

Chapter 5

Termination of Elastic Range of Pressure Insensitive Materials—Isotropic and Anisotropic Initial Yield Criteria

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Abstract In this chapter basic features of isotropic versus anisotropic initial yield criteria are discussed. Two ways to account for anisotropy are presented: the explicit and implicit formulations. The explicit description of anisotropy is rigorously based on well-established theory of common invariants (Sayir, Goldenblat–Kopnov, von Mises, Hill). The implicit approach involves linear transformation tensor of the Cauchy stress that accounts for anisotropy to enhance the known isotropic criteria to be able to capture anisotropy, hydrostatic pressure insensitivity, and asymmetry of the yield surface (Barlat, Plunckett, Cazacu, Khan). The advantages and differences of both formulations are critically presented. Possible convexity loss of the classical Hill'48 yield surface in the case of strong orthotropy is examined and highlighted in contrast to unconditionally stable von Mises–Hu–Marin's criterion. Various transitions from the orthotropic yield criteria to the transversely isotropic ones are carefully distinguished in the light of irreducibility or reducibility to the isotropic Huber–von Mises criterion in the transverse isotropy plane and appropriate symmetry class of tetragonal symmetry (classical Hill's formulation) or hexagonal symmetry (hexagonal Hill's or von Mises–Hu–Marin's). The new hybrid formulation applicable for some engineering materials based on additional bulge test is also proposed.

Keywords Pressure insensitive criteria · von Mises anisotropic criterion · Convexity loss for strong orthotropy · Degeneration of Hill's surface to isotropic von Mises · Unconditionally stable Hu–Marin's criterion · Implicit versus explicit formulation

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5.1 Isotropic Initial Yield Criteria of Pressure Insensitive Materials

In case of isotropic materials limit criteria for elastic range are independent of reference frame. For this reason the isotropic initial yield criteria can be written down in the reduced frame of simple stress tensor invariants $f(J_{i\sigma}, \Pi_i) = 0$ instead of the stress tensor components frame $f(\sigma_{ij}, \Pi_i) = 0$. By contrast, in case of anisotropic materials the stress components frame has to be applied and the common both stress and structural tensor invariants should be used (see Table 5.2). Such a simplification means reduction of the six-dimensional stress space to the three-dimensional space spanned by arbitrary set of three stress invariants ($J_{i\sigma}$; $i = 1, 2, 3$). Symbol Π_i denotes scalar material constants defining termination of the elastic behavior through the yield initiation in a form of micro-slips in ductile material $\Pi_i = k_i^p$ (yield stresses) or through the local microcracks in brittle material $\Pi_i = k_i^d$ (failure limits).

The number of independent material constants Π_i depends on the number of parameters in the equation of limit surface (*yield or failure initiation*) which have to be identified from independent *strength tests*: e.g., the *uniaxial tension* (k_t), the *uniaxial compression* (k_c), and the *pure shear* (k_s). In the simplest case, when conditions of initial yielding or failure are identical for tension and compression and simultaneously the shear is not independent constant the number of material constants reduces to one parameter $k_t = k_c = k$ which corresponds to yield or failure initiation, whereas $k_s = \frac{k}{\sqrt{3}}$. Such a limitation is true for majority of *ductile materials* (metals and metallic alloys). However, in case of *brittle materials* that exhibit different limit stress points for tension and compression (both yield and failure), the limit surface is to be characterized by at least two independent constants $k_t \neq k_c$ and such property is called *strength differential effect*.

Assuming narrower case of the experimentally confirmed for majority of metals independence of yield initiation from hydrostatic pressure $J_{1\sigma}$, we arrive at the limit surface equation being function of the second and the third stress deviator invariants

$$f(J_{2s}, J_{3s}; k_i) = 0 \quad (5.1)$$

Such a narrower class of materials is called *hydrostatic pressure insensitive isotropic materials*.

The above condition depends on both the second and the third stress deviator invariants J_{2s} , J_{3s} but it is independent of the first stress invariant $J_{1\sigma}$. It simply means that the cylindrical limit surface possesses the axis equally inclined to the principal stress axes ($\sigma_1, \sigma_2, \sigma_3$) called the *hydrostatic axis* (Fig. 5.1).

For purpose of further geometric illustration of considered surfaces it is convenient to apply the *Haigh–Westergaard coordinates* [21, 63] ξ , ρ , and θ which represent, respectively: distance along the hydrostatic axis measured from the origin to the current stress point (effect of $J_{1\sigma}$), distance in the *deviatoric plane* measured from the hydrostatic axis and the stress point considered (effect of J_{2s}), and the polar coordinate of the stress point in the deviatoric plane (effect of J_{3s}) (Fig. 5.1). Hence the following definitions of the Haigh–Westergaard coordinates hold

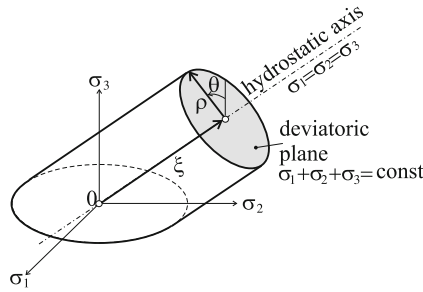


Fig. 5.1 Cylindrical yield surface in the Haigh–Westergaard coordinates

$$\xi = \frac{J_{1\sigma}}{\sqrt{3}}, \quad \rho = \sqrt{2J_{2s}}, \quad \cos(3\theta) = \frac{3\sqrt{3}}{2} \frac{J_{3s}}{(J_{2s})^{3/2}} \quad \text{for } 0 \leq \theta \leq \frac{\pi}{3} \quad (5.2)$$

Roughly speaking, dependence on the first coordinate ξ stands for noncylindricity, the second one ρ comprises size and the third one θ describes asymmetry of the yield surface.

For further consideration it is also convenient to use a concept of the generating curve of limit surface conventionally called the meridian. *Meridians of the limit surface* either yield or failure are curves being intersections of the surface by planes of $\theta = \text{const}$ containing the hydrostatic axis. In case of *rotationally symmetric limit surfaces* all meridians are identical. In a particular case of *cylindrical surface* all cross sections by planes $\xi = \text{const}$ (deviatoric planes) are identical and hence meridians are straight lines.

In more general case of cylindrical but *nonrotationally symmetric surface*, which depends on either the third invariant J_{3s} or alternatively the third coordinate θ , three of all meridians are of the particular importance (Fig. 5.2)

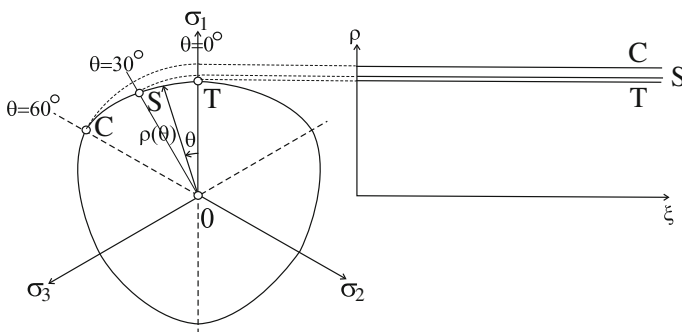


Fig. 5.2 Cross section of the cylindrical limit surface in deviatoric plane $\xi = \text{const}$; points T, S, and C correspond to the tensile k_t , the shear k_s , and the compressive k_c yield points, $k_t \neq k_c$

1. *Tensile meridian* T ($\theta = 0^\circ$)
2. *Shear meridian* S ($\theta = 30^\circ$)
3. *Compressive meridian* C ($\theta = 60^\circ$)

Hence the equation of nonrotationally symmetric cylindrical surface (5.1) can be written as

$$f(\rho, \theta; k_i) = 0 \tag{5.3}$$

where the independence of the position at the hydrostatic axis ξ is obvious.

Note that in a general case Eq. (5.3) represents cylindrical surface, the cross section of which is not necessarily circular $\rho(\theta)$. This property is called the *strength differential effect* or the *tension and compression asymmetry* $k_t \neq k_c$. Summarizing, for isotropic materials considered the 60° *symmetry property* must be fulfilled which means that the curve in the *deviatoric plane* is completely described by the form for the sector $0 \leq \theta \leq \frac{\pi}{3}$ and this form is repeated in the remaining sectors (Fig. 5.2), for details see Chen and Han [9], Ottosen and Ristinmaa [47]. In case of majority of metals yield point stresses for compression and tension do not differ $k_t = k_c = k$ which means that no strength differential effect exists. In other words in the Haigh–Westergaard space arbitrary cross section of a cylindrical yield surface done by any deviatoric plane has to pass through six skeletal points: T_i ($\theta = 0^\circ, 120^\circ, 240^\circ$) and C_i ($\theta = 60^\circ, 180^\circ, 300^\circ$) at constant distance from the origin equal to $\sqrt{\frac{2}{3}}k$. Simultaneously, each of sectorial curve has to pass through three points corresponding to pure shear S_i ($\theta = 30^\circ, 150^\circ, 270^\circ$), see Ottosen and Ristinmaa [47], (Fig. 5.3).

In the simple case of majority of metals and steels the additional assumption of independence of the cross section from the angle θ or alternatively from the

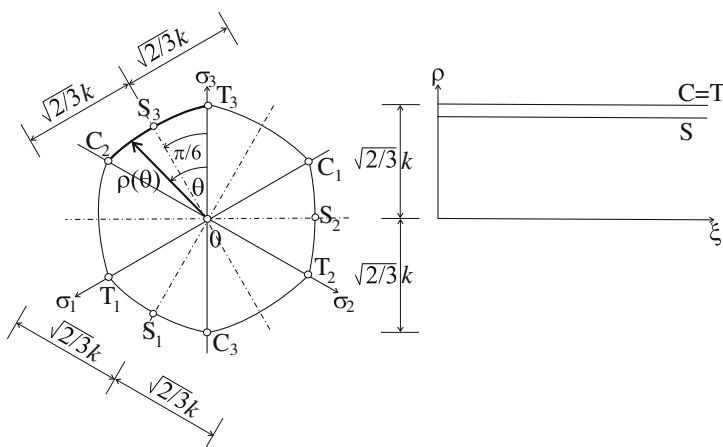


Fig. 5.3 The 60° symmetry property of the yield surface (5.3) in the deviatoric plane

third invariant J_{3s} can be done. In such a case *limit surface* is the *cylindrical* and *rotationally symmetric* simultaneously as follows:

$$f(\rho; k) = \sqrt{3J_{2s}} - k = \sqrt{\frac{3}{2}s_{ij}s_{ij}} - k = \sqrt{\frac{3}{2}}\rho - k = 0 \quad (5.4)$$

In such a way we arrive at the limit case of the unit shear strain energy-based classical *isotropic von Mises criterion* occasionally called the *Huber–von Mises criterion* anticipated by Huber [31], extended by von Mises [43] and interpreted physically by Hencky [22], cf. Ottosen and Ristinmaa [47]. When the engineering notation is used the isotropic von Mises criterion takes the explicitly deviatoric form

$$(\sigma_y - \sigma_z)^2 + (\sigma_z - \sigma_x)^2 + (\sigma_x - \sigma_y)^2 + 6(\tau_{yz}^2 + \tau_{zx}^2 + \tau_{xy}^2) = 2k^2 \quad (5.5)$$

or

$$\sigma_1^2 - \sigma_1\sigma_2 + \sigma_2^2 - \sigma_2\sigma_3 + \sigma_3^2 - \sigma_1\sigma_3 = k^2 \quad (5.6)$$

if principal stresses are used.

In a more general case when the yield criterion depends on both the second and the third Haigh–Westergaard coordinates $f(\rho, \theta)$ or alternatively on both the second and the third deviatoric stress invariants $f(J_{2s}, J_{3s})$ we met the historically earlier cylindrical criterion proposed by Tresca [60]

$$f(\rho, \theta; k) = \sqrt{2}\rho \sin\left(\theta - \frac{\pi}{3}\right) - k = 0 \quad 0 \leq \theta \leq \frac{\pi}{3} \quad (5.7)$$

When the principal stresses are used the classical form of the *Tresca criterion*

$$f(\sigma_1, \sigma_2, \sigma_3; k) = \max(|\sigma_1 - \sigma_2|, |\sigma_2 - \sigma_3|, |\sigma_3 - \sigma_1|) - k = 0 \quad (5.8)$$

clearly corresponds to the hypothesis of maximum shear stress. The Tresca criterion can also be presented in terms of the second and the third stress deviator invariants (1.15), cf. Reuss [49]

$$f(J_{2s}, J_{3s}) = 4J_{2s}^3 - 27J_{3s}^2 - 9k^2J_{2s}^2 + 6k^4J_{2s} - k^6 = 0 \quad (5.9)$$

The Tresca *initial yield surface* is *cylindrical* but *not rotationally symmetric* built on the regular hexagon and the hydrostatic axis Fig. 5.4. It is clear that the Tresca yield surface represents a regular prism inscribed into the Huber–von Mises circular cylinder and possessing six joint meridians seen here as six skeletal points T_1, T_2, T_3 and C_1, C_2, C_3 (Fig. 5.4). The Tresca initial yield surface exhibits the 60° *symmetry property*.

The Tresca limit surface suffers from the existence of edges (tension T_1, T_2, T_3 and compression C_1, C_2, C_3 meridians) in which the normality rule does not hold Fig. 5.5. In order to avoid this deficiency the Hosford and Backhofen [27] and Hosford [28, 29] limit surface can be introduced

Fig. 5.4 Cylindrical initial yield criteria in the deviatoric plane

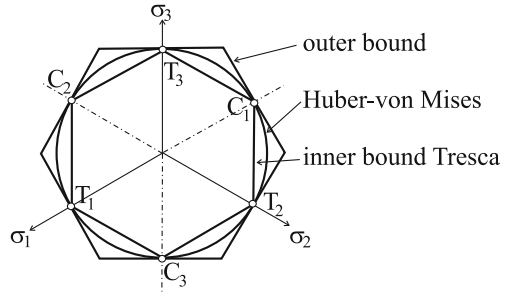
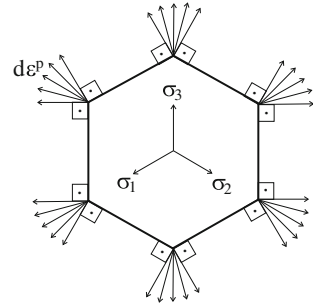


Fig. 5.5 Nonuniqueness of plastic strain increment direction in case of skeletal points for Tresca yield surface



$$|\sigma_1 - \sigma_2|^m + |\sigma_2 - \sigma_3|^m + |\sigma_3 - \sigma_1|^m = 2k^m \tag{5.10}$$

The discussed criterion is commonly called the *Hosford criterion* (1964) although it was earlier suggested by *Hershey* [24] and *Davies* [11]. The exponent m used in the Hosford criterion is an additional material constant that should be chosen according to experimental evidence. The range of this constant exhibits certain limitations and particular cases. It can theoretically change in range $1 \leq m < \infty$, cf. Cazacu and Barlat [7]. In the cases $1 < m < 2$ or $4 < m < \infty$ the initial yield curves are located between the Tresca and the Huber–von Mises loci, whereas for $m = 1$ and $m \rightarrow \infty$ or for $m = 2$ and $m = 4$ the Tresca or the Huber–von Mises yield loci are recovered, respectively. If $2 < m < 4$ the yield curve slightly exceeds the Huber–von Mises loci as shown in Fig. 5.6. If $0 < m < 1$ is chosen a *concave yield curve* is met, which is inadmissible from the *Drucker stability postulate* point of view. According to Hershey, magnitudes $m = 6$ and $m = 8$ well fit experimental findings.

Concluding, the *Tresca* initial yield criterion is the *inner bound* for all *limit curves* of the isotropic materials without the strength differential effect. Note however that there exists wide class of materials which exhibits the strength differential effect hence the Tresca does not have to be treated as the inner bound nevertheless the convexity condition resulting from the Drucker postulate is not violated, see Cazacu and Barlat [7].

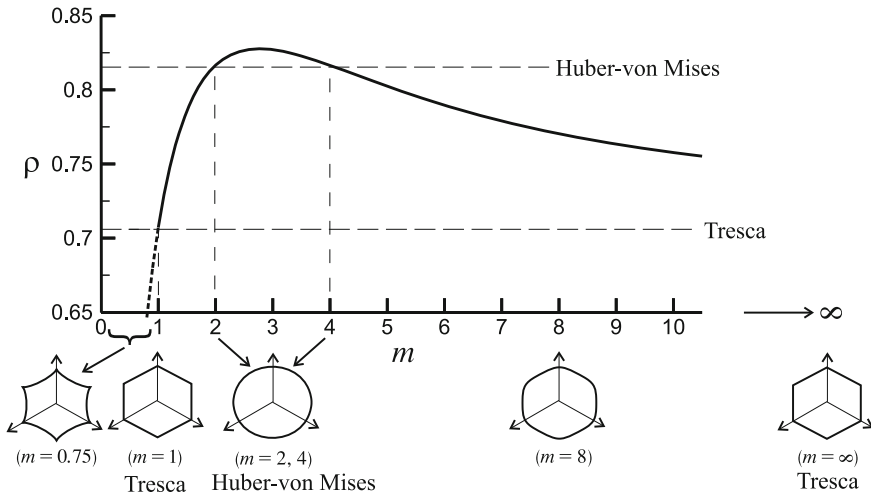


Fig. 5.6 The second Haigh–Westergaard coordinate ρ versus exponent m in Hosford’s criterion (5.10)

It is also possible to derive the *outer bound* for all *limit curves* of the isotropic materials without the strength differential effect which does not violate convexity according to Drucker’s postulate. To this end the *criterion of maximal deviatoric stress* proposed by Schmidt [53], Ishlinsky [32], and Hill [26] can be used

$$f(\sigma_1, \sigma_2, \sigma_3; k) = \max [|\sigma_1 - \sigma_h|, |\sigma_2 - \sigma_h|, |\sigma_3 - \sigma_h|] - \frac{2}{3}k = 0 \quad (5.11)$$

The above equation when rigorously expressed in the Haigh–Westergaard space takes the alternative form

$$f(\rho, \theta; k) = \max \left[\left| \sqrt{\frac{2}{3}}\rho \cos \theta \right|, \left| \sqrt{\frac{2}{3}}\rho \cos \left(\theta + \frac{2\pi}{3} \right) \right|, \left| \sqrt{\frac{2}{3}}\rho \cos \left(\theta - \frac{2\pi}{3} \right) \right| \right] - \frac{2}{3}k = 0 \quad (5.12)$$

In this space the outer bound represents a regular prism circumscribed onto the Huber–von Mises circular cylinder and possessing six joint meridians seen here as six skeletal points T_1, T_2, T_3 and C_1, C_2, C_3 (Fig. 5.4). By contrast to Tresca’s inner bound now six meridians do not coincide with the outer bound prism edges but lie in the middle of walls (Fig. 5.4).

Summarizing the above considerations, the *postulate of inner and outer bounds* of limit surfaces of initial yield in isotropic and tension/compression materials (no strength differential effect included) by Tresca (inner bound) and the criterion of maximal deviatoric stress (outer bound) define the admissible range for all cylindrical

Fig. 5.7 Experimental findings for thin-walled tubes made of steel (○), copper (●), and nickel (□), after Lode [40]

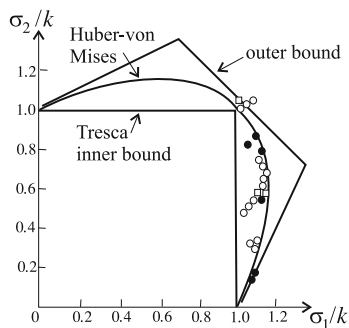
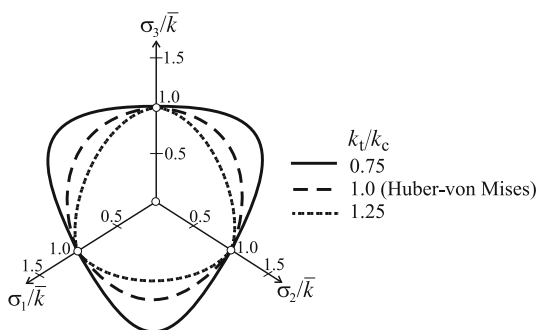


Fig. 5.8 Yield surfaces exhibiting strength differential effect $k_t/k_c = 0.75, 1.0$ (Huber–von Mises), 1.25; described by (5.13), after Cazacu and Barlat [7]



limit surfaces for the class of metals and steels. It directly results from both the 60° symmetry property in the Haigh–Westergaard space as well as the *Drucker convexity assumption*. Hence, all initial yield surfaces of real tension/compression asymmetry insensitive materials have to include tensile T_1, T_2, T_3 and compressive C_1, C_2, C_3 meridians being straight lines equidistant from the hydrostatic axis $\sqrt{\frac{2}{3}}k$. For instance, the Lode [40] experimental findings for thin-walled tubes made of steel, copper, and nickel confirm suitability of the Huber–von Mises and the Tresca criteria for prediction of yield initiation in case of ductile materials under the plane stress state ($\sigma_3 = 0$), see Fig. 5.7. Limit surface dependent on the second and the third stress invariants with the *strength differential effect* accounted for, was used by Raniecki and Mróz [48] when applied to initial yield or phase change surfaces in *NiTi shape memory alloys*

$$f(J_{2s}, J_{3s}) = (J_{2s})^{3n/2} - c(J_{3s})^n - k^{3n} = 0 \tag{5.13}$$

Raniecki and Mróz's criterion (5.13) includes three material constants c, k, n and it is an extension of the *Cazacu and Barlat* [7] *criterion* for $n = 1$ for describing *asymmetry in yielding initiation in pressure insensitive isotropic materials*, Fig. 5.8

$$f(J_{2s}, J_{3s}) = (J_{2s})^{3/2} - cJ_{3s} - k^3 = 0 \tag{5.14}$$

On the other hand, substituting $n = 2$ to Eq. (5.13) we arrive at *Drucker's criterion* [12]

$$f(J_{2s}, J_{3s}) = (J_{2s})^3 - c(J_{3s})^2 - k^6 = 0 \quad (5.15)$$

5.2 Von Mises Anisotropic Criterion

In a general case of material anisotropy, extension of the isotropic yield initiation criteria (Table 5.3) to the *anisotropic yield/failure behavior* (Table 5.4), by the use of *common invariants* of the stress tensor and of the structural tensors of plastic anisotropy (cf. Hill [25], Sayir [52], Betten [5], Źyczkowski [65]), can be shown in a general fashion

$$f(\Pi, \Pi_{ij}\sigma_{ij}, \Pi_{ijkl}\sigma_{ij}\sigma_{kl}, \Pi_{ijklmn}\sigma_{ij}\sigma_{kl}\sigma_{mn}, \dots) = 0 \quad (5.16)$$

where Einstein's summation convention holds.

In such a case, initiation of plastic flow or failure is governed by the *structural tensors of material anisotropy* of even-ranks: $\overset{<0>}{\mathbb{I}} = \Pi$, $\overset{<2>}{\mathbb{I}} = \Pi_{ij}$, $\overset{<4>}{\mathbb{I}} = \Pi_{ijkl}$, $\overset{<6>}{\mathbb{I}} = \Pi_{ijklmn}$, ..., etc., instead of the scalar constants k_i as it is known for isotropic materials. Equation (5.16) owns a general representation, but its practical identification is limited by a large number of required material tests and, additionally, because the components of the structural tensors are temperature dependent, which makes identification much more complicated (cf., e.g., Herakovich and Aboudi [23], Tamma and Avila [59]). Hence, a general form (5.16) is usually more specified and limited for engineering needs.

In a particular case when a general tensorially polynomial form of Eq. (5.16) is assumed (cf. Sayir [52], Kowalsky et al. [37], Źyczkowski [65], Ganczarski and Skrzypek [18]) the *polynomial anisotropic yield criterion* is furnished

$$(\Pi_{ij}\sigma_{ij})^\alpha + (\Pi_{ijkl}\sigma_{ij}\sigma_{kl})^\beta + (\Pi_{ijklmn}\sigma_{ij}\sigma_{kl}\sigma_{mn})^\gamma + \dots - 1 = 0 \quad (5.17)$$

where, if the Voigt notation is used the *structural anisotropy tensors* take corresponding *matrix* forms

$$[\overset{<2>}{\mathbb{I}}] = \begin{bmatrix} \pi_{11} & \pi_{12} & \pi_{13} \\ & \pi_{22} & \pi_{23} \\ & & \pi_{33} \end{bmatrix} \quad (5.18)$$

and

$$[\text{III}] = \begin{array}{c} \langle 4 \rangle \\ \left[\begin{array}{ccc|ccc} \Pi_{11} & \Pi_{12} & \Pi_{13} & \Pi_{14} & \Pi_{15} & \Pi_{16} \\ & \Pi_{22} & \Pi_{23} & \Pi_{24} & \Pi_{25} & \Pi_{26} \\ & & \Pi_{33} & \Pi_{34} & \Pi_{35} & \Pi_{36} \\ \hline & & & \Pi_{44} & \Pi_{45} & \Pi_{46} \\ & & & & \Pi_{55} & \Pi_{56} \\ & & & & & \Pi_{66} \end{array} \right] \end{array} \quad (5.19)$$

The even-rank structural anisotropy tensors $\Pi_{ij}, \Pi_{ijkl}, \Pi_{ijklmn}, \dots$, in Eq. (5.17) are normalized by the common constant Π and $\alpha, \beta, \gamma \dots$, etc., are arbitrary exponents of a polynomial representation. In a narrower case if $\alpha = 1, \beta = 1/2, \gamma = 1/3$, and limiting an infinite form (5.17) to the equation that contains only three common invariants, we arrive at the narrower form known as the *Goldenblat and Kopnov criterion* [19]

$$\Pi_{ij}\sigma_{ij} + (\Pi_{ijkl}\sigma_{ij}\sigma_{kl})^{1/2} + (\Pi_{ijklmn}\sigma_{ij}\sigma_{kl}\sigma_{mn})^{1/3} - 1 = 0 \quad (5.20)$$

which satisfies the dimensional homogeneity of three polynomial components.

Equation (5.20), when limited only to three common invariants of the stress tensor σ and structural anisotropy tensors of even orders: 2nd Π_{ij} , 4th Π_{ijkl} , and 6th Π_{ijklmn} is not the most general one, in the meaning of the representation theorems, which determine the most general irreducible representation of the scalar and tensor functions that satisfy the invariance with respect to change of coordinates and material symmetry properties (cf., e.g., Spencer [56], Rymarz [51], Rogers [50]). However, 2nd, 4th, and 6th order structural anisotropy tensors, which are used in (5.20) or in case if $\alpha = 1, \beta = 1, \gamma = 1$ and the deviatoric stress representation used by Kowalsky et al. [37]

$$h_{ij}^{(1)}s_{ij} + h_{ijkl}^{(2)}s_{ij}s_{kl} + h_{ijklmn}^{(3)}s_{ij}s_{kl}s_{mn} - h^{(0)} = 0 \quad (5.21)$$

are found satisfactory for describing fundamental *transformation modes* of limit surfaces caused by plastic or failure processes, namely: isotropic *change of size*, kinematic *translation* and *rotation*, as well as surface *distortion* (cf. Betten [5], Kowalsky et al. [37]).

In what follows, we shall reduce class of the limit surface from the general tensorially polynomial representation to the forms independent of both the first $\Pi_{ij}\sigma_{ij}$ and the third $\Pi_{ijklmn}\sigma_{ij}\sigma_{kl}\sigma_{mn}$ common invariants, but preserving the most general representation for the second common invariant, according to von Mises [43, 44]. In such a case the 4th *rank tensor of material anisotropy* Π_{ijkl} is, in general, defined by 21 anisotropy modules (but 18 of them independent), since the anisotropy 6×6 matrix $[\text{III}]_{ij}$ (5.19) can completely be populated. Further reduction of the number of modules to 15 will be achieved, when the insensitivity of general von Mises quadratic form with respect to the change of hydrostatic stress will be assumed. In such a way the *general tensorial von Mises criterion* will be reduced to the deviatoric von Mises form defined by 15 anisotropy modules. A choice of 15 anisotropy

modules considered as independent is, in general, not unique (cf. Szczepiński [58], Ganczarski and Skrzypek [17]). However, the 15-parameter *deviatoric von Mises criterion* is sensitive to the change of sign of shear stresses, which may be considered as questionable (cf., e.g., Malinin and Rżysko [42]). Simplest way to avoid a doubtful physical explanation for existence of terms linear for shear stresses τ_{ij} , a reduction of the 15-parameter von Mises equation to the 9-parameter orthotropic von Mises criterion can be done. This form does not satisfy the deviatoric property, but when the constraints of independence of the hydrostatic stress is consistently applied, it is easily reduced to the deviatoric form, known as orthotropic Hill’s criterion, with only 6 independent moduli of orthotropy (cf. Hill [25]).

Limiting ourselves to plastic yield initiation in ductile materials, a consecutive reduction of the general tensorially polynomial anisotropic criterion (5.20) to the form dependent only on the 4th rank common invariant $\sigma_{ij}\Pi_{ijkl}\sigma_{kl}$ holds, as it was proposed in the *von Mises criterion for anisotropic yield initiation* (item D8 in Table 6.3) (cf. von Mises [43, 44]).

$$\sigma_{ij}\Pi_{ijkl}\sigma_{kl} - 1 = 0 \tag{5.22}$$

When the more convenient Voigt’s vector–matrix notation is used, the form equivalent to (5.22) is obtained

$$\{\sigma\}^T [\overset{<4>}{\mathbb{I}}] \{\sigma\} - 1 = 0 \tag{5.23}$$

where only one fourth-rank tensor of plastic anisotropy \mathbb{I} is saved.

Anisotropic von Mises criterion (5.22) or (5.23), being an initial yield criterion of anisotropic material is an extension of the *isotropic Huber–von Mises criterion* (5.4). This is more clear when the Huber–von Mises condition is rewritten in a following fashion

$$\sigma_{ij}\Pi_{ijkl}^{\text{HMH}}\sigma_{kl} - 1 = 0 \tag{5.24}$$

where Π_{ijkl}^{HMH} stands for the isotropic fourth-rank structural tensor whose representation matrix is

$$[\mathbb{I}^{\text{HMH}}] = \frac{1}{k^2} \left[\begin{array}{ccc|ccc} 1 & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 \\ & 1 & -\frac{1}{2} & 0 & 0 & 0 \\ & & 1 & 0 & 0 & 0 \\ \hline & & & 3 & 0 & 0 \\ & & & & 3 & 0 \\ & & & & & 3 \end{array} \right] \tag{5.25}$$

Note however that condition (5.24) comprises stress tensor components σ_{ij} but not stress deviator components s_{ij} as commonly used. However, Eq. (5.24) takes analogous form when stress deviator components s_{ij} are used, namely

$$s_{ij}\Pi_{ijkl}^{\text{HMH}}s_{kl} - 1 = 0 \tag{5.26}$$

since when decomposition of the stress tensor into the deviatoric and the hydrostatic part is done $\sigma_{ij} = s_{ij} + \frac{1}{3}\sigma_{kk}\delta_{ij}$ we arrive at

$$s_{ij}\Pi_{ijkl}^{\text{HMH}}s_{kl} + \left(2s_{ij} + \frac{1}{3}\sigma_{mm}\delta_{ij}\right) \left(\underline{\Pi_{ijkl}^{\text{HMH}}\delta_{kl}}\right) \frac{1}{3}\sigma_{nn} - 1 = 0 \quad (5.27)$$

However, the underlined term in (5.27) is identically equal to zero since the following holds

$$\begin{aligned} \Pi_{11}^{\text{HMH}} + \Pi_{12}^{\text{HMH}} + \Pi_{13}^{\text{HMH}} &= 1 - \frac{1}{2} - \frac{1}{2} = 0 \\ \Pi_{21}^{\text{HMH}} + \Pi_{22}^{\text{HMH}} + \Pi_{23}^{\text{HMH}} &= -\frac{1}{2} + 1 - \frac{1}{2} = 0 \\ \Pi_{31}^{\text{HMH}} + \Pi_{32}^{\text{HMH}} + \Pi_{33}^{\text{HMH}} &= -\frac{1}{2} - \frac{1}{2} + 1 = 0 \end{aligned} \quad (5.28)$$

when the Voigt notation for the Huber–von Mises matrix is used.

The structural 4th rank tensor of plastic anisotropy in Eq. (5.22) must be symmetric: $\Pi_{ijkl} = \Pi_{klij} = \Pi_{jikl} = \Pi_{ijlk}$, if stress tensor symmetry is assumed. Hence, in case if none other symmetry properties are implied, the von Mises plastic anisotropy tensor is defined by 21 modules. However, due to its invariance of the tensorial transformation rule, number of independent anisotropy modules is reduced to 18. Finally, the general *anisotropic von Mises criterion* can be furnished as

$$\begin{aligned} &\Pi_{xxxx}\sigma_x^2 + \Pi_{yyyy}\sigma_y^2 + \Pi_{zzzz}\sigma_z^2 + \\ &2\Pi_{xxyy}\sigma_x\sigma_y + 2\Pi_{yyzz}\sigma_y\sigma_z + 2\Pi_{zzxx}\sigma_z\sigma_x + \\ &4\Pi_{xxyz}\sigma_x\tau_{yz} + 4\Pi_{xxzx}\sigma_x\tau_{zx} + 4\Pi_{xxyx}\sigma_x\tau_{xy} + \\ &4\Pi_{yyyz}\sigma_y\tau_{yz} + 4\Pi_{yyzx}\sigma_y\tau_{zx} + 4\Pi_{yyxy}\sigma_y\tau_{xy} + \\ &4\Pi_{zzyz}\sigma_z\tau_{yz} + 4\Pi_{zzzx}\sigma_z\tau_{zx} + 4\Pi_{zzxy}\sigma_z\tau_{xy} + \\ &8\Pi_{xxyz}\tau_{xy}\tau_{yz} + 8\Pi_{yyzx}\tau_{yz}\tau_{zx} + 8\Pi_{zxyx}\tau_{zx}\tau_{xy} + \\ &4\Pi_{yzyz}\tau_{yz}^2 + 4\Pi_{zxzx}\tau_{zx}^2 + 4\Pi_{xyxy}\tau_{xy}^2 = 1 \end{aligned} \quad (5.29)$$

where Π_{ijkl} denote 21 components of the von Mises plastic anisotropy tensor.

The *von Mises* 6×6 matrix of plastic anisotropy, being symmetric and fully populated matrix representation of the 4th rank anisotropy tensor Π_{ijkl} shown in (5.22), is furnished as follows:

$$\left[\begin{array}{ccc|ccc} \Pi_{11} & \Pi_{12} & \Pi_{13} & \Pi_{14} & \Pi_{15} & \Pi_{16} \\ & \Pi_{22} & \Pi_{23} & \Pi_{24} & \Pi_{25} & \Pi_{26} \\ & & \Pi_{33} & \Pi_{34} & \Pi_{35} & \Pi_{36} \\ \hline & & & \Pi_{44} & \Pi_{45} & \Pi_{46} \\ & & & & \Pi_{55} & \Pi_{56} \\ & & & & & \Pi_{66} \end{array} \right] = \left[\begin{array}{ccc|ccc} \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ & \bullet & \bullet & \bullet & \bullet & \bullet \\ & & \bullet & \bullet & \bullet & \bullet \\ \hline & & & \bullet & \bullet & \bullet \\ & & & & \bullet & \bullet \\ & & & & & \bullet \end{array} \right] \quad (5.30)$$

if engineering vectorial representation of the stress tensor $\{\sigma\}$ is chosen as

$$\{\sigma\} = \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{Bmatrix} = \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{yz} \\ \tau_{zx} \\ \tau_{xy} \end{Bmatrix} \tag{5.31}$$

When the matrix coordinates Π_{ij} (5.30) are consistently defined by the tensorial coordinates Π_{ijkl}

$$\begin{aligned} \Pi_{11} &= \Pi_{xxxx} & \Pi_{22} &= \Pi_{yyyy} & \Pi_{33} &= \Pi_{zzzz} \\ \Pi_{12} &= \Pi_{xxyy} & \Pi_{13} &= \Pi_{xxzz} & \Pi_{23} &= \Pi_{yyzz} \\ \Pi_{14} &= 2\Pi_{xxyz} & \Pi_{15} &= 2\Pi_{xxzx} & \Pi_{16} &= 2\Pi_{xxyy} \dots \\ \Pi_{44} &= 4\Pi_{yzyz} & \Pi_{55} &= 4\Pi_{zxzx} & \Pi_{66} &= 4\Pi_{xyxy} \\ \Pi_{45} &= 4\Pi_{yzzx} & \Pi_{46} &= 4\Pi_{xyyz} & \Pi_{56} &= 4\Pi_{zxyx} \end{aligned} \tag{5.32}$$

we arrive at the general anisotropic von Mises equation equivalent to (5.29)

$$\begin{aligned} &\Pi_{11}\sigma_x^2 + \Pi_{22}\sigma_y^2 + \Pi_{33}\sigma_z^2 + \\ &2(\Pi_{12}\sigma_x\sigma_y + \Pi_{23}\sigma_y\sigma_z + \Pi_{31}\sigma_z\sigma_x + \\ &\Pi_{14}\sigma_x\tau_{yz} + \Pi_{15}\sigma_x\tau_{zx} + \Pi_{16}\sigma_x\tau_{xy} + \\ &\Pi_{24}\sigma_y\tau_{yz} + \Pi_{25}\sigma_y\tau_{zx} + \Pi_{26}\sigma_y\tau_{xy} + \\ &\Pi_{34}\sigma_z\tau_{yz} + \Pi_{35}\sigma_z\tau_{zx} + \Pi_{36}\sigma_z\tau_{xy} + \\ &\Pi_{45}\tau_{yz}\tau_{zx} + \Pi_{46}\tau_{xy}\tau_{yz} + \Pi_{56}\tau_{zx}\tau_{xy}) + \\ &\Pi_{44}\tau_{yz}^2 + \Pi_{55}\tau_{zx}^2 + \Pi_{66}\tau_{xy}^2 = 1 \end{aligned} \tag{5.33}$$

Representation of the anisotropic von Mises condition (5.23) in deviatoric form is not trivial. The von Mises equation in the vector–matrix notation depends on both the deviatoric s and the hydrostatic part $\sigma_h \mathbf{1}$, when stress decomposition $\sigma = s + \sigma_h \mathbf{1}$ is applied, namely

$$\{s\}^T [\overset{<4>}{\mathbb{I}}] \{s\} + \underbrace{\left(2\{s\}^T + \sigma_h \{\mathbf{1}\}^T \right) \left([\overset{<4>}{\mathbb{I}}] \{\mathbf{1}\} \sigma_h \right)}_{= 1} - 1 = 0 \tag{5.34}$$

The tensorial *von Mises* equation (5.34) can further be reduced to the *deviatoric form* independent of the hydrostatic pressure as follows:

$$\{s\}^T [_{\text{dev}}\mathbb{I}] \{s\} - 1 = 0 \tag{5.35}$$

only if the constraint

$$[\overset{<4>}{\mathbb{I}}] \{\mathbf{1}\} = 0 \tag{5.36}$$

is consistently applied. The constraint (5.36) guarantees the deviatoric von Mises equation (5.35) to be represented in the reduced six-dimensional stress space by a cylindrical surface defined by 15 independent anisotropy modules, when six constraints are satisfied

$$\begin{aligned}
 \Pi_{11} + \Pi_{12} + \Pi_{13} &= 0 \\
 \Pi_{12} + \Pi_{22} + \Pi_{23} &= 0 \\
 \Pi_{13} + \Pi_{23} + \Pi_{33} &= 0 \\
 \Pi_{14} + \Pi_{24} + \Pi_{34} &= 0 \\
 \Pi_{15} + \Pi_{25} + \Pi_{35} &= 0 \\
 \Pi_{16} + \Pi_{26} + \Pi_{36} &= 0
 \end{aligned}
 \tag{5.37}$$

However, the final matrix representation (5.30) with (5.37) employed depends on a choice of independent elements. Two of such representations are of special importance.

In the first case, the elements of matrix (5.30) considered as independent are: $\Pi_{12}, \Pi_{13}, \Pi_{23}; \Pi_{15}, \Pi_{16}, \Pi_{24}, \Pi_{26}, \Pi_{34}, \Pi_{35}$ and $\Pi_{44}, \Pi_{55}, \Pi_{66}; \Pi_{45}, \Pi_{46}, \Pi_{56}$, such that the following *first* representation for the *deviatoric von Mises matrix* is furnished

$$\begin{aligned}
 [\text{dev III}] = & \left[\begin{array}{ccc|ccc}
 -\Pi_{12} - \Pi_{13} & \Pi_{12} & \Pi_{13} & & & \\
 & -\Pi_{12} - \Pi_{23} & \Pi_{23} & & & \\
 & & -\Pi_{13} - \Pi_{23} & & & \\
 \hline
 -\Pi_{24} - \Pi_{34} & \Pi_{15} & \Pi_{16} & & & \\
 \Pi_{24} & -\Pi_{15} - \Pi_{35} & \Pi_{26} & & & \\
 \Pi_{34} & \Pi_{35} & -\Pi_{16} - \Pi_{26} & & & \\
 \hline
 \Pi_{44} & \Pi_{45} & \Pi_{46} & & & \\
 & \Pi_{55} & \Pi_{56} & & & \\
 & & \Pi_{66} & & &
 \end{array} \right] \left[\begin{array}{ccc|ccc}
 \circ & \bullet & \bullet & \circ & \bullet & \bullet \\
 & \circ & \bullet & \bullet & \circ & \bullet \\
 & & \circ & \bullet & \bullet & \circ \\
 \hline
 & & & \bullet & \bullet & \bullet \\
 & & & & \bullet & \bullet \\
 & & & & & \bullet
 \end{array} \right]
 \end{aligned}
 \tag{5.38}$$

if constraints (5.37) are applied as follows

$$\begin{aligned}
 \Pi_{11} &= -\Pi_{12} - \Pi_{13}, & \Pi_{14} &= -\Pi_{24} - \Pi_{34} \\
 \Pi_{22} &= -\Pi_{12} - \Pi_{23}, & \Pi_{25} &= -\Pi_{15} - \Pi_{35} \\
 \Pi_{33} &= -\Pi_{13} - \Pi_{23}, & \Pi_{36} &= -\Pi_{16} - \Pi_{26}
 \end{aligned}
 \tag{5.39}$$

In the second case, the elements of matrix (5.30) chosen as independent are: $\Pi_{11}, \Pi_{22}, \Pi_{33}; \Pi_{15}, \Pi_{16}, \Pi_{24}, \Pi_{26}, \Pi_{34}, \Pi_{35}$ and $\Pi_{44}, \Pi_{55}, \Pi_{66}; \Pi_{45}, \Pi_{46}, \Pi_{56}$, hence we arrive at the *second* representation of the *deviatoric von Mises matrix* as follows:

$$\begin{aligned}
 [\text{dev III}] = & \left[\begin{array}{ccc|ccc}
 \Pi_{11} & \frac{1}{2}(\Pi_{33} - \Pi_{11} - \Pi_{22}) & \frac{1}{2}(\Pi_{22} - \Pi_{11} - \Pi_{33}) & & & \\
 & \Pi_{22} & \frac{1}{2}(\Pi_{11} - \Pi_{22} - \Pi_{33}) & & & \\
 & & \Pi_{33} & & & \\
 \hline
 -\Pi_{24} - \Pi_{34} & \Pi_{15} & \Pi_{16} & \bullet & \circ & \circ & \bullet & \bullet \\
 \Pi_{24} & -\Pi_{15} - \Pi_{35} & \Pi_{26} & \bullet & \circ & \circ & \bullet & \\
 \Pi_{34} & \Pi_{35} & -\Pi_{16} - \Pi_{26} & \bullet & \bullet & \bullet & \circ & \\
 \hline
 \Pi_{44} & \Pi_{45} & \Pi_{46} & & \bullet & \bullet & \bullet & \\
 & \Pi_{55} & \Pi_{56} & & & \bullet & \bullet & \\
 & & \Pi_{66} & & & & \bullet &
 \end{array} \right] \quad (5.40)
 \end{aligned}$$

if, instead of (5.39), other substitution is used

$$\begin{aligned}
 \Pi_{12} &= \frac{1}{2}(\Pi_{33} - \Pi_{11} - \Pi_{22}) \\
 \Pi_{13} &= \frac{1}{2}(\Pi_{22} - \Pi_{11} - \Pi_{33}) \\
 \Pi_{23} &= \frac{1}{2}(\Pi_{11} - \Pi_{22} - \Pi_{33}) \\
 \Pi_{14} &= -\Pi_{24} - \Pi_{34} \\
 \Pi_{25} &= -\Pi_{15} - \Pi_{35} \\
 \Pi_{36} &= -\Pi_{16} - \Pi_{26}
 \end{aligned} \quad (5.41)$$

A choice of 15 elements in the von Mises matrix (5.30) considered as independent is not a unique procedure and can result in the different deviatoric von Mises equation forms. In particular, when a more convenient representation (5.38) is substituted for $[\text{dev III}]$ in (5.35) we arrive at the following von Mises equation expressed in the *deviatoric stress space*

$$\begin{aligned}
 & -\Pi_{12} (s_x - s_y)^2 - \Pi_{13} (s_x - s_z)^2 - \Pi_{23} (s_y - s_z)^2 + \\
 & \frac{2 \{ \tau_{yz} [\Pi_{24} (s_y - s_x) + \Pi_{34} (s_z - s_x)] + \tau_{zx} [\Pi_{15} (s_x - s_y) + \Pi_{35} (s_z - s_y)] + \tau_{xy} [\Pi_{16} (s_x - s_z) + \Pi_{26} (s_y - s_z)] + \Pi_{45} \tau_{yz} \tau_{zx} + \Pi_{46} \tau_{xy} \tau_{yz} + \Pi_{56} \tau_{zx} \tau_{xy} \}}{\Pi_{44} \tau_{yz}^2 + \Pi_{55} \tau_{zx}^2 + \Pi_{66} \tau_{xy}^2} = 1
 \end{aligned} \quad (5.42)$$

It is visible that above equation owns the clear deviatoric structure hence, when the tensorial stress space is used instead of the deviatoric one, the analogous equivalent to (5.42) representation of the *deviatoric von Mises equation* is also true in terms of stress components (cf. Szczepiński [58])

$$\begin{aligned}
 & -\Pi_{12} (\sigma_x - \sigma_y)^2 - \Pi_{13} (\sigma_x - \sigma_z)^2 - \Pi_{23} (\sigma_y - \sigma_z)^2 + \\
 & \frac{2 \{ \tau_{yz} [\Pi_{24} (\sigma_y - \sigma_x) + \Pi_{34} (\sigma_z - \sigma_x)] + \tau_{zx} [\Pi_{15} (\sigma_x - \sigma_y) + \Pi_{35} (\sigma_z - \sigma_y)] + \tau_{xy} [\Pi_{16} (\sigma_x - \sigma_z) + \Pi_{26} (\sigma_y - \sigma_z)] \}}{\Pi_{44} \tau_{yz}^2 + \Pi_{55} \tau_{zx}^2 + \Pi_{66} \tau_{xy}^2} + \\
 & \frac{\Pi_{45} \tau_{yz} \tau_{zx} + \Pi_{46} \tau_{xy} \tau_{yz} + \Pi_{56} \tau_{zx} \tau_{xy}}{\Pi_{44} \tau_{yz}^2 + \Pi_{55} \tau_{zx}^2 + \Pi_{66} \tau_{xy}^2} = 1
 \end{aligned} \tag{5.43}$$

Note, that Eqs. (5.42) or (5.43) are defined by 15 elements Π_{ij} . However, the underlined terms are sensitive to change of sign of shear stresses, e.g., $\tau_{yz}(\sigma_y - \sigma_x)$ etc., which is physically questionable and, finally, such terms are consequently omitted in some cases (cf., e.g., Malinin and Rżysko [42]). Nevertheless, the full representation (5.43) might occur useful when the von Mises–Tsai–Wu extension to the brittle-like material is sought for (cf. Tsai and Wu [61]).

5.3 Orthotropic Initial Yield Criteria—The von Mises Orthotropic Criterion, the Hill Deviatoric Criterion

General form of the 21-parameter anisotropic von Mises criterion (5.33) involves none material symmetry property. In a particular case if plastic orthotropy is assumed for the initial yield criterion (5.23), when represented in principal orthotropy axes, the 9-parameter *orthotropic von Mises matrix* (5.30) takes the form

$$\left[{}_{\text{ort}}\mathbb{III} \right] = \left[\begin{array}{ccc|ccc} \Pi_{11} & \Pi_{12} & \Pi_{13} & 0 & 0 & 0 \\ & \Pi_{22} & \Pi_{23} & 0 & 0 & 0 \\ & & \Pi_{33} & 0 & 0 & 0 \\ \hline & & & \Pi_{44} & 0 & 0 \\ & & & & \Pi_{55} & 0 \\ & & & & & \Pi_{66} \end{array} \right] \left[\begin{array}{ccc|ccc} \bullet & \bullet & \bullet & & & \\ & \bullet & \bullet & & & \\ & & \bullet & & & \\ \hline & & & \bullet & & \\ & & & & \bullet & \\ & & & & & \bullet \end{array} \right] \tag{5.44}$$

In such a case the general anisotropic von Mises equation (5.33) is reduced to the narrower 9-parameter *orthotropic von Mises criterion*

$$\begin{aligned}
 & \Pi_{11} \sigma_x^2 + \Pi_{22} \sigma_y^2 + \Pi_{33} \sigma_z^2 + \\
 & 2(\Pi_{12} \sigma_x \sigma_y + \Pi_{23} \sigma_y \sigma_z + \Pi_{31} \sigma_z \sigma_x) + \\
 & \Pi_{44} \tau_{yz}^2 + \Pi_{55} \tau_{zx}^2 + \Pi_{66} \tau_{xy}^2 = 1
 \end{aligned} \tag{5.45}$$

When the *Voigt notation* is used, the 9-parameter orthotropic von Mises criterion takes the form

$$\{\sigma\}^T \left[{}_{\text{ort}}\mathbb{III} \right] \{\sigma\} - 1 = 0 \tag{5.46}$$

that involves definition (5.44). Note that equation (5.46) belongs to the class of *hydrostatic pressure sensitive criteria* (cf. item D8 in Table 6.3 Khan et al. [35, 36]).

In order to achieve pressure insensitive orthotropic criterion we apply a procedure described in Sect. 5.2. If we decompose again the stress tensor into deviatoric and volumetric parts $\sigma = s + \sigma_h \mathbf{1}$ in the orthotropic von Mises equation (5.46) we arrive at the equation analogous to (5.34)

$$\{s\}^T [\text{ortIII}] \{s\} + \frac{\left(2 \{s\}^T + \sigma_h \{\mathbf{1}\}^T\right) ([\text{ortIII}] \{\mathbf{1}\} \sigma_h) - 1}{\sigma_h} = 0 \tag{5.47}$$

Assuming further hydrostatic pressure insensitive form the following holds

$$[\text{ortIII}] \{\mathbf{1}\} = 0 \tag{5.48}$$

which leads to three constraints instead of six in general case of von Mises anisotropic Eq. (5.37)

$$\begin{aligned} \Pi_{11} + \Pi_{12} + \Pi_{13} &= 0 \\ \Pi_{12} + \Pi_{22} + \Pi_{23} &= 0 \\ \Pi_{13} + \Pi_{23} + \Pi_{33} &= 0 \end{aligned} \tag{5.49}$$

In this way the *orthotropic von Mises criterion* (5.46) reduces to the *pressure insensitive criterion* called *Hill's criterion* [25, 26] that contains six independent modules

$$\{s\}^T [\text{III}^H] \{s\} - 1 = 0 \tag{5.50}$$

Hill's matrix $[\text{III}^H]$ appearing in Eq. (5.50) contains six independent modules. A choice of the three independent modules form six involved in Eq. (5.49) is not unique. In what follows two of them are discussed (see two aforementioned forms (5.38) and (5.40)).

In this way we arrive at the following *Hill's matrices*

$$[\text{III}^H] = \left[\begin{array}{ccc|ccc} -\Pi_{12} - \Pi_{13} & \Pi_{12} & \Pi_{13} & & & \\ & -\Pi_{12} - \Pi_{23} & \Pi_{23} & & & \\ & & -\Pi_{13} - \Pi_{23} & & & \\ \hline & & & \Pi_{44} & & \\ & & & & \Pi_{55} & \\ & & & & & \Pi_{66} \end{array} \right] \tag{5.51}$$

$$\left[\begin{array}{c|c} \circ & \bullet \\ \bullet & \bullet \\ \circ & \bullet \\ \bullet & \bullet \\ \circ & \bullet \\ \hline & \bullet \\ & \bullet \\ & \bullet \end{array} \right]$$

or

$$[\mathbb{H}^H] = \left[\begin{array}{ccc|ccc} \Pi_{11} & \frac{\Pi_{33}-\Pi_{11}-\Pi_{22}}{2} & \frac{\Pi_{22}-\Pi_{11}-\Pi_{33}}{2} & & & \\ & \Pi_{22} & \frac{\Pi_{11}-\Pi_{22}-\Pi_{33}}{2} & & & \\ & & \Pi_{33} & & & \\ \hline & & & \Pi_{44} & & \\ & & & & \Pi_{55} & \\ & & & & & \Pi_{66} \end{array} \right]$$

(5.52)

When the engineering notation is used, corresponding representations of the *Hill's criterion* are

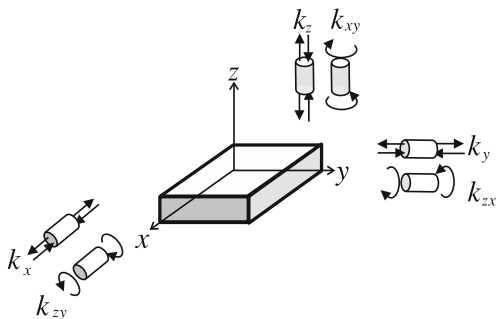
$$- \left[\Pi_{23} (\sigma_y - \sigma_z)^2 + \Pi_{13} (\sigma_z - \sigma_x)^2 + \Pi_{12} (\sigma_x - \sigma_y)^2 \right] + \Pi_{44} \tau_{yz}^2 + \Pi_{55} \tau_{zx}^2 + \Pi_{66} \tau_{xy}^2 = 1 \tag{5.53}$$

or

$$\begin{aligned} & \Pi_{11} \sigma_x^2 + \Pi_{22} \sigma_y^2 + \Pi_{33} \sigma_z^2 + (\Pi_{33} - \Pi_{11} - \Pi_{22}) \sigma_x \sigma_y + \\ & (\Pi_{22} - \Pi_{11} - \Pi_{33}) \sigma_x \sigma_z + (\Pi_{11} - \Pi_{22} - \Pi_{33}) \sigma_y \sigma_z + \\ & \Pi_{44} \tau_{yz}^2 + \Pi_{55} \tau_{zx}^2 + \Pi_{66} \tau_{xy}^2 = 1 \end{aligned} \tag{5.54}$$

Both representations (5.53) or (5.54) describe the same Hill's limit surface, but applying two different choices of six independent elements of the Hill matrices (5.51) or (5.52). In order to calibrate Hill's criterion in the form (5.53) or (5.54) three

Fig. 5.9 Six tests for Hill's criterion calibration



tests of uniaxial tension $\sigma_x = k_x$, $\sigma_y = k_y$, $\sigma_z = k_z$ and three tests of pure shear $\tau_{xy} = k_{xy}$, $\tau_{yz} = k_{yz}$, $\tau_{zx} = k_{zx}$, in directions and planes of material orthotropy (Fig. 5.9), must be performed. These tests allow to express six modules of material orthotropy in Eqs. (5.53) and (5.54) in terms of 3 independent *plastic tension limits* k_x , k_y , k_z (in directions of orthotropy), and 3 independent *plastic shear limits* k_{yz} , k_{zx} , k_{xy} (in planes of material orthotropy). Hence,

$$\begin{aligned} -\Pi_{23} &= \frac{1}{2} \left(\frac{1}{k_y^2} + \frac{1}{k_z^2} - \frac{1}{k_x^2} \right), \quad \Pi_{44} = \frac{1}{k_{yz}^2} \\ -\Pi_{13} &= \frac{1}{2} \left(\frac{1}{k_z^2} + \frac{1}{k_x^2} - \frac{1}{k_y^2} \right), \quad \Pi_{55} = \frac{1}{k_{zx}^2} \\ -\Pi_{12} &= \frac{1}{2} \left(\frac{1}{k_x^2} + \frac{1}{k_y^2} - \frac{1}{k_z^2} \right), \quad \Pi_{66} = \frac{1}{k_{xy}^2} \end{aligned} \quad (5.55)$$

such that *orthotropic Hill's criteria* equivalent to (5.53) or (5.54) can be furnished in terms of plastic anisotropy limits as follows:

$$\begin{aligned} \frac{1}{2} \left(\frac{1}{k_y^2} + \frac{1}{k_z^2} - \frac{1}{k_x^2} \right) (\sigma_y - \sigma_z)^2 + \frac{1}{2} \left(\frac{1}{k_z^2} + \frac{1}{k_x^2} - \frac{1}{k_y^2} \right) (\sigma_z - \sigma_x)^2 + \\ \frac{1}{2} \left(\frac{1}{k_x^2} + \frac{1}{k_y^2} - \frac{1}{k_z^2} \right) (\sigma_x - \sigma_y)^2 + \left(\frac{\tau_{yz}}{k_{yz}} \right)^2 + \left(\frac{\tau_{zx}}{k_{zx}} \right)^2 + \left(\frac{\tau_{xy}}{k_{xy}} \right)^2 = 1 \end{aligned} \quad (5.56)$$

or

$$\begin{aligned} \left(\frac{\sigma_x}{k_x} \right)^2 + \left(\frac{\sigma_y}{k_y} \right)^2 + \left(\frac{\sigma_z}{k_z} \right)^2 - \left(\frac{1}{k_x^2} + \frac{1}{k_y^2} - \frac{1}{k_z^2} \right) \sigma_x \sigma_y - \\ \left(\frac{1}{k_y^2} + \frac{1}{k_z^2} - \frac{1}{k_x^2} \right) \sigma_y \sigma_z - \left(\frac{1}{k_z^2} + \frac{1}{k_x^2} - \frac{1}{k_y^2} \right) \sigma_z \sigma_x + \\ \left(\frac{\tau_{yz}}{k_{yz}} \right)^2 + \left(\frac{\tau_{zx}}{k_{zx}} \right)^2 + \left(\frac{\tau_{xy}}{k_{xy}} \right)^2 = 1 \end{aligned} \quad (5.57)$$

Note that under a particular *plane stress condition*, e.g., in the x, y plane, when $\sigma_z = \tau_{zx} = \tau_{yz} = 0$, both formulas (5.56) and (5.57) reduce to the 4-parameter orthotropic Hill's condition

$$\frac{\sigma_x^2}{k_x^2} + \frac{\sigma_y^2}{k_y^2} - \left(\frac{1}{k_x^2} + \frac{1}{k_y^2} - \frac{1}{k_z^2} \right) \sigma_x \sigma_y + \frac{\tau_{xy}^2}{k_{xy}^2} = 1 \quad (5.58)$$

where initiation of plastic flow in the x, y plane is controlled not only by the in-plane limits k_x, k_y , and k_{xy} , but also by the out-of-plane limit k_z , which may finally lead to inadmissible loss of convexity by the yield surface. This will be discussed in detail in the next section.

Note that in case when $\Pi_{23} = \Pi_{13} = \Pi_{12} = -1/2k^2$ and $\Pi_{44} = \Pi_{55} = \Pi_{66} = 3/k^2$ the orthotropic Hill criterion (5.53) reduces to the *isotropic Huber–von Mises criterion*

$$(\sigma_y - \sigma_z)^2 + (\sigma_z - \sigma_x)^2 + (\sigma_x - \sigma_y)^2 + 6(\tau_{yz}^2 + \tau_{zx}^2 + \tau_{xy}^2) = 2k^2 \quad (5.59)$$

The Hill criterion (5.53) is formulated in the space of principal material directions of orthotropy which in general do not coincide with directions of principal stresses. In the particular case when the coaxiality holds $\sigma_x = \sigma_1, \sigma_y = \sigma_2, \sigma_z = \sigma_3, \tau_{xy} = \tau_{yz} = \tau_{zx} = 0$ we arrive at simplified

$$-\Pi_{23}(\sigma_2 - \sigma_3)^2 - \Pi_{13}(\sigma_3 - \sigma_1)^2 - \Pi_{12}(\sigma_1 - \sigma_2)^2 = 1 \quad (5.60)$$

or when calibration (5.55) is used the explicit form of (5.60) is finally furnished

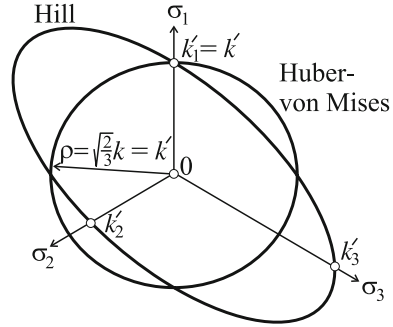
$$\begin{aligned} & \frac{1}{2} \left(\frac{1}{k_2^2} + \frac{1}{k_3^2} - \frac{1}{k_1^2} \right) (\sigma_2 - \sigma_3)^2 + \\ & \frac{1}{2} \left(\frac{1}{k_3^2} + \frac{1}{k_1^2} - \frac{1}{k_2^2} \right) (\sigma_3 - \sigma_1)^2 + \\ & \frac{1}{2} \left(\frac{1}{k_1^2} + \frac{1}{k_2^2} - \frac{1}{k_3^2} \right) (\sigma_1 - \sigma_2)^2 = 1 \end{aligned} \quad (5.61)$$

Hill's condition (5.61) represents cylindrical elliptic surface whose axis coincides with the hydrostatic axis. Nevertheless in some cases, the limit surface loses closed form for high orthotropy degree which may occur when one of following expressions $\frac{1}{k_2^2} + \frac{1}{k_3^2} - \frac{1}{k_1^2}$ elsewhere $\frac{1}{k_3^2} + \frac{1}{k_1^2} - \frac{1}{k_2^2}$ or $\frac{1}{k_1^2} + \frac{1}{k_2^2} - \frac{1}{k_3^2}$ changes the sign. Such behavior is not admissible and a way how to overcome it will be presented in the next section.

It is convenient to express Hill's limit surface by use of the *Haigh–Westergaard coordinates* (cf. Ganczarski and Lenczowski [15])

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \end{Bmatrix} = \frac{\xi}{\sqrt{3}} \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix} + \sqrt{\frac{2}{3}} \rho(\theta) \begin{Bmatrix} \cos \theta \\ \cos(\theta - \frac{2\pi}{3}) \\ \cos(\theta + \frac{2\pi}{3}) \end{Bmatrix} \quad (5.62)$$

Fig. 5.10 Comparison of the Huber–von Mises and the Hill criteria in deviatoric plane applying the Haigh–Westergaard coordinates $\rho(\theta)$ ($k_1 = k$, $k_2 = 0.8k$, $k_3 = 1.5k$)



to finally obtain Hill’s criterion in form $\rho(\theta)$

$$\rho(\theta) = \left[\frac{2}{\left(\frac{1}{k_2^2} + \frac{1}{k_3^2} - \frac{1}{k_1^2}\right) \sin^2\left(\theta + \frac{\pi}{3}\right) + \left(\frac{1}{k_3^2} + \frac{1}{k_1^2} - \frac{1}{k_2^2}\right) \sin^2\left(\theta - \frac{\pi}{3}\right) + \left(\frac{1}{k_1^2} + \frac{1}{k_2^2} - \frac{1}{k_3^2}\right) \sin^2\theta} \right]^{1/2} \quad (5.63)$$

Note that in case if $k_1 = k_2 = k_3 = k$ the *Huber–von Mises circular cylinder* is recovered Fig. 5.10

$$\rho = \sqrt{\frac{2}{3}}k = \text{const} \quad (5.64)$$

5.4 Hill’s Criterion Versus Hu–Marin’s Concept in Case of Strong Orthotropy

Classical orthotropic *Hill’s criterion* [25], despite obvious advantages and wide technical applications, is limited however by some *constraints* of applicability, which are discussed in the present section following [18].

First limitation of applicability range of the classical Hill criterion is established through the inequality bounding the magnitudes of the engineering orthotropy constants k_1 , k_2 , and k_3 in order to avoid *ellipticity loss* of the limit surface in the stress space when the coordinate axes are aligned with the material axes of orthotropy (see, e.g., Ottosen and Ristinmaa [47], Ganczarski and Skrzypek [17, 18]). Such limit bounds put upon the orthotropy limits usually hold in case if the degree of material orthotropy is moderate. For example, if the material ensures the *transverse isotropy*

symmetry, it is shown that the orthotropy degree bounded by the inequality $\frac{k_{\max}}{k_{\min}} < 2$ guarantees ellipticity of the limit surface to be saved. However, if the orthotropy bound is violated, the Hill criterion becomes useless when a possible degeneration of the elliptic cylindrical surface into two concave hyperbolic cylinders occurs, what is inadmissible in the light of *Drucker's* or *Sylvester's stability postulates*.

To illustrate this restriction, we consider two types of true materials for which the classical Hill criterion occurs to be: either useful, if material orthotropy degree is not very high such that the ellipticity property of the limit surface is preserved, or useless if the orthotropy degree is as high as the described limit surface no longer holds the ellipticity requirement. Other words, a physically inadmissible degeneration of the single convex and simply connected elliptical limit surface into two concave hyperbolic surfaces occurs.

The following inequality bounds the *range of applicability* for *Hill's criterion* (cf., e.g., Ottosen and Ristinmaa [47])

$$\frac{2}{k_1^2 k_2^2} + \frac{2}{k_2^2 k_3^2} + \frac{2}{k_3^2 k_1^2} > \frac{1}{k_1^4} + \frac{1}{k_2^4} + \frac{1}{k_3^4} \quad (5.65)$$

For simplicity, a coincidence of the principal stress axes with the material orthotropy axes is assumed in (5.65). In the narrower case of *transverse isotropy* $k_1 = k_2$, condition (5.65) reduces to the simple form

$$\frac{1}{k_3^2} \left(\frac{4}{k_1^2} - \frac{1}{k_3^2} \right) > 0 \quad (5.66)$$

Substitution of the dimensionless parameter $R = 2\left(\frac{k_3}{k_1}\right)^2 - 1$, after Hosford and Backhofen [27], leads to the simplified restriction

$$R > -0.5 \quad (5.67)$$

If the above inequalities (5.65)–(5.67) do not hold, elliptic cross sections of the limit surface degenerate into two hyperbolic branches and the lack of convexity occurs. To illustrate this limitation, the yield curves in two planes: the transverse isotropy (σ_1, σ_2) and the orthotropy plane (σ_1, σ_3) for various R -values, are sketched in Fig. 5.11a, b, respectively. It is observed that when R starting from $R = 3$ approaches the limit $R = -0.5$, the curves change from closed ellipses to two parallel lines, whereas for $R < -0.5$ concave hyperbolas appear.

As example of orthotropic engineering material for which classical Hill's criterion can correctly predict the limit surface, consider first the *OTCz Titanium Alloy*, the mechanical orthotropic properties of which are given in Table 5.1 (cf. Malinin and Rżysko [42]). Note that, for the *OTCz Titanium Alloy*, yield limits in the plane of weak orthotropy 1,2 differ not so much, but the 3 axis is the dominant orthotropy axis. As a consequence, in the plane of weak orthotropy 1,2 Hill's ellipse is slightly rotated towards 2-axis ($\alpha_{12} \approx 45^\circ$), in contrast to the plane of strong orthotropy

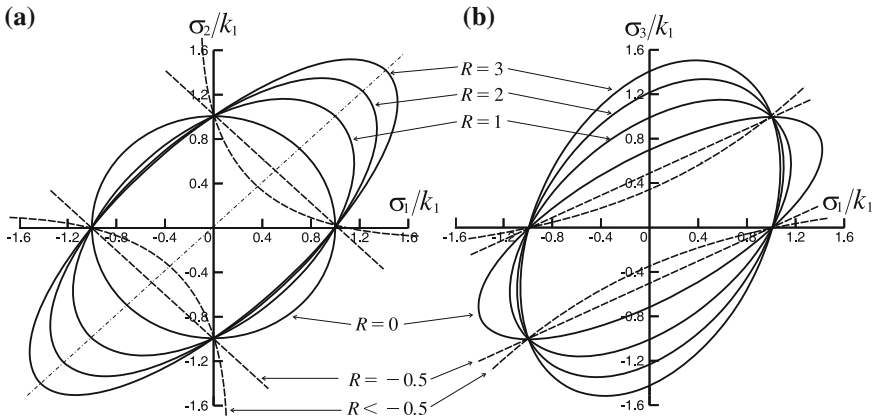


Fig. 5.11 Degeneration of the Hill's limit surface with the magnitude of the Hosford and Backhofen parameter R : **a** transverse isotropy plane, **b** orthotropy plane (after Ganczarski and Skrzypek [18])

Table 5.1 Mechanical properties of orthotropic OTCz Titanium Alloy after Malinin and Rżysko [42]

Yield limits	k_1 [MPa]	k_2 [MPa]	k_3 [MPa]
	490	520	800

1,3, where the rotation of the Hill ellipse is significant ($\alpha_{13} \approx 71^\circ$), as shown in Fig. 5.12a, b, respectively.

In a case of *high orthotropy degree* (observed for majority of the long fiber reinforced composites, for instance: Boron/Al, SiC/Ti, Glass/Epoxy, Graphite/Epoxy, etc., e.g., Herakovich and Aboudi [23], Sun and Vaidya [57], and others), the concept other than Hill's is proposed. This new approach suggests formulation of limit criterion based on the 9-parameter von Mises condition, but enhanced by the Hu–Marin type biaxial orthotropic loading conditions (cf. Hu and Marin [30], Skrzypek and Ganczarski [54]). It will be demonstrated that, even in a case of arbitrarily strong orthotropy (for instance, $k_{\max}/k_{\min} \approx 9$, in case if brass Ł62 is tested) the property of ellipticity is saved.

In general case of strong orthotropy, when the ellipticity condition (5.65) does not hold, the deviatoric Hill criterion (5.56) or (5.57) becomes useless. Hence, in order to describe physically admissible closed and convex limit surface, the more general 9-parameter *orthotropic von Mises equation* (5.44) has to be recalled. In a narrower case of the principal stress axes coinciding with the material orthotropy axes the Eq. (5.45) reads as

$$\Pi_{11}\sigma_1^2 + \Pi_{22}\sigma_2^2 + \Pi_{33}\sigma_3^2 + 2(\Pi_{12}\sigma_1\sigma_2 + \Pi_{23}\sigma_2\sigma_3 + \Pi_{31}\sigma_3\sigma_1) = 1 \quad (5.68)$$

The condition (5.68) is defined by six material parameters only, because $\tau_{23} \equiv \tau_{31} \equiv \tau_{12} \equiv 0$, hence its calibration requires six conditions:

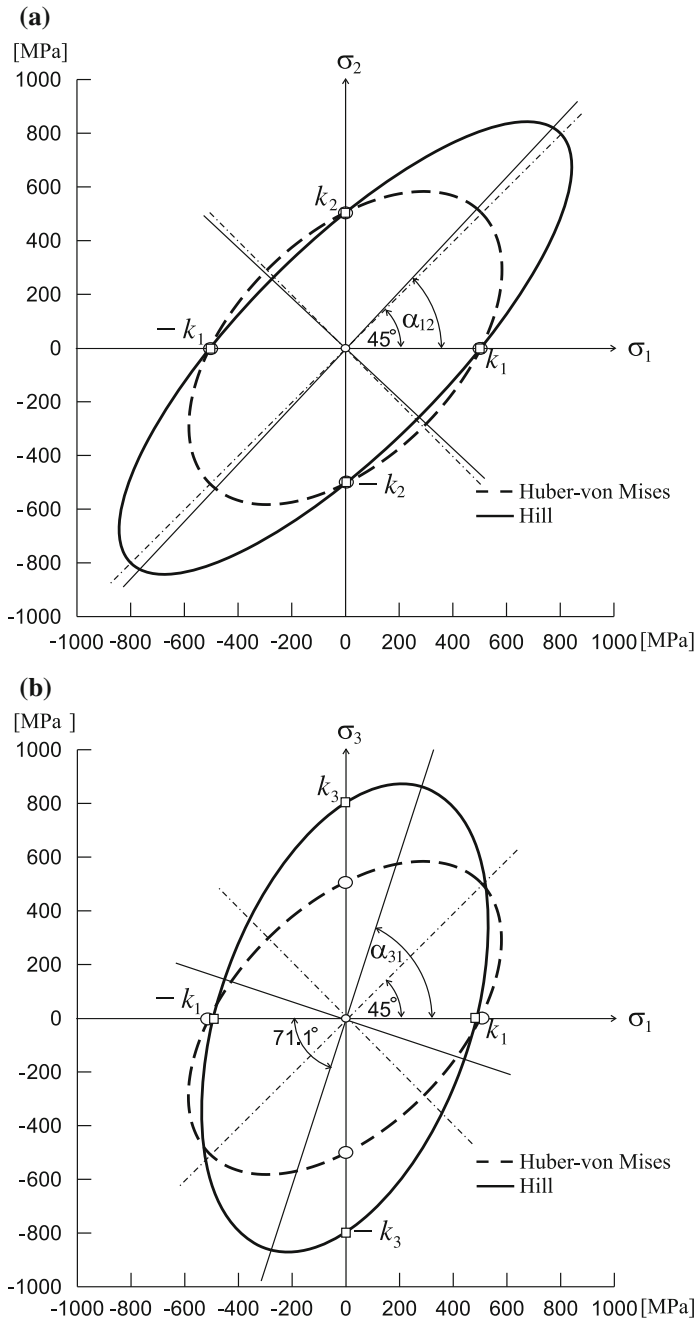
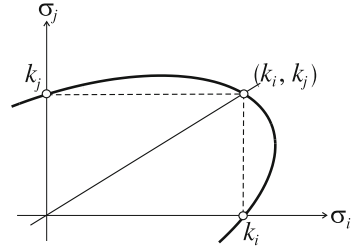


Fig. 5.12 Hill's deviatoric initial yield conditions versus Huber-von Mises' isotropic approximation for the OTCz Titanium Alloy (cf. Table 5.1): **a** the plane of "weak" orthotropy (σ_1, σ_2), **b** the plane of "strong" orthotropy (σ_1, σ_3) (after Ganczarski and Skrzypek [18])

Fig. 5.13 Graphical illustration of biaxial loading conditions (5.70)



three tests of *uniaxial tension* along the orthotropy axes

$$\begin{aligned}
 \sigma_1 = k_1 \quad \sigma_2 = 0 \quad \sigma_3 = 0 &\longrightarrow \Pi_{11} = 1/k_1^2 \\
 \sigma_2 = k_2 \quad \sigma_1 = 0 \quad \sigma_3 = 0 &\longrightarrow \Pi_{22} = 1/k_2^2 \\
 \sigma_3 = k_3 \quad \sigma_1 = 0 \quad \sigma_2 = 0 &\longrightarrow \Pi_{33} = 1/k_3^2
 \end{aligned}
 \tag{5.69}$$

and three orthotropic *biaxial tension loading conditions* (k_i, k_j) cf. Fig. 5.13

$$\begin{aligned}
 \sigma_1 = k_1 \quad \sigma_2 = k_2 \quad \sigma_3 = 0 &\longrightarrow \Pi_{12} = -1/2k_1k_2 \\
 \sigma_1 = k_1 \quad \sigma_3 = k_3 \quad \sigma_2 = 0 &\longrightarrow \Pi_{13} = -1/2k_1k_3 \\
 \sigma_2 = k_2 \quad \sigma_3 = k_3 \quad \sigma_1 = 0 &\longrightarrow \Pi_{23} = -1/2k_2k_3
 \end{aligned}
 \tag{5.70}$$

The similar equibiaxial tension loading conditions are used, e.g., by Khan and Liu [35].

Calibration of the orthotropic von Mises criterion (5.68), performed with conditions (5.69) and (5.70) used, leads to the three-axial extension of the Hu–Marin type criterion (cf. Ganczarski and Skrzypek [16], Skrzypek and Ganczarski [54])

$$\left(\frac{\sigma_1}{k_1}\right)^2 - \frac{\sigma_1\sigma_2}{k_1k_2} + \left(\frac{\sigma_2}{k_2}\right)^2 - \frac{\sigma_2\sigma_3}{k_2k_3} + \left(\frac{\sigma_3}{k_3}\right)^2 - \frac{\sigma_1\sigma_3}{k_1k_3} = 1
 \tag{5.71}$$

The *enhanced Mises–Hu–Marin type criterion* (5.71) is free from Hill’s deficiency even in case of arbitrarily strong orthotropy degree, since it never violates the Drucker stability postulate, which is not guaranteed by Hill-type equations. The Hu–Marin-type Eq. (5.71) can easily be presented in the “*pseudo-deviatoric*” format

$$\left(\frac{\sigma_1}{k_1} - \frac{\sigma_2}{k_2}\right)^2 + \left(\frac{\sigma_2}{k_2} - \frac{\sigma_3}{k_3}\right)^2 + \left(\frac{\sigma_3}{k_3} - \frac{\sigma_1}{k_1}\right)^2 = 2
 \tag{5.72}$$

Three orthotropy limit yield points k_1, k_2 and k_3 establish the proportional *stress/strength axis* of cylindrical Hu–Marin’s surface. Note that this proportional stress/strength axis, which determines a position of the limit surface axis in the principal stress space, is different from the *hydrostatic axis*, but the condition of equal ratios $\sigma_i/k_i = \alpha$ holds at all points belonging to this axis. The extended

von Mises–Hu–Marin type criteria (5.71–5.72) are always “unconditionally stable” criteria, that remain convex even for very strong orthotropy, by contrast to the classical Hill condition in which the possible loss of convexity can be met in the case of highly orthotropic materials. However, the fully deviatoric format of the Hill criteria (5.50–5.56) is lost in the Hu–Marin type format (5.72) where the *hydrostatic pressure insensitivity is relaxed*.

In the particular case of *plane stress state* $\sigma_3 = 0$ the three-parameter enhanced von Mises–Hu–Marin equation (5.71) is reduced to the two-parameter one, as proposed by Hu–Marin [30]

$$\left(\frac{\sigma_1}{k_1}\right)^2 - \frac{\sigma_1\sigma_2}{k_1k_2} + \left(\frac{\sigma_2}{k_2}\right)^2 = 1 \quad (5.73)$$

Comparison of the 2-parameter Hu–Marin plane stress equation (5.73) with the simplified 4-parameter plane stress Hill’s equation (5.58) written for principal stress axes, leads to the 3-parameter form

$$\left(\frac{\sigma_1}{k_1}\right)^2 - \left(\frac{1}{k_1^2} + \frac{1}{k_2^2} - \frac{1}{k_3^2}\right)\sigma_1\sigma_2 + \left(\frac{\sigma_2}{k_2}\right)^2 = 1 \quad (5.74)$$

which becomes identical to the von Mises–Hu–Marin equation (5.73) only if following constraint holds

$$\frac{1}{k_3^2} = \frac{1}{k_1^2} + \frac{1}{k_2^2} - \frac{1}{k_1k_2} \quad (5.75)$$

which is usually not true.

In order to illustrate a suitability of the von Mises–Hu–Marin orthotropic Eq. (5.71), when compared to certain limitations of the Hill deviatoric Eq. (5.58), two engineering materials characterized by different degrees of orthotropy: *OTCz Titanium Alloy* (“weak” orthotropy) and *L62 brass* (“strong” orthotropy) are studied. The results are presented in Fig. 5.14a, b on the planes σ_1, σ_3 and σ_1, σ_2 , respectively. In case of “weak” orthotropy both Hill’s and Hu–Marin’s ellipses differ not so much, and both concepts are recommended (Fig. 5.14a). However, in case of “strong” orthotropy, when the inequality (5.65) is not satisfied, following the Hill concept two concave hyperbolic cylinders are formed by opening of the elliptic cylinder towards the proportional stress/strength axis (Fig. 5.14b). On the other hand, the Hu–Marin type surface saves the ellipticity property regardless of the magnitude of orthotropy degree considered. In other words the *Hu–Marin surface* is “unconditionally stable” which remains convex for very strong orthotropy. It is possible due to three additional constraints (5.70) satisfied for the pairs of orthotropy yield limits (k_1, k_2) , (k_2, k_3) , and (k_3, k_1) . But, it should be pointed out that the Hu–Marin cylindrical surface does not satisfy the condition of deviatoricity, hence this condition in fact should be classified as a specific representative of the hydrostatic pressure sensitive class of materials where the independence of the hydrostatic stress constraint is relaxed.

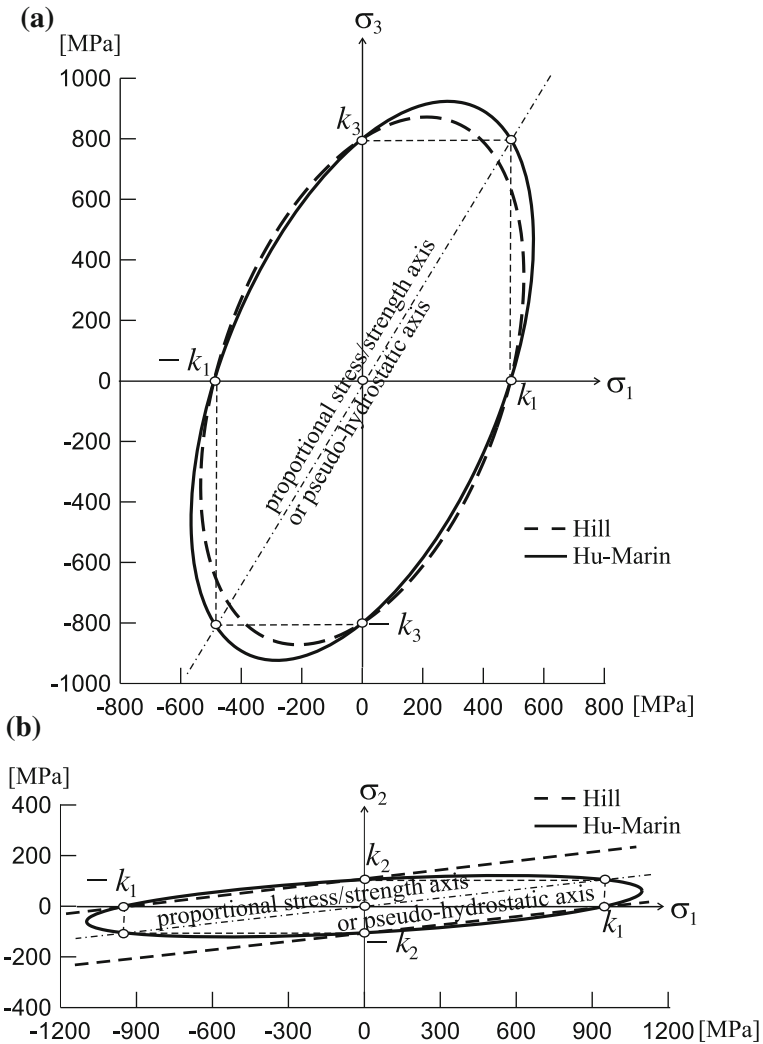


Fig. 5.14 Comparison of the Hill and the Hu–Marín plastic yield criteria for two orthotropic materials of different orthotropy degrees: **a** “weak” orthotropy in case of OTCz titanium alloy ($k_1 = 490$ MPa, $k_2 = 520$ MPa, $k_3 = 800$ MPa), **b** “strong” orthotropy in case of Ł62 brass ($k_1 = 105$ MPa, $k_2 = 120$ MPa, $k_3 = 950$ MPa) (after Ganczarski and Skrzypek [18])

The aforementioned possible *loss of the convexity* of classical *Hill’s criterion* [25] (5.56) in case of highly orthotropic materials is even more pronounced when the *orthotropic generalization of the isotropic Hosford criterion* [27] (5.10) for higher (even) exponents is done

$$\begin{aligned}
 & -\Pi_{23}|\sigma_y - \sigma_z|^m - \Pi_{13}|\sigma_z - \sigma_x|^m - \Pi_{12}|\sigma_x - \sigma_y|^m \\
 & + \Pi_{44}|\tau_{yz}|^m + \Pi_{55}|\tau_{zx}|^m + \Pi_{66}|\tau_{xy}|^m = 1
 \end{aligned} \tag{5.76}$$

Six generalized *orthotropy modules* $\Pi_{23}, \dots, \Pi_{66}$ can be expressed in terms of six yield point stresses k_x, \dots, k_{xy} in analogous fashion as previously discussed manner of calibration for Hill's criterion (5.55), namely

$$\begin{aligned}
 -\Pi_{23} &= \frac{1}{2} \left(\frac{1}{|k_y|^m} + \frac{1}{|k_z|^m} - \frac{1}{|k_x|^m} \right), \quad \Pi_{44} = \frac{1}{|k_{yz}|^m} \\
 -\Pi_{13} &= \frac{1}{2} \left(\frac{1}{|k_z|^m} + \frac{1}{|k_x|^m} - \frac{1}{|k_y|^m} \right), \quad \Pi_{55} = \frac{1}{|k_{zx}|^m} \\
 -\Pi_{12} &= \frac{1}{2} \left(\frac{1}{|k_x|^m} + \frac{1}{|k_y|^m} - \frac{1}{|k_z|^m} \right), \quad \Pi_{66} = \frac{1}{|k_{xy}|^m}
 \end{aligned} \tag{5.77}$$

Note however that in this extended case (different from $m = 2$ and $m = 4$ when the orthotropic Hill is recovered) dimension of the orthotropy modules $\Pi_{23}, \dots, \Pi_{66}$ depends on the value of power m and it is equal to MPa^{-m} .

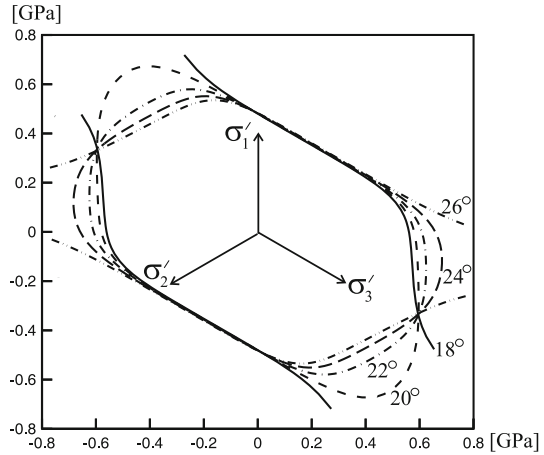
Although the yield criterion defined by Eq. (5.76) with the calibration (5.77) used has been mathematically verified and its convexity has been proven in case of the *planar anisotropy* in the principal stress space if and only if $m \geq 1$ and the orthotropy modules $\Pi_{23}, \dots, \Pi_{66}$ are positive constant coefficients (see Barlat and Lian [2], also Chu [10]), in case of the general orthogonal anisotropy in the six-dimensional stress space convexity is not obvious.

The more general case, when axes of material orthotropy are different from axes of principal stresses, was considered by Ganczarski and Lenczowski [15]. It was shown that, although the limit surface is closed and convex in space of *principal material orthotropy* frame, it occurs that lack of convexity is met when transformation to the space of *principal stress frame* is done in terms of three angles defining the mutual configuration of these two frames. This type of convexity loss was examined for the brass sheet Ł22 the six orthotropic yield points of which are given in Table 5.2 after Malinin and Rżysko [42] who gave three-axial yield point stresses whereas three shear yield point stresses were estimated in [15] using simplified formulas $k_{ij} = \sqrt{\frac{k_{ij}}{3}}$ for $m = 2$ and $k_{ij} = \frac{\sqrt{k_{ij}}}{2}$ for $m = 6, 8$. For simplicity the evolution of the generalized *orthotropic Hosford yield condition* ($m = 8$) with respect to only one of the Euler angles ϑ was considered. It represents a prism of the semi-hexagonal cross section with oval corners as presented in Fig. 5.15. The *loss of convexity* is observed for $18^\circ \geq \vartheta \geq 26^\circ$.

Table 5.2 Yield point stresses for brass Ł22 after Malinin and Rżysko [42]

m	k_x [MPa]	k_y [MPa]	k_z [MPa]	k_{zy} [MPa]	k_{zx} [MPa]	k_{xy} [MPa]
2	120	105	950	182	194	64.8
6 or 8	120	105	950	157	168	56.1

Fig. 5.15 Evolution of the generalized orthotropic Hosford yield condition ($m = 8$) versus the Euler angle $\vartheta = 18^\circ, 20^\circ, 22^\circ, 24^\circ,$ and 26° for brass Ł22, after Ganczarski and Lenczowski [15]



It should be pointed out that the limit criteria considered throughout this section do not exhibit the strength differential effect such that they cannot be recommended as failure criteria for brittle materials where this effect is essential.

5.5 Transversely Isotropic Case—Hill-Type Tetragonal Symmetry Versus Hu–Marin-Type Hexagonal Symmetry Criteria

The second limitation of applicability range of classical Hill’s criterion arises when the transverse isotropy property is considered. In this section it will be shown that, if reduction of Hill’s criterion to the *transverse isotropy symmetry* is performed, the 4-parameter form that satisfies the *tetragonal symmetry class* is furnished (cf., e.g., Voyiadjis and Thiagarajan [62], Sun and Vaidya [57]). This type of symmetry is of particular importance in case of *unidirectional fiber reinforced composites*. In such a case moduli: $k_x, k_y, k_z,$ and k_{xy} are considered as independent (z is the orthotropy axis), which makes impossible to reduce classical Hill’s criterion to the isotropic von Mises condition in the plane of transverse isotropy.

To avoid this *irreducibility to isotropic von Mises*, the new *Hu–Marin-based transversely isotropic criterion* exhibiting *hexagonal symmetry* is proposed instead of *deviatoric transversely isotropic Hill’s criterion* exhibiting *tetragonal symmetry*. It enables to achieve reducibility to the isotropic von Mises condition in the transverse isotropy plane, preserving cylindricity regardless of the magnitude of orthotropy degree.

Finally, it will be demonstrated that, for some composite materials it is necessary to further modify the 3-parameter Hu–Marin-type criterion to the new 4-parameter intermediate-type criterion between classical Hill’s and hexagonal Hu–Marin’s

concepts, taking advantage of the bulge test. This new hybrid-type criterion differs essentially from both the Hu–Marin hexagonal type criterion and the isotropic von Mises criterion in the isotropy plane. *Bulge tests* have been performed and described, e.g., by Jackson et al. [33] with equipment used by Lankford et al. [39]. This new criterion is capable of properly describing the SiC/Ti long fiber reinforced composite examined by Herakovich and Aboudi [23].

Classical Hill’s equation (5.53–5.54), which is expressed in terms of six independent plastic yield limits $k_x, k_y, k_z, k_{yz}, k_{zx}$, and k_{xy} , (5.56) is often too general for engineering applications. Orthotropic structural materials usually exhibit the transversely isotropic symmetry, basically due to either fabrication process or microstructure texture, as often observed in many long parallel fiber reinforced composites. In particular, if in elastic range the transversely isotropic symmetry group holds, it is expected that, also for the plastic yield initiation criterion such a narrower symmetry is true.

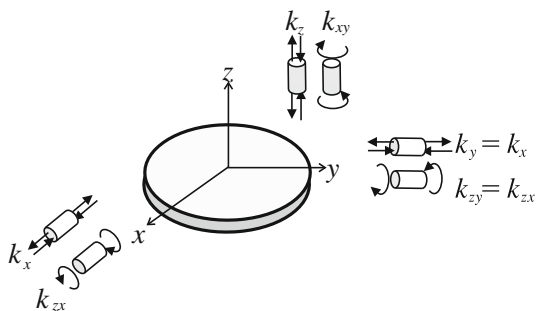
In what follows, a distinction between two symmetry classes of the *transverse isotropy–tetragonal* or *hexagonal type* has to be done. Such distinction is known, e.g., from definitions of Representative Unit Cell used in homogenization methods for *composite materials* (cf., e.g., Berryman [4], Sun and Vaidya [57], etc.).

Assume that the z -axis is the orthotropy axis, whereas x, y is the *transverse isotropy plane*. When applying Eq. (5.54) with calibrations (5.56) or (5.57) and additionally assuming $k_x = k_y \neq k_z, k_{zx} = k_{zy} \neq k_{xy}$, the number of independent limits in transversely isotropic Hill’s equation reduces to four for instance: two axial yield limits k_x and k_z , and two shear yield limits k_{zx} and k_{xy} (Fig. 5.16).

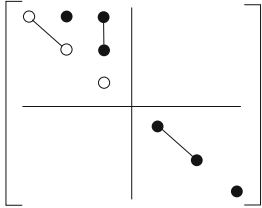
In this way the following is furnished

$$\begin{aligned}
 -\Pi_{13} = -\Pi_{23} &= \frac{1}{2k_z^2}, & \Pi_{44} = \Pi_{55} &= \frac{1}{k_{zx}^2} \\
 -\Pi_{12} &= \frac{1}{k_x^2} - \frac{1}{2k_z^2}, & \Pi_{66} &= \frac{1}{2k_{xy}^2}
 \end{aligned}
 \tag{5.78}$$

Fig. 5.16 Four independent tests for transversely isotropic Hill’s criterion calibration

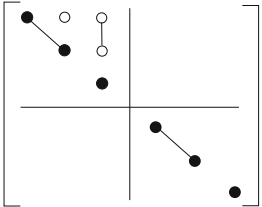


Substitution of (5.78) into (5.51) and (5.52) yields to *transversely isotropic Hill's matrices*

$$[\text{tris III}^H] = \left[\begin{array}{ccc|ccc} -\Pi_{12} - \Pi_{13} & \Pi_{12} & \Pi_{13} & & & \\ & -\Pi_{12} - \Pi_{23} & \Pi_{13} & & & \\ & & -2\Pi_{13} & & & \\ \hline & & & \Pi_{44} & & \\ & & & & \Pi_{44} & \\ & & & & & \Pi_{66} \end{array} \right]$$


(5.79)

or

$$[\text{tris III}^H] = \left[\begin{array}{ccc|ccc} \Pi_{11} & \frac{\Pi_{33} - 2\Pi_{11}}{2} & -\frac{\Pi_{33}}{2} & & & \\ & \Pi_{11} & -\frac{\Pi_{33}}{2} & & & \\ & & \Pi_{33} & & & \\ \hline & & & \Pi_{44} & & \\ & & & & \Pi_{44} & \\ & & & & & \Pi_{66} \end{array} \right]$$


(5.80)

The transversely isotropic 4-parameter Hill criteria corresponding to *orthotropic Hill's criteria* (5.56) and (5.57) take the following representations

$$\frac{(\sigma_y - \sigma_z)^2 + (\sigma_z - \sigma_x)^2}{2k_z^2} + \left(\frac{1}{k_x^2} - \frac{1}{2k_z^2} \right) (\sigma_x - \sigma_y)^2 + \frac{\tau_{yz}^2 + \tau_{zx}^2}{k_{zx}^2} + \frac{\tau_{xy}^2}{k_{xy}^2} = 1$$

(5.81)

or equivalently

$$\frac{\sigma_x^2 + \sigma_y^2}{k_x^2} + \frac{\sigma_z^2}{k_z^2} - \left(\frac{2}{k_x^2} - \frac{1}{k_z^2} \right) \sigma_x \sigma_y - \frac{\sigma_y \sigma_z + \sigma_z \sigma_x}{k_z^2} + \frac{\tau_{yz}^2 + \tau_{zx}^2}{k_{zx}^2} + \frac{\tau_{xy}^2}{k_{xy}^2} = 1$$

(5.82)

Both forms involve four plastic limits k_x, k_z, k_{zx} , and k_{xy} , considered as independent parameters. Underlined factor in (5.82) includes not only k_x but also k_z . The explicitly deviatoric form (5.81) exhibits the similar feature. The plastic state in the transverse isotropy plane x, y is controlled not only by the tensile yield limit in this plane k_x , but also by the out-of-plane tensile yield limit k_z . Concluding, *transversely isotropic Hill's criteria* (5.81) or (5.82) have to be classified as the *tetragonal*

symmetry format (see Table 1.4). The assumption of tetragonal symmetry of the criteria (5.81–5.82) was also considered by Voyiadjis and Thiagarajan [62] in case of directionally reinforced metal matrix composites (Boron–Aluminum). Broader discussion that relates to distinction between the tetragonal versus hexagonal symmetry in the yield/failure criteria will be presented in Sect. 6.5, where additional constraint for case if $\Pi_{66} = -2(\Pi_{13} + 2\Pi_{12})$ is assumed, such that Π_{66} has to be considered as dependent plastic modulus. To this end, if aforementioned constraint postulated by Chen and Han [9] is applied, the equality holds

$$\Pi_{66} = \frac{4}{k_x^2} - \frac{1}{k_z^2} \quad (5.83)$$

instead of (5.78₄) and *transversely isotropic 3-parameter Hill's criteria* corresponding to (5.81) and (5.82) in following format

$$\frac{(\sigma_y - \sigma_z)^2 + (\sigma_z - \sigma_x)^2}{2k_z^2} + \left(\frac{1}{k_x^2} - \frac{1}{2k_z^2} \right) (\sigma_x - \sigma_y)^2 + \frac{\tau_{yz}^2 + \tau_{zx}^2}{k_{zx}^2} + \left(\frac{4}{k_x^2} - \frac{1}{k_z^2} \right) \tau_{xy}^2 = 1 \quad (5.84)$$

or equivalently

$$\frac{\sigma_x^2 + \sigma_y^2}{k_x^2} + \frac{\sigma_z^2}{k_z^2} - \left(\frac{2}{k_x^2} - \frac{1}{k_z^2} \right) \sigma_x \sigma_y - \frac{\sigma_y \sigma_z + \sigma_z \sigma_x}{k_z^2} + \frac{\tau_{yz}^2 + \tau_{zx}^2}{k_{zx}^2} + \left(\frac{4}{k_x^2} - \frac{1}{k_z^2} \right) \tau_{xy}^2 = 1 \quad (5.85)$$

can be written down.

In the particular case of plane stress state in the transverse isotropy plane (x, y) $\sigma_x, \sigma_y, \tau_{xy} \neq 0$ Eqs.(5.81) or (5.82) reduce to (5.58) with additional condition $k_x = k_y$

$$\frac{\sigma_x^2 + \sigma_y^2}{k_x^2} - \left(\frac{2}{k_x^2} - \frac{1}{k_z^2} \right) \sigma_x \sigma_y + \frac{\tau_{xy}^2}{k_{xy}^2} = 1 \quad (5.86)$$

The above form simply means that commonly used “transversely isotropic Hill's criterion” does not coincide in the “transverse isotropy plane” with the *isotropic Huber–von Mises equation*

$$\frac{\sigma_x^2 + \sigma_y^2}{k_x^2} - \frac{\sigma_x \sigma_y}{k_x^2} + 3 \frac{\tau_{xy}^2}{k_{xy}^2} = 1 \quad (5.87)$$

In other words, when the new transversely isotropic yield criterion, that is free from inconsistencies between (5.86) and (5.87) is sought for, the material parameter

preceding term $\sigma_x \sigma_y$ must be equal to $\Pi_{33} - 2\Pi_{11} = 1/k_x^2$ and not depend on k_z and, simultaneously, the material parameter $\Pi_{66} = 3/k_x^2$ must depend on k_x only.

In order to derive the transversely isotropic yield criterion reducible to coincidence with the Huber–von Mises criterion in the isotropy plane, the new *transversely isotropic hexagonal Hu–Marin equation* will be postulated. To obtain this criterion, the general *orthotropic von Mises equation* (5.45), which is not deviatoric, can be calibrated analogously to that presented in (5.69) and (5.70). Namely, when the constraints of transverse isotropy are imposed, we invoke:

the two *tensile tests* in the x - and the orthotropy z -axes and the *shear test* in the orthotropy zx -plane

$$\begin{aligned} \sigma_x = k_x, \quad \sigma_y = \sigma_z = \tau_{xy} = \tau_{yz} = \tau_{zx} = 0 &\longrightarrow \Pi_{11} = \Pi_{22} = 1/k_x^2 \\ \sigma_z = k_z, \quad \sigma_x = \sigma_y = \tau_{xy} = \tau_{yz} = \tau_{zx} = 0 &\longrightarrow \Pi_{33} = 1/k_z^2 \\ \tau_{zx} = k_{zx}, \quad \sigma_x = \sigma_y = \sigma_z = \tau_{xy} = \tau_{yz} = 0 &\longrightarrow \Pi_{44} = \Pi_{55} = 1/k_{zx}^2 \end{aligned} \quad (5.88)$$

and the three *biaxial conditions* for coincidence of appropriate pairs of yield limits

$$\begin{aligned} \sigma_x = k_x, \sigma_y = k_x, \quad \sigma_z = \tau_{xy} = \tau_{yz} = \tau_{zx} = 0 &\longrightarrow \Pi_{12} = -1/2k_x^2 \\ \sigma_x = k_x, \sigma_z = k_z, \quad \sigma_y = \tau_{xy} = \tau_{yz} = \tau_{zx} = 0 &\longrightarrow \Pi_{13} = -1/2k_x k_z \\ \sigma_x = k_x, \tau_{xy} = k_x/\sqrt{3}, \sigma_y = \sigma_z = \tau_{yz} = \tau_{zx} = 0 &\longrightarrow \Pi_{66} = 3/k_x^2 \end{aligned} \quad (5.89)$$

Introduction of (5.88) and (5.89) into *orthotropic von Mises' criterion* (5.45) leads to *transversely isotropic 3-parameter hexagonal Hu–Marin's criterion* as follows:

$$\frac{\sigma_x^2 + \sigma_y^2}{k_x^2} - \frac{\sigma_x \sigma_y}{k_x^2} + \frac{\sigma_z^2}{k_z^2} - \frac{\sigma_y \sigma_z + \sigma_z \sigma_x}{k_z k_x} + \frac{\tau_{yz}^2 + \tau_{zx}^2}{k_{zx}^2} + 3 \frac{\tau_{xy}^2}{k_x^2} = 1 \quad (5.90)$$

or

$$\left(\frac{\sigma_x - \sigma_y}{k_x} \right)^2 + \left(\frac{\sigma_y}{k_x} - \frac{\sigma_z}{k_z} \right)^2 + \left(\frac{\sigma_z}{k_z} - \frac{\sigma_x}{k_x} \right)^2 + 3 \frac{\tau_{yz}^2 + \tau_{zx}^2}{k_{zx}^2} + 6 \frac{\tau_{xy}^2}{k_x^2} = 2 \quad (5.91)$$

Note, that the above conditions correspond to generalized Hu–Marin's equations (5.71) or (5.72) with $k_1 = k_2$, but enhanced by the additional shear terms and referring to optional directions x, y, z . Equations (5.90) or (5.91) reduce to the Huber–von Mises equation (5.87) in case of *plane stress state* in the transverse isotropy plane (x, y), which means that this new criterion can finally be recognized as transversely isotropic hexagonal symmetry von Mises–Hu–Marin's based criterion.

Transversely isotropic conditions—*tetragonal Hill's* (5.71) or (5.72) and *hexagonal Hu–Marin's* (5.90) or (5.91), are examined for given orthotropy degrees $R = 2\left(\frac{k_z}{k_x}\right)^2 - 1 = 2, k_{xy}/k_x = 0.8, k_{(xy)}/k_x = 0.9$, and $k_{zx}/k_x = 0.8$, for following stress states: biaxial normal stresses (σ_x, σ_y) and combined normal with shear stresses (σ_x, τ_{xy}) in the *transverse isotropy plane* (see Fig. 5.17a, b), as well as biaxial normal stresses (σ_x, σ_z) and combined normal with shear stresses (σ_x, τ_{zx})

in the *orthotropy plane* (see Fig. 5.18a, b). It is worth to mention that transversely isotropic Hill’s condition of tetragonal symmetry (5.81) or (5.82) comprises four independent plastic yield limits: k_x , k_z , k_{zx} , and k_{xy} , because shear yield limit in isotropy plane k_{xy} is considered as independent.

Contrarily, transversely isotropic enhanced Hu–Marin-type condition, the symmetry class of which is hexagonal, is defined by three independent yield limits only: k_x , k_z , and k_{zx} , since in-plane shear yield limit k_{xy} must agree with the Huber–von Mises criterion in the isotropy plane $k_{xy} = \frac{k_x}{\sqrt{3}}$. Hence, representation of the *transversely isotropic hexagonal symmetry Hu–Marin-type* constitutive matrix of plasticity is as follows:

$$[\text{tris}^{\text{III}}\text{HM}] = \left[\begin{array}{ccc|c} \frac{1}{k_x^2} & -\frac{1}{2k_x^2} & -\frac{1}{2k_x k_z} & \\ & \frac{1}{k_x^2} & -\frac{1}{2k_x k_z} & \\ & & \frac{1}{k_z^2} & \\ \hline & & & \frac{1}{k_{zx}^2} \\ & & & \frac{1}{k_{zx}^2} \\ & & & \frac{3}{k_x^2} \end{array} \right] \left[\begin{array}{c} \bullet \\ \circ \\ \circ \\ \bullet \\ \bullet \\ \bullet \\ \bullet \\ \circ \end{array} \right] \tag{5.92}$$

The general case of *transversely isotropic 4-parameter tetragonal symmetry Hu–Marin-type* yield criterion that preserves convexity but lost property of reducibility to the isotropic von Mises condition in the plane of transverse isotropy is considered by Voyiadjis and Thiagarajan [62]. The corresponding *matrix* of plasticity used by authors results from the general orthotropic matrix when four independent plastic onset limits are $k_1, k_2 = k_3, k_4 = k_5$, and k_6 (if original notation is saved 1 denotes fiber direction)

$$[\text{tris}^{\text{III}}\text{VT}] = \left[\begin{array}{ccc|c} \frac{4}{9}k_1^2 & -\frac{2}{9}k_1 k_2 & -\frac{2}{9}k_1 k_2 & \\ & \frac{4}{9}k_2^2 & -\frac{2}{9}k_1^2 & \\ & & \frac{4}{9}k_2^2 & \\ \hline & & & \frac{4}{3}(k_1 k_2 + k_4^2) \\ & & & \frac{4}{3}(k_1 k_2 + k_4^2) \\ & & & \frac{4}{3}(k_2^2 + k_6^2) \end{array} \right] \left[\begin{array}{c} \bullet \\ \circ \\ \circ \\ \bullet \\ \bullet \\ \bullet \\ \bullet \\ \bullet \end{array} \right] \tag{5.93}$$

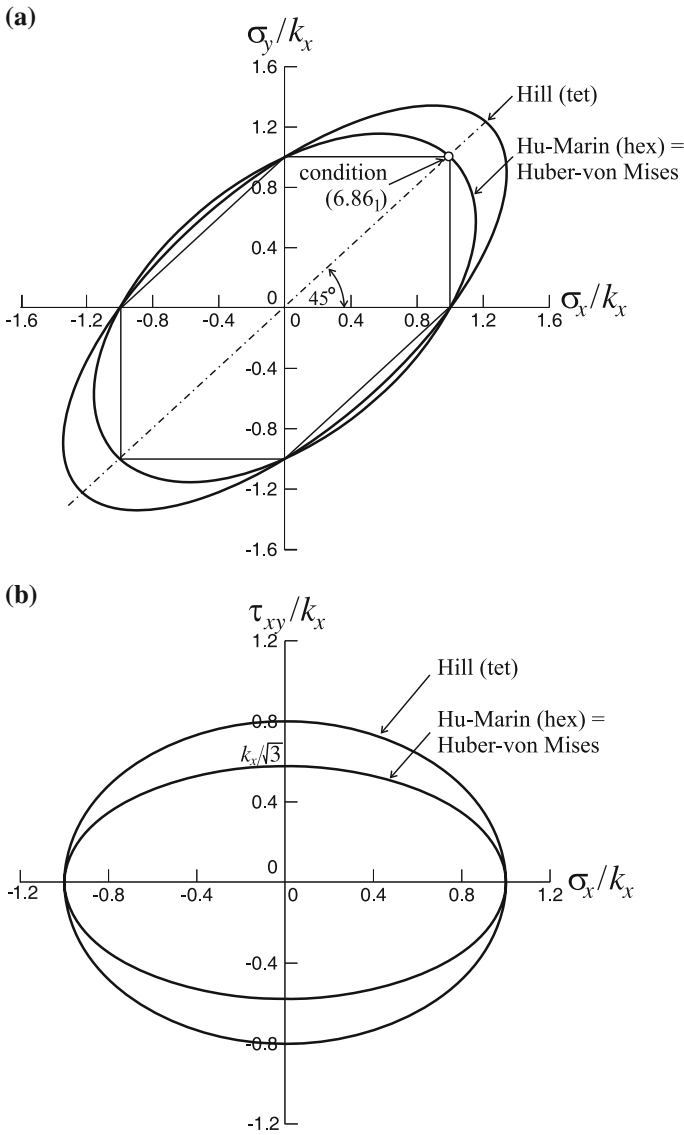


Fig. 5.17 Comparison of transversely isotropic criteria in the transverse isotropy planes: Hill's tetragonal (5.82), Hu-Marín's hexagonal (5.90) and Huber-von Mises' for given magnitudes of orthotropy ratios: $R = 2$, $k_{zx}/k_x = 0.8$, $k_{(xy)}/k_x = 0.9$ in case of 2D states of stress: **a** biaxial normal stresses (σ_x , σ_y) and **b** combined normal with shear stresses (σ_x , τ_{xy}) (after Ganczarski and Skrzypek [18])

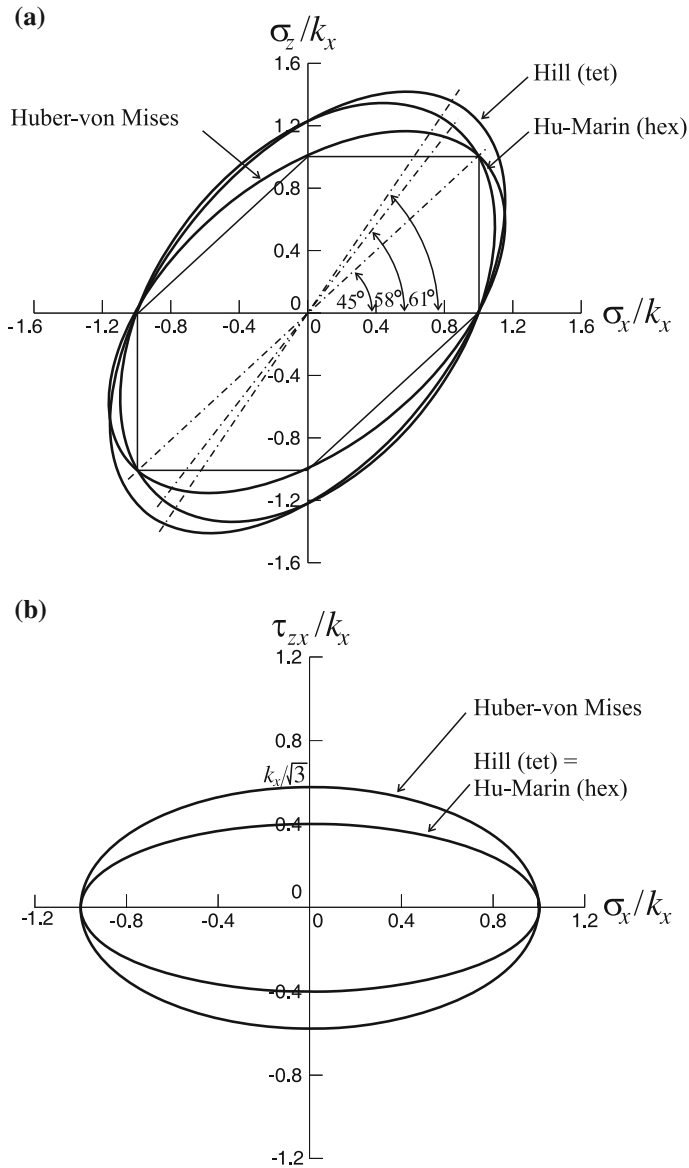


Fig. 5.18 Comparison of transversely isotropic criteria in the orthotropy plane: Hill’s tetragonal (5.82), Hu–Marín’s hexagonal (5.90), and Huber–von Mises for given magnitudes of orthotropy ratios: $R = 2$, $k_{zx}/k_x = 0.8$, $k_{(xy)}/k_x = 0.9$, in case of 2D states of stress: **a** biaxial normal stresses (σ_x , σ_z) and **b** combined normal with shear stresses (σ_x , τ_{zx}) (after Ganczarski and Skrzypek [18])

Introducing for $k_1, k_2, k_4,$ and k_6 the following substitution $\frac{2}{9}k_1^2 = \frac{1}{k_z^2}, \frac{2}{9}k_2^2 = \frac{1}{k_x^2}$ $\frac{2}{3}(k_1k_2 + k_4^2) = \frac{1}{k_{zx}^2}, \frac{2}{3}(k_2^2 + k_6^2) = \frac{1}{k_{xy}^2}$ we end up with format of the *Voyiadjis and Thiagarajan condition* analogous to (5.92) however 4-parameter, where not only $k_x, k_z,$ and k_{zx} but additionally k_{xy} are considered as independent (see doubly underlined terms in (5.90) and (5.94))

$$\frac{\sigma_x^2 + \sigma_y^2}{k_x^2} - \frac{\sigma_x\sigma_y}{k_x^2} + \frac{\sigma_z^2}{k_z^2} - \frac{\sigma_y\sigma_z + \sigma_z\sigma_x}{k_zk_x} + \frac{\tau_{yz}^2 + \tau_{zx}^2}{k_{zx}^2} + \frac{\tau_{xy}^2}{k_{xy}^2} = 1 \quad (5.94)$$

Such criterion is irreducible to the isotropic von Mises type in the plane of isotropy, but it fits the experimental data for *Boron–Aluminum composite* tubular specimen having unidirectional lamina (Dvorak et al. [14] and Nigam et al. [45]).

Both *transversely isotropic criteria: Hill-type of tetragonal symmetry* (5.81) as well as *Hu–Marin-type of hexagonal symmetry* (5.90) describe cylindrical surfaces in space of principal stresses. However, Hill's type limit surface represents elliptical cylinder, the axis of which coincides with the hydrostatic axis, in contrast to enhanced Hu–Marin-type limit surface that represents elliptic cylinder, the axis of which forms a proportional stress/strength axis, different from the hydrostatic axis. It means that enhanced Hu–Marin's condition does not satisfy the deviatoricity property, which is a price for property of reducibility to the Huber–von Mises condition in the isotropy plane, with cylindricity ensured regardless of the magnitude of orthotropy degree.

A choice of appropriate transversely isotropic limit criterion, of either the tetragonal symmetry (5.81) or the hexagonal symmetry (5.90), depends on coincidence with experimental findings for real material. This may often lead to one of the two above considered symmetry classes, but sometimes material limit response is different even from both of them. Note that the shape of limit curves in the transverse isotropy plane is the key to appropriate classification of real transversely isotropic material as exhibiting *tetragonal symmetry* or *hexagonal* or *mixed symmetry* properties.

5.6 Hybrid Formulation of Enhanced Hu–Marin-Type Condition

In what follows a description of new limit criterion of the *hybrid symmetry property* between the tetragonal (5.81) or (5.82) and the hexagonal (5.90) or (5.91) symmetry classes, is proposed. The Hu–Marin type equation of pure hexagonal symmetry property (5.90) or (5.91) comprises three independent material constants $k_x, k_z,$ and k_{zx} . However, real engineering materials of hybrid-type nature are frequently characterized by four independent material constants determined from four tests: two limits in uniaxial tensions k_x and $k_z,$ shear limit in orthotropy plane k_{zx} (5.69) and additionally, in the *biaxial tension* test (bulge test) $k_{(xy)}$ instead of the first of condition

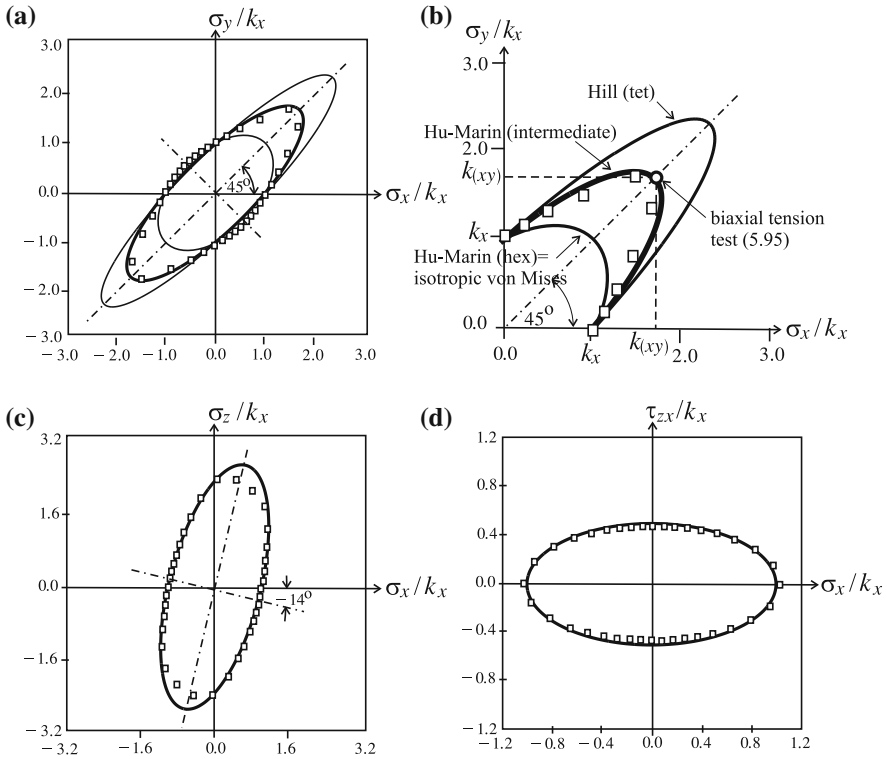


Fig. 5.19 Fitting of the initial yield surface of unidirectional SiC/Ti composite according to Herakovich and Aboudi findings [23] (symbol \square) by the use of transversely isotropic Hu–Marin’s hybrid-type criterion (5.90): **a, b** transverse isotropy plane (σ_x, σ_y) , **c** orthotropy plane (σ_x, σ_z) , **d** orthotropy shear plane (σ_x, τ_{zx}) (after Ganczarski and Skrzypek [18])

(5.70₁), namely

$$\sigma_x = \sigma_y = k_{(xy)} \sigma_z = \tau_{xy} = \tau_{yz} = \tau_{zx} = 0 \longrightarrow \Pi_{12} = -\frac{1}{2k_{(xy)}^2} \quad (5.95)$$

The above condition leads to the hybrid formulation of enhanced Hu–Marin’s condition

$$\frac{\sigma_x^2 + \sigma_y^2}{k_x^2} - \frac{\sigma_x \sigma_y}{\underline{k_{(xy)}^2}} + \frac{\sigma_z^2}{k_z^2} - \frac{\sigma_y \sigma_z + \sigma_z \sigma_x}{k_z k_x} + \frac{\tau_{yz}^2 + \tau_{zx}^2}{k_{zx}^2} + 3 \frac{\tau_{xy}^2}{k_x^2} = 1 \quad (5.96)$$

Equation (5.96) differs from the hexagonal form of Hu–Marin’s condition (5.90) in the underlined term, where the fourth independent material constant $k_{(xy)}$ is taken from the *bulge test* (5.95), additionally to conditions (5.89_{2,3}). The hybrid formulation of 4-parameter *transversely isotropic Hu–Marin’s* condition (5.96) has matrix

representation as follows:

$$[\text{hybr} \text{III}^{\text{HM}}] = \left[\begin{array}{ccc|cc} \frac{1}{k_x^2} & -\frac{1}{2k_{(xy)}^2} & -\frac{1}{2k_x k_z} & & \\ & \frac{1}{k_x^2} & -\frac{1}{2k_x k_z} & & \\ & & \frac{1}{k_z^2} & & \\ \hline & & & \frac{1}{k_{zx}^2} & \\ & & & & \frac{1}{k_{zx}^2} \\ & & & & \frac{3}{k_x^2} \end{array} \right] \left[\begin{array}{c} \bullet \\ \bullet \\ \bullet \\ \circ \\ \bullet \\ \bullet \\ \bullet \\ \bullet \\ \circ \end{array} \right] \quad (5.97)$$

It is illustrated in Fig. 5.19a–d for the *SiC/Ti long fiber reinforced composite* by the use of thick solid line.

The hybrid-type enhanced Hu–Marin criterion is capable of capturing behavior of some *long fiber reinforced composite materials*, that in the transverse isotropy plane exhibit limit response different from both the Hill and the Huber–von Mises materials (cf., e.g., Herakovich and Aboudi [23]).

5.7 Comparison of Four Selected Transversely Isotropic Yield Criteria

Transition from the orthotropic yield criterion (von Mises or Hill) to the *transverse isotropy* is connected with the reduction of independent plastic modules (von Mises 9 → 6, Hill 6 → 4). These independent modules have to be identified by the use of appropriate number of tests and constraints. At present section the detailed discussion of the four selected yield criteria from Sect. 5.5 is performed. To this end we invoke following selected yield criteria, two of them based on the Hill origin and the other two based on the von Mises origin

$$\begin{aligned}
 & \frac{\sigma_x^2 + \sigma_y^2}{k_x^2} + \frac{\sigma_z^2}{k_z^2} - \left(\frac{2}{k_x^2} - \frac{1}{k_z^2} \right) \sigma_x \sigma_y - \frac{\sigma_y \sigma_z + \sigma_z \sigma_x}{k_z^2} \\
 & \qquad \qquad \qquad + \frac{\tau_{yz}^2 + \tau_{zx}^2}{k_{zx}^2} + \frac{\tau_{xy}^2}{k_{xy}^2} = 1 \quad (5.98)
 \end{aligned}$$

$$\begin{aligned}
 & \frac{\sigma_x^2 + \sigma_y^2}{k_x^2} + \frac{\sigma_z^2}{k_z^2} - \left(\frac{2}{k_x^2} - \frac{1}{k_z^2} \right) \sigma_x \sigma_y - \frac{\sigma_y \sigma_z + \sigma_z \sigma_x}{k_z^2} \\
 & \qquad \qquad \qquad + \frac{\tau_{yz}^2 + \tau_{zx}^2}{k_{zx}^2} + \left(\frac{4}{k_x^2} - \frac{1}{k_z^2} \right) \tau_{xy}^2 = 1
 \end{aligned}$$

$$\frac{\sigma_x^2 + \sigma_y^2}{k_x^2} + \frac{\sigma_z^2}{k_z^2} - \frac{\sigma_x \sigma_y}{k_x^2} - \frac{\sigma_y \sigma_z + \sigma_z \sigma_x}{k_z k_x} + \frac{\tau_{yz}^2 + \tau_{zx}^2}{k_{zx}^2} + 3 \frac{\tau_{xy}^2}{k_x^2} = 1$$

$$\frac{\sigma_x^2 + \sigma_y^2}{k_x^2} + \frac{\sigma_z^2}{k_z^2} - \frac{\sigma_x \sigma_y}{k_{(xy)}^2} - \frac{\sigma_y \sigma_z + \sigma_z \sigma_x}{k_z k_x} + \frac{\tau_{yz}^2 + \tau_{zx}^2}{k_{zx}^2} + 3 \frac{\tau_{xy}^2}{k_x^2} = 1$$

The above criteria have been derived in terms of different combinations of engineering modules based on appropriate identification procedures. All four criteria under consideration involve three common tests:

$$\sigma_x = k_x \quad \sigma_z = k_z \quad \tau_{zx} = k_{zx} \quad (5.99)$$

Additional conditions necessary for full identification take different forms.

In case of the classical Hill criterion (5.98₁) the additional fourth condition holds:

$$\tau_{xy} = k_{xy} \quad (5.100)$$

Aforementioned *Hill's* criterion contains four independent parameters and can be classified as *tetragonal symmetry form* (see Table 1.4d).

The second formulation (5.98₂) is also based on Hill's criterion however the additional constraint is imposed on the Π_{66} modulus (see Chen and Han [9])

$$\Pi_{66} = -2(\Pi_{13} + 2\Pi_{12}) = \frac{4}{k_x^2} - \frac{1}{k_z^2} \quad (5.101)$$

such that number of independent parameters is reduced to three k_x , k_z , and k_{zx} as a consequence this *Hill's criterion* exhibits property of *hexagonal symmetry*. However, it is irreducible to the Huber–von Mises criterion in transverse isotropy plane.

The third formulation (5.98₃) inherits the von Mises format hence, if reduction to transverse isotropy is performed, it requires identification of six plastic modules in terms of three independent plastic limits k_x , k_z , and k_{zx} . Therefore, except from three common conditions (5.99) the following additional three must be formulated

$$\sigma_x = \sigma_y = k_x \quad \sigma_x = \sigma_y = k_x \wedge \sigma_z = k_z \quad \tau_{xy} = k_x / \sqrt{3} \quad (5.102)$$

In other words, two biaxial conditions in planes (xy) and (xz) hold and additional condition imposed on Π_{66} exhibits Huber–von Mises reducibility property

$$\Pi_{66} = \frac{3}{k_x^2} \quad (5.103)$$

The fourth formulation (5.98₄) is of specific nature, namely it is based on the von Mises–Hu–Marin criterion and requires the following three additional conditions

$$\sigma_x = \sigma_y = k_{(xy)} \quad \sigma_x = \sigma_y = k_x \wedge \sigma_z = k_z \quad \tau_{xy} = k_x/\sqrt{3} \quad (5.104)$$

Hence, (5.104₁) essentially differs from (5.102₁) since it involves the new independent constant $k_{(xy)}$ established from the *bulge test* in the transverse isotropy plane (see Jackson et al. [33]).

In conclusion it is clear that only the *classical Hill* condition is of *tetragonal symmetry* whereas all three other proposals discussed above have to be classified as *hexagonal symmetry types* even though the reasons of reduction of independent parameters (4 or 3) are of different nature.

5.8 Implicit Formulation of Pressure Insensitive Anisotropic Initial Yield Criteria—Barlat’s and Khan’s Concepts

In this section another approach (implicit formulation) is discussed based on a series of papers developed by Barlat, Planckett, Cazacu, and Khan to mention some names only. The implicit formulation involves the linear transformation of the Cauchy stress tensor σ to the transformed stress $\Sigma = \mathbb{L} : \sigma$ by the use of transformation tensor \mathbb{L} responsible for orthotropy. Such *linear transformation concept* of the stress tensor was first introduced by Sobotka [55] and Boehler and Sawczuk [6]

$$\widehat{\sigma}_{ij} = A_{ijkl}\sigma_{kl} \quad (5.105)$$

where A_{ijkl} stands for a certain dimensionless tensor of anisotropy that satisfies general symmetry conditions $A_{ijkl} = A_{jikl} = A_{ijlk} = A_{klij}$ and the well-known isotropic yield conditions to hold for anisotropic materials as well if σ_{ij} are replaced by $\widehat{\sigma}_{ij}$. This approach is not directly based on the theory of *common invariants* in the sense of Sayir, Goldenblat, Kopnov, Spencer, Boehler, Betten etc. formalism (explicit formulation). According to this *implicit approach* an extension of isotropic initial yield/failure criteria is performed to account for the *tension/compression asymmetry* property and to *material anisotropy frame* (usually orthotropy) by applying the linear transformation to the stress tensor and inserting this transformed stress tensor into the originally isotropic yield/failure criteria.

In the paper by Cazacu et al. [8] authors consider both the isotropic yield criterion for description of *asymmetric yielding*

$$f(J_{2s}, J_{3s}) = (|s_1| - \widehat{k}s_1)^a + (|s_2| - \widehat{k}s_2)^a + (|s_3| - \widehat{k}s_3)^a = 2k^a$$

$$\widehat{k} = \frac{1 - h(\frac{k_t}{k_c})}{1 + h(\frac{k_t}{k_c})} \quad h\left(\frac{k_t}{k_c}\right) = \left[\frac{2^a - 2(\frac{k_t}{k_c})^a}{(2\frac{k_t}{k_c})^a - 2} \right]^{1/a} \quad (5.106)$$

where s_i , $i = 1, \dots, 3$ are the principal values of the stress deviator and f gives the size of the yield locus (isotropic hardening), as well as its extension to include orthotropy by the use of linear transformation of the Cauchy stress deviator $\Sigma = \mathbb{C} : s$ through

$$\mathbb{C} = \left[\begin{array}{ccc|ccc} C_{11} & C_{12} & C_{13} & & & \\ C_{12} & C_{22} & C_{23} & & & \\ C_{13} & C_{23} & C_{33} & & & \\ \hline & & & C_{44} & & \\ & & & & C_{55} & \\ & & & & & C_{66} \end{array} \right] \quad (5.107)$$

which lead to following anisotropic equation

$$(|\Sigma_1| - \widehat{k}\Sigma_1)^a + (|\Sigma_2| - \widehat{k}\Sigma_2)^a + (|\Sigma_3| - \widehat{k}\Sigma_3)^a = 2k^a \quad (5.108)$$

Authors proved convexity of the isotropic yield form (5.106) as well as pressure insensitivity of its orthotropic form (5.108) obtained through the linear transformation to the transformed stress frame. However, the question of convexity of the orthotropic form (5.108) remains open in the light of discussion performed for Hill's (Fig. 5.11) and Hosford's (Fig. 5.15) extensions in case of a highly orthotropic materials.

The proposed yield function appears to be suitable for description of the strong asymmetry and anisotropy observed in *textured Mg-Th* and *Mg-Li* binary alloy sheets and for *titanium 4Al-1/4O₂*, see Cazacu et al. [8]. The orthotropic yield criterion proposed by Cazacu et al. [8] was also investigated in a series of multiaxial loading experiments on *Ti-6Al-4V titanium alloy* by Khan et al. [34].

Extension of Drucker's isotropic yield criterion (5.15) to anisotropy by use of common invariants J_2^0 and J_3^0 is due to Cazacu and Barlat [7], and investigated by Yoshida et al. [64]

$$(J_2^0)^{3/2} - cJ_3^0 - k^3 = 0 \quad (5.109)$$

The constant c in the Eq. (5.109) accounts for the *tension/compression asymmetry* defined as

$$c = \frac{3\sqrt{3}(k_t^3 - k_c^3)}{2(k_t^3 + k_c^3)} \quad (5.110)$$

and belongs to two ranges

$$c \in \begin{cases} \left(0, \frac{3\sqrt{3}}{2}\right) & \text{for } k_t > k_c > 0 \\ \left(-\frac{3\sqrt{3}}{2}, 0\right) & \text{for } 0 < k_t < k_c \end{cases} \quad (5.111)$$

The second and third *common invariants of orthotropy* are defined as

$$\begin{aligned} J_2^0 &= \frac{1}{6} [a_1(\sigma_x - \sigma_y)^2 + a_2(\sigma_y - \sigma_z)^2 + a_3(\sigma_z - \sigma_x)^2] \\ &\quad + a_4\tau_{xy}^2 + a_5\tau_{xz}^2 + a_6\tau_{zy}^2 \\ J_3^0 &= \frac{1}{27} \left\{ (b_1 + b_2)\sigma_x^3 + (b_3 + b_4)\sigma_y^3 + [2(b_1 + b_4) - b_2 - b_3]\sigma_z^3 \right\} \\ &\quad + 2b_{11}\tau_{xy}\tau_{yz}\tau_{zx} + \frac{1}{9} \left\{ 2(b_1 + b_2)\sigma_x\sigma_y\sigma_z - (b_1\sigma_y + b_2\sigma_z)\sigma_x^2 \right. \\ &\quad \left. - (b_3\sigma_z + b_2\sigma_x)\sigma_y^2 - [(b_1 - b_2 + b_4)\sigma_x + (b_1 + b_3 + b_4)\sigma_y]\sigma_z^2 \right\} \\ &\quad - \frac{1}{3} \left\{ \tau_{yz}^2[(b_6 + b_7)\sigma_x - b_6\sigma_y - b_7\sigma_z] \right. \\ &\quad \left. - \tau_{zx}^2[2b_9\sigma_y - b_8\sigma_z - (2b_9 - b_8)\sigma_x] \right. \\ &\quad \left. - \tau_{xy}^2[2b_{10}\sigma_z - b_5\sigma_y - (2b_{10} - b_5)\sigma_x] \right\} \end{aligned} \quad (5.112)$$

The discussed anisotropic criterion was successfully verified for textured *magnesium Mg-Th* and *Mg-Li alloy sheets*. Authors proved convexity of the *enhanced isotropic yield criterion* only for $c(k_t/k_c)$ belonging to the range $[-\frac{3\sqrt{3}}{2}, -\frac{3\sqrt{3}}{2}]$. In case of the anisotropic form of *Cazacu and Barlat's criterion* (5.109) the general proof of convexity for the wide class of highly tension/compression asymmetric and anisotropic materials may not be possible.

More complete representation of J_2^0 and J_3^0 common invariants as well as the extended model (5.109) verification for high-purity α -*titanium* is done by Nixon et al. [46].

Korkolis and Kyriakides [38] applied *anisotropic extension of Hosford's isotropic criterion* (5.10) in terms of principal stress deviator s_1, s_2 in case of plane stress state

$$|s_1 - s_2|^n + |2s_1 + s_2|^n + |s_1 + 2s_2|^n = 2k^n \quad (5.113)$$

Folowing Barlat et al. [3] they introduced anisotropy by use of a concept of two linear transformations $\mathbf{S}' = \mathbb{L}' : \mathbf{s}$ and $\mathbf{S}'' = \mathbb{L}'' : \mathbf{s}$ where \mathbb{L}' and \mathbb{L}'' are transformation tensors introducing anisotropy

$$|S'_1 - S'_2|^n + |2S''_1 + S''_2|^n + |S''_1 + 2S''_2|^n = 2k^n \quad (5.114)$$

Experimental validation of (5.114) is due to Korkolis and Kyriakides [38] applied to *Al-6260-T4* as well as due to Dunand et al. [13], Luo et al. [41] applied to *AA6260-T6 alloys* under classical tensile and butterfly shear tests.

Comparison of two different approaches: explicit formulation based on common invariants and *implicit formulation* composed as extension of isotropic criteria to anisotropy and tension/compression asymmetry leads to the following characteristic features.

The implicit formulation is very advantageous and fruitful in order to build numerical models able to capture experimental evidence for broad class of innovative metallic materials (mainly metal-based alloys) that simultaneously exhibit *tension/compression asymmetry*, *anisotropy*, and *hydrostatic pressure insensitivity*. Apart from these advantages some open questions may be highlighted. Among them there might be mentioned not obvious physical interpretation for the extended criteria based on known isotropic forms enhanced through strength differential sensitivity and orthotropic linear transformation of stress. The general proof of convexity is rather cumbersome and not attached in a complete and convinced form. Although the isotropic equations are understandable, have physical interpretations, and satisfy convexity requirements the transposition of these equations to the transformed stress frame may lead to the loss of convexity.

By contrast use of the explicit approach based on well-established theory of common invariants is more rigorous and so leads to more clear physical interpretation (energy) and convexity of quadratic or poly-quadratic forms. However, this consistent approach leads to major difficulties when numerical implementation and experimental validation are considered. Additional difficulties arise when implementing the explicit approach to more general cases if the material orthotropy frame does not coincide with the principal stress frame. Such more general problem was discussed by Ganczarski and Lenczowski [15] in case of Hill's and orthotropic Hosford's criteria. In such a case it is necessary to transform tensor of structural orthotropy to the frame of principal stress resulting in a possible loss of convexity and even degeneration of an initially closed surface into twofold surface (nonclosed).

5.9 Brief Survey of Commonly Used Pressure Insensitive Isotropic and Anisotropic Initial Yield Criteria

In this section a brief survey of the selected commonly used pressure insensitive initial yield criteria is presented. The survey is focused on following two aspects:

- isotropic versus anisotropic formulation,
- direct versus indirect dependence on the stress invariants or the common invariants.

Special attention is paid for invariant representation of invoked limit criteria. Chosen *isotropic yield criteria* are collected in Table 5.3. All cited criteria depend on the second deviatoric invariant and additionally they may depend on the third deviatoric invariant. Criteria A1, A2, and A3 are written down in the format directly

Table 5.3 Survey of pressure insensitive isotropic yield criteria

A.	Author(s)	Limit criterion
A1	Raniecki and Mróz [48] Eq. (5.13)	$(J_{2s})^{3n/2} - c(J_{3s})^n = k^{3n}$
A2	Cazacu and Barlat [7] Eq. (5.14)	$(J_{2s})^{3/2} - cJ_{3s} = k^3$
A3	Drucker [12] Eq. (5.15)	$(J_{2s})^3 - c(J_{3s})^2 = k^6$
A4	Huber [31], von Mises [43] Eq. (5.4)	$3J_{2s} = k^2$
A5	Tresca [60], Guest [20] Eq. (5.8) Reuss [49] Eq. (5.9)	$\max(\sigma_1 - \sigma_2 , \sigma_2 - \sigma_3 , \sigma_3 - \sigma_1) = k$ $4J_{2s}^3 - 27J_{3s}^2 - 9k^2J_{2s}^2 + 6k^4J_{2s} = k^6$ $ \sigma_1 - \sigma_2 + \sigma_2 - \sigma_3 + \sigma_3 - \sigma_1 = 2k$
A6	Schmidt [53], Ishlinsky [32] and Hill [26] Eq. (5.11)	$\max(\sigma_1 - \sigma_h , \sigma_2 - \sigma_h , \sigma_3 - \sigma_h) = \frac{2}{3}k$
A7	Hershey [24], Davies [11] and Hosford [29] Eq. (5.10)	$ \sigma_1 - \sigma_2 ^m + \sigma_2 - \sigma_3 ^m + \sigma_3 - \sigma_1 ^m = 2k^m$
A8	Cazacu et al. [8] Eq. (5.106)	$(s_1 - \widehat{k}s_1)^a + (s_2 - \widehat{k}s_2)^a + (s_3 - \widehat{k}s_3)^a = 2k^a$

dependent on both invariants. The existence of the third invariant being argument of different power functions enables to capture various asymmetry of the initial yield curve in the deviatoric plane. The particular case of aforementioned *Drucker-like criteria* when $c = 0$ is the classical *Huber–von Mises criterion* A4 in which influence of the third stress invariant is ignored. Another classical *Tresca’s criterion* A5 is written down in the three equivalent formats: the form suggested by Tresca [60] and experimentally validated by Guest [20], explicitly invariant Reuss’ form and the Cazacu and Barlat [7] form being a particular case of *Hosford’s criterion* A7 when $m = 1$. The Tresca criterion represents the regular hexagonal prism in the Haigh–Westergaard space inscribed into the Huber–von Mises circular cylinder (see Fig. 5.4). The *maximal deviatoric stress-based criterion* A6 formulated by Schmidt [53], Ishlinsky [32], and Hill [25] also represents the regular hexagonal prism in the Haigh–Westergaard space, however circumscribed onto the Huber–von Mises circular cylinder (see Fig. 5.4). The Tresca and Schmidt–Ishlinsky–Hill criteria are useful as the *inner and outer bounds* for all isotropic third stress invariant insensitive criteria, however the existence of corners on initial yield surfaces is physically questionable because the uniqueness of plastic strain increment is lost (see Fig. 5.5). The direct generalization of the Tresca criterion A5 by the use of power form that eliminates corners is due to Hershey [24], Davies [11], and Hosford [27]. The exponent m that ensures convexity has to be taken from the range $1 \leq m < \infty$, see Fig. 5.6. The Tresca-like criteria A5, A6, and A7 do not account for the tension/compression asymmetry effect. Another original criterion proposed by Cazacu et al. [8] A8 is relevant to the Drucker criterion A3 in such a sense that it is a homogeneous function of degree a in stresses, the cross section of which represents a “triangle” with rounded corners, see Cazacu et al. [8]. The *strength differential effect* is included and controlled by a parameter $\widehat{k}(\frac{k_t}{k_c})$. The existence of absolute values in the criterion proposed results from a reversible shear mechanisms such as slip, since yielding depends only on the magnitude but not direction of the shear stress, yield criterion

$f(s) = f(-s)$. Other yield criteria accounting for different representations of the second and the third invariants due to Sayir that exhibit discrete 120°-symmetry are discussed by Altenbach et al. [1].

Chosen *anisotropic yield criteria* are collected in Table 5.4. In the item B1 two examples of implementation of *implicit anisotropic extension of the isotropic Drucker yield criterion* (dependent on the second and the third deviatoric stress invariants) referring to works by Cazacu and Barlat [7] and Nixon et al. [46] are presented. The original notation used by the authors is given in Table 5.3. By contrast to original notation in item B1 of Table 5.4 the criterion is rewritten in a frame of transformed stress $\Sigma = \mathbb{L} : \sigma$ instead of the Cauchy stress frame σ . Due to this concept the second J_2^0 and the third J_3^0 *transformed invariants* are expressed in terms of only one fourth-rank *transformation tensor* \mathbb{L} instead of the second-rank $s : \overset{<4>}{\text{dev}}\mathbb{I} : s$ and the third-rank common invariants $s : \overset{<6>}{\text{dev}}\mathbb{I} : s : s$ necessary to be implemented when the *Goldenblat–Kopnov explicit formulation* would be used. The discussed implicit formulation shows essential reduction of the number of material constants that have to be identified in order to capture experimental data (see discussion in Sect. 5.2). Note that the transformation tensor \mathbb{L} exhibits format of the Hill orthotropy matrix however it is dimensionless. When comparing items B2 and B3 corresponding to the deviatoric von Mises criterion (5.43) written in the form suggested by Szczepiński [58] and to the Hill criterion (5.53) [25, 26] different population of corresponding plastic matrices is applied. In case of Hill’s format the terms which are sensitive to

Table 5.4 Survey of pressure insensitive anisotropic yield criteria

B.	Author(s)	Limit criterion
B1	Cazacu and Barlat [7] and Nixon et al. [46] Eq. (5.109)	$\left\{ \frac{1}{2} \text{tr} [(\mathbb{L} : \sigma) \cdot (\mathbb{L} : \sigma)] \right\}^{3/2} - c \frac{1}{3} \text{tr} [(\mathbb{L} : \sigma) \cdot (\mathbb{L} : \sigma) \cdot (\mathbb{L} : \sigma)] = k^3$
B2	Szczepiński [58] Eq. (5.43)	$s : \overset{<4>}{\text{dev}}\mathbb{I} : s = 1$
B3	Hill [25, 26] Eq. (5.53)	$s : \overset{<4>}{\mathbb{I}}^H : s = 1$
B4	Voyiadjis and Thiagarajan [62] Eq. (5.94)	$\sigma : \overset{<4>}{\text{tris}}\mathbb{I}^{\text{VT}} : \sigma = 1$
B5	Skrzypek and Ganczarski [18, 54] Eq. (5.90)	$\sigma : \overset{<4>}{\text{tris}}\mathbb{I}^{\text{HM}} : \sigma = 1$
B6	Ganczarski and Skrzypek [18] Eq. (5.96)	$\sigma : \overset{\text{hybr}}{\text{tris}}\mathbb{I} : \sigma = 1$
B7	Cazacu et al. [8] and Khan et al. [34] Eq. (5.108)	$(\Sigma_1 - \hat{k}\Sigma_1)^a + (\Sigma_2 - \hat{k}\Sigma_2)^a + (\Sigma_3 - \hat{k}\Sigma_3)^a = 2k^a$
B8	Ganczarski and Lenczowski [15] Eq. (5.76)	$a_1 \sigma_y - \sigma_z ^m + a_2 \sigma_z - \sigma_x ^m + a_3 \sigma_x - \sigma_y ^m + a_4 \tau_{yz} ^m + a_5 \tau_{zx} ^m + a_6 \tau_{xy} ^m = 1$
B9	Korkolis and Kyriakides [38] Eq. (5.114)	$ S'_1 - S'_2 ^n + 2S''_1 + S''_2 ^n + S'_1 + 2S''_2 ^n = 2k^n$

change of sign of shear stresses, for instance $\tau_{yz}(s_y - s_z), \dots, \tau_{yz}\tau_{zx}, \dots$ are omitted. It is equivalent to the reduction of a number of independent plastic modules from 15 to 6.

Items B4, B5, and B6 refer to the *transversely isotropic criteria* of initial yield/failure in unidirectionally reinforced Boron–Aluminum fibrous composites. Voyiadjis and Thiagarajan [62] used generally *transversely isotropic tetragonal symmetry* form of the yield criterion. However the experimental data used for calibration based on Dvorak et al. [14] and Nigam et al. [45] were limited to narrower case in which only plane stress state in the orthotropy plane was considered without distinction between the tetragonal and hexagonal symmetries. All three formulas B4, B5, and B6 describe cylindrical limit surfaces in stress space, the axis of which does not coincide with the hydrostatic axis.

The key difference between the Voyiadjis and Thiagarajan formulation B4 and the Skrzypek and Ganczarski approach B5 both related to the transversely isotropic materials, lies in the format of doubly underlined terms in Eqs. (5.94) and (5.90), respectively. Namely when Eq. (5.94) is used the fourth constant k_{xy} is independent and determined from experiment, whereas in Eq. (5.90) the fourth constant is dependent and equals to $k_{xy} = \frac{k_x}{\sqrt{3}}$. In other words, the *Voyiadjis and Thiagarajan criterion* (5.94) is irreducible to the Huber–von Mises criterion in the transverse isotropy plane, whereas the *Skrzypek and Ganczarski criterion* is reducible. This means that the Voyiadjis and Thiagarajan criterion possesses *tetragonal symmetry* whereas the Skrzypek and Ganczarski criterion exhibits *hexagonal symmetry*.

The full reducibility requirement in Eq. (5.90) may occur too restrictive when some composite materials are experimentally tested. In such a case the *hybrid formulation* (5.96) is proposed where $k_{(xy)}$ taken from the bulge test in the transverse isotropy plane is independent leading to 4-parameter *tetragonal format* (k_x, k_z, k_{zx} , and $k_{(xy)}$). The considered criteria B4, B5, and B6 are in fact secondary *pressure sensitive*, however this *sensitivity* property is *inquired* due to preserved cylindricity. The property of cylindricity is predominant and justifies the appearance of criteria B4, B5, and B6 in this section.

To describe both the *asymmetry* between *tension and compression* and the anisotropy observed in hexagonal closed packed metal sheets, Cazacu et al. [8] and Khan et al. [34] proposed extension of isotropic criterion (5.106) to the case of orthotropy represented by item B7. It consists in application of fourth-order *linear transformation operator* on the Cauchy stress tensor expressed by its principal values. The proposed anisotropic criterion was successively applied to the description of the anisotropy and asymmetry of the yield loci of textured *polycrystalline magnesium* and binary *Mg–Th, Mg–Li alloys* and α *titanium*.

Orthotropic generalization of the Hosford criterion (item A7) in which principal axes of material orthotropy do not coincide with principal stress axes was proposed by Ganczarski and Lenczowski [15] in the form of item B8. Next the convexity check of the yield condition was performed in case of the brass sheet of Russian commercial symbol Ł22, that is material of strong orthotropy slightly different from transverse isotropy.

The last criterion (item B9) is another anisotropic *generalization of Hosford's isotropic criterion* (item A7) done by Korkolis and Kyriakides [38] and addressed to Al-6260-T4 tubes inflated under combined internal pressure and axial load.

References

1. Altenbach, H., Bolchoun, A., Kolupaev, V.A.: Phenomenological yield and failure criteria. In: Altenbach, H., Öchsner, A. (eds.) *Plasticity of Pressure-Sensitive Materials*. Springer, Heidelberg (2014)
2. Barlat, F., Lian, J.: Plastic behavior and stretchability of sheet metals. *Int. J. Plast.* **5**(1), 51 (1989)
3. Barlat, F., Brem, J.C., Yoon, J.W., Chung, K., Dick, R.E., Lege, D.J., Pourboghrat, F., Choi, S.-H., Chu, E.: Plane stress function for aluminium alloy sheets—part I: theory. *Int. J. Plast.* **19**, 1297–1319 (2003)
4. Berryman, J.G.: Bounds and self-consistent estimates for elastic constants of random polycrystals with hexagonal, trigonal, and tetragonal symmetries. *J. Mech. Phys. Solids* **53**, 2141–2173 (2005)
5. Betten, J.: Applications of tensor functions to the formulation of yield criteria for anisotropic materials. *Int. J. Plast.* **4**, 29–46 (1988)
6. Boehler, J.P., Sawczuk, A.: Equilibre limite des sols anisotropes. *J. Mécanique* **9**, 5–33 (1970)
7. Cazacu, O., Barlat, F.: A criterion for description of anisotropy and yield differential effects in pressure-insensitive materials. *Int. J. Plast.* **20**, 2027–2045 (2004)
8. Cazacu, O., Planckett, B., Barlat, F.: Orthotropic yield criterion for hexagonal close packed metals. *Int. J. Plast.* **22**, 1171–1194 (2006)
9. Chen, W.F., Han, D.J.: *Plasticity for Structural Engineers*. Springer, Berlin (1995)
10. Chu, E.: Generalization of Hill's 1979 anisotropic yield criteria. In: *Proceedings of the NUMISHEETS'89*, pp. 199–208 (1989)
11. Davies, E.A.: The Bailey flow rule and associated yield surface. *Trans. ASME* **E28**(2), 310 (1961)
12. Drucker, D.C.: Relation of experiments to mathematical theories of plasticity. *J. Appl. Mech.* **16**, 349–357 (1949)
13. Dunand, M., Maertens, A.P., Luo, M., Mohr, D.: Experiments and modeling of anisotropic aluminum extrusions under multi-axial loading—part I: plasticity. *Int. J. Plast.* **36**, 34–49 (2012)
14. Dvorak, G.J., Bahei-El-Din, Y.A., Macheret, Y., Liu, C.H.: An experimental study of elastic-plastic behavior of a fibrous boron-aluminum composite. *Int. J. Mech. Phys. Solids* **36**, 655–687 (1988)
15. Ganczarski, A., Lenczowski, J.: On the convexity of the Goldenblatt-Kopnov yield condition. *Arch. Mech.* **49**(3), 461–475 (1997)
16. Ganczarski, A., Skrzypek, J.: Modeling of limit surfaces for transversely isotropic composite SCS-6/Ti-15-3. *Acta Mechanica et Automatica* **5**(3), 24–30 (2011) (in Polish)
17. Ganczarski, A., Skrzypek, J.: *Mechanics of Novel Materials* (in Polish). Wydawnictwo Politechniki Krakowskiej, Kraków (2013)
18. Ganczarski, A., Skrzypek, J.: Constraints on the applicability range of Hill's criterion: strong orthotropy or transverse isotropy. *Acta Mech.* **225**, 2568–2582 (2014)
19. Goldenblat, I.I., Kopnov, V.A.: *Obobshchennaya teoriya plasticheskogo techeniya anizotropnyh sred*, pp. 307–319. *Sbornik Stroitel'naya Mehanika*, Stroït'zdat, Moskva (1966)
20. Guest, J.J.: On the strength of ductile materials under combined stress. *Philos. Mag.* **50**, 69–132 (1900)
21. Haigh, B.F.: The strain-energy function and the elastic limit. *Eng. Lond.* **109**, 158–160 (1920)
22. Hencky, H.: Zur Theorie plastischer Deformationen und der hierdurch im Material hervorgerufenen Nach-Spannungen. *ZAMM* **4**, 323–334 (1924)

23. Herakovich, C.T., Aboudi, J.: Thermal effects in composites. In: Hetnarski, R.B. (ed.) *Thermal Stresses V*, pp. 1–142. Lastran Corporation Publishing Division, Rochester (1999)
24. Hershey, A.V.: The plasticity of an isotropic aggregate of anisotropic face-centred cubic crystals. *J. Appl. Mech.* **21**(3), 241–249 (1954)
25. Hill, R.: A theory of the yielding and plastic flow of anisotropic metals. *Proc. R. Soc. Lond.* **A193**, 281–297 (1948)
26. Hill, R.: *The Mathematical Theory of Plasticity*. Oxford University Press, Oxford (1950)
27. Hosford, W.F., Backhofen, W.A.: Strength and plasticity of textured metals. In: Backhofen, W.A., Burke, J., Coffin, L., Reed, N., Weisse, V. (eds.) *Fundamentals of Deformation Processing*, pp. 259–298. Syracuse University Press, Syracuse (1964)
28. Hosford, W.F.: Texture Strengthening. *Met. Eng. Q.* **6**, 13–19 (1966)
29. Hosford, W.F.: A generalized isotropic yield criterion. *Trans. ASME* **E39**(2), 607–609 (1972)
30. Hu, Z.W., Marin, J.: Anisotropic loading functions for combined stresses in the plastic range. *J. Appl. Mech.* **22**, 1 (1956)
31. Huber, M.T.: Właściwa praca odkształcenia jako miara wyężenia materiału, *Czas. Techn.* **22**, 34–40, 49–50, 61–62, 80–81, *Lwów, Pisma*, Vol. II, PWN, Warszawa 1956, 3–20 (1904)
32. Ishlinskiĭ, A.Yu.: *Gipoteza prochnosti formoizmeneniya*, p. 46. University, Mekh, Uchebnyye Zapiski Mosk (1940)
33. Jackson, L.R., Smith, K.F., Lankford, W.T.: Plastic flow in anisotropic sheet steel. *Am. Inst. Min. Metall. Eng.* **2440**, 1–15 (1948)
34. Khan, A.S., Kazmi, R., Farrok, B.: Multiaxial and non-proportional loading responses, anisotropy and modeling of Ti-6Al-4V titanium alloy over wide ranges of strain rates and temperatures. *Int. J. Plast.* **23**, 931–950 (2007)
35. Khan, A.S., Liu, H.: Strain rate and temperature dependent fracture criteria for isotropic and anisotropic metals. *Int. J. Plast.* **37**, 1–15 (2012)
36. Khan, A.S., Yu, S., Liu, H.: Deformation enhanced anisotropic responses of Ti-6Al-4V alloy, part II: a stress rate and temperature dependent anisotropic yield criterion. *Int. J. Plast.* **38**, 14–26 (2012)
37. Kowalsky, U.K., Ahrens, H., Dinkler, D.: Distorted yield surfaces—modeling by higher order anisotropic hardening tensors. *Comput. Math. Sci.* **16**, 81–88 (1999)
38. Korkolis, Y.P., Kyriakides, S.: Inflation and burst of aluminum tubes. part II: an advanced yield function including deformation-induced anisotropy. *Int. J. Plast.* **24**, 1625–1637 (2008)
39. Lankford, W.T., Low, J.R., Gensamer, M.: The plastic flow of aluminium alloy sheet under combined loads. *Trans. AIME* **171**, 574; *TP 2238, Met. Techn.*, Aug. 1947
40. Lode, W.: Der Einfluss der mittleren Hauptspannung auf der Fließen der Metalle, *Forschungsarbeiten auf dem Gebiete des Ingenieurwesens*, 303 (1928)
41. Luo, M., Dunand, M., Moth, D.: Experiments and modeling of anisotropic aluminum extrusions under multi-axial loading—part II: ductile fracture. *Int. J. Plast.* **32–33**, 36–58 (2012)
42. Malinin, N.N., Rżysko, J.: *Mechanika Materiałów*. PWN, Warszawa (1981)
43. von Mises, R.: *Mechanik der festen Körper im plastisch deformablen Zustand*, Göttingen Nachrichten. *Math. Phys.* **4**(1), 582–592 (1913)
44. von Mises, R.: *Mechanik der plastischen Formänderung von Kristallen*. *ZAMM* **8**(13), 161–185 (1928)
45. Nigam, H., Dvorak, G.J., Bahei-El-Din, Y.A.: An experimental investigation of elastic-plastic behavior of a fibrous Boron-Aluminum composite. I. Matrix-dominated mode. *Int. J. Plast.* **10**, 23–48 (1933)
46. Nixon, M.E., Cazacu, O., Lebensohn, R.A.: Anisotropic response of high-purity α -titanium: experimental characterization and constitutive modeling. *Int. J. Plast.* **26**, 516–532 (2010)
47. Ottosen, N.S., Ristinmaa, M.: *The Mechanics of Constitutive Modeling*. Elsevier, Amsterdam (2005)
48. Raniecki, B., Mróz, Z.: Yield or martensitic phase transformation conditions and dissipative functions for isotropic, pressure-insensitive alloys exhibiting SD effect. *Acta Mech.* **195**, 81–102 (2008)

49. Reuss, A.: Vereifachte Berechnung der plastischen Formänderungen in der Plastizitätstheorie. *ZAMM* **10**(3), 266–274 (1933)
50. Rogers, T.G.: Yield criteria, flow rules, and hardening in anisotropic plasticity. In: Boehler, J.P. (ed.) *Yielding, Damage and Failure of Anisotropic Solids*, pp. 53–79. Mechanical Engineering Publications, London (1990)
51. Rymarz, Cz.: *Continuum Mechanics* (in Polish). PWN, Warszawa (1993)
52. Sayir, M.: Zur Fließbedingung der Plastizitätstheorie. *Ingenieurarchiv* **39**, 414–432 (1970)
53. Schmidt, R.: Über den Zusammenhang von Spannungen und Formänderungen im Vestigungsgebiet. *Ing.-Arch.* **3**, 215–235 (1932)
54. Skrzypek, J., Ganczarski, A.: Anisotropic initial yield and failure criteria including temperature effect. In: Hetnarski, R.B. (ed.) *Encyclopedia of Thermal Stresses*, vol. A, pp. 146–159. Springer, Dordrecht (2014)
55. Sobotka, Z.: Theorie des plastischen Fließens von anisotropen Körpern. *Z. Angew. Math. Mechanik* **49**, 25–32 (1969)
56. Spencer, A.J.M.: Theory of invariants. In: Eringen, C. (ed.) *Continuum Physics*, pp. 239–353. Academic Press, New York (1971)
57. Sun, C.T., Vaidya, R.S.: Prediction of composite properties from a representative volume element. *Compos. Sci. Technol.* **56**, 171–179 (1996)
58. Szczepiński, W.: On deformation-induced plastic anisotropy of sheet metals. *Arch. Mech.* **45**(1), 3–38 (1993)
59. Tamma, K.K., Avila, A.F.: An integrated micro/macro modelling and computational methodology for high temperature composites. In: Hetnarski, R.B. (ed.) *Thermal Stresses V*, pp. 143–256. Lastran Corporation Publishing Division, Rochester (1999)
60. Tresca, H.: Mémoire sur l'écoulement des corps solides soumis á de fortes pressions. *Comptes Rendus de l'Académie des Sciences* **59**, 754–758 (1864)
61. Tsai, S.T., Wu, E.M.: A general theory of strength for anisotropic materials. *Int. J. Numer. Methods Eng.* **38**, 2083–2088 (1971)
62. Voyiadjis, G.Z., Thiagarajan, G.: An anisotropic yield surface model for directionally reinforced metal-matrix composites. *Int. J. Plast.* **11**, 867–894 (1995)
63. Westergaard, H.M.: On the resistance of ductile materials to combined stresses in two and three directions perpendicular to one another. *J. Frankl. Inst.* **189**, 627–640 (1920)
64. Yoshida, F., Hamasaki, H.M., Uemori, T.: A user-friendly 3D yield function to describe anisotropy of steel sheets. *Int. J. Plast.* **45**, 119–139 (2013)
65. Życzkowski, M.: Anisotropic yield conditions. In: Lemaitre, J. (ed.) *Handbook of Materials Behavior Models*, pp. 155–165. Academic Press, San Diego (2001)