Chapter 1
Introduction to Mechanics of Anisotropic Materials

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Abstract This book is focused on constitutive description of mechanical behavior of engineering materials: both conventional (e.g., polycrystalline homogeneous isotropic or anisotropic metallic materials) and nonconventional ones (e.g., heterogeneous multicomponent usually anisotropic composite materials) fabricated by modern material engineering. Effective material properties at the macrolevel depend on both the material microstructure (isotropic or originally anisotropic in general case) and on dissipative phenomena occurred on fabrication and consecutive loading phase resulting in irreversible microstructure changes (acquired anisotropy). The material symmetry is a background and anisotropy is a core around which the book is formed. In this way a revision of classical rules of enhanced constitutive description of materials is required. The aim of this introductory chapter lies in providing, apart from classical definitions of tensor single invariants, also the choice of state variables necessary to describe irreversible microstructure changes accompanying coupled dissipative phenomena, and basic definitions of common invariants of either two second-order tensors (e.g., stress/strain and damage tensors) or two different-order tensors (e.g., stress/strain and fourth-order structural tensors). Concise classification of anisotropic materials with respect to symmetry of elastic matrices as referred to the crystal lattice symmetry is given, and extended analogy between symmetries of constitutive material matrices (elastic and yield/failure) is also discussed. Next, strain and complementary energy as function of either stress/strain invariants (initial elastic isotropy) or common stress/strain—damage invariants (damage acquired anisotropy) are mentioned. Constitutive equation of linear elasticity in terms of common invariants of strain and structural orthotropic tensors is given. The scope of

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this chapter provides necessary tools for more extended constitutive description of materials which exhibit either virgin anisotropy or damage or phase change acquired anisotropy following microstructure changes.

Keywords Single or common tensor invariants · Material symmetry and constitutive matrices · Virgin or acquired anisotropy · Shear and volumetric change coupling · Strain energy of anisotropic materials · Damage and phase change state variables · Constitutive tensors analogy

1.1 Second-Order Tensors

1.1.1 Stress Tensor and Stress Tensor Invariants

Stress tensor $\boldsymbol{\sigma}$, when mathematical σ_{ij} $i, j = 1, 2, 3$, or $i, j = x, y, z$, and engineering notations are used is furnished as

$$[\sigma_{ij}] = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix} = \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{bmatrix} \tag{1.1}$$

where x, y, z denote cartesian coordinate system.

When symmetry of the stress tensor $\sigma_{ij} = \sigma_{ji}$ is assumed, the stress tensor can be represented as *columnar stress vector* as follows:

$$\{\boldsymbol{\sigma}\} = \{\sigma_{11}, \sigma_{22}, \sigma_{33}, \sigma_{23}, \sigma_{13}, \sigma_{12}\}^{\mathrm{T}} = \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{Bmatrix} \tag{1.2}$$

When the definition of *stress deviator* is assumed as

$$s_{ij} = \sigma_{ij} - \frac{1}{3}\sigma_{kk}\delta_{ij} = \sigma_{ij} - \sigma_{\mathrm{h}}\delta_{ij} = \sigma_{ij} - \frac{1}{3}\mathrm{tr}\,(\boldsymbol{\sigma})\,\delta_{ij} \tag{1.3}$$

where $\sigma_{\mathrm{h}} = \frac{1}{3}\sigma_{kk}$ denotes either hydrostatic or mean stress, while $\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$ denotes Kronecker's symbol, decomposition of the stress tensor into the *stress axiator* and the *stress deviator* takes the following form:

$$\boldsymbol{\sigma} = \sigma_{\mathrm{h}}\mathbf{1} + \boldsymbol{s} \tag{1.4}$$

where absolute notation $\sigma_{\mathrm{h}}\mathbf{1}$ and $\boldsymbol{s}$ are used for the stress axiator and the stress deviator, respectively

$$\begin{aligned}[\sigma_{\mathrm{h}}\mathbf{1}] &= \begin{bmatrix} \sigma_{\mathrm{h}} & 0 & 0 \\ 0 & \sigma_{\mathrm{h}} & 0 \\ 0 & 0 & \sigma_{\mathrm{h}} \end{bmatrix} \\ [\boldsymbol{s}] &= \begin{bmatrix} \sigma_x - \sigma_{\mathrm{h}} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y - \sigma_{\mathrm{h}} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z - \sigma_{\mathrm{h}} \end{bmatrix} = \begin{bmatrix} s_{xx} & s_{xy} & s_{xz} \\ s_{yx} & s_{yy} & s_{yz} \\ s_{zx} & s_{zy} & s_{zz} \end{bmatrix} \end{aligned} \tag{1.5}$$

Classical *stress transformation* rule from i, j to k, l directions is

$$\sigma_{kl} = a_{ki} a_{lj} \sigma_{ij} \tag{1.6}$$

where second-order tensor transformation rule is applied and a_{ki}, a_{lj} denote direction cosines of the transformation from the original frame $i, j = x, y, z$ in the new reference frame $k, l = \xi, \eta, \zeta$. It is possible to distinguish the specific transformation into eigendirections (principal directions) for which the corresponding stress tensor takes the diagonal representation

$$\begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix} \xrightarrow{\text{transformation}} \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix} \tag{1.7}$$

Three *principal stresses* are determined as real roots of the cubic equation, being solution of eigenproblem for the stress tensor $\boldsymbol{\sigma}$

$$\boldsymbol{\sigma} = \lambda \mathbf{1} \tag{1.8}$$

where $\lambda_i = \sigma_1, \sigma_2, \sigma_3$ stand for eigenvalues. These principal stresses are real roots of the characteristic equation of stress tensor $\lambda_i = \sigma_i$

$$\det(\boldsymbol{\sigma} - \lambda \mathbf{1}) = 0 \tag{1.9}$$

which can be rewritten in the equivalent fashion

$$\sigma^3 - I_{1\sigma}\sigma^2 + I_{2\sigma}\sigma - I_{3\sigma} = 0 \tag{1.10}$$

Three coefficients of the characteristic equation (1.10) $I_{1\sigma}, I_{2\sigma}, I_{3\sigma}$ are called the *principal invariants of* the *stress tensor* and may be defined in terms of stress components

$$
\begin{aligned}
I_{1\sigma} &= \mathrm{tr}(\boldsymbol{\sigma}) = \sigma_{ii} = \sigma_{xx} + \sigma_{yy} + \sigma_{zz} && [\mathrm{MPa}] \\
I_{2\sigma} &= \begin{vmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{yx} & \sigma_{yy} \end{vmatrix} + \begin{vmatrix} \sigma_{yy} & \sigma_{yz} \\ \sigma_{zy} & \sigma_{zz} \end{vmatrix} + \begin{vmatrix} \sigma_{zz} & \sigma_{zx} \\ \sigma_{xz} & \sigma_{xx} \end{vmatrix} \\
&= \sigma_{xx}\sigma_{yy} + \sigma_{yy}\sigma_{zz} + \sigma_{zz}\sigma_{xx} - (\sigma_{xy}^2 + \sigma_{yz}^2 + \sigma_{zx}^2) && [\mathrm{MPa}^2] \\
I_{3\sigma} &= \det\boldsymbol{\sigma} = \begin{vmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{vmatrix} = \sigma_{xx}\sigma_{yy}\sigma_{zz} \\
&\quad + 2\sigma_{xy}\sigma_{yz}\sigma_{zx} - (\sigma_{xx}\sigma_{yz}^2 + \sigma_{yy}\sigma_{xz}^2 + \sigma_{zz}\sigma_{xy}^2) && [\mathrm{MPa}^3]
\end{aligned}
\tag{1.11}
$$

Apart from the principal invariants, the *basic stress invariants* also called the *generic stress invariants* are of particular importance, namely

$$
\begin{aligned}
J_{1\sigma} &= \sigma_{ii} = \mathrm{tr}\,(\boldsymbol{\sigma}) && [\mathrm{MPa}] \\
J_{2\sigma} &= \tfrac{1}{2}\sigma_{ij}\sigma_{ji} = \tfrac{1}{2}\mathrm{tr}\,(\boldsymbol{\sigma}^2) && [\mathrm{MPa}^2] \\
J_{3\sigma} &= \tfrac{1}{3}\sigma_{ij}\sigma_{jk}\sigma_{ki} = \tfrac{1}{3}\mathrm{tr}\,(\boldsymbol{\sigma}^3) && [\mathrm{MPa}^3]
\end{aligned}
\tag{1.12}
$$

It is seen that the basic stress invariants can be interpreted as traces of subsequent powers of stress tensor $\boldsymbol{\sigma}$, $\boldsymbol{\sigma}^2 = \boldsymbol{\sigma}\cdot\boldsymbol{\sigma}$, $\boldsymbol{\sigma}^3 = \boldsymbol{\sigma}\cdot\boldsymbol{\sigma}\cdot\boldsymbol{\sigma}$, if appropriate coefficients 1, 1/2, 1/3 are used. Note that the basic invariants differ from the principal invariants, which are coefficients of the characteristic equation (1.10).

The basic stress invariants $J_{1\sigma}$, $J_{2\sigma}$, $J_{3\sigma}$ are expressed in terms of the principal stress invariants $I_{1\sigma}$, $I_{2\sigma}$, $I_{3\sigma}$ as follows:

$$
\begin{aligned}
J_{1\sigma} &= I_{1\sigma} \\
J_{2\sigma} &= \tfrac{1}{2}I_{1\sigma}^2 - I_{2\sigma} \\
J_{3\sigma} &= \tfrac{1}{3}I_{1\sigma}^3 - I_{1\sigma}^2 I_{2\sigma} + I_{3\sigma}
\end{aligned}
\tag{1.13}
$$

The reciprocal relations are

$$
\begin{aligned}
I_{1\sigma} &= J_{1\sigma} \\
I_{2\sigma} &= \tfrac{1}{2}J_{1\sigma}^2 - J_{2\sigma} \\
I_{3\sigma} &= \tfrac{1}{6}J_{1\sigma}^3 - J_{1\sigma}^2 J_{2\sigma} + J_{3\sigma}
\end{aligned}
\tag{1.14}
$$

Decomposition of the stress tensor into the stress axiator (spherical tensor) and the stress deviator (1.3−1.5) leads to the following system of *principal* or *generic invariants of* the *stress deviator*

$$
\begin{aligned}
J_{1s} &= s_{ii} = \mathrm{tr}(\boldsymbol{s}) = 0 && [\mathrm{MPa}] \\
J_{2s} &= \tfrac{1}{2}s_{ij}s_{ji} = \tfrac{1}{2}\mathrm{tr}\,(\boldsymbol{s}^2) && [\mathrm{MPa}^2] \\
J_{3s} &= \tfrac{1}{3}s_{ij}s_{jk}s_{ki} = \tfrac{1}{3}\mathrm{tr}\,(\boldsymbol{s}^3) && [\mathrm{MPa}^3]
\end{aligned}
\tag{1.15}
$$

where, in similar fashion as in Eq. (1.12), subsequent powers of the stress deviator $\boldsymbol{s}, \boldsymbol{s}^2 = \boldsymbol{s} \cdot \boldsymbol{s}, \boldsymbol{s}^3 = \boldsymbol{s} \cdot \boldsymbol{s} \cdot \boldsymbol{s}$ are used. Note that the first basic deviatoric stress invariant J_{1s} is equal to zero according to definition (1.3).

Additionally, some *engineering tensor stress invariants* characterized by the stress dimension homogeneity [MPa], by contrast to the above defined basic invariants of different dimensions [MPa], [MPa^2], [MPa^3] are frequently used as

$$\begin{aligned} \sigma_{\mathrm{h}} &= \tfrac{1}{3} J_{1\sigma} = \tfrac{1}{3}\mathrm{tr}\,(\boldsymbol{\sigma}) = \tfrac{1}{3}\boldsymbol{\sigma} : \mathbf{1} = \tfrac{1}{3}\sigma_{kk} \quad [\mathrm{MPa}] \\ \sigma_{\mathrm{eq}} &= \sqrt{3 J_{2s}} = \sqrt{\tfrac{3}{2} s_{ij} s_{ji}} \qquad\qquad [\mathrm{MPa}] \end{aligned} \tag{1.16}$$

The first of them σ_{h} is easily recognized as the *mean stress* and the second σ_{eq} represents the commonly used stress intensity also called the *effective stress*.

1.1.2 Strain Tensor and Strain Tensor Invariants

Strain tensor $\boldsymbol{\varepsilon} = \varepsilon_{ij}$ when uniform mathematical notation $i, j = 1, 2, 3$ or $i, j = x, y, z$, and the engineering notation are used, is furnished as

$$[\varepsilon_{ij}] = \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{21} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33} \end{bmatrix} = \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz} \end{bmatrix} = \begin{bmatrix} \varepsilon_x & \frac{1}{2}\gamma_{xy} & \frac{1}{2}\gamma_{xz} \\ \frac{1}{2}\gamma_{yx} & \varepsilon_y & \frac{1}{2}\gamma_{yz} \\ \frac{1}{2}\gamma_{zx} & \frac{1}{2}\gamma_{zy} & \varepsilon_z \end{bmatrix} \tag{1.17}$$

where x, y, z denote cartesian coordinate frame.

Transformation of the strain tensor is described in a similar fashion as the stress tensor transformation (1.6), namely

$$\varepsilon_{kl} = a_{ki} a_{lj} \varepsilon_{ij} \tag{1.18}$$

Similarly, the *principal strains* can be obtained by solution of the eigenproblem of the tensor $\boldsymbol{\varepsilon}$

$$\boldsymbol{\varepsilon} = \lambda \mathbf{1} \tag{1.19}$$

or equivalently as solution of characteristic equation of strain tensor

$$\varepsilon^3 - I_{1\varepsilon}\varepsilon^2 + I_{2\varepsilon}\varepsilon - I_{3\varepsilon} = 0 \tag{1.20}$$

Coefficients of the above equation $I_{1\varepsilon}, I_{2\varepsilon}, I_{3\varepsilon}$ denote the *principal invariants of* the small (linearized) *strain tensor* and are defined as the homogeneous, scalar functions of the strain components

$$
\begin{aligned}
I_{1\varepsilon} &= \mathrm{tr}(\boldsymbol{\varepsilon}) = \varepsilon_{ii} = \varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz} \\
I_{2\varepsilon} &= \begin{vmatrix} \varepsilon_{xx} & \varepsilon_{xy} \\ \varepsilon_{yx} & \varepsilon_{yy} \end{vmatrix} + \begin{vmatrix} \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{zy} & \varepsilon_{zz} \end{vmatrix} + \begin{vmatrix} \varepsilon_{zz} & \varepsilon_{zx} \\ \varepsilon_{xz} & \varepsilon_{xx} \end{vmatrix} \\
&= \varepsilon_{xx}\varepsilon_{yy} + \varepsilon_{yy}\varepsilon_{zz} + \varepsilon_{zz}\varepsilon_{xx} - (\varepsilon_{xy}^2 + \varepsilon_{yz}^2 + \varepsilon_{zx}^2) \\
I_{3\varepsilon} &= \begin{vmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz} \end{vmatrix} = \varepsilon_{xx}\varepsilon_{yy}\varepsilon_{zz} + 2\varepsilon_{xy}\varepsilon_{yz}\varepsilon_{zx} \\
&\quad - (\varepsilon_{xx}\varepsilon_{yz}^2 + \varepsilon_{yy}\varepsilon_{xz}^2 + \varepsilon_{zz}\varepsilon_{xy}^2)
\end{aligned}
\tag{1.21}
$$

If symmetry of the strain tensor is assumed equivalent representation of the strain tensor in the form of *columnar strain vector* may be applied as

$$
\{\boldsymbol{\varepsilon}\} = \{\varepsilon_{11}, \varepsilon_{22}, \varepsilon_{33}, \varepsilon_{23}, \varepsilon_{13}, \varepsilon_{12}\}^{\mathrm{T}} = \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \varepsilon_{23} \\ \varepsilon_{13} \\ \varepsilon_{12} \end{Bmatrix} \tag{1.22}
$$

When the definition of the *strain deviator*, analogous to the stress deviator (1.3), is used, we arrive at

$$
e_{ij} = \varepsilon_{ij} - \frac{1}{3}\varepsilon_{kk}\delta_{ij} = \varepsilon_{ij} - \varepsilon_{\mathrm{m}}\delta_{ij} = \varepsilon_{ij} - \frac{1}{3}\mathrm{tr}\,(\boldsymbol{\varepsilon})\,\delta_{ij} \tag{1.23}
$$

where ε_{m} denotes *mean* (volumetric) *strain*. Decomposition of the strain tensor into the *strain axiator* and the *strain deviator* is given according to the scheme

$$
\boldsymbol{\varepsilon} = \varepsilon_{\mathrm{m}}\mathbf{1} + \boldsymbol{e} \tag{1.24}
$$

when the absolute notation was used, where $\varepsilon_{\mathrm{m}}\mathbf{1}$ and $\boldsymbol{e}$ denote the strain axiator and the strain deviator, respectively

$$
\begin{aligned}
[\varepsilon_{\mathrm{m}}\mathbf{1}] &= \begin{bmatrix} \varepsilon_{\mathrm{m}} & 0 & 0 \\ 0 & \varepsilon_{\mathrm{m}} & 0 \\ 0 & 0 & \varepsilon_{\mathrm{m}} \end{bmatrix} \\
[\boldsymbol{e}] &= \begin{bmatrix} \varepsilon_x - \varepsilon_{\mathrm{m}} & \frac{1}{2}\gamma_{xy} & \frac{1}{2}\gamma_{xz} \\ \frac{1}{2}\gamma_{yx} & \varepsilon_y - \varepsilon_{\mathrm{m}} & \frac{1}{2}\gamma_{yz} \\ \frac{1}{2}\gamma_{zx} & \frac{1}{2}\gamma_{zy} & \varepsilon_z - \varepsilon_{\mathrm{m}} \end{bmatrix} = \begin{bmatrix} e_{xx} & e_{xy} & e_{xz} \\ e_{yx} & e_{yy} & e_{yz} \\ e_{zx} & e_{zy} & e_{zz} \end{bmatrix}
\end{aligned}
\tag{1.25}
$$

The *basic* or the *generic strain tensor invariants* are defined in analogous fashion as in Eq. (1.12)

$$\begin{aligned} J_{1\varepsilon} &= \varepsilon_{ii} = \mathrm{tr}\,(\boldsymbol{\varepsilon}) \\ J_{2\varepsilon} &= \tfrac{1}{2}\varepsilon_{ij}\varepsilon_{ji} = \tfrac{1}{2}\mathrm{tr}\left(\boldsymbol{\varepsilon}^2\right) \\ J_{3\varepsilon} &= \tfrac{1}{3}\varepsilon_{ij}\varepsilon_{jk}\varepsilon_{ki} = \tfrac{1}{3}\mathrm{tr}\left(\boldsymbol{\varepsilon}^3\right) \end{aligned} \tag{1.26}$$

The basic strain tensor invariants $J_{1\varepsilon}$, $J_{2\varepsilon}$, $J_{3\varepsilon}$ are expressed in terms of the principal strain invariants $I_{1\varepsilon}$, $I_{2\varepsilon}$, $I_{3\varepsilon}$ as

$$\begin{aligned} J_{1\varepsilon} &= I_{1\varepsilon} \\ J_{2\varepsilon} &= \tfrac{1}{2}I_{1\varepsilon}^2 - I_{2\varepsilon} \\ J_{3\varepsilon} &= \tfrac{1}{3}I_{1\varepsilon}^3 - I_{1\varepsilon}^2 I_{2\varepsilon} + I_{3\varepsilon} \end{aligned} \tag{1.27}$$

The reciprocal relationships are

$$\begin{aligned} I_{1\varepsilon} &= J_{1\varepsilon} \\ I_{2\varepsilon} &= \tfrac{1}{2}J_{1\varepsilon}^2 - J_{2\varepsilon} \\ I_{3\varepsilon} &= \tfrac{1}{6}J_{1\varepsilon}^3 - J_{1\varepsilon}^2 J_{2\varepsilon} + J_{3\varepsilon} \end{aligned} \tag{1.28}$$

The *principal invariants of the strain deviator* may be determined in an analogous way as the principal invariants of the stress deviator (1.15), namely

$$\begin{aligned} J_{1e} &= e_{ii} = \mathrm{tr}(\boldsymbol{e}) = 0 \\ J_{2e} &= \tfrac{1}{2}e_{ij}e_{ji} = \tfrac{1}{2}\mathrm{tr}\left(\boldsymbol{e}^2\right) \\ J_{3e} &= \tfrac{1}{3}e_{ij}e_{jk}e_{ki} = \tfrac{1}{3}\mathrm{tr}\left(\boldsymbol{e}^3\right) \end{aligned} \tag{1.29}$$

1.1.3 Matrix Representation of Stress and Strain Tensors

Stress σ_{ij} and strain ε_{ij} are the second-rank tensors having in general $3^2 = 9$ components, since each of indices i, j runs from 1 to 3. Each of them can be interpreted as linear transformation of a certain vector to another vector. In case of the stress tensor, linear transformation of direction cosines n_j into a traction vector p_i according to rule

$$p_i = \sigma_{ij} n_j \tag{1.30}$$

is written down or

$$\begin{aligned} p_1 &= \sigma_{11}n_1 + \sigma_{12}n_2 + \sigma_{13}n_3 \\ p_2 &= \sigma_{21}n_1 + \sigma_{22}n_2 + \sigma_{23}n_3 \\ p_3 &= \sigma_{31}n_1 + \sigma_{32}n_2 + \sigma_{33}n_3 \end{aligned} \tag{1.31}$$

when the expanded form is used. Applying the *matrix-vector notation* to the above formulae the equivalent form is reached

$$\begin{Bmatrix} p_1 \\ p_2 \\ p_3 \end{Bmatrix} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} \begin{Bmatrix} n_1 \\ n_2 \\ n_3 \end{Bmatrix} \tag{1.32}$$

In Eq. (1.32) the second-rank *stress tensor* is represented by the 3×3 tensor *representation matrix* and analogously, the strain tensor representation matrix

$$[\boldsymbol{\sigma}] = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} \qquad [\boldsymbol{\varepsilon}] = \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{21} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33} \end{bmatrix} \tag{1.33}$$

Due to symmetry conditions of both the stress $\sigma_{ij} = \sigma_{ji}$ and the strain $\varepsilon_{ij} = \varepsilon_{ji}$ tensors, both representation matrices are symmetric, comprising 6 independent components each. When engineering notation is used, replacing $1, 2, 3$ frame by x, y, z cartesian coordinate frame, and introducing appropriate notation $\sigma_{ij} = \tau_{ij}$ and $\varepsilon_{ij} = \frac{1}{2}\gamma_{ij}$ for $i \neq j$, we arrive at

$$[\boldsymbol{\sigma}] = \begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ & \sigma_{yy} & \tau_{yz} \\ & & \sigma_{zz} \end{bmatrix} \qquad [\boldsymbol{\varepsilon}] = \begin{bmatrix} \varepsilon_{xx} & \frac{1}{2}\gamma_{xy} & \frac{1}{2}\gamma_{xz} \\ & \varepsilon_{yy} & \frac{1}{2}\gamma_{yz} \\ & & \varepsilon_{zz} \end{bmatrix} \tag{1.34}$$

1.1.4 Decomposition of Strains

In the case of infinitesimal deformation the total strain ε_{ij} can be expressed as the sum of the *elastic (reversible) strain* $\varepsilon_{ij}^{\mathrm{e}}$, *inelastic (irreversible) strain* $\varepsilon_{ij}^{\mathrm{I}}$, and *thermal strain* $\varepsilon_{ij}^{\mathrm{T}}$:

$$\varepsilon_{rs} = \varepsilon_{rs}^{\mathrm{e}} + \varepsilon_{rs}^{\mathrm{I}} + \varepsilon_{rs}^{\mathrm{T}} \tag{1.35}$$

In the process of deformation, various *microstructural rearrangements* of material structure may take place, for example, the changes in density and configuration of dislocations, the development of microscopic cavities, changes from primary to secondary phase, etc. All these rearrangements may contribute to both reversible and irreversible strains (cf. Abu Al-Rub and Voyiadjis [1]), therefore:

$$\begin{aligned} \varepsilon_{rs}^{\mathrm{e}} &= \varepsilon_{rs}^{\mathrm{E}} + \varepsilon_{rs}^{\mathrm{ed}} + \varepsilon_{rs}^{\mathrm{eph}} + \cdots \\ \varepsilon_{rs}^{\mathrm{I}} &= \varepsilon_{rs}^{\mathrm{p}} + \varepsilon_{rs}^{\mathrm{d}} + \varepsilon_{rs}^{\mathrm{ph}} + \cdots \end{aligned} \tag{1.36}$$

where $\varepsilon_{rs}^{\mathrm{E}}$ is a "pure" elastic strain, and $\varepsilon_{rs}^{\mathrm{ed}}, \ldots, \varepsilon_{rs}^{\mathrm{p}}, \ldots$ are respectively the reversible and irreversible components of the total strain induced by *dissipative phenomenon*

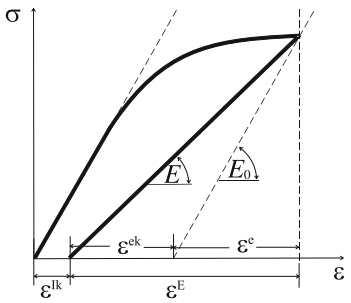


Fig. 1.1 Components of the strain tensor ε_{ij}^{k} induced by kth dissipative phenomenon

(see Fig. 1.1), e.g., *plastic flow*, *damage*, *phase transformation*, etc. For example, in the case of *thermo-elastic-plastic-damage material* the total strain tensor ε_{ij} is expressed as

$$\varepsilon_{rs} = \underbrace{\varepsilon_{rs}^{\mathrm{E}} + \varepsilon_{rs}^{\mathrm{ed}}}_{\varepsilon_{rs}^{\mathrm{e}}} + \underbrace{\varepsilon_{rs}^{\mathrm{Id}} + \varepsilon_{rs}^{\mathrm{p}}}_{\varepsilon_{rs}^{\mathrm{I}}} + \varepsilon_{rs}^{\mathrm{T}} \tag{1.37}$$

while its damage-induced component, $\varepsilon_{rs}^{\mathrm{d}}$, consists of both *reversible* (ed) and *irreversible* (Id) *damage strain* terms:

$$\varepsilon_{rs}^{\mathrm{d}} = \varepsilon_{rs}^{\mathrm{ed}} + \varepsilon_{rs}^{\mathrm{Id}} \tag{1.38}$$

1.2 Fourth-Order Tensors and Matrix Representation

1.2.1 Stiffness and Compliance Matrices—Voigt's Notation

General *linear elasticity equation* for anisotropic material, frequently called the *generalized Hooke law*, takes the forms

$$\varepsilon_{ij} = E_{ijkl}^{-1}\sigma_{kl} \qquad \sigma_{ij} = E_{ijkl}\varepsilon_{kl} \tag{1.39}$$

where the *fourth-rank elasticity tensors*, *stiffness* E_{ijkl} or *compliance* E_{ijkl}^{-1}, are defined, in general by $3^4 = 81$ components, since each of indices i, j, k, l runs through 1, 2, 3. Because of the symmetry of the stress $\sigma_{kl} = \sigma_{lk}$ and the strain $\varepsilon_{ij} = \varepsilon_{ji}$ tensors, both the stiffness and compliance tensors are symmetric with respect to change of indices in pairs $i \leftrightarrow j$ and $k \leftrightarrow l$

$$E_{ijkl} = E_{jikl} = E_{ijlk} \qquad E_{ijkl}^{-1} = E_{jikl}^{-1} = E_{ijlk}^{-1} \tag{1.40}$$

Additionally, because of property of *positive definiteness* of *strain energy* or *complementary energy* the symmetry with respect to change of indices between pairs $ij \leftrightarrow kl$ must also hold

$$E_{ijkl} = E_{klij} \qquad E^{-1}_{ijkl} = E^{-1}_{klij} \tag{1.41}$$

Because of symmetry conditions (1.40) and (1.41) from among 81 components of stiffness or compliance tensors, only 21 are independent. In order to describe the *generalized Hooke's law* (1.39) by use of *vector-matrix Voigt's notation*, stress and strain tensors are written as columnar stress and strain vectors, if the following scheme of change between tensor $i, j = 1, 2, 3$ and vectors $k = 1, 2, \ldots, 6$ indices holds:

$$\begin{array}{cccccccc} ij & 11 & 22 & 33 & 23,32 & 31,13 & 12,21 \\ & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ k & 1 & 2 & 3 & 4 & 5 & 6 \end{array} \tag{1.42}$$

From the above scheme we obtain the following *representations of stress and strain tensors*:

$$\begin{aligned} [\sigma_{ij}] &= \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ & \sigma_{22} & \sigma_{23} \\ & & \sigma_{33} \end{bmatrix} \rightarrow \begin{bmatrix} \sigma_1 & \sigma_6 & \sigma_5 \\ & \sigma_2 & \sigma_4 \\ & & \sigma_3 \end{bmatrix} \rightarrow \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{Bmatrix} \\ [\varepsilon_{ij}] &= \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ & \varepsilon_{22} & \varepsilon_{23} \\ & & \varepsilon_{33} \end{bmatrix} \rightarrow \begin{bmatrix} \varepsilon_1 & \varepsilon_6 & \varepsilon_5 \\ & \varepsilon_2 & \varepsilon_4 \\ & & \varepsilon_3 \end{bmatrix} \rightarrow \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{Bmatrix} \end{aligned} \tag{1.43}$$

Analogous scheme is applied to the first and second pairs of indices of stiffness and compliance tensors

$$\begin{aligned} E_{ijkl} = E_{mn},\ E^{-1}_{ijkl} = E^{-1}_{mn} &\text{ if } m \text{ or } n \text{ go through } 1, 2, 3 \\ 2E_{ijkl} = E_{mn},\ 2E^{-1}_{ijkl} = E^{-1}_{mn} &\text{ if } m \text{ or } n \text{ go through } 4, 5, 6 \\ 4E_{ijkl} = E_{mn},\ 4E^{-1}_{ijkl} = E^{-1}_{mn} &\text{ if both } m \text{ and } n \text{ go through } 4, 5, 6 \end{aligned} \tag{1.44}$$

where appropriate factors 2 or 4 are applied.

For instance, if the axial strain ε_{11} is considered the transformation scheme is as follows:

$$\begin{array}{ccccccccccccc}
\varepsilon_{11} & = & E^{-1}_{1111}\sigma_{11} & + & E^{-1}_{1122}\sigma_{22} & + & E^{-1}_{1133}\sigma_{33} & + & 2E^{-1}_{1123}\sigma_{23} & + & 2E^{-1}_{1113}\sigma_{13} & + & 2E^{-1}_{1112}\sigma_{12} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\varepsilon_1 & = & E^{-1}_{11}\sigma_1 & + & E^{-1}_{12}\sigma_2 & + & E^{-1}_{13}\sigma_3 & + & E^{-1}_{14}\sigma_4 & + & E^{-1}_{15}\sigma_5 & + & E^{-1}_{16}\sigma_6
\end{array} \tag{1.45}$$

In case the shear strain ε_{23} is considered, the following are furnished:

$$\begin{array}{ccccccccccccc}
\varepsilon_{23} & = & E^{-1}_{2311}\sigma_{11} & + & E^{-1}_{2322}\sigma_{22} & + & E^{-1}_{2333}\sigma_{33} & + & 2E^{-1}_{2323}\sigma_{23} & + & 2E^{-1}_{2313}\sigma_{13} & + & 2E^{-1}_{2312}\sigma_{12} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
2\varepsilon_4 & = & 2E^{-1}_{2311}\sigma_1 & + & 2E^{-1}_{2322}\sigma_2 & + & 2E^{-1}_{2333}\sigma_3 & + & 4E^{-1}_{2323}\sigma_4 & + & 4E^{-1}_{2313}\sigma_5 & + & 4E^{-1}_{2312}\sigma_6 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\gamma_4 & = & E^{-1}_{41}\sigma_1 & + & E^{-1}_{42}\sigma_2 & + & E^{-1}_{43}\sigma_3 & + & E^{-1}_{44}\sigma_4 & + & E^{-1}_{45}\sigma_5 & + & E^{-1}_{46}\sigma_6
\end{array} \tag{1.46}$$

Finally, the *generalized Hooke's law* (1.39) is represented in *vector-matrix notation* as follows:

$$\begin{array}{ll}
\varepsilon_i = E^{-1}_{ij}\sigma_j & (i = 1, 2, 3,\ j = 1, 2, \ldots, 6) \\
\gamma_i = E^{-1}_{ij}\sigma_j & (i = 4, 5, 6,\ j = 1, 2, \ldots, 6)
\end{array} \tag{1.47}$$

or

$$\{\boldsymbol{\varepsilon}\} = [\mathbb{E}^{-1}]\,\{\boldsymbol{\sigma}\} \tag{1.48}$$

or equivalently

$$\{\boldsymbol{\sigma}\} = [\mathbb{E}]\,\{\boldsymbol{\varepsilon}\} \tag{1.49}$$

where $[\mathbb{E}]$ or $[\mathbb{E}^{-1}]$ denote representation matrices of elastic stiffness or compliance tensors, whereas $\{\boldsymbol{\varepsilon}\}$ and $\{\boldsymbol{\sigma}\}$ denote the columnar vectors of strain and stress, respectively. When *columnar vectors of stress* and *strain* are used as well as elasticity matrices are explicitly written down, Hooke's law is furnished as

$$\begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \gamma_4 \\ \gamma_5 \\ \gamma_6 \end{Bmatrix} = \left[\begin{array}{ccc|ccc} E^{-1}_{11} & E^{-1}_{12} & E^{-1}_{13} & E^{-1}_{14} & E^{-1}_{15} & E^{-1}_{16} \\ E^{-1}_{21} & E^{-1}_{22} & E^{-1}_{23} & E^{-1}_{24} & E^{-1}_{25} & E^{-1}_{26} \\ E^{-1}_{31} & E^{-1}_{32} & E^{-1}_{33} & E^{-1}_{34} & E^{-1}_{35} & E^{-1}_{36} \\ \hline E^{-1}_{41} & E^{-1}_{42} & E^{-1}_{43} & E^{-1}_{44} & E^{-1}_{45} & E^{-1}_{46} \\ E^{-1}_{51} & E^{-1}_{52} & E^{-1}_{53} & E^{-1}_{54} & E^{-1}_{55} & E^{-1}_{56} \\ E^{-1}_{61} & E^{-1}_{62} & E^{-1}_{63} & E^{-1}_{64} & E^{-1}_{65} & E^{-1}_{66} \end{array}\right] \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \tau_4 \\ \tau_5 \\ \tau_6 \end{Bmatrix} \tag{1.50}$$

or

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \tau_4 \\ \tau_5 \\ \tau_6 \end{Bmatrix} = \left[\begin{array}{ccc|ccc} E_{11} & E_{12} & E_{13} & E_{14} & E_{15} & E_{16} \\ E_{21} & E_{22} & E_{23} & E_{24} & E_{25} & E_{26} \\ E_{31} & E_{32} & E_{33} & E_{34} & E_{35} & E_{36} \\ \hline E_{41} & E_{42} & E_{43} & E_{44} & E_{45} & E_{46} \\ E_{51} & E_{52} & E_{53} & E_{54} & E_{55} & E_{56} \\ E_{61} & E_{62} & E_{63} & E_{64} & E_{65} & E_{66} \end{array}\right] \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \gamma_4 \\ \gamma_5 \\ \gamma_6 \end{Bmatrix} \tag{1.51}$$

where engineering notation for shear stress $\tau_j = \sigma_j$ $(j = 4, 5, 6)$ is used.

It should be mentioned that symmetric *stiffness* $[E_{ij}] = [E_{ji}]$ and symmetric *compliance* $[E_{ij}^{-1}] = [E_{ji}^{-1}]$ *matrices*, both having dimension 6×6, are representation matrices of fourth-rank elasticity tensors E_{ijkl} or compliance E_{ijkl}^{-1}. Transformation of each matrix to another coordinate frame can be performed if the matrix notation (1.51) is replaced by the tensor notation (1.39), or by use of the appropriate transformation matrix $[\mathbb{Q}]$

$$[\mathbb{E}'] = [\mathbb{Q}]^{\mathrm{T}}[\mathbb{E}][\mathbb{Q}] \tag{1.52}$$

For instance, if the stiffness matrix is considered the transformation matrix takes the form

$$[\mathbb{Q}] = \left[\begin{array}{ccc} q_{11}q_{11} & q_{12}q_{12} & q_{13}q_{13} \\ q_{21}q_{21} & q_{22}q_{22} & q_{23}q_{23} \\ q_{31}q_{31} & q_{32}q_{32} & q_{33}q_{33} \\ 2q_{31}q_{21} & 2q_{32}q_{22} & 2q_{33}q_{23} \\ 2q_{31}q_{11} & 2q_{32}q_{12} & 2q_{33}q_{13} \\ 2q_{21}q_{11} & 2q_{12}q_{22} & 2q_{13}q_{23} \end{array}\right.$$
$$\left.\begin{array}{ccc} q_{12}q_{13} & q_{13}q_{11} & q_{12}q_{11} \\ q_{23}q_{22} & q_{23}q_{21} & q_{22}q_{21} \\ q_{33}q_{32} & q_{33}q_{31} & q_{32}q_{31} \\ q_{33}q_{22} + q_{32}q_{23} & q_{33}q_{21} + q_{31}q_{23} & q_{31}q_{22} + q_{32}q_{21} \\ q_{33}q_{12} + q_{32}q_{13} & q_{33}q_{11} + q_{31}q_{13} & q_{31}q_{12} + q_{32}q_{11} \\ q_{13}q_{22} + q_{12}q_{23} & q_{13}q_{21} + q_{11}q_{23} & q_{11}q_{22} + q_{12}q_{21} \end{array}\right] \tag{1.53}$$

the coefficients of which are scalar products of corresponding direction cosines $q_{ij} = n_i n_j$ between both coordinate frames.

1.2.2 The Choice of State Variables

The irreversible rearrangements of the internal structure can be represented by a group of variables describing the current state of *material microstructure*:

$$\{V^k\} = \{V^{\mathrm{p}}, V^{\mathrm{d}}, V^{\mathrm{ph}}, \ldots\} \tag{1.54}$$

where V^k may be scalars, vectors, or even rank tensors. For damage description, in the case where the damaged material remains isotropic, the current state of damage is often represented by the scalar variable $V^{\mathrm{d}} = D$ denoting the volume fraction of cracks and voids $\mathrm{d}V^{\mathrm{d}}$ in the total volume $\mathrm{d}V^0$. *Damage acquired orthotropy* requires a second-order tensor, for example, the classical *Murakami–Ohno* [38] *tensor* $V_{ij}^{\mathrm{d}} = D_{ij}$, see Eq. (1.61). In the most general case of anisotropy the description of *damage* needs to be embodied in an *eight-order tensor* (cf. Cauvin and Testa [6]), while the principle of strain equivalence allows using *fourth-order tensors*, see Sect. 1.2.3. For *phase transformation* analysis the *scalar variable* $V^{\mathrm{ph}} = \xi$ is commonly adopted (cf. Egner and Skoczeń [14]), which denotes the *volume fraction of the* secondary *phase* in the total volume of the two-phase Representative Volume Element. However, a scalar variable is not capable of describing the *acquired anisotropy* due to partially directional nature of the *secondary inclusions* in the *primary matrix*. Therefore, instead of scalar variable a *second-order phase change tensor* can be defined in analogy to the damage tensor:

$$\boldsymbol{V}^{\mathrm{ph}} = \boldsymbol{\xi} = \sum_{i=1}^{3} \xi_i \boldsymbol{n}_i \otimes \boldsymbol{n}_i \tag{1.55}$$

where ξ_i describes the ratio of the secondary phase area $\mathrm{d}A_i^{\mathrm{ph}}$ to the total area $\mathrm{d}A_i^0$ on the principal plane of normal unit vector $\boldsymbol{n}_i$ (cf. Egner [13]). Another group of state variables consists of *internal (hidden) variables* corresponding to the *modifications of loading surfaces*:

$$\begin{aligned}\{h^k\} = \Big\{ & r^{\mathrm{p}}, \alpha_{ij}^{\mathrm{p}}, l_{ijkl}^{\mathrm{p}}, g_{ijklmn}^{\mathrm{p}}, \dots \\ & r^{\mathrm{d}}, \alpha_{ij}^{\mathrm{d}}, l_{ijkl}^{\mathrm{d}}, g_{ijklmn}^{\mathrm{d}}, \dots \Big\}\end{aligned} \tag{1.56}$$

where $r^{\mathrm{p}}, r^{\mathrm{d}}$ correspond to *isotropic expansion* of the *loading surface*, $\alpha_{ij}^{\mathrm{p}}, \alpha_{ij}^{\mathrm{d}}$ affect *loading surface translatoric displacements*, $l_{ijkl}^{\mathrm{p}}, l_{ijkl}^{\mathrm{d}}$ are hardening tensors of the fourth order which includes varying lengths of axes and *rotation of the loading surface*, and $g_{ijklmn}^{\mathrm{p}}, g_{ijklmn}^{\mathrm{d}}$ describe changes of the curvature of the *loading surface (distortion)* related to appropriate dissipative phenomenon (cf. Kowalsky et al. [27], see Fig. 1.2). The complete set of *state variables* $\{V_{\mathrm{st}}\}$ reflecting the current state of the thermodynamic system consists of *observable variables*: elastic (or total) strain tensor $\varepsilon_{ij}^{\mathrm{e}}$ and absolute temperature T, and two groups of *microstructural* $\{V^k\}$ and *hardening* $\{h^k\}$ *state variables*:

$$\{V_{\mathrm{st}}\} = \{\varepsilon_{ij}^{\mathrm{e}}, T; V^{\mathrm{p}}, V^{\mathrm{d}}, V^{\mathrm{ph}}, \dots, h^{\mathrm{p}}, h^{\mathrm{d}}, h^{\mathrm{ph}}, \dots\} \tag{1.57}$$

When *thermo-elastic-plastic-damage* two-phase material is considered, the exemplary set of state variables for a general case of hardening/softening effects induced by different dissipative phenomena is further listed in Table 7.1.

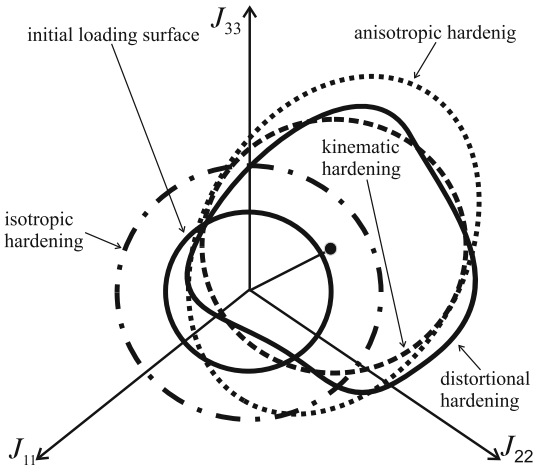


Fig. 1.2 Modifications of the loading surface in the space of thermodynamic conjugate forces $\{J^k\}$ (after [13])

When the material is subjected to *reverse tension-compression cycles*, the unsymmetrical behavior in tension and compression is observed as the *unilateral response* due to partial *crack closure effect*. To describe the phenomenon of the *unilateral damage*, also called the *damage deactivation* or the *crack closure/opening effect*, a decomposition of the stress or strain tensors into the positive or negative projection is usually introduced using the fourth-rank *projection operators* (cf. Krajcinovic [30]; Bielski et al. [4]):

$$\varepsilon_{ij}^{*} = \sum_{I=1}^{3} \kappa(\varepsilon_I) n_{iI}^{(\varepsilon)} n_{jI}^{(\varepsilon)} n_{Ik}^{(\varepsilon)} n_{Il}^{(\varepsilon)} \varepsilon_{kl} = B_{ijkl}^{(\varepsilon)} \varepsilon_{kl} \tag{1.58}$$

where the fourth-rank tensor $B_{ijkl}^{(\varepsilon)}$ is built of directional cosines between the principal and the current spatial systems, $n_{iI}^{(\varepsilon)}$ and $\kappa(\varepsilon_I) = \mathrm{H}(a) + \zeta \mathrm{H}(-a)$, H is a Heaviside function and ζ is a material constant.

1.2.3 Damage and Damage Effect Tensors

So far constitutive description of material has not accounted for influence of damage. Damage means existence of microvoids and microcracks in the material that result in essential deterioration of mechanical properties at the macroscale, such as strength and stiffness or compliance.

In the simplest case when microvoids are spherical and homogeneously distributed in material, damage is described by the scalar damage variable D, usually called the damage parameter, Fig. 1.3

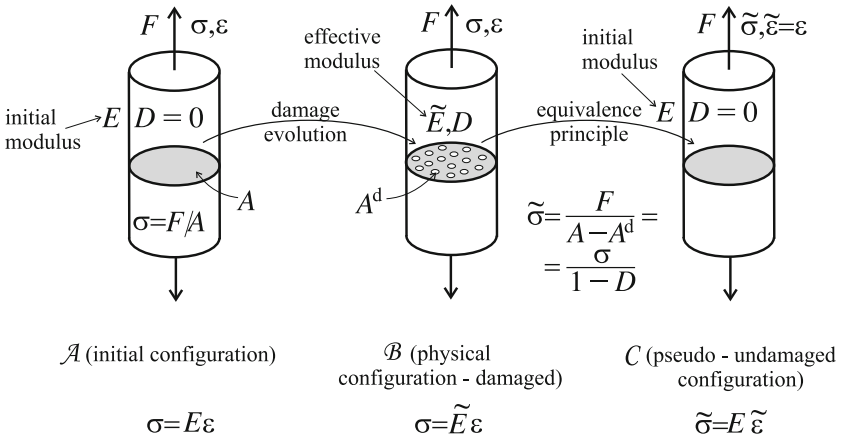


Fig. 1.3 Three configurations used in CDM: **a** initial, **b** damaged, **c** effective pseudo-undamaged

$$D = \frac{A^{\mathrm{d}}}{A} \tag{1.59}$$

Scalar damage variable D, introduced by Kachanov [24] and Rabotnov [46], represents the loss of effective area from the initial A to the damaged A^{d} states. In order to generalize the scalar damage variable to the case when microvoids exhibit clearly directional nature, the *vector damage variable* D_i, is proposed by Davison and Stevens [11], Kachanov [25], Krajcinovic and Fonseka [28]

$$D_i = \frac{A_i^{\mathrm{d}}}{A_i} \quad i = 1, 2, 3 \tag{1.60}$$

Murakami and Ohno [39] introduced more general damage variable defined by the symmetric *second-rank damage tensor* $\boldsymbol{D}$, capable of capturing an *orthotropic damage* nature

$$\boldsymbol{D} = \begin{bmatrix} D_{11} & D_{12} & D_{13} \\ & D_{22} & D_{23} \\ & & D_{33} \end{bmatrix} \tag{1.61}$$

Recently, researches aimed towards correct description of damage mechanism in elastic-brittle *rock-like materials*, *ceramics* or *concrete* led to definition of the *fourth-rank damage tensors*, e.g., Chaboche [8], Krajcinovic [29] or Lubarda and Krajcinovic [35]. Apart from the above-mentioned damage variables possessing clear geometric interpretation other damage variables referring to physical planes, described in details e.g., by Gambarotta and Lagomarsino [15], Seweryn and Mróz [48] should also be mentioned. More general *classifications of damage variables*

were listed in following subject monographs by Krajcinovic [29, 30], Skrzypek and Ganczarski [51], Betten [3] or Murakami [40].

In the frame of continuum damage mechanics (CDM), three configurations are considered: *initial configuration* $\mathcal{A}$ that describes material in undamaged state $D(\mathcal{A}) = 0$, *physical configuration* $\mathcal{B}$ referring to the damaged state $D(\mathcal{B}) \neq 0$, and the equivalent, *fictitious pseudo-undamaged configuration* $\mathcal{C}$ in which real heterogeneous material is substituted by a homogeneous material, free of damage $D(\mathcal{C}) = 0$, as schematically is shown in Fig. 1.3.

The physical (damaged) configuration $\mathcal{B}$ is equivalent to the effective (pseudo-undamaged) configuration $\mathcal{C}$ in a certain sense, for instance, of *strain equivalence* Chaboche [7], *stress equivalence* Taher et al. [54], or *elastic strain energy equivalence* Cordebois and Sidoroff [10]. In physical configuration $\mathcal{B}$ damage state manifests through the effective elasticity modulus $\widetilde{E}$, for instance,

$$\widetilde{E} = E(1 - D) \quad \text{or} \quad \widetilde{E} = E(1 - D)^2 \tag{1.62}$$

where the hypotheses of strain or stress equivalence (first formula) or elastic energy equivalence are used. Contrarily, in the effective configuration $\mathcal{C}$ damage state manifests by the definition of the *effective variables* $\widetilde{\sigma}$, $\widetilde{\varepsilon}$, respectively

$$\widetilde{\sigma} = \sigma \frac{E}{\widetilde{E}}, \quad \widetilde{\varepsilon} = \varepsilon \quad \text{or} \quad \widetilde{\sigma} = \sigma \sqrt{\frac{E}{\widetilde{E}}}, \quad \widetilde{\varepsilon} = \varepsilon \sqrt{\frac{\widetilde{E}}{E}} \tag{1.63}$$

or equivalently

$$\widetilde{\sigma} = \frac{\sigma}{1 - D}, \quad \widetilde{\varepsilon} = \varepsilon \quad \text{or} \quad \widetilde{\sigma} = \frac{\sigma}{1 - D}, \quad \widetilde{\varepsilon} = \varepsilon(1 - D) \tag{1.64}$$

The *damage effect matrix*, being matrix representation of the *damage effect tensor*

$$[\mathbb{M}] = \left[\text{diag}\,\{M_{11}, M_{22}, M_{33}, M_{44}, M_{55}, M_{66}\}\right] \tag{1.65}$$

is expressed in terms of the *damage parameter* D as follows:

$$[\mathbb{M}] = \frac{1}{1 - D}\left[\text{diag}\,\{1, 1, 1, 1, 1, 1\}\right] \tag{1.66}$$

where the diagonal form is applicable.

Damage effect matrix plays an essential role in definitions of the *damage effective stress tensor* $\widetilde{\boldsymbol{\sigma}}$

$$\{\widetilde{\boldsymbol{\sigma}}\} = [\mathbb{M}]\,\{\boldsymbol{\sigma}\} = \left\{\frac{\sigma_x}{1 - D}, \frac{\sigma_y}{1 - D}, \frac{\sigma_z}{1 - D}, \frac{\tau_{yz}}{1 - D}, \frac{\tau_{xz}}{1 - D}, \frac{\tau_{xy}}{1 - D}\right\}^{\mathrm{T}} \tag{1.67}$$

and the *damage effective compliance (stiffness) matrix* $[\widetilde{\mathbb{E}}^{-1}]$

$$[\widetilde{\mathbb{E}}^{-1}] = [\mathbb{M}]^{\mathrm{T}}[\mathbb{E}^{-1}][\mathbb{M}] = \\ = \frac{1}{E(1-D)^2}\left[\begin{array}{ccc|ccc} 1 & -\nu & -\nu & & & \\ & 1 & -\nu & & & \\ & & 1 & & & \\ \hline & & & 1+\nu & & \\ & & & & 1+\nu & \\ & & & & & 1+\nu \end{array}\right] \tag{1.68}$$

For brevity, in all the above equations (1.62–1.68) the assumption of material isotropy in undamaged state ($\mathcal{A}$) is applied.

Assumption of the isotropic damage nature is too strong a simplification since usually microvoids or microcracks are of oval or directional shapes. A proper damage description requires application of orthotropic damage representation (1.61), which under the assumption of the principal damage frame reduces to the diagonal form, where D_1, D_2, D_3 components may be interpreted by reduction of effective areas 1, 2, 3 (1.60), hence

$$\boldsymbol{D} = \begin{bmatrix} D_1 & & \\ & D_2 & \\ & & D_3 \end{bmatrix} \qquad D_i = \frac{A_i^{\mathrm{d}}}{A_i} \qquad i = 1, 2, 3 \tag{1.69}$$

Chosen *representations* of the *damage effect matrix* based on various hypotheses, after Chen and Chow [9], Skrzypek [49], Murakami [40], can be defined as follows:

$$[\mathbb{M}_1] = \begin{bmatrix} \frac{1}{1-D_1} & & & & & & \\ & \frac{1}{1-D_2} & & & & & \\ & & \frac{1}{1-D_3} & & & & \\ & & & \frac{1}{\sqrt{(1-D_2)(1-D_3)}} & & & \\ & & & & \frac{1}{\sqrt{(1-D_3)(1-D_1)}} & & \\ & & & & & \frac{1}{\sqrt{(1-D_1)(1-D_2)}} & \end{bmatrix} \tag{1.70}$$

or

$$[\mathbb{M}_2] = \begin{bmatrix} \frac{1}{1-D_1} & & & & & & \\ & \frac{1}{1-D_2} & & & & & \\ & & \frac{1}{1-D_3} & & & & \\ & & & \frac{1}{1-0.5(D_2+D_3)} & & & \\ & & & & \frac{1}{1-0.5(D_3+D_1)} & & \\ & & & & & \frac{1}{1-0.5(D_1+D_2)} & \end{bmatrix} \tag{1.71}$$

or

$$[\mathbb{M}_3] = \begin{bmatrix} \frac{1}{1-D_1} & & & & & & \\ & \frac{1}{1-D_2} & & & & & \\ & & \frac{1}{1-D_3} & & & & \\ & & & \frac{1}{2}\left(\frac{1}{1-D_2}+\frac{1}{1-D_3}\right) & & & \\ & & & & \frac{1}{2}\left(\frac{1}{1-D_3}+\frac{1}{1-D_1}\right) & & \\ & & & & & \frac{1}{2}\left(\frac{1}{1-D_1}+\frac{1}{1-D_2}\right) & \end{bmatrix} \tag{1.72}$$

The damage effective stress can be defined, for instance in the following two ways, both satisfying symmetry of the effective stress $\widetilde{\boldsymbol{\sigma}}(\boldsymbol{D})$:

$$\{\widetilde{\boldsymbol{\sigma}}\} = [\mathbb{M}_1^{1/2}]^{\mathrm{T}}\{\boldsymbol{\sigma}\}[\mathbb{M}_1^{1/2}] \longrightarrow [\widetilde{\boldsymbol{\sigma}}] = \begin{bmatrix} \frac{\sigma_x}{1-D_1} & \frac{\tau_{xy}}{\sqrt{(1-D_1)(1-D_2)}} & \frac{\tau_{xz}}{\sqrt{(1-D_1)(1-D_3)}} \\ & \frac{\sigma_y}{1-D_2} & \frac{\tau_{yz}}{\sqrt{(1-D_2)(1-D_3)}} \\ & & \frac{\sigma_z}{1-D_3} \end{bmatrix} \tag{1.73}$$

or

$$\{\widetilde{\boldsymbol{\sigma}}\} = [\mathbb{M}_1]^{\mathrm{T}}\{\boldsymbol{\sigma}\}[\mathbb{M}_1] \longrightarrow [\widetilde{\boldsymbol{\sigma}}] = \begin{bmatrix} \frac{\sigma_x}{(1-D_1)^2} & \frac{\tau_{xy}}{(1-D_1)(1-D_2)} & \frac{\tau_{xz}}{(1-D_1)(1-D_3)} \\ & \frac{\sigma_y}{(1-D_2)^2} & \frac{\tau_{yz}}{(1-D_2)(1-D_3)} \\ & & \frac{\sigma_z}{(1-D_3)^2} \end{bmatrix} \tag{1.74}$$

Exemplary effective compliance matrices take the following representations, Skrzypek and Ganczarski [51]:

$$[\widetilde{\mathbb{E}}^{-1}] = [\mathbb{M}_1]^{\mathrm{T}}[\mathbb{E}^{-1}][\mathbb{M}_1] = \frac{1}{E}\left[\begin{array}{ccc|ccc} \frac{1}{(1-D_1)^2} & \frac{-\nu}{(1-D_1)(1-D_2)} & \frac{-\nu}{(1-D_1)(1-D_3)} & & & \\ \frac{-\nu}{(1-D_2)(1-D_1)} & \frac{1}{(1-D_2)^2} & \frac{-\nu}{(1-D_2)(1-D_3)} & & & \\ \frac{-\nu}{(1-D_3)(1-D_1)} & \frac{-\nu}{(1-D_3)(1-D_2)} & \frac{1}{(1-D_3)^2} & & & \\ \hline & & & \frac{1+\nu}{(1-D_2)(1-D_3)} & & \\ & & & & \frac{1+\nu}{(1-D_3)(1-D_1)} & \\ & & & & & \frac{1+\nu}{(1-D_1)(1-D_2)} \end{array}\right] \tag{1.75}$$

or

$$[\widetilde{\mathbb{E}}^{-1}] = \frac{1}{2}\left([\mathbb{M}_2][\mathbb{E}^{-1}] + [\mathbb{E}^{-1}][\mathbb{M}_2]\right) = \frac{1}{E}\left[\begin{array}{ccc|ccc} \frac{1}{1-D_1} & \frac{-\nu}{1-0.5(D_1+D_2)} & \frac{-\nu}{1-0.5(D_1+D_3)} & & & \\ \frac{-\nu}{1-0.5(D_2+D_1)} & \frac{1}{1-D_2} & \frac{-\nu}{1-0.5(D_2+D_3)} & & & \\ \frac{-\nu}{1-0.5(D_3+D_1)} & \frac{-\nu}{1-0.5(D_3+D_2)} & \frac{1}{1-D_3} & & & \\ \hline & & & \frac{1+\nu}{1-0.5(D_2+D_3)} & & \\ & & & & \frac{1+\nu}{1-0.5(D_1+D_3)} & \\ & & & & & \frac{1+\nu}{1-0.5(D_1+D_2)} \end{array}\right] \tag{1.76}$$

In both cases, for the sake of brevity, material isotropy at the undamaged state was assumed.

In a more general case of full damage anisotropy the *fourth-rank damage tensor* D_{ijkl}, built of 21 independent components, should be used.

Following Cauvin and Testa [6] the effective stiffness tensor is defined as

$$\widetilde{\mathbb{E}} = (\mathbb{I} - \mathbb{D}) : \mathbb{E} = \mathbb{R} : \mathbb{E} \tag{1.77}$$

where fourth-rank tensors $\mathbb{R}$ and $\mathbb{D}$ stand for damage effect and damage tensors, respectively. In general case of full *damage anisotropy* the 6×6 *matrix representation* of the *fourth-rank damage tensor* is as follows:

$$[\mathbb{D}] = \begin{bmatrix} D_{11} & D_{12} & D_{13} & D_{14} & D_{15} & D_{16} \\ & D_{22} & D_{23} & D_{24} & D_{25} & D_{26} \\ & & D_{33} & D_{34} & D_{35} & D_{36} \\ & & & D_{44} & D_{45} & D_{46} \\ & & & & D_{55} & D_{56} \\ & & & & & D_{66} \end{bmatrix} \tag{1.78}$$

As a particular case the *orthotropic damage* is considered as example for which the unsymmetric orthotropic damage matrix reduces to

$$[\mathbb{D}] = \begin{bmatrix} D_{11} & D_{12} & D_{13} & & & \\ D_{21} & D_{22} & D_{23} & & & \\ D_{31} & D_{32} & D_{33} & & & \\ & & & D_{44} & & \\ & & & & D_{55} & \\ & & & & & D_{66} \end{bmatrix} \tag{1.79}$$

In the particular case when the orthotropic symmetry of damaged material is considered, the damage tensor takes the following *matrix representation*, after Cauvin and Testa [6], also Ganczarski [17]:

$$[\mathbb{D}] = \begin{bmatrix} D_{1111} & D_{1122} & D_{1133} & & & & \\ D_{2211} & D_{2222} & D_{2233} & & & & \\ D_{3311} & D_{3322} & D_{3333} & & & & \\ & & & 2D_{2323} & & & \\ & & & & 2D_{1313} & & \\ & & & & & 2D_{1212} & \end{bmatrix} \tag{1.80}$$

defined by 12 independent elements, in general nonsymmetric because three elements under diagonal D_{2211}, D_{3311}, D_{3322} are truly independent.

In the narrower case of *transverse isotropy* (in the plane 2, 3), number of independent elements of the tensor D_{ijkl} reduces to 5, namely D_1, D_2, D_3, D_4, D_5

$$[\mathbb{D}] = \begin{bmatrix} D_1 & D_2 & D_2 & & & & \\ D_{2\prime} & D_3 & D_4 & & & & \\ D_{2\prime} & D_4 & D_3 & & & & \\ & & & D_3 - D_4 & & & \\ & & & & D_5 & & \\ & & & & & D_5 & \end{bmatrix} \tag{1.81}$$

Two components $D_{2211} = D_{3311} = D_{2\prime}$ are dependent, and expressed as

$$D_{2\prime} = \frac{1}{1-\nu}\left[D_2 + \nu\left(D_1 - D_3\right) - \nu D_4\right] \tag{1.82}$$

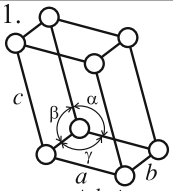
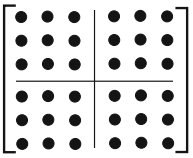
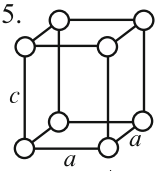
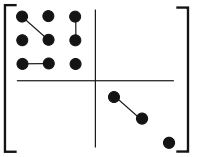
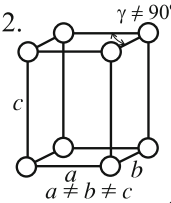
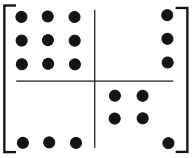
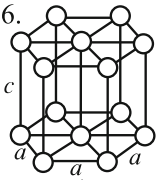
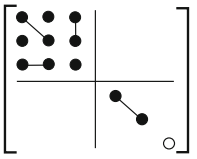
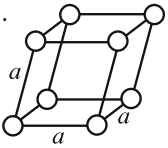
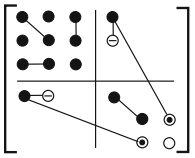
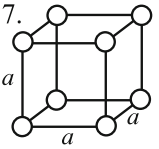
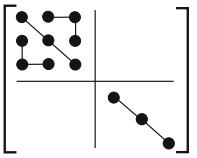
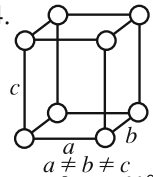
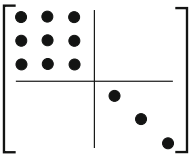
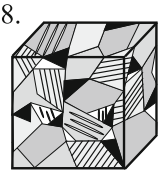
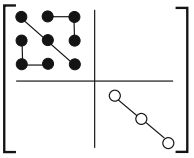
This kind of transverse isotropy will further be classified as *transverse isotropy* case *of hexagonal symmetry* (5 independent components in contrast to another *transverse isotropy of tetragonal symmetry* where all 6 components are truly independent, see Table 1.1).

The 6×6 *transversely isotropic compliance matrix* is of the following form:

$$[\widetilde{\mathbb{E}}^{-1}] = \begin{bmatrix} \frac{1}{E_1} & -\frac{\nu_{12}}{E_1} & -\frac{\nu_{12}}{E_1} & & & & \\ -\frac{\nu_{12}}{E_1} & \frac{1}{E_2} & -\frac{\nu_{23}}{E_2} & & & & \\ -\frac{\nu_{12}}{E_1} & -\frac{\nu_{23}}{E_2} & \frac{1}{E_2} & & & & \\ & & & \frac{1}{G_{23}} & & & \\ & & & & \frac{1}{G_{12}} & & \\ & & & & & \frac{1}{G_{12}} & \end{bmatrix} \tag{1.83}$$

in which damage affected modules expressed in terms of damage variables are

Table 1.1 *Classification of anisotropic elastic materials* with respect to *stiffness matrix symmetry* referring to crystal lattice cf. Nye [42]

Conventional unit cells of space lattices	Stiffness matrix $[\mathbb{C}]$	Conventional unit cells of space lattices	Stiffness matrix $[\mathbb{C}]$
1.  $a \neq b \neq c$ $\alpha \neq \beta \neq \gamma \neq 90°$ triclinic lattice	 anisotropic Hooke's (21 constants)	5.  $a = b \neq c$ $\alpha = \beta = \gamma = 90°$ tetragonal lattice	 transversely isotropic tetragonal Hooke's (6 constants)
2. $\gamma \neq 90°$  $a \neq b \neq c$ $\alpha = \beta = 90°, \gamma \neq 90°$ monoclinic lattice	 monoclinic or oblique Hooke's anisotropy	6.  $a \neq c$ $\alpha = 90°, \gamma = 120°$ hexagonal lattice	 hexagonal Hooke's (5 constants)
3.  $a = b = c$ $\alpha = \beta = \gamma \neq 90°$ rhombohedral lattice	 trigonal anisotropy (6 constants)	7.  $a = b = c$ $\alpha = \beta = \gamma = 90°$ cubic (regular) lattice	 Cubic Hooke's (3 constants)
4.  $a \neq b \neq c$ $\alpha = \beta = \gamma = 90°$ orthorhombic lattice	 orthotropic Hooke's (9 constants)	8.  polycrystal	 isotropy (2 constants)

$$
\begin{aligned}
E_1 &= E\,\frac{(1-D_1)(1-D_3-D_4)-2D_2D_{2'}}{1-D_3-D_4-2\nu D_{2'}}\\
\nu_{12} &= \frac{\nu(1-D_3-D_4)-(1-\nu)D_{2'}}{1-D_3-D_4-2\nu D_{2'}}\\
E_2 &= E\,\frac{(1-D_3+D_4)[(1-D_1)(1-D_3-D_4)-2D_2D_{2'}]}{(1-D_1)(1-D_3-\nu D_4)-\nu D_2(1-D_3+D_4)-(1+\nu)D_2D_{2'}}\\
\nu_{23} &= \frac{(1-D_1)(\nu-\nu D_3-D_4)+\nu D_2(1-D_3+D_4)-(1+\nu)D_2D_{2'}}{(1-D_1)(1-D_3-\nu D_4)-\nu D_2(1-D_3+D_4)-(1+\nu)D_2D_{2'}}\\
G_{23} &= \frac{E}{2(1+\nu)}(1-D_3-D_4)\\
G_{12} &= \frac{E}{2(1+\nu)}(1-D_5)
\end{aligned}
\tag{1.84}
$$

More accurate description of *anisotropic damage* may be provided by use of *fabric tensors*, see Murakami [40], Voyiadjis and Kattan [55], Yun-bing and Xing-fu [56], Lubarda and Krajcinovic [35]. For this reason a unit spherical surface around a given point $P(x)$ in the RVE is considered (see Fig. 1.4), and the *directional distribution* $\xi(\boldsymbol{n})$ of the *microvoid density* on the unit sphere is defined as a *polynomial function* of the *direction vector* $\boldsymbol{n}$

$$\xi(\boldsymbol{n}) = D_0 + D_{ij} f_{ij}(\boldsymbol{n}) + D_{ijkl} f_{ijkl}(\boldsymbol{n}) + \cdots \tag{1.85}$$

Expression (1.85) is a *generalized Fourier series* with respect to the *irreducible tensor bases* $f_{ij}(\boldsymbol{n})$, $f_{ijkl}(\boldsymbol{n})$, ...

$$\begin{aligned} f_{ij}(\boldsymbol{n}) &= n_i n_j - \tfrac{1}{3}\delta_{ij} \\ f_{ijkl}(\boldsymbol{n}) &= n_i n_j n_k n_l - \tfrac{1}{7}(\delta_{ij} n_k n_l + \delta_{ik} n_j n_l + \delta_{il} n_j n_k \\ &\quad + \delta_{jk} n_i n_l + \delta_{jl} n_i n_k + \delta_{kl} n_i n_j) + \tfrac{1}{5\times 7}(\delta_{ij} n_k n_l \\ &\quad + \delta_{ik} n_j n_l + \delta_{il} n_j n_k) \end{aligned} \tag{1.86}$$

The *tensor bases* $f_{ij}(\boldsymbol{n})$, $f_{ijkl}(\boldsymbol{n})$, ... are symmetric with respect to the indices, consist of even-order tensor components, and have vanishing trace.

The tensors $D_0, D_{ij}, D_{ijkl}, \ldots$ characterize the *directional distribution of damage*, and are called *fabric tensors*. For given $\xi(\boldsymbol{n})$ they can be derived by calculating the following integrals (cf. Murakami [40]):

$$\begin{aligned} D_0 &= \tfrac{1}{4\pi} \int_{S^2} \xi(\boldsymbol{n}) \mathrm{d}\Omega \\ D_{ij} &= \tfrac{1}{4\pi} \tfrac{3\times 5}{2} \int_{S^2} \xi(\boldsymbol{n}) f_{ij}(\boldsymbol{n}) \mathrm{d}\Omega \\ D_{ijkl} &= \tfrac{1}{4\pi} \tfrac{3\times 5\times 7\times 9}{2\times 3\times 4} \int_{S^2} \xi(\boldsymbol{n}) f_{ijkl}(\boldsymbol{n}) \mathrm{d}\Omega \end{aligned} \tag{1.87}$$

The even-order tensors $D_0, D_{ij}, D_{ijkl}, \ldots$ represent completely the damage state of the materials, and have been used as the internal state variables in thermodynamic modeling of creep and brittle damage, see Onat and Leckie [43], Lacy et al. [32].

Concluding, it is worth to mention that *virgin material anisotropy* may either manifest from the very beginning of the elastic response when appropriate anisotropic

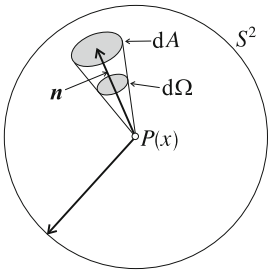


Fig. 1.4 Unit spherical surface to represent directional void distribution

formulation of Hooke's law is required or at damage initiation phase when *damage acquired anisotropy* appears as shown above. In the last case the elasticity matrix at the virgin state may have isotropic nature, whereas after some dissipative process initiates it changes to anisotropic form.

1.3 Common Invariants of the Second-Order and Fourth-Order Tensors

1.3.1 Common Invariants of Two Second-Order Tensors: The Stress/Strain and the Damage Tensors

A fundamental (irreducible) set of common invariants of two second-order tensors comprises 10 invariants. In a particular case when the common *strain-damage space* $(\boldsymbol{\varepsilon}, \boldsymbol{D})$ is considered they are furnished as

$$\begin{aligned}
J_{1\varepsilon} &= \mathrm{tr}(\boldsymbol{\varepsilon}) = \varepsilon_{ii} \\
J_{2\varepsilon} &= \tfrac{1}{2}\mathrm{tr}(\boldsymbol{\varepsilon}\cdot\boldsymbol{\varepsilon}) = \tfrac{1}{2}\varepsilon_{ij}\varepsilon_{ji} \\
J_{3\varepsilon} &= \tfrac{1}{3}\mathrm{tr}(\boldsymbol{\varepsilon}\cdot\boldsymbol{\varepsilon}\cdot\boldsymbol{\varepsilon}) = \tfrac{1}{3}\varepsilon_{ij}\varepsilon_{jk}\varepsilon_{ki} \\
J_{1D} &= \mathrm{tr}(\boldsymbol{D}) = D_{ii} \\
J_{2D} &= \tfrac{1}{2}\mathrm{tr}(\boldsymbol{D}\cdot\boldsymbol{D}) = \tfrac{1}{2}D_{ij}D_{ji} \\
J_{3D} &= \tfrac{1}{3}\mathrm{tr}(\boldsymbol{D}\cdot\boldsymbol{D}\cdot\boldsymbol{D}) = \tfrac{1}{3}D_{ij}D_{jk}D_{ki} \\
J_{1\varepsilon D} &= \mathrm{tr}(\boldsymbol{\varepsilon}\cdot\boldsymbol{D}) = \varepsilon_{ij}D_{ji} \\
J_{2\varepsilon D} &= \mathrm{tr}(\boldsymbol{\varepsilon}\cdot\boldsymbol{\varepsilon}\cdot\boldsymbol{D}) = \varepsilon_{ij}\varepsilon_{jk}D_{ki} \\
J_{3\varepsilon D} &= \mathrm{tr}(\boldsymbol{\varepsilon}\cdot\boldsymbol{D}\cdot\boldsymbol{D}) = \varepsilon_{ij}D_{jk}D_{ki} \\
J_{4\varepsilon D} &= \mathrm{tr}(\boldsymbol{\varepsilon}\cdot\boldsymbol{\varepsilon}\cdot\boldsymbol{D}\cdot\boldsymbol{D}) = \varepsilon_{ij}\varepsilon_{jk}D_{kl}D_{li}
\end{aligned} \tag{1.88}$$

When another *stress-damage* commonly used *space* $(\boldsymbol{\sigma}, \boldsymbol{D})$ is considered the following holds:

$$\begin{aligned}
J_{1\sigma} &= \mathrm{tr}(\boldsymbol{\sigma}) = \sigma_{ii} \\
J_{2\sigma} &= \tfrac{1}{2}\mathrm{tr}(\boldsymbol{\sigma}\cdot\boldsymbol{\sigma}) = \tfrac{1}{2}\sigma_{ij}\sigma_{ji} \\
J_{3\sigma} &= \tfrac{1}{3}\mathrm{tr}(\boldsymbol{\sigma}\cdot\boldsymbol{\sigma}\cdot\boldsymbol{\sigma}) = \tfrac{1}{3}\sigma_{ij}\sigma_{jk}\sigma_{ki} \\
J_{1D} &= \mathrm{tr}(\boldsymbol{D}) = D_{ii} \\
J_{2D} &= \tfrac{1}{2}\mathrm{tr}(\boldsymbol{D}\cdot\boldsymbol{D}) = \tfrac{1}{2}D_{ij}D_{ji} \\
J_{3D} &= \tfrac{1}{3}\mathrm{tr}(\boldsymbol{D}\cdot\boldsymbol{D}\cdot\boldsymbol{D}) = \tfrac{1}{3}D_{ij}D_{jk}D_{ki} \\
J_{1\sigma D} &= \mathrm{tr}(\boldsymbol{\sigma}\cdot\boldsymbol{D}) = \sigma_{ij}D_{ji} \\
J_{2\sigma D} &= \mathrm{tr}(\boldsymbol{\sigma}\cdot\boldsymbol{\sigma}\cdot\boldsymbol{D}) = \sigma_{ij}\sigma_{jk}D_{ki} \\
J_{3\sigma D} &= \mathrm{tr}(\boldsymbol{\sigma}\cdot\boldsymbol{D}\cdot\boldsymbol{D}) = \sigma_{ij}D_{jk}D_{ki} \\
J_{4\sigma D} &= \mathrm{tr}(\boldsymbol{\sigma}\cdot\boldsymbol{\sigma}\cdot\boldsymbol{D}\cdot\boldsymbol{D}) = \sigma_{ij}\sigma_{jk}D_{kl}D_{li}
\end{aligned} \tag{1.89}$$

1.3.2 Common Invariants of Two Different-Order Tensors: The Second Stress/Strain and the Fourth-Order Structural Tensors

The orthotropic material is characterized by three mutually perpendicular symmetry planes determined by three second-rank tensors called the *structural tensors* in terms of which the elastic strain energy $\mathcal{W}$ can be represented as

$$\mathcal{W} = \mathcal{W}(\varepsilon, \boldsymbol{M}^{(1)}, \boldsymbol{M}^{(2)}, \boldsymbol{M}^{(3)}) \tag{1.90}$$

When axes of material orthotropy coincide with axes of reference frame the structural tensors take the simplified forms

$$\boldsymbol{M}^{(1)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & \\ & & 0 \end{bmatrix} \quad \boldsymbol{M}^{(2)} = \begin{bmatrix} 0 & 0 & 0 \\ & 1 & 0 \\ & & 0 \end{bmatrix} \quad \boldsymbol{M}^{(3)} = \begin{bmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ & & 1 \end{bmatrix} \tag{1.91}$$

for which the following holds:

$$\mathbf{1} = \boldsymbol{M}^{(1)} + \boldsymbol{M}^{(2)} + \boldsymbol{M}^{(3)} \tag{1.92}$$

Condition (1.92) means that the structural tensors are mutually dependent. Hence, *elastic strain energy* (1.90) can be represented *in terms of* two *structural tensors* chosen as independent, e.g., $\boldsymbol{M}^{(1)}$ and $\boldsymbol{M}^{(2)}$

$$\mathcal{W} = \mathcal{W}(\varepsilon, \boldsymbol{M}^{(1)}, \boldsymbol{M}^{(2)}) \tag{1.93}$$

Analogously, strain tensor can be written as $\varepsilon = \mathbf{1} \cdot \varepsilon = \varepsilon \cdot \mathbf{1}$, which finally leads to

$$\begin{aligned} \varepsilon &= \varepsilon \cdot \boldsymbol{M}^{(1)} + \varepsilon \cdot \boldsymbol{M}^{(2)} + \varepsilon \cdot \boldsymbol{M}^{(3)} = \varepsilon \cdot \mathbf{1} \\ \varepsilon &= \boldsymbol{M}^{(1)} \cdot \varepsilon + \boldsymbol{M}^{(2)} \cdot \varepsilon + \boldsymbol{M}^{(3)} \cdot \varepsilon = \mathbf{1} \cdot \varepsilon \end{aligned} \tag{1.94}$$

Summing up, the above equations assure symmetry of the strain tensor ε

$$\varepsilon = \frac{1}{2}(\varepsilon \cdot \boldsymbol{M}^{(1)} + \boldsymbol{M}^{(1)} \cdot \varepsilon) + \frac{1}{2}(\varepsilon \cdot \boldsymbol{M}^{(2)} + \boldsymbol{M}^{(2)} \cdot \varepsilon) + \frac{1}{2}(\varepsilon \cdot \boldsymbol{M}^{(3)} + \boldsymbol{M}^{(3)} \cdot \varepsilon) \tag{1.95}$$

The following *representation of elastic strain energy in terms of* 7 *invariants* can be obtained:

$$\begin{aligned} \mathcal{W} = \mathcal{W}\big[&\mathrm{tr}(\varepsilon), \tfrac{1}{2}\mathrm{tr}(\varepsilon \cdot \varepsilon), \tfrac{1}{3}\mathrm{tr}(\varepsilon \cdot \varepsilon \cdot \varepsilon), \\ &\mathrm{tr}(\varepsilon \cdot \boldsymbol{M}^{(1)}), \mathrm{tr}(\varepsilon \cdot \boldsymbol{M}^{(2)}), \mathrm{tr}(\varepsilon \cdot \boldsymbol{M}^{(3)}), \mathrm{tr}(\varepsilon \cdot \varepsilon \cdot \boldsymbol{M}^{(2)})\big] \end{aligned} \tag{1.96}$$

comprising both 3 *single strain invariants* and 4 *common strain and structural tensor invariants*. However, based on (1.95) two first-strain invariants can be represented as

$$\begin{aligned} \mathrm{tr}(\boldsymbol{\varepsilon}) &= \mathrm{tr}(\boldsymbol{\varepsilon} \cdot \boldsymbol{M}^{(1)}) + \mathrm{tr}(\boldsymbol{\varepsilon} \cdot \boldsymbol{M}^{(2)}) + \mathrm{tr}(\boldsymbol{\varepsilon} \cdot \boldsymbol{M}^{(3)}) \\ \mathrm{tr}(\boldsymbol{\varepsilon} \cdot \boldsymbol{\varepsilon}) &= \mathrm{tr}(\boldsymbol{\varepsilon} \cdot \boldsymbol{\varepsilon} \cdot \boldsymbol{M}^{(1)}) + \mathrm{tr}(\boldsymbol{\varepsilon} \cdot \boldsymbol{\varepsilon} \cdot \boldsymbol{M}^{(2)}) + \mathrm{tr}(\boldsymbol{\varepsilon} \cdot \boldsymbol{\varepsilon} \cdot \boldsymbol{M}^{(3)}) \end{aligned} \tag{1.97}$$

whereas the third strain invariant $\frac{1}{3}\mathrm{tr}(\boldsymbol{\varepsilon} \cdot \boldsymbol{\varepsilon} \cdot \boldsymbol{\varepsilon})$ is ignored because strain energy must be a quadratic function of strain $\boldsymbol{\varepsilon}$. For further details see Sect. 1.7.3.

1.4 Classification of Elastic Materials with Respect to Symmetry Groups and Classes

For further considerations, analogy between the *crystal lattice symmetry* groups and classes and corresponding *symmetry* of the *stiffness matrices* defined for crystalline materials might be useful (cf. e.g. Nye [42]). Unit cells of the eight conventional crystal lattices are demonstrated based on Love [34] and Jastrzebski [23], whereas corresponding *constitutive elasticity matrices* are schematically sketched applying Nye's graphics (symbol • refers to independent element, symbol ∘ refers to dependent element, symbols ●——● or ○——○ represent pairs of identical matrix elements, symbols ●——⊙ stand for pairs of elements in which one is doubled (effect of engineering notation applied to shear strain $\gamma_{ij} = 2\varepsilon_{ij}$), whereas symbols ●——⊖ denote pairs of elements of the same absolute value but opposite signs, respectively.

1.4.1 Triclinic Hooke's Anisotropy (21 Constants)

Deformation of representative cube taken of the generally anisotropic material of *triclinic symmetry* subjected to exemplary axial tension along three axes is fully anisotropic. This means that it comprises both anisotropic axial strains (transformation of the cube to a rectangular prism) and anisotropic shear strains (transformation of the rectangular prism to a parallelepiped), as schematically sketched in Fig. 1.5. In such a case of general deformation the *elastic compliance matrix* is fully populated. In other words, all components of the *columnar stress vector* depend on all six components of the *columnar strain vector* (36 combinations). Final representation of compliance matrix for fully *anisotropic (triclinic) material* is as follows:

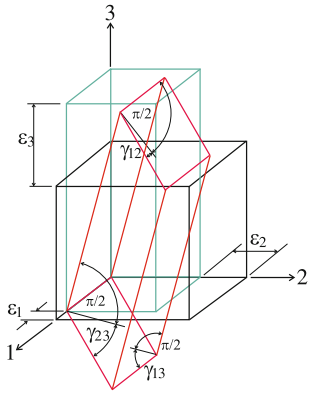


Fig. 1.5 Schematic deformation of representative cube of anisotropic triclinic material under uniaxial tension along three axes

$$\left[\mathbb{E}^{-1}\right] = \left[\begin{array}{ccc|ccc} \frac{1}{E_{11}} & -\frac{\nu_{21}}{E_{11}} & -\frac{\nu_{31}}{E_{11}} & \frac{\eta_{23(1)}}{E_{11}} & \frac{\eta_{31(1)}}{E_{11}} & \frac{\eta_{12(1)}}{E_{11}} \\ -\frac{\nu_{12}}{E_{22}} & \frac{1}{E_{22}} & -\frac{\nu_{32}}{E_{22}} & \frac{\eta_{23(2)}}{E_{22}} & \frac{\eta_{31(2)}}{E_{22}} & \frac{\eta_{12(2)}}{E_{22}} \\ -\frac{\nu_{13}}{E_{33}} & -\frac{\nu_{23}}{E_{33}} & \frac{1}{E_{33}} & \frac{\eta_{23(3)}}{E_{33}} & \frac{\eta_{31(3)}}{E_{33}} & \frac{\eta_{12(3)}}{E_{33}} \\ \hline \frac{\eta_{(1)23}}{G_{23}} & \frac{\eta_{(2)23}}{G_{23}} & \frac{\eta_{(3)23}}{G_{23}} & \frac{1}{G_{23}} & \frac{\mu_{31(23)}}{G_{23}} & \frac{\mu_{12(23)}}{G_{23}} \\ \frac{\eta_{(1)31}}{G_{31}} & \frac{\eta_{(2)31}}{G_{31}} & \frac{\eta_{(3)31}}{G_{31}} & \frac{\mu_{(23)31}}{G_{31}} & \frac{1}{G_{31}} & \frac{\mu_{12(31)}}{G_{31}} \\ \frac{\eta_{(1)12}}{G_{12}} & \frac{\eta_{(2)12}}{G_{12}} & \frac{\eta_{(3)12}}{G_{12}} & \frac{\mu_{(23)12}}{G_{12}} & \frac{\mu_{(31)12}}{G_{12}} & \frac{1}{G_{12}} \end{array}\right] \quad (1.98)$$

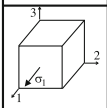
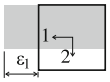
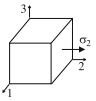
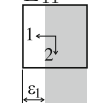
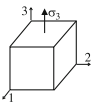
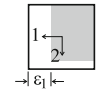
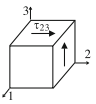
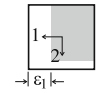
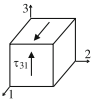
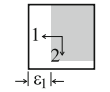
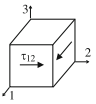
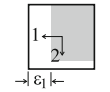
Symmetry of the *elastic compliance matrix* (1.98) results from symmetry of both stress and strain tensors, namely

$$\begin{aligned} \frac{\nu_{ij}}{E_{jj}} = \frac{\nu_{ji}}{E_{ii}} &\longrightarrow \nu_{ij}E_{ii} = \nu_{ji}E_{jj} \\ \frac{\eta_{ij(k)}}{E_{kk}} = \frac{\eta_{(k)ij}}{G_{ij}} &\longrightarrow \eta_{ij(k)}G_{ij} = \eta_{(k)ij}E_{kk} \\ \frac{\mu_{ij(ki)}}{G_{ki}} = \frac{\mu_{(ki)ij}}{G_{ji}} &\longrightarrow \mu_{ij(ki)}G_{ji} = \mu_{(ki)ij}G_{ki} \end{aligned} \quad (1.99)$$

In should be pointed out that the symmetry $\mathbb{E}^{-1}_{ij} = \mathbb{E}^{-1}_{ji}$ holds for elements of compliance matrix but not for corresponding *engineering material constants* $E_{ii}, \nu_{ij}, G_{ij}, \eta_{(i)jk}, \mu_{ij(ki)}$ as shown in (1.100) versus (1.98) (Table 1.2)

$$\left[\mathbb{E}^{-1}\right] = \left[\begin{array}{ccc|ccc} E^{-1}_{11} & E^{-1}_{12} & E^{-1}_{13} & E^{-1}_{14} & E^{-1}_{15} & E^{-1}_{16} \\ E^{-1}_{21} & E^{-1}_{22} & E^{-1}_{23} & E^{-1}_{24} & E^{-1}_{25} & E^{-1}_{26} \\ E^{-1}_{31} & E^{-1}_{32} & E^{-1}_{33} & E^{-1}_{34} & E^{-1}_{35} & E^{-1}_{36} \\ \hline E^{-1}_{41} & E^{-1}_{42} & E^{-1}_{43} & E^{-1}_{44} & E^{-1}_{45} & E^{-1}_{46} \\ E^{-1}_{51} & E^{-1}_{52} & E^{-1}_{53} & E^{-1}_{54} & E^{-1}_{55} & E^{-1}_{56} \\ E^{-1}_{61} & E^{-1}_{62} & E^{-1}_{63} & E^{-1}_{64} & E^{-1}_{65} & E^{-1}_{66} \end{array}\right] \quad (1.100)$$

Table 1.2 Superposition of the strain tensor components of anisotropic material corresponding to subsequent stress tensor components

State	Strains					
	axial			shear		
	ε_1	ε_2	ε_3	γ_{23}	γ_{31}	γ_{12}
						
	$\frac{1}{E_{11}}\sigma_1$	$-\frac{\nu_{21}}{E_{11}}\sigma_1$	$-\frac{\nu_{31}}{E_{11}}\sigma_1$	$\frac{\eta_{23(1)}}{E_{11}}\sigma_1$	$\frac{\eta_{31(1)}}{E_{11}}\sigma_1$	$\frac{\eta_{12(1)}}{E_{11}}\sigma_1$
						
	$-\frac{\nu_{12}}{E_{22}}\sigma_2$	$\frac{1}{E_{22}}\sigma_2$	$-\frac{\nu_{32}}{E_{22}}\sigma_2$	$\frac{\eta_{23(2)}}{E_{22}}\sigma_2$	$\frac{\eta_{31(2)}}{E_{22}}\sigma_2$	$\frac{\eta_{12(2)}}{E_{22}}\sigma_2$
						
	$-\frac{\nu_{13}}{E_{33}}\sigma_3$	$-\frac{\nu_{23}}{E_{33}}\sigma_3$	$\frac{1}{E_{33}}\sigma_3$	$\frac{\eta_{23(3)}}{E_{33}}\sigma_3$	$\frac{\eta_{31(3)}}{E_{33}}\sigma_3$	$\frac{\eta_{12(3)}}{E_{33}}\sigma_3$
						
	$\frac{\eta_{(1)23}}{G_{23}}\tau_{23}$	$\frac{\eta_{(2)23}}{G_{23}}\tau_{23}$	$\frac{\eta_{(3)23}}{G_{23}}\tau_{23}$	$\frac{1}{G_{23}}\tau_{23}$	$\frac{\mu_{(31)23}}{G_{23}}\tau_{23}$	$\frac{\mu_{(12)23}}{G_{23}}\tau_{23}$
						
	$\frac{\eta_{(1)31}}{G_{31}}\tau_{31}$	$\frac{\eta_{(2)31}}{G_{31}}\tau_{31}$	$\frac{\eta_{(3)31}}{G_{31}}\tau_{31}$	$\frac{\mu_{(23)31}}{G_{31}}\tau_{31}$	$\frac{1}{G_{31}}\tau_{31}$	$\frac{\mu_{(12)31}}{G_{31}}\tau_{31}$
						
	$\frac{\eta_{(1)12}}{G_{12}}\tau_{12}$	$\frac{\eta_{(2)12}}{G_{12}}\tau_{12}$	$\frac{\eta_{(3)12}}{G_{12}}\tau_{12}$	$\frac{\mu_{(23)12}}{G_{12}}\tau_{12}$	$\frac{\mu_{(31)11}}{G_{12}}\tau_{12}$	$\frac{1}{G_{12}}\tau_{12}$

Elastic engineering modules of five types can be sorted in the following way, after Lekhnitskii [33]:

- E_{ii}—axial modules (3 *generalized Young's modules*)
- G_{ij}—shear modules for planes parallel to the coordinate planes (3 *generalized Kirchhoff's modules*)
- ν_{ij}—Poisson's ratios characterizing the contraction in the direction of one axis when tension is applied in the direction of another axis (3 *generalized Poisson's coefficients*)

- $\mu_{ij(kl)}$—coefficients characterizing shears in planes parallel to the coordinate planes resulting from shear stresses acting in other planes parallel to the coordinate planes (3 *Chencov's modules*)
- $\eta_{i(jk)}$—mutual influence coefficients characterizing extensions in the directions of the coordinate axes resulting from shear stresses acting in the coordinate planes (9 *Rabinovich's modules*)

The aforementioned modules are listed in Table 1.3. In case of full anisotropy the shear stress acting in one plane results in a shear strain appearing in another plane. This effect is described by the three *Chencov modules*. Hence, the bottom right-hand side block of the *compliance matrix* (1.100) is fully populated, in contrast to the case of isotropy where shear stress acting in one plane results in shear strain in the same plane exclusively. This means that in case of isotropy the considered block of compliance matrix must have the diagonal form.

In order to describe effect of axial stresses on shear strains (upper right-hand side block), as well as effect of shear stresses on axial strains (lower left-hand side block), it is necessary to define 9 additional modules $\eta_{(i)jk}$, called *Rabinovich's modules* where the appropriate symmetry conditions hold (1.99). The total number of discussed modules is equal to 21. However, only 18 of them are truly independent because the compliance matrix $[\mathbb{E}^{-1}]$ has to obey transformation with respect to three Euler angles. It should be pointed out that in general case of anisotropy it is not possible to find any reference frame for which any element of the compliance matrix can be equal to zero. The general case of anisotropy corresponds to the *triclinic symmetry lattice cell* in which all three edges differ from each other and all three angles between them differ from each other and none of them is equal to 90°, as shown in item 1 of Table 1.1.

Table 1.3 *Engineering modules* defining elements of *elastic compliance matrix* (1.98) of fully anisotropic material

Engineering elastic modules	Coupling between		Corresponding axes or planes	Number of coefficients
	Stress	Strain		
E_{11}, E_{22}, E_{33}	Axial	Extension	The same axes $1 \rightarrow 1$, etc.	3
G_{12}, G_{32}, G_{31}	Shear	Shear strain	The same planes $12 \rightarrow 12$, etc.	3
$\nu_{21}, \nu_{31}, \nu_{32}$	Axial	Extension	Different exes $1 \rightarrow 2$, etc.	3
$\mu_{31(23)}, \mu_{12(23)}, \mu_{12(31)}$	Shear	Shear strain	Different planes $13 \rightarrow 23$, etc.	3
$\eta_{23(1)}, \ldots, \eta_{12(3)}$	Shear	Extension	Normal to shear plane $23 \rightarrow 1$, etc.	9

1.4.2 Monoclinic Hooke's Anisotropy (13 Constants)

Among anisotropic materials the narrower group called *monoclinic symmetry* can be distinguished. *Monoclinic or oblique symmetry* corresponds to *monoclinic space lattice cell* symmetry in which all three edges differ from each other, whereas two angles are equal to 90° and one is different, as shown in item 2 of Table 1.1. The corresponding stiffness matrix symmetry characterizes through incomplete population in which only 13 elements are not equal to zero, as shown below.

$$\left[\mathbb{E}^{-1}\right] = \left[\begin{array}{ccc|ccc} \frac{1}{E_{11}} & -\frac{\nu_{21}}{E_{11}} & -\frac{\nu_{31}}{E_{11}} & & & \frac{\eta_{12(1)}}{E_{11}} \\ -\frac{\nu_{12}}{E_{22}} & \frac{1}{E_{22}} & -\frac{\nu_{32}}{E_{22}} & & & \frac{\eta_{12(2)}}{E_{22}} \\ -\frac{\nu_{13}}{E_{33}} & -\frac{\nu_{23}}{E_{33}} & \frac{1}{E_{33}} & & & \frac{\eta_{12(3)}}{E_{33}} \\ \hline & & & \frac{1}{G_{23}} & \frac{\mu_{31(23)}}{G_{23}} & \\ & & & \frac{\mu_{(23)31}}{G_{31}} & \frac{1}{G_{31}} & \\ \frac{\eta_{(1)12}}{G_{12}} & \frac{\eta_{(2)12}}{G_{12}} & \frac{\eta_{(3)12}}{G_{12}} & & & \frac{1}{G_{12}} \end{array}\right] \tag{1.101}$$

In other words, in case of monoclinic symmetry only three of the Rabinovich modules and only one of the Chencov modules are different from zero.

1.4.3 Trigonal/Rhombohedral Hooke's Anisotropy (6 Constants)

Another important narrower case of material anisotropy called trigonal anisotropy can be distinguished. The trigonal anisotropy corresponds to the *rhombohedral cell lattice* in which all three edges are equal to each other and all three angles are equal but different from 90°, as shown in item 3 of Table 1.1. The corresponding *compliance matrix* takes the following representation:

$$\left[\mathbb{E}^{-1}\right] = \left[\begin{array}{ccc|ccc} \frac{1}{E_{11}} & -\frac{\nu_{21}}{E_{11}} & -\frac{\nu_{31}}{E_{11}} & \frac{\eta_{23(1)}}{E_{11}} & & \\ -\frac{\nu_{12}}{E_{22}} & \frac{1}{E_{11}} & -\frac{\nu_{31}}{E_{11}} & -\frac{\eta_{23(1)}}{E_{11}} & & \\ -\frac{\nu_{13}}{E_{33}} & -\frac{\nu_{13}}{E_{33}} & \frac{1}{E_{33}} & & & \\ \hline \frac{\eta_{(1)23}}{G_{23}} & -\frac{\eta_{(1)23}}{G_{23}} & & \frac{1}{G_{23}} & & \\ & & & & \frac{1}{G_{31}} & \frac{2\mu_{12(31)}}{G_{31}} \\ & & & & \frac{2\mu_{(31)12}}{G_{12}} & \frac{2(1+\nu_{12})}{E_{11}} \end{array}\right] \tag{1.102}$$

It is seen that in case of *trigonal symmetry* among *Rabinovich's modules* only two are nonzeroth but in fact only one of them is independent because they only differ in sign. Additionally, only one *Chencov's modulus* is different from zero but in fact it is the dependent modulus due to the specific coupling between components $2E_{14}^{-1} = E_{56}^{-1}$ and $E_{24}^{-1} = -E_{14}^{-1}$ as well as $E_{11}^{-1} = E_{22}^{-1}$, $E_{44}^{-1} = E_{55}^{-1}$, $E_{13}^{-1} = E_{23}^{-1}$ whereas $E_{66}^{-1} = (E_{11}^{-1} - E_{12}^{-1})/2$ must hold. Finally for trigonal symmetry only 6 elements of the compliance matrix are independent, see Berryman [2].

1.4.4 Orthorhombic Hooke's Orthotropy (9 Constants)

The majority of engineering materials exhibit a specific symmetry property, which may result in reduction of the number of nonzeroth elastic modules. It can be done when, for chosen symmetry group or class, some particular material directions are defined in such a way that transformation of the compliance matrix from an arbitrary coordinate frame to the given *structural symmetry frame* leads to the zeroth population of the top right-hand side and the bottom left-hand side blocks of the *compliance matrix* (1.98), and additionally the bottom right-hand side block possesses a diagonal form. In such practically important cases both the nine Rabinovich $\eta_{(i)jk}$ and the three Chencov $\mu_{ij(kl)}$ modules are equal to zero, and consequently, coupling between the shear stresses and elongations does not exist such that shear strains are produced exclusively by the action of stresses at the same planes. In this particular symmetry, called orthotropy, there exist three mutually perpendicular axes (1, 2, 3) that determine the three *material orthotropy planes*. The *orthotropy symmetry* case corresponds to the *orthorhombic lattice* in which all three edges differ each from other but all angles are equal to 90°, as presented in item 4 of Table 1.1.

$$\left[\mathbb{E}^{-1}\right] = \left[\begin{array}{ccc|ccc} \frac{1}{E_{11}} & -\frac{\nu_{21}}{E_{11}} & -\frac{\nu_{31}}{E_{11}} & & & \\ -\frac{\nu_{12}}{E_{22}} & \frac{1}{E_{22}} & -\frac{\nu_{32}}{E_{22}} & & & \\ -\frac{\nu_{13}}{E_{33}} & -\frac{\nu_{23}}{E_{33}} & \frac{1}{E_{33}} & & & \\ \hline & & & \frac{1}{G_{23}} & 0 & 0 \\ & & & & \frac{1}{G_{13}} & 0 \\ & & & & & \frac{1}{G_{12}} \end{array}\right] \left[\begin{array}{ccc|ccc} \bullet & \bullet & \bullet & & & \\ & \bullet & \bullet & & & \\ & & \bullet & & & \\ \hline & & & & & \\ & & & & \bullet & \\ & & & & & \bullet \\ & & & & & & \bullet \end{array}\right] \tag{1.103}$$

The following conditions must hold to assure matrix symmetry:

$$\frac{\nu_{21}}{E_{11}} = \frac{\nu_{12}}{E_{22}} \qquad \frac{\nu_{13}}{E_{33}} = \frac{\nu_{31}}{E_{11}} \qquad \frac{\nu_{23}}{E_{33}} = \frac{\nu_{32}}{E_{22}} \tag{1.104}$$

Finally, in case of orthotropy the number of independent material constants is nine, that is, three *generalized Hooke's modules* E_{11}, E_{22}, E_{33}, 3 *generalized Kirchhoff's modules* G_{12}, G_{23}, G_{31} and three *generalized Poisson's ratios* $\nu_{21}, \nu_{23}, \nu_{31}$.

1.4.5 Tetragonal Hooke's Transverse Isotropy (6 Constants)

For several engineering applications the general orthotropic symmetry model seems too complicated, since additional symmetry conditions frequently appear. Particularly, when conditions of isotropy hold in selected orthotropy plane the so-called transverse isotropy obeys.

In case of so-called *tetragonal symmetry* material properties in the plane (1, 2) satisfy condition of cubic symmetry, see item 5 of Table 1.1

$$E_{11} = E_{22}, \quad G_{13} = G_{23}, \quad \nu_{31} = \nu_{32} \tag{1.105}$$

Hence, in case of transverse isotropy of tetragonal symmetry the number of independent material constants is equal to 6: E_{11}, E_{33}, G_{23}, G_{12}, ν_{21}, ν_{31}. Corresponding crystal lattice is sketched in item 5 of Table 1.1, where *tetragonal lattice* being special case of the *orthorhombic lattice* with $a = b \neq c$ obeys.

When the constraints (1.105) are applied to compliance matrix (1.103) the transverse isotropy tetragonal symmetry case yields

$$\left[\mathbb{E}^{-1}\right] = \begin{bmatrix} \frac{1}{E_{11}} & -\frac{\nu_{21}}{E_{11}} & -\frac{\nu_{31}}{E_{11}} & & & \\ & \frac{1}{E_{11}} & -\frac{\nu_{31}}{E_{11}} & & & \\ & & \frac{1}{E_{33}} & & & \\ & & & \frac{1}{G_{23}} & & \\ & & & & \frac{1}{G_{23}} & \\ & & & & & \frac{1}{G_{12}} \end{bmatrix} \tag{1.106}$$

It follows from the constraints (1.105) that six independent material constants define the *tetragonal symmetry matrix*:

- E_{11}, E_{33}—two Young's modulus in the plane of isotropy and direction perpendicular to this plane,
- ν_{21}, ν_{31}—two Poisson's ratios referring to transverse contraction or swelling caused by tension or compression in direction perpendicular to isotropy plane,
- G_{12}, G_{23}—two different Kirchhoff's modules in the isotropy or orthotropy planes.

1.4.6 Hexagonal Hooke's Transverse Isotropy (5 Constants)

In special case of the transverse isotropy called *hexagonal symmetry* the additional constraint must obey for the shear modulus in the isotropy plane

$$G_{12} = \frac{E_{11}}{2(1+\nu_{21})} \quad \text{or} \quad E_{66}^{-1} = 2\left(E_{11}^{-1} - E_{12}^{-1}\right) \tag{1.107}$$

where modulus G_{12} is expressed in terms of the *transverse Young modulus* E_{11} and *transverse Poisson's ratio* ν_{21}. Hence, in case of the transverse isotropy of hexagonal symmetry the number of independent constants is equal to 5: E_{11}, E_{33}, G_{23}, ν_{21}, ν_{31}. A choice of the five independent material constants from among six can be performed in an optional way, for instance

$$\left[\mathbb{E}^{-1}\right] = \left[\begin{array}{ccc|ccc} \frac{1}{E_{11}} & -\frac{\nu_{21}}{E_{11}} & -\frac{\nu_{31}}{E_{11}} & & & \\ & \frac{1}{E_{11}} & -\frac{\nu_{31}}{E_{11}} & & & \\ & & \frac{1}{E_{33}} & & & \\ \hline & & & \frac{1}{G_{23}} & & \\ & & & & \frac{1}{G_{23}} & \\ & & & & & \frac{2(1+\nu_{21})}{E_{11}} \end{array}\right] \left[\begin{array}{ccc|ccc} \bullet & \bullet & \bullet & & & \\ & \bullet & \bullet & & & \\ & & \bullet & & & \\ \hline & & & \bullet & & \\ & & & & \bullet & \\ & & & & & \circ \end{array}\right] \tag{1.108}$$

Rolled metals, some multi-phase composite materials, basalt, or columnar ice are examples of *transversely isotropic materials*, however, precise distinction between the *tetragonal* or *hexagonal symmetry classes* is often difficult (see for example Gan et al. [16]).

1.4.7 Cubic Hooke's Symmetry (3 Constants)

Further reduction in the number of independent constants leads to cubic symmetry for which the compliance matrix is characterized by three independent material constants $E_{11} = E_{22} = E_{33} = E$, $G_{23} = G_{31} = G_{12} = G$ and $\nu_{21} = \nu_{31} = \nu_{32} = \nu$. Hence, the following form of the compliance matrix is furnished:

$$\left[\mathbb{E}^{-1}\right] = \left[\begin{array}{ccc|ccc} \frac{1}{E} & -\frac{\nu}{E} & -\frac{\nu}{E} & & & \\ & \frac{1}{E} & -\frac{\nu}{E} & & & \\ & & \frac{1}{E} & & & \\ \hline & & & \frac{1}{G} & & \\ & & & & \frac{1}{G} & \\ & & & & & \frac{1}{G} \end{array}\right] \left[\begin{array}{ccc|ccc} \bullet & \bullet & \bullet & & & \\ & \bullet & \bullet & & & \\ & & \bullet & & & \\ \hline & & & \bullet & & \\ & & & & \bullet & \\ & & & & & \bullet \end{array}\right] \tag{1.109}$$

Note that in case of *cubic symmetry* the condition (1.107) does not hold. The corresponding *cubic* or *regular lattice* is shown in item 7 of Table 1.1. A particular example of the cubic symmetry material is *nickel-based single crystal superalloy* widely used

in aircraft engines, especially for turbine blades as discussed by Desmorat and Marull [12]. The cubic symmetry is the narrower symmetry case known from crystallography, see Jastrzebski [23], since fully isotropic crystal lattices are unknown.

1.4.8 Isotropic Hooke's Symmetry (2 Constants)

All the aforementioned symmetry groups have equivalences in existing crystal lattice systems. Nevertheless, even narrower than the cubic symmetry called isotropy is frequently used. The isotropy requires the infinite symmetry group which means that all material directions are equivalent in terms of mechanical, thermal, electric, optical, and magnetic properties. In other words it is not possible to distinguish any specific direction. The isotropy is helpful when describing the majority of *polycrystalline materials* either in a virgin state or artificially fabricated as *particulate composites*, *nano-composites*, etc., see item 8 of Table 1.1.

In an isotropic material physical properties are independent of the reference frame. Hence, any optional *reference frame* x, y, z is sufficient for unique definition of material properties. In order to derive mathematical form of the *Hooke law of isotropic material* it is most convenient to apply superposition of strain components $\{\varepsilon\} = \{\varepsilon_x, \varepsilon_y, \varepsilon_z, \gamma_{yz}, \gamma_{zx}, \gamma_{xy}\}$ caused by subsequent stress components $\{\boldsymbol{\sigma}\} = \{\sigma_x, \sigma_y, \sigma_z, \tau_{yz}, \tau_{zx}, \tau_{xy}\}$ (see Table 1.2). Applying vector-matrix notation the *isotropic Hooke law* takes the form

$$\{\varepsilon\} = \left[\mathbb{E}^{-1}\right]\{\boldsymbol{\sigma}\} \tag{1.110}$$

where the *isotropic compliance matrix* $[\mathbb{E}^{-1}]$ takes the following representation.

$$\left[\mathbb{E}^{-1}\right] = \begin{bmatrix} \frac{1}{E} & \left(\frac{1}{E} - \frac{1}{2G}\right) & \left(\frac{1}{E} - \frac{1}{2G}\right) & & & \\ & \frac{1}{E} & \left(\frac{1}{E} - \frac{1}{2G}\right) & & & \\ & & \frac{1}{E} & & & \\ & & & \frac{1}{G} & & \\ & & & & \frac{1}{G} & \\ & & & & & \frac{1}{G} \end{bmatrix} \begin{bmatrix} \circ & \circ & \circ & & & \\ & \bullet & & & & \\ & & \bullet & & & \\ & & & & & \\ & & & & \bullet & \\ & & & & & \bullet \end{bmatrix} \tag{1.111}$$

It is clear that the elastic isotropic material is uniquely defined by two independent material constants, the choice of which from among E, G, ν is optional. In the above representation diagonal modules E and G are chosen as independent. Hooke's law can also be transformed to the following inverse relation, (see Ottosen and Ristinmaa [44]):

$$\{\boldsymbol{\sigma}\} = [\mathbb{E}]\{\varepsilon\} \tag{1.112}$$

where the *isotropic stiffness matrix* $[\mathbb{E}]$ is defined as

$$[\mathbb{E}] = \frac{E}{(1+\nu)(1-2\nu)} \left[\begin{array}{ccc|ccc} 1-\nu & \nu & \nu & & & \\ & 1-\nu & \nu & & & \\ & & 1-\nu & & & \\ \hline & & & \frac{1-2\nu}{2} & & \\ & & & & \frac{1-2\nu}{2} & \\ & & & & & \frac{1-2\nu}{2} \end{array}\right] \tag{1.113}$$

Format of elastic stiffness matrix (1.113) involves elements all dependent on both E and ν such that the format equivalent to (1.111) cannot be achieved. Explicit separation of the diagonal matrix elements related to shear deformation and the off-diagonal matrix elements related to extension is possible by use of the format expressed in terms of Lamé's constants λ and μ

$$\left[\begin{array}{ccc|ccc} \lambda+2\mu & \lambda & \lambda & & & \\ & \lambda+2\mu & \lambda & & & \\ & & \lambda+2\mu & & & \\ \hline & & & 2\mu & & \\ & & & & 2\mu & \\ & & & & & 2\mu \end{array}\right] \quad \left[\begin{array}{ccc|ccc} \bullet & \bullet & \bullet & & & \\ & \bullet & \bullet & & & \\ & & \bullet & & & \\ \hline & & & \circ & & \\ & & & & \circ & \\ & & & & & \circ \end{array}\right] \tag{1.114}$$

where the classical definitions of Lamé's constants hold

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)} \qquad 2\mu = G \tag{1.115}$$

It is worth to mention that the last format (1.114) can be interpreted by use of Nye graphics ($\bullet$ or $\circ$) where three off-diagonal first quarter elements and three diagonal third quarter elements are considered as independent.

The considered case of elastic isotropy is the only symmetry case for which it is possible to separate effects of *shape* and *volume changes* when *decomposition* of *strain* and *stress tensors* into *deviators* and *axiators* (1.5), (1.25) is used as

$$\varepsilon_{\mathrm{m}}\mathbf{1} = \frac{1}{3K}\sigma_{\mathrm{m}}\mathbf{1} \qquad \boldsymbol{e} = \frac{1}{2G}\boldsymbol{s} \tag{1.116}$$

Two modules in the above pair of relations called the *bulk modulus* K and the *Kirchhoff modulus* G can be expressed in terms of the *Young modulus* E and the *Poisson ratio* ν

$$K = \frac{E}{3\,(1-2\nu)} \qquad G = \frac{E}{2\,(1+\nu)} \tag{1.117}$$

However, in all other cases of material anisotropy (items 1 to 7 in Table 1.1) aforementioned separation of volumetric from shear effects is impossible.

In the particular case of *plane stress state* in the x, y plane strain component ε_z can be expressed in terms of strain components in x, y plane as follows:

$$\sigma_z = 0 \rightarrow \varepsilon_z = -\frac{\nu}{1-\nu}\left(\varepsilon_x + \varepsilon_y\right) \tag{1.118}$$

Finally, *plane stress stiffness matrix* $\mathbb{E}$ can be reduced to the 3×3 matrix

$$\left\{\begin{array}{c}\sigma_x\\ \sigma_y\\ \tau_{xy}\end{array}\right\} = \left[\begin{array}{cc|c}\frac{E}{1-\nu^2} & \frac{\nu E}{1-\nu^2} & 0\\ \frac{\nu E}{1-\nu^2} & \frac{E}{1-\nu^2} & 0\\ \hline 0 & 0 & \frac{E}{2(1+\nu)}\end{array}\right] \left\{\begin{array}{c}\varepsilon_x\\ \varepsilon_y\\ \gamma_{xy}\end{array}\right\} \tag{1.119}$$

1.5 Analogy Between Constitutive Fourth-Order Tensors: The Elastic (Hooke's) and the Yield/Failure (von Mises') of the Same Symmetry

Identification of *material symmetry* in elastic range of deformation (*anisotropy*, *orthotropy*, *transverse isotropy*, *isotropy*, etc.) is a starting point to appropriate description of both the *limit criteria* that control transition from the elastic range into the state connected with energy dissipation (*material damage*, *plastic yield*, *phase change*, etc.) as well as correct constitutive description of deformation processes in nonelastic range. It can be expected that if material in the elastic range exhibits isotropic behavior, then at least in the initial phase of plastic yielding it will approximately save properties of isotropy. The nature of elastic deformation resulting from interatomic distances change in crystal lattice is qualitatively different from the nature of plastic deformation commonly interpreted as plastic microslips considered usually as slips and dislocations between atom layers inside lattice. However, it can be expected that during more advanced plastic deformation certain orientation of plastic slip systems in the particular grains leading to appearance of a *material texture* characterized by an *acquired anisotropy* is observed (metal forming processes like rolling, drawing and press forming, see Mróz and Maciejewski [37]).

On the other hand if material even in *elastic range* is characterized by an *anisotropy* (e.g., *long fiber reinforced composites*, *wood*, *biological tissues*) it can be expected that in *nonelastic range* it will also exhibit *anisotropy*. However, it will be possible decrease of a symmetry class toward more general *plastic anisotropy*, for instance due to gradual evolution of elastic orthotropy. It can be however noticed that in case of dissipative processes different from plasticity (e.g., *material damage* or *failure*) loss of isotropy may be expected just in the elastic range, as observed in *elastic brittle materials* e.g., *ceramics*, *composites*, *concrete*, etc. Additionally, *initiation* and *growth of* other *dissipative processes* connected with *plastic yielding*, *phase change*

or other *structural changes* may result in change in the initial symmetry class. For example, in case of *spheroidal graphite cast iron* which generally exhibits *brittle-ductile behavior* a gradual transition from elastic anisotropy caused by directional damage to a state close to isotropy may be observed.

It can be assumed that features of anisotropy present in the elastic range are in general inherited in nonelastic range if some dissipative processes like plastic yield, damage, failure are present. Notice however that even, in the case when in inelastic range material behaves as isotropic, initiation of inelastic range (plasticity, damage, or failure) may provoke a *material symmetry change*. It was previously discussed that in case of damage evolution the fourth-rank *damage effect tensor* $[\mathbb{M}(\boldsymbol{D})]$ may be used to describe *degeneration* of the *elasticity tensor* $[\widetilde{\mathbb{E}}] = [\mathbb{M}(\boldsymbol{D})]^{\mathrm{T}}[\mathbb{E}][\mathbb{M}(\boldsymbol{D})]$, in a similar fashion effect of other dissipative phenomena such as *plastic yield*, *structural change* due to *phase transformation* may result in anisotropy nucleation and growth.

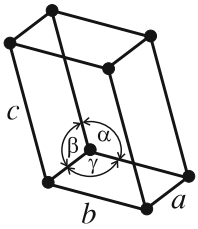
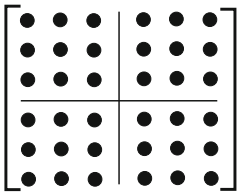
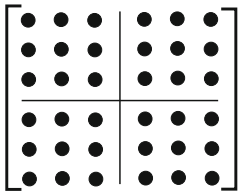
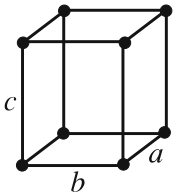
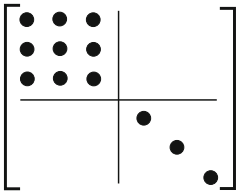
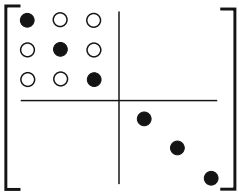
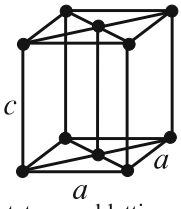
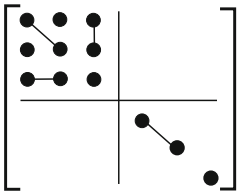
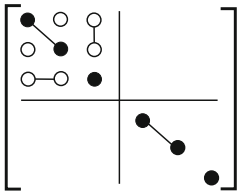
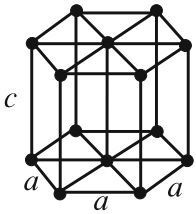
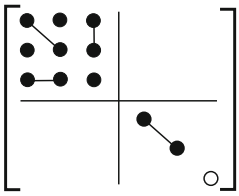
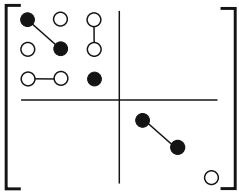
Analogy between *crystal unit cells* of *space lattices* and *constitutive matrices* of *elasticity* and *initiation of plasticity* is presented in Table 1.4. In the fundamental book by Love [34] the analogy between crystal symmetry classes and groups from one side and appropriate forms of elastic strain energy function $\mathcal{W} = \frac{1}{2}\{\boldsymbol{\varepsilon}\}^{\mathrm{T}}[\mathbb{E}]\{\boldsymbol{\varepsilon}\}$ from the other, is demonstrated. In this book an extension of the aforementioned analogy also for symmetry of constitutive matrix of plastic yield initiation $[\mathbb{H}]$ appearing in the *von Mises criterion* $\{\boldsymbol{\sigma}\}^{\mathrm{T}}[\mathbb{H}]\{\boldsymbol{\sigma}\} = 1$ is proposed. Unit cells of the four chosen space lattices have been presented following Jastrzebski [23], whereas corresponding constitutive elasticity matrices have schematically been presented applying Nye [42] graphics (symbol • refers to independent element, symbol ∘ refers to dependent element, whereas symbols •——• or ∘——∘ represent pairs of identical matrix elements).

In case of full anisotropy the complete analogy between the Hooke matrix and the von Mises plasticity matrix holds (21 independent matrix elements in both classes). However, when narrower symmetry groups are considered: *orthotropic*, *transversely isotropic* of *tetragonal* or *hexagonal* classes, it is necessary to notice that elastic matrices are usually defined in stress tensor coordinates, whereas plastic constitutive matrices are often defined in the narrower *stress deviator coordinates*.

Reduction of the *tensorial space* to the *deviatoric* one is always equivalent to imposing additional constraints, hence the number of independent elements of *plasticity matrix* is always lower than the corresponding number of independent elements of *elasticity matrix*. Namely, it is clear that the 6-element orthotropic deviatoric Hill's matrix corresponds to the 9-element orthotropic Hooke's matrix. Similarly, the 4-element *transversely isotropic tetragonal* class *Hill's matrix* corresponds to the 6-element Hooke's matrix, when the independence of Hill's matrix of hydrostatic stress is imposed. Finally, the 3-element *transversely isotropic hexagonal* class *Hu–Marin matrix* corresponds to the 5-element transversely isotropic hexagonal class Hooke matrix. Let us note that pairs of identical matrix elements are arranged in the same way in both matrices of elasticity and plasticity.

Nevertheless, some dependent elements in the plasticity matrix (as represented by symbol ∘) correspond to independent elements of elasticity matrix (sketched by symbol •), but general population of both matrices remains unchanged.

Table 1.4 Analogy between chosen symmetry groups: *triclinic*, *orthorhombic*, *tetragonal* and *hexagonal* symmetry of *Hooke's matrix* and plastic yield initiation *von Mises' matrix*

Conventional unit cells of space lattices	Chosen constitutive matrix symmetry	
	Elastic Hooke's matrix $2\mathcal{W} = \{\varepsilon\}^{\mathrm{T}}[\mathbb{E}]\{\varepsilon\}$	Plastic yield initiation matrix $\{\sigma\}^{\mathrm{T}}[\mathbb{\Pi}]\{\sigma\} = 1$
 triclinic lattice	 Hooke's (21 constants)	 von Mises (21 constants)
 orthorhombic lattice	 orthotropic Hooke's (9 constants)	 deviatorc Hill's (6 constants)
 tetragonal lattice	 transversely isotropic tetragonal Hooke's (6 constants)	 deviatoric transversely isotropic teragonal Hill's (4 constants)
 hexagonal lattice	 transversely isotropic hexagonal Hooke's (5 constants)	 pseudodeviatoric transversely isotropic hexagonal Hu-Marin's (3 constants)

The commonly used term "*transversely isotropic criterion*" may be misleading as long as an additional distinction between the *tetragonal* and the *hexagonal symmetry* is not introduced. The aforementioned distinction is known from the literature dealing with prediction of composite behavior in elastic range and its validation by experiments. For example, Sun and Vaidya [53] examined two types of materials: *Boron/Al composite* and *Graphite/Epoxy composite*, and found that some of them exhibit tetragonal while others hexagonal symmetry classes. However, even this distinction between tetragonal and hexagonal symmetry classes may be insufficient to describe some composite materials, for example, *SiC/Ti unidirectional lamina* examined by Herakovich and Aboudi [19]. This is basically caused by *residual stresses* that appear after cooling-down during *fabrication process*.

The above considerations are limited to the description of initial yield surface only. Generally, it is assumed that during plastic hardening the initial yield surface possessing certain symmetry is rebuilt in an isotropic way, which is generally not true. This question was discussed, e.g., by Malinin and Rżysko [36], who invoked Mursa [41] results for *OTCz Titanium Alloy* that confirms assumption of isotropic nature of plastic hardening. However, Hu and Marin's [22] findings for *Aluminum Alloy* showed anisotropic nature of plastic hardening rather than isotropic.

Nevertheless, the plastic hardening theory is usually taken in an isotropic fashion, e.g., Malinin and Rżysko [36], Ottosen and Ristinmaa [44], Hill [20, 21]. Such approach, although commonly used, may be questionable in light of the aforementioned experimental testing, some of which confirm such assumption, cf. Mursa [41] (*Titanium alloy*) but others contradict it cf. Hu and Marin [22] (*Aluminum alloy*), Kowalewski and Śliwowski [26] (influence of first common invariant).

1.6 Strain Energy and Complementary Energy—The State Potentials for Isotropic or Anisotropic Materials

Material is called *elastic* if its *response* (deformation) is independent of loading history (Fig. 1.6), which means that stress is determined to be strain

$$\sigma_{ij} = \sigma_{ij}(\varepsilon_{kl}) \tag{1.120}$$

or vice versa

$$\varepsilon_{ij} = \varepsilon_{ij}(\sigma_{kl}) \tag{1.121}$$

After the fully closed *loading–unloading cycle* (A-B-A), the initial material state A is recovered, independent of the loading–unloading path, Fig. 1.6b.

When the concept of *strain energy per unit volume* $\mathcal{W}$ [Nm/m^3] is introduced, the following definitions hold:

$$\mathcal{W}(\varepsilon) = \int_0^{\varepsilon} \sigma(\epsilon)\mathrm{d}\epsilon \quad \text{or} \quad \mathcal{W}(\varepsilon_{ij}) = \int_0^{\varepsilon_{ij}} \sigma_{ij}(\epsilon_{kl})\mathrm{d}\epsilon_{ij} \tag{1.122}$$

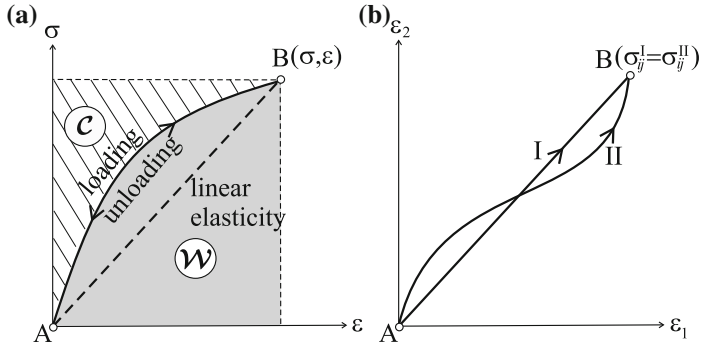


Fig. 1.6 Schematic illustration of elastic material response: **a** strain energy and complementary energy, **b** independence of final state of loading history

in case of the *uniaxial* or the *multiaxial loadings*, respectively. In the following fashion the *complementary energy per unit volume* $\mathcal{C}$ [Nm/m^3] is defined as

$$\mathcal{C}(\sigma) = \int_0^{\sigma} \varepsilon(\varsigma)\mathrm{d}\varsigma \qquad \text{or} \qquad \mathcal{C}(\sigma_{ij}) = \int_0^{\sigma_{ij}} \varsigma_{ij}(\varepsilon_{kl})\mathrm{d}\varsigma_{ij} \tag{1.123}$$

It is seen from Fig. 1.6 that the following is true:

$$\mathcal{C}(\sigma_{ij}) = \sigma_{ij}\varepsilon_{ij} - \mathcal{W}(\varepsilon_{ij}) \tag{1.124}$$

It should be emphasized that in the considered case of pure elastic material both the *strain energy* $\mathcal{W}$ and *complementary energy* $\mathcal{C}$ are independent of loading path but depend on the current state exclusively.

In a more general case, when the deformation process is accompanied by permanent (irreversible) *changes* in *material microstructure*, for instance, resulting from *plastic yielding*, *damage growth*, or *phase transformation* during *martensitic change* or other *irreversible phenomena*, the strain energy and the complementary energy depend on loading history.

In the elastic material for which strain energy depends on the current state only $\mathcal{W}(\varepsilon_{ij})$ but does not depend on strain path

$$\frac{\partial \sigma_{ij}}{\partial \varepsilon_{kl}} = \frac{\partial \sigma_{kl}}{\partial \varepsilon_{ij}} \tag{1.125}$$

the *strain energy* can be used as an invariant *potential function* for the *stresses*

$$\sigma_{ij} = \frac{\partial \mathcal{W}(\varepsilon_{ij})}{\partial \varepsilon_{ij}} \tag{1.126}$$

In a similar fashion the *complementary energy* that depends on the current state only $\mathcal{C}(\sigma_{ij})$ but does not depend on strain path

$$\frac{\partial \varepsilon_{ij}}{\partial \sigma_{kl}} = \frac{\partial \varepsilon_{kl}}{\partial \sigma_{ij}} \tag{1.127}$$

can be used as an invariant *potential function* for the *strain* as follows:

$$\varepsilon_{ij} = \frac{\partial \mathcal{C}(\sigma_{ij})}{\partial \sigma_{ij}} \tag{1.128}$$

In a general case of *nonlinear elastic material* the strain energy and the complementary energy are not equal to each other, $\mathcal{W} \neq \mathcal{C}$, whereas only in the case of linear elastic material the equality $\mathcal{W} = \mathcal{C}$ holds.

In the above considerations the initial state was treated as stress and strain free, point A ($\sigma = 0, \varepsilon = 0$) in Fig. 1.6. In the more general case a residual stress and/or strain are built-in $\mathrm{A}^{\mathrm{res}}$ ($\sigma = \sigma^{\mathrm{res}}, \varepsilon = \varepsilon^{\mathrm{res}}$). This *residual state* may result from *fabrication process* or prior *loading history* in which some irreversible changes of material structure have occurred (e.g., cyclic plasticity) or certain residual stresses or strains have been built-in (e.g., after cooling-down of *long fiber reinforced composite* characterized by different *thermal properties* of *fiber* and *matrix*). Note that in general this residual state is unknown since the whole history of material, which contains complete information about fabrication, its initial machining, as well as concerning unloading process prior to the appearance of this self-balanced residual stress, is unknown.

Consider the process of elastic deformation of material starting from the *residual state* $\mathrm{A}^{\mathrm{res}}(\sigma^{\mathrm{res}}, \varepsilon^{\mathrm{res}})$ toward the final state $\mathrm{B}(\Delta\sigma, \Delta\varepsilon)$, assuming at the beginning uniaxial tension (see Fig. 1.7).

The *increment* of *elastic strain energy* of material corresponding to applied strain $\Delta\mathcal{W}(\Delta\varepsilon)$ in the presence of *residual stress* $\varepsilon^{\mathrm{res}}$ is equal to

$$\Delta\mathcal{W}(\Delta\varepsilon) = \int_0^{\Delta\varepsilon} \Delta\sigma(\Delta\epsilon)\mathrm{d}(\Delta\epsilon) \tag{1.129}$$

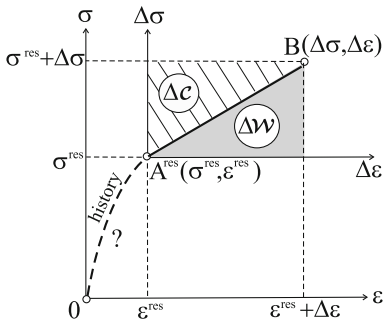


Fig. 1.7 Process of elastic deformation of material with prior residual state included

In the particular case of linear Hooke's law for *isotropic material* it yields

$$\Delta\mathcal{W}(\Delta\varepsilon) = \frac{1}{2}E(\Delta\varepsilon)^2 \tag{1.130}$$

where

$$\Delta\sigma = E\Delta\varepsilon \qquad \Delta\sigma = \sigma - \sigma^{\text{res}} \qquad \Delta\varepsilon = \varepsilon - \varepsilon^{\text{res}} \tag{1.131}$$

obey. In the more general case of *multiaxial deformation state* the strain energy per unit volume of elastic material in the presence of residual stress may be written as

$$\Delta\mathcal{W}(\Delta\varepsilon_{ij}) = \int_0^{\Delta\varepsilon_{ij}} \Delta\sigma_{ij}(\Delta\epsilon_{kl})\mathrm{d}(\Delta\epsilon_{ij}), \tag{1.132}$$

whereas in case, if *linear elastic material* is assumed, the linear relation combining stress and strain increments is furnished as

$$\Delta\sigma_{ij} = E_{ijkl}\Delta\varepsilon_{kl} \tag{1.133}$$

Equation (1.132) represents the increment of elastic energy $\Delta\mathcal{W}$ in the presence of the residual state $\varepsilon_{ij} = \varepsilon_{ij}^{\text{res}} + \Delta\varepsilon_{ij}$, hence

$$\Delta\mathcal{W}(\Delta\varepsilon_{ij}) = \int_0^{\Delta\varepsilon_{ij}} E_{ijkl}\Delta\epsilon_{kl}\mathrm{d}(\Delta\epsilon_{ij}) \tag{1.134}$$

where the *fourth-rank stiffness tensor* E_{ijkl} is used. Note that Eq. (1.134) is true both for *isotropic* and *anisotropic materials* of optional class of symmetry. The stiffness tensor E_{ijkl} comprises complete information defining the elastic material response.

In a similar way, the *complementary energy increment* $\Delta\mathcal{C}$ of elastic material in the presence of residual stress can be written as

$$\Delta\mathcal{C}(\Delta\sigma_{kl}) = \int_0^{\Delta\sigma_{kl}} \Delta\varepsilon_{kl}(\Delta\varsigma_{mn})\mathrm{d}(\Delta\varsigma_{kl}) \tag{1.135}$$

If the *linear elastic material* is assumed we arrive at

$$\Delta\mathcal{C}(\Delta\sigma_{kl}) = \int_0^{\Delta\sigma_{kl}} E_{klmn}^{-1}\Delta\varsigma_{mn}\mathrm{d}(\Delta\varsigma_{kl}) \tag{1.136}$$

where E^{-1}_{klmn} stands for the *compliance tensor* of *elastic* material

$$\Delta\varepsilon_{kl} = E^{-1}_{klmn}\Delta\sigma_{mn} \tag{1.137}$$

The above Eq. (1.137) is an extension of the law of linear elastic material to the case of *existence* of a nonzeroth *residual stress* and *strain* $\Delta\sigma_{ij} = \sigma_{ij} - \sigma^{\text{res}}_{ij}$, $\Delta\varepsilon_{kl} = \varepsilon_{kl} - \varepsilon^{\text{res}}_{kl}$

$$\sigma_{ij} - \sigma^{\text{res}}_{ij} = E_{ijkl}\left(\varepsilon_{kl} - \varepsilon^{\text{res}}_{kl}\right) \tag{1.138}$$

or

$$\varepsilon_{kl} - \varepsilon^{\text{res}}_{kl} = E^{-1}_{klmn}\left(\sigma_{mn} - \sigma^{\text{res}}_{mn}\right) \tag{1.139}$$

When the *vector-matrix notation* is used the fourth-rank elastic tensors E_{ijkl} or E^{-1}_{klmn} can be represented by the symmetric 6×6 matrices: $[\mathbb{E}]$ or $[\mathbb{E}^{-1}]$ called the *stiffness* or the *compliance matrices*, respectively, whereas the tensors $\sigma_{ij} - \sigma^{\text{res}}_{ij}$ or $\varepsilon_{kl} - \varepsilon^{\text{res}}_{kl}$ take the format of *columnar vectors* of *overstress* or *overstrain*, respectively,

$$\begin{Bmatrix} \sigma_{11} - \sigma^{\text{res}}_{11} \\ \sigma_{22} - \sigma^{\text{res}}_{22} \\ \sigma_{33} - \sigma^{\text{res}}_{33} \\ \tau_{23} - \tau^{\text{res}}_{23} \\ \tau_{31} - \tau^{\text{res}}_{31} \\ \tau_{12} - \tau^{\text{res}}_{12} \end{Bmatrix} \quad \begin{Bmatrix} \varepsilon_{11} - \varepsilon^{\text{res}}_{11} \\ \varepsilon_{22} - \varepsilon^{\text{res}}_{22} \\ \varepsilon_{33} - \varepsilon^{\text{res}}_{33} \\ \gamma_{23} - \gamma^{\text{res}}_{23} \\ \gamma_{31} - \gamma^{\text{res}}_{31} \\ \gamma_{12} - \gamma^{\text{res}}_{12} \end{Bmatrix} \tag{1.140}$$

Hence, when the *Voigt notation* is used Eqs. (1.138) and (1.139) can be written in equivalent fashion

$$\left\{\boldsymbol{\sigma} - \boldsymbol{\sigma}^{\text{res}}\right\} = [\mathbb{E}]\left\{\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^{\text{res}}\right\} \tag{1.141}$$

or

$$\left\{\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^{\text{res}}\right\} = [\mathbb{E}^{-1}]\left\{\boldsymbol{\sigma} - \boldsymbol{\sigma}^{\text{res}}\right\} \tag{1.142}$$

1.7 Elastic Strain Energy as Function of Invariants

The stress and the strain invariants are presented in Sect. 1.1. In the present section the *elastic strain energy per unit volume* $\mathcal{W}$ expressed as the scalar product of both these tensors

$$\mathcal{W} = \frac{1}{2}\sigma_{ij}\varepsilon_{ji} \tag{1.143}$$

will also be presented in terms of invariants. In the case of isotropic material three basic invariants of the strain tensor are sufficient for unique representation of the strain energy, whereas in case of elastic material comprising damage the use of common invariants defining internal material microstructure is necessary (see Sect. 1.3).

1.7.1 Elastic Strain Energy of Isotropic Materials

The simplest example of the scalar function of tensorial argument is the elastic strain energy $\mathcal{W}(\boldsymbol{\varepsilon})$. In the case of isotropic material the strain tensor is uniquely determined in terms of three *basic* or *generic strain invariants* (1.26) as follows:

$$\mathcal{W}(\boldsymbol{\varepsilon}) = \mathcal{W}(J_{1\varepsilon}, J_{2\varepsilon}, J_{3\varepsilon}) \tag{1.144}$$

Constitutive law of *elastic* material (1.126) can be written as follows:

$$\sigma_{ij} = \frac{\partial \mathcal{W}}{\partial \varepsilon_{ij}} = \frac{\partial \mathcal{W}}{\partial J_{1\varepsilon}} \frac{\partial J_{1\varepsilon}}{\partial \varepsilon_{ij}} + \frac{\partial \mathcal{W}}{\partial J_{2\varepsilon}} \frac{\partial J_{2\varepsilon}}{\partial \varepsilon_{ij}} + \frac{\partial \mathcal{W}}{\partial J_{3\varepsilon}} \frac{\partial J_{3\varepsilon}}{\partial \varepsilon_{ij}} \tag{1.145}$$

where

$$\frac{\partial J_{1\varepsilon}}{\partial \varepsilon_{ij}} = \delta_{ij} \qquad \frac{\partial J_{2\varepsilon}}{\partial \varepsilon_{ij}} = \varepsilon_{ij} \qquad \frac{\partial J_{3\varepsilon}}{\partial \varepsilon_{ij}} = \varepsilon_{ik}\varepsilon_{kj} \tag{1.146}$$

hence,

$$\sigma_{ij} = \frac{\partial \mathcal{W}}{\partial J_{1\varepsilon}} \delta_{ij} + \frac{\partial \mathcal{W}}{\partial J_{2\varepsilon}} \varepsilon_{ij} + \frac{\partial \mathcal{W}}{\partial J_{3\varepsilon}} \varepsilon_{ik}\varepsilon_{kj} \tag{1.147}$$

Introducing the *Lamé* elastic *constants* $\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)}$ and $\mu = \frac{E}{2(1+\nu)}$ with

$$\frac{\partial \mathcal{W}}{\partial J_{1\varepsilon}} = \lambda \varepsilon_{kk} \qquad \frac{\partial \mathcal{W}}{\partial J_{2\varepsilon}} = 2\mu \qquad \frac{\partial \mathcal{W}}{\partial J_{3\varepsilon}} = 0 \tag{1.148}$$

we arrive at the classical *Hooke law* of the *isotropic* material

$$\sigma_{ij} = \lambda \varepsilon_{kk} \delta_{ij} + 2\mu \varepsilon_{ij} \tag{1.149}$$

Summing up, the *isotropic elastic Hooke material* is uniquely defined by the *strain energy* which depends on the *first* and the *second basic invariants* of the *strain* tensor

$$\mathcal{W} = \frac{1}{2}\lambda (J_{1\varepsilon})^2 + 2\mu J_{2\varepsilon} \tag{1.150}$$

but does not depend on the *third invariant* $J_{3\varepsilon}$.

1.7.2 Strain or Complementary Energy of Elastic-Damage Material—Common Strain-Damage and Stress-Damage Invariants; the Helmholtz or the Gibbs State Potentials

Theory of invariants allows to determine minimal number the basic invariants from which all other tensorial invariants necessary to obtain a sufficiently general

representation of the state equations can be built (cf. e.g., Spencer [52], Rymarz [47]). Usually the strain energy per unit volume $\mathcal{W}(\varepsilon_{ij})$ or the complementary energy per unit volume $\mathcal{C}(\sigma_{ij})$ is taken as the *state potential of elasticity* (see Sect. 1.7.1). As shown in Sect. 1.7.1, in case of elastic isotropy three invariants sufficiently determine both types of energy $\mathcal{W}(J_{i\varepsilon})$ or $\mathcal{C}(J_{i\sigma})$, $i = 1, 2, 3$.

A *scalar function* dependent on a pair of *tensorial arguments*, each of them being the symmetric second-rank tensor, is a more complex case. The representative example of such a case is the *strain energy of damaged material* $\mathcal{W}(\boldsymbol{\varepsilon}, \boldsymbol{D})$. Analogous to the isotropic material (1.144), both tensors $\boldsymbol{\varepsilon}$ or $\boldsymbol{D}$ are determined by their single *basic invariants* $J_{i\varepsilon}$ or J_{iD}, $i = 1, 2, 3$. However, the scalar function dependent on both arguments $\mathcal{W}(\boldsymbol{\varepsilon}, \boldsymbol{D})$ has to be uniquely defined not only by single invariants $J_{i\varepsilon}$ and J_{iD} but also by the *common invariants* $J_{j\varepsilon D}$, $j = 1, 2, 3, 4$. This leads to the format dependent on six single and four common invariants (total 10)

$$\mathcal{W}(\boldsymbol{\varepsilon}, \boldsymbol{D}) = \mathcal{W}(J_{1\varepsilon}, J_{2\varepsilon}, \underline{J_{3\varepsilon}}, J_{1D}, \underline{J_{2D}, J_{3D}}; J_{1\varepsilon D}, J_{2\varepsilon D}, \underline{J_{3\varepsilon D}, J_{4\varepsilon D}}) \tag{1.151}$$

In addition, the strain energy $\mathcal{W}$ has to be a decreasing function with damage growth since energy is released during the *damage nucleation* and *growth*, so it has to be linear with respect to $\boldsymbol{D}$. Hence, the strain energy cannot depend either on the third strain invariant $J_{3\varepsilon}$ and on the two single damage invariants J_{2D}, J_{3D} and also on the two common invariants $J_{3\varepsilon D}$, $J_{4\varepsilon D}$ (underlined arguments in Eq. (1.151)). Based on the above physical reasons the strain energy of elastic damaged material can completely be represented in terms of a combination of five invariants (three single and two common)

$$\mathcal{W}(\boldsymbol{\varepsilon}, \boldsymbol{D}) = \rho\psi(\boldsymbol{\varepsilon}, \boldsymbol{D}) = \rho\psi(J_{1\varepsilon}, J_{2\varepsilon}, J_{1D}, J_{1\varepsilon D}, J_{2\varepsilon D}) \tag{1.152}$$

In this way an invariant representation of the *Helmholtz free energy per unit mass* is furnished and finally applied as the *state potential* that determines the stress state in a unique fashion

$$\boldsymbol{\sigma} = \frac{\partial[\rho\psi(\boldsymbol{\varepsilon}, \boldsymbol{D})]}{\partial\boldsymbol{\varepsilon}} \tag{1.153}$$

Note also that when the representation (1.152) is specified, some *combinations of invariants* are allowed for which the scalar function $\psi(\boldsymbol{\varepsilon}, \boldsymbol{D})$ remains quadratic with respect to $\boldsymbol{\varepsilon}$. Hence, following Murakami and Kamiya [38] the *free energy function* $\rho\psi(\boldsymbol{\varepsilon}, \boldsymbol{D})$ *per unit mass* is furnished as

$$\begin{aligned}\rho\psi(\boldsymbol{\varepsilon}, \boldsymbol{D}) = {} & \tfrac{1}{2}\lambda(J_{1\varepsilon})^2 + 2\mu J_{2\varepsilon} + \eta_1(J_{1\varepsilon})^2 J_{1D} + 2\eta_2 J_{2\varepsilon} J_{1D} \\ & + \eta_3 J_{1\varepsilon} J_{1\varepsilon D} + \eta_4 J_{2\varepsilon D}\end{aligned} \tag{1.154}$$

or

$$\begin{aligned}\rho\psi(\boldsymbol{\varepsilon}, \boldsymbol{D}) = {} & \tfrac{1}{2}\lambda\,(\mathrm{tr}\boldsymbol{\varepsilon})^2 + \mu\mathrm{tr}\,(\boldsymbol{\varepsilon}\cdot\boldsymbol{\varepsilon}) + \eta_1\,(\mathrm{tr}\boldsymbol{\varepsilon})^2\,\mathrm{tr}(\boldsymbol{D}) \\ & + \eta_2\mathrm{tr}\,(\boldsymbol{\varepsilon}\cdot\boldsymbol{\varepsilon})\,\mathrm{tr}(\boldsymbol{D}) + \eta_3\mathrm{tr}(\boldsymbol{\varepsilon})\mathrm{tr}\,(\boldsymbol{\varepsilon}\cdot\boldsymbol{D}) + \eta_4\mathrm{tr}\,(\boldsymbol{\varepsilon}\cdot\boldsymbol{\varepsilon}\cdot\boldsymbol{D})\end{aligned} \tag{1.155}$$

when the equivalent representation is used, e.g., Skrzypek et al. [50].

Remember that the above formulas (1.154) and (1.155) for the Holmholtz free energy refer to the specific case of elastic *anisotropy* which is *acquired* as the result of *damage nucleation* and *growth*. Hence, in a virgin state where damage does not exist the energy representation of the isotropic elastic material has to be recovered, such that symbol $\boldsymbol{\varepsilon}$ has to be referred to the elastic strain $\boldsymbol{\varepsilon}^{\mathrm{e}}$.

In a general 3D case the following matrix representation of the *constitutive equation* with *total formulation* holds:

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{Bmatrix} = \begin{bmatrix} {}^{\mathrm{s}}\widetilde{E}_{11} & {}^{\mathrm{s}}\widetilde{E}_{12} & {}^{\mathrm{s}}\widetilde{E}_{13} & {}^{\mathrm{s}}\widetilde{E}_{14} & {}^{\mathrm{s}}\widetilde{E}_{15} & {}^{\mathrm{s}}\widetilde{E}_{16} \\ & {}^{\mathrm{s}}\widetilde{E}_{22} & {}^{\mathrm{s}}\widetilde{E}_{23} & {}^{\mathrm{s}}\widetilde{E}_{24} & {}^{\mathrm{s}}\widetilde{E}_{25} & {}^{\mathrm{s}}\widetilde{E}_{26} \\ & & {}^{\mathrm{s}}\widetilde{E}_{33} & {}^{\mathrm{s}}\widetilde{E}_{34} & {}^{\mathrm{s}}\widetilde{E}_{35} & {}^{\mathrm{s}}\widetilde{E}_{36} \\ & & & {}^{\mathrm{s}}\widetilde{E}_{44} & {}^{\mathrm{s}}\widetilde{E}_{45} & {}^{\mathrm{s}}\widetilde{E}_{46} \\ & \text{symm.} & & & {}^{\mathrm{s}}\widetilde{E}_{55} & {}^{\mathrm{s}}\widetilde{E}_{56} \\ & & & & & {}^{\mathrm{s}}\widetilde{E}_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon^{\mathrm{e}}_{11} \\ \varepsilon^{\mathrm{e}}_{22} \\ \varepsilon^{\mathrm{e}}_{33} \\ \gamma^{\mathrm{e}}_{23} \\ \gamma^{\mathrm{e}}_{13} \\ \gamma^{\mathrm{e}}_{12} \end{Bmatrix} \tag{1.156}$$

where ${}^{\mathrm{s}}\widetilde{E}_{ij}$ represents *effective elastic-damage secant stiffness matrix*. The damage acquired anisotropy is described by the 6×6 symmetric secant stiffness matrix as follows (cf. Skrzypek et al. [50]):

$$\begin{aligned}
{}^{\mathrm{s}}\widetilde{E}_{11} &= \lambda + 2\mu + 2(\eta_1 + \eta_2)\mathrm{tr}(\boldsymbol{D}) + 2(\eta_3 + \eta_4)D_{11} \\
{}^{\mathrm{s}}\widetilde{E}_{22} &= \lambda + 2\mu + 2(\eta_1 + \eta_2)\mathrm{tr}(\boldsymbol{D}) + 2(\eta_3 + \eta_4)D_{22} \\
{}^{\mathrm{s}}\widetilde{E}_{33} &= \lambda + 2\mu + 2(\eta_1 + \eta_2)\mathrm{tr}(\boldsymbol{D}) + 2(\eta_3 + \eta_4)D_{33} \\
{}^{\mathrm{s}}\widetilde{E}_{12} &= \lambda + 2\eta_1\mathrm{tr}(\boldsymbol{D}) + \eta_3(D_{11} + D_{22}) \\
{}^{\mathrm{s}}\widetilde{E}_{13} &= \lambda + 2\eta_1\mathrm{tr}(\boldsymbol{D}) + \eta_3(D_{11} + D_{33}) \\
{}^{\mathrm{s}}\widetilde{E}_{23} &= \lambda + 2\eta_1\mathrm{tr}(\boldsymbol{D}) + \eta_3(D_{22} + D_{33}) \\
{}^{\mathrm{s}}\widetilde{E}_{44} &= \tfrac{1}{2}\left[2\mu + 2\eta_2\mathrm{tr}(\boldsymbol{D}) + \eta_4(D_{33} + D_{22})\right] \\
{}^{\mathrm{s}}\widetilde{E}_{45} &= \eta_4 D_{12} \\
{}^{\mathrm{s}}\widetilde{E}_{55} &= \tfrac{1}{2}\left[2\mu + 2\eta_2\mathrm{tr}(\boldsymbol{D}) + \eta_4(D_{11} + D_{33})\right] \\
{}^{\mathrm{s}}\widetilde{E}_{46} &= \eta_4 D_{13} \\
{}^{\mathrm{s}}\widetilde{E}_{66} &= \tfrac{1}{2}\left[2\mu + 2\eta_2\mathrm{tr}(\boldsymbol{D}) + \eta_4(D_{11} + D_{22})\right] \\
{}^{\mathrm{s}}\widetilde{E}_{56} &= \eta_4 D_{23} \\
{}^{\mathrm{s}}\widetilde{E}_{14} &= \eta_3 D_{23} \\
{}^{\mathrm{s}}\widetilde{E}_{24} &= {}^{\mathrm{s}}\widetilde{S}_{34} = (\eta_3 + \eta_4)D_{23} \\
{}^{\mathrm{s}}\widetilde{E}_{25} &= \eta_3 D_{13} \\
{}^{\mathrm{s}}\widetilde{E}_{15} &= {}^{\mathrm{s}}\widetilde{S}_{35} = (\eta_3 + \eta_4)D_{13} \\
{}^{\mathrm{s}}\widetilde{E}_{36} &= \eta_3 D_{12} \\
{}^{\mathrm{s}}\widetilde{E}_{16} &= {}^{\mathrm{s}}\widetilde{S}_{26} = (\eta_3 + \eta_4)D_{12}
\end{aligned} \tag{1.157}$$

The alternative formulation based on a concept of the *complementary energy* $\mathcal{C}$ represented by a *scalar function* of the *two tensorial arguments* $\boldsymbol{\sigma}$ and $\boldsymbol{D}$, namely $\mathcal{C}(\boldsymbol{\sigma}, \boldsymbol{D})$, leads to the *Gibbs potential function* per unit mass $\mathcal{G}$ as follows (cf. Hayakawa–Murakami [18], Murakami [40]):

$$\mathcal{C}(\boldsymbol{\sigma}, \boldsymbol{D}) = \rho\mathcal{G}(J_{1\sigma}, J_{2\sigma}, J_{3\sigma}, J_{1D}, J_{2D}, J_{3D}, J_{1\sigma D}, J_{2\sigma D}, J_{3\sigma D}, J_{4\sigma D}) \quad (1.158)$$

where the *crack closure effect* due to compressive stress, originally introduced in Hayakawa–Murakami [18], is omitted.

Repeating the above reasoning for physical nature of the *Gibbs complementary energy* $\mathcal{C}(\boldsymbol{\sigma}, \boldsymbol{D})$, only five of the above aforementioned ten (1.158) common stress and damage invariants can be admitted, namely

$$\mathcal{C}(\boldsymbol{\sigma}, \boldsymbol{D}) = \rho\mathcal{G}(\boldsymbol{\sigma}, \boldsymbol{D}) = \rho\mathcal{G}(J_{1\sigma}, J_{2\sigma}, J_{1D}, J_{1\sigma D}, J_{2\sigma D}) \quad (1.159)$$

Hence, in case of the *elastic isotropic material* in a *virgin state* which changes to *anisotropic* material due to *damage evolution*, the Gibbs state potential takes the following format (cf. Hayakawa and Murakami [18]):

$$\begin{aligned}\rho\mathcal{G}(\boldsymbol{\sigma}, \boldsymbol{D}) = &-\tfrac{\nu}{2E}(\mathrm{tr}\boldsymbol{\sigma})^2 + \tfrac{1+\nu}{2E}\mathrm{tr}(\boldsymbol{\sigma}\cdot\boldsymbol{\sigma}) + \vartheta_1(\mathrm{tr}\boldsymbol{\sigma})^2\,\mathrm{tr}(\boldsymbol{D})\\ &+ \vartheta_2\mathrm{tr}(\boldsymbol{\sigma}\cdot\boldsymbol{\sigma})\,\mathrm{tr}(\boldsymbol{D}) + \vartheta_3\mathrm{tr}(\boldsymbol{\sigma})\mathrm{tr}(\boldsymbol{\sigma}\cdot\boldsymbol{D}) + \vartheta_4\mathrm{tr}(\boldsymbol{\sigma}\cdot\boldsymbol{\sigma}\cdot\boldsymbol{D})\end{aligned} \quad (1.160)$$

which is complementary to (1.155). The matrix representation of secant compliance matrix referring to Hayakawa–Murakami type elastic-plastic-damage material is as follows:

$$\begin{Bmatrix}\varepsilon^{\mathrm{e}}_{11}\\ \varepsilon^{\mathrm{e}}_{22}\\ \varepsilon^{\mathrm{e}}_{33}\\ \varepsilon^{\mathrm{e}}_{23}\\ \varepsilon^{\mathrm{e}}_{13}\\ \varepsilon^{\mathrm{e}}_{12}\end{Bmatrix} = \begin{bmatrix} {}^{\mathrm{s}}\widetilde{E}^{-1}_{11} & {}^{\mathrm{s}}\widetilde{E}^{-1}_{12} & {}^{\mathrm{s}}\widetilde{E}^{-1}_{13} & 0 & 0 & 0\\ & {}^{\mathrm{s}}\widetilde{E}^{-1}_{22} & {}^{\mathrm{s}}\widetilde{E}^{-1}_{23} & 0 & 0 & 0\\ & & {}^{\mathrm{s}}\widetilde{E}^{-1}_{33} & 0 & 0 & 0\\ & & & {}^{\mathrm{s}}\widetilde{E}^{44}_{-1} & 0 & 0\\ & \text{symm.} & & & {}^{\mathrm{s}}\widetilde{E}^{-1}_{55} & 0\\ & & & & & {}^{\mathrm{s}}\widetilde{E}^{-1}_{66}\end{bmatrix}\begin{Bmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{33}\\ \sigma_{23}\\ \sigma_{13}\\ \sigma_{12}\end{Bmatrix} \quad (1.161)$$

where

$$\begin{aligned}
{}^{\mathrm{s}}\widetilde{E}^{-1}_{11} &= \tfrac{1}{E} + 2\mathrm{tr}(\boldsymbol{D})(\vartheta_1+\vartheta_2) + 2D_{11}(\vartheta_3+\vartheta_4)\\
{}^{\mathrm{s}}\widetilde{E}^{-1}_{12} &= -\tfrac{\nu}{E} + 2\vartheta_1\mathrm{tr}(\boldsymbol{D}) + \vartheta_3(D_{11}+D_{22})\\
{}^{\mathrm{s}}\widetilde{E}^{-1}_{13} &= -\tfrac{\nu}{E} + 2\vartheta_1\mathrm{tr}(\boldsymbol{D}) + \vartheta_3(D_{11}+D_{33})\\
{}^{\mathrm{s}}\widetilde{E}^{-1}_{22} &= \tfrac{1}{E} + 2\mathrm{tr}\boldsymbol{D}(\vartheta_1+\vartheta_2) + 2D_{22}(\vartheta_3+\vartheta_4)\\
{}^{\mathrm{s}}\widetilde{E}^{-1}_{23} &= -\tfrac{\nu}{E} + 2\vartheta_1\mathrm{tr}(\boldsymbol{D}) + \vartheta_3(D_{22}+D_{33})\\
{}^{\mathrm{s}}\widetilde{E}^{-1}_{33} &= \tfrac{1}{E} + 2\vartheta_1\mathrm{tr}(\boldsymbol{D}) + \vartheta_3(D_{22}+D_{33})\\
{}^{\mathrm{s}}\widetilde{E}^{-1}_{44} &= \tfrac{1+\nu}{E} + 2\vartheta_2\mathrm{tr}(\boldsymbol{D}) + \vartheta_4(D_{22}+D_{33})\\
{}^{\mathrm{s}}\widetilde{E}^{-1}_{55} &= \tfrac{1+\nu}{E} + 2\vartheta_2\mathrm{tr}(\boldsymbol{D}) + \vartheta_4(D_{11}+D_{33})\\
{}^{\mathrm{s}}\widetilde{E}^{-1}_{66} &= \tfrac{1+\nu}{E} + 2\vartheta_2\mathrm{tr}(\boldsymbol{D}) + \vartheta_4(D_{11}+D_{22})
\end{aligned} \quad (1.162)$$

Note that the Gibbs complementary energy per unit mass refers to elastic strains ε^{e} and is represented in the stress space by the quadratic function of $\boldsymbol{\sigma}$ linear with respect to $\boldsymbol{D}$, in a similar way as the Helmholtz free energy $\rho\psi(\boldsymbol{\varepsilon}, \boldsymbol{D})$ but defined in the *strain space*.

Four material constants η_i appearing in the *Helmholtz state potential* (1.155) as well as four constants ϑ_i appearing in the *Gibbs state potential* (1.160) ($i = 1, 2, 3, 4$) act as additional material constants to the elastic constants of the *virgin elastic isotropic material*: λ, μ or E, ν, defining effect of damage on the state equation. Namely, when the *Helmholtz potential function* $\mathcal{W} = \rho\psi(\boldsymbol{\varepsilon}, \boldsymbol{D})$ is used as the *stress potential* we arrive at the *state equation* $\boldsymbol{\sigma} = \mathbb{E}(\boldsymbol{D}) : \boldsymbol{\varepsilon}$

$$\begin{aligned}\boldsymbol{\sigma} = \frac{\partial(\rho\psi)}{\partial\boldsymbol{\varepsilon}} &= [\lambda \mathrm{tr}(\boldsymbol{\varepsilon}) + 2\eta_1 \mathrm{tr}(\boldsymbol{\varepsilon})\mathrm{tr}(\boldsymbol{D}) + \eta_3 \mathrm{tr}\,(\boldsymbol{\varepsilon}\cdot\boldsymbol{D})]\,\mathbf{1} \\ &+ 2\,[\mu + \eta_2 \mathrm{tr}(\boldsymbol{D})]\,\boldsymbol{\varepsilon} + \eta_3 \mathrm{tr}(\boldsymbol{\varepsilon})\boldsymbol{D} + \eta_4(\boldsymbol{\varepsilon}\cdot\boldsymbol{D} + \boldsymbol{D}\cdot\boldsymbol{\varepsilon})\end{aligned} \tag{1.163}$$

On the other hand, when the formulation based on the *Gibbs potential function* is used as the *strain potential* $\mathcal{C} = \rho\mathcal{G}(\boldsymbol{\sigma}, \boldsymbol{D})$ we obtain the state equation in equivalent form $\boldsymbol{\varepsilon} = \mathbb{E}^{-1}(\boldsymbol{D}) : \boldsymbol{\sigma}$

$$\begin{aligned}\boldsymbol{\varepsilon} = \frac{\partial(\rho\mathcal{G})}{\partial\boldsymbol{\sigma}} &= -\tfrac{\nu}{E}\mathrm{tr}\,(\boldsymbol{\sigma})\,\mathbf{1} + \tfrac{1+\nu}{2E}\boldsymbol{\sigma} + 2\vartheta_1 \mathrm{tr}(\boldsymbol{D})\mathrm{tr}(\boldsymbol{\sigma})\mathbf{1} \\ &+ 2\vartheta_2 \mathrm{tr}(\boldsymbol{D})\boldsymbol{\sigma} : \mathbf{1} + \vartheta_3\,[\mathrm{tr}\,(\boldsymbol{\sigma}\cdot\boldsymbol{D})\,\mathbf{1} + \mathrm{tr}(\boldsymbol{\sigma})\boldsymbol{D}] \\ &+ \vartheta_4\,(\boldsymbol{\sigma}\cdot\boldsymbol{D} + \boldsymbol{D}\cdot\boldsymbol{\sigma})\end{aligned} \tag{1.164}$$

Note however that in the case of *elastic damaged material* constitutive matrices stiffness $[\mathbb{E}(\boldsymbol{D})]$ and compliance $[\mathbb{E}^{-1}(\boldsymbol{D})]$ are rebuilt following *damage evolution* such that originally isotropic elastic material *acquires* an *anisotropy*.

The state equation of elastic damaged material (1.155) was calibrated for the *high strength concrete* by Murakami and Kamiya [38], see also Skrzypek [49] as shown in Table 1.5.

Apart from the constants of isotropic elasticity E, ν (λ, μ) additional four constants η_i ($i = 1, 2, 3, 4$) are shown in Table 1.5.

The state equation of elastic moderate ductility with damage (1.164) was calibrated for *spheroidal graphite cast iron* FCD400 by Hayakawa and Murakami [18], see also Skrzypek [49] as shown in Table 1.6.

Apart from the constants of isotropic elasticity E, ν (λ, μ) additional four constants ϑ_i ($i = 1, 2, 3, 4$) are shown Table 1.6.

Table 1.5 Calibration of six material constants in the constitutive equation of high strength concrete, after Murakami and Kamiya [38])

E (GPa)	ν (–)	η_1 (MPa)	η_2 (MPa)	η_3 (MPa)	η_4 (MPa)
21.4	0.2	−400	−900	100	−23500

Table 1.6 Callibration of six material constants in the constitutive equation of the spheroidal graphite cast iron FCD400, after Hayakawa and Murakami [18]

E (GPa)	ν (–)	ϑ_1 (MPa^{-1})	ϑ_2 (MPa^{-1})	ϑ_3 (MPa^{-1})	ϑ_4 (MPa^{-1})
169	0.285	-3.95×10^{-1}	4.0×10^{-6}	-4.0×10^{-7}	2.50×10^{-6}

The more extended analysis including: crack closure effect under compressive stress, the initial damage threshold, and the subsequent *damage growth* during the hardening phases can be found in Murakami and Kamiya [38], Hayakawa and Murakami [18], Skrzypek et al. [50], Bielski et al. [4], Kuna-Ciskał and Skrzypek [31].

1.7.3 Strain Energy of the Elastic Orthotropic Materials—The Structural Tensors

So far the case of scalar function of second-order tensors expressed in terms of invariants has been discussed. The more general case of a *scalar function of* a *pair of tensorial arguments* being the second-order and the *structural tensors* is considered in this section. The strain energy of orthotropic material $\mathcal{W} = \mathcal{W}(\boldsymbol{\varepsilon}, \boldsymbol{M}^{(i)})$ is the representative example of such a case.

The constitutive equation of *orthotropic hyperelastic material* is obtained by differentiation of the *strain energy function*, cf. Boehler [5]

$$\begin{aligned}\boldsymbol{\sigma} = \frac{\partial \mathcal{W}}{\partial \boldsymbol{\varepsilon}} &= \frac{\partial \mathcal{W}}{\partial J_1}\boldsymbol{M}^{(1)} + \frac{\partial \mathcal{W}}{\partial J_2}\boldsymbol{M}^{(2)} + \frac{\partial \mathcal{W}}{\partial J_3}\boldsymbol{M}^{(3)} \\ &+ \frac{\partial \mathcal{W}}{\partial J_4}(\boldsymbol{\varepsilon}\cdot\boldsymbol{M}^{(1)} + \boldsymbol{M}^{(1)}\cdot\boldsymbol{\varepsilon}) + \frac{\partial \mathcal{W}}{\partial J_5}(\boldsymbol{\varepsilon}\cdot\boldsymbol{M}^{(2)} + \boldsymbol{M}^{(2)}\cdot\boldsymbol{\varepsilon}) \\ &+ \frac{\partial \mathcal{W}}{\partial J_6}(\boldsymbol{\varepsilon}\cdot\boldsymbol{M}^{(3)} + \boldsymbol{M}^{(3)}\cdot\boldsymbol{\varepsilon})\end{aligned} \tag{1.165}$$

where the following definitions of *common invariants* are used:

$$\begin{aligned} J_1 &= \mathrm{tr}(\boldsymbol{\varepsilon}\cdot\boldsymbol{M}^{(1)}) \quad J_2 = \mathrm{tr}(\boldsymbol{\varepsilon}\cdot\boldsymbol{M}^{(2)}) \quad J_3 = \mathrm{tr}(\boldsymbol{\varepsilon}\cdot\boldsymbol{M}^{(3)}) \\ J_4 &= \mathrm{tr}(\boldsymbol{\varepsilon}\cdot\boldsymbol{\varepsilon}\cdot\boldsymbol{M}^{(1)}) \; J_5 = \mathrm{tr}(\boldsymbol{\varepsilon}\cdot\boldsymbol{\varepsilon}\cdot\boldsymbol{M}^{(2)}) \; J_6 = \mathrm{tr}(\boldsymbol{\varepsilon}\cdot\boldsymbol{\varepsilon}\cdot\boldsymbol{M}^{(3)}) \end{aligned} \tag{1.166}$$

and definitions (1.91) hold. Following Boehler [5], in order to determine the *constitutive equation of linear orthotropic material* we choose, (see also Ottosen and Ristinmaa [44])

$$\begin{aligned}
\frac{\partial \mathcal{W}}{\partial J_1} &= \alpha_1 J_1 + \beta_1 J_2 + \beta_2 J_3 \\
\frac{\partial \mathcal{W}}{\partial J_2} &= \alpha_2 J_1 + \alpha_3 J_2 + \beta_3 J_3 \\
\frac{\partial \mathcal{W}}{\partial J_3} &= \alpha_4 J_1 + \alpha_5 J_2 + \alpha_6 J_3 \\
\frac{\partial \mathcal{W}}{\partial J_4} &= \alpha_7 \qquad \frac{\partial \mathcal{W}}{\partial J_5} = \alpha_8 \qquad \frac{\partial \mathcal{W}}{\partial J_6} = \alpha_9
\end{aligned} \tag{1.167}$$

The coefficients $\beta_1, \beta_2, \beta_3$ can be substituted by corresponding coefficients $\alpha_2, \alpha_4, \alpha_5$ in order to satisfy symmetry of *orthotropic stiffness matrix*

$$\beta_1 = \alpha_2, \qquad \beta_2 = \alpha_4, \qquad \beta_3 = \alpha_5 \tag{1.168}$$

The above yields the *constitutive equation* of *linear orthotropic material* by use of *common invariants of strain and structural tensors*

$$\begin{aligned}
\boldsymbol{\sigma} = &\left[\alpha_1 \mathrm{tr}(\boldsymbol{\varepsilon} \cdot \boldsymbol{M}^{(1)}) + \alpha_2 \mathrm{tr}(\boldsymbol{\varepsilon} \cdot \boldsymbol{M}^{(2)}) + \alpha_4 \mathrm{tr}(\boldsymbol{\varepsilon} \cdot \boldsymbol{M}^{(3)})\right] \boldsymbol{M}^{(1)} \\
&+ \left[\alpha_2 \mathrm{tr}(\boldsymbol{\varepsilon} \cdot \boldsymbol{M}^{(1)}) + \alpha_3 \mathrm{tr}(\boldsymbol{\varepsilon} \cdot \boldsymbol{M}^{(2)}) + \alpha_5 \mathrm{tr}(\boldsymbol{\varepsilon} \cdot \boldsymbol{M}^{(3)})\right] \boldsymbol{M}^{(2)} \\
&+ \left[\alpha_4 \mathrm{tr}(\boldsymbol{\varepsilon} \cdot \boldsymbol{M}^{(1)}) + \alpha_5 \mathrm{tr}(\boldsymbol{\varepsilon} \cdot \boldsymbol{M}^{(2)}) + \alpha_6 \mathrm{tr}(\boldsymbol{\varepsilon} \cdot \boldsymbol{M}^{(3)})\right] \boldsymbol{M}^{(3)} \\
&+ \alpha_7 \left(\boldsymbol{\varepsilon} \cdot \boldsymbol{M}^{(1)} + \boldsymbol{M}^{(1)} \cdot \boldsymbol{\varepsilon}\right) + \alpha_8 \left(\boldsymbol{\varepsilon} \cdot \boldsymbol{M}^{(2)} + \boldsymbol{M}^{(2)} \cdot \boldsymbol{\varepsilon}\right) \\
&+ \alpha_9 \left(\boldsymbol{\varepsilon} \cdot \boldsymbol{M}^{(3)} + \boldsymbol{M}^{(3)} \cdot \boldsymbol{\varepsilon}\right)
\end{aligned} \tag{1.169}$$

Equation (1.169) can be rewritten in the classical form at $\boldsymbol{\sigma} = \mathbb{E} : \boldsymbol{\varepsilon}$ when the consecutive *tensor products* $\boldsymbol{\varepsilon} \cdot \boldsymbol{M}^{(i)}$ and their *traces* are defined. For instance,

$$\boldsymbol{\varepsilon} \cdot \boldsymbol{M}^{(1)} = \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ & \varepsilon_{yy} & \varepsilon_{yz} \\ & & \varepsilon_{zz} \end{bmatrix} \cdot \begin{bmatrix} 1 & & \\ & 0 & \\ & & 0 \end{bmatrix} = \begin{bmatrix} \varepsilon_{xx} & 0 & 0 \\ \varepsilon_{xy} & 0 & 0 \\ \varepsilon_{xz} & 0 & 0 \end{bmatrix} \tag{1.170}$$

from where one finds

$$\mathrm{tr}(\boldsymbol{\varepsilon} \cdot \boldsymbol{M}^{(1)}) = \varepsilon_{xx} \tag{1.171}$$

and

$$\boldsymbol{\varepsilon} \cdot \boldsymbol{M}^{(1)} + \boldsymbol{M}^{(1)} \cdot \boldsymbol{\varepsilon} = \begin{bmatrix} 2\varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{xy} & 0 & 0 \\ \varepsilon_{xz} & 0 & 0 \end{bmatrix} \tag{1.172}$$

When the remaining products $\boldsymbol{\varepsilon} \cdot \boldsymbol{M}^{(2)}$ and $\boldsymbol{\varepsilon} \cdot \boldsymbol{M}^{(3)}$ are calculated analogously, the coefficients preceding the components of the strain tensor are grouped, and when the engineering notation is consequently used the *state equation* (1.169) can finally be furnished in the following form:

$$\left\{\begin{array}{c}\sigma_{xx}\\ \sigma_{yy}\\ \sigma_{zz}\\ \hline \tau_{yz}\\ \tau_{zx}\\ \tau_{xy}\end{array}\right\} = \left[\begin{array}{ccc|ccc} E_{11} & E_{12} & E_{13} & & & \\ E_{21} & E_{22} & E_{23} & & & \\ E_{31} & E_{32} & E_{33} & & & \\ \hline & & & E_{44} & & \\ & & & & E_{55} & \\ & & & & & E_{66}\end{array}\right] \left\{\begin{array}{c}\varepsilon_{xx}\\ \varepsilon_{yy}\\ \varepsilon_{zz}\\ \hline \gamma_{yz}\\ \gamma_{zx}\\ \gamma_{xy}\end{array}\right\} \tag{1.173}$$

Subsequent elements of *stiffness matrix* of the *orthotropic elastic material* $[\mathbb{E}]$ are expressed in terms of coefficients α_i as follows:

$$\begin{array}{lll} E_{11} = \alpha_1 + 2\alpha_7 & E_{12} = E_{21} = \alpha_2 & E_{13} = E_{31} = \alpha_4 \\ E_{22} = \alpha_3 + 2\alpha_8 & E_{23} = E_{32} = \alpha_5 & E_{33} = \alpha_6 + 2\alpha_9 \\ E_{44} = \alpha_8 + \alpha_9 & E_{55} = \alpha_9 + \alpha_7 & E_{66} = \alpha_7 + \alpha_8 \end{array} \tag{1.174}$$

Note that the above described procedure of *linear orthotropic elasticity* derivation is based on the *theory of invariant representation* which differs from the conventional approach (1.103). More detailed distinction between different ways of formulating the linear elasticity constitutive laws will be presented in Sect. 1.9.

1.8 Remarks on Irreducible Coupling of Volumetric and Shear Response in Anisotropic Materials

In the general case of *full* material *anisotropy* complete mutual coupling between all stress and strain components holds. In fact, the generalized Hooke law (1.39) with the compliance matrix for general anisotropy taken in the form (1.98) leads to (after Rabinovich [45])

$$\begin{aligned} \varepsilon_{11} &= \frac{1}{E_{11}}(\sigma_{11} - \nu_{21}\sigma_{22} - \nu_{31}\sigma_{33} \\ &\quad + \eta_{23(1)}\tau_{23} + \eta_{31(1)}\tau_{31} + \eta_{12(1)}\tau_{12}) \\ \varepsilon_{22} &= \frac{1}{E_{22}}(-\nu_{12}\sigma_{11} + \sigma_{22} - \nu_{32}\sigma_{33} \\ &\quad + \eta_{23(2)}\tau_{23} + \eta_{31(2)}\tau_{31} + \eta_{12(2)}\tau_{12}) \\ \varepsilon_{33} &= \frac{1}{E_{22}}(-\nu_{13}\sigma_{11} - \nu_{23}\sigma_{22} + \sigma_{33} \\ &\quad + \eta_{23(3)}\tau_{23} + \eta_{31(3)}\tau_{31} + \eta_{12(3)}\tau_{12}) \\ \gamma_{23} &= \frac{1}{G_{23}}\left(\eta_{(1)23}\sigma_{11} + \eta_{(2)23}\sigma_{22} + \eta_{(3)23}\sigma_{33} \right. \\ &\quad \left. + \tau_{23} + \mu_{31(23)}\tau_{31} + \mu_{12(23)}\tau_{12}\right) \\ \gamma_{31} &= \frac{1}{G_{31}}\left(\eta_{(1)31}\sigma_{11} + \eta_{(2)31}\sigma_{22} + \eta_{(3)31}\sigma_{33}\right. \end{aligned} \tag{1.175}$$

$$+\mu_{23(31)}\tau_{23}+\tau_{31}+\mu_{12(31)}\tau_{12}\big)$$
$$\gamma_{12}=\frac{1}{G_{12}}\big(\eta_{(1)12}\sigma_{11}+\eta_{(2)12}\sigma_{22}+\eta_{(3)12}\sigma_{33}$$
$$+\mu_{23(12)}\tau_{23}+\mu_{31(12)}\tau_{12}+\tau_{12}\big)$$

Note that in the above equations elastic extensions $\varepsilon_{11}, \varepsilon_{22}, \varepsilon_{33}$ depend not only on all normal stresses $\sigma_{11}, \sigma_{22}, \sigma_{33}$ but also on all shear stresses $\tau_{23}, \tau_{31}, \tau_{12}$ (through the *generalized Young modules* E_{ii} and the *Rabinovich modules* $\eta_{ij(k)}$), resulting in nonzeroth elements of symmetric constitutive matrix of elasticity in its right top block. Moreover, the shear strains $\gamma_{23}, \gamma_{31}, \gamma_{12}$ depend on all shear stresses $\tau_{23}, \tau_{31}, \tau_{12}$ (through the *generalized Kirchhoff modules* G_{ij} and the *Chencov coefficients* $\mu_{ij(kl)}$) as well as on all normal stresses $\sigma_{11}, \sigma_{22}, \sigma_{33}$ such that the left bottom and the right bottom blocks of the elasticity matrix are fully populated. The above remarks lead in consequence to the conclusion that, in all cases different from isotropy, pure volumetric deformation is inseparable from pure shear deformation. In other words, *irreducibility of elasticity equations* (1.175) into uncoupled *law of volume change* and *law of shape change* holds when the *decomposition of strain* and *stress tensors* into axiator and deviator $\boldsymbol{\varepsilon}=\varepsilon_{\mathrm{m}}\mathbf{1}+\boldsymbol{e}$ and $\boldsymbol{\sigma}=\sigma_{\mathrm{m}}\mathbf{1}+\boldsymbol{s}$ is used.

This impossibility is inevitable even in a narrower case of orthotropy (1.103) in spite of the fact that shear stresses are uncoupled to the extensions and, vice versa, normal stresses do not result in shear strains. In order to trace this let us rewrite (1.103) as

$$\begin{aligned}
\varepsilon_{11}&=\frac{1}{E_{11}}\left(\sigma_{11}-\nu_{21}\sigma_{22}-\nu_{31}\sigma_{33}\right)\\
\varepsilon_{22}&=\frac{1}{E_{22}}\left(-\nu_{12}\sigma_{11}+\sigma_{22}-\nu_{13}\sigma_{33}\right)\\
\varepsilon_{33}&=\frac{1}{E_{33}}\left(-\nu_{13}\sigma_{11}-\nu_{23}\sigma_{22}+\sigma_{33}\right)
\end{aligned}\tag{1.176}$$

$$\gamma_{23}=\frac{\tau_{23}}{G_{23}}\qquad\gamma_{31}=\frac{\tau_{31}}{G_{31}}\qquad\gamma_{12}=\frac{\tau_{12}}{G_{12}}$$

Calculating the unit *volume change* called *dilatation* $\Theta=\varepsilon_{11}+\varepsilon_{22}+\varepsilon_{33}$ we obtain

$$\begin{aligned}
\Theta^{\mathrm{orto}}=3\varepsilon_{\mathrm{m}}&=\frac{1}{E_{11}}\left(\sigma_{11}-\nu_{21}\sigma_{22}-\nu_{31}\sigma_{33}\right)\\
&+\frac{1}{E_{22}}\left(-\nu_{12}\sigma_{11}+\sigma_{22}-\nu_{13}\sigma_{33}\right)\\
&+\frac{1}{E_{33}}\left(-\nu_{13}\sigma_{11}-\nu_{23}\sigma_{22}+\sigma_{33}\right)
\end{aligned}\tag{1.177}$$

or recalling the symmetry of elasticity matrix (1.104) the equivalent form is furnished

$$\Theta^{\text{orto}} = \frac{\sigma_{11}}{E_{11}}(1-\nu_{21}-\nu_{31}) + \frac{\sigma_{22}}{E_{22}}(1-\nu_{12}-\nu_{32}) + \frac{\sigma_{33}}{E_{33}}(1-\nu_{13}-\nu_{23}) \tag{1.178}$$

Note that in case of orthotropy dilatation is expressed not only in terms of the hydrostatic stress $\Theta = \Theta^{\text{izo}}(\sigma_{\text{h}})$ but by the more general function $\Theta = \Theta^{\text{orto}}(\sigma_{11}, \sigma_{22}, \sigma_{33}; E_{ij}, \nu_{ij})$ or $\Theta = \Theta^{\text{orto}}(\sigma_{kk}; \mathbb{E}^{-1}_{ijkl})$.

In the particular case of isotropy when $E_{ij} = E$, $\nu_{ij} = \nu$ the above equations reduce to the classical form

$$\Theta^{\text{izo}} = 3\varepsilon_{\text{m}} = \frac{1-2\nu}{E}(\sigma_{11}+\sigma_{22}+\sigma_{33}) = \frac{3(1-2\nu)}{E}\sigma_{\text{h}} \tag{1.179}$$

or

$$\varepsilon_{\text{m}} = \frac{1}{3K}\sigma_{\text{h}}, \qquad K = \frac{E}{3(1-2\nu)} \tag{1.180}$$

in which dilatation or mean strain ε_{m} depends on hydrostatic stress σ_{h} exclusively.

Contrary to the previous case for *material orthotropy* by use of the following definition of deviatoric strain:

$$\boldsymbol{e} = \boldsymbol{\varepsilon} - \frac{1}{3}\varepsilon_{kk}\mathbf{1} = \mathbb{E}^{-1} : \boldsymbol{\sigma} - \frac{1}{3}\Theta^{\text{orto}}(\sigma_{kk}; \mathbb{E}^{-1}_{ijkl})\mathbf{1} \tag{1.181}$$

we obtain

$$\begin{Bmatrix} e_{11} \\ e_{22} \\ e_{33} \\ e_{23} \\ e_{31} \\ e_{12} \end{Bmatrix} = \left[\begin{array}{ccc|ccc} \frac{1}{E_{11}} & -\frac{\nu_{21}}{E_{11}} & -\frac{\nu_{31}}{E_{11}} & 0 & 0 & 0 \\ -\frac{\nu_{12}}{E_{22}} & \frac{1}{E_{22}} & -\frac{\nu_{32}}{E_{22}} & 0 & 0 & 0 \\ -\frac{\nu_{13}}{E_{33}} & -\frac{\nu_{23}}{E_{33}} & \frac{1}{E_{33}} & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & \frac{1}{G_{23}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{G_{31}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{G_{12}} \end{array}\right] \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{31} \\ \sigma_{12} \end{Bmatrix} - \frac{1}{3}\Theta^{\text{orto}}(\sigma_{kk}; \mathbb{E}^{-1}_{ijkl}) \begin{Bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \tag{1.182}$$

In other words, the *pure shear deformation* obtained by subtracting of the *dilatation* from the full deformation depends also on Θ^{orto}, so separation of these two effects is impossible.

1.9 Cauchy's Elasticity, Hyperelasticity, or Hypoelasticity

In the theory of linear elasticity in case of infinitesimal deformations occurring in isothermal or adiabatic conditions the *constitutive relations* linking tensors of stress and strain can be defined in three equivalent ways:

- According to the *Cauchy formulation* it is assumed that there exists an equilibrium state, called natural state, for which all components of the stress and strain tensors are equal to zero and to which material returns after removing loadings. An environment of natural state obeys unique value relation between stress and strain as

$$\sigma_{ij} = E_{ijkl}\varepsilon_{kl} \tag{1.183}$$

- According to the *Green formulation*, also called *hyperelasticity*, it is postulated an existence of function of elastic strain energy per unit volume $\mathcal{W}$ which is equal to zero in an environment of natural state and such that an increment of work done by stress is equal to an increment of strain energy

$$\sigma_{ij} = \frac{\partial \mathcal{W}}{\partial \varepsilon_{ij}} \qquad \mathcal{W} = \frac{1}{2}\sigma_{ij}\varepsilon_{kl} = \frac{1}{2}E_{ijkl}\varepsilon_{ij}\varepsilon_{kl} \tag{1.184}$$

- According to the third formulation, called *hypoelasticity*, it is postulated an incremental relation of the following form:

$$\mathrm{d}\sigma_{ij} = E_{ijkl}\mathrm{d}\varepsilon_{kl} \qquad \text{or} \qquad \frac{\partial \sigma_{ij}}{\partial t} = E_{ijkl}\frac{\partial \varepsilon_{kl}}{\partial t} \tag{1.185}$$

For all three cases: *Cauchy's*, *hyper-* and *hypoelasticity tensor* E_{ijkl} may depend on temperature but is independent of stress and strain tensors.

Note however that in the general case of nonlinearity constitutive tensors of elasticity or hyperelasticity (1.183) and (1.184) may differ from constitutive tensor of hypoelasticity (1.185). In the first case *tensor representative* matrix $\mathbb{E}$ is the *secant matrix* $[\mathbb{E}] = [{}_{\mathrm{sec}}\mathbb{E}]$, whereas in the other case it is the tangent matrix $[\mathbb{E}] = [{}_{\tan}\mathbb{E}]$.

It is worth to mention that although Cauchy, hyper- and hypoformulations of elasticity are alternative in case of theory of infinitesimal deformations, they may lead to essentially different results after entering the *finite deformation range*. Namely, introducing *definitions* of *finite strains*

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \begin{array}{l} \nearrow \epsilon_{ij} = \dfrac{1}{2}\left(\dfrac{\partial u_i}{\partial X_j} + \dfrac{\partial u_j}{\partial X_i} + \dfrac{\partial u_i}{\partial X_j}\dfrac{\partial u_j}{\partial X_i}\right) \\ \searrow \epsilon_{ij} = \dfrac{1}{2}\left(\dfrac{\partial u_i}{\partial x_j} + \dfrac{\partial u_j}{\partial x_i} + \dfrac{\partial u_i}{\partial x_j}\dfrac{\partial u_j}{\partial x_i}\right) \end{array} \tag{1.186}$$

where ϵ_{ij} and $\in_{ij}$ stand for *Green's* and *Almansi's strain tensors*, respectively, and corresponding stress tensors

$$\sigma_{ij} \longrightarrow \Sigma_{ij} = \frac{\rho_0}{\rho}\frac{\partial X_i}{\partial x_k}\frac{\partial X_j}{\partial x_l}\sigma_{kl} \tag{1.187}$$

where σ_{ij} and Σ_{ij} denote the *Lagrange* and the *second Piola–Kirchhoff stress tensors* instead of formulations (1.183–1.185) we arrive at mutually different formulations

$$\begin{aligned} &\Sigma_{ij} = E_{ijkl}\epsilon_{kl} \\ &\frac{\mathrm{D}\mathcal{W}}{\mathrm{D}t} = \frac{1}{\rho_0}\Sigma_{ij}\frac{\partial \epsilon_{ij}}{\partial t} \\ &\frac{\mathrm{D}\sigma_{ij}}{\mathrm{D}t} - \sigma_{ip}\Omega_{pj} - \sigma_{jp}\Omega_{pi} = E_{ijkl}\dot{\epsilon}_{kl} \end{aligned} \tag{1.188}$$

In case of *hypoelastic material* subjected to finite deformation appropriate constitutive equation $(1.188)_3$ comprises both the symbol of *objective derivative of* the *stress tensor* $\mathrm{D}\sigma_{ij}/\mathrm{D}t$ and an effect of change of stress tensor resulting from rigid rotation which is described by *skew-symmetric spin tensor*

$$\Omega_{ij} = \begin{bmatrix} 0 & \frac{1}{2}\left(\frac{\partial \dot{u}_2}{\partial x_3} - \frac{\partial \dot{u}_3}{\partial x_2}\right) & -\frac{1}{2}\left(\frac{\partial \dot{u}_3}{\partial x_1} - \frac{\partial \dot{u}_1}{\partial x_3}\right) \\ -\frac{1}{2}\left(\frac{\partial \dot{u}_2}{\partial x_3} - \frac{\partial \dot{u}_3}{\partial x_2}\right) & 0 & \frac{1}{2}\left(\frac{\partial \dot{u}_1}{\partial x_2} - \frac{\partial \dot{u}_2}{\partial x_1}\right) \\ \frac{1}{2}\left(\frac{\partial \dot{u}_3}{\partial x_1} - \frac{\partial \dot{u}_1}{\partial x_3}\right) & -\frac{1}{2}\left(\frac{\partial \dot{u}_1}{\partial x_2} - \frac{\partial \dot{u}_2}{\partial x_1}\right) & 0 \end{bmatrix} \tag{1.189}$$

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