Chapter 6
Use of the Differential Calculus for Finding Caustics by Reflection

Definition I. [104] If we imagine that an infinity of rays *BA*, *BM*, and *BD* *(see Figs. 6.1, 6.2)*, which emanate from a radiant point[1] B, are reflected when they encounter a curved line *AMD*, so that the angles of reflection are equal to the angles of incidence, then the line *HFN*, which touches the reflected rays or their prolongations *AH*, *MF*, and *DN*, is called *the Caustic by reflection*.[2]

Corollary I. (§110) *If we prolong HA to* I (see Fig. 6.1)*, so that* $AI = AB$*, and if we evolve*[3] *the caustic HFN beginning at the point* I*, we describe the curve ILK so that the tangent FL is continually (see §75) equal to the portion FH of the caustic plus the straight line HI. Moreover, if we imagine two incident and reflected rays Bm and mF, infinitely close to BM and MF, and if we prolong Fm to* l *and describe the little arcs MO and MR with centers* F *and* B*, then we form the little right triangles MOm and MRm, which are similar and equal. This is because the angles* $OmM = FmD = RmM$*, and furthermore the hypotenuse Mm is common, and so the little sides Om and Rm are equal to each other. Now, because Om is the differential of LM, and Rm is the differential of BM, and because this always happens no matter where we take the point* M*, it follows that* $ML - IA$ *or* $AH + HF - MF$*, the sum (see §96) of all the differentials Om in the portion AM of the curve, is* $= BM - BA$*, the sum (see §96) of the of all the differentials Rm in the same portion AM. Therefore, the portion HF of the caustic HFN is equal to* $BM - BA + MF - AH$*.*

[1] In L'Hôpital (1696) the term *point lumineux* was used, literally meaning "luminous point." We adopt the modern term "radiant point," e.g. Lockwood (1971, pp. 183–185).

[2] A caustic by reflection is sometimes called a "Catacaustic."

[3] I.e., describe the involute of *HFN*; see Chapter 5.

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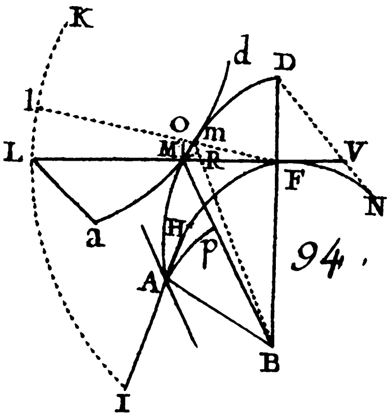

Fig. 6.1 Caustic by Reflection, Concave Case

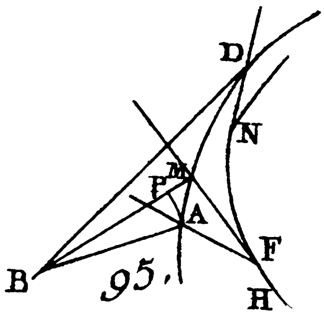

Fig. 6.2 Caustic by Reflection, Convex Case

There could be different cases, according to whether the incident ray *BA* is greater or less than *BM*, and whether the [105] reflected ray *AH* evolves or envelops[4] the portion *HF* to arrive at *MF*. However, we will still prove, as we have just done, that the difference of the incident rays is equal to the difference of the reflective rays, by joining to one of them the portion of the caustic that it evolves before falling on the other. For example *(see Fig. 6.2)*, $BM - BA = MF + FH - AH$; from which we conclude that $FH = BM - BA + AH - MF$.

If we describe the circular arc of *AP* with center *B (see Figs. 6.1, 6.2)*, then it is clear that *PM* is the difference of the incident rays *BM* and *BA*. Moreover, if we suppose that the radiant point *B* becomes infinitely distant from the curve *AMD*, the incident rays *BA* and *BM (see Fig. 6.3)* become parallel and the arc *AP* becomes a straight line perpendicular to these rays.

[4]In L'Hôpital (1696) the verb *enveloper* is used to mean the reverse of the process of describing the involute. For example, in Fig. 6.2, the tangential thread *HA* is laid on the curve *HFN* as one moves from the tangent *HA* to the tangent *FN*, instead of being peeled away in the usual evolution.

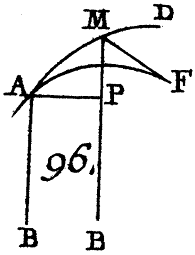

Fig. 6.3 Caustic by Reflection, Infinitely Distant Radiant Point

Corollary II. (§111) *If we imagine that the figure BAMD* (see Fig. 6.1) *is reflected in the same plane, so that the point B falls on the point I, and that therefore the tangent at A to the curve AMD in its first position, still touches it in this new position, and if we make the curve aMd roll on the curve AMD, that is to say on itself, so that the portions aM and AM are always equal, then I say that the point B describes by this motion a kind of roulette ILK whose evolute is the caustic HFN.*

This is because it follows from the generation that:

1. The line *LM* drawn from the describing point *L* to the point of contact *M* is (see §43) perpendicular to the curve *ILK*.
2. *La* or $IA = BA$, and $LM = BM$.
3. The angles made by the straight lines *ML* and *BM* on the common tangent at *M* are equal, and consequently if we prolong *LM* to *F*, the ray *MF* is the reflected ray of the incident ray *BM*.

From this we see that the perpendicular *LF* touches the caustic *HFN*, and because this always happens whenever we take the point *L*, it follows that the curve *ILK* is formed by the evolution of the caustic *HFN*, plus the straight line *HI*.

It follows from this that the portion *FH* or $FL - HI = BM + MF - BA - AH$. [106] This is what we have just demonstrated in another way in the preceding Corollary.

Corollary III. (§112) *If the tangent DN becomes infinitely close to the tangent FM, it is clear that the point of contact N, and the point of intersection V will coincide with the other point of contact F. Thus, to find the point F where the reflected ray MF touches the caustic HFN, we need only find the point of intersection of the infinitely close reflected rays MF and mF. Indeed, if we imagine an infinity of incidence rays infinitely close to one another, we will see a polygon with an infinity of sides, the assemblage of which makes up the caustic HFN born of the intersections of the reflected rays.*

Proposition I.

General Problem. (§113) *Given the nature of the curve AMD* (see Fig. 6.4), *the radiant point B, and the incident ray BM, we wish to find the point F on the reflected ray MF, given in position, where it touches the caustic.*

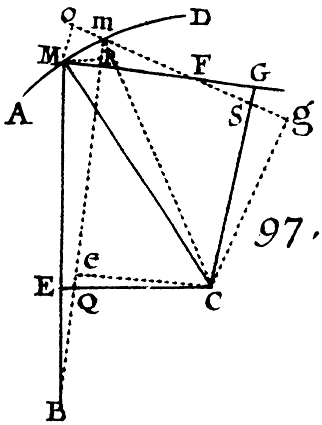

Fig. 6.4 Intersection of Reflected Ray with the Caustic

By the previous chapter, we find the length MC of the radius of the evolute at the point M. We take the infinitely small arc Mm, and draw the straight lines Bm, Cm, and Fm. We describe the little arcs MR and MO with centers B and F, and we drop the perpendiculars CE, Ce, CG, and Cg on the incident and reflected rays. We then denote the given quantities BM by y, and ME or MG by a.

Given this, we prove, as in the first Corollary (see §110), that the triangles MRm and MOm are similar and equal, and thus $MR = MO$. Now, because of the equality of the angles of incidence and reflection, we also have $CE = CG$ and $Ce = Cg$, and consequently $CE - Ce$ or $EQ = CG - Cg$ or SG. Therefore, because of the similar triangles BMR and BEQ, and FMO and FGS, we have [107] $BM + BE(2y - a) : BM(y) :: MR + EQ$ or $MO + GS : MR$ or $MO :: MG(a) : MF = \frac{ay}{2y-a}$.

If the radiant point B falls on the other side of the point E with respect to the point M, or (what is the same thing) if the curve AMD is convex towards the radiant point B, then y changes from positive to negative, and consequently we have $MF = \frac{-ay}{-2y-a}$ or $\frac{ay}{2y+a}$.

If we suppose that y becomes infinite *(see Fig. 6.3)*, that is to say that the point B is infinitely distant from the curve AMD, the incident rays are parallel to one another, and we have $MF = \frac{1}{2}a$, because a is null with respect to $2y$.

Corollary I. (§114) *Because we have found only one value for MF* (see Figs. 6.1, 6.2) *along the radius of the evolute, it follows that a curved line AMD may have but a single caustic HFN by reflection, because (see §80) it has but a single evolute.*

Corollary II. (§115) *When AMD is geometric* (see Fig. 6.4)*, it is clear (see §85) that its evolute is also geometric, that is to say that we find all of its points C geometrically. From this it follows that all points F of its caustic are also determined geometrically* (see Figs. 6.1, 6.2)*, that is to say that the caustic HFN is geometric. However, I say furthermore, that this caustic is always rectifiable, because it is* *clear (see §110) that we may find straight lines equal to any of its portions with the assistance of the curve AMD, which we assume to be geometric.*

Corollary III. (§116) *If the curve AMD* (see Fig. 6.4) *is convex towards the radiant point B, then the value of MF* $\left(\frac{ay}{2y+a}\right)$ *is always positive, and consequently we must take the point F on [108] the same side as the point C with respect to the point M, as we have assumed in making the calculations. From this we see that infinitely close reflected rays are divergent.*

However, if the curve *AMD* is concave towards the radiant point B, the value of $MF\left(\frac{ay}{2y-a}\right)$ is positive when y is greater than $\frac{1}{2}a$, negative when it is less than, and infinite when it is equal. From this it follows that if we describe a circle that has half of the radius of the evolute *MC* as its diameter, then the infinitely close reflecting rays are convergent when the radiant point *B* falls outside of its circumference, they are divergent when falls inside, and finally they are parallel when it falls on the circumference.

Corollary IV. (§117) *If the incident ray BM touches the curve AMD at the point M, we have* $ME(a) = 0$*, and consequently* $MF = 0$*. Because the reflected ray is therefore in the direction of the incident ray, and because the nature of the caustic consists of touching all reflected rays, it follows that it will also touch the incident ray BM at the point M. That is to say that the caustic and the given curve have the same tangent at the point M, which is common to them.*

If the radius *MC* of the evolute is null, we also have $ME(a) = 0$, and consequently $MF = 0$. From this we see that the given curve and the caustic make an angle equal to the angle of incidence between them at their common point M.

If the radius *CM* of the evolute is infinite, then the little arc *Mm* becomes a straight line, and we have $MF = \mp y$, because when $ME(a)$ is infinite, y is null with respect to a. Now, because this value is negative when the point B falls on the same side as the point C with respect to the line *AMD*, and positive when it falls on the opposite side, it follows that the infinitely close reflected rays are always divergent when the line *AMD* is straight.

Corollary V. [109] (§118) *It is clear that given any two of the three points B, C, and F, we easily find the third.*

1. Let the curve *AMD (see Fig. 6.5)* be a Parabola, which has the radiant point B as its focus. It is clear by the elements of the conic sections, that all reflected rays are parallel to the axis, and consequently *MF* is always infinite wherever we suppose the point M to be. Therefore, we have $a = 2y$. From this it follows that if we take *ME* to be twice *MB*, and if we draw the perpendicular *EC*, it cuts *MC* perpendicular to the curve *AMD* at a point C, which is on the evolute of this curve.
2. Let the curve *AMD (see Fig. 6.6)* be an Ellipse, which has the radiant point B as one of its foci. Again, it is clear that all reflected rays *MF* meet in the same point F, which is the other focus. If we denote *MF* by z, we have (see §113)

$z = \frac{ay}{2y-a}$, from which we find the quantity that we wish to find, $ME(a) = \frac{2yz}{y+z}$. However, if the curve *AMD (see Fig. 6.7)* is a Hyperbola, the focus F falls on the other side, and consequently $MF(z)$ becomes negative. From this it follows that we therefore have $ME(a) = \frac{-2yz}{y-z}$ or $\frac{2yz}{z-y}$. This gives the following construction, which also serves for the Ellipse.

Let *ME (see Figs. 6.6, 6.7)* be taken as the fourth proportional to the transverse semi-axis,[5] the incident ray, and the reflected ray. Let *EC* be drawn perpendicular; it cuts the line *MC*, which is perpendicular to the conic section, at a point C that is on the evolute.

Example I. (§119) Let the curve *AMD (see Fig. 6.8)* be a Parabola, whose incident rays *PM* are perpendicular to its axis *AP*. We wish to find the points F on the reflected rays *MF* where they touch the caustic *AFK*.

[110] It is clear that if we draw the radius *MC* of the evolute, and drop the perpendicular *CG* to the reflected ray *MF*, we must (see §113) take *MF* equal to half of *MG*. However, this construction can be shortened, considering that if we draw *MN* parallel to the axis *AP*, and draw the straight line *ML* to the focus L, then the angles *LMP* and *FMN* are equal, because by the property of the parabola $LMQ = QMN$, and by assumption $PMQ = QMF$. If we now add the same angle *PMF* to both sides, the angle *LMF* is equal to the angle *PMN*, that is to say a right angle. Now, we have just demonstrated (see §118, no. 1) that the perpendicular *LH* on *ML* meets the radius *MC* of the evolute at its midpoint H. Therefore, if we draw *MF* parallel and equal to *LH*, it is one of the reflected rays and it touches the caustic *AFK* at F. This is what we were required to find.

If we suppose that the reflected ray *MF* is parallel to the axis *AP*, it is clear that the point F of the caustic will be the furthest possible from the axis *AP*, because the tangent at this point is parallel to the axis. Thus, in order to determine this point on all caustics, such as *AFK*, formed by incident rays perpendicular to the axis of the given curve, we need to only consider that *MP* is equal to *PQ*. This gives $dy = dx$.

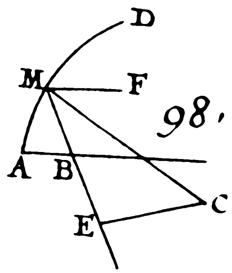

Fig. 6.5 Caustic of the Parabola by Reflection

[5]The transverse axis of an ellipse is the horizontal axis. The transverse of a hyperbola is the line segment joining the vertices. The transverse semi-axis is half the length of the transverse axis.

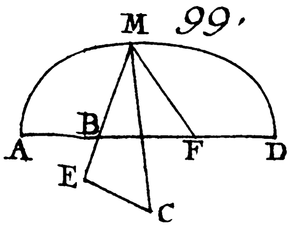

Fig. 6.6 Caustic of the Ellipse by Reflection

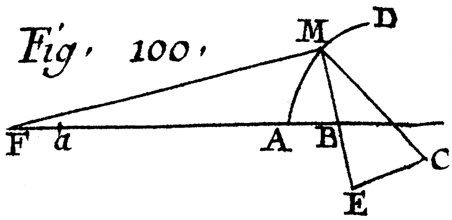

Fig. 6.7 Caustic of the Hyperbola by Reflection

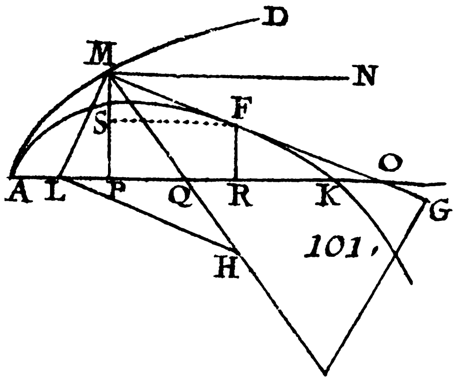

Fig. 6.8 Caustic of the Parabola by Reflection, Incident Rays Perpendicular to the Axis

Let $ax = yy$; we have $dy = \frac{a\,dx}{2\sqrt{ax}} = dx$, from which we conclude $AP(x) = \frac{1}{4}a$, that is to say if the point P falls on the focus L, the reflected ray MF is parallel to the axis. What is also clear is that because in this case MP coincides with LM, MF must also coincide with MN, and LH with LQ. From this we see that MF is therefore equal to ML, and consequently that if we draw FR perpendicular to the axis, we have AR or $AL + MF = \frac{3}{4}a$. We also see that the portion AF of the caustic is equal, in this case, to the parameter, because it is always (see §110) equal to $PM + MF$.

To determine the point K where the caustic AFK meets the axis AP, we must find the value MO, and [111] make it equal to MF, because it is clear that if the point F falls on K, the lines MF and MO become equal to each other. Therefore, denoting the unknown MO by t, the angle PMO cut equally in two by MQ, perpendicular to the curve, gives $MP(y) : MO(t) :: PQ\left(\frac{y\,dy}{dx}\right) : OQ = \frac{t\,dy}{dx}$, and consequently $OP = \frac{t\,dy+y\,dy}{dx} = \sqrt{tt-yy}$, because of the right triangle MPO. Dividing both sides by $t+y$, we find $\frac{dy}{dx} = \sqrt{\frac{t-y}{t+y}}$, from which we conclude $MO(t) = \frac{y\,dx^2+y\,dy^2}{dx^2-dy^2} = MF\left(\frac{1}{2}a\right) = \frac{dx^2+dy^2}{-2ddy}$, because (see §77) $ME(a) = \frac{dx^2+dy^2}{-ddy}$. This gives $dy^2-2y\,ddy = dx^2$, which is used to find the point P such that if we draw the incident ray PM and the reflected ray MF, this latter touches the caustic AFK at the point K where it meets the axis AP.

For the parabola $y = x^{\frac{1}{2}}$, we have $dy = \frac{1}{2}x^{-\frac{1}{2}}\,dx$ and $ddy = -\frac{1}{4}x^{-\frac{3}{2}}\,dx^2$, and substituting these values in the preceding equation, we find $\frac{1}{4}x^{-1}\,dx^2 + \frac{1}{2}x^{-1}\,dx^2 = dx^2$, from which we conclude $AP(x) = \frac{3}{4}$ of the parameter.

To find the nature of the caustic AFK in the manner of *Descartes*,[6] we must find an equation that expresses the relationship of the abscissa $AR(u)$ to the ordinate $RF(z)$, which is done in the following way. Because $MO(t) = \frac{y\,dx^2+y\,dy^2}{dx^2-dy^2}$, we have $PO\left(\frac{t\,dy+y\,dy}{dx}\right) = \frac{2y\,dx\,dy}{dx^2-dy^2}$, and because of the similar triangles MPO and MSF, we form the following proportions: $MO\left(\frac{y\,dx^2+y\,dy^2}{dx^2-dy^2}\right) : MF\left(\frac{dx^2+dy^2}{-2ddy}\right)$ or $-2y\,ddy : dx^2-dy^2 :: MP(y) : MS(y-z) = \frac{dx^2-dy^2}{-2ddy} :: PO\left(\frac{2y\,dx\,dy}{dx^2-dy^2}\right) : SF$ or $PR(u-x) = \frac{dx\,dy}{-ddy}$. We therefore have the following two [112] equations $z = y + \frac{dy^2-dx^2}{-2ddy}$ and $u = x + \frac{dx\,dy}{-ddy}$, which can be used with the equation of the given curve to form a new equation where x and y are no longer present, and which consequently expresses the relationship of $AR(u)$ to $FR(z)$.

When the curve AMD is a parabola, as we have assumed in this example, we find $z = \frac{3}{2}x^{\frac{1}{2}} - 2x^{\frac{3}{2}}$, or (by squaring each side) $\frac{9}{4}x - 6xx + 4x^3 = zz$ and $u = 3x$, from which we derive the equation we wish to find, $azz = \frac{4}{27}u^3 - \frac{2}{3}auu + \frac{3}{4}aau$, which expresses the nature of the caustic AFK. We may remark that PR is always twice AP, because $AR(u) = 3x$. This again gives us a new method for determining the point F that we wish to find on the reflected ray MF.

Example II. (§120) Let the curve AMD *(see Fig. 6.9)* be a semi-circle that has the line AD as its diameter and its center at the point C. Let the incident rays PM be perpendicular to AD.

Because the evolute of the circle is a single point which is its center, it follows (see §113) that if we cut the radius CM equally in two at the point H, and we drop the perpendicular HF to the reflected ray MF, then it cuts this ray at a point F where it touches the caustic AFK. It is clear that the reflected ray MF is equal to half the incident ray PM. From this it follows that:

1. If the point P falls on C, then the point F falls on the midpoint of CB at K.

[6] I.e., to give an equation for the curve.

2. The portion AF is three times MF, and the caustic AFK is three times BK.

We also see that if we make the angle AMC half of a right angle, the reflected ray MF is parallel to AC, and consequently the point F is higher above the diameter AD than any other point of the caustic.

The circle with diameter MH passes through the point F, because the angle HFM is a right angle. If we describe the circle KHG with [113] center C and radius CK or CH, half of CM, then the arc HF is equal to the arc HK. This is because the angle CMF is equal to[7] CMP or HCK, so the arcs $\frac{1}{2}HF$ and HK, which measure these angles in the circles MFH and KHG, are to each other as $\frac{1}{2}MH$ is to HC, the radii of these circles. From this we see that the Caustic AFK is a Roulette formed by the revolution of the mobile circle MFH around the immobile circle KHG, whose origin is at K, and whose vertex is at A.

Example III. (§121) Let the curve AMD *(see Fig. 6.10)* be a circle with the line AD as diameter and the point C as center. Let the radiant point, from which all the incident rays AM emanate, be A, one of the extremities of this diameter.

If we drop the perpendicular CE from the center C to the incident ray AM, it is clear by the property of the circle, that the point E cuts the chord AM into two equal parts, and thus that $ME(a) = \frac{1}{2}y$. We therefore have $MF\left(\frac{ay}{2y-a}\right) = \frac{1}{3}y$, that is to say we must take the reflected ray MF equal to one-third of the incident ray

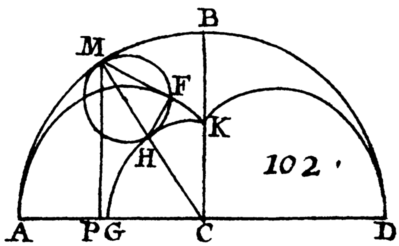

Fig. 6.9 Caustic of the Semi-circle by Reflection, Incident Rays Perpendicular to the Diameter

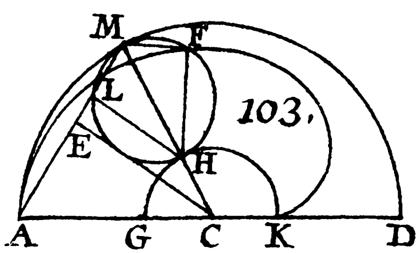

Fig. 6.10 Caustic of the Circle by Reflection, Radiant Point on the Circumference

[7]In L'Hôpital (1696) this was written as CPM, but corrected to CMP in the *Errata*.

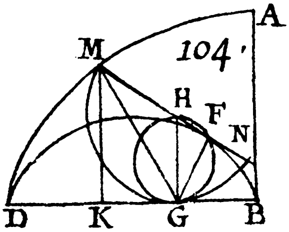

Fig. 6.11 Caustic of the Half-Cycloid by Reflection, Incident Rays Parallel to the Axis

AM. From this we see that $DK = \frac{1}{3}AD$, $CK = \frac{1}{3}CD$, and that (see §110) the caustic $AFK = \frac{4}{3}AD$, as well as that its portion $AF = \frac{4}{3}AM$. If we take $AM = AC$, the reflected ray *MF* is parallel to the diameter *AD*, and consequently the point *F* is the highest possible above this diameter.

If we take $CH = \frac{1}{3}CM$, and we draw *HF* perpendicular to *MF*, the point *F* is on the caustic, for if we draw *HL* perpendicular to *AM*, it is clear that $ML = \frac{2}{3}ME = \frac{1}{3}AM$, because $MH = \frac{2}{3}CM$. The circle with diameter *MH* therefore passes through the point *F* of the caustic, and if we describe another circle *KHG* with center *C* and with radius *CK* or *CH*, it will be equal to it, and the arc *HK* [114] will be equal to the arc *HF*, because in the isosceles triangle *CMA*, the external angle $KCH = 2CMA = AMF$. Consequently the arcs *HK* and *HF*, the measures in the equal circles, are also equal. From this it follows that the Caustic *AFK* is again a Roulette described by the revolution of the mobile circle *MFH* around the immobile circle *KHG*, whose origin is at *K*, and whose vertex is at *A*.

We might also prove this by the following other method. If we describe a roulette by the revolution of a circle equal to the circle *AMD* around that circle, starting at the point *A*, we demonstrated in the second corollary (see §111) that its evolute is the caustic *AFK*. Now (see §100), this evolute is a roulette of the same kind, that is to say that the diameters of the generating circles are equal, and we determine the point *K* by taking *CK* as the third proportional to $CD + DA$ and to *CD*, that is to say equal to $\frac{1}{3}CD$. Therefore, etc.

Example IV. (§122) Let the curve *AMD (see Fig. 6.11)* be an ordinary half-roulette described by the revolution of the semi-circle *NGM* on the straight line *BD*, whose vertex is at *A*, and whose origin at *D*. Let the incident rays *KM* be parallel to the axis *AB*.

Because (see §95) *MG* is equal to half the radius of the evolute, it follows (see §113) that if we draw *GF* perpendicular to the reflected ray *MF*, the point *F* will be on the caustic *DFB*. From this we see that *MF* must be taken equal to *KM*.

If we draw the radii *HG* and *HM* from the center *H* of the generating circle *MGN* to the point of contact *G* and to the describing point *M*, then it is clear that *HG* is perpendicular to *BD*, and that the angle $GMH = MGH = GMK$, from which we see that the reflected ray *MF* passes through the center *H*. Now, the circle with diameter *GH* also passes through the point *F*, because the angle *GFH* is a right

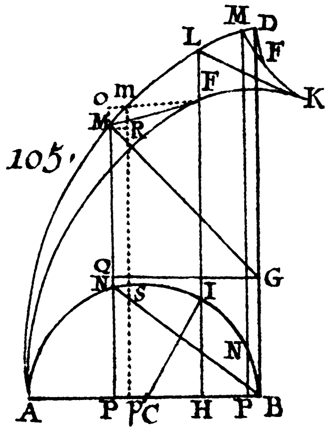

Fig. 6.12 Caustic of the Half-Cycloid by Reflection, Incident Rays Parallel to the Base

angle. Therefore, the arcs GN and $\frac{1}{2}GF$, which measure the same angle GHN, are to each other as the MN is to GH, the diameters [115] of their circles, and consequently the arc $GF = GN = GB$. It is therefore clear that the Caustic DFB is a Roulette described by an entire revolution of the circle GFH on the straight line BD.

Example V. (§123) Let the curve AMD *(see Fig. 6.12)* again be an ordinary half-roulette, whose base BD is equal to the semi-circumference ANB of the generating circle. Additionally, let the incident rays PM now be parallel to the base BD.

If we draw GQ perpendicular to PM, the right triangles GQM and BPN are equal and similar, and therefore $MQ = PN$. From this we see (see §95, 113) that we must take MF equal to the corresponding ordinate PN in the generating semi-circle ANB.

In order that the point F is the furthest possible from the axis AB, it is necessary that the tangent MF at this point be parallel to this axis. The angle PMF is therefore a right angle, its half PMG or PNB is half a right angle, and consequently the point P falls on the center of the circle[8] ANB.

It is worth remarking, that if the point P then continually approaches the extremity B, the point F also moves closer the axis AB up to a certain point K, after which it moves away to D, so that the caustic $AFKFD$ has a cusp at K .

To determine this, I remark (see §110, 111) that the portion $AF = PM + MF$, the portion $AFK = HL + LK$, and the portion KF of the part KFD is $= HL + LK - PM - MF$, from which we see that $HL + LK$ must be the *greatest*. That is why if we denote AH by x, HI by y, and the arc AI by u, we have $HL + LK = u + 2y$, whose differential gives $du + 2\,dy = 0$ and $\frac{a\,dx}{y} + 2\,dy = 0$ by substituting the value $\frac{a\,dx}{y}$

[8] In L'Hôpital (1696) this was written as *AND*.

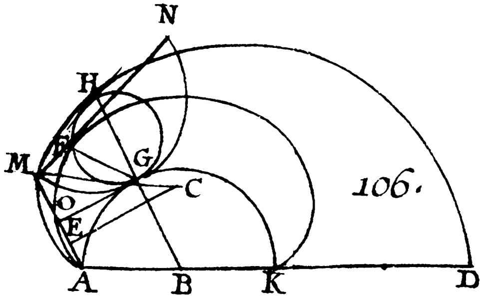

Fig. 6.13 Caustic of an Epicycloid by Reflection

for du. From this we derive $a\,dx = -2y\,dy = 2x\,dx - 2a\,dx$ because of the circle, and consequently $AH(x) = \frac{3}{2}a$.

Corollary. [116] (§124) *The space AFM or AFKFM enclosed by the portions of the curves AF or AFKF, AM, and by the reflected rays MF, is equal to half of the circular space APN. This is because its differential, which is the sector FMO, is equal to half of the rectangle PpSN, the differential of the space APN, because the right triangles MOm and MRm are equal and similar, so MO is equal to MR or NS or Pp, and furthermore MF = PN.*

Example VI. (§125) Let the curve *AMD (see Fig. 6.13)* be the half-roulette formed by the revolution of the circle *MGN* around an equal circle *AGK*, whose origin is at A, and whose vertex is at D. Let the incident rays be *AM*, which all emanate from the point A. The line *BH* that joins the centers of these two generating circles continually passes through the point of contact G, and the arcs *GM* and *GA*, as well as their chords, are always equal. So the angle $HGM = BGA$ and the angle $GMA = GAM$. Now, the angle $HGM + BGA = GMA + GAM$, because if we add the same angle *AGM* to both sides, we form two right angles. Thus, the angle *HGM* is always equal to the angle *GMA*, and consequently also to the angle of reflection *GMF*. From this it follows that *MF* is always passes through the center H of the mobile circle.

If we now drop the perpendiculars *CE* and *GO* on the incident ray *AM*, it is clear that $MO = OA$, and that $OE = \frac{1}{3}OM$, because (see §100) since the point C is on the evolute $GC = \frac{1}{3}GM$. We therefore have $ME = \frac{2}{3}AM$, that is to say $a = \frac{2}{3}y$, and consequently $MF\left(\frac{ay}{2y-a}\right) = \frac{1}{2}y$. From this we see that if we draw *GF* perpendicular to *MF*, the point F is on the caustic *AFK*.

The circle with diameter *GH* passes through the point F, and the arcs *GM* and $\frac{1}{2}GF$, which measure the same angle *GHM*, being [117] to each other as *MH* is to *GH*, the diameters of their circles, the arc *GF* is equal to the arc *GM*, and consequently to the arc *GA*. From this it is clear that the Caustic *AFK* is a Roulette

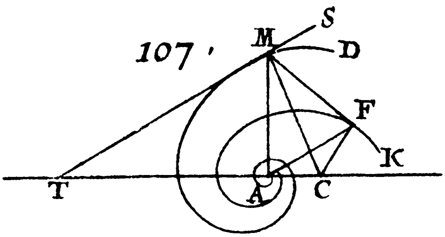

Fig. 6.14 Caustic of the Logarithmic Spiral by Reflection

described by the revolution of the mobile circle *HFG* around the immobile circle *AGK*.

Corollary. (§126) *If we describe a circle with center at the point B, and whose radius is a straight line equal to BH or AK, that has an infinity of straight lines parallel to BD that fall on its circumference, it is clear (see §120) that by reflection the rays form the same caustic AFK.*

Example VII. (§127) Let the curve *AMD (see Fig. 6.14)* be a logarithmic spiral, with incident rays *AM* that all of emanate from the center *A*.

If we draw the straight line *CA* from the extremity *C* of the radius of the evolute perpendicular to the incident ray *AM*, it meets it (see §91) at the center *A*. This is why $AM(y) = a$, and consequently $MF\left(\frac{ay}{2y-a}\right) = y$. The triangle *AMF* is therefore isosceles, and because the angles of incidence and reflection, *AMT* and *FMS*, are equal to each other, it follows that the angle *AFM* is equal to the angle *AMT*. From this it is clear that caustic *AFT* is a logarithmic spiral which differs from the given *AMD* only in its position.

Proposition II.

Problem. (§128) *Given the caustic by reflection HF* (see Fig. 6.15) *with its radiant point B, we wish to find an infinity of curves, such as AM, of which it is the caustic by reflection.*

Take the point *A* at will on any tangent *HA* to be one of the points on the curve *AM* that we wish to find. [118] We describe the circular arc *AP* with center *B* and interval *BA*, and with any other interval *BM*, we describe another circular arc. If we take $AH + HE = BM - BA$ or *PM*, we evolve the caustic *HF* beginning at the point *E*, and we describe by this motion the curved line *EM* that cuts the arc of the circle described from the radius *BM* at a point *M* that is (see §110) on the curve *AM*. This is because by the construction, $PM + MF = AH + HF$.

Alternately, if we attach a thread *BMF* by its extremities at *B* and to *F*, then we make this thread tight by means of a stylus placed at *M*, and we make it move so that we envelop the caustic *HF* with the part *MF* of this thread, it is clear that by this motion the stylus describes the curve *MA* that we wish to find.

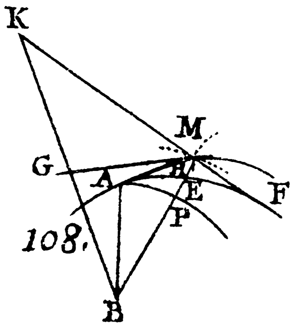

Fig. 6.15 The Inverse Problem: Finding the Original Curve from its Caustic

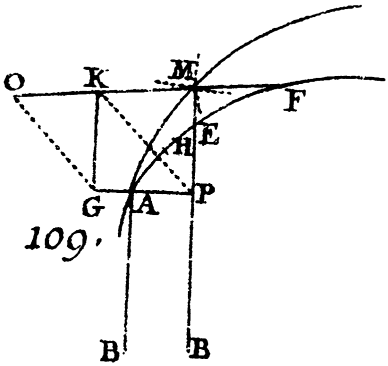

Fig. 6.16 The Inverse Problem: Radiant Point at Infinity

Alternate Solution. (§129) If we draw at will a tangent *FM*, other than *HA*, we wish to find a point *M* on it, such that $BM + MF = BA + AH + HF$. This is done in the following manner.

Let *FK* be taken $= BA + AH + HF$, and if we divide *BK* in the middle at *G*, then let the perpendicular *GM* be drawn; it meets the tangent *FM* at the point *M* that we wish to find. This is because $BM = MK$.

If the point *B* is infinitely far from the curve *AM (see Fig. 6.16)*, that is to say that the incident rays *BA* and *BM* are parallel to a straight line given in position, the first construction will still hold, by considering that the circular arcs described from the center *B* become straight lines perpendicular to the incident rays. However, this latter construction becomes useless, which is why we must substitute it with the following.

Let *FK* be taken $= AH + HF$. If we find the point *M*, so that *MP* is parallel to *AB* and perpendicular to *AP*, and is equal to *MK*, it is clear (see §110) that this point is on the curve *AM* that we wish to find, because $PM + MF = AH + HF$. Now, this is done as follows.

Let *KG* be drawn perpendicular to *AP*. If we take $KO = KG$, let *KP* be drawn parallel to *OG* and *PM* parallel to *GK*, I say that the point *M* is the one we wish

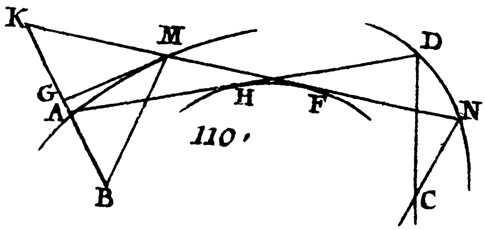

Fig. 6.17 An Application of the Inverse Problem: Constructing a Focus

to find. [119] This is because *GKO* and *PMK* are similar triangles, so that we have $PM = MK$, because $GK = KO$.

If the caustic *HF* is a single point, the curve *AM* becomes a conic section .

Corollary I. (§130) *It is clear that the curve that passes through all the points K is formed by the evolution of the curve HF beginning at A, and that its nature changes as the point A changes its position on the tangent AH. Thus, because the curves AM are all born from these curves by the same construction, which is geometric, it follows (see §108) that they are of a different nature from each other, and that they are geometric only when the caustic HF is geometric and rectifiable.*

Corollary II. (§131) *Given a curved line DN* (see Fig. 6.17) *with a radiant point C, we wish to find an infinity of lines, such as AM, so that the reflected rays DA and NM meet at a given point B, after being reflected again upon meeting any such line AM.*

If we imagine that the curve *HF* is the caustic of the given curve *DN*, formed by the radiant point C, it is clear that this line *HF* must also be the caustic of the curve *AM* having the given point B as its radiant point, so that $FK = BA + AH + HF$, and $NK = BA + AH + HF + FN = BA + AD + DC - CN$, because (see §118) $HD + DC = HF + FN + NC$. This gives the following construction.

If we take the point A at will on any reflected ray to be one of the points on the curve *AM* that we wish to find and, on any other reflected ray *NM* that we wish, we take the part $NK = BA + AD + DC - CN$, then we find the point M that we wish as above in §129.