Chapter 7
Use of the Differential Calculus for Finding Caustics by Refraction

Definition. [120] If we imagine that an infinity of rays *BA*, *BM*, and *BD* (see Fig. 7.1), which emanate from the same radiant point *B*, are refracted[1] when they encounter a curved line *AMD*, by approaching or moving away from its perpendiculars *MC*, so that the sines *CE* of the angles *CME* of incidence are always to the sines *CG* of the angles *CMG* of refraction in the same given ratio as *m* to *n*, then the curved line[2] *HFN* (see Fig. 7.2) which touches all the refracted rays or their prolongations *AH*, *MF*, and *DN* is called *the Caustic by refraction*.[3]

Corollary. (§132) *If we envelop*[4] *the caustic HFN beginning at the point A, we describe the curve ALK so that the tangent LF plus the portion FH of the caustic is continually equal to the same straight line AH. Moreover, if we imagine another tangent Fml infinitely close to FML, with another incident ray Bm, and if we describe the little arcs MO and MR with centers F and B, then we form two little right triangles MRm and MOm, which are similar to two others MEC and MGC, pair by pair, because if we remove the same angle EMm from the right angles RME and CMm, then the remaining angles RMm and EMC are equal. Similarly, if we remove the same angle GMm from the right angles GMO and CMm, the remaining angles OMm and GMC are equal. This is why Rm : Om :: CE : CG :: m : n. Now, because Rm is the differential of BM and Om is the differential of LM, it follows (see §96) that BM − BA, the sum of all the differentials Rm in the portion*

[1] In L'Hôpital (1696) the term *rompre* is used, literally meaning "to break." We consistently translate the term *rayon rompu* as "refracted ray."

[2] In L'Hôpital (1696) the curved line was given as *FHN*, but corrected in the *Errata*.

[3] A caustic by refraction is sometimes called a "Dicaustic."

[4] I.e., describe the involute in reserve order, see §110, Footnote 4.

R.E. Bradley et al., *L'Hopital's Analyse des infiniments petits*, Science Networks. Historical Studies 50, DOI 10.1007/978-3-319-17115-9_7

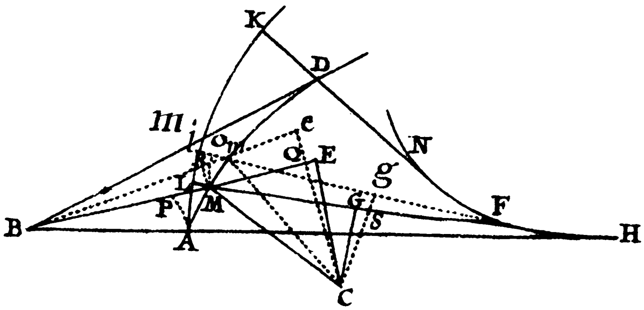

Fig. 7.1 Caustic by Refraction, Convex Case

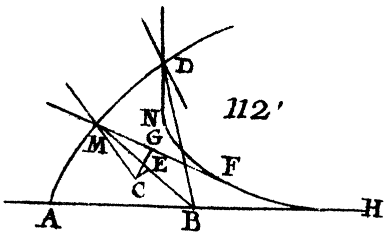

Fig. 7.2 Caustic by Refraction, Concave Case

AM of the curve, is to ML or AH − MF − FH, the sum of all the differentials Om in the same portion [121] AM, as m is to n, and consequently the portion $FH = AH - MF + \frac{n}{m}BA - \frac{n}{m}BM$.

There could be different cases, according to whether the incident ray *BA* is greater or less than *BM*, and whether the refracted ray *AH* envelops or evolves the portion *HF*. However, we will still prove, as we have just done, that the difference of the incident rays is to the difference of the refracted rays (by joining to one of them the portion of the caustic that it evolves before falling on the other) as m is to n. For example (see Fig. 7.2), $BA - BM : AH - MF - FH :: m : n$, from which we conclude that $FH = AH - MF + \frac{n}{m}BM - \frac{n}{m}BA$.

If we describe the circular arc *AP* with center *B* (see Fig. 7.1), then it is clear that *PM* is the difference of the incident rays *BM* and *BA*. Moreover, if we suppose that the radiant point *B* becomes infinitely distant from the curve *AMD*, the incident rays *BA* and *BM* become parallel and the arc *AP* becomes a straight line perpendicular to these rays.

Proposition I.

General Problem. (§133) *Given the nature of the curve AMD* (see Fig. 7.1)*, the radiant point B, and the incident ray BM, we wish to find the point F on the refracted ray MF, given in position, where it touches the caustic by refraction.*

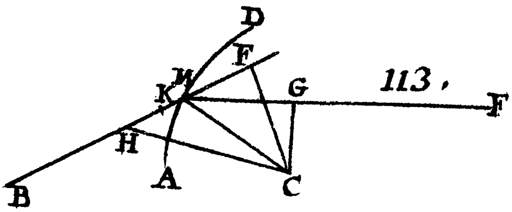

Fig. 7.3 Intersection of the Refracted Ray with the Caustic

We find (see Ch. 5) the length MC of the radius of the evolute at the given point M. We take the infinitely small arc Mm, and draw the straight lines Bm, Cm, and Fm. We describe the little arcs MR and MO with centers B and F, and we drop the perpendiculars CE, Ce, CG, and Cg to the incident and refracted rays. We denote the given quantities BM by y, ME by a, MG by b, and the little arc MR by dx.

Given this, the similar right triangles MEC and MRm, MGC and MOm, and BMR and BQe give $ME(a) : MG(b) :: MR(dx) : MO = \frac{b\,dx}{a}$ and [122] $BM(y) : BQ$ or $BE(y+a) :: MR(dx) : Qe = \frac{a\,dx + y\,dx}{y}$. Now, by the property of the refraction $Ce : Cg :: CE : CG :: m : n$. Consequently, $m : n :: Ce - CE$ or $Qe\left(\frac{a\,dx+y\,dx}{y}\right) : Cg - CG$ or $Sg = \frac{an\,dx + ny\,dx}{my}$. Thus, because of the similar right triangles FMO and FSg, we have $MO - Sg\left(\frac{bmy\,dx - any\,dx - aan\,dx}{amy}\right) : MO\left(\frac{b\,dx}{a}\right) :: MS$ or $MG(b) : MF = \frac{bbmy}{bmy - any - aan}$. This gives the following construction.

Let the angle $ECH = GCM$ (see Fig. 7.3) be constructed towards CM, and let $MK = \frac{aa}{y}$ be taken towards B. I say that if we make $HK : HE :: MG : MF$, then the point F is on the caustic by refraction.

Because of the similar triangles CGM and CEH, we have $CG : CE :: n : m :: MG(b) : EH = \frac{bm}{n}$. From this we conclude that $HE - ME$ or $HM = \frac{bm - an}{n}$, $HM - MK$ or $HK = \frac{bmy - any - aan}{ny}$, and consequently $HK\left(\frac{bmy - any - aan}{ny}\right) : HE\left(\frac{bm}{n}\right) :: MG(b) : MF = \frac{bbmy}{bmy - any - aan}$.

It is clear that if the value of HK is negative, the value of MF is also negative, from which it follows that the point M falls between the points G and F, when the point H is between the points K and E.

If the radiant point B falls on the side of the point E (see Fig. 7.2),[5] or (what is the same thing) if the curve AMD is concave on the side of the radiant point B, then y changes from positive to negative, and consequently we have $MF = \frac{-bbmy}{-bmy + any - aan}$ or $\frac{bbmy}{bmy - any + aan}$. The construction remains the same.

If we suppose that y becomes infinite, that is to say that the radiant point B is infinitely distant from the curve AMD, then the incident rays are parallel to each other, and we have $MF = \frac{bbm}{bm - an}$, because the term aan is null [123] with respect to

[5]In L'Hôpital (1696), the reference here was to figures 7.1 and 7.3.

the other two, bmy and any, and because $MK\left(\frac{aa}{y}\right)$ therefore vanishes, we need only make $HM : HE :: MG : MF$.

Corollary I. (§134) *We demonstrate, in the same way as for the caustic by reflection (see §114), that a curved line AMD has only one caustic by refraction, given the ratio of m to n. This caustic is always geometric and rectifiable when the given curve AMD is geometric.*

Corollary II. (§135) *If the point E falls on the other side of the perpendicular MC with respect to the point G, and if CE is equal to CG, then it is clear that the caustic by refraction changes to a caustic by reflection. Indeed, we have* $MF\left(\frac{bbmy}{bmy-any\mp aan}\right) = \frac{ay}{2y\mp a}$, *because* $m = n$, *and a changes from negative to positive, and it also becomes equal to b. This agrees with what we proved in the previous chapter.*

If m is infinite with respect to n, then it is clear that the refracted ray MF falls on the perpendicular CM, so that the Caustic by refraction becomes the Evolute. Indeed, we have $MF = b$, *which in this case becomes MC, that is to say that the point F falls on the point C, which is on the evolute.*

Corollary III. (§136) *If the curve AMD is convex with radiant point B, and the value of MF* $\left(\frac{bbmy}{bmy-any-aan}\right)$ *is positive, it is clear that we must take the point F on the same side as the point G with respect to the point M, as we have supposed from making the calculations. On the contrary, if it is negative, we must take it on the opposite side. It is the same when the curve AMD is concave towards the point B, however it should be noted that in this case [124]* $MF = \frac{bbmy}{bmy-any+aan}$. *From this it follows that infinitely close refracted rays are convergent when the value of MF is positive in the first case, and negative in the second case, and, on the contrary, they are divergent when the value of MF is negative in the first case, and positive in the second. Given this, it is clear that:*

1. If the curve *AMD* is convex towards the radiant point B, and m is less than n, or if it is concave towards this point, and m is greater than n, then infinitely close refracted rays are always divergent.
2. If the curve *AMD* is convex towards the radiant point B, and m is greater than n, or if it is concave towards this point and m is less than n, then infinitely close refracted rays are convergent, when $MK\left(\frac{aa}{y}\right)$ is less than $MH\left(\frac{bm}{n} - a \text{ or } a - \frac{bm}{n}\right)$, divergent when MK is greater, and parallel when it is equal. Now, because $MK = 0$ when the incident rays are parallel, it follows that in this case infinitely close refracted rays are always convergent.

Corollary IV. (§137) *If the incident ray BM touches the curve AMD at the point M, then we have* $ME(a) = 0$, *and consequently* $MF = b$. *This shows that the point F therefore falls on the point G.*

If the incident ray *BM* is perpendicular to the curve *AMD*, then the straight lines $ME(a)$ and $MG(b)$ each become equal to the radius of the evolute *CM*, because they

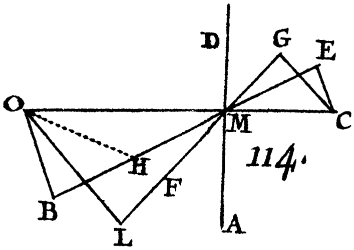

Fig. 7.4 Caustic of a Straight Line by Refraction

coincide with it. We therefore have $MF = \frac{bmy}{my - ny \mp bn}$, which becomes $\frac{bm}{m-n}$ when the incident rays are parallel to each other.

If the refracted ray MF touches the curve AMD at the point M, then we have $MG(b) = 0$. From this we see that the caustic therefore touches the given curve at the point M.

[125] If the radius of the evolute CM is null, then the straight lines $ME(a)$ and $MG(b)$ are also equal to zero. Consequently, the terms aan and $bbmy$ are null with respect to the other terms bmy and any. From this it follows that $MF = 0$, and therefore that the caustic has the point M in common with the given curve.

If the radius of the evolute CM is infinite, then the straight lines $ME(a)$ and $MG(b)$ are also infinite. Consequently, the terms bmy and any are null with respect to other terms aan and $bbmy$, so that we have $MF = \frac{bbmy}{\mp aan}$. Now (see §133), because this quantity is negative when we suppose that the point F falls on the other side of the point B, with respect to the line AMD, and on the contrary it is positive when we suppose that it falls on the same side, it follows (see §136) that we must take the point F on the same side of the point B, that is to say that infinitely close refracted rays are divergent. It is clear that the little arc Mm thus becomes a straight line, and that the preceding construction no longer holds. We may substitute the following one for it, which can be used to determine the points of caustics by refraction when the line AMD is straight.

Draw BO perpendicular to the incident ray BM (see Fig. 7.4), meeting the straight line MC perpendicular to AD at O. If we draw OL perpendicular to the refracted ray MG, and make the angle BOH equal to the angle LOM, then we have $BM : BH :: ML : MF$. I say that the point F is on the caustic by refraction.

Because the right triangles MEC and MBO are similar and the right triangles MGC and MLO are also similar, no matter what magnitude we suppose CM to have, and consequently when it becomes infinite, we still have[6] $ME(a) : MG(b) :: BM(y) : ML = \frac{by}{a}$. Additionally, because the triangles OLM and OBH are similar,

[6]In L'Hôpital (1696) the connective between $BM(y)$ and ML was missing.

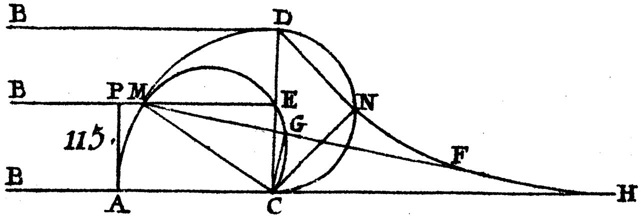

Fig. 7.5 Caustic of a Quarter Circle by Refraction, Convex Case

we also have[7] $OL : OB(n : m) :: ML\left(\frac{by}{a}\right) : BH = \frac{bmy}{an}$. From this we see that $BM(y) : BH\left(\frac{bmy}{an}\right) :: ML\left(\frac{by}{a}\right) : MF\left(\frac{bbmy}{aan}\right)$.

Corollary V. [126] (§138) *It is clear that given any two of the three points B, C, and F, we can easily find the third one.*

Example I. (§139) Let the curve *AMD* (see Fig. 7.5) be a quarter of a circle that has the point C as its center. Let the incident rays *BA*, *BM*, and *BD* be parallel to each other and perpendicular to *CD*. Finally, let the ratio of m to n be as 3 is to 2, which is the ratio for rays of light passing from air into glass. Because the evolute of the circle *AMD* is the point C, which is its center, it follows that if we describe a semi-circumference *MEC*, which has the radius *CM* as its diameter, and if we take the chord $CG = \frac{2}{3}CE$, then the line *MG* is the refracted ray, on which we determine the point F, as we demonstrated above (see §133).

To find the point H where the incident ray *BA*, perpendicular to *AMD*, touches the caustic by refraction, we have (see §137) $AH\left(\frac{bm}{m-n}\right) = 3b = 3CA$.[8] Moreover, if we describe a semi-circumference *CND* with the radius *CD* as its diameter, and if we take the chord $CN = \frac{2}{3}CD$, then it is clear (see §137) that the point N is on the caustic by refraction because the incident ray *BD* touches the circle *AMD* at the point D.

If we draw *AP* parallel to *CD*, then it is clear (see §132) that the portion $FH = AH - MF - \frac{2}{3}PM$, so that the entire caustic $HFN = \frac{7}{3}CA - DN = \frac{7-\sqrt{5}}{3}CA$.

If the quarter circle *AMD* (see Fig. 7.6) is concave towards the incident rays *BM*, and the ratio of m to n, is as 2 is to 3, we take the chord $CG = \frac{3}{2}CE$ on the semi-circumference *CEM* that has the radius *CM* as its diameter, and we draw the refracted ray *MG*, on which we determine the point F by the general construction of §133.

[127] We have (see §137) $AH\left(\frac{bm}{m-n}\right) = -2b$, that is to say that *AH* is on the side (see §136) of the convexity of the quarter circle *AMD*, and twice the radius *AC*. If we

[7]In L'Hôpital (1696) the parentheses around $\frac{by}{a}$ were missing.

[8]In L'Hôpital (1696) the parentheses following *AH* were omitted.

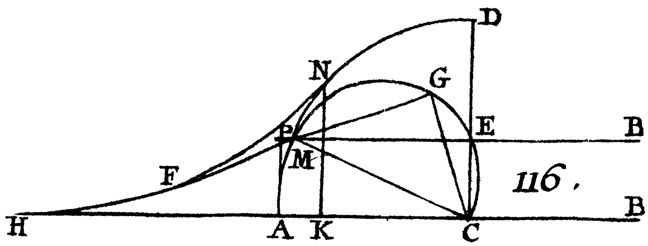

Fig. 7.6 Caustic of a Quarter Circle by Refraction, Concave Case

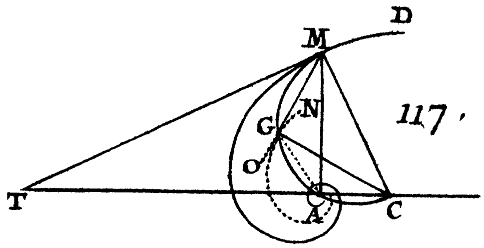

Fig. 7.7 Caustic of the Logarithmic Spiral by Refraction Spiral

suppose that CG or $\frac{3}{2}CE$ is equal to CM, then it is manifest that the refracted ray MF touches the circle AMD at M, because then the point G coincides with the point M. From this it follows that if we take $CE = \frac{2}{3}CD$, the point M falls on the point N, where the caustic HFN (see §137) touches the quarter circle AMD. However, when CE is greater than $\frac{2}{3}CD$, the incident rays BM can no longer be refracted, that is to say pass from glass into the air, because it is impossible for CG, perpendicular to the refracted ray MG, to be greater than CM, so that all the rays that fall on the part ND are reflected.

If we draw AP parallel to CD, then it is clear (see §132) that the portion $FH = AH - MF + \frac{3}{2}PM$, so that if we draw NK parallel to CD, the entire caustic $HFN = 2CA + \frac{3}{2}AK = \frac{7-\sqrt{5}}{2}CA$.

Example II. (§140) Let the curve AMD (see Fig. 7.7) be a logarithmic spiral, which has the point A as its center, from which all the incident rays AM emanate.

It is clear (see §91) that the point E falls on the point A, that is to say that $a = y$. Thus, if we substitute y in the place of a in $\frac{bbmy}{bmy-any+aan}$, the value (see §133) of MF when the curve is concave on the side of the radiant point, then we have $MF = b$. From this we see that the point F falls on the point G.

If we draw the straight line AG and the tangent MT, the angle AGO, supplementary to the angle AGM, is equal the angle AMT. This is because in the circle whose diameter is the line CM, that passes through the points A and G, the angles AGO and AMT each has as measure of half of the same arc AM. Therefore, it is clear that

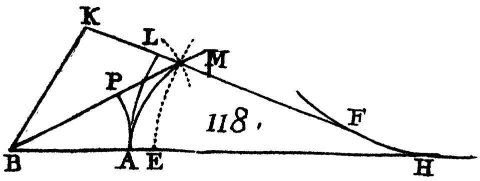

Fig. 7.8 Caustic by Refraction, Inverse Problem

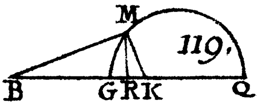

Fig. 7.9 Caustic by Refraction, Inverse Problem, Alternate Solution

the caustic *AGN* is the same [128] logarithmic spiral as the given *AMD*, and that it only differs in its position.

Proposition II.

Problem. (§141) *Given the caustic by refraction HF* (see Fig. 7.8) *with its radiant point B and the ratio of m to n, we wish to find an infinity of curves, such as AM, for which HF is the caustic by refraction.*

Take the point A at will on any tangent *HA* as one of the points on the curve *AM*. Describe the circular arc *AP* with center B and interval *BA*, and another circular arc with any other interval *BM*. Taking $AE = \frac{n}{m}PM$, we describe a curved line *EM* by enveloping the caustic *HF*, that cuts the circular arc described on the interval *BM* in a point M, which is on the curve we wish to find. This is because (see §132), $PM : AE$ or $ML :: m : n$.

Alternate Solution. (§142) On any tangent *FM*, other than *HA*, we wish to find the point M such that $HF + FM + \frac{n}{m}BM = HA + \frac{n}{m}BA$. This is why if we take $FK = \frac{n}{m}BA + AH - FH$, and we find a point M on *FK*, such that $MK = \frac{n}{m}BM$, this (see §132) will be the point that we wish to find. Now, this can be done by describing a curved line *GM* (see Fig. 7.9) such that when we draw the straight lines *MB* and *MK* from any of its point M to the given points B and K, they are always to each other in the same ratio as m is to n. It is therefore only a matter of finding the nature of this place.[9]

To this end, let *MR* be drawn perpendicular to *BK* and denote the given *BK* by a, and the indeterminates *BR* by x and *RM* by y. The right triangles *BRM* and *KRM* give $BM = \sqrt{xx + yy}$ and $KM = \sqrt{aa - 2ax + xx + yy}$, [129] so that to satisfy the condition of the problem, we must have $\sqrt{xx + yy} : \sqrt{aa - 2ax + xx + yy} :: m : n$.

[9]I.e., the place of the general point M of the curved line *GM*.

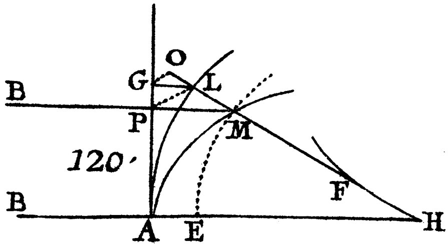

Fig. 7.10 Caustic by Refraction, Inverse Problem, Radiant Point at Infinity

From this we conclude $yy = \frac{2ammx - aamm}{mm - nn} - xx$, which is a place on the circle that we construct as follows.

Let us take $BG = \frac{am}{m+n}$ and $BQ = \frac{am}{m-n}$, and let the semi-circumference GMQ be described with diameter GQ; I say this is the required place. Because we have QR or $BQ - BR = \frac{am}{m-n} - x$ and RG or $BR - BG = x - \frac{am}{m+n}$, the property of the circle, which gives $QR \times RG = RM^2$, yields $yy = \frac{2ammx - aamm}{mm - nn} - xx$ in analytic terms.

If the incident rays BA and BM (see Fig. 7.10) are parallel to a straight line given in position, the first solution will still hold, but this latter becomes useless, and we may substitute it with the following.

Let us take $FL = AH - HF$, and draw LG parallel to AB and perpendicular to AP. We take $LO = \frac{n}{m} LG$, and draw LP parallel to GO, and PM parallel to GL. It is clear (see §132) that the point M is the one that we wish to find; because $LO = \frac{n}{m} LG$, so it follows that $ML = \frac{n}{m} PM$.

If the caustic by refraction FH meets in a point, then the curves AM become the Ovals of *Descartes*, which have caused such a stir among Geometers.[10]

Corollary I. (§143) *We prove as we did for the caustics by reflection (see §130) that the curves AM have different natures, and they are not geometric except when the caustic by refraction HF is geometric and rectifiable.*

Corollary II. (§144) *Given a curved line AM* (see Fig. 7.11) *with the radiant point B, and the ratio of m to n, we wish to find an [130] infinity of lines such as DN, so that the refracted rays MN break again when they encounter these lines DN to meet at a given point C.*

If we imagine that the curved line HF is the caustic by refraction of the given curve AM, formed by the radiant point B, then it is clear that this same line HF must also be the caustic by refraction of the curve DN that we wish to find, having the given point C as its radiant point. This is why (see §132)

[10]The Ovals of Descartes, or Cartesian Ovals, are a quartic curve (Lockwood 1971, p. 188).

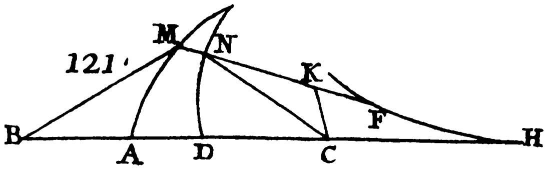

Fig. 7.11 Using the Inverse Construction to Focus Refracted Rays

$$\frac{n}{m}BA + AH = \frac{n}{m}BM + MF + FH \quad \text{and}$$

$$NF + FH - \frac{n}{m}NC = HD - \frac{n}{m}DC.$$

Consequently

$$\frac{n}{m}BA + AH = \frac{n}{m}BM + MN + HD - \frac{n}{m}NC,$$

and transposing as usual,

$$\frac{n}{m}BA - \frac{n}{m}BM + \frac{n}{m}DC + AD = MN + \frac{n}{m}NC.$$

This gives the following construction.

Take the point D at will on any refracted ray AH as one of the points of the curve DN that we wish to find. On any other refracted ray MF we take the part $MK = \frac{n}{m}BA - \frac{n}{m}BM + \frac{n}{m}DC + AD$, and find the point N, as above (see §142), such that $NK = \frac{n}{m}NC$. It is then clear (see §132) that the point M will be on the curve DN.

General Corollary.

For the Three Precceding Chapters. (§145) *It is manifest (see §80, 85, 107, 108, 114, 115, 128, 129, 134, 143) that a curved line can have only one evolute, only one caustic by reflection, and only one caustic by refraction, given the radiant point and the ratio of sines. These lines are always geometric and rectifiable when the given curve is geometric. On the other hand, the same curved line may be the evolute, or one or the other caustic in the same ratio of sines, with the same position of the radiant point, of an*[11] *infinity of very different lines, which are only geometric when the given curve is geometric and rectifiable.*

[11]In L'Hôpital (1696), the indefinite article *une* was omitted, but this was corrected in the *Errata*.