p = 1$ (Fig. 9).

It is straightforward to show that the rectangle on BC and BE is similar to the original one and that AE is the positive root of the polynomial $x^2 - px - 1$. We call this root Φ_p, so that $\Phi = \Phi_1$ is now a member of the family $\{\Phi_p : p > 0\}$; the members of this family are called *generalized golden numbers* [6]. One gets immediately

$$\Phi_p^2 = p\Phi_p + 1$$
$$\frac{1}{\Phi_p} = \Phi_p - 1$$
$$\Phi_p = \sqrt{1 + p\sqrt{1 + p\sqrt{1 + \cdots}}}$$

and

$$\Phi_p = p + \frac{1}{p + \frac{1}{p+\cdots}}$$

Setting $p = 1$ in the above expressions we recover (4)–(7). From this point of view, Φ's properties are nothing special—infinitely many numbers have quite similar properties.

One might argue that Φ has, say, a wonderful relation to the pentagon, given by (8), which is not shared by any of the Φ_p. But consider a regular n-gon with odd $n \geq 5$ and side 1 (Fig. 10).

Letting AC be the longest segment connecting two vertices and AB the second longest one, it is a neat exercise to show that we recover (8) in exactly the same form. Hence, if $AB = p$ then $AC = \Phi_p$; the particular case $n = 5$ (when $AB = 1$ is the side of the pentagon) gives us $AC = \Phi$. We now see that Φ's relation to the pentagon is just a special case of a much more general phenomenon.

It turns out that there is Φ_p in an equilateral triangle too; it suffices to consider Fig. 11, where ABC is an inscribed equilateral triangle and M is such that $\frac{CM}{AM} = p$; again, the reader should have no trouble showing that $\frac{MN}{NP} = \Phi_p$.

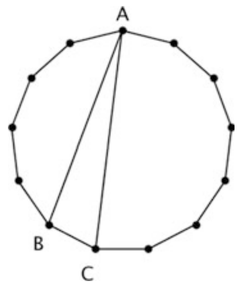


Fig. 10 A polygon with an odd number of sides and Φ_p

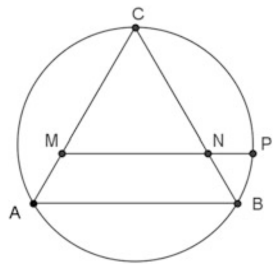


Fig. 11 Φ_p in an equilateral triangle

But certainly Φ's relation with the logarithmic spiral is a special one! Not quite. Logarithmic spirals come in a family indexed by p and can be constructed from any $\Phi_p \times 1$ rectangle, in exactly the same way we constructed the golden spiral. In Fig. 12 we present some examples; the numbers below the spirals are the corresponding values of Φ_p. The top right spiral is the golden one.

In passing, we note that logarithmic spirals are not, in general, tangent to the sides of their generating rectangles. In fact, there is only one such; we leave it to the reader to find out the corresponding p.

It is simply not true that all spirals in nature are golden ones. Even the prime such "example", the nautilus shell, is false, as can be seen below (hand fitted spiral, but certainly quite far from the golden one!) (Fig. 13).

Finally, what about the relation between Φ and the Fibonacci numbers? Well, the fact is that if $a, b > 0$ are arbitrary and the sequence (a_n) is given by $a_1 = a$, $a_2 = b$ and $a_n = a_{n-1} + a_{n-2}$ for $n \geq 2$, then $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \Phi$; this happens just because of the special form of the recursion, whose associated polynomial is $x^2 - x - 1$. More generally, given arbitrary $a, b > 0$, one defines the sequence (a_n) by $a_1 = a$, $a_2 = b$ and $a_n = pa_{n-1} + a_{n-2}$; the reader can then show that $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \Phi_p$.

We could go on a long way like this, but time and space force us to stop here. Hopefully, the message is clear: most, if not all, of what is said about Φ and its "unique" properties is false, and can be refuted by elementary mathematics and willingness to check "facts".

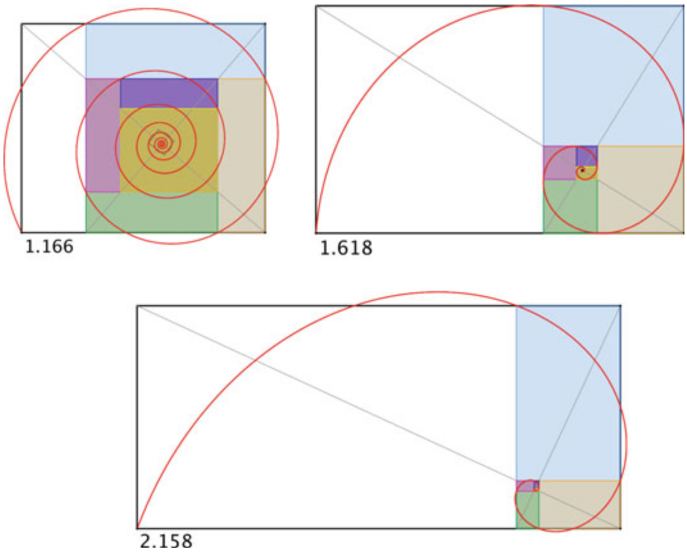


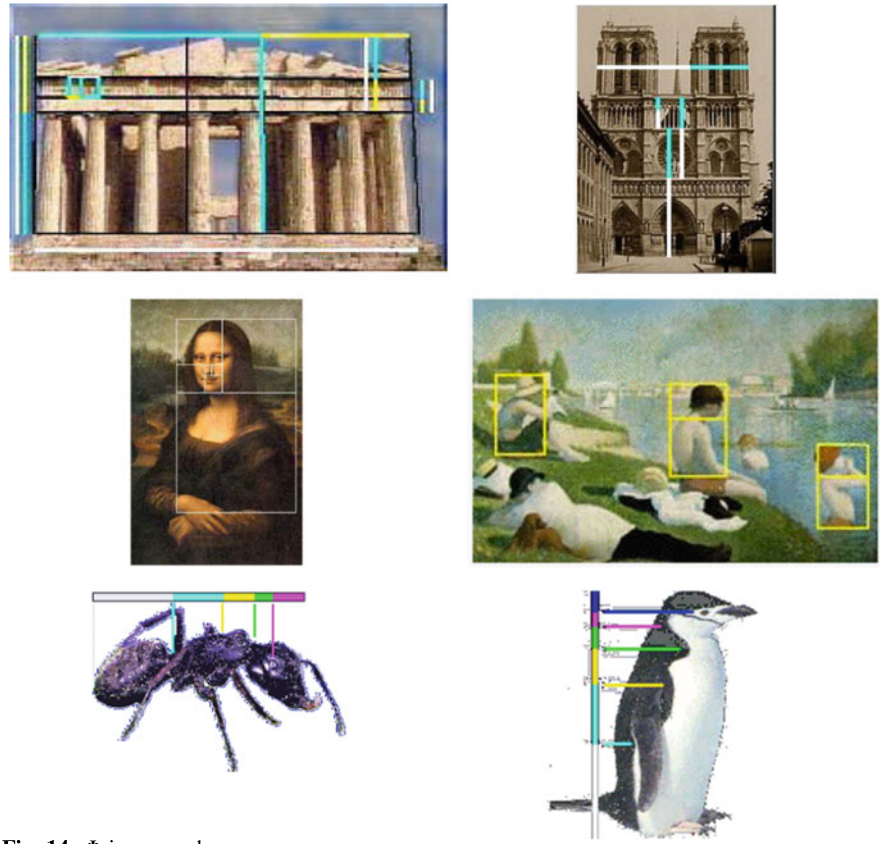
Fig. 12 Some logarithmic spirals

Fig. 13 The nautilus spiral

How to Find Φ

The Φ cult had its origins in [19], from which we extract the following quotation.

> [The golden number is] the universal law in which is contained the ground-principle of all formative striving for beauty and completeness in the realms of both nature and art, and which permeates, as a paramount spiritual ideal, all structures, forms and proportions,



◀ **Fig. 14** Φ is everywhere

> whether cosmic or individual, organic or inorganic, acoustic or optical; which finds its fullest realization, however, in the human form.

In this section we present a few pictures (Fig. 14), chosen among uncountable similar ones, to show that, once you believe in what the quotation says, the quest for Φ can go a long way. In all these pictures, the claim is that Φ is there. No details will be given; we just call the reader's attention to the following, often in combination, which allow for easy fudging (so that $\frac{5}{3}$, say, can be taken as Φ);

- arbitrary placement of points, lines, rectangles and spirals;
- arbitrary thickness of points and lines used as basis for measurements;
- measurements of monuments eroded by time and of objects in photographs distorted by perspective .

One could go on forever like this, but it is time to move on to the conclusion of this article.

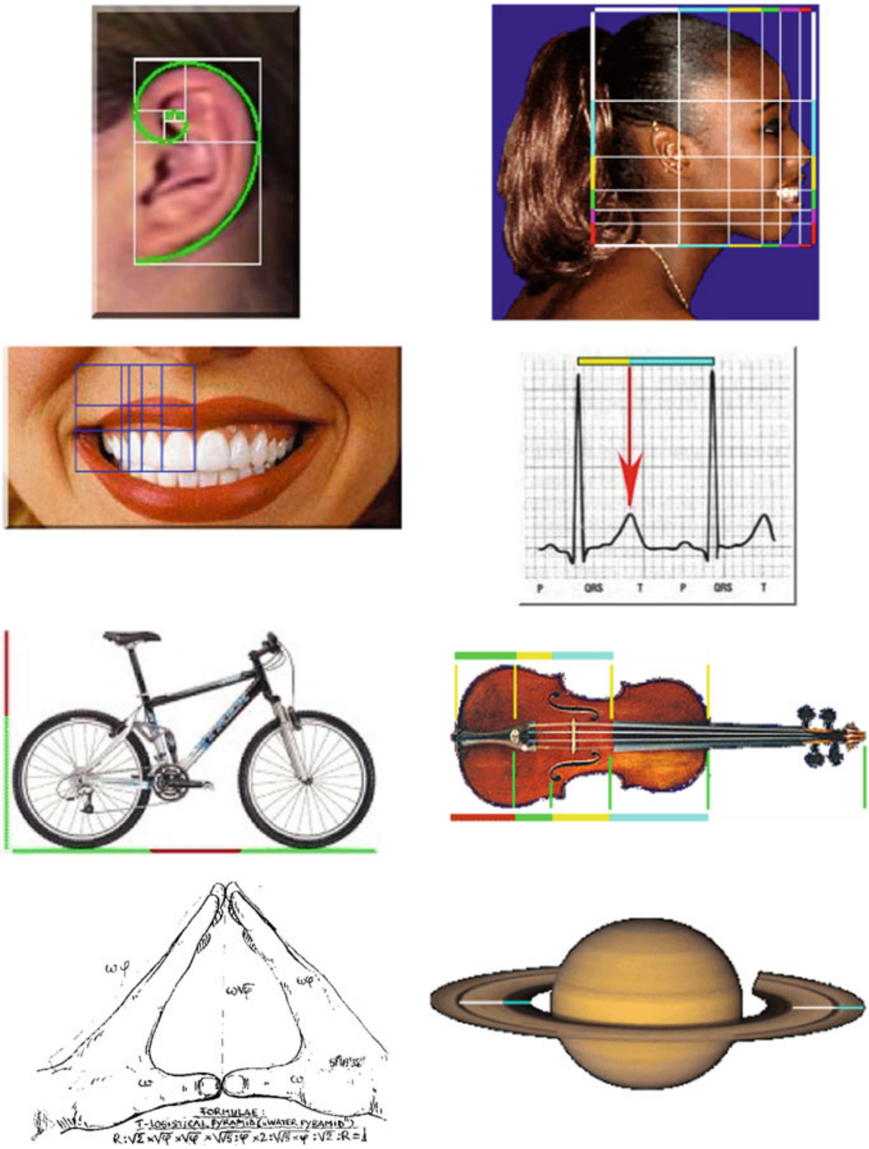


Fig. 14 (continued)

Φ, Beauty and Spiritual Development

Let's have a look the following exercise sheet,[3] intended for middle school kids (Fig. 15).

The title is *Finding the Gold*. In it, one is asked to make some measurements with a ruler between previously marked points and to compute a few quotients. The text on the lower left is worth transcribing: "Your answers to the above ratios should be near the Golden Ratio 1.618. If you're very far off on any one of them, recheck both your measurements and your calculations." And this to three decimal places!

This is the voice of authority speaking; it's here, you better find it, if you don't it's your mistake, no questioning allowed. This may sound harmless enough, but it is not. If you are a mathematician, Math teacher or scientist and say something that involves numbers, people will believe you; this power can and has been misused.

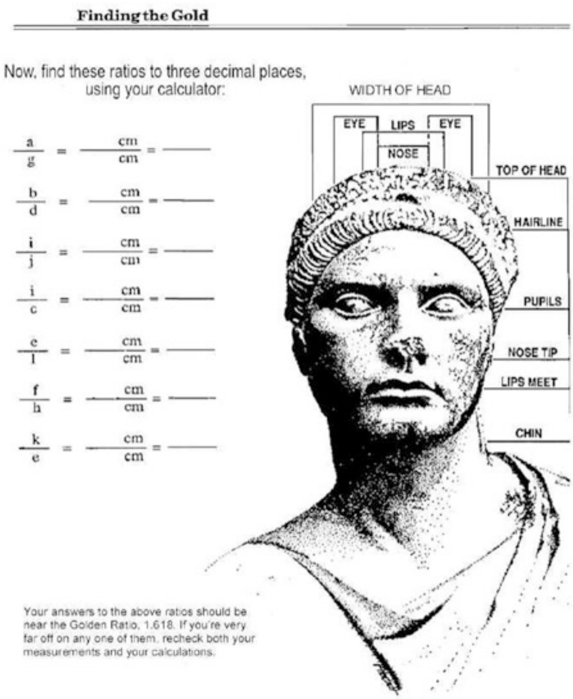


Fig. 15 Φ is here, you better believe it

[3]Unfortunately, the author lost the reference to the source of this illustration.

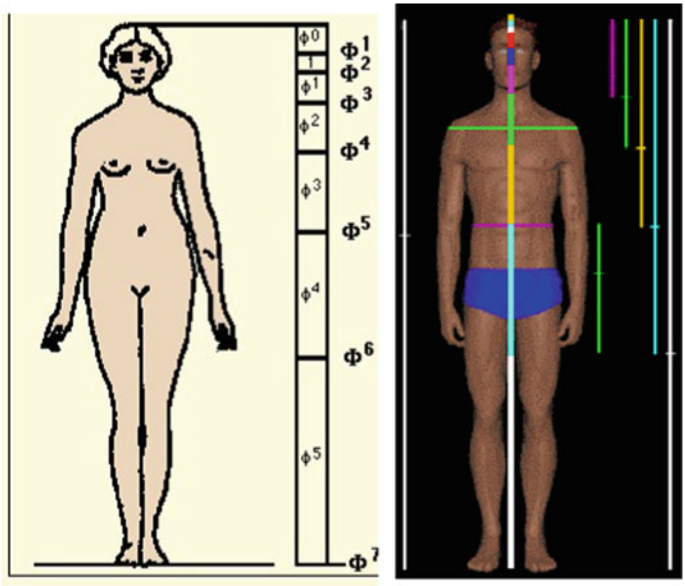


Fig. 16 Φ and beauty (look at the position of her navel!)

On the harmless side, one finds the following in [1], translated to the best of this author's ability:

> Anything that breaks in half can be repaired, but if it reaches the $^1/_\Phi$ mark no repair will be possible (do you believe this?).
>
> A fruit tree will have the most succulent fruit when it reaches $^1/_\Phi$ of its total load.
>
> A woman's cycle lasts 28 days, therefore $^1/_\Phi$ of 28 will be 17.5 days, when fertilization is guaranteed.

Still (apparently) harmless is the famous connection between Φ and beauty. As the theory goes, you should measure your navel ratio, i.e., the ratio in which your navel divides your height (from feet to head, in this order). The closer this ratio is to Φ, the more beautiful you are. Of course a navel, having a diameter, is not a point—a lot of fudging can be done here (Fig. 16).

Well, this is silly enough. But what if "beauty" is replaced by "spiritual development"? Still sounds silly, but then one finds [17], written by someone who speaks with "authority"; even high school algebra and geometry can intimidate people. The idea is as follows. Humankind is in a permanent state of spiritual development, the degree of which in a given race can be measured by the navel ratio of this race's women. This ratio is always less than Φ, since (of course!) Φ represents perfection (Fig. 17).

Fig. 17 Perfect “spiritual development”

A few freehand drawings of women of various races (rigorously true to life, we are told) follow, a horizontal line showing where the subject is divided in the golden ratio, so as to provide a visual estimation of how far the navel is from the ideal position—in other words, how far from the ideal spiritual development the corresponding race is. These drawings are shown in Fig. 18.

Can the reader guess which races are at the bottom of the spiritual development scale? Of course! Jews and Blacks, no doubt about it. The author takes pains to tell us so, in case we did not notice: “… cet écart sur la divine proportion est surtout accusé chez la Juive (Fig. M) et chez la jeune Négrille de l’Afrique équatoriale (Fig. K)”.[4]

Now this is certainly not harmless. One is reminded of the uses of I.Q. tests to typify people as morons; those interested should read [8]. The idea is the same; one ranks people by a number (the score in I.Q. tests or the navel ratio) to which is attributed a meaning invested with some sort of authority; this number, invariably, will be used to vindicate existing prejudice and social divisions.

In [8] it is pointed out that the I.Q. rank had as one of its consequences the forced sterilization of some of those characterized as morons, as well as the establishment of immigration quotas in the USA; these quotas caused the death of thousands in concentration camps during World War II. Paraphrasing [8], one can say that sometimes Math is more powerful than swords.

[4]“This deviation from the golden ratio is particularly noticeable in the Jewess (Fig. M) and in the young Negro woman from Equatorial Africa (Fig. K)”.

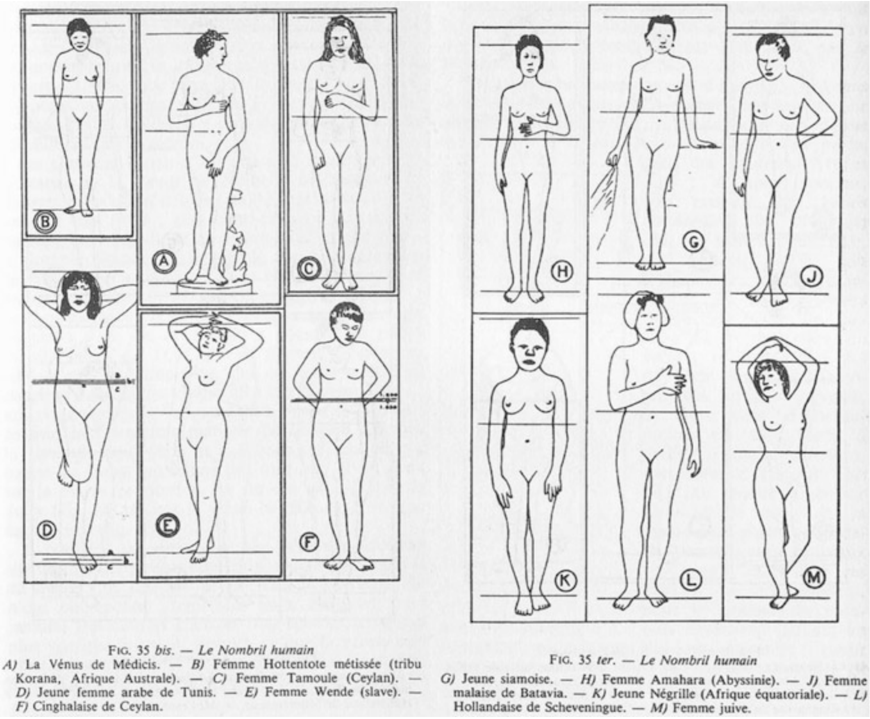


Fig. 18 Comparison of the "spiritual development" of various races

It is our responsibility, as teachers and mathematicians, to fight the use of Mathematics as a language of power. A good place to start is to tell people that Φ, golden number cultists notwithstanding, is not the "key to the living world" and that most of what is said about it is bogus. The author hopes that this article will help people to do so.

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The International Assessment of Mathematical Literacy: PISA 2012 Framework and Items

Kaye Stacey

Abstract The OECD PISA international survey of mathematical literacy for 2012 is based on a new Framework and has several new constructs. New features include an improved definition of mathematical literacy; the separate reporting of mathematical processes involved in using mathematics to solve real world problems; a computer-based component to assess mathematical literacy as it is likely to be encountered in modern workplaces; and new questionnaire items targeting mathematics. Procedures for quality assurance that arise in the preparation of an assessment for use in many countries around the world are illustrated with some items and results from the 2011 international field trial. The paper will provide background for the interpretation of the results of the PISA 2012 survey, which are to be published in December 2013.

Keywords Mathematical literacy · Assessment · Comparative studies · Computer-based assessment · Achievement · Mathematical competencies

Introduction

This paper reports on work done in preparation for the OECD 2012 Programme for International Student Assessment (PISA) survey of mathematical literacy. PISA surveys are conducted every three years. The first was in 2000, so that the 2012 survey is the fifth in the series. The first results from the 2012 survey will appear in December 2013. As well as making inter-country comparisons and linking achievement data to information on schools and teaching, it is now possible to examine trends in achievement over about a decade. In each cycle, the major focus of the survey rotates through reading literacy, scientific literacy and mathematical literacy. The 2012 survey focuses on mathematical literacy, for the first time since 2003.

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Because of this focus, a large number of new mathematical literacy items have been created and trialled for 2012, the Framework which specifies the nature of the assessment has been revised and an optional additional computer-based assessment has been developed. The questionnaires for students and schools will emphasise mathematics. In 2012, a random sample of students in at least 67 countries (including all 33 of the OECD member countries) participated in the main survey, with many undertaking the optional components, including computer-based assessment of mathematical literacy (CBAM), general problem solving, financial literacy, parent and teacher questionnaires and a student questionnaire on familiarity with ICT. There is a great deal of information freely available about PISA, past and present. The official OECD website (http://www.pisa.oecd.org) includes general descriptions of the project, official reports, links to all released items from previous cycles, and secondary analyses of data on topics of interest. The MyPisa website https://mypisa.acer.edu.au hosted by the Australian Council of Educational Research which led the International Consortium of contractors for PISA 2012, contains or links to copies of student and school questionnaires, national and international reports, research publications, technical manuals and all released items (e.g. http://pisa-sq.acer.edu.au and http://cbasq.acer.edu.au). It is possible to download data bases and manuals for analysis, or to submit a query to an automated data analysis service. In addition to official sites, there are many reports of scientific procedures (e.g. Turner and Adams 2007), secondary analyses of PISA data and many reports with a policy or local focus (see, for example, Oldham 2006; Stacey and Stephens 2008; Stacey 2010, 2011).

In this paper, I will first outline briefly some of the changes and developments for the PISA survey in 2012 that have been the concern of the Mathematics Expert Group (MEG). The outcome of this work is summarised in the new PISA 2012 Mathematics Framework (OECD 2013). The constructs of the framework and the item development process will be illustrated by discussing three items from the PISA 2012 field trial. Some examples of items from the new computer-based assessment of mathematics are then discussed. The paper finishes with a brief discussion of some new aspects of the student questionnaire.

New Framework and New Emphases

The purpose of the Framework is to describe the rationale of the assessment and to define its constructs, describe the components for reporting and specify the nature of the items and the proportion of each type. This section will report on improvements to the definition of mathematical literacy, and describe the underlying constructs of the assessment.

The new PISA Mathematics Framework was accepted by the PISA Governing Board in 2010 (OECD 2010). The final official version was formally published after data collection was complete (OECD 2013). With the guidance of the Mathematics Expert Group, the Framework was prepared under contract by ACER and Achieve (www.achieve.org) who organised feedback on drafts from over 170 experts in 40 countries. This illustrates one of the many ways in which PISA surveys tap into expertise from around the world.

Definition of Mathematical Literacy

The new Framework has clarified the definition of mathematical literacy, including emphasising the fundamental role that mathematics plays. The intention of having a revised definition is to clarify the ideas underpinning mathematical literacy so that they can be more transparently operationalised, whilst retaining strong continuity with the past definition so that the survey outcomes provide clear evidence of trends in educational outcomes.

Mathematical literacy is still seen as the understanding of mathematics that is central to a young person's preparedness to successfully manage life in modern society, from simple everyday activities to preparing for professional employment. Even more than when the PISA project was first devised, a growing proportion of problem situations encountered in work and life require some level of understanding of mathematics, mathematical reasoning and use of tools with a mathematical aspect. The notion of mathematical modelling (especially through de Lange's theorisation of mathematisation) was the cornerstone of the PISA Framework for mathematics from the start (OECD 2003) and it remains so. Now it has been more explicitly drawn into the new definition, with explicit reference to the component processes of modelling namely (i) formulating real world problems mathematically, (ii) employing mathematics to solve the mathematically formulated problem and then (iii) interpreting and evaluating the mathematical results in real world terms.

The new PISA 2012 definition of mathematical literacy is as follows.

> Mathematical literacy is an individual's capacity to formulate, employ, and interpret mathematics in a variety of contexts. It includes reasoning mathematically and using mathematical concepts, procedures, facts, and tools to describe, explain, and predict phenomena. It assists individuals to recognise the role that mathematics plays in the world and to make the well-founded judgments and decisions needed by constructive, engaged and reflective citizens. (OECD 2013, p. 25)

Mathematical literacy is intended to be a construct applicable to all ages and all levels of expertise. The new definition and discussion in the Framework more directly addresses the misconception that mathematical literacy is synonymous with minimal, or low-level, knowledge and skills. This misconception has been evident in some critiques of PISA. The Framework clarifies that there is a continuum of mathematical literacy from low levels to high levels, not a cut-off point above which one is mathematically literate. (Note, however, that cut-off points are used to report and compare the percentage of students in countries who have mathematical

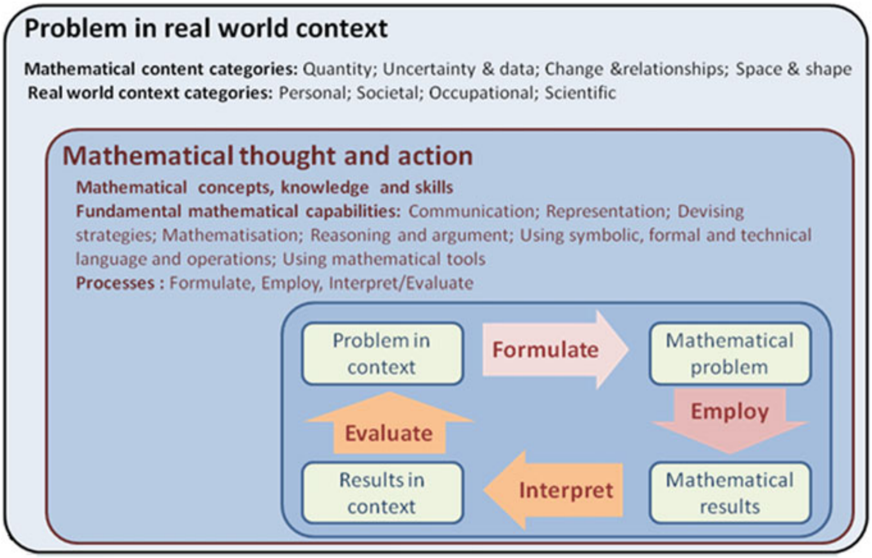


Fig. 1 A model of mathematical literacy in practice. (OECD 2013, p. 26)

literacy that is judged to be adequate for various purposes.) Of course, useful assessment of 15-year olds must take into account relevant characteristics of these students, including the mathematical content they are likely to know, so only a certain band of the whole mathematical literacy continuum will actually be assessed in the PISA survey.

The Mathematics Expert Group developed the diagram in Fig. 1 to illustrate the roles of the various components of the Framework. Consistent with the definition of mathematical literacy, PISA items are almost all set in a real world context (indicated by the outside box) although to make a good assessment there is variation in the degree to which solving the problems requires engagement with the context. Stillman (1998) described three degrees of engagement with real world context that problems may require. In some problems, the context acts as no more than a 'border' around a clear mathematical problem. PISA has a few of these problems. More commonly in PISA, the real context acts as a 'wrapper' around the mathematical problem. Students have to engage with the context sufficiently so that they can unwrap it to find the mathematical problem inside and when finished, wrap up the solution so that it is also presented in real world terms. PISA also has some problems in Stillman's third category, where considerations of the real world context and considerations from the mathematical core are inseparable, woven together in a 'tapestry' during the problem solving. Somewhat similar gradations are also described as first to third order context by de Lange (1987).

Content and Context Categories

The personal, societal, occupational and scientific *context categories* point to the areas of life where mathematical literacy is required (from the everyday to the professional) and from which item contexts are drawn. The Framework specifies the percentage of PISA items to be in these (and other) categories, to ensure a balance of interests in the assessment. The context categories have been renamed (a simplification) from the previous Framework, and there is some minor regrouping. The very few intra-mathematical items (without a real world context) are now allocated to the scientific category.

Still in the outside "Problem in real world context" box, PISA items can also be categorised according to the 4 major mathematical themes which underlie the situations and structures involved. In the 2003 Framework, these categories were called the 'overarching ideas', but they are now called *mathematical content categories*, although the construct is the same. There has been minor renaming, so that 'Uncertainty' is now 'Uncertainty and data' in order to more formally recognise that dealing directly with data (not just uncertainty about data) is a key ability for citizens making judgements and decisions. Feedback gathered by the Achieve consultation on the draft Framework highlighted the difficulty that some experts around the world had with this point, so hopefully the renaming will communicate more clearly. The content categories classify the items according to major mathematical structure and themes inherent in the mathematical core of the problem. These are the underlying ideas which have inspired the various branches of mathematics. For example, change is a theme encountered in many contexts, which can be handled with rates, functions or calculus. The *Space and shape* description describes geometry and measurement as foundational to activities such as perspective drawing, creating and reading maps, transforming shapes with or without technology, dealing with images of three dimensional scenes and representing objects and shapes. Again, the Framework specifies the proportion of items belonging to each content category. Because some countries have differential performance in different content categories, decisions about the proportion of items from each category can affect overall country rankings, and therefore need public decisions.

An important addition to the Framework is a move towards a more explicit description of mathematical content that is appropriate for assessment of 15 year old students. Expert advice was moderated by a survey conducted by Achieve (www.achieve.org) of published curricula from 11 countries. The aim was to clarify what 15 year olds will have had the opportunity to learn and also what countries deem realistic and important preparation for students approaching entry into the workplace or higher education. As the examples below show, the content is described in broad terms, in contrast to the careful curriculum analysis used in the TIMSS tests (timssandpirls.bc.edu). It is not the intention to test any school curriculum, but to work from the problem situations that arise, moderated by realistic expectations of

15 year olds around the world. An extract from the 'change and relationships' content category is as follows:

> *Algebraic expressions*: Verbal interpretation of and manipulation with algebraic expressions, involving numbers, symbols, arithmetic operations, powers and simple roots. *Equations and inequalities*: Linear and related equations and inequalities, simple second-degree equations, and analytic and non-analytic solution methods. (OECD 2013, p. 36)

This indicates to item writers that problems such as the following (these are not real PISA items) would be suitable from the point of view of the equation solving involved. Example (a) is linear and example (b) results in an equation that is easily converted to a linear equation. Of course these problems are very likely to be solved by students without formally setting up equations.

a. How far you can go in a taxi for \$50, if the 'flag fall' is \$3.20 and the charge is \$1.20 per km? (Equation $3.20 + 1.20x \leq 50$)
b. How long it will take to drive 5 km at an average speed of 32 km/hr? (Equation $5/t = 32$).

The content descriptions are intended to guide item writers, rather than be prescriptive, and they are not intended to make a curriculum. It is recognised that there is not a one-to-one relationship between mathematical topics and the content categories. Furthermore, because items arise in real contexts, very often they do not fit neatly into just one category.

Fundamental Mathematical Capabilities

Dealing with a problem in the PISA assessment necessarily involves mathematical thought and action. As depicted in the middle box of Fig. 1, this is conceptualised as having three components. The first is *Mathematical concepts, knowledge and skills*: the knowledge base, consisting of mathematical concepts, known facts, and skills in performing mathematical actions (the topics described above). Second, activating the knowledge base involves the seven *fundamental mathematical capabilities* (FMC) for mathematical action. These capabilities are derived from the mathematical competencies of the 2003 Framework, and indeed the original PISA Framework of 1999. They originated in work done by Mogens Niss (a long-standing member of the Mathematics Expert Group) and colleagues (Niss 1999, 2003; Niss and Højgaard 2011). In the new Framework, these have been renamed to fit better with terminology in other OECD assessments, and simplified based on empirical work conducted with the Consortium and Mathematics Expert Group (Adams 2012; Turner et al. 2011; Turner 2012). Given the importance of new technology for doing mathematics, and the new possibilities of the CBAM, *Using mathematical tools* (i.e. physical and electronic devices) is an additional FMC.

The FMC describe the various processes that are increasingly seen as central to an individual's understanding of mathematical ideas and capacity to apply his or her mathematical knowledge. Evidence of this recognition can be seen in the formal curriculum statements of various educational jurisdictions around the world. Empirical evidence of the centrality and importance of these capabilities to mathematical performance would be established if a strong relationship can be found between item difficulty (measured empirically) and ratings of items on the capabilities. Consortium staff and members of the MEG have been engaged in this work; describing four graduated levels of operation of each of the capabilities and judging the extent to which successfully answering PISA questions demands their activation. It is recognised that the chosen capabilities overlap to some extent, and that they frequently operate in concert and interact with each other; nevertheless the rating procedure has been to treat each competency as distinctly as possible. Early statistical work reduced the number of competencies (earlier name) to the present number and examined inter-rater reliability, leading to improved level descriptions in an iterative process (Turner 2012; Turner et al. 2011). Ongoing work will test the power of the ratings according to the refined levels to predict item difficulty in the new PISA study (Adams 2012). It is expected that the completed scheme will have wide application in understanding the capabilities that underpin mathematical literacy and creative mathematical activity more generally. The scheme could be used by item developers in many situations, and as a guide for teachers to demonstrate both capabilities to emphasise and levels of development through which their students may progress. The FMC with their level descriptions are also used to describe performance at the 6 levels of the proficiency scale, which (along with point scores) are used for reporting PISA results.

New Reporting of Processes

The third part of mathematical thought and action (still in the middle box of Fig. 1) are the processes of solving problems involving mathematical literacy. In previous cycles, mathematical literacy has been reported on one overall proficiency scale and also by the ‘overarching ideas’ (now called the mathematical content categories). However in 2012 reporting will also be against the processes of using mathematics to solve real world problems. Figure 1 lists the processes and the inner box sketches a very simplified description of how they are related to mathematical literacy. This sketch is a simplification of a model that appeared in the 2003 Mathematics Framework (OECD 2003). A real world problem first needs to be transformed into a mathematical problem. This is the *formulate* process (full name *Formulating situations mathematically*.), indicated by the top arrow. The mathematical problem is solved by the *employ* process (full name *Employing mathematical concepts, facts, procedures, and reasoning)*. This is indicated by the rightmost arrow. The mathematical results that are produced then need to be translated into real world terms and judged for their adequacy. If inadequate, the problem situation may need to be reformulated. Together these two steps make the *interpret process* (full name

Interpreting, applying and evaluating mathematical outcomes) which is indicated by the bottom and left arrows of the mathematical modelling cycle in the diagram.

In 2012, PISA mathematical literacy results will be reported as overall score, scores for each of the four mathematical content categories and, for the first time, scores for each of the three processes. It is hoped that this additional reporting structure will provide useful and policy-relevant results. The scores on the *formulate* scale should show how effectively students are able to recognise and identify opportunities to use mathematics in problem situations and then provide the necessary mathematical structure needed to formulate that contextualised problem into a mathematical form. The *employ* scale should indicate how well students are able to perform computations and manipulations and apply the concepts and facts they know to arrive at a mathematical solution to a problem formulated mathematically. The *interpret* scale should indicate how effectively students are able to reflect upon mathematical solutions or conclusions, interpret them in the context of a real-world problem, and evaluate whether the results or conclusions are reasonable. Students' facility at applying mathematics to problems and situations is dependent on all three of these processes, and an understanding of their effectiveness in each process can help inform both policy-level discussions and decisions being made closer to the classroom level.

The field trial results have been analysed by Ray Adams (personal communication). It shows that there are very high correlations between the student scores on the three process scales, especially for the paper-based assessment. This indicates a redundancy in the information. However, they are comparable to the correlations between component scales for other domains in previous field trials, and these have proved to be useful for country comparisons and comparisons of identified subgroups of students. In part, the high correlations may be explained by the difficulty of allocating items to only one of the processes (multiple allocation is not advisable for the data processing algorithms). There is a tension between writing items with strong face validity for mathematical literacy (which tend to involve multiple stages of the modelling cycle) and writing items which can be unequivocally allocated to one of the three processes. This difficulty mirrors the difficulty of allocating items to one content category or one context category. Another factor is that the significant time limits on answering a PISA question means that items often hone in somewhat artificially on a specific aspect, so that the relationship of the item to the modelling cycle is less clear. Despite these inevitable 'boundary disputes' for categorisation, the first analysis shows some country-level spread across processes. The first glimpse of the field trial data also shows potentially interesting differences in the performance of countries on the process scales derived from paper-based tests and computer-based items. The field study showed that items classified as *formulate* (i.e. items where this was judged to be the main source of cognitive demand) tended to be harder than other items. This observation concurs with the lament of many teachers: "my students cannot do word problems". For the test as a whole to give reliable results for students of all abilities, items should be selected so that there is a good range of difficulty for items of every type, so easy *formulate* items were in demand when selecting items.

Item Development

The 2012 mathematics assessment required 72 new items to be created for the nine-yearly in-depth study of mathematical literacy, alongside 36 link items from earlier surveys to calculate trends. New items, proposed by teams around the world, went into a large pool which was approximately halved for the field trial and halved again for the main study in the light of empirical results. Before selection for the field trial, items were subject to intensive scrutiny by the MEG, by external experts organised by Achieve, and by the national teams in every country. Items were generally very well received by these reviewers, indicating the substantial imagination and expertise of the writing teams. The field trial showed that all but a few of the items had very good statistical properties, so there were good options for making a final selection that meets the Framework's multiple criteria of balance across the 4 context, 4 content and 3 process categories, with items of a wide range of difficulty in each cell of the matrix.

Although many items are publically released from previous cycles (see, for example, OECD 2009), I will illustrate some of the points about item development with three new items of one unit, which went into the field trial in 2011 but were not selected for the 2012 main study. There are many reasons for an item not being selected. These include differential performance in countries (likely to be due to cultural or linguistic factors); lack of support from national teams; statistical anomalies at the field trial (e.g. poor discrimination, worrying gender or country differences, too hard or too easy); occasionally a logical or mathematical content flaw identified late in the process; or simply because it is surplus to requirements.

For the 2012 survey, as in the past, a disproportionately large number of the harder items were not selected, because an efficient test should bunch the difficulty of the items near the (measured) ability of most of the students, and the items in the field trial tended to be relatively hard for the students. It is a waste of time and money to include in PISA surveys, items which only a small percentage of students can do, since they do not contribute meaningfully to the reported results, which are at the level of country and identified subgroups such as by gender. Item writers, even those who are very experienced, tend to create relatively too many items which have very low success rates.

Many people (myself included) who examine the mathematics in the PISA items feel that it is not of a 'high standard' for Year 10 students, and there was an attempt in the field trial to ensure that this concern was addressed. However, in evaluating the nature and level of mathematics in the PISA items, it is important for everyone concerned to focus on whether the tested mathematics is important—whether knowing it makes a significant contribution to students' mathematical literacy as they require it in their lives. A trap is to confuse significant mathematics with hard mathematics. Percentage calculations and the associated proportional reasoning, for example, are clearly of significance in mathematical literacy, although these are usually learned long before students reach 15 year old and so appear 'easy' in curriculum terms. That being said, it is still of concern that the success rates in PISA items are in general surprisingly low, even in countries that are highly ranked. There is a great deal of room for improvement in mathematical achievement even in countries that do well. One hypothesis is that the low success rates on PISA items stem from the mathematical literacy focus, presenting them in contexts. Using the scheme for rating the influence of the fundamental capabilities may provide a research tool to help in understanding this.

In order to match item difficulty better with student ability, in 2012 some countries which have had low achievement in the past have chosen to use an item pool which includes relatively more easy items. This will enable more reliable measures to be made at the lower end of the achievement scale, and so enhance the usefulness of their results for intra-country policy development. Complex statistical procedures, using a Rasch model place individual students on one common scale, despite the fact that they do different booklets (subsets) of items, and then combine student level data to create comparable country scores. Because of the complex processing involved, individual students cannot be given meaningful feedback on their performance immediately after completing the test.

Example: Chinese Lamp (PM999)

This section illustrates some of the quality control measures and the application of the constructs of the PISA framework. This example illustrates a new unit containing three items from the field trial that, because of unpredicted poor statistical performance (low discrimination), will not be used in the main study and hence can now be released. *Chinese Lamp* was a unit that was well regarded by all the reviewers, and so can be used to illustrate the nature of mathematical literacy items, and the processes of item development and categorisation. Its poor statistical performance was not predicted, showing the importance of the field trial. The *Chinese Lamp* items were generally approved by the national PISA teams, with the average "priority for inclusion" at 3.5, 3.4 and 3.2 on a 5 point scale across countries. Item statistics reported below are based on over 6000 students from the 25 OECD countries whose results had been fully analysed at the time when selection of items for the main study was undertaken. The sampling for the field trial is not strictly controlled as it is in the main study, but previous experience indicates the field trial results are a good guide to later main study results.

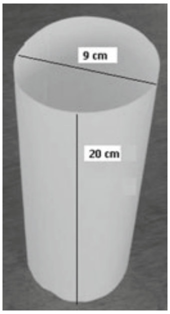
The three items in the unit *Chinese Lamp* (PM999) shown in Fig. 2 (stem) and Fig. 3 (items) belong to the Shape and Space content category in the personal context category. With only a minor change of the cover story, the unit could be altered to belong to the occupational context category. The first two items (Questions 1 and 2 in Fig. 3) were allocated to the mathematical process category 'formulate' and Question 3 was allocated to 'interpret'. This will be discussed below.

Figure 2 retains the notes that assist translators to distinguish those parts of the text which are intended to be common language descriptions (in this case both 'outside angle' and the description of the folding) from those which have specific mathematical terminology which needs to be translated precisely. The PISA project has very high standards of translation, with multiple procedures requiring

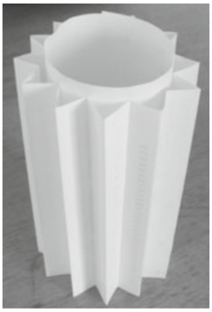
Chinese Lamp

Mira is organising a party at her house and would like to make Chinese lamps. Each lamp is made of two sheets of special paper: an internal sheet rolled in a cylinder (Picture 1) and a second sheet folded to create 12 triangles (Picture 2). The diameter of the internal paper cylinder is 9 cm and its height is 20 cm. The triangle shapes in Picture 2, that are created by the folding, form approximately equilateral triangles.

Translation Note: In this unit please retain metric units throughout.
Translation Note: Please adapt the terminology if you have a specific expression for this kind of folding. For example, in FRE it is "feuille pliée en accordéon".



Picture 1



Picture 2

Fig. 2 Chinese Lamp (PM999Q01) unit stem (internal document of Mathematics Expert Group)

translating into the third language from very carefully matched originals in both English and French, then comparing the third language product as well as back and cross translations. The process is illustrated in Fig. 4.

Even though mathematics is often thought of as a universal language, PISA items present many translation challenges. A major source of the challenge is the relatively informal language that is authentically used in mathematical literacy items to describe real world contexts contrasting with formal mathematical language. The *Chinese Lamp* provides examples of this ('triangle shapes', 'sheets are 20 cm high', 'outside angle'). Differences also come in mathematical descriptions. English, for example, has many ways of expressing a ratio: "three times longer than" (Fig. 3), the more correct 'three times as long as"), "a ratio of 3:1" or "triple". Some languages do not have such a wide range of alternatives. Consequently, in some items students need to do less translation from colloquial language to formal mathematics when they undertake the assessment in some languages than in others. To some extent, this is guarded against in the field trial by checking for unexpected country differences.

Question 1 (see Fig. 3) requires spatial reasoning to identify how the paper will be folded to make the cylinder and to link the length of paper to the perimeter (circumference) of a circle through the cross-sectional view plus overlap for gluing. The perimeter must then be calculated approximately. This item is allocated to the 'formulate' process because it was judged that the main cognitive demand came from the spatial reasoning rather than from the calculation of the circumference

Question 1

In Mira's favourite hobby shop they sell several sheet sizes. The sheets are all 20 cm high but have different lengths.

Which of the following is the smallest length Mira could buy in order to make the paper cylinder (see Picture 1)? (Note that at least 0.5 cm extra is needed for gluing.)

a) 20 cm
b) 30 cm
c) 40 cm
d) 50 cm
e) 60 cm

Question 2

In a finished Chinese lamp (see Picture 2), how many times longer is the folded sheet of paper compared to the length of the internal sheet rolled in a cylinder?

a) Approximately 1.5 times
b) Approximately 2 times
c) Approximately 3 times
d) Approximately 12 times

Question 3

Mira wants to create another Chinese lamp in a similar style.

Which of the following changes to the lamp shown in Picture 2 will affect the length of the folded sheet of paper?

Circle "Yes" or "No" for each change.

a) Keep the same size paper cylinder, change the size of the outside angle in each of the 12 triangles of the folded sheet of paper from about 60° to about 30°. Yes / No
b) Keep the same size paper cylinder, increase the number of equilateral triangles of the folded sheet of paper from 12 to 20. Yes / No
c) Change the diameter of the internal paper cylinder, keep the folded sheet of paper with 12 equilateral triangles. Yes / No

Translation Note: Please do not translate "outside angle" as "external angle" as this term has a different and precise mathematical meaning.

Fig. 3 Chinese Lamp questions with correct answers underlined (internal document of Mathematics Expert Group)

from the diameter. Had the latter been judged the more demanding aspect, the item would have been allocated to the 'employ' process.

This item was a middle difficulty item in the field trial with 45.63 % of students correct (b). Another 35 % of students were approximately equally spread between the answers of (a) 20 cm and (c) 40 cm, and about 7 % selected each of (d) 50 cm and (e) 60 cm. This item had low discrimination. One indicator of this is that the 7 % of students who selected (e) had higher average proficiency (as judged by the rest of the test) than students selecting the correct answer. Their ability was considerably higher than students who selected (d), which is an indicator that there are quite different reasons for making these two choices. Because the aim of the PISA survey is to measure students' mathematical literacy as effectively and efficiently as possible, items of low discrimination are discarded. In a school assessment or public examination, there are other goals of assessment to consider—such as testing across a curriculum and illustrating to teachers important aspects of mathematics in

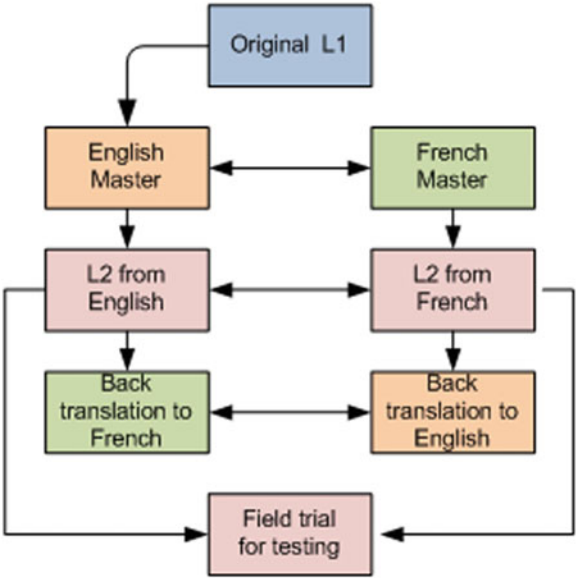


Fig. 4 Quality assurance processes in translation

accordance with the Mathematical Sciences Examination Board's "Learning Principle" (Stacey and Wiliam 2013). However, in an international assessment which is kept secure, the considerations are different. The same consideration applies to items which assess the more advanced mathematics which many of us (including myself) feel should be well within the capability of well educated 15 year olds. If the proportion of students who answer correctly is very small, then the item makes only a tiny contribution to the main aim of the PISA test, which is to get good estimates of the mathematical literacy of specified groups across countries.

There was a moderate gender difference favouring girls on this item and also on Question 2, but not on Question 3 where the sexes performed equally. Items with large gender differences in the field trial are not used in the main survey.

Question 2 (see Fig. 3) again requires considerable spatial reasoning, to identify the importance of the view from the top and to see the 12 nearly equilateral triangles, and then to see that the circumference of the cylinder is made up of one side of each 'equilateral triangle'. The folded paper is therefore made of two of the three sides of each equilateral triangle and so is close to twice the length of the circumference of the cylinder. Again this item is categorised as 'formulate', identifying the spatial reasoning as a more important source of cognitive demand than employing knowledge of equal side length of equilateral triangles. The percentages of students correct [option (b)] was 42.22 %, whilst about 20 % of students chose each of options (a) and (c). Only 9.71 % chose option (d), where the 12 times in the response superficially matches the 12 triangles in the diagram. This item also has low discrimination. Perhaps this is because it can be solved by the relatively complex spatial reasoning above, or by simpler methods such as direct estimation of comparative length from the diagram, or even by measuring the folds in the picture with a ruler (about 1 cm each on my copy).

Question 3 (see Fig. 3) is a complex multiple choice item. As with almost all complex multiple choice items, a response to this item was only scored correct if all parts are right. This format significantly reduces the chance of randomly choosing a correct response, in this case to 1 in 8. In total, 24.21 % of students were correct (i.e. all parts right), and about 60 % were right on one or two parts, but scored no credit for the whole item. Again, the item has low discrimination. For example, the correlation of the score on this item with the score on the rest of the test was 0.06. Statistics for individual parts are not available. I guess that the second part was the most difficult. This item was classified as 'interpret' but arguments could also be made for a classification as 'formulate'. It is important to note that the 'interpret' process is not about interpreting the meaning of the statements (i.e. receptive communication) or reading mathematical representations (e.g. understanding a graph), but about putting mathematical outcomes into real world terms, and evaluating the adequacy of solutions. It refers to the shift back from the mathematical world into the real world.

Computer-Based Assessment of Mathematics (CBAM)

A major initiative for the 2012 survey is the introduction of the optional computer-based assessment of mathematics CBAM. This follows the development of assessment of electronic reading (2009) and computer-based assessment of scientific literacy (2006). In CBAM, specially designed PISA units are presented on a computer, and students respond on the computer. They are also able to use pencil and paper as they wish to assist their thinking processes.

Behind the introduction of all of these computer-based assessments is a long term intention to shift all PISA assessment from paper-based to computer-based. Initial plans for 2015 are to do this. Developing expertise and infrastructure is therefore valuable. However, there are specific reasons why computer-based assessment is important for mathematics and for mathematical literacy. First, computers are now so commonly used in the workplace and in everyday life that a level of competency in mathematical literacy in the 21st century includes using digital mathematical tools. Hoyles et al. (2002) note that mathematical literacy in the workplace is now completely intertwined with computer literacy, at all levels of the workforce. Doing mathematics with the assistance of a computer is now part of mathematical literacy.

A second consideration is that the computer provides a range of opportunities for designers to write test items that are more interactive, authentic and engaging, and which may move mathematics assessment away from the current strong reliance on verbal stimuli and responses, enabling different student abilities to be tapped and also recorded (Stacey and Wiliam 2013). These opportunities include using new item formats (e.g., drag-and-drop or hotspots), presenting students with flexible access to real-world data (such as a large, sortable dataset), moving stimuli and simulations, representations of three-dimensional objects that can be rotated, or to use colour and graphics to make the assessment more engaging. The effect of the latter apparently trivial factor has been immediately recognised during item development. In 'cognitive laboratories' where individual students are interviewed solving the items, stronger engagement and willingness to experiment with solutions was frequently noted. By permitting a wider range of response types, CBAM could also give a more rounded picture of mathematical literacy, for example in spatial items and dynamic items involving change over time.

It is to be hoped that future PISA cycles may feature more sophisticated computer-based items, as developers and item writers become more fully immersed in computer-based assessment and the infrastructure develops. Indeed, PISA 2012 represents only a starting point for the possibilities of the computer-based assessment of mathematics. In the future, I hope to see assessment of proficiencies at the interface of mathematics and ICT. Examples would include making a chart from data (e.g., pie chart, bar chart, line graph of a large data set) or producing graphs of functions to answer questions about them. In the present assessment, due to expectations of what 15 year olds around the world may be able to manage now, using generic mathematical tools does not go much beyond using the types of on-screen calculators that are found on websites for tasks such as ordering tickets or converting currencies.

A key challenge is to distinguish the mathematical and mathematics-with-ICT demands of a PISA computer-based item from demands unrelated to mathematical proficiency such as using a mouse, understanding basic conventions such as clicking on arrows to move to a new screen. The latter category of demand needs to be minimised. Reports from the field trial of CBAM (and later the main study) indicated that this was satisfactorily achieved.

In 2015, mathematics will be a minor domain assessed only for trend purposes. It is expected that this assessment will be delivered with computers, meaning that items will be presented on screen and responses will be input with the keyboard or mouse. This computer-delivery of what were the 'paper-based items' of 2012 is not the same as the 2012 CBAM.

Examples from the CBAM Field Trial

In the field trial for 2012 CBAM, computer-based assessments were administered to 39,970 students in 52 locales (a locale is a language by country combination) across 43 countries in 30 languages, with considerable operational success. For CBAM, the field trial tested 86 items, from which the final pool of 45 items were to be selected. Each student will do about a quarter of these in the main study, with computer records of time taken being used to select the subsets.

Four sample units (translated into 14 languages) are available on the website (cbasq.acer.edu.au). These items have not been included in the main study, either because of statistical anomalies from the field trial, a flaw in the item, or because they are surplus to requirements.

The two sample units *Graphs* (CM010) and *Car cost calculator* (CM013) provide examples of items that could equally well be paper-based items, items where computer presentation enables a new response type, and items which require use of some computational power of the computer. The first item of the unit *Graphs* asks students, in multiple choice format, to identify a real world situation that could relate to a given bar graph. In this case, the bar graph shows regular cycling, and the correct choice is maximum monthly temperature of a city, rather than temperature of a cup of coffee, weight of a baby, or diminishing coal reserves. There were 49.53 % correct in the 15 OECD countries of the CBAM field trial. This could be a paper-based item: it does not need computer presentation.

The second and third items take advantage of the computer presentation for enhanced responses and automated marking. Here the computer-based format offers both new opportunities and increased convenience. Students have to construct bar graphs, by dragging prepared bars onto the graph to match the real world situation. Item 3 is shown in Fig. 5, partially completed with three bars already dragged onto the axes. The instruction is “Drag and position each of the bars onto the Time axis to show how Jenny’s yearly income changed over the 10-year period.” Answering in paper-based format would be very impractical, because students would have to draw bars of precise heights and would be likely to need several trials, and starting again by erasing drawn bars would be awkward. Also scoring by hand would be time consuming. Mathematically, this is a difficult item, because the information about the constant annual increase needs to be carefully considered, by using qualitative reasoning about the constant gradient property of constant increase.

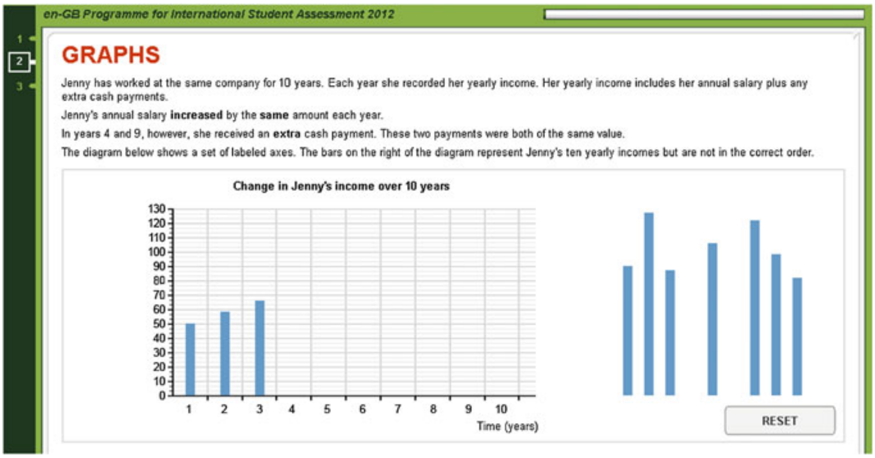


Fig. 5 Partially completed Item 3 of *Graphs* (CM010Q03) (ACER 2012)

Only 10.57 % of students were correct, taking an average of 167 s to respond. The capacity to track additional characteristics of computer-based responses, such as response time or number of attempts, can provide researchers and test developers with additional insights into students' performance (Stacey and Wiliam 2013). This capacity will grow as infrastructure for computer-based tests is refined in the future. One reason why this item has been released is that it had low discrimination in 6 of 15 OECD countries in the field trial.

Some of the computer-based items have students directly use the computational power of a computer, both for calculation and exploring the mathematical structure inherent in a real world context. Items requiring use of specific mathematically-able software (e.g. to program a spreadsheet, or use a generic tool to plot of graph) have not been used at this early stage of CBAM, because of the very varied abilities of students around the world to use it. However, there are a great number of user-friendly calculators available on websites around the world to assist consumers, so mimicking this provides an opportunity for mathematical thinking in a very realistic setting.

The unit *Car Cost Calculator* (CM013) (Fig. 6) provides an example of this. Dragging the car on the graphic causes the distance and monthly cost of travelling to work and back by car to change. The first item requires direct use of the car cost calculator followed by other calculation (perhaps with an ordinary or on-screen calculator). It asks what percentage of car travel cost would be saved by buying

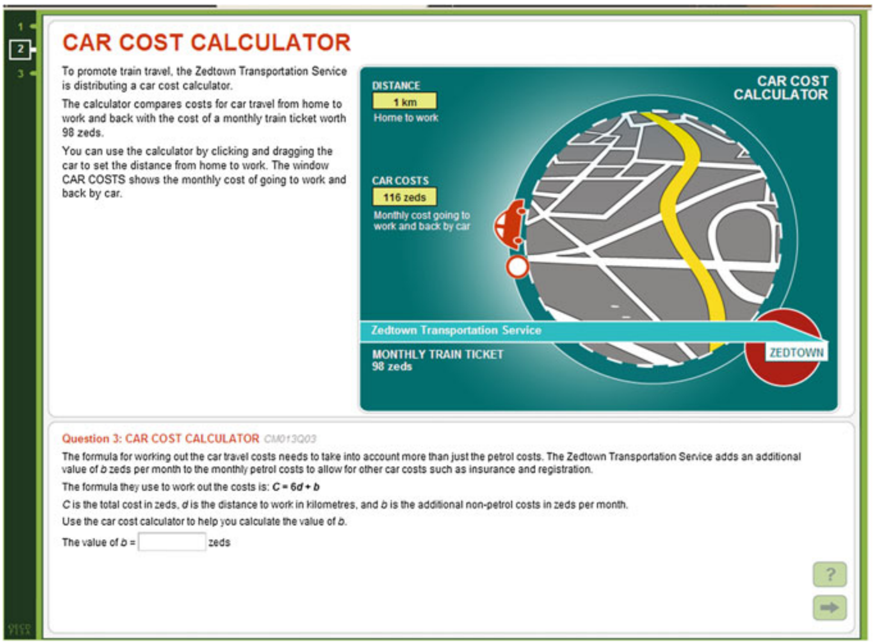


Fig. 6 CBAM item CM013Q03 (ACER 2012)

a monthly transport ticket for a person travelling 15 km (36.46 % correct). The second item could as well be paper-based, asking students to select a formula for working out petrol costs as a function of distance to work, given appropriate data (18.80 % correct, with low discrimination in most countries). The third item (see Fig. 6) requires obtaining sensible data from the car calculator to work with algebra. On average, the National Program Managers of the participating countries expressed approval for all of these items to be included (average scores between 3.4 and 3.9 on a 5 point scale) which indicates that they judged the mathematical content and real world setting to both be appropriate for the 15 year old students in their countries. The poor discrimination was not predicted by the expert teams (NPM, item writers, MEG members, Achieve consultants).

Student Questionnaire

In 2012, the questionnaires for students and schools (20–30 min) contain many general items about school as well as items that focus on mathematics, prepared by the PISA questionnaire expert group and Consortium staff. There are questions about the mathematics learning environment, students' opportunity to learn mathematical literacy at school, their interest in mathematics and their willingness to engage in it. Responses can be related to achievement scores. *Opportunity to Learn* items relate to student experience with applied mathematics problems, student-reported familiarity with mathematical concepts by name, and student experience in class or tests with PISA style items. *Interest in mathematics* relates to present and future activity: mathematics at school, perceived usefulness in real life and intentions to undertake further study and/or mathematics-oriented careers. *Willingness to engage in mathematics* taps into emotions of enjoyment, confidence and (lack of) mathematics anxiety, and the self-related beliefs of self-concept and self-efficacy. There is international concern about interest and willingness because of a decline in the percentage of students who are choosing mathematics-related future studies, whereas at the same time there is a growing need for graduates from these areas. A recent analysis of the subsequent progress of young Australians who scored poorly on PISA at age 15 found that those who

> recognise the value of mathematics for their future success are more likely to achieve this success, and that includes being happy with many aspects of their personal lives as well as their futures and careers. (Thomson and Hillman 2010, p. 31)

The study recommends that a school focus on the practical applications of mathematics may go some way to improving the outlook for these low-achieving students.

Conclusion

The aim of this paper was to outline some aspects of the PISA 2012 mathematical literacy assessment, and to highlight some of the theoretical and practical changes that have been introduced. These changes and more are thoroughly discussed in a recent book (Stacey and Turner 2015). The new Framework has strong continuity with the past, so that trends can be measured, but it has also addressed earlier critiques, by some clarification and simplification, by emphasising the centrality of mathematics more strongly, and placing the multiple components of the Framework into a holistic picture. A major strength of PISA lies in the rigorous procedures for item development, review, trialling and selection, for translation and coding, for sampling in the main survey and expert data analysis for reporting the results. It is hoped that insight into some of these strengths has been provided above. It is expected that the additional reporting categories of PISA 2012 (the three processes) will enhance the usefulness of the results for public policy development and provide further insights into how the nature of the mathematics provision in schools affects mathematical literacy. On-going work in rating the influence of the fundamental mathematical capabilities on the total cognitive demand of items (checked against empirical measures as they become available) may prove useful to item developers for PISA and other assessments, and provide new research findings into what makes mathematical literacy items difficult. The new initiative of computer-based assessment of mathematics is an important one, which opens up new avenues for probing mathematical literacy with and without technological assistance. I look forward to the release of the first results at the end of 2013 and the many subsequent secondary analyses which will follow.

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Applications and Modelling Research in Secondary Classrooms: What Have We Learnt?

Gloria A. Stillman

Abstract This paper focuses on my 20 year program of research into the teaching and learning of applications and modelling in secondary classrooms. The focus areas include the impact of task context and prior knowledge of the task context during the solution of applications and modelling tasks, mathematical modelling in secondary school, and metacognition and modelling and applications. Some of the analysis tools used in this research are also presented.

Keywords Modelling · Secondary school · Applications · Task context · Prior knowledge · Metacognition

Introduction

Research into teaching and learning through mathematical modelling and applications has been strong for several decades now (Blum et al. 2007; Kaiser et al. 2011) and a feature in regular lectures at recent congresses. However, in many countries it would be true to say, "there is still a substantial gap between the forefront of research and development in mathematics education, on the one hand, and the mainstream of mathematics instruction, on the other" (Blum 1993, p. 7) when it comes to this area of teaching and learning. My colleagues and I have been researching in this area for many years. In my work I have focused on a range of significant issues that impact on the field of applications and modelling in mathematics education, and I take this opportunity to reflect on some of these from the perspective of my own work and that of others. The focus areas include the impact of task context and prior knowledge during applications and modelling tasks, mathematical modelling in school particularly in secondary school, and metacognition and modelling and applications.

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Given the various idiosyncrasies associated with some localised curricular initiatives within and between countries the meanings and interpretations ascribed to terms such as *applications* and *mathematical modelling* in my work will first be clarified. These meanings are consistent with those adopted by the International Community for the Teaching of Mathematical Modelling and Applications (ICTMA), which is an Affiliated Study Group of ICMI.

> Various intermediate stages exist between completely structured word problems and open modelling problems where the structuring must be supplied entirely by the modeller. One such stage involves contexts where the aim of the problem is well defined, where the problem is couched in everyday language, but where some additional mathematical information must be inferred on account of the real world setting in which the problem is presented. This is a level between textbook word problems and modelling problems contextualised fully within real-life settings (Stillman and Galbraith 1998, p. 158).

These I call applications tasks. With *applications* the direction (mathematics → reality) is the focus. With *mathematical modelling* the reverse direction (reality → mathematics) becomes the focus. The model has to be built through idealising, specifying and mathematising the real world situation. Both types of task have their place in school classrooms.

The term mathematical modelling when used in curricular discussions and implementations is often interpreted differently. One interpretation sees mathematical modelling as motivating, developing, and illustrating the relevance of particular mathematical content whilst a second views the teaching of modelling as a goal for educational purposes not a means for achieving some other mathematical learning end. My own approach sees the second interpretation as encompassing the first. Both approaches agree that modelling involves some overall process that involves formulating, mathematisating, solving, interpreting, and evaluating as essential components.

Impact of Task Context and Prior Knowledge During Applications Tasks

A plethora of meanings is conveyed by the word *context* in mathematics education. Two of these are *situation context*, the "context for learning, using and knowing mathematics" (Wedege 1999, p. 207) and *task context*, "representing reality in tasks, word problems, examples, textbooks, teaching materials" (p. 206). Although the influence of the situation context on student solutions cannot be denied, the focus here is on the effects of task context using the meaning above.

The location of mathematical tasks in meaningful contexts for both teaching and assessment purposes can be enriching according to Van den Heuvel-Panhuizen (1999) as accessibility is enhanced contributing to "transparency" (p. 136) of tasks, and by providing students with solution strategies inspired "by their imagining themselves in the situation[s]" (p. 136) portrayed in the tasks. On the other hand, the use of tasks embedded in familiar contexts is not always supportive of students' solution attempts and may also create difficulties, particularly in assessment. Students sometimes refuse to engage with the intended mathematical interpretation of the problem by appealing to plausible alternative realistic scenarios that resolve the task non-mathematically (see Gravemeijer 1994). At other times they ignore the task context entirely and therefore exclude their "real-world knowledge and realistic considerations" (Van den Heuvel-Panhuizen 1999, p. 137).

Results from a study by Busse and Kaiser (2003) indicate that task context effects can be very individual and unpredictable. They found that at times emotional involvement with issues which students associated with the situation portrayed in a task context had a distracting effect. A rich store of knowledge about the task context also can hinder as it expands the real-world associations a student is able to activate. These extraneous associations were occasionally used by students to accept incorrect results when alternative realistic scenarios were suggested to explain a spurious result. Other students in their study reported the use of contexts of interest to them as being highly motivational, whilst others used knowledge of the task context to correctly verify results. Busse (2011) concludes that it might be useful "to use the notion of *contextual idea* … to indicate the mental representation of the real-world context *offered* in the task" (p. 42). He points out that such ideas are dynamic, changing as the student works on the task.

Abstraction Within Versus Abstraction Away from the Task Context

Students often have great difficulty formulating adequate mathematical representations of applications tasks for a variety of reasons. The traditional view of the mathematisation of a task context necessitates the extraction of the inherent mathematics from the situation through a process of abstraction. An alternative viewpoint involves "abstracting *within*, not *away from*," the task context (Noss and Hoyles 1996, p. 125). In approaching the task, students are viewed as having the potential to activate a web of connections between the situation described in the task statement, their mental models of that situation (both contextual and mathematical) (see Fig. 1) and the written mathematical model which they construct (Noss and Hoyles 1996).

As applications tasks are presented to students as written text, text comprehension strategies come into play. Nathan et al. (1992) theorise that for the comprehension and solving of worded problems to be successful three mental representations of the problem need to be constructed (a) a *textbase* from the textual input in the problem statement, (b) a *situational model* of the events described in the *task statement* and inferred or elaborated from it using the task solver's general knowledge base, and (c) a model of its mathematical structure (the *problem model*). Thus, to understand an applications task in order to construct a *mental situational model* (Nathan et al. 1992), a student must possess sufficient resources to comprehend the situation described

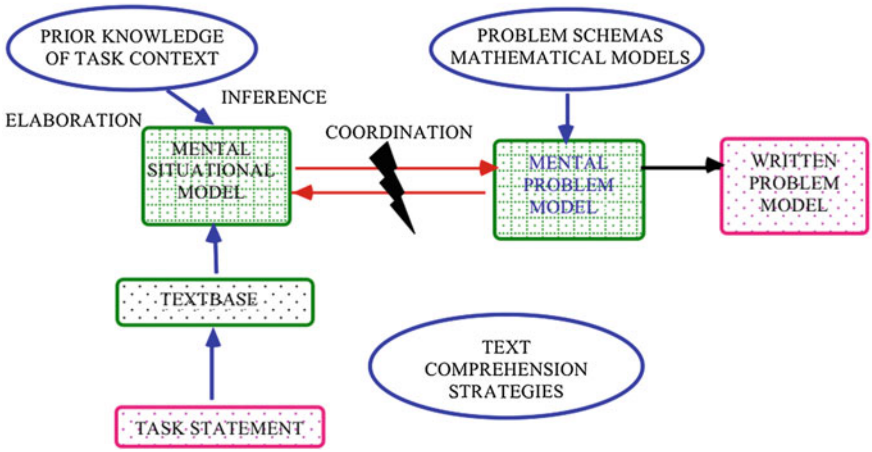

Fig. 1 Abstraction within or away from task context (Stillman 2002b)

together with the appropriate strategies to generate the necessary *inferences* and *elaborations* to fully specify the situation being modelled or mathematised. The situational model draws on the student's prior knowledge to fill in the gaps in the description presented (see Fig. 1). Prior knowledge of task contexts can be from (a) vicarious experiences in other academic subjects (*academic* knowledge), (b) general encyclopaedic knowledge of the world (*encyclopaedic* knowledge), or (c) truly experiential knowledge developed from personal experiences outside school or in practical school subjects (*episodic* knowledge). Episodic prior knowledge is personal but it may be derived directly from actions or from observation of actions. This derivation can have implications for the role of episodic prior knowledge in model formulation (see Stillman 2000, for further details).

According to Nathan et al. (1992), a set of *problem schemas* guide construction of a *mental problem model* from the mental situation model. This is where abstraction away from the context can occur with the mental situational model becoming divorced from the mental problem model instead of the two being integrated if coordination fails (i.e., jagged arrow in Fig. 1). If this occurs, the task solver could use mathematical procedures that are invalid within the task context. "There is not necessarily any smooth transition from the situation model" to the problem model and this might not even be possible if the student does not possess the necessary mathematical schemas, selects inappropriate ones, or a mathematical representation of the relationship involved does not exist (MacGregor and Stacey 1998, p. 58). In these instances the student must recognise the need to discard the situational model (MacGregor and Stacey 1998), make new inferences, and construct another model.

In a study of a group of Australian senior secondary school students (Years 11 and 12) solving applications tasks (Stillman 2002a, b), many students appeared to use a general problem solving schema rather than specific schemas when attempting the tasks, except when the problem was recognised as parallel to one they had attempted previously. Different students in the study reasoned within the task context for the purposes of (a) initial comprehension and problem representation, (b) progressing throughout the solution, (c) verifying the final solution and/or (d) recovering from errors. Other students ignored or backgrounded extra-mathematical knowledge. This is a sensible tactic with injudicious problems where the context is not authentic to reality or in problems where the context acts as a border around the mathematics and the two are completely separable. However, even though a task may be appropriately classified as a border problem, in practice the separation process may not be obvious for a student inexperienced in applying mathematics to real situations. Mathematically equivalent contextualised and decontextualized forms of such tasks give rise to quite different solution success rates. The contextualised task involves sorting out the order in which contextual cues are to be applied, determining relevancy, coordinating and integrating. The integration of information is the critical aspect of the difference in cognitive demand of the two task versions which may account for differences in success rates in mathematically equivalent contextualised and decontextualized task forms when mathematical and language competency are adequate (see Stillman 2001, for examples). A student who was able to abstract the mathematics away from the contextual detail would not have to address the high cognitive load that is imposed by the need of students with little previous experience in doing so, having to integrate data cues.

Engagement with the Task Context

The impact of engagement with the task context of these applications tasks on the students' performance on the tasks was also investigated (see Stillman 1998b). Moderate to high engagement with a task context was not often associated with poor performance which was more likely to be associated with no to low engagement. High engagement with task context was not a necessary condition for success as the degree of engagement necessary for success appeared to be task specific. Students identified a sense of realism and having an objective to work towards as facilitators of their engaging with task context. Many students in the study were unable to engage with the context of an applications task to any significant degree and only a few of these students were successful at solving them. Engagement with task context alone was not of sufficient explanatory power to account for all the patterns in the data. Other factors clearly came into play. To develop the meta-knowledge associated with the successful modelling of situations in their environment, students need tasks that require them to engage with the context in order to solve them successfully. They need experience in abstracting within, not away from the situation described in the task statement so engagement with a task context continues throughout the solution process. This requires the setting of tasks that allow this and the modelling of this process by teachers.

Task Accessibility

A cognitive/metacognitive framework developed in Stillman (2002a) proved useful in identifying and examining the conditions that facilitated or impeded task access for the students in the study through an analysis of students' responses to the tasks (see Stillman 2004). When the conditions facilitating task access were examined some conditions reduced the difficulty level of a task for particular students whilst others were related to reduced complexity of a task. Personal conditions that reduced difficulty were personal attributes of the task solver such as being able to visualise, possessing metacognitive knowledge that encouraged task access, or prior knowledge of the task context. Other conditions reducing task difficulty were attributes of the task that were susceptible to individual variation when a particular student interacted with the task (e.g., how recent a particular piece of mathematics required in the task had been studied). On the other hand, facilitating conditions associated with tasks being of lower complexity occurred when particular task attributes were present [e.g., the presence of salient cues, Kaplan and Simon (1990), in the form of trigger words or visual features]. Similarly, impeding conditions that increased the difficulty of a task for particular individuals often resulted from the interaction of a student's personal attributes with the attributes of the task (e.g., reluctance to make assumptions, interference from prior knowledge) but sometimes were purely personal (e.g., possessing metacognitive personal knowledge that discouraged access). Impeding conditions that were associated with increased complexity of the task were task attributes such as the mathematics or the goal of the task not being obvious, the need to integrate given and derived contextual information in order to construct a mental representation of the situation described in the task, or the need to make assumptions in order to formulate a mathematical model.

Task difficulty varied from student to student whilst task complexity was fixed as it was determined by task attributes. This is in agreement with Williams' (2002) distinction between these two terms. It is foreshadowed, however, that these attributes may be related to particular solution methods rather than the task per se (e.g., one solution approach may require a deeper level of integration of information than another). Personal attributes of the student also appear to act as intervening conditions between task complexity and task difficulty. These would explain the different consequences that occur (e.g., whether or not impeding conditions were overcome) when different students attempt tasks of the same complexity.

For students to benefit from facilitating conditions in applications tasks, they need: a well-developed repertoire of cognitive and metacognitive strategies as well as a rich store of mathematical concepts, facts, procedures, and experiences; vicarious general encyclopaedic knowledge of the world and word meanings; and truly experiential knowledge from personal experiences outside school or in more practical school subjects. In particular, a variety of retrieval, recognition, mental imaging, perceptual, and integration strategies together with metacognitive strategies for monitoring, regulating, and coordinating the use of these cognitive strategies is necessary (see Stillman 2004, for further details). The tasks used in this study were highly reading-oriented and thus, as suggested by Nathan et al. (1992), relied on students (a) accessing a good store of relevant prior knowledge for generating the inferences and elaborations necessary for understanding the situation fully and (b) having good comprehension skills to enable the student to specify a valid problem model for the task through the application of mathematical procedures. In some instances use of both cognitive and metacognitive strategies was enhanced by students (e.g., in an ice hockey task) pretending to be in the situation (cf. Van den Heuvel-Panhuizen 1999). This facilitation of access was also enhanced by the development of metacognitive knowledge which encouraged students to engage with the task (e.g., knowledge of task structure that has facilitated access to applications tasks for this task solver previously). However, once a modest degree of skill has been achieved in accessing complex tasks such as applications, coordination and integration of multiple representations, further cues, and mathematical processes and procedures become critical as the solution attempt progresses.

Failure of a student to possess a well-developed strategic repertoire or rich store of mathematical, encyclopaedic, semantic, or experiential knowledge can lead to conditions that facilitate one student's task access becoming impeding conditions for another. For example, a task about road construction, with apparent obvious mathematical and contextual cues for one student, may be inaccessible for another who does not have the appropriate knowledge base or fails to activate an appropriate one because of a poorly developed strategic repertoire. If the student does have appropriate knowledge and strategic bases but fails to activate either initially, an initial period of difficulty may be experienced but then overcome. At other times, particular attributes of an applications task such as unusual wording or the required mathematical model not being obvious can impede access. These difficulties can be overcome by students possessing and activating a well-developed strategic store together with an appropriate knowledge base. A wide variety of cognitive strategies that include information organising and representing, attention focussing, and visualising are necessary for overcoming the potential array of impeding conditions a student may encounter in attempting to access an applications task. Effective strategy use is enhanced by an equally rich and varied store of metacognitive strategies (Stillman 2004).

Modelling in Secondary Schooling

Modelling in schooling has two concurrent purposes—(a) to solve a particular problem at hand, and (b) over time to develop modelling skills, that empower students to describe and solve problems in their personal and social worlds. These purposes have characterised my work in the field. Figure 2 provides a useful launching point for issues discussed in relation to these. The task begins with the

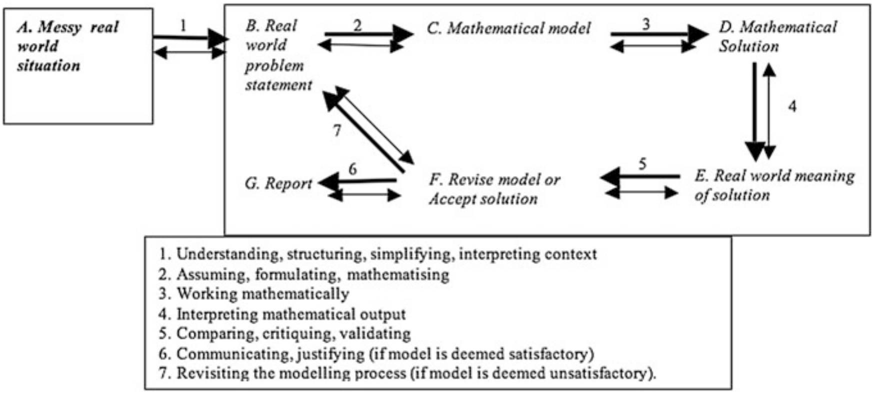

Fig. 2 Modelling cycle from Stillman et al. (2007)

messy real world situation [A]. The respective entries B-G represent stages when particular products (e.g., mathematical model [C] or decisions about acceptance or rejection of the model [F]) have been produced in the modelling process. The thicker arrows signify transitions between stages. The overall solution process is described by following these arrows clockwise around the diagram from the top left. Either the successful modelling outcome is reported or a further cycle begins if evaluation indicates an unsatisfactory solution. The kinds of mental activity modellers engage in as they attempt to undertake the transitions are given by the broad descriptors of cognitive activity 1–7.

The light double-headed arrows emphasise that thinking within the modelling process is neither linear nor unidirectional as confirmed empirically (e.g., Borromeo Ferri 2006, Oke and Bajpai 1986). The light arrows indicate reflective metacognitive activity as widely recognised and articulated by many researchers (e.g., Stillman 2011; Tanner and Jones 1993). Such reflective activity can look both forwards and backwards with respect to stages in the modelling process. The double-headed arrows are indicative rather than exhaustive as in theory they connect every pair of stages but diagrammatic clarity precludes inclusion of them all as was done previously in Stillman (1998a, p. 145) where a regulatory mechanism was included in a process rather than a product diagram as here.

Such a modelling diagram serves many purposes as articulated in Stillman and Galbraith (2012). For example, it defines and identifies key foci for research with respect to individuals learning mathematical modelling. Thus, with colleagues I have developed a research tool from the diagram and researched factors seen as blockages to progress, when students have difficulty in making transitions between stages as will be overviewed below. I now briefly indicate how my work relates to aspects of the structure in Fig. 2.

Issues in Formulating Models

A key part of the modelling process is the transition between stages B and C. This transition has been known as a "bottleneck" in modelling for quite some time (e.g., Hickman 1986). Negotiating this transition is the most demanding part of many modelling tasks. It separates modelling as real world problem solving from other educational approaches using the term 'modelling'. Helping students make this transition is a continuing major teaching and research priority. Blum and Leiß (2005), as cited in Leiß and Wiegand (2005) add a 'real model' between B and C. Clearly, there is a connection to worded applications tasks abstraction models that were based on text comprehension models. As "in applied mathematics one does not distinguish a real model from a mathematical model, but regards the transition from real life situation into a mathematical problem as a core of modelling" (Kaiser 2005, p. 100), my colleagues and I have maintained the transition as shown. This has proved a useful basis for researching the transition nevertheless.

Development of a Framework for Identifying Blockages in Transitions

A new research tool was developed with colleagues from Fig. 2 and refined in attempting to address our goals of (a) identifying and classifying critical aspects of modelling activity within transitions between stages in the modelling cycle, and (b) identifying pedagogical insights for implementation through task design and organisation of learning. Initially, the tool consisted of an empty frame of the transitions between stages as shown in Fig. 2. Using preliminary data analysis an emergent framework was developed (Galbraith and Stillman 2006; Galbraith et al. 2007) for identifying potential places where student blockages could occur in these transitions. Details for the second transition are shown in Fig. 3. These are illustrated with data from tasks used at year 9 level. Generic elements are in ordinary type, instantiations for a particular task in small capitals. Continuing analysis of task implementations led to further refinements such as the construct, level of intensity

2. REAL WORLD PROBLEM STATEMENT → MATHEMATICAL MODEL

2.1 Identifying dependent and independent variables [TOTAL RUN LENGTH AND DISTANCE FROM CORNER]
2.2 Representing formulae in terms of 'knowns' [LENGTH EXPRESSED IN TERMS OF FIELD EDGE DISTANCES]
2.3 Realising independent variable must be uniquely defined [*X*-CANNOT BE DISTANCE FROM BOTH A AND B]
2.4 Making relevant assumptions [LINEAR MODEL APPROPRIATE EVEN WHEN DATA POINTS APPEAR TO FOLLOW CURVE]
2.5 Choosing technology to enable calculation [RECOGNISING HAND METHODS ALONE ARE IMPRACTICAL]
2.6 Choosing technology to automate formulae for multiple cases [LISTS HANDLE MULTIPLE X-VALUES]
2.7 Choosing technology to produce graphical representation of model [SPREADSHEET OR GRAPHING CALCULATOR]
2.8 Choosing to use technology to verify algebraic equation [RECOGNISING GRAPHING CALCULATOR FACILITY TO GRAPH *L* VERSUS *X*]
2.9 Perceiving a graph can be used on function graphers but not data plotters to verify an algebraic equation [GRAPHING CALCULATOR CAN PRODUCE GRAPH OF FUNCTION TO FIT POINTS – SPREADSHEET CANNOT]

Fig. 3 Second transition in Galbraith et al. (2007) framework of a blockage, to explain the robustness of particular blockages to change and to identify student or teacher interventions to overcome these (Stillman et al. 2010; Stillman 2011).

The Framework systematically documents activities and content for modeller competence in order to successfully apply mathematics at their level. The insights obtained into student learning can inform our understanding of how students act when approaching modelling problems. Furthermore, by identifying difficulties with generic properties, the possibility arises to anticipate where, in given problems, blockages of different types might be expected. This understanding in turn contributes to teacher planning and task design, in particular the identification of prerequisite knowledge and skills, preparation for intervention at key points if required, and scaffolding of significant learning episodes. For example, blockages of low intensity appear to be able to be resolved by students themselves engaging in genuine reflection so tasks can be designed to allow this to occur (see Galbraith et al. 2010). Blockages of high intensity, on the other hand, might need direct teacher intervention to facilitate revision of mental models by students (see Stillman 2011).

Assumptions and Role of Technology in Modelling

The modelling diagram raises two other areas that have evolved over time. The first is the role of assumptions. Assumptions were conceived originally as confined to the initial setting up of a mathematical model. Whilst they play a major role in formulation, it is now realised they occur throughout the whole modelling process (see examples in Galbraith 1996; Galbraith and Stillman 2001; Galbraith et al. 2010). Galbraith and Stillman (2001), for example, identified three different classes of assumption. Firstly, *assumptions associated with model formulation* are those which have traditionally been called ‘assumption’. Secondly, *assumptions associated with mathematical processes* are also identified as important in the solution process, for example, that domain requirements for mathematical functions invoked in the solution are satisfied by real world values existing within the problem context. Thirdly, *assumptions associated with strategic choices in the solution process* influence the progress of a solution being central in providing global choices to the modeller, and determining the direction of a solution path. Typically they are required when an interim result creates a temporary impasse unforeseen at outset. This can occur either from a mathematical impasse, or when evaluating a model against the real context. Increased sensitivity to the different types and pervasive nature of assumptions often highlight the possibility of a different approach when modelling.

The second area to benefit from reflection on Fig. 2 is the role of technology in modelling. Within an approach to modelling as mathematical content, appropriate use of technology is essential. With colleagues I have explored the intersections in Fig. 4. Galbraith et al. (2007) argue that for beginning modellers the mathematics required for solution needs to be within the range of known and practised

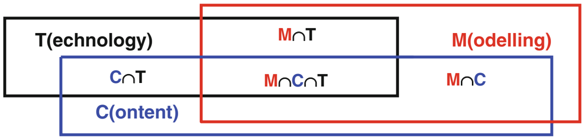

Fig. 4 Interactions between modelling, mathematics content, and technology (Galbraith et al. 2007)

knowledge and techniques even if it is not known exactly which mathematics is appropriate for a given situation. Additionally, the presence of a technology-rich teaching and learning environment impacts on the modelling, changing the mathematics accessible to students.

The use of electronic technologies in real-world settings such as analysis tools (e.g., graphing and CAS calculators) and real-world interfaces (e.g., image digitisers) can reduce the cognitive demand of these tasks. This can be achieved through supplementation and reorganisation of human thought (Tikhomirov 1981) by carrying out routine arithmetic calculations, algebraic manipulations, graph sketching, acting as an external store of interim results, or overlaying visual images (e.g., digital photographs of real phenomena) within an interactive coordinate system to facilitate analysis. However, use of these technologies has the potential to bring in a degree of complexity as they transform classroom activity and allow new forms of activity to occur. Regulation of this complexity affords teachers a further opportunity to mediate the cognitive demand of lessons involving real world contexts through careful crafting of tasks (Stillman et al. 2004). Figure 5 provides questions teachers could consider in relation to complexity of technology use to inform decisions regulating cognitive demand over time. These are particularly important in an introductory modelling environment where there is the intention of

COMPLEXITY OF TECHNOLOGY USE

General Attributes	***Dimensional Ranges***
LEVEL OF COMPLEXITY	simple … complex
Specific Attributes	
How many electronic technologies are involved?	1…many
How are these technologies used?	analysis tool, real-world interface
How much technological knowledge is required?	little…a lot
How easy is the technology to use?	easy…very difficult
How obscure is the choice of techniques?	fairly obvious…fairly obscure
How complex is each technique?	quite simple…quite complex
How complex is the combination of techniques?	all quite simple…most quite complex
How visible are the links between techniques?	fairly apparent…quite obscure
How many steps are involved?	1…many
How many features of the technology are involved?	1…many
What amount of guidance is given?	none…high
How much decision making is necessary?	none…a lot
How many representations can the technology provide?	1…many

Fig. 5 Complexity of technology use in real world settings (Stillman et al. 2004, p. 494)

a progression from "supplantive modelling where the modelling structure is supplied by the task setter to generative modelling where the students generate the modelling themselves" (p. 490).

Metacognitive Activity and Applications and Modelling

Figure 2 highlights metacognitive activity as permeating every aspect of the modelling process as proposed by Stillman (1998a, b). Earlier work (Andrews and McLone 1976) foreshadowed a regulatory mechanism validating processes and products throughout the modelling cycle. On-going research has confirmed this metacognitive activity. Stillman and Galbraith (1998) inferred from a previous study of applications tasks (Stillman 1993), that applications teaching should focus on reducing the time students spend on orientation activities by "developing cognitive skills that facilitate more effective problem representation and analysis, and by promoting the development of metacognitive strategy knowledge" (p. 185) to facilitate appropriate decision making during orientation. Since this time there has been a limited number of studies in the area.

Three types of metacognitive failure during problem solving were identified by Goos (2002), namely, (a) *lack of progress*, (b) *error detection*, and (c) *anomalous results*. She called these red flag situations. In mathematical modelling such red flags may be triggered by incorrect mathematics, or outcomes that, while mathematically accurate, are inconsistent with real world aspects of the problem. Goos typified three prevalent forms of metacognitive failure: (a) *Metacognitive blindness* when a red flag situation is not recognised so no appropriate action is taken; (b) *Metacognitive vandalism* when a perceived red flag results in drastic actions that fail to address the difficulty but alter or invalidate the problem itself; (c) *Metacognitive mirage* where solvers take unnecessary actions derailing a solution, perceiving a non-existent difficulty. Two more classifications have been added from our work (e.g., Stillman et al. 2010). *Metacognitive misdirection* describes a potentially relevant but inappropriate response to a perceived red flag that represents inadequacy, not vandalism. *Metacognitive impasse* occurs when progress stalls, and no amount of reflective thinking or strategic effort by the problem solver (s) alone is able to release the blockage. All five forms of metacognitive failure have been identified in my modelling work with students (see Stillman 2011).

Meta-Metacognition

Given that metacognitive activity is concentrated at the transitions in Fig. 2, how pedagogy addresses the fostering of associated metacognitive competencies is crucial to producing consistently able modellers. A key teacher task in supervising mathematical modelling is monitoring progress of individuals or groups, and intervening strategically. One needs to evaluate student performance of metacognitive activities—whether, for example, a student is unwilling to revise a mental model despite mounting evidence this is needed. In considering if student metacognitive activity is appropriate, or if appropriate is adequate, teachers are reflecting on metacognitive activity itself, both situation specific and with respect to its role in the overall modelling process; that is, they may be thought of as undertaking mental activity that is *meta-metacognitive* in nature (Stillman 2011).

How teachers generally undertake such *meta-metacognition* in relation to student activities and subsequently act is crucial to how mathematical modelling is nurtured or stifled in their classrooms generally. Students' capacity to develop skills in making transitions between modelling stages and to release blockages, depends critically upon how they are facilitated and supported in learning and applying the modelling process, and the metacognitive strategies central to it. Subsequently this depends upon the perceptiveness and skill with which teachers assess, mediate, and provide for student metacognitive activity. This extends beyond intervening to help with the solution of a specific problem, to ensuring that intervention also contributes to the long term goal of developing modelling competency.

Conclusion

Throughout this paper I have focused on issues that are both productive and challenging that I have researched with respect to mathematical applications and modelling in secondary schooling. By conducting research in classrooms about issues perceived by teachers as of concern I have endeavoured in my research program to close the gap alluded to by Blum (1993). In addition, I have begun researching the new generation of teachers in whose hands is future mainstream classroom instruction. By understanding how these future teachers are formed mathematically and pedagogically might give use more purchase on scaling up changed practices throughout educational systems.

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Mathematics Competition Questions: Their Pedagogical Values and an Alternative Approach of Classification

Tin Lam Toh

Abstract In this paper, it is argued that the role of the various mathematics competitions could be expanded beyond helping the nation in identifying and developing the mathematically gifted students. Through an examination of some mathematics competition questions, it was identified that these competition questions could serve to help the general student population to (1) acquire mathematical problem solving processes through engaging in or developing a problem solving model; (2) learn mathematics beyond the constraint of the school mathematics curriculum; (3) deepen students' understanding of school mathematics; and (4) acquire mathematical techniques rendered obsolete by evolving technology. With the availability of vast resources on competition questions, an alternative approach to classifying competition questions based on the functions they could serve in the usual mathematics classroom is proposed.

Keywords Mathematics competition · Problem solving · Classification

Introduction

The origin of mathematics competitions is not easy to trace. As early as the year 1885, it was reported that a mathematics competition for primary school students was held in Romania. Within a few decades after the First International Congress of Mathematicians in 1897, several countries began organizing their own mathematical competitions (Kenderov 2006).

The most prestigious mathematics competition, the International Mathematical Olympiad (IMO), was first held in 1959 in Romania. With the exception of 1980, this competition has taken place annually. It first emerged as a small-scale mathematics

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contest with participants from several European countries and gradually transformed to a large-scale contest with more than 80 participating countries. According to Reiman (2005a, b, c), “wherever mathematics education has reached a moderate level, sooner or later the country has turned up at the IMO” (p 1).

To better identify talents and prepare those mathematically gifted for the IMO, mathematical societies were set up in various countries. These societies organized various mathematical competitions at the national and regional levels. Such activities appear to have impacted the mathematics education in these countries, at least for the mathematically gifted students. According to Reiman (2005a, b, c):

> [M]athematics Olympiad has enriched the publishing activity in several countries. Math-clubs have been formed on a large scale and periodicals have started… if the educators regard the competitions not as ultimate aims, but as ways to introduce and endear pupils to mathematics, then their pedagogical benefit is undeniable (p 1).

Roles of Mathematics Competition

Literature abounds with studies on the use of mathematics competitions in developing the mathematically gifted students (for example, see Bicknell 2008; Campbell and Walberg 2010; Kalman 2002). Mathematicians, competition trainers and mathematics educators are generally of the opinion that mathematics competitions are instrumental in improving “the mathematical thinking and technical ability in solving mathematical problems” (Xu 2010, p. v) for the higher ability students. Researchers have also identified the possibility for cooperation across mathematics and other disciplines at the international level through various forms of mathematics competitions (France and Andzans 2008).

Expanding beyond the role of mathematics competition on the mathematically talented, some researchers propose the use of competition as an enriching problem solving experience for the general student population (for example, Grugnetti and Jaquet 2005). Some organizers of mathematics competitions have transcended the objective of talent identification and development; they are making the competitions more inviting to the general student population by designing their competition around the core content of the common school mathematics (for example, Swetz 1983).

Mathematics Competition Questions and Their Classification

Over the years, many mathematics competition questions have been generated and volumes of compilations of these questions from various regions and countries have been published. As these mathematics competitions generally do not have a fixed “syllabus”, mathematicians have attempted to design the “curriculum” of these competitions based on the content of these questions from the past years. Books designed to help students learn these specific elementary mathematics content have been published (for example, Xu 2010, 2012).

As an illustration, the three volumes of IMO questions (Reiman 2005a, b, c) consist of almost two hundred problems from the competition from 1959 to 2004, together with their official solutions. It is an easy reference for interested readers to identify the questions based on the year of the competition. Effort has been made by Reiman (2005a, b, c) to identify the elementary mathematical content knowledge required for the individual competition questions by the inclusion of a Glossary of Theorems with reference to the number of the individual questions that require the theorem. From one mathematician's perspective, "the overwhelming majority [of the problems] is interesting and rewarding; together they more or less cover the usual syllabus—chapters of elementary mathematics." The solutions of the IMO problems generally require creative multiple-step strategies and additional elementary mathematics content knowledge, which is not usually covered in the school mathematics curriculum.

The Canadian Mathematical Society (CMS) published a compilation of problems and solutions from the Canadian mathematical Olympiad (CMO) from 1969 to 1993 (Wright 1993), and the problems and solutions of the later years are available online. The problems were classified according to the year of the competition. In the publication, the executive director commented that it is their hope to "challenge and inspire each new generation of talented young mathematicians" (Wright 1993). The problems of CMO, unlike the IMO, are of varying level of difficulty, and the solutions of these problems are of varying length—from one-step solution to those requiring multiple steps. Some problems require elementary mathematics content knowledge that is beyond the school curriculum while others are slight extensions of the usual school mathematics curriculum.

Classification is the process of assigning elements or units to classes according to some criteria. According to the literature on classification (for example, Soergel 1985), an object can be classified according to its content or the functions it serves. The publication of the compilation of the questions from the IMO and other mathematics competitions was an initial step to classification.

The classification of the competition problems begins with the functions such a classification could serve in mind. It is evident that the majority of the compilation of the past competition questions is done with the mathematically gifted in mind.

In Singapore, the "five-year series" competition guide (compilation of past five years of mathematics competition problems from the Singapore Mathematical Olympiad) "serve[s] as a guide and an exercise for students who are preparing for future [competitions]. As most of the problems are more challenging than those encountered in the classrooms, this book is a useful supplement for mathematics students." (Chua et al. 2011). While publishing the competition questions of the previous years classified according to the years of the competition could primarily serve a complete record purpose and to prepare the mathematically gifted students for the competition, the editors felt that these competition questions can be used beyond merely preparing students for mathematics competitions.

> For the skilful mathematics teachers who intend to develop mathematical problem solving skills and heuristics, the range and repertoire of problems presented can be adopted and integrated into the teaching and learning process. The mathematical origins and context of some of these problems also provide the potential for further extensions and which can be developed into mathematical investigations for mathematics project work (Chua et al. 2011 p. i).

However, the subsequent pages of the book do not seem to offer any suggestion as to how these problems can be integrated into classroom teaching and learning of mathematics.

Other countries such as Hong Kong have published their mathematics competition questions classified by the year of the competition. Larger countries such as China have also published their competition questions by the regions and the year of the competition (for example, Xiong and Lee 2007).

Mathematicians have examined all the past years' problems of IMO to create an unofficial syllabus of the competition. The elementary mathematics content knowledge required for the competition has been identified through this process. Besides identifying the content for the competition, content that has never occurred in the competition was also identified. For example, "there are no problems at all from the theory of probability, for example, and complex numbers hardly show up" (Reiman 2005a).

Based on the "syllabus" identified for the IMO, classification of competition questions by topics has emerged. For instance, Vietnam (2010) has classified selected problems from the Vietnamese Mathematical Olympiad (VMO) by topics: Algebra, Analysis, Number Theory, Combinatorics and Geometry. Classifying the questions by topics helps the potential competitors to focus on the individual topics more easily, although it is likely that one might not have seen the complete set of competition problems. In the preface to the Vietnamese compilation, it was stated that the publication (Le and Le 2010) not only took into consideration the classification of the topics but also the "level of difficulty". However, there was no justification on how the "level of difficulty" was determined. As most of the questions are beyond the usual school mathematics curriculum and are generally difficult for average student population, it is not unreasonable to assume that this book was published with the higher achieving students in mind.

Classification of the mathematics competition questions by the year of competition or by the essential elementary mathematics content knowledge described above generally serve the very restrictive purpose of preparing the mathematically talented. Undeniably, many mathematics competition problems are beyond the ability of typical students as the solution of these problems may require creative multiple-step strategies and additional elementary mathematics content knowledge (not usually covered in the usual school mathematics curriculum).

An examination of the huge resource of mathematics competition questions over the past years clearly reveals that there are significantly many questions that can be of pedagogical value for the general student population in the usual mathematics lessons. Some of these problems are slight extension of the school mathematics curriculum and could be useful even to deepen students' understanding of school mathematics. In fact, it is not an exaggeration to state that these questions helped teachers to identify the blind-spots in their teaching—aspects of mathematics teaching aligned to the spirit of the mathematics curriculum that teachers might have overlooked in their teaching.

This paper proposes an alternative classification of these competition questions, tapping on the pedagogical values of these questions through examining some past year competition questions, mainly from the Singapore Mathematical Olympiad (SMO), and suggestions for using these questions in the usual classroom teaching are proposed. An alternative classification of the mathematics competition questions is offered based on the mathematical purposes the questions can serve.

Problem Solving Experience

Some educators and mathematicians are beginning to use mathematics competition questions beyond the mere objective of identifying and developing the mathematically talented for the IMO or other mathematics competitions. Many mathematics competition questions could be suitable to expose students to the rich process of mathematical problem solving, which is the heart of the mathematics curriculum in many countries around the world. According to the Singapore Mathematical Society, besides identifying talents in mathematics, one of the main objectives of the Singapore Mathematical Olympiad is to "enhance students' [i.e. the general student population] problem-solving skills that they learnt in school" (Tay et al. 2011). Holton (2010), through imparting the mathematical content knowledge required for the IMO on discrete mathematics, introduced mathematical problem solving processes (which he believes to be the essence of mathematics) and used scaffoldings by providing hints on the use of appropriate problem solving heuristics for selected IMO problems on discrete mathematics. Motivated by this approach, Toh et al. (2011) developed a problem solving module (teaching *about* problem solving), which emphasizes the processes of problem solving. The seventeen problems in the module were each selected to highlight different aspects of problem solving to be introduced at different junctures of the course. Many of these problems were selected from mathematics competition questions (for the complete collection of the problems, see http://math.nie.edu.sg/mprose).

Most mathematics educators would agree that a student learning mathematics requires a problem solving model to which he or she can refer, especially when progress in solving a mathematics problem is not satisfactory. Good mathematics students would presumably have built up their own models of problem solving.

A problem solving model that is made explicit to students should be helpful in guiding them in learning mathematics and in regulating their attempts in solving a (non-routine) problem. Even a good mathematics student may find the structured approach of learning a problem solving model useful. As Schoenfeld (1985) recounts in the preface to his book *Mathematical Problem Solving* about Polya's book *How to Solve It* (1945):

> In the fall of 1974 I ran across George Polya's little volume, *How to Solve It*. I was a practising mathematician … My first reaction to the book was sheer pleasure. If, after all, I had discovered for myself the problem-solving strategies described by an eminent mathematician, then I must be an honest-to-goodness mathematician myself! After a while, however, the pleasure gave way to annoyance. These kinds of strategies had not been mentioned at any time during my academic career. Why wasn't I given the book when I was a freshman, to save me the trouble of discovering the strategies on my own?

Many problems in the various mathematics competitions provide a suitable platform to enrich students' learning of mathematical problem solving and could help them acquire a problem solving model or facilitate them to develop their own model.

The characteristics of this category of problems that are suitable for mathematical problem solving are those that "use little knowledge and have many variations. They are flexible and diverse" (Yu and Lin 2010). This is illustrated with examples from the past years problems from the junior category (ages 13 and 14) from the Singapore Mathematical Olympiad (SMO) (Fig. 1).

Although Q1 and Q2 can be classified under the advanced undergraduate mathematics topics on Number Theory, these problems can be solved without resort to sophisticated mathematical content knowledge; students can solve these problems using problem solving processes (which are emphasized in the mathematics curriculum of countries that recognize the centrality of problem solving in mathematics). In other words, students can solve these problems even without learning any additional content beyond their school mathematics curriculum.

The subsequent discussion is based on Polya's problem solving model (although any other sound problem solving model is equally viable). The essential features of Polya's problem solving model are shown in Fig. 2.

Consider how the above questions can be solved by problem solving processes instead of using content knowledge on Number Theory. In solving Q1 and Q2, *understanding the problem* is the first important stage. In these two examples, students need to understand the terms *last two digits* in Q1 and *how many zeros* in Q2, ideas that are rarely emphasized in the usual school mathematics curriculum.

Further, these two problems provide the opportunity to emphasize *devising a plan* and *carrying out the plan* (using appropriate heuristics). For example, a useful plan to solve either of the two questions would be to start from simpler (or smaller) cases, observe the pattern and make conjectures. These heuristics are transferable to many other problems from other topics in mathematics as well.

Q1. The last two digits of 9^{2004} is
(A) 21 (B) 81 (C) 09 (D) 61 (E) 01
(SMO 2004 Junior)

Q2. How many zeros does the number 50x 49 x 48 x … x 3 x 2 x 1 end with?
(A) 8 (B) 9 (C) 10 (D) 11 (E) 12
(SMO 2005 Junior)

Fig. 1 Two SMO questions for problem solving experience

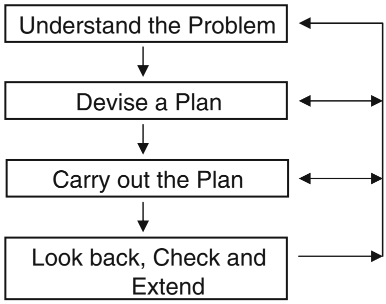

Fig. 2 Polya's problem solving model

A rigorous approach to solving Q1 and Q2 would be to use Number Theory knowledge: knowledge about modulo arithmetic for Q1 and the special formula $\sum_{k=1}^{\infty} \left\lfloor \frac{n}{5^k} \right\rfloor$ for the "number of zeros in n!" for Q2. If one demands that the rigorous approach is the only correct method, these two questions would not be appropriate for the general student population. On the other hand, the "heuristic" approach would provide an enriching learning experience for students, through which they could acquire the skills of mathematical problem solving.

Anecdotal evidence in mathematics classrooms shows that generally teachers do not emphasize *looking back* (e.g. checking the reasonableness of the solution) in solving mathematics problems. Some mathematics competition questions provide rich opportunities for this process. Consider Q3 and Q4 in Fig. 3, which are from SMO Senior (for ages 15 and 16).

The legitimacy of cancelling identical algebraic terms from both sides of a given equation and $\sqrt{a^2} = a$ for all real values of a are two very common misconceptions among secondary school students (Toh 2006). These two problems demonstrate the importance of checking the reasonableness of one's answers obtained from standard mathematical procedures. This type of problems provides the opportunity to stress the importance of looking back at one's solution of a mathematics problem. Such a reflection could create a cognitive conflict in students and thereby challenge them to

Multiple Choice Question

Q3. How many real numbers x satisfy the equation $\dfrac{x^2 - x - 6}{x^2 - 7x - 1} = \dfrac{x^2 - x - 6}{2x^2 + x + 15}$?

(A) 4 (B) 3 (C) 2 (D) 1 (E) 0

(SMO 2004 Senior)

Q4. Let $a < 0$. Find $\sqrt{a^2} + \sqrt{(1 - a^2)}$ in terms of a.

(A) 1 (B) -1 (C) $2a - 1$ (D) $1 - 2a$ (E) None

(SMO 1996 Senior)

Fig. 3 Two multiple-choice questions from SMO re-examine their understanding of concepts and skills acquired in school mathematics. Teachers could further convert this category of questions into appropriate mathematical tasks to enrich students' learning experience.

Learning Beyond the School Mathematics Curriculum

Students' involvement in the various mathematics competitions has resulted in the emergence of additional training lessons to prepare their students for mathematics competitions. For instance, in Singapore, these preparatory lessons have evolved into *mathematics enrichment lessons,* which more students (that is, students who are not preparing for the mathematics competitions) are allowed to attend.

As there is generally no fixed "syllabus" for these competitions, students are typically exposed to content knowledge beyond the usual school mathematics curriculum (but which is manageable to them). Challenging students to extend their learning beyond the limits of the school mathematics curriculum is an important step to getting them excited about the subject. This could arouse their interest and develop their confidence in the subject (Toh 2011). However, mathematics teachers might have neglected this aspect in their usual mathematics classrooms because of the large amount of mathematics content in the curriculum and limited curriculum time. The mathematics enrichment lessons could serve to fill this gap by providing students the opportunity to learn mathematics content that is beyond the curriculum. Figure 4 provides two examples of this category of questions.

Solving questions such as those in Fig. 4 requires either new techniques (as in Q5, Gauss' method of pairing) or new formulas (as in Q6), which most school students could manage. Interestingly, these questions demonstrate the "power" of mathematics as important tools to simplify cumbersome numerical and algebraic expressions.

Extension of learning beyond the curriculum should not be solely aimed at building up additional "cognitive resources" (Schoenfeld 1985) for the students. It can be seen as a motivation for the students to appreciate mathematics and a means to introduce them into advanced mathematical thinking, which is seldom emphasized in the school mathematics.

Q5. Let $f(x)=\dfrac{x^{2010}}{x^{2010}+(1-x)^{2010}}$. Find $f\left(\dfrac{1}{2011}\right)+f\left(\dfrac{2}{2011}\right)+f\left(\dfrac{3}{2011}\right)+\ldots\ldots+f\left(\dfrac{2010}{2011}\right)$.
(SMO 2010 Senior)

Q6. Find $\sqrt{14^3+15^3+16^3+\ldots\ldots+24^3+25^3}$.
(SMO 2010 Senior)

Fig. 4 Two competition questions that are typically beyond school curriculum

Deepening Students' Understanding of School Mathematics

This category of problems stresses skills or mathematical concepts that are usually not emphasized in the usual school mathematics curriculum, although the contexts and the mathematical content of the questions are within the school curriculum. Mastering these skills or concepts would necessarily deepen students' understanding and appreciation of school mathematics. Figure 5 shows a sample of four questions: three from the SMO and one from the Australian Mathematics Competition (AMC).

Students would already have learned angle and length properties of triangles at a very early age. However, officially the triangle inequality relation among the sides of a triangle is not usually tested. Testing the constructability of a triangle given three lengths extends the learning experience for students within the usual mathematics curriculum, hence the value of Q7.

Students usually solve or manipulate an algebraic equation procedurally. The focus of teaching algebra in the usual mathematics classroom is frequently procedural manipulation at the expense of student understanding (Lee 2005, p 44). It is thus not uncommon that many students may not have acquired an appreciation or deep understanding of school algebra. In this sense, Q8 introduces an alternative skill of dealing with mathematical equations—that of recognizing what each of the algebraic terms (including both symbol and operation) in an equation represents. This question compels the solver to re-examine the procedural algebra they have learned and hence

Q7. In a triangle ABC, it is given that AB = 1cm, BC = 2007 cm and AC = a cm, where a is an integer. Determine the value of a.
(SMO 2007 Junior)

Q8. Let a, b, c, d be integers and $(a^2+b^2)(c^2+d^2)=29$. Then the value of $a^2+b^2+c^2+d^2$ is ________.
(SMO 1996 Junior)

Q9. Find the value of $\dfrac{2007^2+2008^2-1993^2-1992^2}{4}$.
(SMO 2007 Junior)

Q10. While attempting to solve a quadratic equation, Christobel inadvertently interchanged the coefficient of x^2 with the constant term, causing the equation to change. She solved this different equation accurately. One of the roots she got was 2 and the other was a root of the original equation. Find the sum of the squares of the two roots of the original equation.
(AMC 1998)

Fig. 5 Four competition questions from SMO and AMC deepens their understanding of school algebra. Q9 demonstrates the power of algebra through inviting them to evaluate arithmetical expressions solely through algebraic identities without the use of calculating devices.

While students have learned how to solve a quadratic equation at the lower secondary level (age 13–14), they might not have been exposed to the analytic aspect of the algebraic procedure. Q10 challenges students to think out of the box and coerces them to examine the significance of *interchanging the coefficient of* x^2 *and the constant term*. In fact, this problem is also a rich problem to introduce students to appreciate the usefulness of problem solving model (Toh 2012).

Mathematical Techniques (Rendered "Obsolete" by Technology)

With the evolving school mathematics curriculum and related pedagogy for classroom teaching, some mathematical techniques within the usual mathematics curriculum could be rendered "obsolete". This is especially true for tasks that can easily be completed using technology. It is not surprising that mathematicians and mathematics educators lament that "the official curriculum in mathematics today is far from the level of the 1980s" (France and Andzans 2008).

However, the techniques required to complete some mathematics problems (without the use of technology) could yield creative use of the standard mathematical algorithms or deeper understanding of school mathematical knowledge. Figure 6 provides a set of this category of questions.

In the situation where computing tools are allowed for all high-stake national examinations, it is understandable that Q11 and Q12 will not be examination items, as they would be meaningless calculator pressing activities. However, without the use of calculators, these two questions require the creative use of algebraic manipulation. Solving this category of questions provides students with an opportunity to examine their algebraic procedures critically in appropriate situations.

Q11. Suppose $a = \sqrt{6} - 2$ and $b = 2\sqrt{2} - \sqrt{6}$. Then

(A) $a > b$ (B) $a = b$ (C) $a < b$

(D) $b = \sqrt{2}\,a$ (E) $a = \sqrt{2}\,b$

Q12. Simplify $144\left(\sqrt{7+4\sqrt{3}} + \sqrt{7-4\sqrt{3}}\right)$.

Fig. 6 Sample of three SMO questions that are considered "obsolete"

An Alternative Approach of Classifying Mathematics Competition Questions Based on Their Pedagogical Values

Classifying mathematics competition questions by the years of the competition provides a good avenue for one to identify the source of the competition. It is easy for mathematicians, educators and even students to identify the original source of the question. Classifying the competition questions by topics is a natural progression from the classification of the questions by years, as the classification further streamlines the topics of mathematical content knowledge required to prepare students for the competition. Generally, these two approaches of classification help prepare the mathematically gifted for the competition, but might not be meaningful for the general student population.

As described in the previous section, there can be significant pedagogical value in mathematics competition questions for the general student population in the usual mathematics classrooms that mathematics educators and mathematicians are beginning to explore. This means the enormously vast resource on mathematics competition questions should be "user-friendly" so as to be of greater benefit to both the general student population and the teachers (who do not necessarily teach the mathematically talented). Thus, an alternative classification of these problems is proposed based on the mathematical and learning purposes they can serve in the usual mathematics classroom for the general student population.

Category 1: Questions that provide rich learning experiences related to the various aspects of mathematical problem solving experience (exemplified by Q1, Q2, Q3 and Q4 of Figs. 1 and 3). A typical feature of this category of questions is that the question does not require excessive mathematical content knowledge.

Category 2: Questions that stretch students beyond the content of the school curriculum (exemplified by Q5 and Q6). The content of the questions is not covered in the school curriculum, but the questions demonstrate the "power" of mathematics for example in the simplification of complex mathematical expressions.

Category 3: Questions that deepen students' understanding of school mathematics (exemplified by Q7, Q8, Q9 and Q10). The content of the questions is within the school mathematics curriculum but not the main focus, and the questions are rather dissimilar to the type of questions in the high-stake school examinations.

Category 4: Questions that emphasize particular mathematical techniques which are otherwise made obsolete by technology (exemplified by Q11 and Q12). The content of the questions are within the school mathematics curriculum. The questions are easily obtained using computing tools, but the answers are not easily forthcoming without these tools—creative use of mathematical content knowledge is expected.

According to the classical view of classification, categories used for classifying objects should be clearly defined, mutually exclusive and collectively exhaustive. In the four categories proposed above, it is clear that classifying the mathematics competition problems may not meet these criteria. In fact, as in any other classification in real life, the classifications will be much fuzzier than people might think (Rosch 1978). Indeed, it is "possible to name a population of objects that people would in general agree to call chairs that have no two binary features in common." (Bowker 1998). As an illustration, from one perspective, Q10 deepens students' understanding of solving algebraic equations. One could also argue that the same problem could also be used to develop students' problem solving skills (Toh 2012).

Further, it is likely that the proposed categories are not "exhaustive", as there are competition questions that do not fall under these categories. Mathematics educators can often identify the existence of a category of problems whose solutions require very specialized techniques that are generally not transferrable to a wide category of problems or do not enrich students' learning. In view of the purpose of such problems, it does not serve any purpose to further fine-tune the classification.

We would want to reiterate that the proposed classification is targeted at tapping the vast resource of mathematics competition questions for the daily mathematics classroom teaching, after recognizing the pedagogical value of some of the questions. The classifications can also provide food for thought for mathematics teachers on their pedagogy, and help teachers to identify the "blind spot" they might have overlooked in their mathematics teaching.

Conclusion

This paper introduces the idea that some mathematics competition problems can be useful in general mathematics classroom instruction to the general student population. Through identifying the broader purposes these problems can serve, an alternative classification of mathematics competition problems is proposed. Hopefully, this could spur further research into the usefulness of mathematics competition problems to the wider community beyond the mathematically gifted.

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What Does It Mean to Understand Some Mathematics?

Zalman Usiskin

Abstract Mathematical activity involves work with concepts and problems. Understanding mathematical activity in mathematics education is different for the policy maker, the mathematician, the teacher, and the student. This paper deals with the understanding of a concept in mathematics from the standpoint of the student learner. We make the case for the existence at least five dimensions to this understanding: the skill-algorithm dimension, the property-proof dimension, the use-application (modeling) dimension, the representation-metaphor dimension, and the history-culture dimension. We delineate these dimensions for two concepts: multiplication of fractions, and congruence in geometry.

Keywords Curriculum · Mathematical understanding · Fractions · Congruence · Mathematical concepts

Introduction

To understand mathematics *as a whole* would entail a discussion of the roles mathematics plays in everyday personal affairs, in schooling (e.g., as a sorter), in occupations, in other fields such as physics, and in its existence as a discipline studied for its own sake. In contrast, this paper is primarily concerned with what it means to understand *some* mathematics, which generally means to begin with a bit of mathematics and to subject it to detailed analysis, usually from the perspective of the learning of that bit.

Many individuals have discussed mathematical understanding and these pages could be entirely filled by a list of works on the subject. However, my perspective is slightly different from many who have written on the subject. It comes from the standpoint of a curriculum developer, from decades of writing materials for students

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that attempt to lead them to understand the mathematics they are being asked to learn. This perspective falls somewhere between Freudenthal's *Didactical Phenomenology of Mathematical Structures* (1983) and Hiebert and Carpenter's chapter on learning and teaching with understanding in the *Handbook of Research on Mathematics Learning and Teaching* (1992).

Understanding School Mathematics

Throughout the 20th century psychologists and mathematics educators wrestled with what it means to "understand" a bit of mathematics. In the 1960s, one definition of "learning" was "a change in behavior", which meant that understanding was allied with certain actions. This represented a behaviorist view of learning. Indeed, there is a common saying attributed to Confucius, which in English is sometimes translated as: "I hear and I forget. I see and I remember. I do and I understand." Yet we often hear it said that students can "do" certain mathematics but not understand what they are doing. This roughly parallels the difference between what in psychology are sometimes called *behaviorism* and *cognitivism*. We in education act both as behaviorists and cognitivists. We view "understanding" as something that goes on in the brain without external actions yet we want students to exhibit their understanding by responding to tasks we present before them. Specifically, as behaviorists, we want students to answer questions correctly and sometimes do not care how they got their answers. As cognitivists, we want to know what students are thinking as they work with mathematics and we ask students to show their work.

In the 1970s, Skemp (1976) used the phrases *instrumental understanding* and *relational understanding*, essentially meaning *procedural understanding* and *conceptual understanding*. He wrote, "I now believe that *there are two effectively different subjects being taught under the same name,* '*mathematics*'." Skemp's dichotomy is, I believe, now the most common broad delineation of what is meant by mathematical understanding.

My view of understanding evolved rather independently of Skemp. I agree with Skemp that instrumental and relational understanding are different but I do not agree that they are different subjects. I view them as different aspects of understanding the same subject. Also, I believe there are more than two aspects or types of understanding, as different from each other as Skemp's two types, but all different aspects of understanding mathematics. For reasons I explain later, I call these aspects *dimensions of understanding*.

Since we are speaking of *mathematical* understanding, it is helpful to have some idea of the extent of the subject. In this discussion, mathematics is an activity involving objects and the relations among them; these objects may be abstract or abstractions from real objects. The activity consists of *concepts* and *problems or questions*: mathematicians employ and invent concepts to answer questions and problems; mathematicians pose questions and problems to delineate concepts. So a full understanding of mathematics requires an understanding both of concepts and of problems and what it means to invent mathematics.

It is natural for mathematics educators to view mathematical understanding from the standpoint of the learner, whether that learning is for use in life, for use on a job, for personal enjoyment, or for a test. But the *full* or complete understanding of mathematics in schools requires more than the learning perspective. It includes the understanding of mathematics also from the standpoints of *educational policy* and the *teaching* of mathematics. For mathematicians, the understanding of mathematics includes an understanding from the standpoint of those who *invent* or *discover* new mathematics.

Educational policy towards mathematics includes the selection of content to be covered in school, who should encounter that content, and when. In the selection of content, a fundamental question concerns what constitutes mathematics. Is statistics mathematics? Is physics mathematics? Is formal logic a part of mathematics? In general, when if ever does applied mathematics cease to be mathematics? Should telling time be a part of the mathematics curriculum? What about reading tables of data or locating one's home town on a map of a country? What about doing a logic puzzle such as a Sudoku puzzle? What about a discussion of lucky numbers and favorite numbers and unlucky numbers? Is computing using a calculator *doing* mathematics or *avoiding* it? Is conjecturing *mathematics* or is it *proto-mathematics*, that is, not the real thing but leading up to the real thing. These questions bring out differences among us in what we think mathematics is, and differences in what we think is *real* or *good* mathematics. A full paper could be devoted to these questions but it is not the focus here.

The understanding of mathematics from the teacher's perspective overlaps the learner's perspective but is not the same, and is discussed at the end of this paper.

The understanding of the invention or discovery of mathematics from the mathematician's perspective has been the subject of many books, of which the classics by Hadamard (1945), Hardy (1940), and Polya (1962) are probably the most well-known, at least in the West. The understanding of mathematical invention would not be complete without consideration also of the inventors, mathematicians themselves, through the many available biographies. The recent book by Hersh and John-Steiner (2010) falls into this broad category.

On problem solving, Polya's *How To Solve It* (1957) has long been a seminal work. It is significant that the first of his four steps of problem solving is *understanding the problem.* Polya and others since have treated this subject in such detail that I have nothing significant to add. For this reason, in this paper, I concentrate on the understanding of concepts.

To exemplify understanding, a vast array of concepts might be considered, ranging from general concepts such as number, function, point, linearity, or structure, to specific concepts such as the mean of a set, the Pythagorean Theorem, the solving of a linear equation, and many concepts in between. Here I have picked two dissimilar concepts: *multiplication of fractions*, an arithmetic operation, and *congruence*, a geometric relation.

Finally, as the last demarcation of the topic to be discussed here, let us indicate what does *not* constitute understanding. We say that someone *does not understand* a bit of mathematics when that person acts *blindly* or *incorrectly* to the prompts in the situation.

First Example: Understanding the Multiplication of Fractions

Vocabulary

Mathematics is, among its many other attributes, a language of discourse. It is both a written language and a spoken language, for—particularly in school mathematics—there are words for virtually all the symbols. Familiarity with this language is a precursor to all understanding. A person cannot begin to understand multiplication of fractions without knowing what a fraction is and what it looks like, and that multiplication is an operation which, given two numbers, produces a third. The vocabulary of fractions is interesting and not at all trivial. In English, the word "fraction" has many different meanings, all used in classroom discourse: (1) a number between 0 and 1, as in "He earns only a fraction of what she earns"; (2) a number that is not an integer, as in "I want an answer that is a fraction"; (3) an indicated quotient of two numbers using a bar—or a slash /, as in "an expression of the form a/b"; and (4) in the glossary of the U.S. Common Core State Standards, "a number expressible in the form a/b where *a* is a whole number and *b* is a positive whole number". In general, dealing with the written and spoken vocabulary of a concept is an essential part of its understanding that transcends all aspects of that understanding.

Skill-Algorithm Understanding of Multiplication of Fractions

If a random person on the street would be asked, "Do you understand the multiplication of fractions?", a typical response might be, "Yes, you multiply the numerators and denominators to get the answer." To the world outside academia, understanding is often equated with getting the right answer. Knowing how to get an answer is the essence of the procedural understanding of the multiplication of fractions or any other concept. Because algorithms are often done (and supposed to be done) automatically, applying a procedure is often contrasted with understanding rather than considered an aspect of understanding.

However, there is much more to procedural understanding than merely applying an algorithm. With regard to the multiplication of fractions, the procedure seems very simple. If we are confronted with calculation (1),

$$\frac{2}{3} \times \frac{4}{5}, \tag{1}$$

we merely multiply numerators and denominators to obtain the product $\frac{8}{15}$.

However, the values of the numerators and denominators can alter what we do. In (1), change the 4 to a 6, as shown in (2),

$$\frac{2}{3} \times \frac{6}{5}, \tag{2}$$

and we may divide the 3 and 6 by 3 and thus get $\frac{2}{1} \times \frac{2}{5}$, and now multiply numerators and denominators to obtain the product $\frac{4}{5}$. Or we may divide the 3 into the 6 and write 1 and 2. Some people cross out the 3 and 6 in the process. These variants of the algorithm used in (1) are different enough to require days of instruction in a typical classroom.

Change the 4 in calculation (1) to a 3, and we think about it even another way.

$$\frac{2}{3} \times \frac{3}{5}, \tag{3}$$

We ignore the 3s (some people cross them out) and just write down $\frac{2}{5}$.

Change the $\frac{4}{5}$ to 4 and there is another algorithm.

$$\frac{2}{3} \times 4 \tag{4}$$

I multiply the 2 by 4 and write down $\frac{8}{3}$. Some students feel the necessity to replace the 4 with $\frac{4}{1}$ and then they treat the problem as if were of type (1) and multiply numerators and denominators to obtain $\frac{8}{3}$.

Change the $\frac{4}{5}$ to 60 and there is still another algorithm.

$$\frac{2}{3} \times 60 \tag{5}$$

Now we may divide 3 into 60 and then multiply the quotient 20 by 2. Or, since the numbers are so simple, you might multiply 2 by 60 and then divide by 3.

Change the $\frac{4}{5}$ to $\frac{3}{2}$ and there is still another algorithm.

$$\frac{2}{3} \times \frac{3}{2} \tag{6}$$

We recognize that these numbers are reciprocals and immediately write down the product 1.

Change the $\frac{4}{5}$ to $1\frac{4}{5}$ and there is still another algorithm.

$$\frac{2}{3} \times 1\frac{4}{5} \tag{7}$$

The mixed number (what a term that is!) $1\frac{4}{5}$ needs to be changed to the improper fraction (another unfortunate term!) $\frac{9}{5}$ and then the algorithm used in (1) above is applied.

With more than two fractions to be multiplied, combinations of these strategies are applied.

Multiplication of fractions is perhaps the simplest algorithm in all of elementary school arithmetic. And yet the skillful arithmetician has at least seven different ways of multiplying two fractions, depending on the numbers involved in the situation.

Skill is sometimes thought of as a lower order form of thinking. Accordingly, procedural understanding is often viewed as not as deep as, or contrasted with, conceptual understanding. I believe the understanding of procedures is not lower-level. Skillful people exhibit skill-algorithm understanding of the multiplication of fractions when they do it and get the right answer. They exhibit a higher form of this type of understanding when they know many ways of getting the right answer (that is, know different algorithms) and decide on a particular algorithm because it is more efficient than others, and can check their answers using a different method than employed to get the answer. Many people possess this type of understanding because so much time is spent in school working on the skill. They possess *skill-algorithm understanding*.

Property-Proof Understanding of Multiplication of Fractions

For many people, understanding has a completely different meaning than obtaining the correct answer in an efficient manner. They feel a person doesn't *really* understand something unless he or she can identify the mathematical properties that underlie *why* the way of obtaining the answer works. "Understanding" is contrasted with "doing". This kind of understanding is often found in courses in mathematics for elementary school teachers.

For example, for the multiplication problem $\frac{2}{3} \times \frac{4}{5}$, we wish to justify the rule

For any numbers a, b, c, and d with b ≠ 0 and d ≠ 0, $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$,

by showing how it follows logically from a set of simpler properties.

$\frac{a}{b} \cdot \frac{c}{d} = \left(a \cdot \frac{1}{b}\right) \cdot \left(c \cdot \frac{1}{d}\right)$	(definition of division)
$= \mathrm{a} \cdot \left(\frac{1}{b} \cdot \mathrm{c}\right) \cdot \frac{1}{d}$	(associative property of multiplication)
$= \mathrm{a} \cdot \left(\mathrm{c} \cdot \frac{1}{b}\right) \cdot \frac{1}{d}$	(commutative property of multiplication)
$= (\mathrm{a} \cdot \mathrm{c}) \cdot \left(\frac{1}{b} \cdot \frac{1}{d}\right)$	(associative property of multiplication)
$= (\mathrm{ac}) \cdot \left(\frac{1}{bd}\right)$	(uniqueness of multiplicative inverse; each is the inverse of *bd*)
$= \frac{ac}{bd}$	(definition of division)

This aspect of understanding, *property-proof understanding*, is obviously quite different from skill-algorithm understanding. From this mathematical derivation of the rule, a student learns that multiplication of fractions is not arbitrary but a property deducible from more general properties of multiplication and division. The derivation also shows the importance of thinking of $\frac{a}{b}$ as a $\cdot \frac{1}{b}$, the relevance of reciprocals, and so on.

Technically, the proof above that $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$ only accounts for the algorithm in case (1) mentioned above, a multiplication involving two fractions with no common factors in numerators and denominators. To justify case (2), in which the denominator of one fraction is a factor of the numerator of the other fraction, as in $\frac{2}{3} \times \frac{6}{5}$, we must show that $\frac{a}{b} \cdot \frac{bc}{d} = \mathrm{a} \cdot \frac{c}{d}$. A full justification of the ways in which we multiply fractions requires proofs also for cases (3) through (7). Thus the complexity of the algorithms is matched by a complexity of the mathematical underpinnings.

I have never seen anyone take the seven different cases of multiplying fractions shown above and subject each of them to careful mathematical analysis—and for good reasons—the cases are the tip of an iceberg of infinitely many variations. So any analysis would always be incomplete, and the payoff in increased proficiency that might come from a large number of proofs is unlikely to reflect the extra time that it takes to handle all of the cases.

Some people believe that if you understand these sorts of mathematical derivations and use language correctly, then you will be more skillful. This was the hope of many who worked with the "new math" in the 1950s and 1960s. But it was found that the transfer from understanding properties to understanding skill was not automatic. Skill requires practice, and also requires flexibility to choose among various possible algorithms, a quality that the mathematical derivations do not convey.

It is still the case today that some people will say that a person does not *really* understand arithmetic until he or she knows the mathematical theory behind it. It is also the case that when people contrast *procedural understanding* with *conceptual understanding*, they are often contrasting skill-algorithm understanding with the understanding that comes from mathematical properties. However, there is more to understanding than these two facets.

Use-Application Understanding of Multiplication of Fractions

A person may know *how* to do something and may know *why* his or her method works, but—particularly to people who use mathematics in their daily lives and on the job—a person does not fully understand the multiplication of fractions unless he or she understands *when* to multiply fractions. I would like to call this the "use-modelling understanding" but the word "model" has too many meanings and could be confusing. So I call this kind of understanding *use-application understanding*.

Use-application understanding is different from the previous understandings. Many people who can multiply fractions do not know of any place where they could use it. Students can even know both algorithms and mathematical properties associated with a concept without knowing its uses; this is a common situation in mathematics classrooms worldwide.

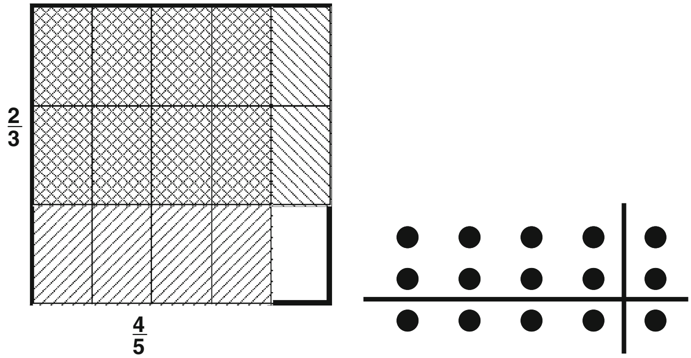
That there are students who can multiply fractions who cannot use them tells us that understanding of applications does not come automatically. We have only begun to realize that uses can be taught and that must be taught before most students realize what to do. Here are some examples of situations that might lead to that multiplication. That these five situations represent five different *types* of applications, not merely five different application contexts, can be seen by examining the units of measure (or lack of units) involved.

1. A rectangular region on a farm is $\frac{2}{3}$ km by $\frac{4}{5}$ km. What is its area? (A measure is multiplied by a measure.)
2. If an animal travels at a rate of 2 km in 3 h (i.e., at $\frac{2}{3}$ km per hour), how many kilometers will it travel in 48 min (i.e., $\frac{4}{5}$ h) (A rate, a measure with a derived unit, is multiplied by a measure.)
3. If two independent events have probabilities $\frac{2}{3}$ and $\frac{4}{5}$, what is the probability both will occur? (Two scalars are multiplied.)
4. If a segment on a sheet of paper is $\frac{4}{5}$ unit long and is put in a copy machine at $\frac{2}{3}$ its original length, what will be its final length? (A scalar is multiplied by a measure.)
5. If something is on sale at 1/3 off (i.e., at $\frac{2}{3}$ its original price) and you get a 20 % discount (to $\frac{4}{5}$ of the sale price) for opening a charge account, your cost is what part of the original price? (A scalar is multiplied by an unknown measure; then the product is multiplied by a scalar.)

We spend large amounts of time teaching arithmetic paper-and-pencil skills, including weeks on multiplication and division of fractions alone. We spend relatively little time teaching students how to apply these operations with fractions. As a result, performance on application is lower than performance on skill and we are led to believe that application is more difficult than skill. I would argue that some application is harder than some skill, but some skill is harder than some application.

Representation-Metaphor Understanding of Multiplication of Fractions

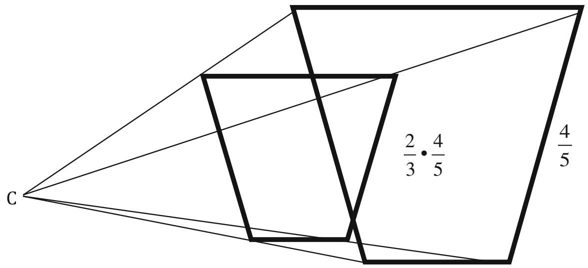
Even these three types of understanding do not encompass the entire scope of what it means to understand a mathematical concept. To cognitive psychologists with whom I have discussed this topic, the three types of understanding discussed so far do not convey the real true understanding of mathematics. From psychology we obtain the notion that a person does not really understand a concept unless he or she can represent the concept in some way. For some, that way must be with concrete objects; for others, a pictorial representation or metaphor will do. Here are two representations of $\frac{2}{3} \times \frac{4}{5}$.



The left representation derives from the area use-application situation (1) above. In it, $\frac{2}{3}$ and $\frac{4}{5}$ are side lengths of a rectangle and the product is the area. We represent $\frac{2}{3}$ by splitting the square horizontally into 3 parts and shading the top two parts. We represent $\frac{4}{5}$ by splitting the same square vertically into 5 parts and shading the 4 parts on the left with a different shading than used for $\frac{2}{3}$. This splits the square into 15 rectangles, 8 of which have both shadings, a picture of $\frac{8}{15}$.

A discrete version of the area representation is with an array of dots. At the right above, $\frac{2}{3}$ is represented by putting 2 of 3 dots above a horizontal line; $\frac{4}{5}$ is represented by putting 4 of 5 dots to the left of a vertical line; and the product consists of the 8 of 15 dots that are both above the horizontal and to the left of the vertical.

A third representation is quite different and views the $\frac{2}{3}$ and $\frac{4}{5}$ not as equal partners but the $\frac{2}{3}$ as operating on the $\frac{4}{5}$. In this representation, suggested by application situation (4) above, we begin with any geometric figure (below, the larger trapezoid) on which there is a segment of length $\frac{4}{5}$. Then we draw segments from some point (below, point C) to the vertices of the trapezoid. Then, on each segment from C, we pick points $\frac{2}{3}$ of the way to the vertices of the trapezoid. Connecting those segments results in an image trapezoid whose sides have $\frac{2}{3}$ the length of the corresponding sides of the original trapezoid.



Although this representation seems like a lot of work for such a simple arithmetic operation, it has wide applicability.

A person can have a rather deep knowledge of multiplication of fractions even if the person has never seen these representations. Millions of youngsters have acquired skills, learned the mathematical underpinnings, and developed the ability to apply mathematics without touching any concrete materials or seeing any sort of representation of a particular piece of mathematics. Thus concrete or pictorial representations do not have to precede acquisition of the other types of understandings, even though some people appeal to representations to advise us in the same way that we were told about knowing properties; if students are brought carefully to understand (in the representational sense) what they are doing, then they will ultimately be better at skill.

One of the principles advocated by some mathematics educators today is to delay the study of certain aspects of "formal arithmetic" in the elementary school, until these concrete and representational understandings are established. The rationale given for all this is that before these ages students either cannot or do not *really* understand what they are doing; that they need conceptual buildup before they can understand. Little is done to define what is meant by a concept or what is really meant by understanding. Little is done to acknowledge the vast numbers of students who gain other understandings without going through these stages. Little is done to analyze the possible effects of such practices if the theory is wrong. It may be as unfruitful to wait for this kind of understanding to move on as it is to wait for skill understanding to move on.

The understanding that comes from representations can be quite useful. Statisticians know the effects of a good graph. Mathematicians often use graphical and diagrammatic representations. More and more graphing is used in algebra and higher mathematics because graphs convey so much information; and with function graphing technology we do not have to work so hard to obtain this understanding.

It is not difficult to apply these four dimensions of understanding to other arithmetic topics and to the understanding of topics from algebra and analysis. For instance, with linear equations in algebra, it is easy to distinguish skills, properties, and uses, and the common representation is with graphs. So, for the second example, I have picked a concept that is quite different, congruence and congruent figures in geometry. The four dimensions of understanding still fit, but a fifth dimension of understanding, quite different from the other four, appears.

Second Example: Congruence in Geometry

We begin with the vocabulary of congruence. We think of a *figure* as referring to any set of points. For this discussion, we restrict ourselves to congruence in the plane.

Informally, two figures are *congruent* if one can be placed on top of the other. This approach comes to us from Euclid and might be termed *dynamic*, for it speaks of a movement of one figure to another. In contrast, in most secondary schools in the world, the approach in the later grades is *static*, and it is common to have individual criteria for one or more specific types of figures. The following definitions are commonly found today.

> Two *segments* are congruent if and only if they have the same length.
> Two *angles* are congruent if and only if they have the same measure.
> Two *triangles* are congruent if and only if there is a correspondence between their vertices with corresponding sides having the same length and corresponding angles having the same measure.
> Two *circles* are congruent if and only if they have the same radius.

Both static and dynamic approaches can be obtained by employing the language of geometric transformations. Here is a dynamic characterization of congruence using transformations that can be traced back at least to Felix Klein.

> Two figures are congruent if and only if one can be reflected, rotated, and/or translated to the position of the other.[1]

Transformations also provide static models for dynamic actions by ignoring any intermediary positions of the figure and concentrating only on the points of the preimage figure and the points of the image figure.

> Two figures are congruent if and only if there is a distance-preserving transformation (isometry) that maps one figure onto the other.

Skill-Algorithm Understanding of Congruence

The skills of geometry relate to drawing and visualizing and are as old as any other aspects of formal geometry. The first theorem in Euclid's *Elements* is the construction of an equilateral triangle, and the steps in that construction constitute the first algorithm of many in this genre. Specifically with respect to congruence, a child's first skill-algorithm understanding is likely with tracing a figure or using a stencil to create a congruent one. Dealing with the puzzle found in some

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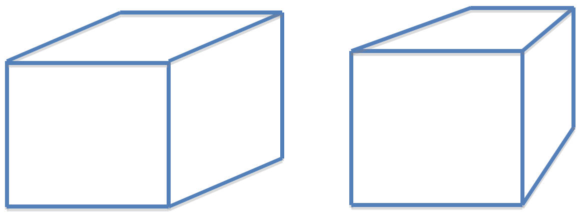
[1]It is understood that composites of these transformations are allowed. In particular, congruent figures may be related by glide reflections.

newspapers and magazines where two complex drawings are shown and a person needs to locate the differences is part of this understanding. An older child may draw or construct reflection, rotation, translation, or glide reflection images of figures, or combine congruent figures to create a tessellation of the plane. Given two congruent figures, a student might be asked to describe in visual terms which transformation maps one onto the other.

This particular aspect of understanding is more difficult in three dimensions than in two. There visualization plays a greater role. What does a figure look like when viewed from a different angle? What are the possible plane sections of the figure?



cube without perspective　　cube with perspective

Artists and designers often understand this aspect of geometry quite well. Whereas it is common in mathematics to draw a stylized cube, in which the front and back look congruent to the viewer, an artist will use perspective to draw a cube the way it would look in the real world, with the back of the cube smaller than the front because it is farther away.

Skill-algorithm understanding in geometry is similar to that in arithmetic and algebra in that technology today is now commonly used when accuracy and speed are desired. The use of software such as dynamic geometry programs and computer-assisted design (CAD) falls under this dimension of understanding geometry.

Property-Proof Understanding of Congruence

The property-proof dimension of understanding is the aspect of congruence that is given most priority in secondary schools. It includes the side-angle-side and other conditions that cause two triangles to be congruent, and the use of these conditions to deduce properties of lines, angles, triangles, quadrilaterals, and other polygons. This dimension also includes the properties of each of the isometries, the properties of symmetric figures that result from the congruence of a figure with itself, relationships between the various isometries (e.g., the composite of two reflections over intersecting lines is a rotation), and relations between congruence and similarity. We derive formulas for the areas of triangles, special quadrilaterals, and regular polygons from the fact that congruent figures have the same area. These and other derivations are also part of the property-proof understanding of congruence.

Advanced aspects of this understanding involve such things as the statements about congruence that need to be assumed to form a complete postulate set for Euclidean geometry, and the notion that in hyperbolic non-Euclidean geometry the only similar figures are congruent ones. The Banach-Tarski paradox in 3-dimensional geometry exemplifies the difficulties that can arise when the idea of congruence is applied to very complicated figures.

Use-Application Understanding of Congruence

Uses of 2-dimensional and 3-dimensional congruence abound. In textbook discussions of geometry, we often see pictures of railroad trestles or buildings using congruent triangles, congruent rectangular window frames, and other examples of congruence of the particular types of figures that are studied. But congruence is far more ubiquitous than this. Printing makes use of congruent letters and symbols, and copies of pages are congruent. Mass production of machines makes their parts congruent and simplifies repairs or replacements. From eating utensils to chairs, tiles to bricks, congruent objects are everywhere in our lives, so common we neglect to mention them.

Each of the types of isometries have their own applications. Reflections model mirrors. Rotations are intimately connected with turns. Translations can be thought of as slides. Glide reflections connect consecutive footprints a person might leave when walking. Problems of optimal packing bring into play areas and volumes of congruent figures. It is rather obvious that a person can describe and use these applications of congruence without being acquainted with the formal properties and proofs associated with congruence, and vice-versa. Likewise the drawing and visualizing skills and algorithms are quite independent of these other dimensions of understanding congruence.

Representation-Metaphor Understanding of Congruence

Geometry is the study of sets of points and visual patterns. These points may take on many forms: idealized locations, as in Euclidean geometry; ordered pairs, 3-tuples, or n-tuples, as in coordinate geometry and linear algebra; data points as in statistics; nodes as in networks; dots or tiny squares as in the pixels on computer screens or some paintings. The variety of these uses of points underlies the geometric representations of so many arithmetic and algebraic concepts. But these geometric ideas do not fall under the representation-metaphor understanding of geometry because they start outside geometry.

The representation-metaphor dimension of geometry goes the other way; it involves representations of geometric ideas that are not necessarily geometric and are often algebraic. We describe lines and some other geometric figures, such as the conic sections, by equations. We place 2-dimensional geometric figures on a coordinate plane. For younger students, we may use a representation on a geoboard.

Specifically with respect to congruence, we represent isometries by formulas for the coordinates of the image (x′, y′) of a point (x, y) under them. These formulas might be described algebraically or represented arithmetically by a matrix. Equivalently, we can place a figure on the complex plane and describe any isometry by a formula involving complex numbers. These are not simple representations, but they are powerful and they are used in describing movements of figures we see on computer screens.

Sometimes we represent one type of geometry by another, as when we describe locations on a subway system by a diagram, or the Königsberg Bridge Problem by a network as Euler did, or a translation by a vector.

A Taxonomy of Mathematical Understanding

The four dimensions of understanding detailed above have certain common qualities. Each dimension of understanding has supporters for whom that dimension is preeminent, and who believe that the other dimensions do not convey the real essence of the understanding of mathematics. Each dimension has aspects that can be memorized. Skills, names of properties, connections between mathematics and the real world, and even work with representations can be memorized. They also have potential for highest level of creative thinking: the invention of algorithms, the proofs that things work, the discovery of new applications for old mathematics, the development of new representations or metaphors.

The four dimensions of understanding are relatively independent in the sense that they can be, and are often, learned in isolation from each other, and no particular dimension need precede any of the others. Some believe mathematics should begin with real world situations; others with skills; others with concrete materials; and still others believe you should develop the mathematical theory first and let everything else come from that.

It is because of the relative independence of skill-property understanding, property-proof understanding, use-application understanding, and representation-metaphor understanding from each other, that I believe that the understanding of mathematics is a multi–dimensional entity, in the sense that there are independent components that constitute what might be called “real true”, “complete”, or “full” understanding.

History-Culture Understanding

There is, I believe, at least one other dimension to this understanding, one that is not usually found in school mathematics but is a part of a full understanding of mathematics. It is the *history-culture dimension*. How and why did a certain bit of mathematics arise? How has it developed over time? How is it treated in different cultures? Those who study the history of mathematics or cross-cultural mathematics obtain an understanding of mathematical concepts that is different from any of the understandings we discussed so far. It is a fifth dimension.

Some important aspects of mathematics are primarily located in this dimension. Among these are that:

Mathematics is invented, or discovered.
Mathematics has grown over time and so has the number and variety of its uses.
Both formal and informal mathematics are not necessarily the same in all cultures.
The truths of mathematics are relative truths, deduced from definitions and a small number of postulates.
There exists "recreational mathematics", that is, mathematics done for fun.

With respect to congruence, the history-culture dimension includes understanding the work of Euclid and its significance, and also the work of Fermat and Descartes to describe figures with coordinates, the Erlanger Program of Felix Klein, and Hilbert's use of the SAS Postulate in his *Foundations of Geometry*. Much of this is found in books on the history of mathematics and is relatively well-known. From a cultural standpoint, we would want to include the tilings that are found in the Alhambra and other Islamic sites. We might include the artwork of Maurits Escher that brought home to many of us the idea that the notion of congruence can be applied to figures such as fish, lizards, and birds.

I do not know much about the history of the *multiplication* of fractions, but most of us do have some cultural-historical understanding of fractions themselves. The first fractions were those for halves, thirds, and fourths. Over 3000 years ago, the Egyptians represented other fractions as sums of unit fractions. Simon Stevin, in his invention of decimals in the late 1500s called them *decimal fractions*, and some places still use that term. The first use of the bar to represent fractions seems to be among Arab mathematicians well over 1000 years ago but their common use did not appear until the 16th century (Flegg 2002, pp. 74–75; Cajori 1928, p. 310). A sign very much like the slash for fractions first appeared in Mexico in the late 1700s (Cajori 1928, p. 313). Even today the symbols are not the same everywhere. In some places, the fraction a/b is represented by a:b, while in other places the symbol a:b represents a ratio that is mathematically not identical to a fraction. For mathematics education, the cultural history of fractions represents a dimension of understanding of the concept that is considered particularly important to those who believe in a genetic approach to learning, that is, a progression of learning activities that parallels the historical development of the subject. The cultural history of a mathematical concept is also central to ethnomathematics. And, as with the other dimensions of understanding, there are those who believe that history-culture understanding underlies the best learning sequence for a mathematical concept.

What Is a "Concept"?

In this analysis of understanding, "multiplication of fractions" has been identified as a concept. Since the word *concept* is often used as a counterpart to skill, why do I consider multiplication of fractions to be a concept and not a skill? The reason is that I believe a *concept* is something that lends itself to be analyzed by these dimensions of understanding. A concept has associated skills, properties, uses, representations and history. A concept is, in the language of this paper, multi-dimensional.

In contrast, an algorithm or a proof or a model or a representation is not by itself a concept. However, by connecting the various dimensions of understanding, one can take any of these and turn it into a concept. For example, the long division algorithm is not a concept, but if one analyzes this algorithm for its mathematical underpinnings, finds uses for it beyond just obtaining answers to division problems, represents it, and discusses its history and variants in different cultures, then long division becomes more than an algorithm; it becomes a concept.

Because concepts involve all of these dimensions of understanding, it is often the case that a view of a concept does not neatly fit into one of the dimensions. For instance, the proving that the algorithms for multiplying fractions mentioned earlier are valid, or justifying the straightedge-and-compass construction of a triangle congruent to a given triangle might be viewed as straddling the skill-algorithm and proof-property dimensions. But carrying out the algorithms is so different from justifying them that it seems clear that they involve quite different dimensions of understanding.

There is thinking that does not fit this multi-dimensional conception of understanding. A notion that a student needs to be able to perform multiplication before he can understand how to use that multiplication, or that a student must be able to draw reflection images before being able to use them is in effect saying that these two things, doing and using, are in the same dimension of understanding, with using more advanced than doing. Likewise, the notion that a student must see concrete representations of ideas before learning the theory also does not fit this multi-dimensional conceptualization. Ordering ideas or concepts in terms of difficulty is only appropriate if these items are in the same dimension.

This multi-dimensional approach to understanding also conflicts with uni-dimensional Rasch models of evaluation, where items of all different kinds are placed on the same linear scale. It also brings into question statistical reliability tests that are used to throw out items that do not act like other items, such as items that higher-scoring students answer incorrectly in greater numbers than would be expected by their scores. It is quite possible that such items are merely in different dimensions of understanding and some students understand those aspects better than others.

This multi-dimensional approach to understanding also conflicts somewhat with the organization of knowledge found in Bloom's *Taxonomy of Educational Objectives* (Krathwohl et al. 1964). In the taxonomy, there are six levels—from lowest to highest: knowledge, comprehension—a synonym for understanding, application, analysis, synthesis, and evaluation. At the level of knowledge would be the ability to do multiply fractions; at the level of comprehension are the mathematical underpinnings and representations; at the third level would be applications. I believe there is no such ordering. The experience that led to this belief was an attempt I had made in the middle 1970s to do a first-year algebra course in which the mathematics developed from real-world applications, not from the field properties. Students using these materials often knew how to apply algebra before they had the paper-and-pencil skills to carry out the application. Bloom (a colleague of mine in the same department) wanted to call such application higher-order thinking, at the third level of his taxonomy. I argued with him that something could be changed from higher order to lower order if you worked on it every day. It made me realize that one goal of mathematics instruction is to change higher-order activities into lower-order ones. This is why it is so difficult to teach problem-solving and proof and the invention of algorithms and new models.

On the other hand, a common view of "understanding" is that understanding involves connecting ideas, and so it should come as no surprise that for many decades we have known that applications and concrete representations can increase the learning of skills, that skills and properties taught together are better than either taught alone.

Applications of the Multi-dimensional Framework

For over a quarter century, we have been using this multi-dimensional framework to help guide our development of the University of Chicago School Mathematics Project materials for secondary schools. Items on tests at the ends of units are identified with one of the first four dimensions (skills, properties, uses, or representations)—the cultural-historical dimension is not tested. Chapter review questions are also sorted into one of the dimensions. Some of the authors who work on our writing teams are stronger at instructing about the pure mathematics of properties and proof, while others are better at one of the other dimensions of understanding. By keeping a watchful eye on these dimensions, we feel that our materials become richer and reach more students than if we did not implement this broad perspective.

"Understanding" mathematics is also important in the new (2010) Common Core State Standards for Mathematics (CCSSM) in the United States. The words "understand", "understands", "understanding", and "understandings" appear 263 times in the document. The standards for the multiplication of fractions involve several dimensions of understanding. Here is a part of one of the standards at grade 5:

> *Apply and extend previous understandings of multiplication to multiply a fraction or whole number by a fraction.*
> *Interpret the product (a/b) × q as a parts of a partition of q into b equal parts; equivalently, as the result of a sequence of operations a × q ÷ b. For example,*

use a visual fraction model to show (2/3) × 4 = 8/3, and create a story context for this equation. Do the same with (2/3) × (4/5) = 8/15. (In general, (a/b) × (c/d) = ac/bd.)

Here we interpret the standard for asking for the dimensions of understanding as noted in parentheses:

Interpret the product (a/b) × q as a parts of a partition of q into b equal parts (Vocabulary); *equivalently, as the result of a sequence of operations a × q ÷ b* (Property-Proof). *For example, use a visual fraction model to show (2/3) × 4 = 8/3* (Representation-Metaphor)*, and create a story context for this equation* (Use-Application). *Do the same with (2/3) × (4/5) = 8/15. (In general, (a/b) × (c/d) = ac/bd.* (Skill-Algorithm).

We can conclude that students are asked to have a broad understanding of fractions.

Here is the standard dealing with congruence at grade 8.

Understand congruence using physical models, transparencies, or geometry software.
1. *Verify experimentally the properties of rotations, reflections, and translations* (Skill-Algorithm or Property-Proof):
   (a) *Lines are taken to lines, and line segments to line segments of the same length.*
   (b) *Angles are taken to angles of the same measure.*
   (c) *Parallel lines are taken to parallel lines.*
2. *Understand that a two-dimensional figure is congruent to another if the second can be obtained from the first by a sequence of rotations, reflections, and translations* (Property-Proof); *given two congruent figures, describe a sequence that exhibits the congruence between them* (Skill-Algorithm).
3. *Describe the effect of dilations, translations, rotations, and reflections on two-dimensional figures using coordinates* (Representation-Metaphor).

Although this analysis of the standard suggests that the standard does not include the use-application dimension, the mention of physical models at the top of the standard could be interpreted as including applications of rotations, reflections, and translations.

In all the grades K-12, there are 385 individual standards in the Common Core. They can be viewed as being at four levels: K-5, 6-8, 9-12 for all students, and 9-12 for students who might go into the STEM areas, that is, science, technology, engineering, or mathematics. Table 1 shows my categorization of these standards into the five dimensions, vocabulary, and problem solving.

Table 1 Frequencies of dimensions of understanding in the CCSSM

	Totals	K-5	6-8	9-12	9-12 STEM
Vocabulary	51	21	18	11	1
Skill-algorithm	186	88	37	38	23
Property-proof	196	67	47	61	21
Use-application	147	55	47	32	13
Representation-metaphor	118	46	30	32	10
History-culture	0	0	0	0	0
Problem solving	13	7	1	2	3
Number of standards	385	148	81	113	43
Totals		284	180	176	71

Understanding Mathematics from the Teacher's Perspective

The teacher is an applied mathematician whose field of application involves the classroom and the student. Like other applied mathematicians, in order to apply the mathematics, the teacher needs to have a good deal of knowledge about the field itself—that is, the about the educational process—as well as about mathematics. Thus, the understandings that a teacher needs involve more than mathematical understandings. The teacher also must take into account students, classrooms, teaching materials, and the necessities of explaining, motivating, and reacting to students.

In 2005, the Mathematical Sciences Research Institute in Berkeley held a conference on the mathematics a teacher needs to know. In advance, the organizers, who included Deborah Ball and Hyman Bass, identified eight tasks they felt required a knowledge of mathematics. I have added to their list and provide the following as four realms of understanding mathematics from the teacher's perspective. I call these realms and not dimensions because operationally they are clearly very much interrelated.

For the first realm, there is a substantial literature. The phrase that identifies it, *pedagogical content knowledge*, was introduced by Shulman (1986).

Pedagogical content knowledge:
designing and preparing for a lesson,
analyzing student errors,
explaining and representing ideas new to students,
responding to questions that learners have about what they are learning.

The second realm deals with applying the understanding of mathematical concepts.

Concept analysis:
engaging students in proof and proving,
choosing and comparing different representations for a specific mathematical procedure or concept,

choosing and using mathematical definitions,
explaining why concepts arose and how they have changed over time,
dealing with the wide range of applications of the mathematical ideas being taught.

The third realm deals with the understanding of problems and problem-solving.

Problem analysis:
examining different student solution methods,
engaging students in problem solving,
discussing alternate ways of approaching problems with and without calculator and computer technology,
offering extensions and generalizations of problems.

The fourth realm integrates the other three.

Connections and generalizations to other mathematics:
comparing different textbook treatments of a mathematical procedure or topic,
extending and generalizing properties and mathematical arguments,
explaining how ideas studied in school relate to ideas students may encounter or have encountered in other mathematics study,
realizing the implications for student learning of spending too little or too much time on a given topic.

It is clear that teachers need understandings that go far beyond those of students.

Summary

Understanding a piece of mathematics from the standpoint of mathematics education is different for the policy maker, the mathematician, the teacher, and the student. The policy maker needs to understand the importance of that piece to the student at a given time and place. The mathematician needs to understand the potential for the invention of new concepts, the consideration of new and previously unsolved problems, and the discovery of new results. The teacher needs to have a variety of understandings related to pedagogy, concepts, problems, and connections and generalizations of what is done in the classroom.

Mathematical activity consists of *concepts* and *problems or questions*: we employ and invent concepts to answer questions and problems; we pose questions and problems to delineate concepts. The central person in mathematics education is the student, and, primarily because there exist well-known treatises on the understanding of problem solving, this paper has mainly dealt with the understanding of a concept in mathematics from the standpoint of the student, that is, the learner's standpoint.

We might say that a learner has full understanding of a mathematical concept if he or she can deal effectively with the skills and algorithms associated with the concept, with properties and mathematical justifications (proofs) involving the concept, with uses and applications of the concept, with representations and metaphors for the concept, and with the history of the concept and its treatment in different cultures. Although these aspects are obviously connected when attached to a particular concept, I call them *dimensions of understanding* because each aspect can be mastered relatively independently of the others.

All but the last of these dimensions are important in the teaching and evaluation of mathematics learning. We have delineated these dimensions for two concepts: multiplication of fractions, and congruence in geometry. An actualization of this framework in a full curriculum for grades 6–12 can be found in the materials of the University of Chicago School Mathematics Project, and the results of an analysis are shown here for the recent standards of the Common Core State Standards in Mathematics.

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Conflicting Perspectives of Power, Identity, Access and Language Choice in Multilingual Teachers' Voices

Lyn Webb

Abstract Teachers in the Eastern Cape, South Africa teach mainly in English, which is not their home language. In order to elicit their inner voices about language conflicts and contradictions in their classrooms they were encouraged to write poetry about their perceptions of the impact of language in their lives. The most prevalent contradiction they expressed was the power and dominance of English juxtaposed against the subordination of their home languages. English gave them access to education and upward employment mobility, whereas they were excluded from various discourses when they used their home languages. Their home languages legitimised and defined their identities, but appeared to be negated in an educational and economic environment. Since the necessity for pupils to become fluent in English conflicted with the pupils' difficulties in understanding content knowledge expressed in English, the teachers faced a choice between teaching in English (for access to social goods) or their home language (for epistemological access), or both. The use of poetry evoked feelings and emotions that may not have been as obvious, or as evocative, if other data-gathering methods had been used. It appears that the self-reflection embodied in the poetry gave the teachers a sense of empowerment, self-realisation and solidarity. Certain parts of this chapter have been reported elsewhere (Webb 2012).

Keywords Multilingual teachers · Language · Power · Access · Identity · Language choice · Poetry

Introduction

In the Eastern Cape those who were disadvantaged during the apartheid era could still be marginalised—not because of their colour, but because of their lack of fluency in English. The struggle for language rights in South Africa is interconnected

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with the struggle for human rights and in the past language was used as a means of domination and separation (Setati 2008; Webb 2002). The majority of teachers in the Eastern Cape teach, and pupils learn, in a language that is not their home language. Although there are 11 official languages in South Africa the hegemonic position of English has resulted in English being the preferred language for schools and business, as it is a passport to social goods (Gee 2008; Setati 2008), tertiary education, fulfilling jobs and positions of influence and power. This implies that English may already have become a weapon in the struggle for power between the different socio-economic and political groups, regardless of colour.

Traditional views of power have negative connotations including oppression, control and authority, whereas the post-modern view is far more positive, since power is envisaged as being productive, creative, effective, active and part of everyday life (Albertyn et al. 2001). Empowerment embodies the sense that power comes from within, where the locus of control is moved from powerful others to within the empowered people. Empowered people would thus be intrinsically motivated and in control of their own lives. At a micro-level, empowerment refers to the way that an individual views identity including, among other aspects, issues of dignity and personal responsibility (Albertyn et al. 2001).

The aim of this chapter is to uncover possible conflicts and contradictions that multilingual teachers in the Eastern Cape perceive as they teach through the medium of English, which is a foreign language to many of their pupils. In order to enable teachers to confront the realities of the language environments of their classrooms and to allow them to explore their own attitudes and emotions concerning the impact of language, it has been essential to create spaces in which teachers feel free to share their "language stories", to reflect on the multilingual realities of their classrooms, to take cognisance of the power language wields over issues of their own cultural identity, to assess the access language affords to the realisation of their dreams, and to consider the effect the choice of language has on the effectiveness of their teaching.

In this chapter I report on research in which teachers from rural, peri-urban and urban schools in the Eastern Cape were encouraged to reflect on their own experiences and write poetry in which they share their perceptions about the force language exerts on their lives. Their voices evocatively recreate episodes in their lives and the conflicts and contradictions they encounter. The main constructs that emerged from the poetry could be distilled down to issues of power, access, identity and language choice. Using poetry as a data collection method seemed to be an effective way of creating awareness and penetrating to the deepest levels where teachers felt valued, respected and empowered. The poetry written by the teachers, the commentaries on their perceptions, interpretations and conclusions have been reported elsewhere (Webb 2010, 2012).

Current Literature

From the reading and analysis of the teachers' poetry, I identified constructs that threaded through the poetic narratives. The most pervading constructs concerning language were that teachers are faced with conflicts concerning issues of power (dominant or subordinate), access (included or excluded), identity (legitimised or negated) and language choice (English or home language) in multilingual classrooms (Janks 2010). It is necessary for teachers to recognise where their own perspectives lie on the four continua so that they can identify their own lived reality in order to orchestrate and effect change.

Power—Dominant or Subordinate

Fairclough (2001: 39) states that power in discourse concerns "powerful participants controlling and constraining the contributions of non-powerful participants". He maintains that "the whole social order of discourse is put together and held together as a hidden effect of power" and that one dimension of this power is the elevation of one social dialect to the position of a standard or "national" language (Fairclough 2001: 47). Gee (2008) warns that mainstream dominant discourses, and particularly school-based discourses, privilege those who have mastered them (mainstream/insiders) and do significant harm to those who have not (non-mainstream/outsiders). Lerman (2001) reinforces Gee's (2008) standpoint that people are positioned in practices as powerful or powerless according to the structure of the discourse and the personal histories, including cultural backgrounds, of the participants.

Teachers in multilingual schools do not only engage in specialist content-based discourse where the pupils need access to particular 'communities of practice' to understand the academic content of the lesson (Wenger 1998), but also where pupils need to make sense of the spoken and written words in which the content is presented to them—in English. The pupils are thus in a subordinate position as the "unknowing" in both content knowledge and English knowledge. This places them in a negative, unvalued space and they are excluded from access to fulfilling expectations of higher education goals.

Access—Included or Excluded

Gee (2008) maintains that cultural models exclude people (in this case English second language pupils) from participating comfortably in academic discourses as they do not know the "rites of passage" for entry to the "club" (Gee 2008). This view is supported by Setati (2008) who posits that the political nature of language pressurises teachers to use English as they are faced with the conflict of providing access to English as well as to knowledge, in this instance mathematical knowledge. She maintains that the desire for access to English has the dominant role, so teachers teach in English (hoping that pupils will learn to communicate in English) with the result that procedural knowledge of skills and practices is foregrounded over conceptual, reasoning mathematical knowledge (Setati 2008).

In South Africa the dominant mainstream cultural model in schools and business is expressed through the medium of English, which marginalises the non-mainstream teachers and pupils. The conflict that teachers in the Eastern Cape have to resolve is how to provide access to the dominant discourse (in this case English) while still valuing the discourses and diversity that the pupils bring to the classroom (Janks 2010). Research has shown that in order to become fluent in a second or third language, pupils must have a strong foundation in their first language (Heugh 2008; Westcott 2004). The foundation includes verbal proficiency as well as vocabulary and grammar skills. Multilingual pupils who do not have a working knowledge of their first language could be prevented from accessing another language.

Identity—Legitimised or Negated

Da Ponte (2009) maintains that the intersection between three circles of teachers' knowledge, teachers' practice and teachers' identity creates a common ground where optimum teaching takes place. He believes that teachers' identity encompasses teachers' perceptions, perspectives, values, norms and ways of being. If teachers do not have a strong sense of their own identity the balance among knowledge, practice and identity is tipped and the common ground for optimum teaching will be decreased. Janks (2010) maintains that African children's sense of identity is compromised when having to learn through the medium of English, if it is not their home language. The pressure for English competence has resulted in a devaluing of pupils' sense of identity. The majority of schools in the Eastern Cape choose English as the dominant language of learning and teaching (LoLT), with a resultant lack of literacy competence in the pupils' home language. In this chapter it will be illustrated how teachers eloquently portray their dismay that their cultural identity is being diminished.

Language Choices—English or Home Language

Janks (2010: 129) describes an occasion where she asked pupils to draw their school playground and to describe the playground in written English text. One of the pupils' drawings is rich with visual representations of children playing and moving, whereas the written description is limited to one- or two-word phrases.

She observes that the children are "mute, robbed of language"; however, if they are encouraged to discuss their reasoning in their vernacular "there is a flood of ideas as the children are rescued from the silence imposed by English". Neville Alexander, in an interview from the motion picture Sink or Swim (Westcott 2004), maintains:

Because of the hegemonic position of English in the world today, because it is the key to social mobility, to upward social mobility, people understandably and justifiably want their children to learn English in South Africa. What most people don't understand is that it does not follow, therefore, that they will acquire the best command of English if they are taught from day one through the medium of English. That does happen, of course, but it happens only under very specific conditions, conditions which do not exist in most South African schools, certainly not in most black schools.

Research has shown that teachers hold strong views as to whether they should teach in English only, or in the pupils' home language, or whether to code-switch by using two languages in the course of a single utterance (Moschkovich 2007). Code-switching is an accepted practice in Eastern Cape multilingual classrooms. Teachers could play the role of discourse guides to steer pupils on the winding path from informal communication in their home language to formal academic language in English in order to allow pupils access to the dominant academic discourse (Setati and Adler 2002).

Research Design

The research reported in this chapter forms part of a larger mixed method study that was conducted in the field of mathematics education over a period of four years. After the presentation at ICME-12 the research was published (Webb 2012). The qualitative part of the study was based on the interpretive paradigm. The purpose of the study was to ascertain whether the introduction of structured dialogue (exploratory talk) in the pupils' home language could enhance multilingual pupils' reasoning and numeracy skills in the Eastern Cape. The first objective of the larger study was to identify Eastern Cape teachers' perceptions about language issues in their classes. In order to attain this objective a questionnaire elicited quantitative results, and reflective writing and the crafting of poetry triangulated these results qualitatively.

Sample

I focused on 176 practising teachers who were registered off-campus students of the Nelson Mandela Metropolitan University and travelled to lectures in Port Elizabeth (28), Queenstown (34), Mthatha (55), King William's Town (32) and Kokstad (29). The teachers were all from previously disadvantaged schools and spoke either

Afrikaans or isiXhosa as their home language. This sample was chosen as the teachers taught at a range of schools in rural, peri-urban and urban locations situated throughout the Eastern Cape and were thus considered to be representative of the area.

Method

Hooijer and Fourie (2009) conducted a study using questionnaires to ascertain how multilingual teachers experience teaching in classrooms where the pupils' home language is not the LoLT. They concluded that teachers found their task difficult and required educational and emotional support. The questionnaire responses in this study were adequate but not rich. I endeavoured to use a triangulating data gathering instrument by encouraging the teachers to express their emotions serendipitously through poetry without question prompts. They could write in any language they preferred. Because a translation would lessen the impact of the poetry, an isiXhosa-speaking academic, Kazeka, was asked to read the poems aloud and reflect verbally on the emotions the poems evoked in her and the impact they made on her. The transcriptions of her commentaries are reported after the poems that are written in isiXhosa (Webb 2012).

Teachers' Voices

The constructs mentioned in the poems often represented two ends of a continuum. The main themes identified were power, access and identity; however, in addition, pride was pitted against feelings of inferiority and cultural capital was contrasted with cultural dissipation. In some poems a variety of constructs were mentioned, so it was a complex task to separate the poems into discrete topics. Perhaps the power of the teachers' voices is sufficient to convey their message and emotion.

One prevalent contradiction that was expressed was the disparity between the teachers' pride in their home language as opposed to their need for fluency in English. The pride and comfort their home language affords them is palpable, but also the sense that their home language is inferior and useless as a tool to access social goods. The following two poems express, differently, the same emotions:

Bilingualism
I think and dream in isiXhosa
My home language
I love isiXhosa
Although I can't communicate worldwide
It is my roots
My culture
My identity

It is my *Ubuntu*, I love my home language.

I study in English
Language international
Recognised worldwide
Power, secret, comfort
Employment, status, relief
All represented in English
I respect my second language
It makes me feel literate.

Bilingualism, strange word,
Existing within one me.

Last Lesson on a Friday
Oh, it is English lesson again
I don't like the lesson
The tenses again?
Is – was, go – went, write – wrote
All in my head?
Oh, it's very boring, especially on a
Friday afternoon!
But, what can I do without it?
Nothing…
It is the key to open the doors of life
Come, English, come!
I want to learn more!

The poems above echo Setati's (2008) point that teachers preferably align themselves with English rather than with epistemological access to content knowledge. Being literate seems to have value only if it relates to being English-literate.

The tone of some of the poetry calls to mind the praise poetry in Xhosa oral tradition, where the *imbongi*, or praise singer, spontaneously lauds a dignitary. Once again contradictions are foremost in the poet's mind. She wants to be fluent in English, but finds it an extremely difficult language to master. This results in a sense of inferiority and exclusion, an example of not knowing the "rites of passage" for entry to the "club" (Gee 2008).

Our Foreign Language
Oh English, our medium of instruction
What a good language it is
What a bad language it is
It is difficult for us to speak it
It is difficult for us to write it
Because it is not our mother tongue

Without it, no jobs
Without it, no education
Without it, no tourism
It is the medium of communication

Some feel inferior to others
Some feel unhappy to talk to others
Some criticise us for using our mother tongue
But our benefits belong to it.

Officials, educators, learners and stakeholders come together
Draw up a language policy which could help every South African child!

When writing in their home language the teachers expressed themselves lyrically using metaphors to express passion, but the underlying theme of disempowerment prevails. The child's helplessness is rooted in not understanding English. Aspirations can only be achieved if the child is fluent in English as opposed to isiXhosa. The question is implicit: Is one only educated if one can speak English?

Isikhalo Somntwana
Isikhalo Somntwana
Ingaba ndinantoni?
Ndiva isikhalo,
Isikhalo somntwana,
Esithi andisiva 'isilungu'
Ndiva isiXhosa!
Ndiva isiXhosa!!

Kodwa mandifunde isiLungu
Mandifunde! Mandifunde!!
Ndoyaphi na?
Ndothini na?
Ngaphandle kwesilungu
Ulwimi lwezwe lonke.

Sona sikuvulela
Iingcango eziya
Enkululekweni.
Mandifunde! Mandifunde!!
The sky is the limit.

Commentary from Kazeka:

Basically this poem is saying, the cry of a child. '*Isikhalo somntwana*' is the cry of a child and this young man, in the first paragraph, is saying 'What will I be?' I hear the cry of the child who is saying, This cry is I don't understand English. Everyone keeps on saying I don't understand English but I understand isiXhosa – this is what I hear, I don't hear English, I hear isiXhosa but everyone keeps on saying I must learn English because it will open the doors to opportunity, especially in a democratic situation. So when everyone is drumming into him that he must be educated, be educated. Where will he go if he doesn't know this English? And then he says he must be educated, he must be educated and the sky is the limit. I would love to speak to this writer because… does he mean that being educated is only in English? It is quite a … quite a … it moves you to your essence. It is that type of poem.

The repetition of "The sky is the limit" in the previous poem and the next emphasises the contradiction between the power of English and the teachers' pride in their home language. In the previous poem the teacher sees proficiency in English as a panacea for all ills, an "open sesame" for fame and fortune; whereas in the next poem the teacher sees his own language as a future force. The combination of English and isiXhosa in the poem emphasises the dual identity of many South Africans, and the language choices that they face daily.

My language – isiXhosa
Oh my beloved language
Oh my African language
Respected by the African speaking nation
amaXhosa, the sons of the African nation

I'm glad I'm black. I'm an African
I'm cheerful I can identify myself
I can write, read and speak isiXhosa
Awu axakekile amaXhosa ngengxoxo yakwaXhosa
My language, powerful, almighty language
You rose above whilst you were
Brutally murdered, tortured, destroyed
By those who were in power

Be strong my language, fear no one
Now is your time
The sky is the limit
IsiXhosa sama Xhosa – AkwaXhosana

Commentary from Kazeka:

Well, in 'My Language isiXhosa', this poet is using both languages, I think that it also represents the contemporary identity of South Africa right now. It is in the second paragraph where he says '*axakekile amaXhosa*'. He is saying the isiXhosa speaking nation is busy talking and negotiating all the issues of isiXhosa tradition. In the last paragraph where he says '*isiXhosa*', he is saying the language isiXhosa; '*samaXhosa*' meaning of the isiXhosa speaking people '*akwaXhosa*'—it is just an expressive idiom to say '*akwaXhosa*', meaning it belongs to you, it belongs to you.

Teachers identified their inner core of self with the spoken language, which legitimised their very being. Their essence, identity and pride are bound to their home language—"What am I without my mother tongue?"

Ulwimi lwam
Ndiyintoni na?
Ngaphandle kolwimi lwam?
Ndizingca ngani na?
Ngaphandle kolwimi lwam?
Ndakuziva njani na
Ngaphandle kolwimi lwam?
Ulwimi lwam ngundoqo kum.
Ndiyazingca ngalo.

Commentary from Kazeka:

In this poem, *Ulwimi lwam*, this poet is saying my tongue (my language). In the first line he asks this question, What am I? '*Ndiyintoni*?' What am I without my mother tongue? And then in the third line he says '*Ndizingca ngani na*', What am I? What is my pride? '*Ngaphandle kolwimi lwa*', without my tongue, '*Ndizakuziva* njani na?' How will I hear myself? And then he says in the last two lines '*Ulwimi lwam ngundoqo kum*', meaning my tongue is the essence of who I am. I pride myself in my mother tongue.

It is not only the isiXhosa-speakers who feel that their language is being superseded by English. Afrikaans-speakers share a sense of loss of identity and betrayal. Their children do not feel any pride in their home language and, in fact, eschew it. The negation of their cultural identity is the net result.

My Verlore Taal
My ma het my leer praat
In 'n taal wat sy lief het.
In Afrikaans het sy gesê, "Staan op,"
Terwyl die polisieman sê, "Lê plat!"

My pa het jou gebruik om te leer sing.
"Slaap, my baba. Slaap soet," het hy snags gesing.
Ek het dit vir my kind snags sag gesing,
Maar hy het "Tula, Tata, tula!" hard gesing.
Afrikaans, jy is in my bloed en siel.
Vir jou sal ek my lewe af baklei.
Ek het jou van my voorvaders gekry
En aan my kinders probeer oordra. Tog onsuksesvol.

My commentary:

The poet confides that his mother, who loved the language, taught him Afrikaans. He graphically mentions the political struggle in South Africa where his mother was metaphorically holding her head high while policemen were harassing her ("Lie down flat on the ground"). His own father sang him to sleep in Afrikaans, but when he sings to his son he is told to be quiet ("*Tula*", ironically an isiXhosa word). Afrikaans is in his soul and he will fight to defend the language, as it comes from his ancestors, but he is unsuccessful in passing this pride on to his children.

The personification of a language indicates that the poet felt a kinship, tantamount to a friendship, with the language. The loss of a language is like experiencing the loss of a friend; it is keenly felt:

Afrikaans, my taal
Afrikaans praat met my,
Sing met my,
Dans met my,
Afrikaans waar is jy?

Jou stem is stil.
Jou kinders veg.
Weggeneem teen jou wil.
Ek's reg ... jy's weg!

My commentary:

The poet compares Afrikaans with a friend and asks the language to sing and dance with him, but is bewildered as the friend's voice is quiet and he seems to have gone away. He concedes that there is strife in the friend's family and that Afrikaans has been forcibly taken away. He ends, 'I am right… you are gone!'

Teachers feel that the emphasis on English dissipates the eloquence with which pupils speak, read and write their home language. The effect thereof is not constrained to language only, as it insidiously seeps into a dissipation of history, cultural capital and traditions. Janks (2010: 116) expresses this same sentiment in prose. She maintains that proficiency in English increases children's linguistic capital but collaborates "in the destruction of their own instruments of expression". This sentiment is echoed in another of the teacher's poems:

Ulwimi LwakwaNtu
Yintoni midaka yakuthi ningalunakanga nje ulwimi lwenu
Ningasoze nisigqibe isivakalisi ningaphawulanga
kulwimi lwesiNgesi
Abantwana abakwazi ukufunda ulwimi lwabo
Andisathethi ke ngokulubhala, kunzima

Ningakhe nijonge kwezinye iintlanga nje?
Kuba zona zineqhayiya ngeelwimi zazo.
Sesiyilahlile nemveli yethu
Kuba kaloku sisityeshele isiXhosa sethu.

Commentary from Kazeka:

Ah, beautiful! This poet is saying '*Ulwimi lakwaNtu'*. *'kwaNtu'* is Bantu-speaking people. He is basically saying in the first stanza, Why have we let this language go? Then, he says in our days it is difficult for even isiXhosa speaking people, basically traditional people to even finish a sentence without including English, and then he says in the third line, the children can't even read their own language, never mind write it. He is saying in exasperation, You know it is difficult – '*kunzima'*. And then in the second paragraph he says, Why not look at other languages, '*kuba zona zineqhayiya ngamalwimi alo'*, which means people from other languages, like German societies, for example. They pride themselves on their languages and he says that we have lost even our traditions, and our culture and our history because we are looking down on '*isiXhosa sethu'* (our Xhosa). We are looking down on our language. Very powerful, very powerful…

Reiterating the sense of language being intertwined with one's identity, the next poem metaphorically equates mother tongue to the evocation of memories of comfort, satiation and smoky safety when recalling times when one's mother cooked mealie-meal porridge for breakfast on the family hearth.

Umbongo Ngolwimi
Ulwimi lwenkobe sisiXhosa
Sinxibelelana ngolwimi
Isizwe ngasinye siyazingca
Ngolwimi lwaso

Zilishumi elinanye iilwimi
Ezisemthethweni apha eMzantsi Afrika.
Sifunda ngolwimi

Lwazi ulwimi lwakho!
Luxabise ulwimi lwakho
Luthethe ungagqwidizi!

Commentary from Kazeka:

'*Ulwimi inkobe*' is your mother tongue. If you had to translate 'mother tongue' from direct English to direct Xhosa, it would actually mean '*ulwimi lwakwamamakho*', but it is not said like that. That is the wonderful thing about the Xhosa language, it is more experiential. They describe the experience and come up with the language. So when they say '*ulwimi lwenkobe*', '*lwenkobe*' is that corn that your mother makes usually in traditional society. So it refers to that mealie-meal, that corn, that your mother makes, that is why it is '*ulwimi lwenkobe*' meaning your mother tongue. It is attaching the experience of making your corn with your mother. Isn't it beautiful? And then he goes on to say, she or he, goes on to say '*lwimi lwenkobe sisiXhosa*' is how we come to understand each other. It is how we come to communicate with one another and languages and nations. '*Ngasinye Ngolwimi lwabo*', meaning nations take pride in their languages and then he says in the second paragraph '*Zilishumi elinanye*', meaning there are eleven languages under the constitution, under the law of South Africa. So language is part of our identity in that we study in language. Then it goes in the last paragraph to just encourage you to know your language, value your language, speak it without shame.

The following poet intrinsically suggests the political nature of language and the rallying cry that it can evoke. This reiterates Setati's (2008) supposition that language is always political.

isiXhosa
Lulwimi lwam lwenkobe
Luyandonwabisa ndakulila
Lundinika yonke endiyifunayo
Luyandithuthezele ndakuxakwa
Phambili ngesiXhosa phambili!

Commentary from Kazeka:

Well in this poem she is saying (it is a lovely short poem and to the point) '*Lulwimi lwam lwenkobe*.' This is my mother-tongue. It comforts me when I am crying. It gives me everything that I need, I take it spiritually. It, *ja*, it paves the way of wisdom when I am lost in my way And then it says '*phambili ngesiXhosa phambili!*' Forward with this Xhosa forward! That sounds like an ANC rally. Well it goes to show that even language is politicized. It is a political discourse also …

The following poem describes the challenges of language in the classroom vividly and addresses the issues and difficulties prevalent in multilingual classes. The teacher shows how pupils are excluded from the dominant educational discourse, not because they do not understand the concepts, but because they do not understand the vocabulary of the question.

What do you see in the picture?
What do you see in the picture?
No one answered
No one understands the question
What do you see in the picture?

Punish those blind fools, who do not answer,
Give them six lashes on the buttocks
Punish them again and again and again,
What do you see in the picture?

All books are wide open,
All eyes are looking at the book,
All minds know what is in the picture.
No one understands the question,

No one answers the question.
No one would have been punished
Had the question been phrased in Xhosa,
Had the answer been said in Xhosa!
Ubona ntoni emfanekisweni?

If the question, "What do you see in the picture?" had been translated into isiXhosa, "*Ubona ntoni emfanekisweni?*" the poet maintains that the frustration and angst in classrooms would dissipate. The sense in the poem echoes Gee (2008) who notes that cultural models, including language choice, exclude certain pupils from participating in academic discourse.

Conclusion

Teachers in the Eastern Cape are doubly challenged in school classrooms. Firstly they have to grapple with their pupils' lack of competence in English (the LoLT) and secondly they have to impart content knowledge. The power of language in this instance leaves the pupils in the position of the disenfranchised as they are denied access not only to English (and all the glittering possibilities of advancement the language proffers) but also to sense-making opportunities for learning, as they are not the insiders in the communities of practice of academic discourse.

The aim of this study was to enable multilingual teachers to reflect on their lived realities and to expose possible conflicts and contradictions that they face when teaching in a language that is not the pupils' home language. The poetry unlocked emotions that the teachers had not necessarily dealt with in an open platform. By writing their own poems, and by reading poems written by other teachers, the teachers came to the self-realisation that their thoughts and feelings were valued. They were able to create a common ground among them where there was a sense of safety, trust and solidarity. Through self-reflection they realised that their conflicting perceptions were shared, so they were intrinsically motivated to take responsibility of their own situations and could shift from being controlled to being in control.

Poetry has a unique quality in that it can convert feeling into form through the evocative role of language. The multilingual teachers wrote of the conflicts and contradictions that they experience in their lives and in their classrooms. The themes of power, identity, access and language usage were inextricably interwoven in the poems. The urgency for pupils to gain proficiency in English was considered to be of paramount importance. In contrast, some poems revealed the deeply suppressed suffering which the marginalisation and disempowering of their mother-tongue, both isiXhosa and Afrikaans, has had on their cultural identities. With this came the realisation of how they were colluding in the distancing of pupils from their own identities and vernacular competence, as well as from access to the learning process through their insistence on English and denial of the academic value of isiXhosa. Perhaps the most passionate calls were to retain pride in their cultural heritage. By writing poems in isiXhosa and Afrikaans the teachers demonstrated that they felt they could express emotions in their home languages more eloquently than they could in English, and implicitly realised the constrictions experienced by their pupils in class.

Once teachers acknowledge the conflicts and contradictions they face in multilingual teaching and learning, the next step is to research language choices and strategies that could creatively utilise the power and impact of languages, both English and their home languages, so that access is enabled and identity retained and celebrated.

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Mathematics at University: The Anthropological Approach

Carl Winsløw

Abstract Mathematics is studied in universities by a large number of students. At the same time it is a field of research for a (smaller) number of university teachers. What relations, if any, exist between university research and teaching of mathematics? Can research "support" teaching? What research and what teaching? In this presentation we propose a theoretical framework to study these questions more precisely, based on the anthropological theory of didactics. As a main application, the links between the practices of mathematical research and university mathematics teaching are examined, in particular in the light of the dynamics between "exploring milieus" and "studying media".

Keywords University mathematics · Tertiary · Anthropological theory of the didactical

A Personal Introduction

To make my choice of subject and viewpoint in this lecture more transparent I will start by introducing my background briefly, not because it is particularly interesting or unique, but for two other reasons: (1) I feel it could and should become more common for mathematicians to specialize in didactics; (2) to explain the choices I have made as a function of my background.

In short, I am a mathematician. Mathematics is the scholarly field in which I learned to do research, and in which I have continued to work even if my subjects and the scope of my work has evolved over time. It should be noted up front that unlike what is common in social and natural sciences, research in mathematics is usually not concerned with what is commonly understood as observations, experiments or data. My thesis dealt with automorphisms on inclusions of von Neumann

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algebras (so-called *subfactors*), objects which can neither be observed nor measured. Indeed, developing, combining and exploring mathematical theories and methods is what I have learned to do professionally and what I continue to do.

Mathematicians do not only think for themselves. They communicate and in particular, they publish and teach. As an article author, conference presenter and instructor, and later university professor, I had (as does any mathematician) to think about mathematics from the point of view of didactics, that is: how to communicate mathematical ideas (my own and those of others). And as a teacher, you are not only faced with the task of convincingly transmitting your ideas to other mathematicians, but also with the task of "empowering" students to do what they need to be able to do in a more or less autonomous fashion. This turns out to be a challenge which most mathematicians find quite interesting and difficult. And it is at the heart of the deep relationship between mathematics and its didactical realm.

The relation between mathematical scholarship and mathematics teaching is so fundamental that Brousseau (1997) begins his seminal book on the "theory of didactical situations in mathematics" by explaining the relationship between the work of the mathematician, the work of the student, and the work of the teacher: the teacher has to *reconstruct* situations of learning for the students, given (often very old and transformed) *products* of what is, essentially, the mathematicians work (namely, mathematical knowledge in generalized and decontextualised form). And, while this implies that there is a kind of inversion problem at the centre of the work of the teacher, such problems are by no means foreign to what mathematicians themselves have to do all the time, not only when they act as teachers but also in their function as researchers, since one may say that

> 80 percent of mathematics research consists in reorganizing, reformulating, and "problematizing" mathematics that has already been "done", by the researcher himself or by others (Brousseau 1999).

In fact, mathematics is—perhaps more than any other science—one in which important progress may be based on simplifying, generalizing, combining or even reformulating previous work, and in which, therefore, there is no sharp boundary between "presenting" and "developing" knowledge, or between the learning of the mathematician and the learning of the student (even if the former is certainly supposed to go beyond what is presently known by the community at large or by his students). We will return to this last point later in this paper.

I was fortunate enough to discover Brousseau's work around 1999, during one of my first searches for relevant knowledge on teaching. It was important for me to realize that teaching mathematics is essentially about reconstructing and developing situations in which mathematical ideas and methods become operational for students—and that solving these *didactical tasks* has an important and non trivial mathematical dimension. To solve them requires (hard) thinking about the mathematical content. At the same time it also requires observational knowledge about the students who are supposed to learn the content, and this evidently requires an empirical dimension which, in terms of research, goes beyond the mathematical domain.

In fact it also requires theoretical tools which are not in themselves mathematical, but which allow human activity and institutional phenomena to be modeled. This need for non-mathematical tools leads many didacticians to draw on human and social sciences such as psychology and anthropology. But these sciences have not developed the sharp tools needed to capture essential parts of a mathematical activity in a teaching context, in a way that will bring about functional knowledge and not just general beliefs. Thus, "applying anthropology" (and for other reasons, psychology) poses several severe problems; we refer to Chevallard (1990) for an interesting review of early work in this direction. We now turn to outline some elements of an anthropological theory of the didactical (ATD), developed by Chevallard and others, in order to deal with these problems, with a specific view on the institutional context of the university. In the rest of this paper, we refer to it as "the anthropological theory".

The Anthropological Theory and University Mathematics

One of the major challenges for mathematicians and mathematics teachers alike is that they have little or no precise means to describe mathematics itself. Moreover, it is highly questionable whether "mathematics" makes sense as a homogenous whole, smoothly developed over time and across institutional and cultural boundaries which are, to some extent, obvious to the practitioner. Most evidently, school mathematics appears to be *different* in essential respects from the way mathematics is thought of in university—for instance, numbers and shapes usually appear somewhat naturalised in the first setting, while they are presented and treated as theorised (for instance, axiomatised) structures in the second. These remarks are far from precise and satisfactory descriptions of course. The historical event of "new mathematics" in the interface between university and school mathematics contributed to spur interest in the difference and sharper descriptions of it.

Institutions, Knowledge and Didactic Transposition

Even natural numbers are no longer "natural" (or God-made, as Kronecker reportedly said) to the contemporary mathematician; and explicit constructions or axiomatizations characterize the way in which familiar numerical objects such as 27, $\sqrt{2}$ or e are situated in the wider mathematical universe, along with triangles, Hilbert spaces, and so on. Most mathematicians are not overly concerned with the philosophical issues surrounding the foundations of their subject (see e.g. Davies and Hersh 1981). But even if the importance of formalization varies from one subfield to another, it is unquestionable that contemporary mathematics has developed more precise and explicit descriptions of its objects through formalizations and abstractions which are far from common sense notions of numbers, shapes and so on. It can also be argued that this difference has grown immensely over the past two centuries—where, at the same time, mathematics has become a school subject taught to entire populations world wide. This reflects the progressive growth of two kinds of *institutions*: universities and schools, each established to circulate and develop *knowledge*.

The anthropological theory begins with clarifying what "knowledge" and "institutions" mean—and in particular what they mean for the teaching and learning of mathematics. First of all, *knowledge* is taken in a quite wide sense, roughly "shared human ways to act and react to specific challenges"—including practical techniques, explanations, theoretical notions and so on. We return to a more detailed model of mathematical knowledge in the following subsection. Secondly, as the term "shared" used above indicates, it is fundamental to human knowledge that it is developed and circulated in communities which possess a great deal of regularity, even if the members of the community change (think, for instance, of a community of research mathematicians or of researchers working on a subfield of mathematics). These "invariant communities" serve as habitats of knowledge as they develop, sustain and transmit the knowledge. Such habitats of knowledge—are called *institutions*. Notice here that the definitions of *knowledge* and of *institution* are 'solidaric' in the sense that one cannot define them independently: in particular knowledge attains its status as knowledge *relative to an institution*.

In contemporary educational systems, knowledge from institutions "outside the school" is specified as "knowledge to be taught" in school institutions—and it subsequently is transformed into "taught objects" by the school institution. This two-step process is what is called *didactic transposition* in the anthropological theory (Chevallard 1991, p. 39). The word transposition means that the process involves "moving" knowledge between institutions—and thereby, inevitably, adapting and modifying the knowledge according to the constraints of the receiving institution. As a case study, Chevallard and Johsua (1991) provide a detailed account of the transposition of the modern mathematical notion of *distance* (as defined in metric space theory) into the French secondary Schools of the early 1970s.

Mathematical Organisations

Up to now we have used rather broad and soft terms. We now come to what I perceive as the core of the anthropological theory of didactics, namely its tools to model mathematical and didactical knowledge (cf. Chevallard 1999, 2002, which we outline and interpret in the sequel). Let's consider what we loosely called the "notion" of *distance*, just mentioned. And what does it mean to "master" it? Well, first of all, it involves that you recognize a distance when you see it—that is, you can determine whether a given entity is one. More broadly, you can solve *tasks* involving distances—that is, you can apply certain *techniques*, defined as means to accomplish tasks. A technique usually solves a whole family of tasks—a *type of tasks* T, defined as those tasks which can be defined by a given technique τ. The couple (T, τ) is called a *practical block*; the two elements define each other. This is the minimal entity of *practical knowledge*. In the context of distance, we might for instance think of T as "compute the distance between two points in the plane", which—when given the coordinates—can be accomplished by the technique τ associated with the usual distance formula. In more elementary contexts, the points might be simply marked on a piece of paper, and the technique could be to execute a measurement with a marked ruler. And in more advanced ones, we might have different practice blocks, corresponding to other distances, such as the L^2-distance on the function space $L^2(0,1)$. The notion of *context* for a practical block can also be made more precise: it consists of some *technology*, a discourse about the techniques, which explains how to apply and distinguish a whole set of techniques. For instance, to distinguish and explain the two techniques for computing distance which applies in the Euclidean plane and in $L^2(0,1)$. At a higher level of discourse, technologies are developed, explained, related and justified in and by a *theory*, which in our case could be the theory of metric spaces and in particular include the definition of metric distance. For a given set of practical blocks, a technology θ and a theory Θ form together what is called a *theoretical block* (θ, Θ). Together, the practical and theoretical blocks form a *praxeology* $(T, \tau, \theta, \Theta)$. These form the "atoms" of mathematical practice and discourse; whenever faced with a task, the mathematician—or the student—will seek to identify it with a type of task and hence with a technique, which is then applied to solve the task; he might go on to explain and justify his choice of technique within a technology, and he might even be able to explain how this technology can be explained and justified within a theory. Of course, none of these derivatives of the task are universal or uniquely defined by the task, but in a given institution some technique, some technology and some theory may appear natural, privileged or even optimal to users. This, certainly, would be the case for the distance formula, when faced with the task of finding the distance between two points given by coordinates, in many secondary schools.

Institutions are habitats of *praxeologies*, and these do not occur as independent atoms. Because technologies (e.g. explaining how to compute distances in different types of tasks) may serve to relate several practical blocks, mathematical practices are unified by technologies; a *local mathematical organisation* is a family of praxeologies defined by sharing one technology. Similarly a host of local mathematical organisations may be unified by one theory, to form a *regional mathematical organisation.* The power of modern mathematical theories resides exactly in unifying a host of previously unrelated local organisations, so that for instance the two inequalities

$$|a| - |b| \leq |a - b| \tag{1}$$

(holding for real numbers a and b) and

$$\sqrt{\int_X |f|^2 d\mu} - \sqrt{\int_X |g|^2 d\mu} \leq \sqrt{\int_X |f - g|^2 d\mu} \tag{2}$$

(valid for L^2-functions f and g on a measure space (X, μ)) could be justified from one and the same principle (the triangle inequality in the definition of a metric). Notice that the two inequalities—with explanations of their range of validity—could themselves appear as technologies to explain and justify calculation techniques, corresponding to a type of tasks, arising in more or less distant practices. The progressive development of still more encompassing theories and technologies—and with them, of regional mathematical organisations—is not only an essential part of the history of mathematics but also of the curriculum of mathematics students.

To make such developments meaningful and functional to students, they need to be (or in fact, remain) rooted in tasks which are somehow simplified or at least related through the unification of the practice blocks in which they live. Of course, historical developments are often more complex than what one can (or wants) to let students experience in an undergraduate course. An important recent area of didactical design is the use of *instrumented* techniques (based on computer algebra systems) as a means to facilitate the access of students to coherent theoretical blocks of real analysis (see Gyöngyösi et al. 2011).

Didactic Organisations and Didactic Co-determination

Related to any mathematical organisation which the students have to learn, we have *didactic tasks* for the teachers, which can be of various kinds and types but which are always linked to the challenge of establishing conditions and experiences which allow students to become familiar—at some level—with a specific mathematical organisation. As for mathematical tasks, didactic tasks are solved by *didactic techniques* (giving rise to *types* of didactic tasks) and also when we explain, relate and justify those techniques, we are using *didactic technologies and theories*. In short, *didactic organisations* arise in close association to a mathematical organisation to be taught to a group of students. The theoretical level may, indeed, be less formalised and well articulated than in the case of mathematical organisations; to improve that is part of the vocation of didactics as a scientific field. At this point, we content ourselves to give one example and to explain what is meant by the co-determination of mathematical and didactic organisations.

Consider again the notion of distance. In a first course on metric space theory, as the one studied and developed by Grønbæk and Winsløw (2007), it is a basic challenge to allow students to master the regional mathematical organisation, unified by the theory of metrics and relating technologies associated with (apparently distant) techniques as reflected in (1) and (2) above. An important type of didactic task is, then, to device (or choose) *mathematical tasks* which somehow require the student to make use of the abstract notion of metric distance. One technique is to simply give examples of spaces M, familiar to the student, together with a real valued function d on $M \times M$, and ask if the axioms are satisfied; this, indeed, is a type of mathematical tasks. A less immediate technique is to build such tasks into mathematical tasks where the validity of the axioms can be used as a technique—perhaps even before the axioms are formulated in general. Indeed, (1) and (2) can be viewed as a (slightly) concealed and (clearly) contextualised form of the triangle inequality which, together with other forms, might serve to develop the students' appreciation of its importance in general. The explanation and justification of different didactic techniques in this case would depend on wider theoretical ideas —or results, or beliefs—about how to teach the mathematical organisation in question, perhaps unifying a number of other didactical organisations as well. However, to be useful, a didactic theory would need to apply—and hence be directly associated with—the concrete didactic practices the teacher could implement, and not least with the mathematical organisations this (theoretically or empirically) enables students to develop. As an example, Grønbæk and Winsløw (2007, Appendix 1) developed a hierarchy of specific competency goals, related to the mathematical contents and specific ways in which it should be mastered by students, to articulate the distinction between mathematical tasks whose accomplishment corresponds to just one such goal, and more advanced tasks which accomplish (parts of) several goals.

This brings us to the final theoretical point: while it is clear that didactic organisations make no sense independently of specific mathematical organisations *to be taught*, it is also clear that the mathematical organisations *actually realised* with students depend crucially on the didactic organisation developed by the teacher. And, as it develops, the students' mathematical organisations are observed by the teacher and the didactic organisation may be adapted according to these observations. There is, thus, a fundamental *co-determination* between mathematical and didactic organisation in a teaching setting, and this co-determination is an essential mechanism of the didactic transposition. First of all, the "knowledge" which is transposed is now modelled much more explicitly by mathematical organisations, with its four levels ranging from tasks to theory. Moreover, the transposition as it is realised *within an institution*—through teaching—involves two distinct forms of mathematical organisation: the one to be taught, and the one actually taught and learned. The means of this part of the transposition—called the *internal didactic transposition*—is modelled as a didactic organisation, involving both practices and theories which, moreover, is *co-determined* with the mathematical organisation developed by students and teachers together, and observable—in principle—in the space-time of their interaction.

University Mathematics and School Mathematics

While the anthropological theory makes sense—and was developed for—the teaching of mathematics in school institutions (for a general public), we now turn to the other fundamental question of this paper: what, if anything, distinguishes university mathematics teaching from the teaching of mathematics in school? And, assuming that important differences exist, what are their consequences for the didactical organizations to be developed in a university institution?

Didactic Transposition in Universities

Just as school institutions vary from one society to another, there is a considerable difference in how universities operate both in different places and over time (cf. Madsen and Winsløw 2009, 741–742), and in particular how decisions are made on mathematical knowledge to be taught in different programmes. However, it appears evident that university teachers, as a collective within the institution, tend to have a higher autonomy than school teachers, in terms of deciding the contents and methods to be used in a given teaching unit. This autonomy, however, may not always be so much bigger in practice as it is in formal principle. For instance, one of our respondents in a recent study (Madsen and Winsløw 2009) on university teachers' praxeologies in research and in teaching, who is a senior mathematician, claimed that

> I think we should emphasise the [mathematical] work method more, I think, you can easily go through especially the bachelor programme without learning the work method, just learning mathematics in a mechanical way (…) There is a syllabus, a description of the course, this we must teach, also because other courses can see, we build on this course, and there they learned this and that, in our system it doesn't work if students in different years learn different things in the same course.

Within an undergraduate programme, course units—within and outside pure mathematics—build upon each other, and the result is often that the teacher finds himself obliged to teach a large—perhaps too large—number of mathematical organizations, typically local ones. So the phenomenon of *thematic autism* was identified by Barbé et al. (2005) in the context of Spanish high school, and can be described as a didactic organisation which presents one local mathematical organisation after the other, in rapid succession, but with few or none of the mathematically relevant bonds (at the level of theory, sometimes even at the level of technology) between them. This phenomenon can easily be found also in universities, especially in introductory courses such as calculus, linear algebra and statistics, meant for a large number of students.

So, internal didactic transposition is more comprehensive in universities, although it has some of the same effects as the constraints which schools have imposed from outside (e.g. the requirements of a national curriculum). However university teachers may react collectively to malfunctions of the mathematical organisations to be taught in a given context, for instance in the case of a perceived overload in a course unit, inefficient sequencing of the organisations, etc. In most universities, they also have a larger command over the *assessment apparatus*, that is the means to evaluate students' command of the target mathematical organisations. In practice, this gives considerable freedom in terms of didactical organisations and it is often possible for the teacher to align new elements in didactical organisations (e.g. teaching techniques, new types of mathematical tasks for students, etc.) with new methods in assessment; this in fact was crucial in the ambitious redesign of real analysis teaching presented by Grønbæk and Winsløw (2007). For this reason, I believe it is both appropriate and useful to consider evaluation techniques, along with their technological and theoretical components, as part and parcel of didactic organisations. This is particularly relevant in university mathematics where evaluation techniques can usually be described with high degrees of precision, although more advanced techniques (involving more involved qualitative judgments) are more difficult—and therefore more important—to describe and justify, especially to students.

The Teaching-Research Nexus

However we cannot understand the institutional conditions of university didactical organisations without paying attention to the presence in universities of another kind of praxeological organisation, namely those arising from *research tasks*—such families of praxeologies are called *research organisations* (Madsen and Winsløw 2009, p. 747). While the disciplinary core of these research tasks is of course related to the mathematical organisations taught in university, didactic tasks and research tasks are different in many respects, some of them obvious and general (e.g. reward structures, time perspective etc.). At the same time they are commonly carried out by the same people—university faculty. An impressive literature exists on the general conditions for the interplay between didactic organisations and research organisations (for reference, see Madsen and Winsløw 2009). As in this literature, we shall refer to the highly complex interplay between research and didactic praxeologies as the *teaching-research nexus.* It can be considered at hugely different levels, from the individual faculty member to worldwide trends (e.g., in the correlation between quantitative measures of universities' performance on research and teaching). The general "higher education" literature does not ignore that the teaching-research nexus can be expected to evolve quite differently, depending on what discipline is taught. But the term "general" also means exactly that it does not study this specificity in a systematic way. Thus the occasional pertinence of its results, for the case of university mathematics teaching and research, remains accidental. As a matter of fact, the comparative case study of Madsen and Winsløw (2009) demonstrated that the teaching-research nexus in this field, and at the level of the individual faculty member, is hugely different from what is found in a relative scientific discipline (physical geography).

As a university discipline, mathematics is indeed quite special. It can even be questioned if it is one single discipline. Besides study programmes on pure mathematics, which are in fact quite similar in content and methods across the world, we have a huge variety of other study programmes in which more or less wide mathematical organisations are taught—engineering, business, natural science, and teacher education, to name but the most important. The research organisations which are relevant to consider for those other study programmes are not always concerned with pure mathematics, and indeed the teachers often have other fields—related to the character of the study programme—as their research specialty. At the institutional level, this can give rise to overt or latent conflicts of interests, since allocation of resources to research in a given field is very often tied to the volume of teaching carried out by researchers of that field. These issues are certainly of importance when it comes to explaining some of the important constraints that weigh on university mathematics in general, and what we said above about the disciplinary specificity of the teaching-research nexus could be expected to hold also when it comes to teachers of similar (or identical) mathematical organisations, but with different research fields. However, to my knowledge, we have little research evidence, for instance, of how the nexus differs in the case of physicists and mathematicians teaching calculus to physics students, and how this affects the didactic organisations and the research organisations of those teachers.

In my own research, I have mainly considered the classical case of university mathematics teaching delivered by researchers of pure mathematics to students whose field of study is pure mathematics. But at least at the bachelor level, this does not imply that the students are necessarily aiming to become researchers in pure mathematics—and in most contexts, this will be possible only for a small minority. Also, at the undergraduate level, the mathematical organisations taught (e.g. from calculus or linear algebra) will not often be very close, and almost never identical, to the mathematical organisations which the researchers develop in their research. This means that the research organisation of the teacher cannot be used as a clear guideline for the design of the mathematical organisations to be taught, or for the associated (co-constructed) didactic organisations. At the same time, there is a widespread belief among mathematicians (evidenced in the study by Madsen and Winsløw 2009) that an *implicit nexus* may still exist, at the level of didactic and research practices. What this means is hinted at by the term "work method" used by our informant in the quote given above; one could use other terms like "modality", "approach" or "style" to indicate the idea that the teacher strives to develop didactic techniques that allow students to work with mathematical organisations in ways that are "similar" to the work of the researcher. Moreover developing such didactic techniques is experienced, by the informants, as somewhat "similar" to research work. This is clarified by examples, such as the following explanation of a mathematician teaching first year calculus (Winsløw and Madsen 2008, 2384f):

> We give them relatively open tasks. (…) That is, where it is not just an exercise, with a question and a specific point in the text book to refer to and a unique answer. (…) For instance (…) we ask them to compute π by using the formula for a function, like Arctangent or something, which gives π up to some decimal point, and then use the Taylor series at a point where they know it [*the value of the function*]. That I would say is a completely standard exercise. They also learn to estimate the error. But then we can go on and ask, can you with certainty find the first 100 decimals in π, using this method. Or when can they be certain that they found the first 100 decimals. (…) And that we give them as an open task which we don't even ourselves completely, well of course we could, but we don't even consider beforehand if we know precisely what the perfect solution would be, because we are not after the perfect solution, we are after *them* thinking about what problems are involved in this task. [*explains the central difficulty of evaluating the error term*] (…) if they just explore this *problematique* we feel they have achieved a lot. And that we think reminds us of our own research, as for the processes (…) that type of exercises we give a lot. (…) We take a standard exercise, and open it up a bit (…) to see how far can you go on this type of task. (…) they need to get the experience that here they have to explore a domain by themselves. (…) That in a way you could call a kind of research. (…) One thing is to prove a theorem with all available mathematical methods. But to ask yourself, can you

> prove this theorem by just using the following methods, it is in itself a mathematical question. (…) [*similarly*] we asks ourselves, can we solve this *without* using this or that result.

We notice here that a didactic technique, related to the construction of mathematical tasks for students, is central to the teachers' explanation of how his teaching relates to his research. The phrase "take a standard exercise, and open it up a bit" becomes part of en emergent didactic technology where the central aim and justification is to get the students "thinking about what are the problems involved in this task". More "open" tasks are ones for which the student has not be given a standard technique—as in the case of a "standard task"—but where the student has to "explore" the problems it contains, and presumably associate some of those with known techniques. It is also interesting that the teacher needs not to have solved such a task "completely" but could consider the possibility of working along with the students "to see how far can you go on this type of task". The expectation that teachers know "the perfect solution" in advance is indeed common in mathematics teaching, at least at school level; but it is of course a radical difference from the mathematical practice of the researcher. It is clear that one cannot just ignore the expectations—and responsibilities—related to didactic organisations; indeed all informants stress a number of constraints (examinations and syllabus volume in particular) which excludes any excesses in having students "explore a domain by themselves". But the quote indicates at least one direction that university teachers of mathematics pursue, in view of developing didactic techniques to induce students into "research-like modes" of mathematical work.

Another important point in the quote is that the construction of the mathematical task—using the didactic technique just discussed—is considered, by the teacher, "a kind of research". This in principle brings us back to the "task of the teacher" explained by Brousseau as the "inverse" of the task of the researcher, namely to establish a problem situation for the student, corresponding to established knowledge. In fact, for the didactic technique considered above, there is a fine dynamics between the way you "open" a standard exercise, and the difficulties of solving it *using the constrained set of techniques the students know.* And indeed, it is fully acceptable in mathematics research journals to prove a new proof of a known result, especially if the new proof is more elementary or "direct" in the sense of building on a smaller set of technical prerequisites. This kind of mathematical research practice may thus be particularly interesting also as a model for "the teachers' work" to construct problems for students, even if the average school teacher will not often develop entirely new ones. It also shown that the implicit nexus very often comes down to differences in *degree* and *domain*, rather than in *principle*, between the mathematical experience of researchers and school teachers.

The difference in principle between the tasks of research organisations and didactic organisations is, evidently, the criteria for solutions: *novelty and interest* to the international research community, and *efficiency* in teaching a mathematical organisation to students. The implicit nexus gives up both criteria for research products, and focus on processes which can, indeed be similar, as explained.

However, there are also important differences in those processes which have not been mentioned above, and which relate to the ways in which *established* knowledge is used and studied in the researchers' work; these will be discussed in the next section. We stress that the problems of transition to and within university mathematics teaching cannot be considered solely on the basis of work modalities, but need to be addressed through a study (and design) of the exact mathematical praxeologies of students, as explained above.

The Dialectics of Media and Milieus in University Mathematics

The term "research" was used above to indicate the work of the mathematician, aiming at constructing mathematical knowledge which is *new* in the sense that it has not been previously published, and which is of interest to the international research community. Certainly this involves thinking about challenging tasks, combining and developing new techniques etc. In the introduction, we stressed the importance, in this activity, of consulting existing works—publications etc. While this part of the mathematicians work may seem, at first, much closer to what students do, there are also crucial differences in common practice.

The term "research" is sometimes used in a different sense, to indicate an activity of problem solving, carried out without consulting literature or other resources not given with the problem. To avoid confusion we shall avoid, here, this use of the term research, and focus on establishing a clearer distinction between two fundamentally different kinds of "arenas" for the search for knowledge (for students and researchers alike), called *media* and *milieus*. Both are external resources—material or immaterial—which an individual may use for developing his knowledge, e.g. to solve a task. Chevallard (2007) defines the difference by the presence of an intention (to inform, represent, etc.) in the media, and the absence of intention in a milieu (which operates like a kind of "mathematical nature"). Here the intention is understood as relative to the specific knowledge which the person seeks. For instance, in a didactical situation, the answer to the problems at stake are not given in the milieu, it has to found by the students themselves, by adapting their knowledge to the milieu (Brousseau 1997, 40). By contrast, the teacher—as well as books, Internet pages and fellow students—may act as media, to the extent they provide the solution to the problem.

This definition is admittedly a bit vague but the distinction is rather operational in practice. For example, a student may stumble upon the formula $\tan(x/2) = (1 - \cos x)/\sin x$ in an Internet forum (a media, in which indeed intentions to instruct etc. are present). If the student wants to test if this holds, it may include watching a few cases with his calculator (milieu), trying some cases with paper and pencil (milieu), accessing other Internet pages or books (media), etc.

To seek knowledge in a milieu is therefore fundamentally different from seeking it in a media, because in the first case, the person has to adapt his knowledge to the *problem* in order to construct a solution, and in the latter case, he adapts his knowledge to a solution which is offered. As we have already noted in the introduction, both modes of mathematical activity form part and parcel of the work of the mathematician. How does it occur in university teaching of mathematics?

First of all, both are certainly present in any university course. The student will study media, typically a text book, in which a lot of instruction is given. During lectures, the teacher acts similarly, as entirely media, even he places himself in small milieus (with knowledge to be found), for instance to demonstrate a solution method. At the other end of the scale, the teacher can create a challenging milieu for students without any indications of a solution. More typically, the teacher offers very restrained media and relatively small milieus (like "End of chapter exercises", where the key techniques can be found in the preceding chapter).

Two American Propositions: The Moore Method and Undergraduate Research

There are many attempts to improve the quality and scope of milieus presented to students in didactic organisations of university mathematics. Besides offering more challenging and "open" exercises or problems, as mentioned above and in many other didactical designs (e.g. Grønbæk and Winsløw 2007), we have also ambitious formats to teach mathematical theory as a kind of problem solving. One remarkable example is the "Moore method" (se e.g. Chalice 1995). A classical Moore course just provides students with key definitions and results, while everything else—in particular proofs—have to be constructed by the students. To Moore, the exclusion of media appears as a principle of primary importance:

> Moore encouraged competition. Do not read, do not collaborate - think, work by yourself, beat the other guy. Often a student who hadn't yet found the proof of Theorem 11 would leave the room while someone else was presenting the proof of it - each student wanted to be able to give Moore his private solution, found without any help. Once, the story goes, a student was passing an empty classroom, and, through the open door, happened to catch sight of a figure drawn on a blackboard. The figure gave him the idea for a proof that had eluded him till then. Instead of being happy, the student became upset and angry, and disqualified himself from presenting the proof. That would have been cheating - he had outside help! (Halmos 1985, p. 258)

The Moore method promotes, in an original way, certain similarities with the work of the mathematician, namely the experience of establishing formally, by brute force, a result whose validity is ensured. However, even more than the average university course, it omits completely the experience and familiarity with consulting relevant media.

Today, the device of "undergraduate research" is probably more widespread in the USA than the method of Moore. The programs of different universities differ, but generally it is an activity which teachers (as directors of the "research") and students undertake voluntarily, during some months or a whole year. For students, working individually or in small teams, the format involves choosing a mathematical subject or "problem" (more or less open), exploring it in the literature, and then producing a product (paper) which reflects the work done. The shape of these texts is normally that of an article by researcher, complete with a bibliography that reflects an activity of study, motivated by the selected problem. There are many online journals that publish only such items, either from a single university or national publications such as the Pi Mu Epsilon Journal, which regularly publishes research subjects (see e.g. Ahlin and Reiter 2010). Conferences are held where students present and discuss their work in addition to the experience, which is also important in mathematicians' research practice.

In many mathematics programs, at least in Denmark, the master thesis has features which resemble the American undergraduate research format. It requires students to present a topic based on journals and research monographs. This could, a priori, appear as a mere task of study and reconstruction of media. In reality, it is not so. Let me share my personal experience. For my master thesis, I was asked to produce a presentation of key findings in an article by Connes (1973). The proofs in this text are terse, at least for a student (with a lot of "we easily see that", etc.), so I often found myself in a situation similar to the students of a Moore course, having essentially the result and the task to prove it. And when I had to resort after all to my thesis supervisor, he often ended up doing for me what I had not managed to do: construct a proof directly without worrying about what the text said, except that the result was true. That is to say that what appears to be a study of media often transforms into the search for knowledge in a (terse) milieu. It is almost a general rule that the study of mathematics journal papers involves autonomous construction of solutions, which may indeed lead to new knowledge (as alternative forms of the proof, for example). In other words, the study of professional "research" media in a didactical organisation almost automatically leads to a dynamics of media and milieus which is, locally, close to the work of the mathematician authors of those media. Of course, the mathematical organisations exposed in research journals are typically beyond the reach of undergraduate students.

An emergent challenge for university teaching of mathematics, in the view of the examples we have just considered, is therefore on the side of the study of media, more than on constructing challenging milieus. Except for optional and advanced contexts, it seems that the autonomy of students with respect to accessing and choosing media remains very low, and so in that sense, the students' work is rarely close to mathematical research practice.

An Example from an Undergraduate Mathematics Course

To illustrate that the dilemma is one of entire programs, rather than individual courses, allow me to mention a recent experience from teaching a course of the capstone type, aimed at easing the transition from university mathematics to high school teaching. The example is related to a topic which has recently become important in the Danish high school mathematics curriculum, but which is barely present in the undergraduate mathematics programme. It is that of linear regression as a tool for empirical modelling of data. The reason why linear (and other types of) regression has become more common in the secondary curriculum is, besides the uses in extra-mathematical contexts, the ease with which one may compute the "best fit" model from data, using technological devices (a handheld calculator or a personal computer). However, the computations are still based on a mathematical method to find the optimal model. In the case of linear regression on a set of data $(x_1, y_1), \ldots, (x_n, y_n)$, the linear model is defined by the line $y = ax + b$ where a and b are chosen such that the square sum error

$$S(a,b) = \sum_{k=1}^{n} (y_k - ax_k - b)^2$$

is minimized. The task for the students of the course was to produce an explanation of the formulae for a and b in terms of the data, accessible to high school students. The formulae appear in some Danish high school text books, without explanation. Most of the students used hints or proofs from university text books, all based on finding the (unique) critical point of S and proving that it is a minimum, using the Hesse matrix and an argument based on the sharp Cauchy-Schwartz inequality. To find a solution which is accessible to high school students requires a more intensive and determined study activity. Depending on the media identified, this has to be combined with autonomous work to simplify and explain the method (an elaborate "completion of the square", similar to the proof of the quadratic formula).

Institutional Constraints to the Dialectics of Media and Milieus

In addition to professional mathematics journals, a wide variety of mathematical resources, at all levels and within all subjects, is available on the Internet. An obstacle to making use of them appears in the programs and more specifically, in the institutional contract, which implies that the contents of a university program are defined as pieces of text (typically extracts from a text book). The effects of this contract depend heavily on the institutional importance, in universities, of student evaluations. Unlike what is true for the mathematician researcher, student success is measured frequently and most often by individual examinations (written or oral), where the use of external media amounts to an act of fraud.

We cannot deny the fundamental difference between success criteria related on the one hand, to the production of new and interesting knowledge, and on the other hand, to individual performance within a few hours. The university institution has an obligation to society to be able to affirm the capacity of each graduate, and the modular organization implies that this affirmation is ultimately based on the evaluation of the students' performance within each module. Students, on their side, have a personal interest in ensuring that these assessments are based on transparent and affordable criteria. The common model is well known: a written exam where the challenge in terms of milieus (to what type of task does this exercise belong) is as limited as the study (on which page in the manual can I find the corresponding technique). But for the maintenance of an internal institutional contract between students and institution, the absence of strong links with the work of the mathematician is not a problem.

It is perhaps not so in terms of an external contract between the university and society. Especially with regard to the mathematical discipline, it can be maintained that the reason for keeping research mathematicians as teachers must be based on the contribution to teaching of their knowledge and experience as mathematicians. There are indeed other profiles of university teachers of mathematics, including mathematics didacticians and researchers in related disciplines (physics, economics, etc.). On the other hand, under the direction of the mathematician-scientist, students should be able to venture far beyond notes or textbooks—while textbooks dominate, at present, the vast majority of university courses in mathematics.

It is time to mobilize new efforts and new ideas to teach students, from the undergraduate level on, that mathematical knowledge is not only communicated through textbooks, and that research is not limited to acts of individual force before a piece of white paper. To achieve this, we must also establish new institutional contracts facilitating the integration of devices like "undergraduate research" in the regular curriculum, even for prospective teachers (for the latter, perhaps the work of the research didactician is as relevant as that of the mathematician). But without any substantial contribution to university teaching of universitary knowledge production, the university becomes just a pretentious school.

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Use of Student Mathematics Questioning to Promote Active Learning and Metacognition

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Abstract Asking questions is a critical step to advance one's learning. This lecture will cover two specific functions of training students to ask their own questions in order to promote active learning and metacognition. The first function is for students to ask themselves mathematical questions so that they learn to think like mathematicians who often advance knowledge by asking new questions and trying to solve them. This is also called problem posing, an important component of the "look back" step in the Polya's problem solving framework. The second function is for students to ask their teachers learning questions during lessons when they do not understand certain parts of the lessons. Students who are hesitant to ask learning questions need to be inducted into the habit of doing so, and a simple tool called Student Question Cards (SQC) can help to achieve this objective. These SQC cover four types of mathematics-related learning questions: meaning, method, reasoning, and applications. In a pilot study involving Grades 4 and 7 Singapore students, every student was given a set of these laminated cards. During lessons, the teacher paused two or three times and required the students to select questions from SQC to ask to clarify their doubts. This reverses the normal roles of teacher and students during classroom interactions. Teachers and students in this pilot study expressed mixed responses to the use of SQC. These two functions of student mathematics questioning have the potential to promote active learning of mathematics among school students through strengthening their metacognitive awareness and control. To realize this potential, teachers need to pay due attention to the science, technology, and art of student questioning.

Keywords Student questions · Problem posing · Metacognition · Buddha · Confucius · Socratic dialogue

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Main Theme

> The most sophisticated understanding often starts with a childlike sense of wonder, the simplest of questions, and an honest, irrepressible desire to know. (Balibar 2008, p. x)

Children are naturally curious about themselves and their environment. A natural way with which they try to satisfy their curiosity is to ask questions. The main theme of this regular lecture is to argue that student questioning should be made a more prominent part of classroom teaching and learning of mathematics than is currently practiced in many countries.

There are many reasons why questioning is important in knowledge construction and its learning. Student questioning can serve two different but related functions. The first function is to help students think like mathematicians by posing their own mathematics questions and trying to solve them. This could lead to "new" knowledge constructed by the students at their levels of mathematical maturity. The second function is to train students to develop the good learning habit of asking their teachers mathematics-related questions about things that they do not understand in the lessons. This learning function reverses the normal roles of teacher and students during classroom interactions: instead of "teacher asks questions and students answer them," the new role becomes "students ask questions and the teacher answers them." These two inter-related functions help to prepare students to become better problem solvers, as noted by Brooks (2002): "Students who stop questioning because they have finished all the problems are not tomorrow's problem solvers" (p. 53).

The following sections will present examples of how these two functions may be achieved using recent studies conducted in Singapore. The lecture will conclude with the observation that student questioning, just like any other teaching techniques, requires that teachers pay due attention to the *science, technology*, and *art* of using it in their lessons.

The next section situates the functions of questioning within the historical traditions in East and West education.

Use of Questioning in East and West Education

It is commonly held nowadays that students in the Confucian Heritage Culture (CHC) are passive learners and teachers in this culture are authorities not to be questioned by their students. This was not the case in ancient China. The Chinese expression for learning or knowledge is made up of two characters, meaning to learn (学) and to ask questions (问). Etymologically this term can be traced to the *Book of Change* (I Ching, 易经): the superior person accumulates knowledge through learning and validates it through asking questions (君子学以聚之, 问以辩之). In the *Book of History* (*Shang Shu*, 尚书 or *Shu Jing* 书经), edited by Confucius (551–479 BC) and used by him as a text, one finds the well-known

Chinese proverb, 好问则裕; whoever asks questions will become abundant (in knowledge). The benefit of asking questions is clearly illustrated by the story of Zheng Zi (曾子 or 曾参) (505–435 BC), one of Confucius' eminent disciples. In current educational terms, Zheng was a "low ability" student but he was diligent in his studies and believed in the efficacy of asking questions in learning: If you do not know and yet do not ask, your mind will become stuck (弗知而不問焉, 固也). One way to ask questions is explained in the *Book of Rites* (*Li Ji*, 礼记): Those skillful in asking begin as if attacking hard wood: begin with the easy parts, then move on to the hard ones. Given time, the various parts can be separated and resolved (善问者, 如攻坚木, 先其易者, 后其节目, 及其久也, 相说以解). Unfortunately, this helpful advice that students should be active inquirers has not been adhered to over thousands of years of teaching practice in China, Korea, and Japan, leading to passive learning in these CHC societies.

The Buddha (ca 563–483 BC) is highly revered as a great teacher among his followers, and the Buddhist canon (*Sutras*) shows that many of his teachings were in response to the questions asked by his disciples and other seekers. One of the most significant questions posed to the Buddha was recorded in the *Kalama Sutta.* The question was how one would be able to discern the truth of the many doctrines preached by wandering ascetics. Note that the questioners were not asking for specific doctrines to believe it; rather it was a question about the processes of inquiry and knowledge construction. To this question, the Buddha first praised the Kalamas for being in doubt about a very important and yet perplexing matter, and his answer was essentially "know for yourselves" after proper investigation. The huge collection of Chinese Zen Buddhist writings includes numerous exchanges when the masters answered questions asked by followers and seekers. Thus, asking questioning has been widely practised as a form of learning in ancient Eastern societies. This effective activity seems to have been neglected in many contemporary Asian classrooms. For example, in December 2011, a survey of 300 Korean school children conducted by *Chosun Ilbo* found that 42 % of them never asked questions in class, and 45 % were scolded or ignored by their teachers when they asked questions.[1] Indeed, a lack of student questioning is also noted in other countries as well (e.g., Boaler and Humphreys 2005; Dillon 1998).

In the West, the ancient Greek philosopher, Socrates (ca 470–399 BC), was well known for developing the teaching technique named after him, namely the Socratic dialogue. In this one-to-one dialogue, he would ask the student (usually a male) a series of leading questions to help him think critically and arrive at the answers himself. The underlying epistemological assumption is that the answers are already innate in the mind of the student, and thus, the teacher's role is that of a midwife (*maieutics*) to help him "give birth" to that knowledge. If the student has not been taught the knowledge prior to the dialogue, then his knowledge must have come from his past lives! In the most widely cited example of this technique, Socrates asked a series of questions to lead Meno's slave to "discover" that the area of a

[1] http://english.chosun.com/site/data/html_dir/2011/12/07/2011120700522.html.

large square is double that of a smaller one.[2] Beck (2006) contrasted the two different teaching methods used by Confucius and Socrates in this way: "Confucius expected his students to make some effort if he was to help them" (p. 329), including to ask questions, whereas "Socrates usually only required that they answer his questions" (ibid.).

The famous mathematician Hadamard (1905/2004) called Socratic dialogue the "heuristic method" and he attempted to extend it from one-to-one exchange to larger groups. Proponents of student-centered teaching in recent years often include Socratic questioning as a key method to have students think deeply and justify their answers. However, the eminent philosopher Bertrand Russell (1872–1970) noted that Socrates had used in the Meno's dialogue "leading questions which any judge would disallow" (1961, p. 110). He maintained that the discovery of scientific and mathematical knowledge can hardly "be elicited from a previously ignorant person by the method of question and answer" (ibid). Thus, it is unlikely that the Socratic dialogue can help school students to come up with mathematical ideas such as prime numbers and algorithms such as completing the square, although it might be used to solicit students' views about how mathematics is used in their daily lives. Unlike the original spirit of Socratic dialogue, the "teacher asks questions and students answer them" mode has evolved in the West to become a key component of direct instruction to find out what students know about the lessons after they have been taught rather than to induce students to "recover" the knowledge from an innate source.

A lesson from this brief sketch is that asking questions has been acknowledged historically in East and West as an active technique to construct knowledge and to learn it, irrespective whether the questioning is done by the teacher or the student. The next section will deal with the function of student questioning in knowledge construction.

Student Questioning I: Construct New Knowledge

Importance of Asking Mathematical Questions

Albert Einstein commented that "The important thing is not to stop questioning. Curiosity has its own reason for existing".[3] Curiosity is the innate motivator of a person to search for knowledge, and asking questions is an important initial step to satisfy this curiosity. Hence, it should be encouraged and become an important part of normal lessons.

First of all, students must be stimulated to see how mathematics can help them know about the world. This involves helping them to habitually adopt a mathematical lens in their observations of everyday situations. With well-developed

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[2]http://www.cut-the-knot.org/proofs/half_sq.shtml.

[3]http://rescomp.stanford.edu/~cheshire/EinsteinQuotes.html.

mathematical literacy, they can form considered opinions about statements that cite quantities and make informed decisions about real-life situations that involve numbers and spatial relationships. For example, when they hear that "people use only 10 % of their brain capacity," they should realize that *percent* is defined with respect to a reference base and ask what might be 100 % of brain capacity.[4] Since it is not clear what is the brain's capacity and how to measure it, the above claim does not add new knowledge about ourselves. Pondering over such questions will help students (and adults as well) to *de*construct *mis*leading information to avoid being *mis*led by *mis*use of mathematics or statistics (e.g., Huff 1982). However, students may not have developed such a critical mindset. This seems to be the case in relation to understanding messages conveyed in graphs. In her doctorate study, Wu (2005) administered a questionnaire to about 900 Grades 7 to 9 students in Singapore about their critical views toward statistical graphs. About one third of them hardly asked themselves questions about the reliability of the data, whether the graph was suitable for the given data, or whether the graph supports claims made in the accompanying report. This lack of critical analysis through asking relevant questions is a weakness that can prevent students from learning about the world through quantitative data.

Mathematics develops when mathematicians ask questions and try to solve them. Classic mathematics texts often consist of lists of questions to solve. For example, the Chinese classic, *Nine Chapters on the Mathematical Art* (九章算术, ca 200 BC) is a collection of problems about taxation, pricing, land measurements and other situations, their solutions, and explanations of the procedures used. This is now recognized as the problem posing approach in the construction of knowledge. Indeed, the recent KOM Project of Denmark (Niss and Højgaard 2011) stressed that "being able to ask and answer questions in and with mathematics" (p. 53) is one of the eight mathematics competencies in the Danish curriculum reform. This problem posing approach has several advantages. First, students are more likely to become engaged working on their own questions. This will help to deepen their mathematical skills and thinking. They may have posed problems that cannot be solved, and this encounter, supported by discussion with their teachers, demonstrates to the students a common situation called unsolved problems (conjectures) faced by working mathematicians. Indeed, to tackle interesting conjectures has spurred the development of much of mathematics. This experience may change students' beliefs about the nature of mathematical thinking. Finally, student questions may reveal "a good deal about what learners are attending to and what they are aware of" (Mason and Johnston-Wilder 2004, p. 310). What is revealed may go beyond mathematical skills, as illustrated by the following example.

A class of Grade 8 students in Brunei Darussalam had just completed some work on quadratic equations using a multi-modal thinkboard (Wong 1999). They were required to write "stories" that apply quadratic equations. In Fig. 1, the student

[4]See Chabris and Simons (2010) for further questions one can ponder over about this and other misleading claims.

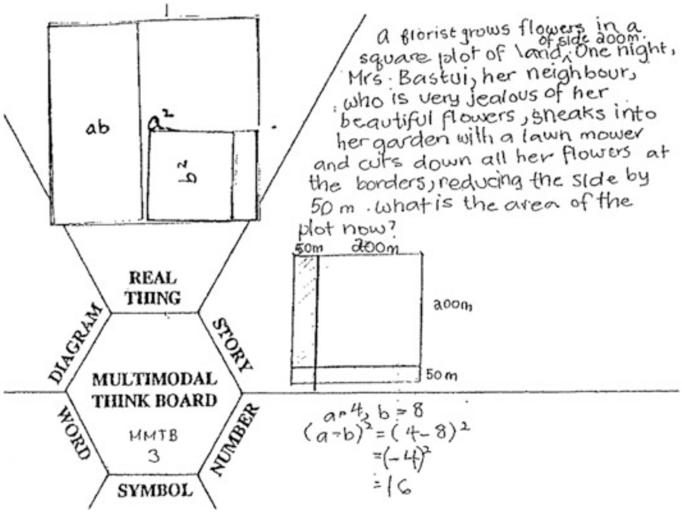

Fig. 1 A multi-modal "story" with embedded values

wrote a story that revealed feelings of jealousy about neighbors. These may or may not be her true feelings, but such stories can help teachers understand the values and interests held by their students. In recent years, the inculcation of values in mathematics instructions has gained some attention among mathematics educators (e.g., Bishop and Seah 2008; Winter 2001; Wong 2003), and teachers ought to be aware of this development and build their capacity to work on this integration.

Ways to Incorporate Problem Posing into Mathematics Lessons

School students can learn to pose their own questions under two conditions in mathematics lessons. First, the Polya's (1957) framework of problem solving has "look back" as the last of a 4-step process. Having solved a problem, students are advised to think of using the same result or method to solve other problems, either given to the students by the teacher or posed by the students themselves. Wilson et al. (1993) had proposed a helpful framework that includes problem posing after the "look back" step so that it loops back to the first step of the Polya's framework to begin another cycle of problem solving. Secondly, the "problem posing" approach has generated much research interest among mathematics educators (e.g., Brown and Walter 1993, 2005; Silver 1994). A typical approach is to teach students to ask "what if" and "what if not" questions in order to generate new problems. Students learn to make different types of changes: numbers (larger or different types of numbers), conditions of the problem (generalize or specialize), the contexts or storylines, representations (verbal, pictorial, or concrete), the operations (reverse the direction of the operations used in the original problem), and so on. These different ways can give rise to problems of different levels of sophistication and difficulty.

In the Japanese Problem Solving Approach explicated by Isoda and Katagiri (2012), a teacher may begin with a problem, say, compute the value of 37 × 3; after completing this, the students then pose their own problems which they called *problematic*. Through this process, the teacher hopes to change the belief of the students from "just solving a task given by [the teacher] to posing problems by themselves in order to learn and develop their mathematics" (p. 5).

In a recent doctorate study, Chua (2011) asked a sample of 480 Grade 9 students in Singapore to pose problems based on given stems. These students had not been trained to pose problems. About 20 % of the posed problems were non-solvable because of ambiguity in the wording of the problems or contradictions in the information provided by the students; see Fig. 2a, which includes the wrong property of interior angles of parallel lines. The solvable problems, on the other hand, were classified along four dimensions: (i) problem types (relational or direct recall), (ii) problem information (edit information, add object, over-conditioning, implicit assumption), (iii) solution types (multi-step, algebra), and (iv) domain knowledge (e.g., area, arc length, or algebra). An example of a direct recall problem is shown in Fig. 2b, and about 46 % of the solvable problems were of this type. Chua (2011) recommended that pre-service teacher education prepares student teachers to deal with problem posing.

For my mathematics methodology course for the BSc(Ed) program at the National Institute of Education, student teachers for the secondary level completed an assignment about problem solving. Each of them was given a problem suitable for secondary school students and had to solve it using two different ways. Then they planned a problem solving lesson for that problem, including posing an extension problem. Most of the problems posed by these student teachers involved simple changes to numbers and context. One of the given problems is as follows:

> To play a treasure hunt game, 90 boys and 105 girls formed mixed groups with the same proportion of boys and girls in each group. Find the greatest number of groups that could be formed. In this case, how many boys and girls are there in each group?

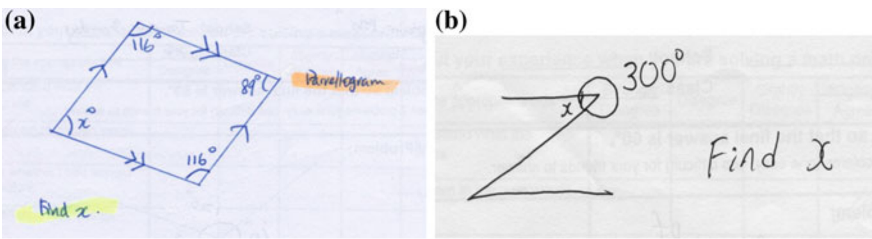

Fig. 2 Problems posed by Singapore secondary schools (Chua 2011). **a** Non-solvable problem. **b** Direct recall

The student teacher who was given this problem posed her problem by adding a third option, changing the numbers, and casting it within the context of racial fairness, which is an important, socio-cultural value related to what is called the National Education in Singapore (Wong 2003). This is a good example of posing problems that include both mathematics content and values, as noted above about the Brunei case.

> In a nationwide treasure hunt in Singapore, to ensure racial fairness, 108 Chinese, 72 Malays, 96 Indians formed mixed groups with the same proportion of each race in each group. Find the greatest number of groups that can be formed. In this case, how many people of each race are there in each group?

Another technique to develop problem posing ability is to give students an answer and ask them to formulate possible questions (Herrington et al. 1994). Yet another activity is to ask students to play the examiner's role. Let them submit their questions (with answers) about the recent lessons, circulate these questions among the students, and select some of them for the upcoming test. This will help them revise relevant mathematics knowledge and skills to be tested, leading to mastery as well as better test scores. To summarize, these diverse activities aim to use problem posing as one form of student questioning to help students master and construct knowledge new to them and to appreciate the role of questioning in mathematical work.

Student Questioning II: Learning in the Classrooms

Prevalent Practice of Teachers as Questioners

Learning any new mathematics requires students to engage in various behaviors within and outside mathematics classrooms. This section will focus only on student questioning as a desirable learning behavior in mathematics lessons.

Teachers often ask questions in lessons to achieve different purposes, one of which is to find out what their students have understood from the lessons, although such questioning often ends up with finding out what the students do *not* know. In the latter case, Einstein felt that it would be a waste of teacher's time and that the teacher should discover through questioning what the students are capable of knowing.[5] There is already a large literature in general education (e.g., Kerry 2002; Wragg and Brown 2001) and mathematics education (e.g., Schuster and Anderson 2005; Small 2009; Watson and Mason 1998; Wilburne 2004) about how to improve teacher questioning techniques. These include asking a combination of closed questions and higher order questions, using wait times to stimulate deeper thinking, providing immediate and meaningful feedback, and moving beyond the standard IRF (Teacher Initiates, Student Responds, Teacher Feedback) sequence found in

[5] http://einstein.biz/quotes.

many lessons. Numerous studies have described the types of questions teachers often ask, usually closed ones, and their effects on student achievement (e.g., Hattie 2009, 2012). It is generally believed that by participating in these teacher-initiated questioning, the students will become more active and engaged in their learning compared to just listening and taking notes. However, at times the students may just guess at what the teacher wants to hear rather than reveal their true answers in front of their classmates for fear of being wrong or ridiculed. They may have developed uncanny strategies to cope with this fear or to avoid engaging deeply with the teacher's questions. This is illustrated by the story of Ruth narrated by Holt (1964). Holt tried to make Ruth "think" about a mathematics problem by asking her some questions. She remained silent, and Holt had to ask simpler and more pointed questions until he found "one so easy that she would feel safe in answering it" (p. 38). In this case, Ruth had a rather "cunning" strategy to make the teacher "do the work for her" (ibid.). This episode also shows the importance of creating a positive learning environment such that students feel safe to articulate their true understanding and feelings during lessons.

In a recent large-scale survey of Grade 9 students (n = 1166) in Singapore, Hogan et al. (2012) asked the students to rate how frequently their teachers asked 15 types of questions on a 5-point scale. They found that the most frequently asked question was "What is the correct answer to this question" (mean = 3.69), a closed question, whereas the least frequently asked question was "What makes this a reasonable guess?" (mean = 3.03), which is more conceptual in nature. They noted that "teachers do not ask one kind of question and neglect others" (p. 187), but the relationship between the types of questions asked in lessons and student learning was not reported. Besides focusing on teachers as questioners, educators should consider reversing the roles of teachers and students in this classroom interaction, and this is explored in the next section.

Scant Discussion About Students as Questioners

Consider this typical scenario in many lessons: Having given some explanation, the teacher asks the class: "Any questions?" or "OK, everybody understands?" It is very likely that there is no response from the students.

In contrast to the comprehensive coverage of the practice and research about teacher questioning, there is scant discussion about student questioning as a learning technique and its impacts on learning outcomes. Similarly, there is inadequate practical guidance on how to carry out student questioning in lessons in general and mathematics lessons in particular. For example, in Kerry's book (2002), there are 58 pages on questioning and only seven pages deal with student questions. Furthermore, "student questions" does not appear in the index of a recent international book about expertise in mathematics instruction (Li and Kaiser 2011), although student questions and discussions are mentioned at several places (for examples, on pages 109, 111, 170). Perhaps, student questioning may have been embedded as an "invisible" part of classroom discussion; if so, it should be made more "visible" to the teachers and students alike. Other books about exemplary practices for mathematics teachers (Posamentier et al. 2007) and pedagogy books in general (Feldman and McPhee 2008; Marzano 2007) also do not deal with student questioning. The title of the book by Godinho and Wilson (2006) is about *helping your pupils to ask questions,* but most of the contents are about teacher questions. On the other hand, in a comprehensive review of student questioning in science education, Chin and Osborne (2008) concluded that "the explicit teaching of questioning skills to students can lead to improved performance on a range of science-related tasks" (p. 34) and noted that "there is still a lot of scope for pedagogies that exploit the potential value of students' questions" (p. 35). Boaler and Humphreys (2005) wrote that "[m]ost teachers welcome student questions in their classes and regard questions as valuable in the service of learning" (p. 72) and cited studies to show that students, especially those at higher levels, rarely ask questions. Fortunately, some educators recognize this undesirable situation and have made recommendations to deal with it. The following sections review some of these writings about student questioning.

About 20 years ago, Dillon (1988) noted that "few student questions are asked [in class] and even fewer are answered" (p. 8). In a chapter entitled "Pedagogy of Student Questions," the author suggested that teachers should welcome questions from the class, provide time for them to ask questions, and find ways to sustain the asking. This requires changing the culture of the classrooms so that the students know that it is safe to ask questions in front of their classmates.

Morgan and Saxton (2006) discussed six types of pressures on teachers that reduce the opportunities for their students to ask questions: personal, right-answer, curriculum, classroom environment, control, and standardized testing pressure (p. 100). Walsh and Sattes (2005) also made similar comments and noted that students ask fewer than 5 % of the questions found in lessons. Attempts to promote student questioning need to take these pressures into account.

The PEEL (Project for Enhancing Effective Learning) project was a comprehensive teacher-led action research that began in Melbourne in 1985 and has since spread to many schools in Australia and other countries.[6] It aims to inculcate in secondary school students good learning behaviors. For example, students are encouraged to seek assistance by telling their teachers when and what they do not understand and asking their teachers why they go wrong (Mitchell 1992, p. 63). Similar good learning behaviors were also discussed by Postman and Weingartner (1969) many years ago. They proposed a "Question Curriculum" and noted: "Once you have learned how to ask questions—relevant and appropriate and substantial questions—you have learned how to learn and no one can keep you from learning whatever you want or need to know" (p. 34). This highlights that questioning is a desirable attribute for life-long learning, which nowadays is considered a critical attribute of workers of the 21st century.

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[6] http://www.peelweb.org/index.cfm?resource=about.

Focusing on mathematics study skills in particular, Ooten and Moore (2010) discussed seven types of questions that students can ask in mathematics classes. For examples, she suggested that the students prepare a prioritized list of questions, ask often, and show their work in public. Likewise, Sammons (2011) suggested that students classified their questions into "right there," "think and search," "thick and thin questions" and so on (p. 126) because these questions found to work in literacy lessons may be adopted for mathematics instruction.

Foster (2011) explained how he began a mathematics lesson by writing on the board questions generated by the students and letting them decide which questions to tackle. He provided several interesting examples related to linear equations. Since this is a new learning experience for the students, he recommended patience on the part of the teacher.

Piccolo et al. (2008) described a study that covers both teacher-generated and student-initiated questioning as part of mathematics discourse at middle school level. They found that "teacher talk was dominant and student talk was mainly a response to teacher questioning" (p. 404) and called for "further research on how better to provide students with the skills and mathematical competence to ask and engage in rich mathematical discourse with teachers" (ibid.). This is in line with what Commeyras (1995) believed: "Children are naturally rich with questions, but when teachers take primary responsibility for questioning, student questioning becomes something to be taught" (p. 105). Chin and Osborne (2008) also made a similar suggestion from their research about science instruction. King (1992) proposed a guided student-generated questioning strategy under which the students learn to use a set of generic questions that are domain independent, for example, "What is the difference between … and …?" In the Japanese "Hatsumon" system, the teacher prepares a set of questions to ask in mathematics lessons to promote students' thinking by themselves, so that they can "develop the ability to ask these questions of themselves and to learn how to think for themselves" (Isoda and Katagiri 2012, p. 121). This approach might lead to stronger metacognitive control among the students.

Despite some attempts to bring student questioning into classroom lessons as reviewed above, few studies have been conducted about this technique. To help fill this gap in a small way, my colleague and I (Wong and Quek 2006, 2009, 2010) conducted a pilot study to help students learn to ask questions in mathematics lessons, coupled with the reversal of roles mentioned earlier. This SQC (Student Question Cards) technique is discussed below.

SQC (Student Question Cards) Technique: Issues and Procedure

The aim of this pilot study was to investigate to what extent student questioning could be incorporated into normal mathematics lessons to help students become

Meaning	Method
M1: What do you mean by ….	Md1: Can you show us how to do this problem in another way?
M2: What is the difference between …. and ….	Md2: Can you explain/show us this step (….) again?
M3: Can you use a diagram to show ….	Md3: What will you do next?
M4: (Your own question)	Md4: (Your own question)
Reasoning	**Application**
R1: Why do you do that ….?	A1: Why do we study this topic (….)?
R2: What happens if you change …. to ….?	A2: How do we use this (….) in everyday life?
R3: (Your own question)	A3: (Your own question)

Fig. 3 Questions used in student question cards (SQC)

more metacognitive about their learning. Two Grade 4 and two Grade 7 teachers volunteered to participate in this study in 2005. Four issues related to student questioning as a learning behavior in the classrooms are addressed.

Issue 1: Be specific. Students are able to ask mundane questions such as “Do we have to underline our answers?” or “When do we have to hand in our homework?” But they are poor in asking questions targeted at specific contents. Some of them would ask over-encompassing questions such as, “Teacher, can you explain everything from the beginning again?” Thus, their ability to ask specific mathematics questions when they do not understand is not inborn (despite the claims of Socrates) and has to be developed. In our investigation, we created a set of standard questions that focus on four key aspects of mathematical learning: Meaning, Method, Reasoning, and Application. These questions are given in Fig. 3.

For each of the four aspects, the research team devised two or three questions in simple English that can be asked of most topics, where the students fill in the ellipses (…) with words related to the specific lesson. The (Your own question) option allows students to frame questions in their own words. While these specific questions serve as a scaffold at the initial stage to promote student questioning, it is hoped that with sufficient practice the scaffolding using these question prompts can be steadily faded so that the students will become better at asking their own questions in a format that is specific and relevant to the mathematics that has just been taught. These questions are printed on laminated cards with one aspect per card; the label of each aspect appears on one side and the sample questions on the reverse side of the same card. These laminated cards were given to the students during the pilot study.

Issue 2: Reduce apprehension. Some students are apprehensive about asking questions in front of their classmates for fear of looking “stupid” because of their “dumb” questions. This might be a universal or an Asian face-saving concern. This issue can be partially alleviated by letting students choose questions from the “approved” list of questions in SQC.

Table 1 Frequency of questions asked by pupils at QT1 and QT2 (Quadratic factorization)

Aspects	Questions	QT1	QT2
Meaning M1	What do you mean by … ?	8	
Meaning M2	What is the difference between … and … ?	3	8
Method Md1	Can you explain/show us this step again?	8	
Method Md2	Can you show us how to do this problem in another way?	3	8
Reasoning R1	Why do you do that?	4	4

Issue 3: Set aside time for student questioning. This refers to the situation where many students have been “conditioned” to answer questions but not to ask them. As pointed out by Dillon (1988), time must be specifically created in lessons to allow students to ask questions. During the trial, the participating teachers planned their lessons to include pauses called Question Times (QT) after about 15 min of teaching or class activity. During these QTs, the students were asked to refer to the SQC to find a question to ask about that part of the lesson prior to the QT. This was the scaffolding part, and doing it two or three times per lesson was an intensive way to help bring about this new habit. The ways in which the teachers applied this technique might influence how this new habit develops among their students.

The use of SQC and QTs is illustrated below using a vignette of a lesson on quadratic factorization with two question times, QT1 and QT2. The result is given in Table 1.

Prior to QT1, the teacher reviewed the techniques for factorizing expressions such as $18ac + 12ab$ and $9 - 25x^2$. The two frequently chosen questions after QT1 were about meaning of factorization (M1) and to repeat the explanation about the technique (Md1). Prior to QT2, the teacher called on some students to factorize the expressions $x^4 - y^4$ and $3x^2 - 48$ on the board. This was followed by a discussion about their solutions. The two frequently chosen questions after that were about comparing the examples (M2) and alternative solutions (Md2). In both pauses, the Reasoning question (R1) was chosen by a few students. This short vignette shows that the questions that were selected by the students depend on the topic being taught prior to the question time.

The four participating teachers were found to handle the QTs in different ways. They tended to select students who were normally quiet or hardly asked questions in class. At times they focused on one particular aspect; for example, Reasoning questions because they were working on justification of certain results. If over several lessons, the teacher could go through all the four aspects, this should help students become familiar with these key aspects of mathematical thinking. One teacher found that her students were able to insert the appropriate terms in the prompts (…), while the students of another teacher had difficulty with this, i.e., they just asked “what do you mean by *that*” without being able to state precisely what is “that.” These diverse ways of using the SQC provided many opportunities for students to learn from how their classmates had asked questions, hopefully fostering their own skills to ask specific mathematics questions.

Issue 4: Teacher learning. Well-prepared teachers do not have difficulty answering questions related to how to solve the problems they assign to their students because they have worked out the solutions beforehand. However, the SQC technique includes questions other than Methods, and this can be challenging to teachers who are not strong in mathematics content knowledge (the Meaning and Reasoning aspects) or how mathematics can be applied in real-life contexts (the Application aspect). Indeed, Application questions were frequently asked by the students in this study, suggesting that they were eager to know why they had to study the topics. The teachers seemed to have difficulty answering these Application questions. Some of them were also caught off guard by questions about different representations (Meaning), for example, "can you draw a diagram to show …?" (M3). When the teachers cannot give appropriate answers to these "unexpected" questions on the spot, this provides a meaningful albeit challenging opportunity for them to learn more about the mathematics they are teaching by discussing these questions with their colleagues or looking up relevant references after class. They may then incorporate new insights into their subsequent lessons. Indeed, as Hattie (2009) noted, "the biggest effects on student learning occur when teachers become learners of their own teaching, and when students become their own teachers" (p. 22). The SQC technique provides a concrete and systematic tool for both teachers and students to learn about mathematics by playing these newly reversed roles with regards to questioning. A significant finding of this pilot study is that this technique of student questioning is teachable, although more effort is needed to find effective ways to lead students from scaffolding to eventual mastery of the technique.

SQC (Student Question Cards) Technique: Responses from Teachers and Students

Teachers' feedback. Table 2 summarizes feedback from the four teachers with respect to the use of SQC. They agreed that it could break the monotony of the lessons and make the students more active. Some of them may try it occasionally in the future. One concern is how to get the students to pay more attention to the questions asked by their classmates and the answers given by the teacher.

Students' feedback. The students responded to a questionnaire about their experience using the 6-point Likert scale: 1 = Strongly Disagree; 6 = Strongly Agree. The mean scores are reported in Table 3. Their responses were slightly positive, although the two secondary classes were less positive than the primary classes. The students found the cards easy to use, but they did not quite enjoy using it. In their open-ended responses, some students expressed positive opinions about being able to choose questions to ask and having the chances to ask questions. Some of them were worried that the teacher might pick on them to ask questions, while others were disappointed when they were not chosen. Some found this

Table 2 Teachers' feedback to Student Question Cards (SQC)

<table>
<tr><th></th><th>T1 (Primary)</th><th>T2 (Primary)</th><th>T3 (Secondary)</th><th>T4 (Secondary)</th></tr>
<tr><td rowspan="2">Usefulness in helping students understand lesson</td><td>Through the questions they asked, realised most students did not understand topic</td><td rowspan="2">Generally quiet and unresponsive class but now more active; thought they were having fun, game-like</td><td rowspan="2">Yes, if students asked correct questions</td><td>Good to use cards at beginning of topic and in remedial class</td></tr>
<tr><td>QT breaks monotony of teaching so students more engaged/likely to listen</td><td>Students tended not to listen when their friends were asking questions</td></tr>
<tr><td>Integrated into practice?</td><td>Not yet a repertoire in her teaching but will continue to use question cards</td><td>Cannot use for every lesson; might use QT now and then</td><td>Yes, but students asking questions could only be sustained when it is a school culture</td><td>Will have QT even if cards not used. Some teachers already doing it but not as structured and detailed</td></tr>
</table>

Table 3 Students' feedback to Student Question Cards (SQC)

Questionnaire items	T1 (P)	T2 (P)	T3 (S)	T4 (S)
Q18: The question cards were easy to use	4.24	5.04	4.03	3.74
Q20: I could understand the questions on the cards	4.12	4.58	4.32	4.00
Q22: This "question cards" method helped me to understand mathematics better	4.10	4.91	3.45	3.16
Q21: I would like my teacher to use this "question cards" method to teach mathematics	3.96	4.04	3.18	3.11
Q19: I enjoyed using these question cards to learn	3.76	4.64	3.61	3.03

practice time-consuming (they were also asked to write down in a short checklist what question they had chosen at each QT) and preferred to simply raise their hands if they wished to ask questions.

Conclusion

I wish to conclude this lecture by considering the *science, technology,* and *art* of student questioning, which is still an under-utilized teaching and learning technique.

The *science* of student questioning should build on sound theories and be supported by empirical data. Relevant theories to justify the use of student questioning include metacognition, self-regulated learning, and inquiry learning.

Gibboney and Webb (1998) mentioned another theoretical point: "unless the learner herself raises questions, no meaningful learning can occur" (p. 32). To what extent these theories are able to support learning via student questioning has to be tested against empirical findings. Evidence is available about the efficacy of problem posing (e.g., Brown and Walter 1993, 2005), but research about students playing the role of questioners in the classrooms and the impacts of student-generated examples and questions on various measures of learning outcomes has hardly started. Teachers can spearhead this by conducting their own collaborative action research with colleagues or engage in lesson study about applying this technique in their lessons.

Scientific experiments require the use of robust tools. This is the *technology* of student questioning.

The SQC is just one tool of this technology. It is easy to use and covers the key aspects of mathematical learning. It is also flexible enough to allow teachers to decide how it fits into their teaching styles as reported above. Postman and Weingartner (1969) suggested that teachers initially focus on the quantity of questions generated by the students "to get them to begin formulating questions" (p. 185) and only later to help them develop criteria "by which the quality of a question can be evaluated" (p. 186). This is similar to the approach taken by Foster (2011) in his lessons. More tools that can promote student questioning about mathematics and their learning and innovative processes of using these tools need to be further developed and their impacts investigated empirically. Some relevant research questions are:

- Do students who are regularly engaged in student questioning have better recall and understanding of the topics, as measured by traditional tests or alternative assessment modes?
- How much more time do teachers take when they use student questioning tools in their lessons compared to when not using these tools?
- What percentages of students believe at the end of a topic that student questioning is important to their mastery of the topic?
- To what extent are students able to learn from question-and-answer exchanges that they do not participate in?

Technology is a critical bridge between the *science* and *art* of teaching. The *art* of student questioning refers to how individual teachers decide to use or not to use it for the target students. The *art* depends on personal styles and reflection, and this takes many hours of practice to develop. Given that the science and technology of student questioning are still underdeveloped at this stage, teachers have to develop their own art of using this technique through trial and error, action research, personal reflection, and working with peers within a community of professional learning.

Teacher questioning will always be an important feature of mathematics lessons. It is necessary to find out to what extent the current practice of teacher as questioner and students as answerers may jeopardize the nurturing of curiosity in students. The loss of curiosity is aptly described as "children enter school as question marks and leave as periods" (Postman and Weingartner 1969, p. 67). Student questioning is offered here as a complement to teacher questioning and assigned mathematics problems with the aim to generate stronger students' curiosity about mathematics and to place this curiosity at the center of their learning. Training them to develop and apply the habit of asking questions about the Meaning, Method, Reasoning, and Application of mathematics is one route to stimulate and sustain that curiosity. This effort could lead them to become more effective learners of mathematics now and into the future.

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The Examination System in China: The Case of Zhongkao Mathematics

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Abstract Examination is a critical issue in education system in China. Zhongkao is a kind of graduation examination of junior high school, and at the same time, the entrance examination to senior high school. This paper describes the structure, features and changes in zhongkao mathematics papers in China based on a detailed analysis of 48 selected zhongkao mathematics papers from eight regions in recent six years. It is observed that the zhongkao mathematics papers stress computation, reasoning, and relations among different mathematics topics, but are less emphasized on applications of mathematics in real context. There are obvious region differences in zhongkao mathematics papers, with regions from west economic zone relatively less demanding and regions from east and central economic zones more demanding. Changes like more process-oriented questions and more real context questions are found. Examples of examination items are given to illustrate the identified features and changes.

Keywords Zhongkao mathematics · Examination · Features · Changes · Junior high school graduates

Introduction

China is the birthplace of examination system. The imperial examination was started in 597 during the Sui Dynasty, and was banned in 1905 during the Qing dynasty (Li and Dai 2009; Zhang 1996). It lasted for about 1300 years. With the influence of the long existence of the imperial examination system, examination is of great importance in China. It attracts attention from parents, educators, teachers, students, policy makers and so on. It is a big issue in education.

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18-	University, college, high vocational schools, employment	
Gaokao		
15-17	Senior high school, secondary vocational school	
Zhongkao		
6-15	13-15 (12-15)	Junior high school
	6-12 (6-11)	Elementary school
Nine-year compulsory education		

Fig. 1 School education system in China

There are two significant examinations for students in school education in China, which are called "zhongkao" and "gaokao". Figure 1 shows the school education system in China. Students start their nine-year compulsory education usually at six years old. Most of them stay at elementary school for six years, and junior high school for three years. In some districts like Shanghai, students stay at elementary school for five years and junior high school for four years. At the end of Grade Nine, all students take zhongkao, which is summative assessment of the nine-year compulsory education, and more importantly, the entrance examination to senior high school. Nearly 90 % of junior high school graduates continue their study. About half of them go to senior high schools, and the other half enter secondary vocational schools (Ministry of Education of China 2010a). The results of zhongkao decide whether students go to key senior high school, ordinary senior high school or vocational school. At the end of three-year senior high school study, students take gaokao, which is the entrance examination to universities. About 80 % of senior high school graduates are promoted to tertiary education (Ministry of Education of China 2010a). The results of gaokao decide whether senior high school graduates go to key university, ordinary university, college, or other high education institutes.

China is a developing country. High quality education resources are still lacking. In order to be promoted into key senior high schools and key universities, Chinese students have to spend a lot of time and effort to prepare for zhongkao and gaokao. Many teachers tend to teach what are examined in zhongkao and gaokao only. Parents buy a plenty of exercise books for their kids to practice. In many districts, the three-year senior high school course is reduced to be taught in two years, and teachers spend a whole year to help students review the content of the examined subjects in gaokao. Entering good high schools or universities becomes the most important goal for many students' learning.

In order to change the above examination-oriented phenomenon, the ministry of education of China makes efforts, including making reforms on zhongkao and gaokao. The present paper focuses on zhongkao mathematics. It intends to depict a comprehensive picture of zhongkao mathematics papers, discussing its features and changes in light of a detailed analysis of the selected zhongkao mathematics examination papers from 2006 to 2011.

Basic Information About Zhongkao and Zhongkao Mathematics

Purpose of Zhongkao

In the No. 2 document issued by Ministry of Education of China (2005), the purpose of zhongkao is stated as to provide complete and accurate information on junior high school students' achievement in the required subjects. The result of zhongkao has two functions. One is to evaluate whether or not students have reached the standard for graduation, and the other is to select qualified students for senior high school study.

Organization of Zhongkao

Zhongkao is organized by local educational administrative department (Wang et al. 2004). The question-setting group for zhongkao mathematics consists of three to four mathematics teachers and teaching-research staff. The question-reviewing group is made of mathematics education specialists and senior teachers, and most of them having previous experiences of question-setting. The marking group is composed of local core teachers. They are trained strictly before marking so as to keep the marking process fair, impartial and impersonal.

Number of Students Who Take Zhongkao

Every year there is a large number of students taking zhongkao. According to the published statistics in the website of the Ministry of Education of China (2010b), there are about 18–20 million junior high school graduates every year from 2006 to 2010.

Date and Key Subjects for Zhongkao

Zhongkao normally takes place in mid-June. Chinese, Mathematics, and English are the three key subjects in zhongkao. Hence, these three subjects attract more attention from school principals, students, parents and researchers.

Principle for Question-Setting in Zhongkao

The No. 2 document issued by Ministry of Education of China (2005) indicates that question-setting should comply with subject curriculum standards, strengthen the relationship between questions and students' real life, and attach importance to students' mastery of mathematical knowledge and skills, especially their ability to apply what they have learnt to solve problems in real context. Catchy questions are not allowed in zhongkao.

Assessment Requirements Stated in Mathematics Curriculum Standard

The curriculum reform for basic education in China was initiated in 2001. The mathematics curriculum standard of compulsory education (experimental version) was first implemented in September 2001 in 42 national level experimental districts (Ma et al. 2009). In 2005, the curriculum was implemented nation-wide. Therefore, in 2008 all the Grade 9 students from the whole country take zhongkao based on the new curriculum.

The mathematics curriculum standard of compulsory education (experimental version) states that the main purpose of assessment is to get comprehensive information about students' learning experiences, to promote students' learning and to improve teachers' teaching (Ministry of Education of China 2001).

The standard highlights the importance of process assessment. It says that "the assessment on mathematics learning should pay attention to students' learning results and their learning process" (Ministry of Education of China 2001, p. 2). The standard proposes portfolio assessment, interview, classroom observation and project work as new approaches to evaluate mathematics learning, attaches great importance to problem posing and problem solving abilities, and show respect to individual's difference so as to make every student to enjoy success in mathematics learning. For example, open-ended problems with various tackling approaches can be developed, so that students with different abilities can solve the problem at different levels. It is hoped that these ideas on assessment could be manifest in zhongkao mathematics.

Research Questions

This paper aims to describe the characteristics of zhongkao mathematics papers in China in recent years. More specifically, it intends to answer the following three research questions.

1. What is the structure of zhongkao mathematics papers?
2. What are the features of zhongkao mathematics papers?
3. What are the changes in zhongkao mathematics papers?

Methodology

Samples

A total number of 48 zhongkao mathematics papers from eight province-level regions from 2006 to 2011 were selected for analysis as shown in Table 1.

The eight regions were chosen for three reasons. First, they represent different categories of province-level regions. China mainland has 22 provinces, five autonomous regions, and four centrally-administered municipalities. The selected regions are made of six provinces, one autonomous region, and one centrally-administered municipality. Second, they locate at different part of China as stated in the table. Third, they are from different economic zones, with three from eastern, another three from central and the other two from western zone. Eastern zone is regarded as affluent and developed area in China, followed by central zone and western zone in a descending order of economical level. The number of province-level regions in eastern, central and western economic zones is twelve, nine and ten respectively (Zhang 2010).

Shanghai was selected for one more reason. Among 65 countries and regions, Shanghai 15-year-old students ranked at the top in mathematics in the 2009 administration of the Program for International Student Assessment (PISA). The scores from Shanghai are by no means representative of all of China. The result from an analysis of Shanghai zhongkao mathematics paper might give some implication to Shanghai students' good performance in PISA mathematics, and on the other hand, the result from a cross-region comparison on zhongkao mathematics papers could show what other regions in China require junior high school graduates to achieve compared to Shanghai.

Table 1 Characteristics of the eight selected province-level region

Name	Category	Location	Economic zone
Shanghai	Centrally-administered municipality	East China	Eastern
Hebei	Province	North China	Eastern
Guangdong	Province	Southern China	Eastern
Anhui	Province	East China	Central
Jilin	Province	Northeast China	Central
Henan	Province	Central China	Central
Yunnan	Province	Southwest China	Western
Xinjiang	Autonomous region	Northwest China	Western

Zhongkao mathematics papers developed by the educational administrative departments at province level rather than at prefecture-level were selected for analysis. This is because many prefectures in some provinces usually take zhongkao mathematics paper at province level as their examination paper. Therefore, zhongkao mathematics paper at province level is more representative in terms of the number of students who take it.

Hence, the 48 zhongkao mathematics papers from eight province-level regions from 2006 to 2011 were used for analysis. The characteristics of the selected papers give some confidence to the external validity of the subsequent analyses to be reported.

Brief on the Composite Difficulty Model

The composite difficulty model, developed by Bao (2002a, b), is used as a framework to analyze zhongkao mathematics papers. The model comprises five factors, as shown in Fig. 2. Each factor has different levels.

The cognitive demand factor has three levels, knowing, understanding and investigating. Knowing is about memorization of mathematical facts and performing routine mathematical procedures; understanding relates to flexibly applying mathematical concepts, formulas, rules, and theories, using routine mathematical methods to obtain definite results, and justifying and arguing mathematical facts; investigating involves making mathematical conjectures and building up mathematical models, and its problem solving strategies are not readily available. Tasks at this level are usually open-ended, non-routine, and exploratory in nature.

The levels of the context factor are set by the distance of the material used to students. The closest is personal life (e.g., house working, school tour), next is public life (e.g., bank interest, sales of goods), and the most distant context for students is scientific ones (e.g., physical experiment). None context items refer to those intra mathematics questions.

The levels of computation and reasoning are set based on the complexity of the operations involved. The four levels of computation factor in an increasing order of complexity are none, numerical, simple symbolic (computation involving one or two steps) and complex symbolic (computation involving three or above three

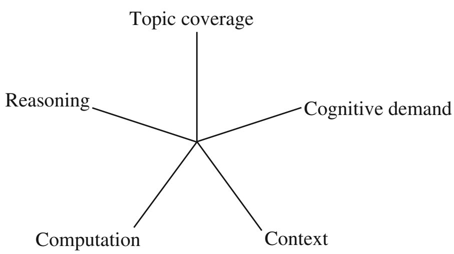


Fig. 2 The composite difficulty model

steps). The levels of reasoning factor were none, simple (reasoning involving one or two steps) and complex (reasoning involving three or above three steps) previously. Considering the fact that many items in zhongkao mathematics requiring reasoning more than three steps, the third level was divided into two, which were moderate (reasoning involving three or four steps) and complex (reasoning involving five or above five steps).

The levels of topic coverage indicate the number of mathematical topics in a single item. In the original model, this factor has three levels, single topic, two topics, and three or above three topics. Here the third level was separated into two, three topics and four or above four topics. It was found that many items in zhongkao mathematics papers contain more than three topics. This minor adjustment could provide more accurate information.

Table 2 gives the revised levels of each factor. The difficulty index of each factor in the composite difficulty model was estimated by the formula below (Bao 2004).

$$d_i = \frac{\sum_j n_{ij} d_{ij}}{n} \quad \left(\sum_j n_{ij} = n \quad i = 1, 2, 3, 4, 5 \quad j = 1, 2, \ldots \right)$$

where di (i = 1, 2, 3, 4, 5) corresponds to the five factors; d_{ij} represents the power index of the j-th level of the i-th factor as shown in Table 2; n_{ij} is the total number

Table 2 Levels under each factor in the composite difficulty model

Factor	Levels	Power index
Cognitive demand	Knowing	1
	Understanding	2
	Investigating	3
Context	None	1
	Personal	2
	Public	3
	Scientific	4
Computation	None	1
	Numerical	2
	Simple symbolic (one or two steps)	3
	Complex symbolic (three or above three steps)	4
Reasoning	None	1
	Simple (one or two steps)	2
	Moderate (three or four steps)	3
	Complex (five or above five steps)	4
Topic coverage	Single topic	1
	Two topics	2
	Three topics	3
	Four or above four topics	4

of items which belong to the j-th level of the i-th factor, and the sum of n_{ij} is n (total items). For example, it was found that in 2011 the number of items at the level of knowing, understanding and investigating are 66, 197 and 17 respectively. Hence, the number of items is 280, and the index of cognitive demand in 2011 is calculated by expression $(66 \times 1 + 197 \times 2 + 17 \times 3) \div (66 + 197 + 17) \approx 1.83$.

This composite difficulty level was used to compare mathematics curriculum (Bao 2004) and mathematics examination papers (Bao 2006). It is a useful tool in comparing mathematical tasks. It could provide important information of the five factors on zhongkao mathematics papers and its quantitative nature helps to trace the changes in zhongkao mathematics papers in recent years and across different province-level regions.

Data Coding

The selected 48 zhongkao mathematics papers were coded at the paper level and at the item level. At the paper level, information of region, year, number of questions, number of items, examination duration, and total marks were recorded. A question might have several items. The number of items shows the actual working load for students.

At the item level, information of question type (multiple choice, blank-filling, or solution-seeking), content (algebra, geometry, statistics, or synthesization), cognitive demand (knowing, understanding, investigating), context (none, personal, public, scientific), computation (none, numerical, one-step symbolic, two-step symbolic, three-step symbolic, etc.), reasoning (none, one-step, two-step, three-step, etc.), topic coverage (single, two, three, four, etc.) were coded. The codes are 1 or 0. If the item applies the category, then code 1 is given, otherwise, code 0. For solution-seeking items, the steps of reasoning or computation required are counted based on reference answers provided by the question-setting group. In addition, mathematics topics involved for each item were listed out. For example, the item in Fig. 3 was coded as blank-filling (question type), geometry (content), none (context), understanding (cognitive demand), none (computation), two-step (reasoning), two topics (topic coverage), and perimeter and figure folding (mathematical topics).

As shown in the figure, the side of an equilateral triangle $\triangle ABC$ is 1 *cm*, D and E are points on the side AB and AC respectively. Fold $\triangle ADE$ along DE, and point A moves to A' which is outside $\triangle ABC$. The perimeter of the shaded figure is ____ *cm*.

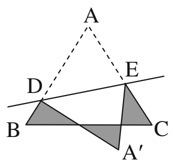


Fig. 3 An item from Hebei 2009

In the course of coding, non-traditional and non-routine questions were recorded in order to help identify features and changes of zhongkao mathematics papers.

Results

Structure of Zhongkao Mathematics

Structure of zhongkao mathematics is discussed from two aspects, the structure of question type and the structure of content. Generally speaking, Zhongkao mathematics papers contain three types of questions, which are multiple choice, blank-filling and solution-seeking questions. Multiple choice questions usually appear first, followed by blank-filling questions and solution-seeking questions in sequence. The ratio of the numbers of these three types of items is about 2:3:5 on average, showing that the number of items requiring answers only and the number of items requiring answers and solutions are about the same.

The distribution of question type across the eight regions is quite different. Four of the regions, which are Anhui, Jilin, Yunnan, Xinjiang, follow the general trend. Shanghai, Hebei, and Henan have more items requiring answers only (63, 61 and 56 %, respectively), while Guangdong has more items requiring answers and solutions (61 %).

In terms of content structure, about 75 % of the total items focus on merely algebra or geometry, the classical content area in school mathematics curriculum, with algebra items (39 %) a bit more than geometry items (36 %). Another 15 % of the total items deal with statistics, a relatively new content area introduced in the mathematics curriculum standard of compulsory education in 2001. The last 10 % items synthesized at least two of the three content areas, and the most of these items relate algebra and geometry. The combination of statistics and algebra or statistics and geometry is rare.

Regarding the content distribution across regions, Shanghai, Jilin, Henan, and Yunnan follow the general trend, while the other four regions vary differently. Guangdong has more geometry items (42 %) than algebra items (35 %). Hebei has more synthesized items (16 %), while Anhui and Xinjiang have less synthesized items (4 and 3 % respectively).

Synthesized items are usually more difficult than items related to only one content area, and they normally appear as the last question(s) in zhongkao mathematics paper. There are a plenty of articles and books discussing how to tackle synthesized questions (e.g., Ma et al. 2011). A typical synthesized problem could be investigating the existence of points satisfying certain conditions in a graph of a function, or finding the relationship between certain variables in a varying process of a geometrical figure or geometrical figures. Figure 4 shows an example. It is the last question in 2010 Henan zhongkao mathematics paper.

In the plane coordinate system, a parabola goes through three points A (- 4, 0), B (0, - 4) and C (2, 0).

(1) Find the formula of the parabola.

(2) M is a moving point in the parabola in the third quadrant. The x-coordinate of M is m, and the area of $\triangle AMB$ is S. Find the relationship between S and m by expressing S in terms of m, and the maximum value of S.

(3) If P is a moving point in the parabola and Q is a moving point in the line of $y = -x$, explore Q's coordinates so that P, Q, B, O could be the four vertices of a parallelogram.

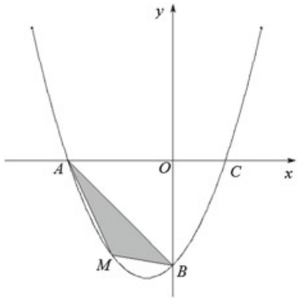


Fig. 4 The last question of Henan 2010

Features of Zhongkao Mathematics

The features of zhongkao mathematics are summarized based on the results of data coding. The general conclusion is that on one hand zhongkao mathematics in China in recent years inherit its traditional emphasis on basic knowledge and basic skills, and on the other hand it takes on a new look which reflects the requirement from the mathematics curriculum standard for basic education issued in 2001.

On Average About 3 min Assigned to Each Item

The average question number and item number of the selected 48 zhongkao mathematics papers are 24 and 35, respectively. The number of questions is in the range from 22 (Guangdong) to 28 (Jilin). It is relatively stable across the eight regions compared to the number of items. The number of items varies across regions, with Hebei having 43 items on average as the most and Anhui having 31 items as the least.

Three of the eight province-level regions, which are Henan, Shanghai and Guangdong, set the examination duration as 100 min, and the other five regions 120 min. Table 3 shows the average time assigned to each item across regions and years. It suggests that a student has about 3 min to answer each item in zhongkao mathematics. There is not much difference in the average time assigned to each item across years. However, it could be observed from the table that students from Anhui, Yunnan and Xinjiang seem to have more time for each item and students from Hebei and Jilin have less time.

Table 3 Average time assigned to each item (min)

	2006	2007	2008	2009	2010	2011	On average
Shanghai	2.9	2.9	3.1	2.9	3.1	2.9	3.0
Hebei	2.6	2.7	2.8	2.9	2.9	3.0	2.8
Guangdong	2.6	3.3	3.2	3.2	2.9	3.0	3.0
Anhui	3.5	4.1	4.0	3.8	3.9	3.8	3.8
Jilin	2.9	2.9	2.9	3.1	2.9	2.9	2.9
Henan	3.4	3.2	3.3	3.2	2.9	3.0	3.2
Yunnan	3.5	3.6	3.9	4.1	3.4	3.4	3.7
Xinjiang	4.0	3.4	4.0	3.5	3.5	3.8	3.7
On average	3.1	3.2	3.4	3.3	3.2	3.2	3.2

Most Items at the Understanding Level

Concerning cognitive demand, the percentages of items at the knowing, understanding and investigating level are 24, 70 and 6 %, respectively. This result is quite different from Bao (2006). He found that the percentages of the items, from Suzhou zhongkao mathematics papers from 1999 to 2001, at the knowing, understanding and investigating levels are 42, 58 and 0 %. Although the two findings are based on different samples, some rough description of the changes in Zhongkao mathematics could be made. It could be inferred that in general the number of items relying on memorization or routine mathematical procedures is decreasing, and the number of items involving applying mathematical knowledge, making connections between different mathematical objects, creating own strategies or making conjectures is increasing. This reflects the ideas advocated in the new mathematics curriculum.

Figure 5 gives two items taken from zhongkao mathematics papers. Although these two items both involve algebraic manipulations, one is coded as at the knowing level and the other at the understanding level. This is because the first item requires only routine mathematical procedure while the second one requires an understanding of the concept of domain and being able to apply this understanding to the given expression.

1. Simplify the expression $\left(\frac{x^2}{x-1}-\frac{2x}{1-x}\right)\div\frac{x}{x-1}$, and then find its value when $x=\sqrt{3}+1$. (from Xinjiang 2010)

2. Simplify the expression $(1-\frac{1}{x-1})\div\frac{x^2-4x+4}{x^2-1}$, and then choose a suitable integer x from the range $-2\leqslant x\leqslant 2$ to find the value of the expression. (from Henan 2011)

Fig. 5 Items from Xinjiang 2010 and Henan 2011

In the plane coordinate system, the coordinates of three given points are A_1 (1, 1), A_2 (0, 2), and A_3 (-1, 1), respectively. An electronic frog is at the origin point. At the first time, the frog jumps from the origin point to its symmetrical point P_1 with A_1 as the center of symmetry, the second time the frog jumps from P_1 to its symmetrical point P_2 with A_2 as the center of symmetry, the third time the frog jumps from P_2 to its symmetrical point P_3 with A_3 as the center of symmetry, …, following this rule, the frog continues jumping with A_1, A_2 and A_3 as the centers of symmetry. When the frog jumps the 2009 times, the coordinates of the point P_{2009} are (_______ , _______).

Fig. 6 An item from Yunnan 2009

Investigating items mostly appear at the end of each type of questions. Figure 4 shows a synthesized question having three items. The last item of the question is at the investigating level. The problem-solving strategy to this item is not readily available. It requires students to analyze the meaning of the item and come up with their own method. Figure 6 gives a blank-filling item which is also at investigating level. It is the last blank-filling question in Yunnan 2009 paper. This item requires students to find out the underlying law from concrete examples like P_1, P_2, P_3 and more if needed, and then apply this law to the special case of P_{2009}.

The eight regions vary in the levels of cognitive demand. Xinjiang, Guangdong, Anhui, Jilin, Hebei, Henan and Yunnan generally follow the global trend, while Shanghai performs quite differently. The zhongkao mathematics papers from Shanghai have about 43 % items at knowing level and 54 % items at the understanding level, showing Shanghai pays more emphasis on basic mathematical facts and procedures.

Most Items Not Having Real Context

The majority items in zhongkao mathematics do not have real context. In general, the ratio of items not having real context to those having real context is about 2:1. Among the items having real context, almost all of them relate to personal life or public life.

Among the eight regions, Shanghai and Guangdong are the top two regions having more intra mathematics items, with the percentages of 82 and 76 %, Xinjiang and Hebei are the bottom two regions, with the percentages of 53 and 58 %.

Attach High Importance to Computation and Reasoning

Computing and reasoning abilities are strongly emphasized in mathematics curriculum in China since 1960s (Zhang and Song 2009). This tradition has been kept till now as reflected in the zhongkao mathematics papers.

In Rt$\triangle ABC$, $\angle C = 90^\circ$, $\angle B = 50^\circ$, point D is on the side BC, $BD = 2CD$. $\triangle ABC$ is rotated $m(0 < m < 180)$ degree anti-clockwise about point D. If point B happened to be on the original sides of Rt$\triangle ABC$, then $m =$ _________.

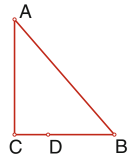


Fig. 7 An item from Shanghai 2011

In general, about two thirds of the items in zhongkao mathematics require computation. Among those items requiring computation, 40 % deals with numerical computations, 35 % is about simple symbolic computations (one or two steps), and the remaining 25 % is about complex symbolic computations (three and more than three steps). For example, items shown in Fig. 5 require at least three steps of symbolic computation. This distribution varies across the eight regions, with Anhui and Xinjiang have bigger percentages of items not requiring computation (40 and 43 % respectively), and Jilin has the biggest percentage of items requiring three or above three steps of symbolic computations (26 %).

Regarding the reasoning factor, the data shows that more than three-fourths of the items need mathematical reasoning. Among these items, nearly 60 % involve one or two steps of reasoning, about 25 % require three or four steps of reasoning, and the remaining 15 % deal with five or more steps of reasoning. Figure 7 shows a blank-filling geometrical item from Shanghai 2011. Seven steps of reasoning are needed to solve this item. Among the eight regions, Xinjiang is the least demanding on complex reasoning, with only 3 % of the total items requiring five or above five steps of reasoning. Oppositely, 85 % of the total items in Henan require reasoning, and 17 % require five or above five steps of reasoning.

Highlight Relations Among Different Mathematics Topics

Topic coverage factor indicates the number of topics involved in a single item. In general, about 35 % of the total items relate to a single topic, and the rest 65 % items deal with at least two topics. Among those items relating to more than one topic, 45 % relates to two topics, 27 % deal with three topics, and the remaining 28 % of items relate to four or above four topics. Items in Figs. 3, 4, 5, 6 and 7 are all examples of items involving more than one topic. However, the distribution of topic coverage varies greatly among the eight regions. If the eight regions are arranged in a descending order according to its percentages of items with single topic, Xinjiang, Shanghai and Yunnan are at the top in the list, with the percentages of 49, 46 and 40 %, and Henan and Jilin are at the bottom, with the percentages of 23 and 26 %.

Changes in Zhongkao Mathematics

Changes in zhongkao mathematics are summarized from two aspects, changes after the new curriculum reform, and changes in recent years.

Changes After the Curriculum Reform

A comparison between the data reported by Bao (2006) and this study is made to trace changes after the curriculum reform implemented since 2001. It should be noted that the data of Bao (2006) are zhongkao mathematics papers of Suzhou from 1999 to 2001. Suzhou is a city located at Jiangsu province in east China. The comparison below can just give a rough estimation of the changes.

Table 4 shows the difficulty indices of the five factors between Bao (2006) and this study. The difficulty indices in Bao (2006) have been re-calculated based on the revised levels of the five factors shown in Table 2. Figure 8 shows the data in a diagram. The table and the diagram show that compared to Suzhou data, the indices of computation and topic coverage from this study decreases, the indices of cognitive demand and context increase, and the index of reasoning remains almost constant. This implies a more balanced composite difficulty level. These changes indicate that zhongkao mathematics is putting more emphasis on assessing students' abilities to solve problem in real context and to flexibly apply what they have

Table 4 Comparison of composite difficulty levels between Suzhou (1999–2001) and the eight regions in this study (2006–2011)

	Cognitive demand	Context	Computation	Reasoning	Topic coverage
This study	1.82	1.51	2.21	2.20	2.19
Suzhou	1.58	1.27	2.60	2.26	2.57

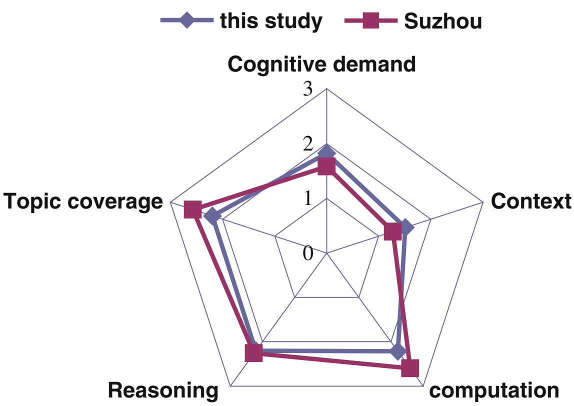
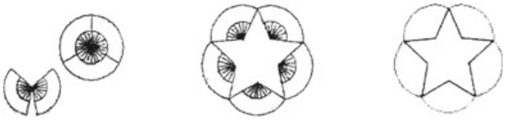


Fig. 8 Comparison of composite difficulty levels between Suzhou (1999–2001) and this study (2006–2011)

The pictures below, made of ten same folding fans, are named "butterfly around flower" (picture 1) and "plum blossom" (picture 2). There is no overlap of the folding fans in both pictures. The size of the five acute angles in the "plum blossom" picture is (   )

A. 36° B. 42° C. 45° D. 48°



Picture 1 Picture 2

Fig. 9 An item from Anhui 2006

Exploration: In figures 1 to 3, the area of $\triangle ABC$ is a.

As shown in figure 1, extend the side BC to D to make $CD=BC$, and connect points D and A. If the area of $\triangle ACD$ is S_1, then S_1 =________ (express S_1 in terms of a)

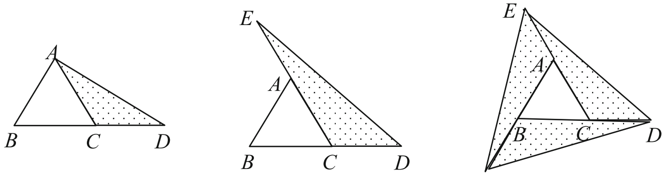


Figure 1 Figure 2 Figure 3

As shown in Figure 2, extend the side BC to D to make $CD=BC$, and extend the side CA to E to make $AE=CA$, and then connect points D and E. If the area of $\triangle DEC$ is S_2, then S_2 =________ (express S_2 in terms of a). Please give reasons.

On the basis of figure 2, extend AB to F to make $BF=AB$, and then connect points F and D, F and E to obtain $\triangle DEF$ as shown in figure 3. If the area of the shaded figure is S_3, then S_3=________ (express S_3 in terms of a)

Discovery: we extend the sides of $\triangle ABC$ to double its length, and then connect the extreme points to obtain $\triangle DEF$. We address this process as "extend $\triangle ABC$ once". We can find that the area of $\triangle DEF$ is ____ times the area of the original triangle $\triangle ABC$.

Application: Last year flowers were planted in the ground having the shape of $\triangle ABC$ as shown in figure 4. The area of $\triangle ABC$ is 10m^2. This year more flowers are going to be planted. We extend $\triangle ABC$ twice, the first time from $\triangle ABC$ to $\triangle DEF$, the second time from $\triangle DEF$ to $\triangle MGH$. Find the area of the shaded figure in figure 4.

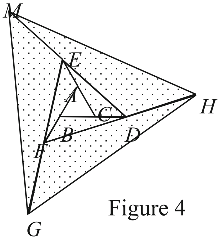


Figure 4

Fig. 10 An item from Hebei 2006

learnt to a new situation, rather than memorization of mathematical facts and procedures only. This is also manifested by the items in zhongkao mathematics papers. Two examples are given.

The item in Fig. 9 does not explicitly give the known. Instead, it requires students to find them out from the two beautiful given pictures. Students need to study the two pictures to find the relationships among different shapes. The question in Fig. 10 leads students to experience the process of problem solving, from exploring a new property by making use of basic geometrical knowledge, to generalizing the property, and to applying the property to solve a real problem. This question is not only to assess students' mathematical ability, but also to guide their mathematical learning.

Changes in the Recent Years

The results shown in Table 5 and Fig. 11 indicate that the difficulty indices of the five factors are generally stable across the recent years. It could be observed that the difficulty index of context is decreasing slightly and the indices of topic coverage and computation are increasing a little bit. Zhongkao is an important examination. The stability of the composite difficulty levels of zhongkao mathematics is of benefit to students' preparing for the examination.

Table 5 Summary of composite difficulty level across years

	Cognitive demand	Context	Computation	Reasoning	Topic coverage
2006	1.86	1.62	2.18	2.22	2.10
2007	1.83	1.54	2.17	2.17	2.08
2008	1.77	1.60	2.21	2.13	2.11
2009	1.84	1.48	2.26	2.24	2.28
2010	1.81	1.47	2.13	2.18	2.19
2011	1.83	1.38	2.31	2.26	2.38

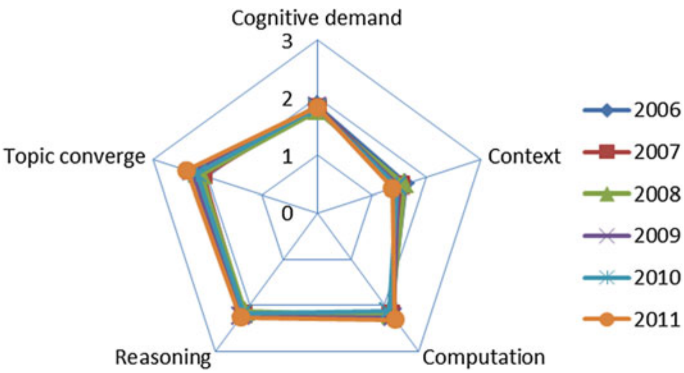


Fig. 11 Composite difficulty levels across years

Conclusions

From the analysis of the selected zhongkao mathematics papers and the comparison with the data reported by Bao (2006), three conclusions could be drawn. First, zhongkao mathematics stress computation, reasoning, and relations among different mathematics topics, but not so care about applications of mathematics in real context. This actually reflects characteristics of mathematical problem solving activities in China (Bao 2009). The tradition of mathematics education in China stresses solid foundation of basic knowledge and basic skills. It is believed that creative abilities and individual development is based on a sound mastery of basic knowledge and skills.

Second, under the background of the implementation of the mathematics curriculum since 2001, zhongkao mathematics remains to keep the good tradition of emphasis on computation, reasoning and connections among mathematical topics, and presents some new features, like more process-oriented questions and more real context questions. This leads to positive changes on mathematics classroom teaching. It is hoped that these changes could relieve students from the heavy burden of coping with the examination, and become the master of their own learning.

Third, there are obvious region differences in zhongkao mathematics as reported sporadically in the paper. Generally speaking, regions from west economic zone like Xinjiang are relatively less demanding in terms of computation, reasoning, and topic coverage factors, while regions from east and central zones like Guangdong, Jilin and Henan are more demanding in these factors. Shanghai has the lowest indices in context and cognitive demand factors, and roughly an average score in computation, reasoning and topic coverage factors. This suggests that a sound mastery of basic mathematical knowledge and skills could be a possible reason for Shanghai students' good performance in PISA mathematics 2009. It also gives some confidence that students from other regions might have a good chance to perform well in an international test.

However, the region differences might result in equity issue which should be taken seriously into consideration. There are other issues in zhongkao mathematics. Calculators are allowed to be used in zhongkao mathematics in some districts like Xinjiang. How to set questions in zhongkao mathematics to fit the use of calculator is a big challenge. Moreover, the mathematics curriculum standard of compulsory education (2011 version) (Ministry of Education of China 2011) has just been published. It brings new ideas in mathematics education. For example, in addition to basic knowledge and basic skills, basic methods of thinking and basic activity experiences are proposed. Hence, the traditional two basics now become four basics. How to make zhongkao mathematics to reflect the new ideas in the standard is not an easy work. What can be expected is that zhongkao mathematics will continue to stress good traditions of Chinese mathematics teaching and learning, and at the same time adapt to the new development of mathematics education.

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Mapping Mathematical Leaps of Insight

Caroline Yoon

Abstract Mathematical leaps of insight—those Aha! moments that seem so unpredictable, magical even—are often the result of a change in perception. A stubborn problem can yield a surprisingly simple solution when one changes the way one looks at it. In mathematics, these changes in perception are usually structural: new insights develop as one notices new mathematical objects, attributes, relationships and operations that are relevant to the problem at hand. This paper describes a novel analytical approach for studying these insights visually using "mathematical SPOT diagrams" (SPOT: Structures Perceived Over Time), which display evidence of the mathematical structures students perceive as they work on problems. SPOT diagrams are used to compare the conceptual development of two pairs of participants, who investigate whether a gradient (derivative) graph yields information about the relative heights of points on its antiderivative; one participant pair experiences a leap of insight, whereas the other does not. Each pair's SPOT diagrams reveal key differences in the structural features they attend to, which can account for the disparate outcomes in their conceptual development.

Keywords Mathematical insight · Calculus · SPOT diagrams · Mathematical structure

Introduction

Mathematical leaps of insight are often described on affective and aesthetic levels—the rush of understanding something that was previously incomprehensible (Liljedahl 2005), the pleasure of a beautiful, elegant solution (Hadamard 1954; Poincaré 1956). On a mathematical level, leaps of insight involve shifts in perceived structure, whereby students notice and create new mathematical structures to

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replace older ones. This paper proposes a way of visualising these leaps of insight via “SPOT diagrams” (SPOT: Structures Perceived Over Time), which display evidence of the mathematical structures students perceive as they work on challenging mathematical tasks. This novel analytical approach is partly inspired by powerful visual techniques for studying neural activity, such as functional magnetic resonance imaging. Just as neuroimaging techniques enable neuroscientists to document and measure neural activity visually, so are SPOT diagrams intended to provide a visual way of studying shifts in the mathematical structures students perceive over time.

Not all shifts in perceived mathematical structure qualify as a *leap* of insight. Sometimes, students will elaborate on an existing mathematical structure by adding new mathematical objects, attributes, operations and relationships, like a variation on a theme. By contrast, a leap of insight requires a significant reorganisation of mathematical structure that goes beyond incremental additions to an extant structure. I use SPOT diagrams to compare the size and quality of the structures two pairs of participants perceive while working on a calculus task. Both pairs investigate whether a given graph provides any information about the relative heights of various points on its antiderivative graph. However, one pair experiences a leap of insight with respect to this question, whereas the other pair does not. An analysis of each pair’s SPOT diagrams reveals some subtle yet key differences in the objects and relationships each pair attended to, which may account for the stark contrast in their respective conceptual development.

Before presenting these particular SPOT diagrams, I begin by situating this analytical approach within the literature in terms of its theoretical foundations and methodological considerations. I then present and analyse the two contrasting SPOT diagrams mentioned above, and end by discussing the research value of SPOT diagrams and comparing them to other diagrammatic approaches for studying mathematical conceptual development.

Mathematical Structure Perceived Over Time

Pure mathematics can be characterised as the study of structures (Shapiro 1997; Mason 2004), which are made up of mathematical objects (such as counts, measures, sets), attributes (e.g. few, large, open), operations (e.g. combine, enlarge, invert), and relationships (e.g. greater than, equivalent to, isomorphic). To illustrate this characterisation, Shapiro writes:

> Group theory studies not a single structure but a type of structure, the pattern common to collections of objects with a binary operation, an identity element thereon, and inverses for each element. Euclidean geometry studies Euclidean-space structure; topology studies topological structures, and so forth (1997, p. 73).

Applied mathematics and mathematical modelling also study structures in the real world:

> Mathematical models are distinct from other categories of models mainly because they focus on structural characteristics (rather than, for example, physical, biological, or artistic characteristics) of systems they describe (Lesh and Harel 2003, p. 159).

My analytical approach is influenced by this view of mathematics as the study of structures. Consequently, the SPOT diagram approach operationally defines students' mathematical conceptual development as the mathematical structures they perceive and describe.

It is important to differentiate my use of the term "structure" as a description of the *mathematics* students attend to, from the way it is sometimes used in learning theory literature to describe mental representations of *mathematical knowledge*. For example, Skemp (1987) describes mathematical understanding as the development of mental schemas, which are interconnected networks or structures, and Piaget (1970) describes conceptual development as the growth of children's cognitive structures. SPOT diagrams are not a theory of learning, but rather a methodological tool for identifying the mathematics students attend to during a problem. The approach addresses questions such as: *What mathematical structures have students constructed and perceived as relevant to the problem? What objects, operations, attributes and relationships are they attending to?* Consequently, the term "structure" refers to the content of students' mathematical perceptions rather than the representation of those perceptions in their mind.

My use of the term "perceive" is not meant to suggest that these mathematical structures are "in" the problem, hidden for students to find. Instead, it reflects an emphasis on the personal, idiosyncratic mathematical structures that students *themselves* construct, manipulate and bring to bear on the problem, rather than any formal mathematical structure that students *should* find. I acknowledge that the personal, idiosyncratic structures a student perceives can only be identified through the observable signs the student creates and works with. According to Arzarello et al. (2009), these signs or semiotic bundles may involve spoken language, gestures, written text, symbols, diagrams, graphs and physical artefacts. Consequently, I analyse the semiotic bundles that students produce while working on a task as *evidence* of the mathematical structures they perceive.

Collecting Evidence of Perceived Mathematical Structures

The data used in this paper were collected as part of a project that investigates students' construction of calculus concepts (Yoon et al. 2011a). Twelve participants worked in pairs on four calculus tasks of 1-h duration each. Ten participants were undergraduate students, two were secondary school mathematics teachers, and all

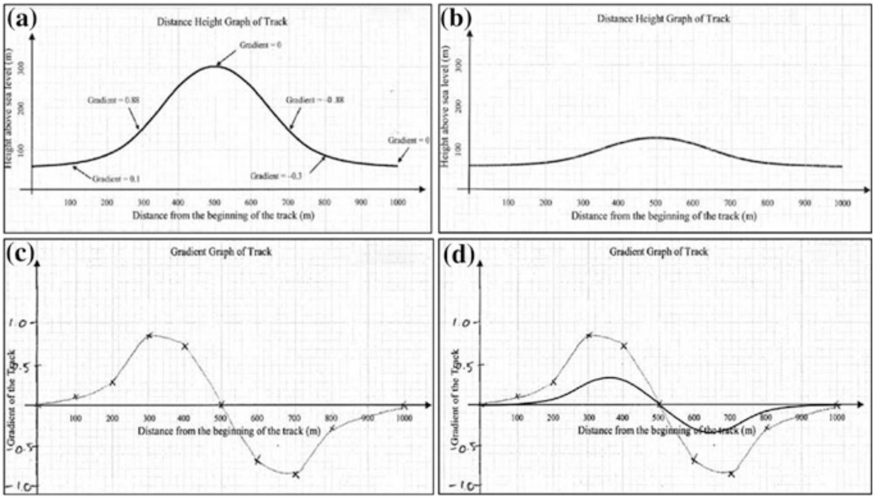


Fig. 1 Graphs of two tramping tracks (**a**, **b**) and their gradient graphs (**c**, **d**)

had basic school-level knowledge of calculus at the time. Two pairs of participants are featured in this paper: Cam and Sid, undergraduate students who did not know each other prior to the study, and Ava and Noa, the two teachers, who knew each other well. The participant pairs worked in the presence of a researcher who clarified task instructions but refrained from giving mathematical direction. The pairs were videotaped and audiotaped on each of the four tasks, which generated 24 verbal transcripts. These transcripts were annotated to include the gestures, inscriptions and nonverbal cues that participants performed or created. This paper reports on the shifts in perceived structure identified from the transcripts and video footage from Cam and Sid's and Ava and Noa's work on the first task—the tramping activity. These sessions are described in a previous paper (Yoon et al. 2010), but this is the first time they have been analysed using SPOT diagrams.

The tramping activity begins with a warmup, where students are given a distance-height graph of a tramping (hiking) track (Fig. 1a), and are instructed to calculate gradients of the track at certain points in order to sketch the gradient graph (derivative) of the track (Fig. 1c). Students are then shown a graph of a similar, flatter tramping track (Fig. 1b), and are asked to sketch the gradient graph of this track (Fig. 1d) without calculating any actual gradients. Next, the problem statement (see Fig. 2) asks students to design a method for drawing the distance-height graph of any tramping track from its gradient graph. This task is mathematically equivalent to finding the graphical antiderivative of a function presented graphically. Students are asked to explain their method in the form of a letter addressed to hypothetical clients (the O'Neills) and to use their method to find features of a

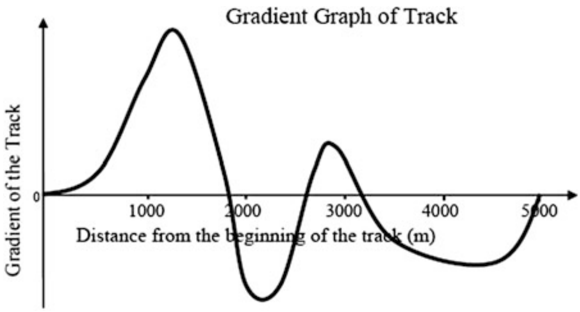


Fig. 2 Graph of a tramping track's gradient and the problem statement

specific track whose gradient graph is given in Fig. 2. Examples of reasonably accurate graphs of the tramping track can be found in Fig. 5a, b.

Analysing Evidence of Students' Mathematical Structures

I used a combination of epistemic analysis and data mining to develop a coding scheme to analyse the structures students perceive during the tramping activity. I began by developing an initial list of codes to identify the mathematical objects, attributes, operations and relationships that could be encountered during the activity, based on my observation of one group's work (Ava and Noa's). Then, two research assistants and I entered into design cycles (Kelly et al. 2008) of implementing, testing and revising the coding scheme on three transcripts, which were analysed in conjunction with the associated video and images of student work. We coded portions of transcripts independently then met to compare and identify coding discrepancies. I used these discrepancies to inform revisions in the coding scheme, which we again tested through independent coding, comparisons, and further revisions. Following this, a third research assistant who had not been involved in the process joined us in further design cycles with three more transcripts. This measure was taken to enhance the reusability of the coding scheme.

The design process lasted six months, during which the design team met more than 20 times to compare and discuss our coding. The process was repeated to develop a coding scheme for another antiderivative task (the fourth in the sequence, which is described in Yoon et al. 2011b), which informed further revisions of the coding scheme for the tramping activity. All of the six transcripts of the first task were then recoded using the final coding scheme (which consists of over 100 codes), and the mathematical structures were represented in network diagrams. Table 1 shows a small selection of the codes from the final scheme that were used to identify mathematical objects and attributes that students perceive during the first task. Antidifferentiation was the main mathematical operation in this task, but the scheme also included combining, multiplying and enlarging. The list of mathematical relationships included equivalence, comparisons and logical relationships.

Table 1 A selection of objects and attributes from the coding scheme

	Objects	Attributes
Related to the gradient graph	y-value	Large, small, zero, bigger than, etc.
	Change in y-value	Increasing, decreasing, constant
	Sign of y-value	Positive, negative
	Maxima/Minima	Number, order, location
	x-axis intercepts	Number, order, location
	Size of bumps/dips on g-graph	Big, small, relative size
Related to the tramping track (antiderivative)	Steepness of slope	Steep, gentle, flat, steeper than, etc.
	Change in steepness of slope	Getting steeper, getting gentler, no change
	Direction of slope	Up, down, flat
	Absolute height	High, low, at or below sea level, etc.
	Points of inflection	Location, number, order
	Horizontal distance of track	Long, short, longer than, shorter than, etc.

Structures Perceived Over Time by Two Participant Pairs

The two pairs (Cam and Sid; Ava and Noa) were chosen because they were both unsure at the outset of the problem as to whether the gradient graph yielded information about the relative heights of points on its antiderivative. Consequently, both pairs had potential for genuine conceptual development in this particular area, and they spent some time investigating questions such as "Is the track's valley is at sea level or at sea level?" and "How high is the second summit in relation to the first?" Cam (and to a lesser extent, Sid) developed a productive, mathematically correct way use features of the gradient graph to determine relative heights of points on the tramping track, which constituted a mathematical leap of insight. In contrast, Ava and Noa did not, but decided that there was no way of telling the height of the track from its gradient graph.

The total time both pairs spent on this particular problem (regarding the track's height) totalled less than 10 min each. Like most students who work on the tramping problem, both pairs spent most of their time determining how y-value attributes of the gradient graph corresponded to the gradient of the tramping track. They successfully ascertained that positive/negative/zero y-values on the gradient graph corresponded to uphill/downhill/flat sections on the track, and that small/large y-values on the gradient graph corresponded to shallow/steep slopes on the track. This paper focuses only on structures the pairs perceived while investigating relative heights of points on the tramping track. These structures are described then displayed diagrammatically for each pair in Figs. 4 and 6.

Structures Perceived by Cam (and to Lesser Extent, Sid)

Cam and Sid's first rough sketch of the tramping track shows the two peaks and valley corresponding (although not very precisely) to the *x*-axis intercepts in the gradient graph (see Fig. 3a). While drawing the track from left to right, they assume that the valley is at sea level (#1 in Fig. 4), and don't question the height of the track until they draw the second peak. Initially, Cam reasons that since the *y*-value of the second local maximum on the gradient graph is one-third that of the first local maximum, the second peak of the tramping track must be one-third the height of the first peak (#2 in Fig. 4). A few seconds later, he retracts this observation, saying that the *y*-value of maxima on the gradient graph corresponds to the slope, not the height of the track (#3 in Fig. 4). Cam draws the second ascent of the track to show a shallower gradient than that of the first ascent (see Fig. 3a).

While drawing this second ascent, he wonders aloud, "What's the area under a gradient graph mean, doesn't it mean something as well? … Isn't that distance (*referring to the height of the antiderivative*)?" (#4 in Fig. 4). Cam locates the chapter in his textbook that describes area under the curve, but then closes the book, saying, "that's for speed". Even though the textbook confirms that the area under the curve of a graph corresponds to the vertical displacement of its antiderivative, Cam dismisses the textbook's treatment of the area under the curve because it is set in the context of speed and time graphs, which he doesn't relate to the tramping problem. He returns to the graph of the track, and draws the final descent from the

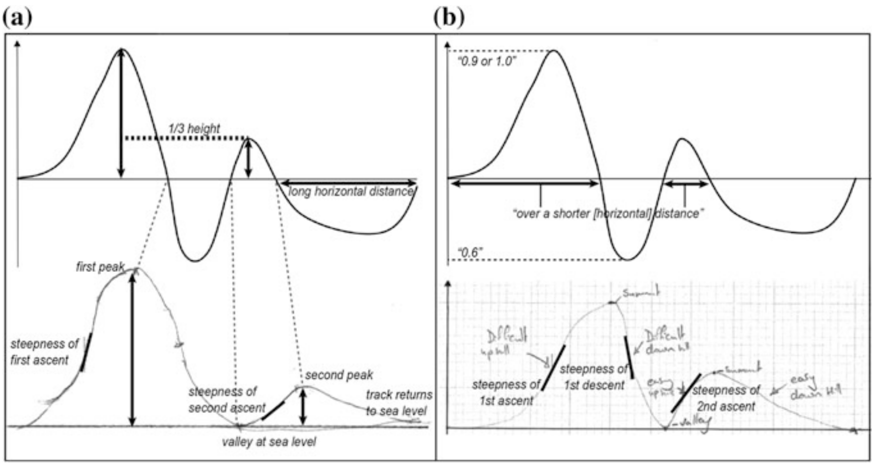


Fig. 3 Cam and Sid's first (**a**) and second (**b**) attempts to draw the tramping track

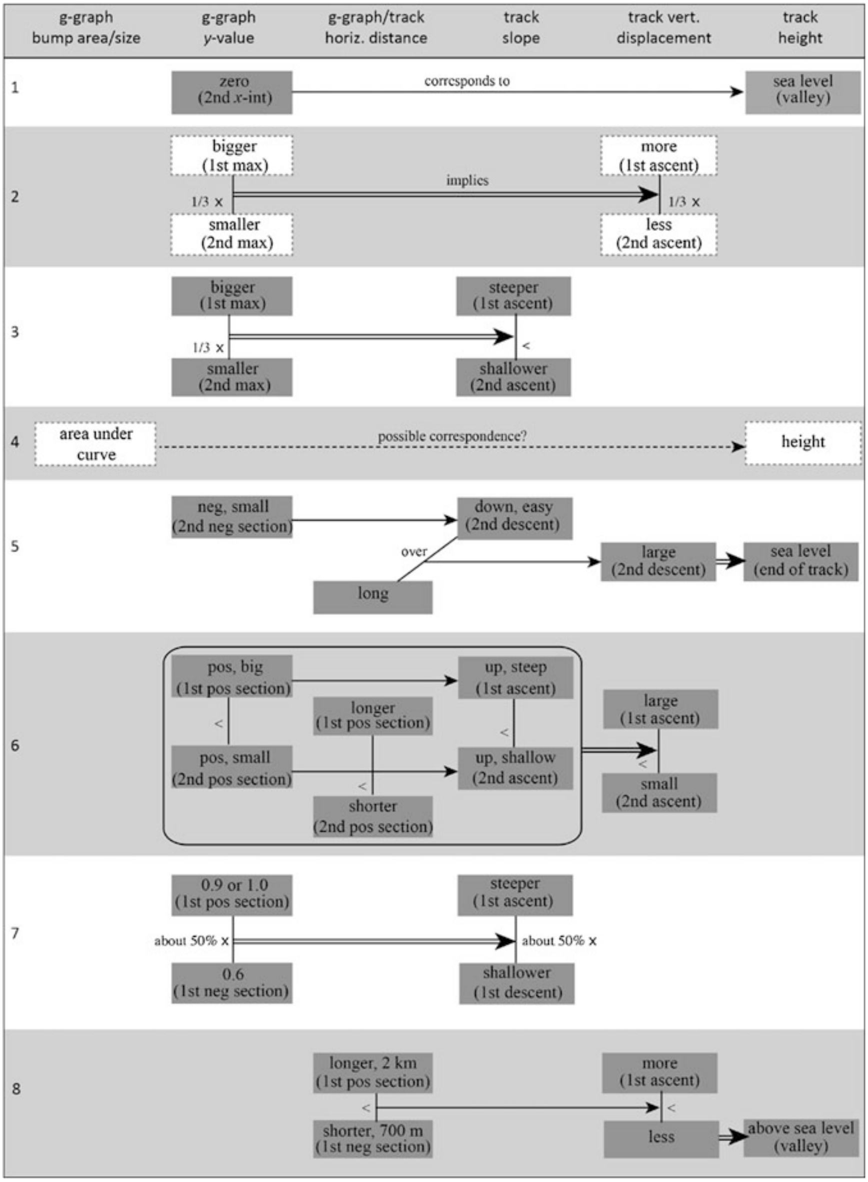


Fig. 4 Diagrams of Cam's structures perceived over time

second summit, reasoning that it should be an "easy slope" downhill, ending at sea level since it occurs over a long horizontal distance (#5 in Fig. 4).

Later, Sid redraws the graph of the track (see Fig. 3b) to correct the horizontal misalignment in Cam's first rough sketch attempt. Cam makes a number of

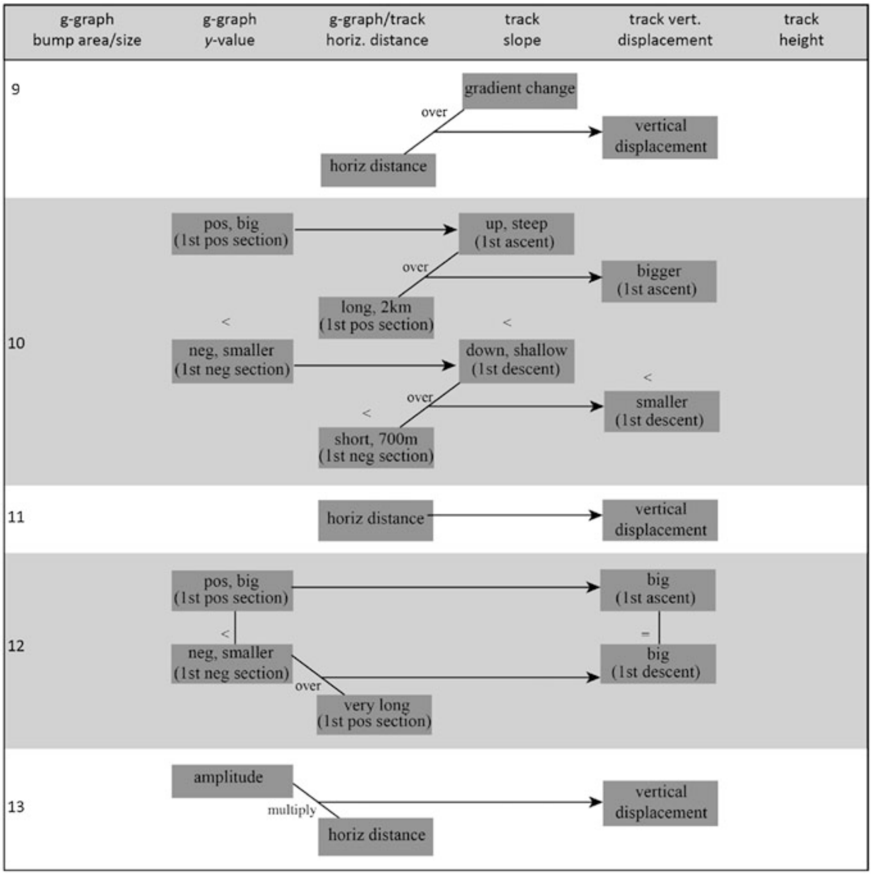


Fig. 4 (continued)

observations about the second ascent: it does not go "very far" (it is unclear whether this refers to the horizontal distance or vertical displacement); it doesn't get as steep as the first; and it is over a shorter horizontal distance than the first. Neither Cam nor Sid explicitly state how this influences the vertical heights of the two peaks, but Sid draws the second ascent significantly lower in height and shallower in steepness than the first ascent to Cam's satisfaction, which suggests they may have a tenuous understanding of the relationships (#6 in Fig. 4). Cam then repeats his earlier observation about the second descent occurring over a long horizontal distance, and therefore reaching sea level through a gradual downhill slope (#5 in Fig. 4).

Cam and Sid then realise that the graph they have drawn doesn't accurately reflect the gradients of the first ascent and descent. Cam remarks that the *y*-value of the first maximum is "about 0.9 or 1.0" (according to some scale), whereas that of the first minimum is "0.6" (absolute value). He reasons that the first descent should be about 50 % of the steepness of the first ascent (#7 in Fig. 4), when in fact they

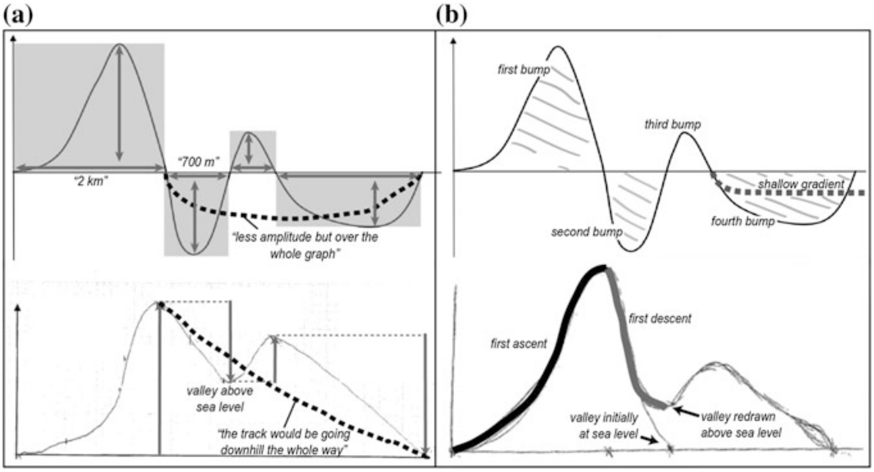


Fig. 5 **a** Cam and Sid's final attempt to draw the tramping track. **b** Ava and Noa's attempt at drawing the tramping track

had drawn it much steeper, and suggests this was because they had neglected to take into account the difference in horizontal distance (#8 in Fig. 4):

> I think what misled us is this is over a shorter distance (*traces pencil along x-axis of first positive section on gradient graph*). So maybe this (*points to valley on tramping track*) doesn't come all the way back down to the ground, it might only come that far (*marks valley above sea level*).

This is the first time that Cam specifically suggests that the horizontal distance impacts the vertical displacement of the track in general. He then redraws the graph (see Fig. 5a).

When trying to explain how the clients could avoid making the same mistake that they did, Cam suggests that the clients should "take into account the [horizontal] distance covered as the gradient is changing" (#9 in Fig. 4), and compares the horizontal distance covered over the first ascent (2 km) to the horizontal distance covered over the first descent (700 m). He explains that since the latter is smaller than the former, the track's vertical displacement over the first descent won't match that of the first ascent (#10 in Fig. 4), and adds that in general, the horizontal distance of a gradient graph "indicates the height climbed or descended" on the track (#11 in Fig. 4).

Sid disagrees, saying the amplitude of the gradient graph alone indicates the vertical displacement of the first ascent and descent—that the track will not reach ground level at the end of the first descent because the amplitude of the first negative section of the gradient graph is smaller than that of the first positive section. Cam counters that it is a combination of both amplitude and horizontal distance, pointing out that if the first minimum had "less amplitude but over the whole graph (*referring to the horizontal distance*), [the track] would be going downhill the whole way (*down to sea level*)" (Fig. 5a). In doing so, Cam argues that the amplitude and the horizontal distance of the gradient graph can influence the height of the track in competing directions (#12 in Fig. 4).

Cam generalises this observation, saying "What about for any graph, just general graphs… it's kind of a factor of amplitude times by how far it's gone (*traces pen along x-axis of gradient graph*) will give you an indication of how far you're going to go, how far you're going to drop." Thus, he ends by proposing that the multiplication of the two object attributes (vertical amplitude and horizontal distance on a gradient graph) gives a measure of the vertical displacement on the track (#12 in Fig. 4).

Structures Perceived by Noa and Ava

Noa and Ava first consider the height of the tramping track when Ava draws the valley of the track. She initially draws the valley at sea level (this is visible in Fig. 5b), but questions this position, saying "Am I here, should I be here?" (#1 in Fig. 6). Noa acknowledges that the valley may indeed be above sea level but doesn't offer a way to determine this. Ava suggests that the sizes of the "bumps" on the gradient graph (see bumps on gradient graph in Fig. 5b) may offer a way of addressing the question. She says that since the first bump is bigger than the second bump, the first ascent on the track is steeper than the first descent, and consequently, the first ascent must have a larger vertical displacement than the first descent (#2 in Fig. 6). When Noa disagrees, Ava argues, "If you've gone steeper, you must have gone higher" (#3 in Fig. 6), to which Noa replies "Oh I see what you're saying. Steeper for longer must be higher. Yes." (#4 in Fig. 6). Ava redraws the valley of the track to sit above sea level, saying, "This (*points to the second bump on the gradient graph*) is smaller than that (*points to the first bump*), so we probably haven't gone back down to zero, so change that." (#5 in Fig. 6).

When Ava and Noa later revisit the question of the height of the tramping track, Noa comments that they are using "anecdotal evidence of big hill (*points to graph in* Fig. 1a), big bump (*points to* Fig. 1c); small hill (*points to* Fig. 1b), small bump (*points to* Fig. 1d)" (#6 in Fig. 6). This comment shifts their attention to the derivative relationship between the hill sizes in the warmup. Ava wonders how they can find the height of the track from the gradient function and says, "If you've got the gradient function how do you get back to the initial function? You need to integrate. Right, so we could integrate this function to provide a value? Yes, we could, couldn't we?" while shading in the area under the bumps in Fig. 5b (#7 in Fig. 6). Noa agrees and they try to recall their knowledge of integration. However, their discussion of integration is rooted in the context of distance, time, and speed, rather than the context of slope and height, and they abandon this pursuit at the interviewer's suggestion after failing to link their knowledge of integration to the tramping problem.

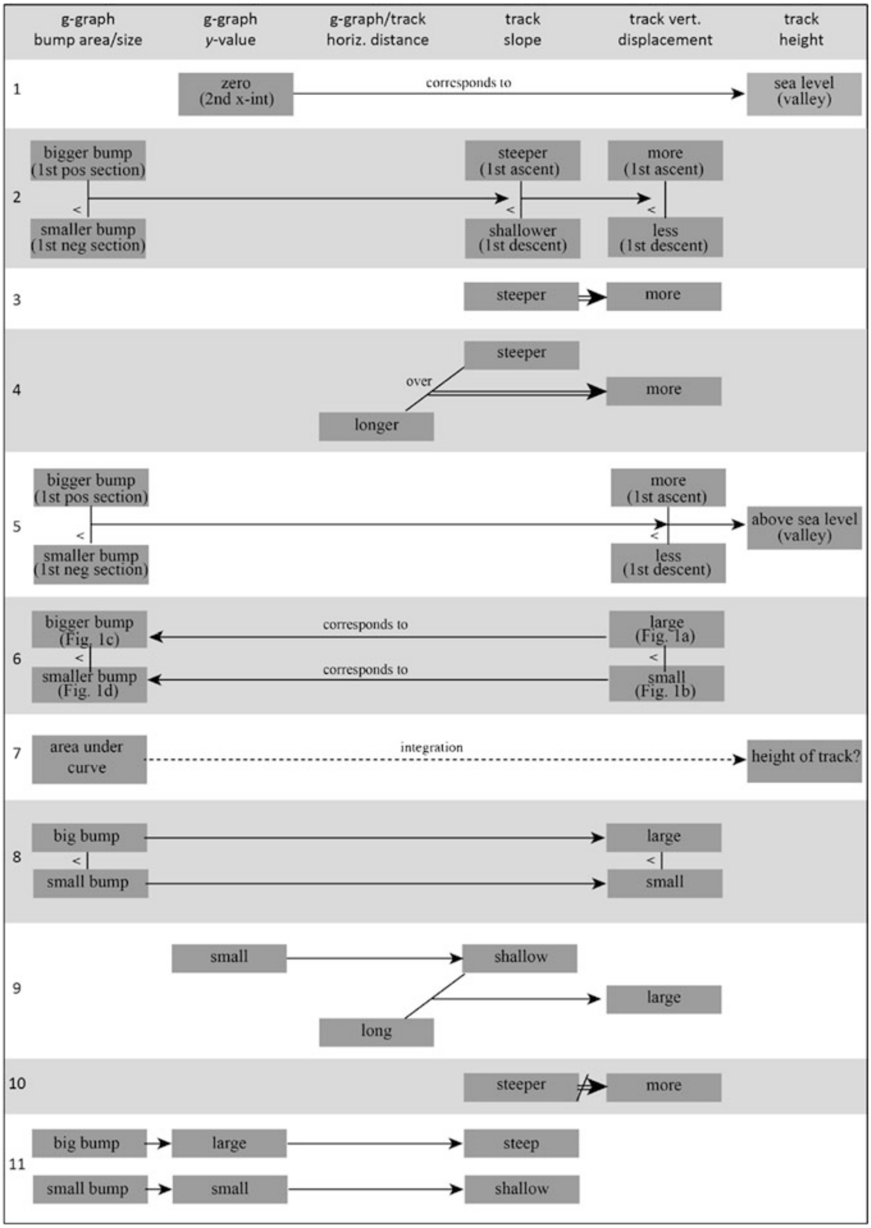


Fig. 6 Diagrams of Noa and Ava's structures perceived over time

Ava restates their previous assertion that the size of bumps on the gradient graph indicates the vertical displacement in the track (#8 in Fig. 6), which is based on their previous assertion that track steepness impacts vertical displacement (#3 in Fig. 6).

Noa rejects this, saying, "You could have [a bump] that's just got a shallow gradient but just goes on for ages (*traces shallow bump on gradient graph in* Fig. 5b), so you could end up being really high, but not really steep to get there." This demonstrates an appreciation of the interaction between horizontal distance (#9 in Fig. 6) and steepness in determining track height. However, Noa uses this insight incorrectly as a counterexample of their previous rule (#3 in Fig. 6), rather than expanding their previous rule to include horizontal distance.

Noa and Ava end the task by asserting that the bumps on the gradient graph indicate the steepness, but not the vertical displacement of the track (#10 in Fig. 6). They list four variations on the theme: "If it's a big bump (*on gradient graph*), you will have a steep climb. If it's a steep valley (*on gradient graph*), you will have a steep downhill. If it's a flat valley (*on gradient graph*)… you will have a gentle downhill… If it's a small bump (*on gradient graph*), the hill will be a gentle gradient" (#11 in Fig. 6). Thus, they end by attending to a smaller set of objects and relationships than they initially started out with, and eschew the possibility of determining track height altogether.

Noa and Ava's conceptual development with respect to the question of the track's height is starkly different to Cam's. Cam experiences a leap of insight when he creates his final structure (#13 in Fig. 4), in which he realises that the horizontal distance multiplied by the amplitude of positive or negative sections of the gradient graph indicates the vertical displacement over corresponding portions of the tramping track. His final structure is both a combination of earlier structures (#3, #6, #7 and #8 in Fig. 4) and a distillation of the common relationship underlying other structures (#10 and #12 in Fig. 4). In contrast, Noa and Ava do not experience such a leap of insight: they decide that the gradient graph does not provide information about the height of the tramping track, and end their investigation of this concept by reverting back to the more conservative rule (which they had previously established): that the amplitude of a gradient graph indicates the steepness of the tramping track. Their final structure (#11 in Fig. 6) abandons rather than builds on earlier structures that relate to the height of the antiderivative (#3, #4 or #9 in Fig. 6), even though these structures bear striking similarities to the ones Cam used to generate his leap of insight.

Comparing Leaps of Insight Through Spot Diagrams

Why do Cam's shifts of attention yield a leap of insight whereas Noa and Ava's do not? Figure 7 shows a side-by-side comparison of the SPOT diagrams depicted in Figs. 4 and 6: the diagrams are simplified to retain the relationships, objects and operations, but do not include details of the object attributes or locations on the graph (although this information can be easily located in Figs. 4 and 6 as the structures are numbered similarly). This side-by-side comparison offers clues into subtle differences in the types of structure that Cam, Noa and Ava perceived—

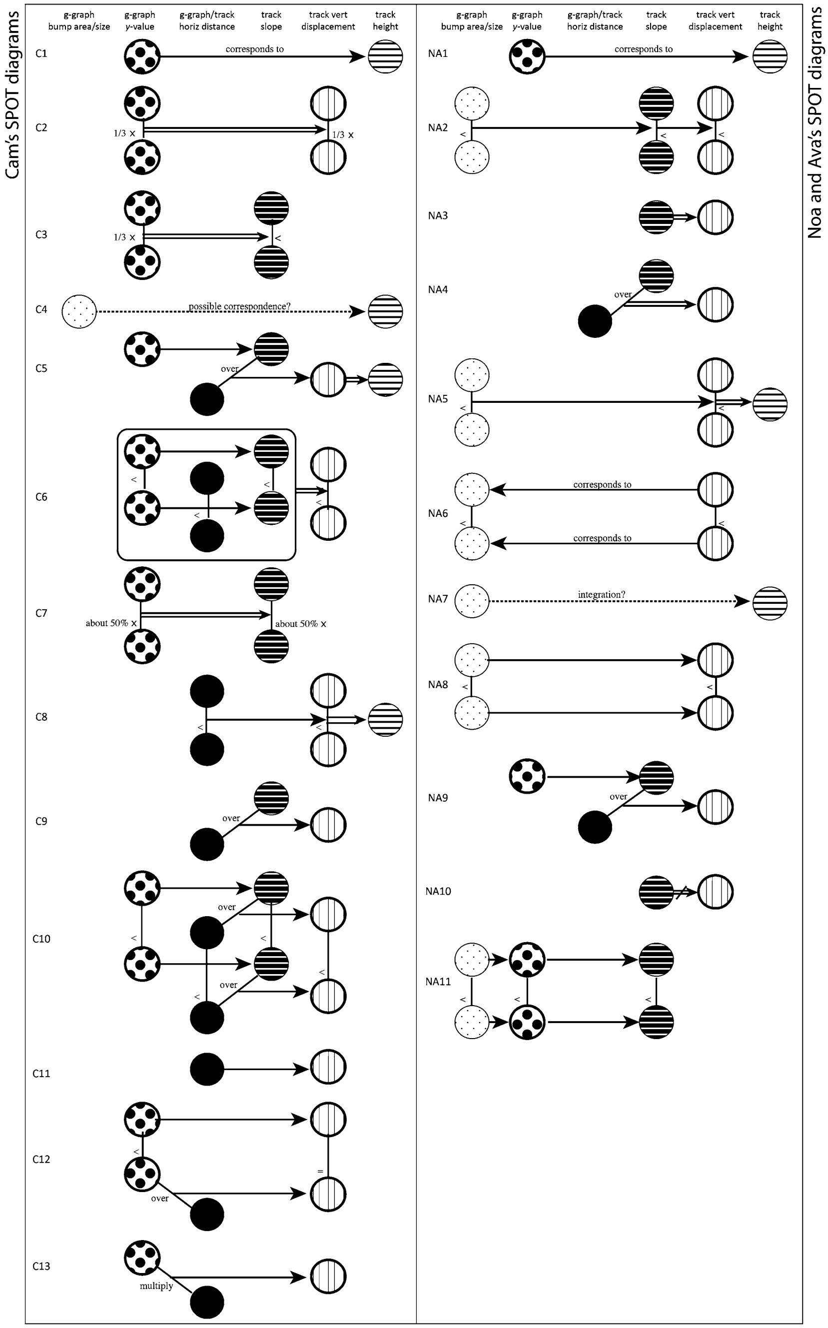
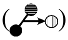
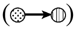


Fig. 7 Comparison of Noa's (and Ava's) and Cam's SPOT diagrams

differences in the objects they attended to and the kinds of relationships they considered—which help explain the different outcomes in their conceptual development.

One key difference is the number of times they attended to the size of "bumps" on the gradient graph, an object denoted in Fig. 7 by the symbol ◌. Cam only considers this object once briefly, whereas Noa and Ava focus on it repeatedly. Noa and Ava's notion of "bump size" is ill-defined—they do not break it down into its constituent parts of horizontal distance and amplitude, but talk about it initially in amorphous terms. At the end, they interpret gradient bump size as indicating that the track will have a steep slope "somewhere", effectively reducing bump size to the vertical amplitude of the bump. Noa and Ava could have increased their chances of experiencing a leap of insight had they shifted attention towards the smaller constituent parts of "bump size". Indeed, Cam focuses on these "lower-level" objects, eventually showing how they work together to indicate vertical displacement on the track, although he never articulates their connection to the higher-level object of bump size. Noa and Ava's persistent attention to the higher-level (and ill-defined) object of bump size may have prevented them from understanding how it relates to vertical displacement, even though this was one of their first hypotheses (see structure NA2 in Fig. 7). Their limited understanding of the hypothesis may explain why they were ready to abandon it in the face of Noa's "counterexample" in structure NA9.

Another significant difference is in the frequency with which both parties attended to the object of horizontal distance, indicated in Fig. 7 by the symbol ●. Cam's structures contain many such black spots. Sometimes, these are connected with the track slope object (denoted by the symbol ⊜), which together correspond to the track's vertical displacement (denoted by the symbol ⦶) as in structures C5, C9 and C10. But other times he focuses on how horizontal distance alone affects track vertical displacement (C11), or simply compares differences in horizontal distance (C6). Cam also attends to the horizontal distance in multiple parts of the graph: the second downhill portion of the track (C5); the first and second positive sections of the gradient graph (C6); the first ascent and descent in the track (C8, C10, C12). Cam's attending to this object in so many different guises, and combinations with other objects may have enabled him to distil its impact on the track's vertical displacement, and integrate it into his final structure (C13). In contrast, Noa only considers the horizontal distance on two occasions: first to extend the claim that a steeper climb means a higher vertical displacement (NA4) and second, as a counterexample to the same claim. Noa never considers the horizontal distance as something that could affect the track's height in its own right, which could have prevented her from ascertaining how it works together with slope to impact track height.

This last observation is curious—why does Noa not perceive the relationships between the horizontal distance, track slope and track height as a valid structure to be considered in its own right? Indeed, it is the same basic structure that Cam distils in his conceptual development, and it explains the driving question about the height of the track. One explanation can be found in the kinds of relationships that Noa and Ava consider. Whereas Cam seems comfortable considering the impact two factors () had on track height, Noa and Ava seem to be searching for a single factor () to map onto track height. When they establish that neither steepness nor "bump size" alone determine track height, they abandon the search for a single factor altogether even though the structure that served as a counterexample (NA9) involves two factors impacting on track height. It seems that while Noa and Ava find this more complex structure useful for extending or countering the single-factor relationships, it is not considered a possible alternative to be considered in its own right. In contrast, Cam builds intricate networks of relationships, showing how changes in multiple object attributes could influence others (C5, C10, C12 and to some degree, C6). The two-factor relationship () underlies each of these complex structures, and is distilled and formalised into the final structure C13.

The SPOT analysis described in this paper only compares two case studies for the purpose of demonstrating the analytical technique. Even so, the analysis yields some useful insights into how attending to certain types of mathematical objects and relationships can affect one's mathematical conceptual development. Noa and Ava's persistent attention to bump size shows that attending to sophisticated objects does not necessarily help generate new insights: sometimes decomposing such objects into more lower-level objects and relationships (like those Cam attends to) can be more productive. Also, overly focusing on one type of relationship (like Noa and Ava's search for a single factor to map to track height) can prevent one from noticing other, more powerful relationships (like Cam's two-factor relationship). These results are part of a larger research programme dedicated to mapping students' mathematical conceptual shifts via SPOT diagrams in order to understand how such shifts come about, and how to facilitate their occurrence in practice. Further systematic SPOT diagram analyses of greater numbers of case studies using different participants, different tasks and different mathematical structures will likely yield even richer insights into the nature of mathematical conceptual development.

Spot Diagrams and Other Diagrammatic Approaches

Although a number of other diagrammatic approaches exist for analysing mathematical conceptual development, SPOT diagrams are distinct in three key ways. First, SPOT diagrams focus on a much finer grain size than the units of analysis typically focused on by other approaches. For example, Williams' (2007) diagrams of "spontaneous abstraction" use numbered circles to represent students' thought processes as they generate new insights into linear functions. The grain size of these numbered circles roughly corresponds to that of the numbered structures in Cam's and Noa and Ava's SPOT diagrams. However, the numbered structures in the SPOT diagrams are further broken down into the fundamental mathematical objects, attributes, operations and relationships that the participants perceive, which comprise these larger structures. SPOT diagrams allow researchers to perform a much finer-grained analysis than most other diagrams: indeed the image of a “spot” evokes the attention to detail with which the structures are displayed, focusing on the fine-grained mathematical objects, attributes, operations and relationships perceived by students.

Ron et al. (2010) use a similar grain size to SPOT diagrams in their abstract diagrams of “partially correct constructs”: they decompose these constructs to their constituent elements to show why they are only partially correct. For example, they show diagrams of hypothetical constructs that have missing elements, incompatible elements and disconnected elements. However, their diagrams are abstract depictions of hypothetical constructs, rather than personalised depictions of specific students’ perceived structures, like Cam’s and Noa and Ava’s SPOT diagrams. This highlights a second distinctive characteristic of SPOT diagrams—their function as an empirical tool for displaying specific students’ perceived structures.

A third key feature of SPOT diagrams is that they are not tied to a particular theory of learning, only a particular perspective on mathematics as structure. Kidron and Dreyfus (2010) use an “interacting parallel constructions diagram” to show the branching, combining, interrupting and resuming nature of a mathematician’s epistemic actions as she learns about bifurcations of a particular dynamical system. Both the interacting parallel constructions diagram and Williams’ (2007) spontaneous abstraction diagrams are situated within the nested epistemic actions (RBC) model (Hershkowitz et al. 2001). In contrast, SPOT diagrams do not display the epistemic processes that lead to the structures students perceive, and consequently they can be used to study conceptual development empirically from multiple perspectives on learning.

Future Development of Spot Diagrams

A number of developments are planned to enhance the empirical functionality of SPOT diagrams in the future. Currently, SPOT diagrams are static, using the vertical orientation of the page (up to down) to represent the passage of time. One drawback of static diagrams is they show discrete snapshots of the structures attended to at different time, but do not show how one structure might grow into another structure. Nor do they show where a person agrees with a particular structure, then changes their mind to reject it later. An immediate goal is to display SPOT diagrams in a dynamic format (on screen), so that researchers can observe evidence of students’ mathematical structures evolving over time.

SPOT diagrams will also be developed into interactive displays, where researchers can zero in on particular structural features of interest. For example, one might click to highlight only those structures that feed into a person’s understanding of points of inflection, or double click an object to see how it was constructed from lower-level objects and relationships earlier on. Finally, a long-term goal is to create a software programme to automate the construction of these dynamic, interactive, SPOT diagrams so that other researchers can use them to study leaps of insight in other mathematical domains.

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