Where Join Preservation Fails in the Bounded Turing Degrees of C.E. Sets

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Abstract. We will look at the question for which bounded Turing reducibilities r and r' such that r is stronger than r' join preservation holds, i.e. for which r and r' every join in the computably enumerable (c.e.) r-degrees is also a join in the c.e. r'-degrees. We will also have a look at the corresponding question for meets. We will consider the class of monotone admissible (uniformly) bounded Turing reducibilities, i.e. the reflexive and transitive Turing reducibilities with use bounded by a function that is contained in a (uniformly computable) family of strictly increasing computable functions. This class contains for example ibT- and cl-reducibility. We will show that join preservation does not hold for cl and any admissible uniformly bounded Turing reducibility. We will show that, on the other hand, for all monotone admissible bounded Turing reducibilities r and r' such that r is stronger than r', meet preservation holds.

1 Introduction

Various notions of reducibilities stronger than Turing reducibility have been studied in computability theory, e.g. the so called classical strong reducibilities: one-one reducibility (1-reducibility), many-one reducibility (m-reducibility), truth-table reducibility (tt-reducibility), and weak truth-table reducibility (wtt-reducibility) (see e.g. Odifreddi [13]). More recently, one has started to look at the so called strongly bounded Turing reducibilities: identity bounded Turing reducibility (ibT-reducibility) and computable Lipschitz reducibility (cl-reducibi lity) which are defined in terms of Turing functionals where the use is bounded by the identity function and the identity function plus a constant and which were introduced by Soare [14] and Downey, Hirschfeldt, and LaForte [9,10], respectively. cl-reducibility is not only a notion of relative complexity but can also be viewed as a notion of relative randomness and is hence important in the field of algorithmic randomness (see the monograph [8] by Downey and Hirschfeldt for more background). The degree structures of the strongly bounded Turing reducibilities on the c.e. sets have been studied intensively. Barmpalias [5] showed that the partial ordering $(\mathbf{R}_{\mathrm{cl}}, \leq)$ of the c.e. cl-degrees has no maximal elements;

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Fan and Lu [12] showed that there are maximal pairs hence the partial orderings of the ibT- and cl-degrees are not upper semilattices, and Barmpalias and Lewis [6] and Day [7] showed that these partial orderings are not dense. Ambos-Spies, Bodewig, Kräling, and Yu [3] embedded the nonmodular lattice N5 into the c.e. ibT- and cl-degrees thereby showing that these partial orderings are not distributive, and Ambos-Spies [1] proved some global results, e.g. showed that the first order theories of the partial orderings of the c.e. ibT- and cl-degrees are undecidable. Recently, Ambos-Spies [2] introduced a more general class of bounded Turing reducibilities, the uniformly bounded Turing reducibilities. A reducibility r is a (uniformly) bounded Turing reducibility ((u)bT-reducibility) if there is a family $\mathcal{F}$ of (uniformly) computable functions such that, for all sets A and B, A is r-reducible to B if and only if A is Turing reducible to B with use bounded by some function f in $\mathcal{F}$. We call a (uniformly) bounded Turing reducibility admissible if it is reflexive and transitive and we call it monotone if it is induced by a family of strictly increasing functions. Examples of monotone admissible ubT-reducibilities are the strongly bounded Turing reducibilities ibT and cl as well as the linearly bounded and the primitive recursively bounded Turing reducibilities. An example of an admissible monotone bT-reducibility which is not uniformly bounded is wtt-reducibility. Here, we will only look at the monotone admissible bT-reducibilities.

If a reducibility r is stronger than a reducibility r', of course, every upper r-bound for some sets A and B is also an upper r'-bound for A and B and the same holds for lower bounds. But this does not necessarily imply that least upper r-bounds (joins) have to be a least upper r'-bounds, too. Again, the same holds for greatest lower bounds (meets). Here, we ask the question for which reducibilities r and r', joins and meets in the c.e. r-degrees are preserved in the c.e. r'-degrees. We say r-r' join (meet) preservation holds if, for all noncomputable c.e. sets A, B, and C such that the r-degree of C is the join (meet) of the r'-degrees of A and B, it holds that the r'-degree of C is the join (meet) of the r'-degrees of A and B, too.

For most of the classical reducibilities mentioned above, the structure of the c.e. degrees is an upper semilattice where the join of the degrees of two sets A and B is induced by the effective disjoint union $A \oplus B$. So, for two such reducibilities where r is stronger than r', of course, r-r' join preservation holds. So, for example, m-tt join preservation, tt-wtt join preservation and wtt-T join preservation hold. For reducibilities r whose degree structures are not an upper semilattice with join induced by the effective disjoint union, the question of r-r' join preservation is less obvious. For the classical strong reducibilities, 1-reducibility is an example of such a reducibility, but, as one can easily show (see Lemma 2 below), 1-m join preservation holds. It easily follows that r-r' join preservation holds for all classical strong reducibilities where r is stronger than r'. For the (uniformly) bounded Turing reducibilities, the question of join preservation is less straightforward. Ambos-Spies, Ding, Fan, and Merkle [4] showed that ibT-cl join preservation holds and Ambos-Spies, Bodewig, Kräling, and Yu (see [1]) showed that cl-wtt join preservation holds, too. This may lead one to conjecture that – just as in case of the classical strong reducibilities – r-r' join preservation holds for any monotone admissible (u)bT-reducibilities where r is stronger than r', too. As we will show here, however, this is not the case. In fact, for $r =$ ibT,cl and for *any* monotone amissible ubT-reducibility r' which is strictly stronger than cl, r-r' join preservation fails (see Theorem 1 below).

We complement our main result by considering meet preservation in the monotone admissible bt-reducibilities, too. There we generalize the result in [4] that ibT-cl meet preservation holds by showing that indeed, r-r' meet preservation holds for all monotone admissible bT-reducibilities r and r' such that r is stronger than r' (see Lemma 5).

So, for the monotone admissible (uniformly) bounded Turing reducibilities, meet preservation holds in general while, in some instances, join preservation fails. For the classical reducibilities, i.e. the strong reducibilities together with Turing reducibility, the converse is true. There join preservation holds in general, whereas, as Downey and Stob [11] showed, wtt-T meet preservation fails.

2 Preliminaries

A reducibility r is *admissible* if it is reflexive and transitive. For two reducibilities r and r', we say that r is *stronger* than r' (denoted by $r \preceq r'$) if, for all sets A and B, from $A \leq_r B$, it follows that $A \leq_{r'} B$, and r is *strictly stronger* than r' ($r \prec r'$) if $r \preceq r'$ and $r \neq r'$.

Definition 1. *For two admissible reducibilities r and r', we say that r-r'* join preservation *holds (in the c.e. degrees) if, for any noncomputable c.e. sets A, B, and C,*

$$deg_r(A) \vee deg_r(B) = deg_r(C) \Rightarrow deg_{r'}(A) \vee deg_{r'}(B) = deg_{r'}(C)$$

holds. Otherwise, we say that r-r' join preservation fails. Similarly, r-r' meet preservation *holds (in the c.e. degrees) if, for any noncomputable c.e. sets A, B, and C,*

$$deg_r(A) \wedge deg_r(B) = deg_r(C) \Rightarrow deg_{r'}(A) \wedge deg_{r'}(B) = deg_{r'}(C)$$

holds and r-r' meet preservation fails otherwise.

Let $\{\Phi_e^X : e \geq 0\}$ be a fixed enumeration of all Turing functionals obtained by Gödelization of the oracle Turing machines. Then, we obtain an enumeration $\{\Phi_e^{X,f} : e \geq 0\}$ of all f-bounded Turing functionals by bounding the use of each Φ_e^X on input x by $f(x)$ (by making the computation divergent in case of longer oracle queries). For any pair of sets A and B, A is f-bounded Turing reducible to B (denoted by $A \leq_{f-\mathrm{T}} B$) if and only if there is an e such that $A = \Phi_e^{B,f}$. By letting $f = id$, we obtain an enumeration $\{\hat{\Phi}_e^X\}$ of all identity bounded Turing functionals.

We call a reducibility r a *bounded Turing reducibility* (*bT-reducibility*) if there is a family $\mathcal{F}$ of computable functions such that $A \leq_r B$ if and only if

$A \leq_{f-\mathrm{T}} B$ for some function $f \in \mathcal{F}$; in this case we say that r is *induced* by $\mathcal{F}$. If $\mathcal{F}$ is uniformly computable, r is called a *uniformly bounded Turing reducibility* (*ubT-reducibility*). We call a bounded Turing reducibility *monotone* if it is induced by a family $\mathcal{F}$ which consists only of strictly increasing functions. Note that ibT and cl are ubT-reducibilities which are induced by $\mathcal{F}_{\mathrm{ibT}} = \{id\}$ and $\mathcal{F}_{\mathrm{cl}} = \{id + e : e \geq 0\}$, respectively.

Lemma 1 (Ambos-Spies [2]). *Let r and r' be admissible ubT-reducibilities. Then, $r \preceq r'$ if and only if there are uniformly computable families $\mathcal{F}$ and $\mathcal{F}'$ that induce r and r', respectively, such that $\mathcal{F} \leq^* \mathcal{F}'$, i.e. for every function $f \in \mathcal{F}$, there is a function $f' \in \mathcal{F}'$ such that $f(x) \leq f'(x)$ for almost all $x \in \omega$.*

3 Join Preservation

It is a straightforward observation that r-r' join preservation holds for reducibilities r and r' such that r is stronger than r' and such that the structures of the c.e. r-degrees and of the c.e. r'-degrees form upper semilattices with join induced by the effective disjoint union. We will now observe (by giving an example) that r-r' join preservation may hold even if the structure of the c.e. r-degrees does not form an upper semilattice.

Lemma 2. 1-m *join preservation holds.*

Proof. Given c.e. sets A_0, A_1, and B such that

$$deg_1(A_0) \vee deg_1(A_1) = deg_1(B) \tag{1}$$

holds, we have to show that $deg_{\mathrm{m}}(A_0) \vee deg_{\mathrm{m}}(A_1) = deg_{\mathrm{m}}(B)$ holds, too. As we know that $deg_{\mathrm{m}}(A_0) \vee deg_{\mathrm{m}}(A_1) = deg_{\mathrm{m}}(A_0 \oplus A_1)$, we only have to show that $B =_{\mathrm{m}} A_0 \oplus A_1$. It is obvious that $A_i \leq_1 A_0 \oplus A_1$ via $f_i(x) = 2x + i$ for $i = 0, 1$, so, it follows from (1) that $B \leq_1 A_0 \oplus A_1$, hence $B \leq_{\mathrm{m}} A_0 \oplus A_1$. On the other hand, if we fix g_i such that $A_i \leq_1 B$ via g_i for $i = 0, 1$, it follows that $A_0 \oplus A_1 \leq_{\mathrm{m}} B$ via g where $g(2x + i) = g_i(x)$ for all $x \geq 0$ and for $i = 0, 1$. ☐

More examples of reducibilities r and r' where the structure of r does not form an upper semilattice but where r-r' join preservation still holds have been given in the bounded Turing degrees.

Lemma 3 (Ambos-Spies, Ding, Fan, and Merkle [4]; Ambos-Spies [1]). ibT-cl*,* ibT-wtt*, and* cl-wtt *join preservation hold.*

This result might lead to the assumption that cl-r join preservation holds for all reducibilities r with cl $\preceq r \preceq$ wtt, but this is not the case. We will now show that cl-r join preservation even fails for *all* admissible monotone ubT-reducibilities with cl $\prec r$.

Theorem 1. *Let r be a monotone admissible ubT-reducibility such that* cl $\prec r$*. Then, for $r' =$* ibT, cl, *r'-r join-preservation fails.*

Proof. By Lemma 3, ibT-cl join preservation holds. So, it is enough to prove the theorem for $r' = \text{ibT}$. Since, by $\text{cl} \prec r$, any upper ibT-bound for two sets A_0 and A_1 is also an upper r-bound for A_0 and A_1, it suffices to construct c.e. sets A_0, A_1, B, and C such that $deg_{\text{ibT}}(A_0) \vee deg_{\text{ibT}}(A_1) = deg_{\text{ibT}}(B)$ and such that $A_0, A_1 \leq_r C$ but $B \not\leq_r C$. Let $\mathcal{F}$ be a uniformly computable admissible family of strictly increasing functions such that r is induced by $\mathcal{F}$. As $\mathcal{F}$ is uniformly computable, we can fix a computable function f such that $f \geq^* h$ for all $h \in \mathcal{F}$. As $\text{cl} \prec r$, hence $r \not\preceq \text{cl}$, $\mathcal{F} \not\leq^* \{id+e : e \geq 0\}$ holds, so, there is a function $g \in \mathcal{F}$ such that $\{g\} \not\leq^* \{id + e : e \geq 0\}$, i.e. for any $e \geq 0$, $g(x) > x + e$ for infinitely many x. Since g is strictly increasing, this implies that for all $e \geq 0$, $g(x) > x+e$ for all but finitely many x, so, $id + e \leq^* g$ for all $e \geq 0$. So, in order to complete the proof, it suffices to show that the following lemma holds.

Lemma 4. *Let g be a strictly increasing computable function such that $id + e \leq^* g$ for all e and let f be any computable function (in particular, f can be chosen as above). Then, there are c.e. sets A_0, A_1, B and C such that the following hold.*

$$deg_{\text{ibT}}(A_0) \vee deg_{\text{ibT}}(A_1) = deg_{\text{ibT}}(B) \tag{2}$$

$$A_0, A_1 \leq_{g\text{-T}} C \tag{3}$$

$$B \not\leq_{f\text{-T}} C. \tag{4}$$

Proof. We will enumerate c.e. sets A_0, A_1, B, and C such that (2) to (4) hold using a tree argument. The construction will use ideas introduced in the proof that the nondistributive lattice N5 can be embedded into the partial orderings $(\mathbf{R}_{\text{ibT}}, \leq)$ and $(\mathbf{R}_{\text{cl}}, \leq)$ in [3]. Our notation will be the same as in that proof. To guarantee that (3) holds and that B is an upper ibT-bound for A_0 and A_1, we will satisfy the following global *permitting* (or *coding*) *requirement* for $i = 0, 1$.

$$(x \searrow_{s+1} A_i \Rightarrow \exists y \leq x(y \searrow_{s+1} B)) \;\&\; (x \searrow_{s+1} A_i \Rightarrow \exists y \leq g(x)(y \searrow_{s+1} C)) \tag{5}$$

To guarantee that B is in fact the least upper ibT-bound for A_0 and A_1, i.e. that (2) holds, we will meet the following *join requirements* for $e \geq 0$.

$$\mathcal{Q}_e : A_0 = \hat{\Phi}_{e_1}^{W_{e_0}} \;\&\; A_1 = \hat{\Phi}_{e_2}^{W_{e_0}} \Rightarrow B \leq_{\text{ibT}} W_{e_0} \;\; (e = \langle e_0, e_1, e_2 \rangle).$$

Finally, we will satisfy condition (4) by meeting the *nonordering requirements*

$$\mathcal{P}_e : B \neq \Phi_e^{C,f}$$

for $e \geq 0$. Before we give the actual construction, we will explain the ideas underlying the strategies for meeting the individual requirements and how to combine them.

As the join requirements $\mathcal{Q}_e$ are conditional requirements whose hypotheses are not decidable, we have to guess on the correctness of the hypotheses.

We define the length of agreement between A_0 and $\hat{\Phi}_{e_1}^{W_{e_0}}$ and between A_1 and $\hat{\Phi}_{e_2}^{W_{e_0}}$ at stage s by letting

$$l(e,s) = \max\{x : \forall y < x(A_{0,s}(y) = \hat{\Phi}_{e_1,s}^{W_{e_0,s}}(y) \;\&\; A_{1,s}(y) = \hat{\Phi}_{e_2,s}^{W_{e_0,s}}(y))\}.$$

Since the $\hat{\Phi}$ are bounded functionals, $\lim_{s\to\infty} l(e,s) \leq \infty$ exists and the following holds.

$$(A_0 = \hat{\Phi}_{e_1}^{W_{e_0}} \;\&\; A_1 = \hat{\Phi}_{e_2}^{W_{e_0}}) \Leftrightarrow \lim_{s\to\infty} l(e,s) = \infty \Leftrightarrow \limsup_{s\to\infty} l(e,s) = \infty. \tag{6}$$

In the following, we call a join requirement $\mathcal{Q}_e$ *infinitary* if its hypothesis is true (i.e., if $\lim_{s\to\infty} l(e,s) = \infty$) and we call $\mathcal{Q}_e$ *finitary* otherwise. The strategy for meeting the join requirements is the join strategy used by Ambos-Spies, Bodewig, Kräling, and Yu in [3]. For meeting an infinitary join requirement $\mathcal{Q}_e$, we guarantee $B \leq_{\mathrm{ibT}} W_{e_0}$ by permitting (up to some computable subset of B). We work with a computable set $S = \{s_n : n \geq 0\}$ of $\mathcal{Q}_e$-*expansionary stages*, i.e., $s_0 < s_1 < s_2 < \ldots$ and $l(e,s_0) < l(e,s_1) < l(e,s_2) < \ldots$. We ensure that numbers put into B between stages $s_n + 1$ and $s_{n+1} + 1$ are greater than $s_n + 1$. So, it suffices to guarantee that if a number x enters B at a stage $s+1$ where $s \in S$ and $x < l(e,s)$ then a number $\leq x$ will be enumerated into W_{e_0} after stage s. This change in W_{e_0} is forced by putting a sufficiently small number into A_0 or A_1. As one can easily check, this is achieved by guaranteeing the following.

$$\begin{gathered} x \searrow_{s+1} B \;\&\; x < l(e,s) \Rightarrow \exists y < \min(x', l(e,s))(y \searrow_{s+1} A_0 \text{ or } y \searrow_{s+1} A_1) \\ \text{where } x' = \mu z(z > x \;\&\; z \notin W_{e_0,s}) \end{gathered} \tag{7}$$

For meeting the nonordering requirements $\mathcal{P}_e$, we will use the Friedberg-Muchnik strategy. For a fixed unused number x, we ensure $B(x) \neq \Phi_e^{C,f}(x)$ by waiting for a stage s such that $\Phi_{e,s}^{C_s,f}(x) = 0$. Then, at stage $s+1$, we put x into B and, in order to preserve the computation $\Phi_{e,s}^{C_s,f}(x)$, we impose a restraint of length $f(x)+1$ on C, thereby ensuring

$$B(x) = 1 \neq 0 = B_s(x) = \Phi_{e,s}^{C_s,f}(x) = \Phi_e^{C,f}(x). \tag{8}$$

In the presence of the join requirements and the global permitting requirement, this strategy needs some amendments. To describe the potential conflicts, consider the situation in which we wish to meet requirement $\mathcal{P}_e$ and simultaneously satisfy the global permitting requirement (5) and follow the join strategy (7) for a single infinitary join requirement $\mathcal{Q}_{e'}$ of higher priority.

Now, when we put a number x into B at stage $s+1$ in order to guarantee (8), then, according to (7), we have to put a number $y < x'$ into A_0 or A_1 at stage $s+1$ where

$$x' = \mu z(z > x \;\&\; z \notin W_{e_0',s}).$$

(In our case, we choose to put y into A_1.) If we do so, then, as long as $x \leq y$, this is consistent with the first part of condition (5). But, for the second part of this condition, we have to put a number $z \leq g(y)$ into C. In case that $z \leq f(x)$, however, this will injure the restraint imposed on C in order to preserve the computation $\Phi_{e,s}^{C_s,f}(x)$. In order to overcome this problem, we will make sure that we can find a number y such that $f(x) < y < x'$ where y is not yet in A_1 and the interval $[y, g(y)]$ is not yet completely enumerated into C. (Then putting y into A_1 and some new number z with $y \leq z \leq g(y)$ into C makes the enumeration of x into B compatible with (5) and (7).)

For that matter, we will assign a sufficiently long interval I_n of unused numbers to $\mathcal{P}_e$. I_n will contain finitely many candidates $x_{n,k}$ for a possible attack on $\mathcal{P}_e$ where these numbers are chosen so that $x_{n,k+1} > f(x_{n,k})$ and $g(x_{n,k}) \geq x_{n,k} + k + 2$ for all k. (Note that the latter can be achieved since, by choice of g, $g(y) > y + k + 2$ for all sufficiently large y; also note that $g(x_{n,k}) \geq x_{n,k} + k + 2$ implies $g(y) \geq y + k + 2$ for all $y \geq x_{n,k}$.) We will arrange that, for some k (and some stage s), $(x_{n,k}, x_{n,k+1}] \subseteq W_{e'_0,s}$ where $x_{n,k}$ is not in B_s, $x_{n,k+1}$ is not in $A_{1,s}$ and the interval $[x_{n,k+1}, g(x_{n,k+1})]$ is not completely contained in C_s. (Hence, for $x = x_{n,k}$ and $y = x_{n,k+1}$, $y < x'$ whence we can ensure (8) and simultaneously obey (5) and (7) by putting $x_{n,k}$ into B, $x_{n,k+1}$ into A_1, and some unused number from the interval $[x_{n,k+1}, g(x_{n,k+1})]$ into C at stage $s+1$.) In order to ensure $(x_{n,k}, x_{n,k+1}] \subseteq W_{e'_0}$ for some k, we will successively and in decreasing order put numbers w from I_n into A_0 at stages $s+1$ where $l(e,s)$ is greater than the endpoint of I_n. This forces $W_{e'_0}$ to respond by enumerating more and more numbers from I_n (or smaller ones). As we will argue, this implies that, at some point s, there will be an interval $(x_{n,k}, \ldots x_{n,k+1}] \subset I_n$ such that the enumeration of the numbers $\geq x_{n,k} + 1$ from I_n into A_0 has forced all the numbers $x_{n,k}+1, \ldots, x_{n,k+1}$ into $W_{e'_0}$. (In the actual construction, all the numbers actually have to be forced simultaneously into all sets $W_{e'_0}$ attached to the infinitary higher priority join requirements, but we will show that this can be achieved.) So we can use $x_{n,k}$ for an attack on $\mathcal{P}_e$ – provided that $x_{n,k} \notin B_s$, $x_{n,k+1} \notin A_{1,s}$ and $[x_{n,k+1}, g(x_{n,k+1})] \not\subseteq C_s$.

The latter, however, is not trivially true, since to make the enumeration of w into A_0 compatible with (5) simultaneously we have to put a trace $w_B \leq w$ into B and a trace $w_C \leq g(w)$ into C. So whenever we put w into A_0, then, simultaneously we put w into B (which is compatible with (7) since w goes simultaneously into A_0) and a number from the interval $[w, g(w))$ into C. Since we put only numbers $w > x_{n,k}$ into A_0 this procedure also puts only numbers $> x_{n,k}$ into B and no numbers into A_1 hence guarantees $x_{n,k} \notin B_s$ and $x_{n,k+1} \notin A_{1,s}$. To ensure that $[x_{n,k+1}, g(x_{n,k+1})] \not\subseteq C_s$, however, we have to choose the trace $w_C \in [w, g(w))$ to be put into C carefully. Here we let $w_C = w + k' + 1$ for the unique k' such that $w \in (x_{n,k'}, x_{n,k'+1}]$. Note that, by choice of the numbers $x_{n,k'}$ this ensures that $w_C \leq g(w)$. On the other hand, this ensures that $x_{n,k+1} + k + 2$ is not enumerated into C since, for $w \leq x_{n,k+1}$, $w_C \leq w + k + 1$ while, for $w > x_{n,k+1} < x_{n,k+1} + k + 2$, $w_C \geq w + (k+1) + 1 > x_{n,k+1} + k + 2$.

This completes the discussion of the basic conflicts among the different goals of the construction and how these conflicts can be resolved. We now turn to the actual construction.

We implement the guesses about which of the join requirements are infinitary on the full binary tree $T = \{0,1\}^{<\omega}$. A node α codes a guess about the first n join requirements $\mathcal{Q}_0, \dots, \mathcal{Q}_{n-1}$ where, for $e < n$, $\alpha(e) = 0$ codes the guess that $\mathcal{Q}_e$ is infinitary and $\alpha(e) = 1$ codes the guess that $\mathcal{Q}_e$ is finitary. So the *true path* $f : \omega \to \{0,1\}$ of the construction is defined by

$$f(e) = \begin{cases} 0 & \text{if } A_0 = \hat{\Phi}_{e_1}^{W_{e_0}} \;\&\; A_1 = \hat{\Phi}_{e_2}^{W_{e_0}} \\ 1 & \text{otherwise.} \end{cases}$$

For each node α of length e there is a strategy $\mathcal{P}_\alpha$ for meeting requirement $\mathcal{P}_e$ which is based on the guess α. We will show that the strategy $\mathcal{P}_{f \upharpoonright e}$ on the true path will succeed in meeting $\mathcal{P}_e$.

At any stage s of the construction we have an approximation δ_s of $f \upharpoonright s$, i.e., a guess which of the first s join requirements are infinitary. For the definition of δ_s, first we inductively define α-*stages* for each node α as follows. Each stage $s \geq 0$ is a λ-stage. If s is an α-stage, then we call s α-*expansionary* if $l(|\alpha|, s) > l(|\alpha|, t)$ for all α-stages $t < s$, and we call s an $\alpha 0$-stage if s is α-expansionary and an $\alpha 1$-stage if s is an α-stage but not α-expansionary. Now, for each $s \geq 0$, let $\delta_s \in T$ be the unique α of length s such that s is an α-stage. So, the node δ_s represents the guess at which of $\mathcal{Q}_0, \dots \mathcal{Q}_{s-1}$ are infinite which is made at the end of stage s. It easily follows from (6) that the true path is the leftmost path visited infinitely often in the construction.

Claim 1 (True Path Lemma). $f = \liminf_{s \to \infty} \delta_s$, i.e., for any α, $\alpha \sqsubset f$ if and only if $\alpha \sqsubset \delta_s$ for infinitely many s and there are only finitely many s such that $\delta_s <_L \alpha$.

The intervals I_n which might be assigned to the strategies for meeting the nonordering requirements are inductively defined as follows, where the nth interval I_n consists of $n(x_{n,0}+1)$ subintervals $I_{n,k} = (x_{n,k}, x_{n,k+1}]$.

$$\begin{aligned} x_{0,0} &= \mu x(g(x) \geq x + 2) \\ x_{n,k} &= \mu x(x > f(x_{n,k-1}) \;\&\; g(x) \geq x + k + 2) \\ &\quad\quad \text{for} \;\; n \in \omega \;\; \text{and} \;\; 1 \leq k \leq n(x_{n,0}+1) \\ x_{n+1,0} &= \mu x(x > x_{n,n(x_{n,0}+1)} + n(x_{n,0}+1) + 2 \;\&\; g(x) \geq x+2) \text{ for } n \in \omega \\ I_{n,k} &= (x_{n,k}, x_{n,k+1}] \quad \text{for} \;\; n \in \omega \;\; \text{and} \;\; 0 \leq k \leq n(x_{n,0}+1) - 1 \\ I_n &= \bigcup_{k=0}^{n(x_{n,0}+1)-1} I_{n,k} \end{aligned}$$

Note that this definition ensures that $x_{n,k+1} > f(x_{n,k})$, $g(w) \geq w + k + 2$ for $w \in I_{n,k}$ and $g(w) < x_{n+1,0}$ for $w \in I_n$.

For a node α of length e, we call a number $x \in I_n \cup \{x_{n,0}\}$ α-*safe* at stage s if

$$x = x_{n,k} \;\; \text{for some} \;\; k \;\; \text{with} \;\; 0 \leq k \leq n(x_{n,0}+1) - 1 \tag{9}$$

$$x \notin B_s, x_{n,k+1} \notin A_{1,s} \;\; \text{and} \;\; x_{n,k+1} + k + 2 \notin C_s, \;\; \text{and} \tag{10}$$

$$\forall e'([e' < e \,\&\, \alpha(e') = 0] \Rightarrow I_{n,k} \subseteq W_{e'_0,s}) \tag{11}$$

hold where $e' = \langle e'_0, e'_1, e'_2 \rangle$.

Using the above definitions, the construction of the sets A_0, A_1, B, and C is as follows where stage 0 is vacuous (i.e., $A_{0,0} = A_{1,0} = B_0 = C_0 = \emptyset$).

Stage $s+1$. A strategy $\mathcal{P}_\alpha$ with $|\alpha| = e$ *requires attention* at stage $s+1$ if $\alpha \sqsubseteq \delta_s$, $\mathcal{P}_\alpha$ is not satisfied at the end of stage s, and one of the following cases applies.

(i) No interval is assigned to $\mathcal{P}_\alpha$ at the end of stage s.
(ii) Interval $I_n = \left(x_{n,0}, x_{n,n(x_{n,0}+1)}\right]$ is assigned to $\mathcal{P}_\alpha$ at the end of stage s,

$$\forall e'([e' < e \,\&\, \alpha(e') = 0] \Rightarrow l(e', s) > x_{n,n(x_{n,0}+1)}) \tag{12}$$

holds, no number $x \in I_n \cup \{x_{n,0}\}$ is α-safe at stage s, and $I_n \not\subseteq A_{0,s}$.
(iii) Interval I_n is assigned to $\mathcal{P}_\alpha$ at the end of stage s, (12) holds, and there is a number $x \in I_n \cup \{x_{n,0}\}$ such that x is α-safe at stage s and $B_s(x) = \Phi_{e,s}^{C_s,f}(x) = 0$.

Fix α minimal such that $\mathcal{P}_\alpha$ requires attention (as $\mathcal{P}_{\delta_s}$ requires attention, there is such an α). Declare that $\mathcal{P}_\alpha$ *receives attention* or *becomes active*, initialize all strategies $\mathcal{P}_\beta$ with $\alpha < \beta$ (i.e., if an interval is assigned to $\mathcal{P}_\beta$ then cancel this assignment and if $\mathcal{P}_\beta$ had been satisfied before, then declare $\mathcal{P}_\beta$ to be unsatisfied), and perform the following action according to the case via which $\mathcal{P}_\alpha$ requires attention.

(i) For the least $n > e, s$ such that the interval I_n has not been assigned to any strategy before, assign I_n to $\mathcal{P}_\alpha$.
(ii) Let y be the greatest number in $I_n \setminus A_{0,s}$. Put y into A_0 and B and, for the unique k such that $y \in I_{n,k}$, put $y + k + 1$ into C.
(iii) Let x be the greatest α-safe number in $I_n \cup \{x_{n,0}\}$ such that $B_s(x) = \Phi_{e,s}^{C_s,f}(x) = 0$. Let k be the unique number such that $x = x_{n,k}$. Put x into B, $x_{n,k+1}$ into A_1, and $x_{n,k+1}+k+2$ into C. Then, declare $\mathcal{P}_\alpha$ to be *satisfied*.

This completes the construction. We will prove a series of claims to show that the construction satisfies all of our requirements. The claims will essentially be the same as in the proof of Theorem 3.2 in [3]. The first of these claims is straightforward and we omit the proof.

Claim 2. Every strategy $\mathcal{P}_\alpha$ on the true path (i.e., $\alpha \sqsubset f$) is initialized only finitely often and requires attention only finitely often. Moreover, for any such strategy, there is an interval I_n which is permanently assigned to it.

Claim 3. The global permitting requirement (5) is satisfied.

Proof. It is crucial to note that numbers from $I_n \cup \{x_{n,0}\} \cup \{g(x) : x \in I_n\}$ can be enumerated into any of the sets under construction at stage $s+1$ only by the strategy to which I_n is assigned at this stage. So, it follows by a straightforward induction that if a strategy $\mathcal{P}_\alpha$ acts via (ii) at stage $s+1$ then, for the number

y there, neither y is in B_s nor $y+k+1$ is in C_s. And, similarly, if a strategy $\mathcal{P}_\alpha$ acts via (iii) at stage $s+1$ then neither $x_{n,k}$ is in B_s nor $x_{n,k+1}$ is in $A_{1,s}$ nor $x_{n,k+1}+k+2$ is in C_s where the latter follows from our observations preceding the construction. This easily implies the claim, since a number x is enumerated into A_0 at some stage $s+1$ only if some strategy $\mathcal{P}_\alpha$ acts at stage $s+1$ via (ii), hence $x \in I_{n,k}$ for some k and, at stage $s+1$, x is enumerated into B and $x+k+1$ is enumerated into C where $x+k+1 \leq g(x)$ by choice of $I_{n,k}$; and since a number x is enumerated into A_1 at some stage $s+1$ only if some strategy $\mathcal{P}_\alpha$ acts at stage $s+1$ via (iii), hence $x = x_{n,k+1}$ for some n,k and, at stage $s+1$, $x_{n,k} < x_{n,k+1}$ is enumerated into B and $x_{n,k+1}+k+2$ is enumerated into C where by choice of $x_{n,k+1}$, $x_{n,k+1}+k+2 \leq g(x)$.

Claim 4. The join requirements $\mathcal{Q}_e$ are met.

Proof. The argumentation is very similar to the one in the proof of Claim 5 in the proof of Theorem 3.2 in [3]. We fix $e = \langle e_0, e_1, e_2 \rangle$ and assume w.l.o.g. that $\mathcal{Q}_e$ is infinitary, so, $\alpha 0 \sqsubset f$ for $\alpha = f \restriction e$. Hence there are infinitely many $\alpha 0$-stages. By Claims 1 and 2, we can fix an $\alpha 0$-stage $s_0 > e$ such that no strategy $\mathcal{P}_\beta$ with $\beta \leq \alpha 0$ becomes active after this stage. Let $S = \{s_n : n \geq 0\}$ be the set of the $\alpha 0$-stages $\geq s_0$. Then, S is computable, $s_0 < s_1 < s_2 < \ldots$, and $l(e, s_0) < l(e, s_1) < l(e, s_2) < \ldots$. So, as explained in the discussion of the strategy for meeting the requirements $\mathcal{Q}_e$, it suffices to show that (7) holds for $s \in S$. But this is immediate by construction since at a stage $s_m + 1$ only a strategy $\mathcal{P}_\beta$ with $\alpha 0 \sqsubseteq \beta$ may act. Namely, if $\mathcal{P}_\beta$ acts via (ii) then the number x enumerated into B is simultaneously enumerated into A_0 and if $\mathcal{P}_\beta$ acts via (iii) then the claim follows from the corresponding action by β-safeness of the number x put into B.

Claim 5. The nonordering requirements $\mathcal{P}_e$ are met.

Proof. For fixed e, assume for a contradiction that $\mathcal{P}_e$ is not met. Exactly as in [3], we can then argue that for $\alpha = f \restriction e$, an interval I_n becomes permanently assigned to $\mathcal{P}_\alpha$ at some stage s_1+1, that there is no number $x \in I_n \cup \{x_{n,0}\}$ that is α-safe at any stage $s' > s_1$, and that all numbers in I_n are enumerated into A_0 in decreasing order after stage s_1+1 according to clause (ii) in the definition of requiring and receiving attention. As in [3], for $x \in I_n$, let $t_x > s_1$ be the α-stage such that x is enumerated into A_0 at stage $t_x + 1$. Then (12) holds for $s = t_x$. So, for $x \in I_n$ and for any infinitary higher priority join requirement $\mathcal{Q}_{e'}$, $W_{e'_0, t_x} \restriction x+1 \neq W_{e'_0, t_{x-1}} \restriction x+1$. So if we let J be the set of the numbers e'_0, such that

$$J = \{e'_0 : \exists e'_1, e'_2 : (\langle e'_0, e'_1, e'_2 \rangle < e \;\&\; \mathcal{Q}_{\langle e'_0, e'_1, e'_2 \rangle} \text{ is infinitary}\},$$

then

$$\forall j \in J \, \forall x \in I_n (W_{j,t_x} \restriction x+1 \subset W_{j,t_{x-1}} \restriction x+1). \tag{13}$$

Now, for $x \in I_n$ and $j \in J$, let

$$w_j(x) = |W_{j,t_x} \restriction x+1| \text{ and } w_J(x) = \sum_{j \in J} w_j(x),$$

and call x *unsaturated* if $x \notin W_{j,t_x}$ for some $j \in J$. By definition, $|J| \leq e$ and $w_j(x) \leq x+1$, hence

$$w_J(x_{n,0}) \leq e(x_{n,0}+1). \tag{14}$$

As in [3], we will now argue that this bound is not compatible with (13) and the fact that there are no α-safe numbers in $I_n \cup \{x_{n,0}\}$. As shown in [3], it follows from (13) that

$$w_J(x_{n,0}) \geq |\{x \in I_n : x \,\text{is unsaturated}\}|. \tag{15}$$

Now, it suffices to give a lower bound on the number of unsaturated numbers in I_n that contradicts (14). For a number $x_{n,k} \in I_n \cup \{x_{n,0}\}$ with $0 \leq k \leq n(x_{n,0}+1)-1$, (9) and (10) hold for $t_x = s$. So, since there are no α-safe numbers in $I_n \cup \{x_{n,0}\}$ after stage s_1+1, (11) must fail for $t_x = s$. It follows that at least one number in $I_{n,k}$ must be unsaturated for every k. As there are $n(x_{n,0}+1)$ many subintervals $I_{n,k}$ in I_n each of which must contain at least one unsaturated number and as $e < n$ by construction, it follows that there are at least $(e+1)(x_{n,0}+1)$ unsaturated numbers in I_n, which, together with (15), leads to the desired contradiction.

This completes the proof of Lemma 4. □

4 Meet Preservation

In contrast to Theorem 1, meet preservation holds for the monotone admissible bounded Turing reducibilities in general. This is immediate by the following lemma which generalizes the observation in [4] that ibT-cl and cl-wtt meet preservation hold.

Lemma 5. *Let r and r' be monotone admissible bounded Turing reducibilities induced by $\mathcal{F}$ and $\mathcal{F}'$, respectively, such that r is stronger than r'. Then, r-r' meet preservation holds.*

Proof. The proof is essentially the same as the one for the results in [4]. Let A_0, A_1, and B be c.e. sets such that

$$deg_r(A_0) \wedge deg_r(A_1) = deg_r(B) \tag{16}$$

holds. As r is stronger than r', B is also an upper r'-bound for A_0 and A_1, so, it suffices to show that for a given c.e. set C such that $C \leq_{r'} A_0, A_1$, $C \leq_{r'} B$ holds. Fix functions $f_i \in \mathcal{F}'$ such that $C \leq_{f_i-\mathrm{T}} A_i$ for $i = 0, 1$. Since r' is admissible, as shown in [2], we may assume that $\mathcal{F}'$ is closed under composition, so, $f_0 \circ f_1 = f \in \mathcal{F}'$. As r' is monotone, we may also assume that f_0 and f_1 are strictly increasing, so, $\max(f_0, f_1) \leq f$. It follows that $C \leq_{f-\mathrm{T}} A_0, A_1$. Let $C_f = \{f(x) : x \in C\}$ be the f-shift of C. Then, $C_f \leq_{\mathrm{ibT}} A_0, A_1$. As ibT is stronger than r, $C_f \leq_r A_0, A_1$, so, by (16), $C_f \leq_r B$, hence $C_f \leq_{r'} B$. We know that $C \leq_{f-\mathrm{T}} C_f$, hence by $f \in \mathcal{F}'$, $C \leq_{r'} C_f$, so, by transitivity of r', $C \leq_{r'} B$. □

5 Open Problems

Contrasting previous positive results on join preservation in the bounded Turing degrees (see Lemma 3) we have shown that r-r' join preservation fails for the strongly bounded Turing reducibilities $r = $ ibT,cl and any monotone admissible uniformy bounded Turing reducibility r' with $cl \prec r'$. This naturally leads to the question of a classification of the monotone admissible bounded Turing reducibilities r and r' for which r-r' join preservation holds. Moreover, one may consider nonmonotone reducibilities, too. For the latter, a classification of the bT-reducibilities for which meet preservation holds is open, too.

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