

Chapter 1

In Which We Give the Rules of This Calculus



Analysis

of the

Infinitely Small

First Part

On the Differential Calculus

Definition I. Those quantities are called *variable* which increase or decrease continually, as opposed to *constant* quantities that remain the same while others change. Thus in the parabola the ordinate and the abscissa are variable quantities, as opposed to the parameter, which is a constant quantity.¹

¹In L'Hôpital (1696) the terms *appliquée* for ordinate and *coupée* for abscissa are used, literally meaning “applied” and “cut.”

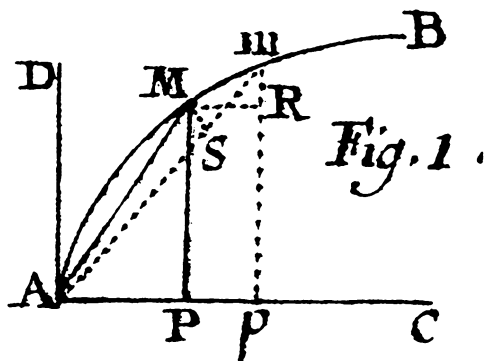


Fig. 1.1 Definition of the Differential

Definition II. [2] The infinitely small portion by which a variable quantity continually increases or decreases is called the *Differential*.² For example, let AMB be an arbitrary curved line (see Fig. 1.1) which has the line AC as its axis or diameter,³ and has PM as one of its ordinates. Let pm be another ordinate, infinitely close to the first one. Given this, if we also draw MR parallel to AC , and the chords AM Am , and describe the little circular arc MS of the circle with center A and radius AM , then Pp is the differential of AP , Rm the differential of PM , Sm the differential of AM , and Mm the differential of the arc AM . Furthermore, the little triangle MAm , which has the arc Mm as its base is the differential of the segment AM , and the little region $MPpm$ is the differential of the region contained by the straight lines AP and PM , and by the arc AM .

Corollary. (§1) *It is evident that the differential of a constant quantity is null or zero, or (what amounts to the same thing) that constant quantities do not have a differential.*

Note. In what follows, we will make use of the symbol d to denote the differential of a variable quantity that is expressed by a single letter and, in order to avoid confusion, the letter d will not be used in any other way in the following calculations. If, for example, we denote AP by x , PM by y , AM by z , the arc AM by u , the curvilinear region APM by s , and the segment AM by t , then dx denotes the value of Pp , dy that of Rm , dz that of Sm , du that of the little arc Mm , ds that of the little region $MPpm$, and dt that of the little curvilinear triangle MAm .

²In L'Hôpital (1696), the same word *différence* is used for both the differential and the difference of ordinary subtraction. In this translation, it is consistently translated as "differential" when the difference is infinitely small.

³In L'Hôpital (1696), these terms are used interchangeably for the x -axis. Technically, the term *diameter* should only be used in the sense for a curve that is symmetric about the axis.

Postulate I.⁴ (§2) We suppose that two quantities that differ by an infinitely small quantity may be used interchangeably, or (what amounts to the same [3] thing) that a quantity which is increased or decreased by another quantity that is infinitely smaller than it is, may be considered as remaining the same. We suppose, for example, that we may take Ap for AP , pm for PM , the region Apm for the region APM , the little region $Mppm$ for the little rectangle $MppR$, the little sector AMm for the little triangle AMS , the angle pAm for the angle PAM , and so forth.

Postulate II.⁵ (§3) We suppose that a curved line may be considered as an assemblage of infinitely many straight lines, each one being infinitely small, or (what amounts to the same thing) as a polygon with an infinite number of sides, each being infinitely small, which determine the curvature of the line by the angles formed amongst themselves. We suppose, for example, that the portion Mm of the curve and the arc MS of the circle may be considered to be straight lines on account of their infinite smallness, so that the little triangle mSM may be considered to be rectilinear.

Note. In what follows, we will normally suppose that the final letters of the alphabet, z, y, x , etc., denote variable quantities, and conversely, that the first letters a, b, c , etc., denote constant quantities, so that as x becomes $x + dx$, y, z , etc., become $y + dy, z + dz$, etc., and a, b, c , etc. remain as a, b, c , etc. (see §1).

Proposition I.

Problem. (§4) *To take the differential of several quantities added together or subtracted from one another.*⁶

Let $a + x + y - z$ be the expression whose differential we are to take. If we suppose that x is increased by an infinitely small quantity, i.e. that it becomes $x + dx$, then y [4] becomes $y + dy$, and z becomes $z + dz$. As for the constant a , it remains a (see §1). Thus, the given quantity $a + x + y - z$ becomes $a + x + dx + y + dy - z - dz$, and the differential, which is found by subtracting the former from the latter, is $dx + dy - dz$. It is similar for other expressions, giving rise to the following rule.

Rule I. For quantities added or subtracted.

We take the differential of each term in the given quantity and, keeping the signs the same, we add them together in a new quantity which is the differential that we wish to find.

Proposition II.

Problem. (§5) *To take the differential of a product composed of several quantities multiplied together.*

⁴Compare to Postulate 1 on p. 187.

⁵Compare to Postulate 2 on p. 187.

⁶Compare to the section “On the Addition and Subtraction of Differentials” on p. 187.

1. The differential of xy is $y dx + x dy$. This is because y becomes $y + dy$ while x becomes $x + dx$, and consequently xy becomes $xy + y dx + x dy + dx dy$, which is the product of $x + dx$ and $y + dy$. The differential is $y dx + x dy + dx dy$, that is to say (see §2) $y dx + x dy$. This is because the quantity $dx dy$ is infinitely small with respect to the other terms $y dx$ and $x dy$: If we were to divide $y dx$, for example, and $dx dy$ by dx we would find, on the one hand y , and on the other dy , which is the differential of the former, and is consequently infinitely smaller than it. From this it follows that the differential of the product of two quantities is equal to the product of the differential of the first of these two quantities with the second, plus the product of the differential of the second quantity with the first.
2. The differential of xyz is $yz dx + xz dy + xy dz$. This is because in considering the product xy as a single quantity, it is necessary, as we have just proven, to take the product of its differential $y dx + x dy$ with the second quantity z (which gives $yz dx + xz dy$), plus the product of the differential dz [5] of the second quantity z by the first xy (which gives $xy dz$). Consequently the differential of xyz is seen to be $yz dx + xz dy + xy dz$.
3. The differential of $xyzu$ is $uyz dx + uxz dy + uxy dz + xyz du$. This is proved as in the previous case, by considering xyz as a single quantity. It is similar for other cases to infinity,⁷ from which we form the following rule.

Rule II. For multiplied quantities.

The differential of a product of several quantities multiplied together is equal to the sum of the products of the differential of each of the quantities multiplied by the product of the others.⁸

Thus, the differential of ax is $x 0 + a dx$, that is $a dx$. The differential of $\overline{a + x} \times \overline{b - y}$ is $b dx - y dx - a dy - x dy$.

Proposition III.

Problem. (§6) *To take the differential of any fraction.*¹⁰

The differential of $\frac{x}{y}$ is $\frac{y dx - x dy}{yy}$. This is because, if suppose that $\frac{x}{y} = z$, we have $x = yz$. Because these two variable quantities x and yz must always be equal to one another, whether increasing or decreasing, it follows that their differentials, that is to say their increments or decrements, are equal to each other. It then follows (see §5) that $dx = y dz + z dy$, and so

$$dz = \frac{dx - z dy}{y} = \frac{y dx - x dy}{yy},$$

⁷In L'Hôpital (1696) the expression "to infinity" is frequently employed meaning roughly "of all orders."

⁸Compare this with the product rule as discussed on p. 189.

⁹In L'Hôpital (1696), the overline is used for grouping terms. We note, however, that a negative sign does not seem to distribute through terms grouped under an overline; see §13 ff.

¹⁰Compare to the section "On the Differentials of Divided Quantities" on p. 189.

substituting the value $\frac{z}{y}$ for z . This is what we wished to find, and we form the following rule.

Rule III. For divided quantities, or fractions.

The differential of any fraction is equal to [6] the product of the differential of the numerator by the denominator minus the product of the differential of the denominator by the numerator all divided by the square of the denominator.

Thus, the differential of $\frac{a}{x}$ is $\frac{-a dx}{xx}$ and the differential of $\frac{x}{a+x}$ is $\frac{a dx}{aa+2ax+xx}$.

Proposition IV.

Problem. (§7) *To take the differential of any power, whether perfect or imperfect,¹¹ of a variable quantity.¹²*

In order to give a general rule that applies to perfect and imperfect powers, it is necessary to explain the analogy that we find among their exponents.

If we consider a geometric progression¹³ whose first term is unity and whose second term is any quantity x , and if under each term we write its exponent, it is clear that these exponents form an arithmetic progression.

Geom. prog.: 1, x , xx , x^3 , x^4 , x^5 , x^6 , x^7 , etc.

Arith. prog.: 0, 1, 2, 3, 4, 5, 6, 7, etc.

If we continue the geometric progression below unity and the arithmetic progression below zero, the terms of the one are the exponents of the corresponding terms in the other. Thus, -1 is the exponent of $\frac{1}{x}$, -2 that of $\frac{1}{xx}$, etc.

Geom. prog.: x , 1, $\frac{1}{x}$, $\frac{1}{xx}$, $\frac{1}{x^3}$, $\frac{1}{x^4}$, etc.

Arith. prog.: 1, 0, -1 , -2 , -3 , -4 , etc.

If we introduce a new term in the geometric progression, we must introduce a similar term in the arithmetic progression to be its exponent.

Thus, \sqrt{x} has $\frac{1}{2}$ as its exponent, $\sqrt[3]{x}$ has $\frac{1}{3}$, $\sqrt[5]{x^4}$ has $\frac{5}{4}$, $\frac{1}{\sqrt{x^3}}$ has $-\frac{3}{2}$, $\frac{1}{\sqrt[3]{x^5}}$ has $-\frac{5}{3}$, $\frac{1}{\sqrt{x^7}}$ has $-\frac{7}{2}$, etc., so that the expressions [7] \sqrt{x} and $x^{\frac{1}{2}}$, $\sqrt[3]{x}$ and $x^{\frac{1}{3}}$, $\sqrt[5]{x^4}$ and $x^{\frac{4}{5}}$, $\frac{1}{\sqrt{x^3}}$ and $x^{-\frac{3}{2}}$, etc., signify the same thing.

Geom. prog.: 1, \sqrt{x} , x .

Arith. prog.: 0, $\frac{1}{2}$, 1.

Geom. prog.: 1, $\sqrt[3]{x}$, $\sqrt[3]{xx}$, x .

Arith. prog.: 0, $\frac{1}{3}$, $\frac{2}{3}$, 1.

Geom. prog.: 1, $\sqrt[5]{x}$, $\sqrt[5]{xx}$, $\sqrt[5]{x^3}$, $\sqrt[5]{x^4}$, x .

Arith. prog.: 0, $\frac{1}{5}$, $\frac{2}{5}$, $\frac{3}{5}$, $\frac{4}{5}$, 1.

¹¹As we will see on p. 6 (L'Hôpital 1696, p. 8), a "perfect" power is an integer and an "imperfect" power is a fraction.

¹²Compare to the section "On the Differentials of Surd Quantities" on p. 190.

¹³Compare the discussion that follows with Bernoulli's discussion of geometric and arithmetic progressions on p. 190.

$$\begin{array}{l}
 \text{Geom. prog.:} \quad \frac{1}{x}, \quad \frac{1}{\sqrt{x^3}}, \quad \frac{1}{xx}. \\
 \text{Arith. prog.:} \quad -1, \quad -\frac{3}{2}, \quad -2. \\
 \text{Geom. prog.:} \quad \frac{1}{x}, \quad \frac{1}{\sqrt[3]{x^4}}, \quad \frac{1}{\sqrt[3]{x^5}}, \quad \frac{1}{xx}. \\
 \text{Arith. prog.:} \quad -1, \quad -\frac{4}{3}, \quad -\frac{5}{3}, \quad -2. \\
 \text{Geom. prog.:} \quad \frac{1}{x^3}, \quad \frac{1}{\sqrt{x^7}}, \quad \frac{1}{x^4}. \\
 \text{Arith. prog.:} \quad -3, \quad -\frac{7}{2}, \quad -4.
 \end{array}$$

From this we see that just as \sqrt{x} is the geometric mean between 1 and x , so also is $\frac{1}{2}$ the arithmetic mean between their exponents zero and 1, and just as $\sqrt[3]{x}$ is the first of two means that are geometrically proportional between 1 and x , so also is $\frac{1}{3}$ the first of two means that are arithmetically proportional between their exponents zero and 1. It is the same for the others. Now, two things follow from the nature of these two progressions:

1. That the sum of the exponents of any two terms of the geometric progression is the exponent of the term that is their product. Thus, x^{4+3} or x^7 is the product of x^3 by x^4 , $x^{\frac{1}{2}+\frac{1}{3}}$ or $x^{\frac{5}{6}}$ is the product of $x^{\frac{1}{2}}$ by $x^{\frac{1}{3}}$, $x^{-\frac{1}{3}+\frac{1}{5}}$ or $x^{-\frac{2}{15}}$ is the product of $x^{-\frac{1}{3}}$ by $x^{\frac{1}{5}}$, etc. Similarly, $x^{\frac{1}{3}+\frac{1}{3}}$ or $x^{\frac{2}{3}}$ is the product of $x^{\frac{1}{3}}$ by itself, that is to say its square, x^{+2+2+2} or x^6 is the product of x^2 by x^2 by x^2 , that is to say its cube, $x^{-\frac{1}{3}-\frac{1}{3}-\frac{1}{3}-\frac{1}{3}}$ or $x^{-\frac{4}{3}}$ is the fourth power of $x^{-\frac{1}{3}}$, and so on for other powers. From this it is clear that the double, triple, etc., of the exponent of any term in the geometric progression is the exponent of the square, the cube, etc., of this term, and consequently that the half, the third, etc., of the exponent of any term of the geometric progression is the exponent of the square root, the cube root, etc., of this term;
2. That the difference of the exponents of any two terms of the geometric progression is the exponent of [8] the quotient of the division of these terms. Thus, $x^{\frac{1}{2}-\frac{1}{3}} = x^{\frac{1}{6}}$ is the exponent of the quotient of the division of $x^{\frac{1}{2}}$ by $x^{\frac{1}{3}}$, and $x^{-\frac{1}{3}-\frac{1}{4}} = x^{-\frac{7}{12}}$ is the exponent of the quotient of the division of $x^{-\frac{1}{3}}$ by $x^{\frac{1}{4}}$, from which we see that multiplying $x^{-\frac{1}{3}}$ by $x^{-\frac{1}{4}}$ is the same thing as dividing $x^{-\frac{1}{3}}$ by $x^{\frac{1}{4}}$. It is the same for the others.

This being well understood, there are two cases that may arise.

The first case is that of a perfect power, that is to say when the exponent is a whole number.¹⁴ The differential of xx is $2x dx$, of x^3 is $3xx dx$, of x^4 is $4x^3 dx$, etc. Because the square of x is nothing but the product of x by x , its differential is $x dx + x dx$ (see §5), that is to say $2x dx$. Similarly, the cube of x being nothing but the product of x by x by x , its differential is $xx dx + xx dx + xx dx$ (see §5), that is to say $3xx dx$. However, because it is thus for other powers up to infinity, it follows that if we suppose that m indicates whatever whole number we might wish, the differential of x^m is $mx^{m-1} dx$.

¹⁴This case is proved by Bernoulli on p. 188, without using the Product Rule.

If the exponent is negative, we find that the differential of x^{-m} or of $\frac{1}{x^m}$ is

$$\frac{-mx^{m-1} dx}{x^{2m}} = -mx^{-m-1} dx.$$

The second case is that of an imperfect power, that is to say when the exponent is a fractional number. Suppose we are to take the differential of $\sqrt[n]{x^m}$, or $x^{\frac{m}{n}}$, where $\frac{m}{n}$ denotes any fractional number. We let $x^{\frac{m}{n}} = z$ and, raising both sides to the power n , we have $x^m = z^n$. Taking differentials as we have explained in the previous case, we find that $mx^{m-1} dx = nz^{n-1} dz$, and so

$$dz = \frac{mx^{m-1} dx}{nz^{n-1}} = \frac{m}{n} x^{\frac{m}{n}-1} dx,$$

or $\frac{m}{n} dx \sqrt[n]{x^{m-n}}$, by substituting the value $nx^{m-\frac{m}{n}}$ in place of nz^{n-1} . If the exponent is negative, we find that the differential of $x^{-\frac{m}{n}}$ or $\frac{1}{x^{\frac{m}{n}}}$ is

$$\frac{-\frac{m}{n} x^{\frac{m}{n}-1} dx}{x^{\frac{2m}{n}}} = -\frac{m}{n} x^{-\frac{m}{n}-1} dx.$$

[9] This gives us the following general rule.

Rule IV. For powers, both perfect and imperfect.

The differential of any power, whether perfect or imperfect, of a variable quantity is equal to the product of the exponent of this power by the same quantity raised to a power diminished by unity, and multiplied by its differential.

Thus, if we suppose that m denotes either a whole or fractional number, whether positive or negative, and x denotes any variable quantity, the differential of x^m is always $mx^{m-1} dx$.

Examples. The differential of the cube of $ay - xx$, that is to say of $\overline{ay - xx^3}$, is $3 \times \overline{ay - xx^2} \times \overline{a dy - 2x dx} = 3a^3yy dy - 6aaxxy dy + 3ax^4 dy - 6aayyx dx + 12ayx^3 dx - 6x^5 dx$.

The differential of $\sqrt{xy + yy}$, or of $\overline{xy + yy}^{\frac{1}{2}}$, is

$$\frac{1}{2} \times \overline{xy + yy}^{-\frac{1}{2}} \times \overline{y dx + x dy + 2y dy},$$

or

$$\frac{y dx + x dy + 2y dy}{2\sqrt{xy + yy}}.$$

The differential of $\sqrt{a^4 + axyy}$, or of $a^4 + axyy$ ^{1/2}, is

$$\frac{1}{2} \times \overline{a^4 + axyy}^{-\frac{1}{2}} \times \overline{ayy dx + 2axy dy},$$

or

$$\frac{ayy dx + 2axy dy}{2\sqrt{a^4 + axyy}}.$$

The differential of¹⁵ $\sqrt[3]{ax + xx}$, or of $ax + xx$ ^{1/3}, is

$$\frac{1}{3} \times \overline{ax + xx}^{-\frac{2}{3}} \times \overline{a dx + 2x dx},$$

or

$$\frac{a dx + 2x dx}{3\sqrt[3]{ax + xx}^2}.$$

The differential of¹⁶ $\sqrt{ax + xx + \sqrt{a^4 + axyy}}$, or of

$$\overline{ax + xx + \sqrt{a^4 + axyy}}^{\frac{1}{2}},$$

is

$$\frac{1}{2} \times \overline{ax + xx + \sqrt{a^4 + axyy}}^{-\frac{1}{2}} \times \overline{a dx + 2x dx + \frac{ayy dx + 2axy dy}{2\sqrt{a^4 + axyy}}},$$

or

$$\frac{a dx + 2x dx}{2\sqrt{ax + xx + \sqrt{a^4 + axyy}}} + \frac{ayy dx + 2axy dy}{2\sqrt{a^4 + axyy} \times 2\sqrt{ax + xx + \sqrt{a^4 + axyy}}}.$$

[10] According to this rule and the rule of fractions (see §7 and 6), the differential of¹⁷

$$\frac{\sqrt[3]{ax + xx}}{\sqrt{xy + yy}}$$

¹⁵Bernoulli gave this example on p. 192.

¹⁶Bernoulli gave a similar example on p. 192.

¹⁷Bernoulli gave this example on p. 192.

is¹⁸

$$\frac{\frac{a dx + 2x dx}{3\sqrt[3]{ax+xx}} \times \sqrt{xy + yy} + \frac{-y dx - x dy - 2y dy}{2\sqrt{xy+yy}} \times \sqrt[3]{ax + xx}}{xy + yy}.$$

Remark. (§8) It is appropriate to note carefully that in taking differentials, we always suppose that, as one of the variables x increases, the others, y, z , etc., also increase. That is to say, as x becomes $x + dx$, then y, z , etc., become $y + dy, z + dz$, etc. This is why if it happens that some of them increase while others decrease, it is necessary to regard the differentials as negative quantities with respect to those others which we suppose are increasing, and consequently to change the signs of the terms containing the differentials of those which are decreasing. Thus, if we suppose that x increases and y and z decrease, that is to say that x becomes $x + dx$ and y and z become $y - dy$ and $z - dz$, and if we wish to take the differential of the product xyz , it is necessary in the differential $xy dz + xz dy + yz dx$ already found (see §5), to change the signs of the terms which contain dy and dz . This gives the differential $yz dx - xy dz - xz dy$ that we wished to find.



¹⁸In L'Hôpital (1696), the + sign separating the terms of the numerator was omitted.