# Quantum Game Players Can Have Advantage Without Discord

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**Abstract.** The last two decades have witnessed a rapid development of quantum information processing, a new paradigm which studies the power and limit of "quantum advantages" in various information processing tasks. Problems such as when quantum advantage exists, and if existing, how much it could be, are at a central position of these studies. In a broad class of scenarios, there are, implicitly or explicitly, at least two parties involved, who share a state, and the correlation in this shared state is the key factor to the efficiency under concern. In these scenarios, the shared *entanglement* or *discord* is usually what accounts for quantum advantage. In this paper, we examine a fundamental problem of this nature from the perspective of game theory, a branch of applied mathematics studying selfish behaviors of two or more players. We exhibit a natural zero-sum game, in which the chance for any player to win the game depends only on the ending correlation. We show that in a certain classical equilibrium, a situation in which no player can further increase her payoff by any local classical operation, whoever first uses a quantum computer has a big advantage over its classical opponent. The equilibrium is fair to both players and, as a shared correlation, it does not contain any discord, yet a quantum advantage still exists. This indicates that at least in game theory, the previous notion of discord as a measure of non-classical correlation needs to be reexamined, when there are two players with different objectives.

## 1 Introduction

Quantum computers have exhibited tremendous power in algorithmic, cryptographic, information theoretic, and many other information processing tasks, compared with their classical counterparts. Meanwhile, for a large number of problems, quantum computers are not able to offer much advantage over classical ones. When and why quantum computers are more powerful are always at a central position in studies on quantum computation and quantum information processing. A particularly interesting class of scenarios is when there are, implicitly or explicitly, at least two parties involved who share a state, the


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correlation in this state is the key factor. What accounts for the quantum advantage is often *entanglement*, one of the most distinctive characters of quantum information. Indeed, it has been showed that a quantum algorithm with only slight entanglement can be simulated efficiently by a classical computer [Vid03]. In certain potential applications of quantum algorithms, it is also shown that entangled measurement is necessary for the existence of efficient quantum algorithms [HMR+10].

Recently people started to realize that entanglement is not always a necessary resource needed for generating quantum correlations. It has been found that *discord*, another unique character of quantum states, also plays an important role in quantum information processing [OZ01]. Discord is a relaxed version of entanglement—states with positive entanglement must also have positive discord, but there are states with positive discord but zero entanglement. People has discovered cases where quantum speed-up exists without entanglement involved, and discord is considered to be responsible for the quantum advantage [DSC08]. Till today, discord is widely considered as necessary for the existence of quantum advantages.

In this paper, we reexamine this notion from the perspective of game theory [OR94]. Game theory studies the situation in which there are two or more players with possibly different goals. There are two broad classes of games, one is strategic-form (or normal-form) games, in which all players make their choice simultaneously; a typical example is Rock-Paper-Scissors. The other class is extensive-form games, in which players make their moves in turn; a typical example is chess.

The research on quantum games began about one decade ago, starting with two pioneering papers.[1] The first one [EWL99] aimed to quantize a specific strategic-form game called *Prisoners' Dilemma* [EWL99], and it unleashed a long sequence of follow-up works in the same model. Despite the rapid growth of literature, controversy also largely exists [BH01, vEP02, CT06], which questioned the meaning of the claimed quantum solution, the ad hoc assumptions in the model, and the inconsistency with standard settings of classical strategic games. Recently a new model was proposed for quantizing general strategic-form games [Zha12]. Compared with [EWL99], the new model corresponds to the classical games more precisely, and has rich mathematical structures and game-theoretic questions; also see later theoretical developments [KZ12, WZ13, JSWZ13, PKL+15].

Back to the early stage of the development of quantum game theory, the other pioneering paper was [Mey99], which demonstrated the power of using quantum strategies in an extensive-form game. More specifically, Meyer considered the quantum version of the classical Penny Matching game. The basic setting is as follows. There are two players, and each has two possible actions on one bit: Flip it or not. Starting with the bit being 0, Player 1 first takes an action, and

[1] Note that there is also a class of "nonlocal games", such as CHSH or GHZ games [BCMdW10], where all the players have the *same* objective. But general game theory focuses more on situation that the players have *different* objective functions, and the players are selfish, each aiming to optimize her own objective function only.

then Player 2 takes an action, and finally Player 1 takes another action, and the game is finished. If the bit is finally 0, then Player 1 wins; otherwise Player 2 wins. It is not hard to see that if Player 2 flips the bit with half probability, then no matter what Player 1 does, each player wins the game with half probability. Now consider the following change of setting: The bit becomes a qubit; the first player uses a quantum computer in the sense that she can perform any quantum admissible operation on the bit; the second player uses a classical computer in the sense that she can perform either Identity or the flip operation $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. In this new setting, Player 1 can win the game with certainty! Her winning strategy is simple: she first applies a Hadamard gate to change the state to $|+\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$, and then no matter whether Player 2 applies the flip operation or not, the state remains the same $|+\rangle$, thus in the third step Player 1 can simply apply a Hadamard gate again to rotate the state back to $|0\rangle$. This shows that a player using a quantum computer can have big advantage over one using a classical computer.

Despite a very interesting phenomena it exhibits, the quantum advantage is not the most convincing due to a fairness issue. After all, the quantum player takes two actions and the classical player takes just one. And the order of "Player 1 $\rightarrow$ Player 2 $\rightarrow$ Player 1" is also crucial for the quantum advantage. One remedy is to consider normal-form games, in which the players give their strategies *simultaneously*, thus there is no longer the issue of the action order. Taking the model in [Zha12], two players play a complete-information normal-form game, with a starting state $\rho$ in systems $(A_1, A_2)$, and $A_i$ being given to Player $i$. A classical player can only measure her part of the state in the computational basis, followed by whatever classical operation $C$ (on the computational basis). In previous works [EWL99, Mey99, ZWC+12] the classical player is usually assumed to be able to apply any classical operation on computational basis (such as $X$-gate), followed by a measurement in the computational basis. A classical operations there is implicitly assumed to be unitary, so the operation in the matrix form is a permutation matrix. Here we allow classical player to measure first and then perform any classical operation, which gives her more power since the second-step classical operation does need to be unitary. Indeed, in Meyer's Penny Matching game, in the second step Player 2 could measure the state first and then randomly set it to be $|0\rangle$ or $|1\rangle$ each with half probability. Then in the third step, Player 1's Hadamard gate will change the state to $|+\rangle$ or $|-\rangle$, in either case, Player 1 could win with only half probability.

Even if we now enlarge the space of possible operations of the classical player, we will show examples where the quantum player has advantage of winning the game. Furthermore, the examples have the following nice properties respecting the fairness of the game:

1. If both players are classical, then both get expected payoff 0, and $\rho$ is a correlated equilibrium in the sense that any classical operation $C$ by one player cannot increase her expected payoff.

2. Suppose that one player remains classical and the other player uses a quantum computer. To illustrate the power of using quantum strategies, we cut the classical player some slack as follows. The classical player can (1) pick one subsystem, $A_1$ or $A_2$, of $\rho$, leaving the other subsystem to the quantum player, and (2) "take side" by picking one of the two payoff matrices, leaving the other to the quantum player.

Examples were found that even with the advantage of taking side and taking part of the shared state, the classical player still has a disadvantage compared to the quantum player. Consider the canonical $2 \times 2$ zero-sum game with the payoff matrices being

$$U_1 = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \quad \text{and} \quad U_2 = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}. \tag{1}$$

*Quantum game with entanglement.* Each player $i$ owns a 2-dimensional Hilbert space, and they share the quantum state

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|+0\rangle + |-1\rangle) = \frac{1}{\sqrt{2}}(|0+\rangle + |1-\rangle), \tag{2}$$

where $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$, and $|-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$. It is not difficult to verify that if both players measure their parts in the computational basis, then each gets payoff 1 and $-1$ with equal probability, resulting an average payoff of zero for both players. This is a correlated equilibrium for classical operations.

Now suppose that Player 1 employs a quantum computer. Since the state is symmetric, it does not matter which part Player 2, the classical player, chooses. Let us assume that Player 2 chooses part 2, and the payoff matrix $U_2$. Then Player 1 can apply the Hadamard transformation on her qubit, followed by the measurement in computational basis. The state immediately before the measurement is $|\psi'\rangle = (|00\rangle + |11\rangle)/\sqrt{2}$. Therefore the measurement in the computational basis gives Player 1 and Player 2 payoff 1 and $-1$, respectively, with certainty. In other words, Player 1 wins with certainty, whereas she could only win with half probability when using a classical computer.

In this example where the quantum player has an advantage, the state shared by players is highly entangled, which motivates the following natural question: Is entanglement necessary for quantum advantage in the game? It turns out that the answer is no. Consider the example below.

*Quantum Game with Discord.* The payoff matrices are the same as before, but the quantum state shared by players is the following.

$$\rho = \frac{1}{4}(|+\rangle\langle+| \otimes |0\rangle\langle 0| + |0\rangle\langle 0| \otimes |+\rangle\langle+| + |-\rangle\langle-| \otimes |1\rangle\langle 1| + |1\rangle\langle 1| \otimes |-\rangle\langle-|). \tag{3}$$

This state is separable and thus does not have any entanglement. It can be checked that if the players measure this state in computational basis, the probability of getting each of the four possible outcomes is 1/4. Thus the overall payoff of each player is zero, and it can be verified that it is a classical correlated equilibrium.

In the quantum setting, again without loss of generality assume that the classical computer picks the second part of $\rho$ and the second payoff matrix. The quantum player can again perform a Hadamard operation on her system, resulting in a new state

$$\rho' = \frac{1}{4}(|0\rangle\langle 0| \otimes |0\rangle\langle 0| + |+\rangle\langle +| \otimes |+\rangle\langle +| + |1\rangle\langle 1| \otimes |1\rangle\langle 1| + |-\rangle\langle -| \otimes |-\rangle\langle -|). \quad (4)$$

Measuring the new state, the quantum player gets state $|00\rangle$, $|01\rangle$, $|10\rangle$, $|11\rangle$ with probability 3/8, 1/8, 1/8, 3/8 respectively. As a result, her winning probability increases from 1/2 to 3/4; in other words, she gets an expected payoff of 1/2.

Note that the quantum state in Eq. (4) is separable, and there is no any entanglement, but the quantum player still gets a quantum advantage. Thus, entanglement is not necessary for quantum advantage to exist in this game. Note that, however, the state in Eq. (4) has a positive discord. As we have mentioned, it was known that in some scenarios, it is discord, rather than entanglement, that produces non-classical correlations. So the above example confirms this traditional notion in the new game-theoretic setting.

These two examples were also experimentally verified recently [ZWC+12]. The present paper makes further studies on the foregoing notion by asking the following fundamental question.

> *Is discord necessary for quantum advantage to exist in games where players share a symmetric state?*

It is tempting to conjecture that the answer is Yes. In the rest of the paper, we will show that, first, discord is indeed necessary for any quantum advantage to exist in a 2-player games where each player has $n = 2$ strategies. We will then show that when $n \geq 3$, however, there are games where the quantum player has a positive advantage even when the shared symmetric state has zero discord.

## 2 Preliminaries

Suppose that in a classical game there are $k$ players, labeled by $\{1, 2, \ldots, k\}$. Each player $i$ has a set $S_i$ of strategies. To play the game, each player $i$ selects a strategy $s_i$ from $S_i$. We use $s = (s_1, \ldots, s_k)$ to denote the *joint strategy* selected by the players and $S = S_1 \times \ldots \times S_k$ to denote the set of all possible joint strategies. Each player $i$ has a utility function $u_i : S \to \mathbb{R}$, specifying the *payoff* or *utility* $u_i(s)$ of Player $i$ on the joint strategy $s$. For simplicity of notation, we use subscript $-i$ to denote the set $[k] - \{i\}$, so $s_{-i}$ is $(s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_k)$, and similarly for $S_{-i}$, $p_{-i}$, etc. In this paper, we will mainly consider 2-player games.

Nash equilibrium is a fundamental solution concept in game theory. Roughly, it says that in a joint strategy, no player can gain more by changing her strategy, provided that all other players keep their current strategies unchanged. The precise definition is as follows.

**Definition 1.** *A* pure Nash equilibrium *is a joint strategy* $s = (s_1, \ldots, s_k) \in S$ *satisfying that*

$$u_i(s_i, s_{-i}) \geq u_i(s_i', s_{-i}), \qquad \forall i \in [k], \forall s_i' \in S_i.$$

Pure Nash equilibria can be generalized by allowing each player to independently select her strategy according to some probability distribution, leading to the following concept of *mixed Nash equilibrium*.

**Definition 2.** *A* (mixed) Nash equilibrium (NE) *is a product probability distribution* $p = p_1 \times \ldots \times p_k$*, where each* $p_i$ *is a probability distributions over* $S_i$*, satisfying that*

$$\sum_{s_{-i}} p_{-i}(s_{-i}) u_i(s_i, s_{-i}) \geq \sum_{s_{-i}} p_{-i}(s_{-i}) u_i(s_i', s_{-i}), \ \forall i \in [k], \ \forall s_i, s_i' \in S_i \ \textit{with} \ p_i(s_i) > 0.$$

A fundamental fact proved by Nash [Nas51] is that every game with a finite number of players and a finite set of strategies for each player has at least one mixed Nash equilibrium.

There are various further extensions of mixed Nash equilibria. Aumann [Aum74] introduced a relaxation called *correlated equilibrium*. This notion assumes an external party, called Referee, to draw a joint strategy $s = (s_1, ..., s_k)$ from some probability distribution $p$ over $S$, possibly correlated in an arbitrary way, and to suggest $s_i$ to Player $i$. Note that Player $i$ only sees $s_i$, thus the rest strategy $s_{-i}$ is a random variable over $S_{-i}$ distributed according to the conditional distribution $p|_{s_i}$, the distribution $p$ conditioned on the $i$-th part being $s_i$. Now $p$ is a correlated equilibrium if any Player $i$, upon receiving a suggested strategy $s_i$, has no incentive to change her strategy to a different $s_i' \in S_i$, assuming that all other players stick to their received suggestion $s_{-i}$.

**Definition 3.** *A* correlated equilibrium (CE) *is a probability distribution* $p$ *over* $S$ *satisfying that*

$$\sum_{s_{-i}} p(s_i, s_{-i}) u_i(s_i, s_{-i}) \geq \sum_{s_{-i}} p(s_i, s_{-i}) u_i(s_i', s_{-i}), \ \forall i \in [k], \forall s_i, s_i' \in S_i.$$

The above statement can also be restated as

$$\mathbb{E}_{s_{-i} \leftarrow \mu|s_i}[u_i(s_i, s_{-i})] \geq \mathbb{E}_{s_{-i} \leftarrow \mu|s_i}[u_i(s_i', s_{-i})]. \tag{5}$$

where $\mu|s_i$ is the distribution $\mu$ conditioned on the $i$-th component being $s_i$. Notice that a classical correlated equilibrium $p$ is a classical Nash equilibrium if $p$ is a product distribution.

Correlated equilibria captures natural games such as the Traffic Light and the Battle of the Sexes ([VNRET97], Chap. 1). The set of CE also has good mathematical properties such as being convex (with Nash equilibria being some of the vertices of the polytope). Algorithmically, it is computationally benign for finding the best CE, measured by any linear function of payoffs, simply by solving a linear program (of polynomial size for games of constant players). A natural learning dynamics also leads to an approximate CE ([VNRET97], Chap. 4) which we will define next, and all CE in a graphical game with $n$ players and with $\log(n)$ degree can be found in polynomial time ([VNRET97], Chap. 7).

## 3 Quantum Game Without Discord

In this section, we will address the question proposed at the end of the first section. Suppose that a game has two players and both of them have $n$ strategies. In other words, each player holds an $n$-dimensional quantum system. Recall that we also require the shared quantum state $\rho \in H \otimes H$ be symmetric, so that swapping the two systems does not change the state. It is not hard to derive from the general criteria of zero-discord state [DVB10] that these quantum states $\rho$ have the form of

$$\rho = \sum_{i,j=0}^{n-1} p(i,j)|\psi_i\rangle\langle\psi_i| \otimes |\psi_j\rangle\langle\psi_j|, \tag{6}$$

where $\{|\psi_i\rangle\}$ is a set of orthogonal basis of the $n$-dimensional Hilbert space $H$, and $P = [p(i,j)]_{ij} \in \mathbb{R}_+^{n\times n}$ is a symmetric matrix with nonnegative entries satisfying that $\sum_{ij} p(i,j) = 1$. (In general, we use the upper case letter $P$ to denote the matrix and the lower case letter $p$ to denote the corresponding two-variate distribution $p(i,j)$.) We sometimes also write the state as

$$\rho = \sum_i p_1(i)|\psi_i\rangle\langle\psi_i| \otimes \sigma_i \tag{7}$$

where $p_1(i) = \sum_j p(i,j)$ is the marginal distribution on the first system, and $\sigma_i = \sum_j \frac{p(i,j)}{p_1(i)}|\psi_j\rangle\langle\psi_j|$ (if $p_1(i) = 0$ then let $\sigma_i = |0\rangle\langle 0|$).

Consider the following game as a natural extension of the Penny Matching game in Sect. 1. The payoff matrices are

$$U_1 = nI - J \quad and \quad U_2 = -U_1, \tag{8}$$

where $J$ is the all-one matrix. Intuitively, whoever takes the first matrix bets that the two $n$-sided dice give the same side, and the other player bets that the two dice give different sides. We first show that there is a unique correlated equilibrium in the game.

**Lemma 1.** *The game given by Eq.* (8) *has only one classical correlated equilibrium $Q = J/n^2$.*

*Proof.* According to the definition of correlated equilibrium, if a distribution $q$ on $[n] \times [n]$ is a classical correlated equilibrium, then the following relationships hold:

$$\sum_j q(i,j)U_1(i,j) \geq \sum_j q(i,j)U_1(i',j), \qquad \forall i,i' \in \{0,1,...,n-1\}, \tag{9}$$

and

$$\sum_i q(i,j)U_2(i,j) \geq \sum_i q(i,j)U_2(i,j'), \qquad \forall j,j' \in \{0,1,...,n-1\}. \tag{10}$$

Plugging the definition of $U_1$ and $U_2$ into the above inequalities, one can verify that $Q = J/n^2$ is the only solution.

Recall that $\rho = \sum_i p_1(i)|\psi_i\rangle\langle\psi_i|\otimes\sigma_i$. Since $\rho$ is symmetric, it does not matter which part the classical player, Player 2, chooses to hold. For the convenience of discussions, let us assume that the classical player takes the second part. We use $\mathtt{supp}(p)$ to denote the support of a distribution $p$, *i.e.,* the set of elements with non-zero probability. The next lemma gives a sufficient and necessary condition for the existence of quantum advantage.

**Lemma 2.** *Suppose that measuring the state $\rho$ gives a classical correlated equilibrium for the game given in Eq.* (8)*. Then Player 1 (who is quantum) does not have any advantage if and only if*

$$\langle i|\sigma_j|i\rangle = 1/n, \quad \forall i \in \{0, 1, ..., n-1\} \text{ and } j \in \mathtt{supp}(p_1). \tag{11}$$

*Proof. "Only if"*: Assume that Player 1 first measures her part in the orthonormal basis $\{|\psi_i\rangle\}$. Note that this does not affect the state. If outcome $j$ occurs, then Player 1 knows that the state of Player 2 is $\sigma_j$. We consider which utility matrix in Eq.(8) Player 1 has. In the first case, Player 1 takes the utility matrix $U_1$. It is not hard to see that her optimal strategy is to replace her part $|\psi_j\rangle$ by $|i\rangle$, where $i$ is a maximizer of $\max_i\langle i|\sigma_j|i\rangle$. Thus Player 1 has a strict positive advantage if and only if there is some $i$ and $j$, where $j \in \mathtt{supp}(p_1)$, with $\langle i|\sigma_j|i\rangle > 1/n$, which is equivalent to saying that there is some $i$ and $j \in \mathtt{supp}(p_1)$ with $\langle i|\sigma_j|i\rangle \neq 1/n$.

Similarly, if Player 1 takes the utility matrix $U_2$, then her optimal strategy is to replace $|\psi_j\rangle$ with $|i\rangle$, where $i$ is a minimizer of $\min_i\langle i|\sigma_j|i\rangle$. Thus Player 1 has a strict positive advantage if and only if there is some $i$ and $j$ with $\langle i|\sigma_j|i\rangle < 1/n$, which is again equivalent to saying that there is some $i$ and $j$ with $\langle i|\sigma_j|i\rangle \neq 1/n$.

*"If"*: Player 2 measures her part in the computational basis, yielding the state

$$\frac{1}{n}\sum_{i,j} p_1(j)|\psi_j\rangle\langle\psi_j| \otimes |i\rangle\langle i|.$$

Now whatever quantum operation Player 1 applies, the probability of observing the same bits (*i.e.,* the state after the measurement is $|ii\rangle$ for some $i$) is $1/n$, with the expected payoff of 0 for both players.

Though the above lemma gives a sufficient and necessary condition, it is still not always clear whether quantum advantage could exist for any symmetric state $\rho$ with zero discord. Next we will further the study by considering a related matrix $M \in \mathbb{R}_+^{n\times n}$, whose $(i, j)$-th entry is defined to be

$$M(i, j) = |\langle i|\psi_j\rangle|^2. \tag{12}$$

It turns out that the rank of $M$ is an important criteria to our question. In the rest of this section, we will consider two cases, depending on whether $M$ is full rank or not.

### 3.1 Case 1: $M$ Is Full-Rank

We will first show that if $M$ is full-rank, then the quantum player cannot have any advantage.

**Theorem 3.** *Suppose that the two players of the game Eq. (8) share a symmetric state $\rho$, measuring which gives a classical correlated equilibrium. Then Player 1 (who is quantum) does not have any advantage if $M$ in Eq. (12) is full-rank.*

*Proof.* By Lemma 1, for any $0 \le k, j \le n-1$ we have

$$\sum_{i=0}^{n-1} p_1(i)|\langle k|\psi_i\rangle|^2 \cdot \langle j|\sigma_i|j\rangle = \frac{1}{n^2}$$

Summing over $j$, we obtain another equality

$$\sum_{i=0}^{n-1} p_1(i)|\langle k|\psi_i\rangle|^2 = \frac{1}{n}.$$

Combining these two equalities, we have

$$\sum_{i=0}^{n-1} |\langle k|\psi_i\rangle|^2 \cdot p_1(i)\left(\langle j|\sigma_i|j\rangle - \frac{1}{n}\right) = 0.$$

Define a matrix $A = [a(i,j)]_{ij} \in \mathbb{R}^{n\times n}$ by $a(i,j) = p_1(i)\left(\langle j|\sigma_i|j\rangle - \frac{1}{n}\right)$. Then the above equality is just $\sum_i M(k,i)a(i,j) = 0$ for all $k, j$. In other words, we have $M \cdot A = 0$. Since the matrix $M$ is assumed to be full-rank, we have $A = M^{-1}0 = 0$. The conclusion thus follows by Lemma 2.

Two corollaries are in order. First, note that $M$ is full-rank for a generic orthogonal basis $\{|\psi_i\rangle\}$, it is generically true that no discord implies no quantum advantage.

**Corollary 4.** *If a set of orthonormal basis $\{|\psi_i\rangle\}$ is picked uniformly at random, then with probability 1, the quantum player does not have any advantage.*

The second corollary considers the case of $n = 2$, which is settled by the above theorem completely. Indeed, when $n = 2$, the rank of $M$ is either 1 or 2. The rank-2 case is handled by the above theorem. If the rank is 1, it is not hard to see that the only possible $M$ is $M = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$. In this case, for any $i$ and any $k$ it holds that

$$\langle k|\sigma_i|k\rangle = \langle k|\Big(\sum_j p(j|i)|\psi_j\rangle\langle\psi_j|\Big)|k\rangle = \sum_j p(j|i)|\langle k|\psi_j\rangle|^2 = \frac{1}{2}\sum_j p(j|i) = \frac{1}{2}.$$

Applying Lemma 2, we thus get the following corollary.

**Corollary 5.** *There is no quantum advantage for the game defined in Eq. (1) on any symmetric state $\rho$ with zero discord.*

### 3.2 Case 2: $M$ Is Not Full Rank

Somewhat surprisingly, the quantum player *can* have an advantage when $M$ is not full-rank. In this section we exhibit a counterexample for $n = 3$. In this case, recall that the payoff matrices are

$$U_1 = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} \quad \text{and} \quad U_2 = \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix}. \tag{13}$$

We consider the following quantum state,

$$\rho = \sum_{i,j=0}^{2} p(i,j) |\psi_i\rangle\langle\psi_i| \otimes |\psi_j\rangle\langle\psi_j|, \tag{14}$$

where

$$|\psi_0\rangle = \frac{1}{\sqrt{2}} \left(|0\rangle + |1\rangle\right), \quad |\psi_1\rangle = \frac{1}{\sqrt{2}} \left(|0\rangle - |1\rangle\right), \quad |\psi_2\rangle = |2\rangle. \tag{15}$$

It is not hard to calculate $M$:

$$M = \begin{pmatrix} 1/2 & 1/2 & 0 \\ 1/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{16}$$

which has rank 2. Define

$$P = \begin{pmatrix} 4/9 & 0 & 0 \\ 0 & 0 & 2/9 \\ 0 & 2/9 & 1/9 \end{pmatrix}. \tag{17}$$

It can be easily verified that if the two players measure the state in computational basis, the probability distribution yielded is uniform, which is a classical Nash equilibrium.

Now suppose that Player 1 uses a quantum computer. One can verify that the condition in Lemma 2 does not hold. For a concrete illustration, let us consider the protocol in Lemma 2 again. Player 1 first measures in the basis $\{|+\rangle, |-\rangle, |2\rangle\}$. With probability 4/9, she observes $|+\rangle$, then changes it to $|0\rangle$. Player 2's state is also $|+\rangle$ in this case, thus a measurement in the computational basis gives the $|00\rangle$ and $|01\rangle$ each with half probability. Thus Player 1's payoff in this case is $2 \cdot \frac{1}{2} - 1 \cdot \frac{1}{2} = \frac{1}{2}$. The second case is that Player 1 observes $|-\rangle$, which happens with probability 2/9, and Player 2's state is $|2\rangle$ for sure. Player 1 changes her part to $|2\rangle$, and gets payoff 2. The third case is that Player 1 observes $|2\rangle$, which happens with probability 1/3, leaving Player 2 $\sigma_3 = (2/3)|1\rangle\langle 1| + (1/3)|2\rangle\langle 2|$. Player 1 then changes her qubit to $|1\rangle$, collides with Player 2's outcome with probability 1/3, thus Player 1's payoff is $2 \cdot \frac{1}{3} - 1 \cdot \frac{2}{3} = 0$. On average, the quantum player has a payoff of $(4/9)(1/2) + (2/9) \cdot 2 + (1/3) \cdot 0 = 2/3$.

It should be pointed out that the matrix $P$ achieving the quantum advantage of 2/3 is not unique. For example, the following matrix also works with the same effect:

$$P = \begin{pmatrix} 2/9 & 2/9 & 0 \\ 0 & 0 & 2/9 \\ 1/9 & 1/9 & 1/9 \end{pmatrix}. \tag{18}$$

### 3.3 Optimization

In this subsection, we show that the 3-dimensional example in the above subsection is actually optimal for $M$ defined in Eq. (16). Actually the theorem below shows more. Note that if the rank of $M$ is 1, it is easy to prove that $M$ must be the uniform matrix, and the quantum advantage must be zero, thus in the following we suppose the rank of $M$ to be 2.

**Theorem 6.** *Suppose that measuring the state $\rho$ gives a classical correlated equilibrium. Suppose the columns of $M$ are $M^0, M^1$ and $M^2$. Without loss of generality, suppose $M^0 = xM^1 + (1-x)M^2$, where $0 \le x \le 1$. Then the quantum advantage*

$$QA \le \frac{1}{3} + \frac{1}{3x_b}, \ \textit{where } x_b = \max\{x, 1-x\}. \tag{19}$$

*Proof.* By Lemma 1, for any $0 \le k, l \le 2$ we have $\sum_{i,j=0}^{2} p(i,j)|\langle k|\psi_i\rangle|^2 \cdot |\langle l|\psi_j\rangle|^2 = \frac{1}{9}$., which turns out to be equivalent to $M \cdot P \cdot M^T = \frac{J}{9}$. Noting that $M \cdot (J/9) \cdot M^T = J/9$, we know that $P$ can be expressed as

$$P = \frac{J}{9} + \bar{P}, \tag{20}$$

where $M \cdot \bar{P} \cdot M^T = 0$. By straightforward calculation, one can show that Eq. (20) indicates $M \cdot \bar{P} = 0$. Considering the form of $M$, $\bar{P}$ can now be expressed as

$$\bar{P} = \begin{pmatrix} k_0 & k_1 & k_2 \\ -k_0 x & -k_1 x & -k_2 x \\ -k_0(1-x) & -k_1(1-x) & -k_2(1-x) \end{pmatrix}, \tag{21}$$

where $k_0, k_1$ and $k_2$ are real numbers.

According to the discussion above, we know that the maximal quantum advantage is

$$QA = \sum_{i=0}^{2} p_1(i)[2 \cdot \langle l_i|\sigma_i|l_i\rangle - 1 \cdot (1 - \langle l_i|\sigma_i|l_i\rangle)], \tag{22}$$

where $l_i = \max_l \langle l|\sigma_i|l\rangle$. Then it holds that

$$\begin{aligned} QA &= 3\sum_{i=0}^{2} p_1(i) \cdot \langle l_i|\sigma_i|l_i\rangle - 1 = 3\sum_{i,j=0}^{2} p(i,j)|\langle l_i|\psi_j\rangle|^2 - 1 \\ &= 3\sum_{i,j=0}^{2} \left(\frac{1}{9} + \bar{p}(i,j)\right)|\langle l_i|\psi_j\rangle|^2 - 1 = 3\sum_{i=0}^{2}\left(\sum_{j=0}^{2} \bar{p}(i,j)|\langle l_i|\psi_j\rangle|^2\right), \end{aligned}$$

where $\bar{p}(i,j)$ is the element of $\bar{P}$. At the same time, it can be obtained that $l_i = \max_l \sum_j \bar{p}(i,j)|\langle l|\psi_j\rangle|^2$. Besides, recall that the rank of $M$ is 2, then there must be one row of $M$, say $M_2$, has the form of $aM_0 + (1-a)M_1$, where $M_0$ and $M_1$ are the other two rows of $M$, and $0 \leq a \leq 1$. Then it can be known that every $l_i$ must be 0 or 1. Based on the form of $\bar{P}$, we have that $l_0 \neq l_1 = l_2$. Without loss of generality, we suppose $l_0 = 0$, and $l_1 = l_2 = 1$. Then

$$\begin{aligned} QA &= 3\sum_{j=0}^{2} \bar{p}(0,j)|\langle 0|\psi_j\rangle|^2 + 3\sum_{j=0}^{2}(\bar{p}(1,j)+\bar{p}(2,j))|\langle 1|\psi_j\rangle|^2 \\ &= 3\sum_{j=0}^{2} \bar{p}(0,j)|\langle 0|\psi_j\rangle|^2 - 3\sum_{j=0}^{2} \bar{p}(0,j)|\langle 1|\psi_j\rangle|^2. \end{aligned}$$

Note that $\bar{P}+J/9$ is a matrix with nonnegative elements. Thus, for any $0 \leq i \leq 2$, if $k_i \geq 0$ we have $-k_i x \geq -\frac{1}{9}$ and $-k_i(1-x) \geq -\frac{1}{9}$, and if $k_i < 0$, we have $-k_i \leq \frac{1}{9}$. And the above inequality indicates that if $0 < x < 1$, $k_i \leq \frac{1}{9x}$ and $k_i \leq \frac{1}{9(1-x)}$, which is equivalent to $k_i \leq \frac{1}{9x_b}$. Actually, this also holds when $x = 0$ or $x = 1$. Therefore, we obtain that

$$\begin{aligned} QA &= 3\sum_{j=0}^{2} \bar{p}(0,j)|\langle 0|\psi_j\rangle|^2 - 3\sum_{j=0}^{2} \bar{p}(0,j)|\langle 1|\psi_j\rangle|^2 \\ &= 3\sum_{j=0}^{2} k_j|\langle 0|\psi_j\rangle|^2 - 3\sum_{j=0}^{2} k_j|\langle 1|\psi_j\rangle|^2 \leq 3\cdot\frac{1}{9x_b} + 3\cdot\frac{1}{9} = \frac{1}{3} + \frac{1}{3x_b}, \end{aligned}$$

where the relationship $\sum_j |\langle 0|\psi_j\rangle|^2 = \sum_j |\langle 1|\psi_j\rangle|^2 = 1$ is utilized.

Go back to the example in the above subsection. Note that for $M$ in Eq. (16) we have $M^0 = 1\cdot M^1 + 0\cdot M^2$(thus in order to utilize Theorem 6, we need to adjust the order of the columns). Thus we can choose $x = 0$, and then $x_b = 1$. As a result, the discussion above shows that $QA \leq 2/3$, which means the choice of $P$ in Eq. (17) is optimal for $M$ in Eq. (16).

***Open Problems.*** From the mathematical perspective, some questions remain open. Two of them are listed as below: (1) What is the maximum gain in a zero-sum $[-1,1]$-normalized game[2] on a state in symmetric subspace without entanglement? (2) What is the maximum gain in a zero-sum $[-1,1]$-normalized game on a state in symmetric subspace without discord?

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[2] A game is $[-1,1]$-normalized if all utility functions have ranges within $[-1,1]$.

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## References


[Aum74] Aumann, R.: Subjectivity and correlation in randomized strategies. J. Math. Econ. **1**, 67–96 (1974)

[BCMdW10] Buhrman, H., Cleve, R., Massar, S., de Wolf, R.: Nonlocality and communication complexity. Rev. Mod. Phys. **82**, 665–698 (2010)

[BH01] Benjamin, S., Hayden, P.: Comment on "quantum games and quantum strategies". Phys. Rev. Lett. **87**(6), 069801 (2001)

[CT06] Cheon, T., Tsutsui, I.: Classical and quantum contents of solvable game theory on hilbert space. Phys. Lett. A **348**, 147–152 (2006)

[DSC08] Datta, A., Shaji, A., Caves, C.M.: Quantum discord and the power of one qubit. Phys. Rev. Lett. **100**, 050502 (2008)

[DVB10] Dakic, B., Vedral, V., Brukner, C.: Necessary and sufficient condition for nonzero quantum discord. Phys. Rev. Lett. **105**, 190502 (2010)

[EWL99] Eisert, J., Wilkens, M., Lewenstein, M.: Quantum games and quantum strategies. Phys. Rev. Lett. **83**(15), 3077–3080 (1999)

[HMR+10] Hallgren, S., Moore, C., Roetteler, M., Russell, A., Sen, P.: Limitations of quantum coset states for graph isomorphism. J. ACM **57**(6), 1–33 (2010)

[JSWZ13] Jain, R., Shi, Y., Wei, Z., Zhang, S.: Efficient protocols for generating bipartite classical distributions and quantum states. IEEE Trans. Inf. Theor. **59**(8), 5171–5178 (2013)

[KZ12] Kerenidis, I., Zhang, S.: A quantum protocol for sampling correlated equilibria unconditionally and without a mediator. In: Kawano, Y. (ed.) TQC 2012. LNCS, vol. 7582, pp. 13–28. Springer, Heidelberg (2012)

[Mey99] Meyer, D.: Quantum strategies. Phys. Rev. Lett. **82**(5), 1052–1055 (1999)

[Nas51] Nash, J.: Non-cooperative games. The Ann. Math. **54**(2), 286–295 (1951)

[OR94] Osborne, M.J., Rubinstein, A.: A Course in Game Theory. MIT Press, Cambridge (1994)

[OZ01] Ollivier, H., Zurek, W.H.: Quantum discord: a measure of the quantumness of correlations. Phys. Rev. Lett. **88**, 017901 (2001)

[PKL+15] Pappa, A., Kumar, N., Lawson, T., Santha, M., Zhang, S., Diamanti, E., Kerenidis, I.: Nonlocality and conflicting interest games. Phys. Rev. Lett. **114**, 020401 (2015)

[vEP02] van Enk, S.J., Pike, R.: Classical rules in quantum games. Phys. Rev. A **66**, 024306 (2002)

[Vid03] Vidal, G.: Efficient classical simulation of slightly entangled quantum computations. Phys. Rev. Lett. **91**, 147902 (2003)

[VNRET97] Vazirani, V., Nisan, N., Roughgarden, T., Éva, T.: Algorithmic Game Theory. Cambridge University Press, New York (1997)

[WZ13] Wei, Z., Zhang, S.: Full characterizing quantum correlated equilibria. Quantum Inf. Comput. **13**(9–10), 0846–0860 (2013)

[Zha12] Zhang, S.: Quantum strategic game theory. In: Proceedings of the 3rd Innovations in Theoretical Computer Science, pp. 39–59 (2012)

[ZWC+12] Zu, C., Wang, Y., Chang, X., Wei, Z., Zhang, S., Duan, L.: Experimental demonstration of quantum gain in a zero-sum game. New J. Phys. **14**(033002), 39–59 (2012)