Chapter 12

Inverse Sturm–Liouville Problems

We will need representations of solutions of the Sturm–Liouville equation and algorithms for recovering its potential q from two of its spectra, corresponding to two distinct sets of separated boundary conditions. These results are due to [178], see also [177], [180]. For the convenience of the reader and easy reference we recall these results from V.A. Marchenko [180], thereby adapting them to our notation and considering Sturm–Liouville problems on intervals $[0, a]$ with arbitrary $a > 0$. Other presentations of the inverse Sturm–Liouville problem can be found, e. g., in [177], [235], [282], [80].

12.1 Riemann's formula

This section is a rewrite of [180, Section 1.1]. The main improvement is that we allow for integrable potentials. This generalization is an exercise in [180] and there is no proof in [180].

Lemma 12.1.1. *Let $\alpha < \beta$ be real numbers and put*

$$D_0 = \{(\xi, \eta, \xi_0, \eta_0) : \alpha \le \eta_0 \le \eta \le \xi \le \xi_0 \le \beta\}.$$

Let q_0 be a locally integrable function on $\mathbb{R}^2$. For $f \in L_\infty(D_0)$ define

$$(Tf)(\xi, \eta, \xi_0, \eta_0) = \int_\xi^{\xi_0} \int_{\eta_0}^{\eta} q_0(\sigma, \tau) f(\sigma, \tau, \xi_0, \eta_0)\, d\tau\, d\sigma, \quad (\xi, \eta, \xi_0, \eta_0) \in D_0. \tag{12.1.1}$$

Then T is a bounded linear operator on $L_\infty(D_0)$ and $I + T$ is invertible. Denoting by 1 the function which is identically 1 on D_0, it follows that $g = (I + T)^{-1}1$ is the unique solution of $f = 1 - Tf$ on D_0. The function g is continuous.

Proof. Clearly, T is a linear operator, and Fubini's theorem shows that T maps $L_\infty(D_0)$ into itself. For each $M \in \mathbb{R}$, the standard norm on $L_\infty(D_0)$ is equivalent

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to the weighted norm given by

$$\|f\|_M = \operatorname{ess\,sup}\{|f(\xi,\eta,\xi_0,\eta_0)|e^{-M((\xi_0-\xi)+(\eta-\eta_0))} : (\xi,\eta,\xi_0,\eta_0) \in D_0\}.$$

For $f \in L_\infty(D_0)$ and $(\xi,\eta,\xi_0,\eta_0) \in D_0$ we estimate

$$\begin{aligned}
&|(Tf)(\xi,\eta,\xi_0,\eta_0)|e^{-M((\xi_0-\xi)+(\eta-\eta_0))} \\
&\qquad \le \int_\xi^{\xi_0} \int_{\eta_0}^{\eta} |q_0(\sigma,\tau)|\,|f(\sigma,\tau,\xi_0,\eta_0)|e^{-M((\xi_0-\xi)+(\eta-\eta_0))}\,d\tau\,d\sigma \\
&\qquad \le \int_\xi^{\xi_0} \int_{\eta_0}^{\eta} |q_0(\sigma,\tau)|e^{-M((\sigma-\xi)+(\eta-\tau))}\,d\tau\,d\sigma\,\|f\|_M.
\end{aligned}$$

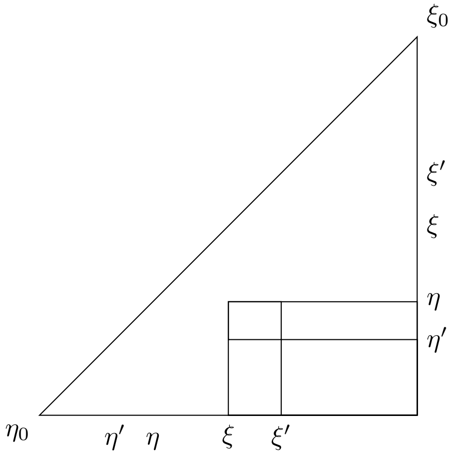
Letting $\chi_{\xi,\eta,\xi_0,\eta_0}$ be the characteristic function of the set $\{(\sigma,\tau) : \eta_0 \le \tau \le \eta \le \xi \le \sigma \le \xi_0\}$, and observing that $\chi_{\xi',\eta',\xi_0',\eta_0'}(\sigma,\tau) \to \chi_{\xi,\eta,\xi_0,\eta_0}(\sigma,\tau)$ as $(\xi',\eta',\xi_0',\eta_0') \to (\xi,\eta,\xi_0,\eta_0)$ for almost all (σ,τ), it follows from Lebesgue's dominated convergence theorem that the function f_M defined by

$$\begin{aligned}
f_M(\xi,\eta,\xi_0,\eta_0) &:= \int_\xi^{\xi_0} \int_{\eta_0}^{\eta} |q_0(\sigma,\tau)|e^{-M((\sigma-\xi)+(\eta-\tau))}\,d\tau\,d\sigma \\
&= e^{M(\xi-\eta)} \int_\xi^{\xi_0} \int_{\eta_0}^{\eta} |q_0(\sigma,\tau)|e^{-M(\sigma-\tau)}\,d\tau\,d\sigma
\end{aligned}$$

is continuous on D_0. For $(\xi,\eta), (\xi',\eta')$ with $\eta_0 \le \eta' \le \eta \le \xi \le \xi' \le \xi_0$ we have

$$f_0(\xi,\eta,\xi_0,\eta_0) - f_0(\xi',\eta',\xi_0,\eta_0) \ge \int_\xi^{\xi'} \int_{\eta'}^{\eta} |q_0(\sigma,\tau)|\,d\tau\,d\sigma,$$

as can be easily seen from the following sketch:



Since f_0 is a continuous function on the compact set D_0, f_0 is uniformly continuous, and it follows that there is $\delta > 0$ such that for all $(\xi, \eta, \xi_0, \eta_0), (\xi', \eta', \xi_0, \eta_0) \in D_0$ with $0 \le \eta - \eta' \le \delta$ and $0 \le \xi' - \xi \le \delta$ we have

$$\int_{\xi}^{\xi'} \int_{\eta'}^{\eta} |q_0(\sigma, \tau)|\, d\tau\, d\sigma \le \frac{1}{4}.$$

Now let $(\xi, \eta, \xi_0, \eta_0) \in D_0$ and put $\eta' = \max\{\eta - \delta, \eta_0\}$ and $\xi' = \min\{\xi + \delta, \xi_0\}$. Denoting the rectangle with opposite vertices (ξ, η_0) and (ξ_0, η) by D_1 and the rectangle with opposite vertices (ξ, η') and (ξ', η) by D_2, it follows that $\sigma - \xi \ge \delta$ or $\eta - \tau \ge \delta$ for $(\sigma, \tau) \in D_1 \setminus D_2$. Hence we obtain

$$\begin{aligned} f_M(\xi, \eta, \xi_0, \eta_0) &\le \frac{1}{4} + \int_{\xi}^{\xi_0} \int_{\eta_0}^{\eta} |q_0(\sigma, \tau)| e^{-M\delta}\, d\tau\, d\sigma \\ &\le \frac{1}{4} + e^{-M\delta} \int_{\alpha}^{\beta} \int_{\alpha}^{\sigma} |q_0(\sigma, \tau)|\, d\tau\, d\sigma \\ &\to \frac{1}{4} \text{ as } M \to \infty. \end{aligned}$$

Hence we may choose M such that

$$f_M(\xi, \eta, \xi_0, \eta_0) \le \frac{1}{2}, \quad (\xi, \eta, \xi_0, \eta_0) \in D_0.$$

Combining the above estimates we have shown that T is a contractive operator. Hence the operator $(I + T)$ is invertible, and the unique solution g of $f = 1 - Tf$ has the representation

$$g = (I + T)^{-1} 1 = \sum_{j=0}^{\infty} (-T)^j 1.$$

Since clearly T maps continuous function into continuous function, since 1 is continuous and since the set of continuous functions $C(D_0)$ is closed in $L_\infty(D_0)$, it follows that g is continuous. □

Lemma 12.1.2. *Let q_0 be a locally integrable function on $\mathbb{R}^2$, let η_0 and ξ_0 be real numbers with $\eta_0 < \xi_0$ and let $D(\xi_0, \eta_0) = \{(\xi, \eta) : \eta_0 \le \eta \le \xi \le \xi_0\}$. Then the problem*

$$r_{\xi\eta} - q_0 r = 0 \quad \text{on } D(\xi_0, \eta_0), \tag{12.1.2}$$

$$r(\xi_0, \eta) = r(\xi, \eta_0) = 1 \quad \text{for } \xi, \eta \in [\xi_0, \eta_0], \tag{12.1.3}$$

has a unique continuous solution r on $D(\xi_0, \eta_0)$. Furthermore, r_ξ, r_η, $r_{\xi\eta}$ and $r_{\eta\xi}$ exist and belong to $L_1(D(\xi_0, \eta_0))$, and $r_{\eta\xi} = r_{\xi\eta}$. If q_0 is continuously differentiable, then r has continuous second derivatives.

Proof. From Lemma 12.1.1 we know that the integral equation

$$r(\xi,\eta) = 1 - \int_{\xi}^{\xi_0}\int_{\eta_0}^{\eta} q_0(\sigma,\tau)r(\sigma,\tau)\,d\tau\,d\sigma. \tag{12.1.4}$$

has a solution g, and we write

$$r(\xi,\eta) = r(\xi,\eta;\xi_0,\eta_0) = g(\xi,\eta,\xi_0,\eta_0).$$

The existence of r_ξ and $r_{\xi\eta}$, their properties, and (12.1.2) and (12.1.3) easily follow from (12.1.4), so that this r is indeed a solution of (12.1.2) and (12.1.3).

Conversely, if r is a continuous solution r of problem (12.1.2), (12.1.3) where the partial derivates in (12.1.2) exist, integration of (12.1.2) with respect to η and taking into account that (12.1.3) implies $r_\xi(\xi,\eta_0) = 0$ for all $\xi \in [\eta_0,\xi_0]$ gives

$$r_\xi(\xi,\eta) = \int_{\eta_0}^{\eta} q_0(\xi,\tau)r(\xi,\tau)\,d\tau. \tag{12.1.5}$$

Integration with respect to ξ and (12.1.3) lead to (12.1.4). With fixed ξ_0 and η_0, the operator T from the proof of Lemma 12.1.1 becomes a contraction T_{ξ_0,η_0} on $L_\infty(D(\xi_0,\eta_0))$, and the uniqueness of the solution r of (12.1.2), (12.1.3) follows. □

For real x_0 and y_0 with $y_0 \geq 0$ let D be the triangular region whose vertices are (x_0,y_0), $(x_0-y_0,0)$, $(x_0+y_0,0)$. We put $\xi_0 = x_0+y_0$, $\eta_0 = x_0-y_0$ and

$$q_0(\xi,\eta) = \frac{1}{4}\left[q_1\left(\frac{\xi+\eta}{2}\right) - q_2\left(\frac{\xi-\eta}{2}\right)\right], \quad \eta_0 \leq \eta \leq \xi \leq x_0. \tag{12.1.6}$$

The linear transformation $\xi = x+y$, $\eta = x-y$, maps the triangle with vertices (x_0,y_0), $(x_0-y_0,0)$, $(x_0+y_0,0)$ into the triangle with vertices (ξ_0,η_0), (η_0,η_0), (ξ_0,ξ_0), that is, it maps the triangle D to the triangle $D(\xi_0,\eta_0)$ defined in Lemma 12.1.2. Then let $(\xi,\eta) \mapsto r(\xi,\eta;\xi_0,\eta_0)$ be the solution according to Lemma 12.1.2 and define

$$R(x,y;x_0,y_0) = r(x+y,x-y;x_0+y_0,x_0-y_0), \quad (x,y) \in D. \tag{12.1.7}$$

The following theorem is a generalization of Riemann's theorem as stated in [180, Theorem 1.1.1].

Theorem 12.1.3. *Let q_1 and q_2 be locally integrable on $\mathbb{R}$ and let φ and ψ be continuous functions on $\mathbb{R}$. Let $u \in W_1^2(D)$ be a solution of*

$$u_{xx} - q_1(x)u = u_{yy} - q_2(y)u \tag{12.1.8}$$

such that u_x and u_y are continuous on D. Assume that u satisfies the initial conditions

$$u(x,0) = \varphi(x),\ u_y(x,0) = \psi(x), \quad x_0 - y_0 \leq x \leq x_0 + y_0. \tag{12.1.9}$$

Then

$$u(x_0, y_0) = \frac{\varphi(x_0 + y_0) + \varphi(x_0 - y_0)}{2} + \frac{1}{2} \int_{x_0 - y_0}^{x_0 + y_0} \Big(\psi(x) R(x, 0; x_0, y_0) - \varphi(x) R_y(x, 0; x_0, y_0) \Big)\, dx. \quad (12.1.10)$$

Proof. We are going to use the transformation $\xi = x + y$, $\xi_0 = x_0 + y_0$, $\eta = x - y$, $\eta_0 = x_0 - y_0$. Expressing u as function $\tilde{u}$ in these new variables, i. e., $\tilde{u}(\xi, \eta) = u(x, y)$, we get

$$u_{xx} = \tilde{u}_{\xi\xi} + 2\tilde{u}_{\xi\eta} + \tilde{u}_{\eta\eta},$$
$$u_{yy} = \tilde{u}_{\xi\xi} - 2\tilde{u}_{\xi\eta} + \tilde{u}_{\eta\eta}.$$

Observing that the continuity of the partial derivatives of u and hence of $\tilde{u}$ gives $\tilde{u}_{\xi\eta} = \tilde{u}_{\eta\xi}$, we obtain

$$\tilde{u}_{\xi\eta} = \frac{1}{4}(u_{xx} - u_{yy}) = \frac{1}{4}(q_1(x) - q_2(y))u = q_0 \tilde{u}. \quad (12.1.11)$$

We recall from Lemma 12.1.2 that (12.1.2), (12.1.3) with q_0 given by (12.1.6) has a unique solution r. Multiplying equations (12.1.11) and (12.1.2) by r and $\tilde{u}$, respectively, and then subtracting the second equation from the first equation we obtain

$$\tilde{u}_{\xi\eta} r - \tilde{u} r_{\xi\eta} = 0. \quad (12.1.12)$$

Observing

$$\frac{\partial}{\partial \eta}(\tilde{u}_\xi r) = \tilde{u}_{\xi\eta} r + \tilde{u}_\xi r_\eta, \ \ \frac{\partial}{\partial \xi}(\tilde{u}_\eta r) = \tilde{u}_{\eta\xi} r + \tilde{u}_\eta r_\xi,$$
$$\frac{\partial}{\partial \eta}(\tilde{u} r_\xi) = \tilde{u}_\eta r_\xi + \tilde{u} r_{\xi\eta}, \ \ \frac{\partial}{\partial \xi}(\tilde{u} r_\eta) = \tilde{u}_\xi r_\eta + \tilde{u} r_{\eta\xi},$$

we conclude that

$$\tilde{u}_{\xi\eta} r - \tilde{u} r_{\xi\eta} = \frac{1}{2} \left(\frac{\partial}{\partial \eta}(\tilde{u}_\xi r - \tilde{u} r_\xi) + \frac{\partial}{\partial \xi}(\tilde{u}_\eta r - \tilde{u} r_\eta) \right) \quad (12.1.13)$$

Integrating both sides of (12.1.12) over $D(\xi_0, \eta_0)$ and taking (12.1.13) into account we get

$$\iint\limits_{D(\xi_0, \eta_0)} \left[\frac{\partial}{\partial \eta}(\tilde{u}_\xi r - \tilde{u} r_\xi) + \frac{\partial}{\partial \xi}(\tilde{u}_\eta r - \tilde{u} r_\eta) \right] d\xi\, d\eta = 0. \quad (12.1.14)$$

By Fubini's theorem we can integrate componentwise, and therefore the left-hand side is the sum of the two integrals

$$\begin{aligned} I_1 &:= \int_{\eta_0}^{\xi_0} [(\tilde{u}_\xi(\xi,\xi)r(\xi,\xi) - \tilde{u}(\xi,\xi)r_\xi(\xi,\xi)) \\ &\qquad - (\tilde{u}_\xi(\xi,\eta_0)r(\xi,\eta_0) - \tilde{u}(\xi,\eta_0)r_\xi(\xi,\eta_0))]d\xi \\ &=: I_{11} - I_{12}, \\ I_2 &:= \int_{\eta_0}^{\xi_0} [(\tilde{u}_\eta(\xi_0,\eta)r(\xi_0,\eta) - \tilde{u}(\xi_0,\eta)r_\eta(\xi_0,\eta)) \\ &\qquad - (\tilde{u}_\eta(\eta,\eta)r(\eta,\eta) - \tilde{u}(\eta,\eta)r_\eta(\eta,\eta))]d\eta \\ &=: I_{21} - I_{22}. \end{aligned}$$

Here we have used that, e. g., $\frac{\partial}{\partial\eta}(\tilde{u}_\xi r - \tilde{u} r_\xi)$ is integrable with respect to η for almost all ξ and that $\tilde{u}_\xi r - \tilde{u} r_\xi$ is continuous with respect to η for these ξ in view of (12.1.5) and the continuity assumption on u and its partial derivatives.

Integrating by parts and observing that $r(\xi,\eta) = 1$ if $\xi = \xi_0$ or $\eta = \eta_0$ we get

$$\begin{aligned} I_{12} &= 2\int_{\eta_0}^{\xi_0} \tilde{u}_\xi(\xi,\eta_0)r(\xi,\eta_0)\,d\xi - \tilde{u}(\xi_0,\eta_0)r(\xi_0,\eta_0) + \tilde{u}(\eta_0,\eta_0)r(\eta_0,\eta_0) \\ &= 2\int_{\eta_0}^{\xi_0} \tilde{u}_\xi(\xi,\eta_0)\,d\xi - \tilde{u}(\xi_0,\eta_0) + \tilde{u}(\eta_0,\eta_0) \\ &= \tilde{u}(\xi_0,\eta_0) - \tilde{u}(\eta_0,\eta_0) \\ I_{21} &= 2\int_{\eta_0}^{\xi_0} \tilde{u}_\eta(\xi_0,\eta)r(\xi_0,\eta)\,d\eta - \tilde{u}(\xi_0,\xi_0)r(\xi_0,\xi_0) + \tilde{u}(\xi_0,\eta_0)r(\xi_0,\eta_0) \\ &= 2\int_{\eta_0}^{\xi_0} \tilde{u}_\eta(\xi_0,\eta)\,d\eta - \tilde{u}(\xi_0,\xi_0) + \tilde{u}(\xi_0,\eta_0) \\ &= \tilde{u}(\xi_0,\xi_0) - \tilde{u}(\xi_0,\eta_0). \end{aligned}$$

For $\xi = \eta$ we have $x + y = x - y$, so that $y = 0$ and $x = \xi$. Hence

$$\begin{aligned} I_{21} - I_{12} &= -2\tilde{u}(\xi_0,\eta_0) + \tilde{u}(\xi_0,\xi_0) + \tilde{u}(\eta_0,\eta_0) \\ &= -2u(x_0,y_0) + u(x_0+y_0,0) + u(x_0-y_0,0) \\ &= -2u(x_0,y_0) + \varphi(x_0+y_0) + \varphi(x_0-y_0). \end{aligned} \tag{12.1.15}$$

From

$$\begin{aligned} u_y(x,y) &= \tilde{u}_\xi(x+y,x-y)\frac{\partial\xi}{\partial y} + \tilde{u}_\eta(x+y,x-y)\frac{\partial\eta}{\partial y} \\ &= \tilde{u}_\xi(x+y,x-y) - \tilde{u}_\eta(x+y,x-y) \end{aligned} \tag{12.1.16}$$

and the same equation for R defined by (12.1.7) and r we find

$$\begin{aligned}
I_{11} - I_{22} \\
&= \int_{\eta_0}^{\xi_0} [\tilde{u}_\xi(\xi,\xi)r(\xi,\xi) - \tilde{u}(\xi,\xi)r_\xi(\xi,\xi) - \tilde{u}_\eta(\xi,\xi)r(\xi,\xi) + \tilde{u}(\xi,\xi)r_\eta(\xi,\xi)]\,d\xi \\
&= \int_{\eta_0}^{\xi_0} [\tilde{u}_\xi(\xi,\xi) - \tilde{u}_\eta(\xi,\xi)]r(\xi,\xi)\,d\xi - \int_{\eta_0}^{\xi_0} \tilde{u}(\xi,\xi)[r_\xi(\xi,\xi) - r_\eta(\xi,\xi)]\,d\xi \\
&= \int_{\eta_0}^{\xi_0} u_y(x,0)R(x,0)\,dx - \int_{\eta_0}^{\xi_0} u(x,0)R_y(x,0)\,dx \\
&= \int_{x_0-y_0}^{x_0+y_0} \psi(x)R(x,0)\,dx - \int_{x_0-y_0}^{x_0+y_0} \varphi(x)R_y(x,0)\,dx. \qquad (12.1.17)
\end{aligned}$$

Recall that $I_1 + I_2 = 0$ by (12.1.14). Hence the sum of (12.1.15) and (12.1.17) is zero, and solving this equation for $u(x_0, y_0)$ completes the proof. □

For the solution r of (12.1.2), (12.1.3) we have already used the four variable notation $r(\xi,\eta;\xi_0,\eta_0)$, and r_ξ will denote the derivative with respect to the first variable, even if the first variable is denoted by a different symbol.

Corollary 12.1.4. *Let q_1 and q_2 be locally integrable on $\mathbb{R}$ and let φ be a continuously differentiable function on $\mathbb{R}$. Let $u \in W_1^2(D)$ be a solution of*

$$u_{xx} - q_1(x)u = u_{yy} - q_2(y)u \qquad (12.1.18)$$

such that u_x and u_y are continuous on D. Assume that u satisfies the initial conditions

$$u(x,0) = \varphi(x),\ u_y(x,0) = \varphi'(x), \quad x_0 - y_0 \le x \le x_0 + y_0. \qquad (12.1.19)$$

Let r be the unique solution of (12.1.2), (12.1.3) *with q_0 given by* (12.1.6). *Then*

$$u(x_0,y_0) = \varphi(x_0+y_0) - \int_{x_0-y_0}^{x_0+y_0} \varphi(x)r_\xi(x,x;x_0+y_0,x_0-y_0)\,dx. \qquad (12.1.20)$$

Proof. By Theorem 12.1.3, u has the representation (12.1.10) with $\psi = \varphi'$. Recall that R has been defined in (12.1.7), where r is the unique solution of (12.1.2), (12.1.3) with q_0 given by (12.1.6). Since $x_0 + y_0 = \xi_0$ and $x_0 - y_0 = \eta_0$, it follows that

$$R(x_0 \pm y_0, 0; x_0, y_0) = r(x_0 \pm y_0, x_0 \pm y_0; x_0 + y_0, x_0 - y_0) = 1.$$

As we have argued in the proof of Theorem 12.1.3, we may used integration by parts to arrive at

$$\begin{aligned}
\int_{x_0-y_0}^{x_0+y_0} \varphi'(x)R(x,0;x_0,y_0)\,dx &= \varphi(x_0+y_0) - \varphi(x_0-y_0) \\
&\quad - \int_{x_0-y_0}^{x_0+y_0} \varphi(x)R_x(x,0;x_0,y_0)\,dx.
\end{aligned}$$

From (12.1.16) for R_y and the corresponding formula for R_x we see that

$$R_x(x,0;x_0,y_0) + R_x(x,0;x_0,y_0) = 2r_\xi(x,x;x_0+y_0,x_0-y_0).$$

Substitution of these identities into (12.1.10) gives (12.1.20). □

12.2 Solutions of Sturm–Liouville problems

Lemma 12.2.1 ([180, Lemma 1.4.3]). *Let $(a_k)_{k=-\infty}^{\infty}$ be a sequence of complex numbers of the form $a_k = \frac{2\pi}{a}k + b + h_k$, where $b \in \mathbb{C}$ and $h_k = O(k^{-1})$ for $k \to \pm\infty$, let $f \in L_2(0,a)$ and let*

$$\tilde{f}(\lambda) := \int_0^a f(x)e^{-i\lambda x}\,dx$$

be its Fourier transform. Then

$$\tilde{f}(a_k) = \tilde{f}\left(\frac{2\pi}{a}k + b\right) + k^{-1}g(k)$$

with $\left(\tilde{f}\left(\frac{2\pi}{a}k+b\right)\right)_{k=-\infty}^{\infty} \in l_2$ and $(g(k))_{k=-\infty}^{\infty} \in l_2$.

Proof. From the equality

$$\tilde{f}(a_k) = \int_0^a f(x)e^{-i(\frac{2\pi}{a}k+b)x}e^{-ih_kx}\,dx = \int_0^a f(x)e^{-i(\frac{2\pi}{a}k+b)x}[1 - ih_kx + O(h_k^2)]\,dx$$

it follows that

$$\tilde{f}(a_k) = \tilde{f}\left(\frac{2\pi}{a}k+b\right) + h_k\tilde{f}'\left(\frac{2\pi}{a}k+b\right) + O(h_k^2) = \tilde{f}\left(\frac{2\pi}{a}k+b\right) + k^{-1}g(k),$$

where

$$g(k) = kh_k\tilde{f}'\left(\frac{2\pi}{a}k+b\right) + k^{-1}O(k^2h_k^2).$$

Since

$$\tilde{f}\left(\frac{2\pi}{a}k+b\right) = \int_0^a f(x)e^{-ibx}e^{-2i\frac{\pi}{a}x}\,dx$$

and

$$\tilde{f}'\left(\frac{2\pi}{a}k+b\right) = -i\int_0^a f(x)xe^{-ibax}e^{-2i\frac{\pi}{a}kx}\,dx$$

are the Fourier coefficients of the functions $x \mapsto f(x)e^{-ibx}$ and $x \mapsto -if(x)xe^{-ibx}$, which belong to $L_2(0,a)$, Bessel's inequality implies that

$$\sum_{k=-\infty}^{\infty}\left|\tilde{f}\left(\frac{2\pi}{a}k+b\right)\right|^2 < \infty, \quad \sum_{k=-\infty}^{\infty}\left|\tilde{f}'\left(\frac{2\pi}{a}k+b\right)\right|^2 < \infty,$$

and hence also that $\sum\limits_{k=-\infty}^{\infty}|g(k)|^2 < \infty$, because by assumption, $\sup\limits_{k\in\mathbb{Z}}|kh_k| < \infty$. □

Definition 12.2.2 ([281, Section 2.5]). An entire function ω of exponential type $\leq \sigma$ is said to belong to the Paley–Wiener class $\mathcal{L}^\sigma$ if its restriction to the real axis belongs to $L_2(-\infty, \infty)$.

Remark 12.2.3. For an entire function ω, let $\omega_e = \frac{1}{2}(\omega + \breve{\omega})$ and $\omega_o = \frac{1}{2}(\omega - \breve{\omega})$ be the even and odd parts of ω, where $\breve{\omega}(\lambda) = \omega(-\lambda)$. Clearly, ω belongs to $\mathcal{L}^\sigma$ if and only if ω_e and ω_o belong to $\mathcal{L}^\sigma$. We denote the sets of even and odd functions in $\mathcal{L}^\sigma$ by $\mathcal{L}_e^\sigma$ and $\mathcal{L}_o^\sigma$, respectively.

Lemma 12.2.4 (Plancherel's theorem). *The function ω belongs to $\mathcal{L}^\sigma$ if and only if it is of the form*

$$\omega(\lambda) = \int_0^\sigma \xi(t) \cos \lambda t \, dt + i \int_0^\sigma \zeta(t) \sin \lambda t \, dt, \quad \lambda \in \mathbb{C},$$

where $\xi, \zeta \in L_2(0, \sigma)$. Furthermore, $\lim_{|\lambda| \to \infty} e^{-|\operatorname{Im} \lambda| a} |\omega(\lambda)| = 0$ if $\omega \in \mathcal{L}^a$.

Proof. The first statement can be found in [263, Theorems 48 and 50] or [281, Theorem 2.18], while the second statement easily follows from [180, Lemma 1.3.1]. □

Remark 12.2.5. In the notation of Lemma 12.2.4,

$$\omega_e(\lambda) = \int_0^\sigma \xi(t) \cos \lambda t \, dt, \quad \omega_o(\lambda) = i \int_0^\sigma \zeta(t) \sin \lambda t \, dt, \quad \lambda \in \mathbb{C}.$$

Consider the Sturm–Liouville equation

$$y'' - q(x)y + \lambda^2 y = 0 \tag{12.2.1}$$

on the interval $(0, a)$, where $0 < a < \infty$, $q \in L_2(0, a)$ is a real-valued function and λ is a complex parameter. Let $e_0(\lambda, x)$ denote the solution of equation (12.2.1) with initial data

$$e_0(\lambda, 0) = 1, \; e_0'(\lambda, 0) = -i\lambda. \tag{12.2.2}$$

Theorem 12.2.6 ([180, Theorem 1.2.1]). *The solution $e_0(\lambda, \cdot)$ of the initial value problem* (12.2.1), (12.2.2) *admits the representation*

$$e_0(\lambda, x) = e^{-i\lambda x} + \int_{-x}^x \tilde{K}(x, t) e^{-i\lambda t} dt, \; 0 \leq x \leq a, \tag{12.2.3}$$

where

$$\tilde{K}(x, t) = -r_\xi(t, t; x, -x), \; 0 \leq x \leq a, \quad |t| \leq x, \tag{12.2.4}$$

and r is the function defined in Lemma 12.1.2 *with*

$$q_0(\xi, \eta) = -\frac{1}{4} q\left(\frac{\xi - \eta}{2}\right). \tag{12.2.5}$$

Proof. The function

$$u(x,y) = e^{-i\lambda x} e_0(\lambda, y)$$

belongs locally to W_2^2, is continuously differentiable for $-\infty < x < \infty$, $0 \le y \le a$, and solves the Cauchy problem

$$u_{xx} = u_{yy} - q(y)u \tag{12.2.6}$$

with initial conditions

$$u(x,0) = e^{-i\lambda x},\ u_y(x,0) = -i\lambda e^{-i\lambda x}.$$

Corollary 12.1.4 gives that the value of the function u at (x_0, y_0) is given by

$$e^{-i\lambda x_0} e_0(\lambda, y_0) = e^{-i\lambda(x_0+y_0)} - \int_{x_0-y_0}^{x_0+y_0} r_\xi(x, x; x_0+y_0, x_0-y_0) e^{-i\lambda x}\, dx.$$

Letting $x_0 = 0$ we get

$$e_0(\lambda, y_0) = e^{-i\lambda y_0} - \int_{-y_0}^{y_0} r_\xi(x, x; y_0, -y_0) e^{-i\lambda x}\, dx.$$

An obvious change in notation now proves (12.2.3). Since (12.2.6) is equation (12.1.18) with $q_1 = 0$ and $q_2 = q$, (12.2.5) follows from (12.1.6). □

Proposition 12.2.7.

$$e_0(\lambda, x) = e^{-i\lambda x} + \int_0^x \frac{\sin\lambda(x-t)}{\lambda} q(t) e_0(\lambda, t)\, dt. \tag{12.2.7}$$

Proof. It is easy to see that

$$y(x) = e^{-i\lambda x} + \int_0^x \frac{\sin\lambda(x-t)}{\lambda} g(t)\, dt \tag{12.2.8}$$

is the solution of the differential equation $y'' + \lambda^2 y = g$ with $g \in L_2(0,a)$ subject to the initial condition $y(0) = 1$, $y'(0) = -i\lambda$. For $g = qe_0(\lambda, \cdot)$ it follows that y is the unique solution of (12.2.1), (12.2.2). Since this solution is $e_0(\lambda, \cdot)$, the equation (12.2.7) follows. □

Lemma 12.2.8. *Let $p \in \mathbb{N}_0$ and $q \in W_2^p(0,a)$. Then $\tilde{K}$ defined in Theorem* 12.2.6 *is continuous, and for all $0 \le x \le a$ and $|t| \le x$,*

$$\tilde{K}(x,t) = \frac{1}{2}\int_0^{\frac{x+t}{2}} q(u)\, du + \int_0^{\frac{x+t}{2}} \int_0^{\frac{x-t}{2}} q(\alpha+\beta) \tilde{K}(\alpha+\beta, \alpha-\beta)\, d\beta\, d\alpha. \tag{12.2.9}$$

Furthermore, $\frac{\partial^{p_1}\partial^{p_2}}{\partial x^{p_1}\partial t^{p_2}} \tilde{K}(a, \cdot) \in W_2^{p+1-p_1-p_2}(-a,a)$ whenever $p_1 + p_2 \le p+1$. If q is real valued, then also $\tilde{K}$ is real valued.

Proof. From (12.2.4), (12.1.4) and Lemma 12.1.1 we conclude that $\tilde{K}$ is continuous. If q is real valued, then also q_0 given by (12.2.5) is real valued, and hence the real part of r also satisfies (12.1.4). From the uniqueness of the solution of (12.1.4) we conclude that r is real valued. Substituting (12.2.3) into (12.2.7) we arrive at

$$\int_{-x}^{x} \tilde{K}(x,t)e^{-i\lambda t}dt = \int_0^x \frac{\sin\lambda(x-t)}{\lambda}q(t)e^{-i\lambda t}dt + \int_0^x \frac{\sin\lambda(x-t)}{\lambda}q(t)\int_{-t}^{t}\tilde{K}(t,\xi)e^{-i\lambda\xi}\,d\xi\,dt. \tag{12.2.10}$$

Next we express the right-hand side of the above equation as a Fourier transform. Since

$$\frac{\sin\lambda(x-t)}{\lambda}e^{-i\lambda\xi} = \frac{1}{2}\int_{\xi-(x-t)}^{\xi+(x-t)} e^{-i\lambda u}\,du, \tag{12.2.11}$$

it follows that

$$\int_0^x \frac{\sin\lambda(x-t)}{\lambda}q(t)e^{-i\lambda t}dt = \frac{1}{2}\int_0^x q(t)\int_{2t-x}^{x} e^{-i\lambda u}\,du\,dt = \frac{1}{2}\int_{-x}^{x} e^{-i\lambda u}\int_0^{\frac{x+u}{2}} q(t)\,dt\,du. \tag{12.2.12}$$

Using equation (12.2.11) once more, we obtain the equality

$$\int_0^x \frac{\sin\lambda(x-t)}{\lambda}q(t)\int_{-t}^{t}\tilde{K}(t,\xi)e^{-i\lambda\xi}\,d\xi\,dt = \frac{1}{2}\int_0^x q(t)\Big(\int_{-t}^{t}\tilde{K}(t,\xi)\int_{\xi-(x-t)}^{\xi+(x-t)} e^{-i\lambda u}\,du\,d\xi\Big)\,dt.$$

Interchanging variables, the inner double integral becomes

$$\int_{-t}^{t}\tilde{K}(t,\xi)\int_{\xi-(x-t)}^{\xi+(x-t)} e^{i\lambda u}\,du\,d\xi = \int_{-x}^{x} e^{i\lambda u}\int_{\max\{-t,u-(x-t)\}}^{\min\{t,u+(x-t)\}}\tilde{K}(t,\xi)\,d\xi\,du.$$

Consequently,

$$\int_0^x q(t)\int_{-t}^{t}\tilde{K}(t,\xi)\int_{\xi-(x-t)}^{\xi+(x-t)} e^{-i\lambda u}\,du\,d\xi\,dt = \int_{-x}^{x} e^{-i\lambda u}\int_0^x q(t)\int_{\max\{-t,u-(x-t)\}}^{\min\{t,u+(x-t)\}}\tilde{K}(t,\xi)\,d\xi\,dt\,du.$$

It follows that

$$\int_0^x \frac{\sin\lambda(x-t)}{\lambda}q(t)\int_{-t}^{t}\tilde{K}(t,\xi)e^{-i\lambda\xi}\,d\xi\,dt = \frac{1}{2}\int_{-x}^{x} e^{-i\lambda t}\int_0^x q(u)\int_{\max\{-u,t-(x-u)\}}^{\min\{-u,t+(x-u)\}}\tilde{K}(u,\xi)\,d\xi\,du\,dt. \tag{12.2.13}$$

Equations (12.2.12) and (12.2.13) show that (12.2.10) leads to

$$\int_{-x}^{x} \tilde{K}(x,t)e^{-i\lambda t}dt = \frac{1}{2}\int_{-x}^{x}\Big(\int_0^{\frac{x+t}{2}} q(u)du + \int_0^x q(u)\int_{\max\{-u,t-(x-u)\}}^{\min\{u,t+(x-u)\}} \tilde{K}(u,\xi)\,d\xi\,du\Big)e^{-i\lambda t}\,dt.$$

Taking the inverse Fourier transform, we arrive at

$$\tilde{K}(x,t) = \frac{1}{2}\int_0^{\frac{x+t}{2}} q(u)du + \frac{1}{2}\int_0^x q(u)\int_{\max\{-u,t-(x-u)\}}^{\min\{u,t+(x-u)\}} \tilde{K}(u,\xi)\,d\xi\,du. \quad (12.2.14)$$

Performing the change of variables

$$u+\xi = 2\alpha \text{ and } u-\xi = 2\beta$$

in this integral, the region of the double integral,

$$-u \le \xi \le u, \quad t-(x-u) \le \xi \le t+(x-u), \quad 0 \le u \le x,$$

becomes

$$0 \le \alpha, \quad 0 \le \beta, \quad 2\beta \le x-t, \quad 2\alpha \le x+t, \quad 0 \le \alpha+\beta \le x,$$

where the last condition is redundant since it follows from the first four. Hence (12.2.9) follows. □

From [180, Corollary after Theorem 1.2.1, Theorem 1.2.2 and (1.2.18)] we obtain

Theorem 12.2.9.

1. *If the potential q of the Sturm–Liouville equation*

$$-y''(x) + q(x)y(x) = \lambda^2 y(x), \quad x \in (0,a), \quad (12.2.15)$$

belongs to $L_2(0,a)$, then the solutions of the initial value problems $s(\lambda,0)=0$, $s'(\lambda,0)=1$ and $c(\lambda,0)=1$, $c'(\lambda,0)=0$ can be expressed as

$$s(\lambda,x) = \frac{\sin\lambda x}{\lambda} + \int_0^x K(x,t)\frac{\sin\lambda t}{\lambda}\,dt, \quad (12.2.16)$$

$$c(\lambda,x) = \cos\lambda x + \int_0^x B(x,t)\cos\lambda t\,dt, \quad (12.2.17)$$

where $K(x,t) = \tilde{K}(x,t) - \tilde{K}(x,-t)$, $B(x,t) = \tilde{K}(x,t) + \tilde{K}(x,-t)$, and $\tilde{K}(x,t)$ is the unique solution of the integral equation

$$\tilde{K}(x,t) = \frac{1}{2}\int_0^{\frac{x+t}{2}} q(s)\,ds + \int_0^{\frac{x+t}{2}}\int_0^{\frac{x-t}{2}} q(s+p)\tilde{K}(s+p,s-p)\,dp\,ds \quad (12.2.18)$$

on the triangular region $\{(x,t) \in [0,a] \times [-a,a] : |t| \le x\}$. *In particular, it is true that* $K(x,0) = 0$ *and*

$$B(x,x) = K(x,x) = \tilde{K}(x,x) = \frac{1}{2}\int_0^x q(s)ds. \tag{12.2.19}$$

2. *If* $q \in W_2^n(0,a)$, *then* K *and* B *have partial derivatives up to* $(n+1)$*th order which belong to* $L_2(0,a)$.

Proof. We observe that

$$s(\lambda,x) = \frac{e_0(-\lambda,x) - e_0(\lambda,x)}{2i\lambda} \tag{12.2.20}$$

for all $x \in [0,a]$ and all nonzero complex numbers λ since both functions solve the initial value problem (12.2.15) subject to the boundary condition $y(0) = 0$, $y'(0) = 1$, see (12.2.1) and (12.2.2) for e_0. Theorem 12.2.6 shows that

$$\begin{aligned} s(\lambda,x) &= \frac{\sin\lambda x}{\lambda} + \frac{1}{2i\lambda}\int_{-x}^x \left(\tilde{K}(x,t)e^{i\lambda t} - \tilde{K}(x,t)e^{-i\lambda t}\right) dt \\ &= \frac{\sin\lambda x}{\lambda} + \frac{1}{2i\lambda}\int_0^x \left([\tilde{K}(x,t) - \tilde{K}(x,-t)]e^{i\lambda t} - (\tilde{K}(x,t) - \tilde{K}(x,-t)e^{-i\lambda t}\right) dt \\ &= \frac{\sin\lambda x}{\lambda} + \int_0^x [\tilde{K}(x,t) - \tilde{K}(x,-t)]\frac{\sin\lambda t}{\lambda}\, dt, \end{aligned}$$

which proves (12.2.17). Similarly, (12.2.18) follows from

$$c(\lambda,x) = \frac{e_0(\lambda,x) + e_0(-\lambda,x)}{2}. \tag{12.2.21}$$

The properties of K and B now follow immediately from their definition and Lemma 12.2.8. □

Observe that $-\sin^{(j+1)}$ is an antiderivative of $\sin^{(j)}$ for all $j \in \mathbb{N}_0$. We also note that K is an odd function with respect to t, whence $\frac{\partial^j}{\partial t^j}K(a,0) = 0$ and $\frac{\partial^j}{\partial t^j}K_x(a,0) = 0$ for all even nonnegative integers. Similarly, since B is an even function with respect to t, it follows that $\frac{\partial^j}{\partial t^j}B(a,0) = 0$ and $\frac{\partial^j}{\partial t^j}B_x(a,0) = 0$ for all odd nonnegative integers. Then integration by parts and differentiation with respect to x, respectively, followed by n integrations by parts in (12.2.16) and (12.2.17) leads to

Corollary 12.2.10. *If $n \in \mathbb{N}_0$ and $q \in W_2^n(0,a)$, then*

$$s(\lambda,a) = \frac{\sin\lambda a}{\lambda} - \sum_{j=0}^{n} \frac{\partial^j}{\partial t^j} K(a,a) \frac{\sin^{(j+1)}\lambda a}{\lambda^{j+2}} + \int_0^a \frac{\partial^{n+1}}{\partial t^{n+1}} K(a,t) \frac{\sin^{(n+1)}\lambda t}{\lambda^{n+2}}\, dt, \tag{12.2.22}$$

$$s'(\lambda,a) = \cos\lambda a + K(a,a)\frac{\sin\lambda a}{\lambda} - \sum_{j=1}^{n} \frac{\partial^{j-1}}{\partial t^{j-1}} K_x(a,a) \frac{\sin^{(j)}\lambda a}{\lambda^{j+1}} + \int_0^a \frac{\partial^n}{\partial t^n} K_x(a,t) \frac{\sin^{(n)}\lambda t}{\lambda^{n+1}}\, dt, \tag{12.2.23}$$

$$c(\lambda,a) = \cos\lambda a + \sum_{j=0}^{n} \frac{\partial^j}{\partial t^j} B(a,a) \frac{\sin^{(j)}\lambda a}{\lambda^{j+1}} - \int_0^a \frac{\partial^{n+1}}{\partial t^{n+1}} B(a,t) \frac{\sin^{(n)}\lambda t}{\lambda^{n+1}}\, dt, \tag{12.2.24}$$

$$c'(\lambda,a) = -\lambda\sin\lambda a + B(a,a)\cos\lambda a + \sum_{j=1}^{n} \frac{\partial^{j-1}}{\partial t^{j-1}} B_x(a,a) \frac{\sin^{(j-1)}\lambda a}{\lambda^{j}} + \int_0^a \frac{\partial^n}{\partial t^n} B_x(a,t) \frac{\sin^{(n+1)}\lambda t}{\lambda^{n}}\, dt, \tag{12.2.25}$$

where $\frac{\partial^{n+1}}{\partial t^{n+1}} K(a,\cdot)$, $\frac{\partial^n}{\partial t^n} K_x(a,\cdot)$, $\frac{\partial^{n+1}}{\partial t^{n+1}} B(a,\cdot)$, $\frac{\partial^n}{\partial t^n} B_x(a,\cdot)$ belong to $L_2(0,a)$.

Corollary 12.2.11. *The entire functions $\lambda \mapsto \lambda s(\lambda,a)$, $s'(\cdot,a)$, $c(\cdot,a)$, and $\lambda \mapsto \lambda^{-1}c'(\lambda,a)$ are sine type functions of type a.*

Proof. The first term of the representation in Corollary 12.2.10 of each of these functions can be estimated as

$$\frac{1}{4}e^{|\operatorname{Im}\lambda|a} \le |\sin\lambda a| \le e^{|\operatorname{Im}\lambda|a} \quad \text{or} \quad \frac{1}{4}e^{|\operatorname{Im}\lambda|a} \le |\cos\lambda a| \le e^{|\operatorname{Im}\lambda|a}$$

for sufficiently large $|\operatorname{Im}\lambda|$, whereas the remaining terms satisfy

$$O(\lambda^{-1})e^{|\operatorname{Im}\lambda|a}.$$

Hence each of these functions is a sine type function by Proposition 11.2.19. □

12.3 Representations of some sine type functions

Let b and c be real numbers. Then the solution ω of the initial value problem

$$y'' + \left(\lambda^2 - 2ib\lambda - c\right)y = 0, \quad x \in (0,a), \quad y(\lambda,0) = 0, \quad y'(\lambda,0) = 1, \tag{12.3.1}$$

has the representation $\omega(\lambda,x) = \tau(\lambda)^{-1}\sin\tau(\lambda)x$, and $\omega'(\lambda,x) = \cos\tau(\lambda)x$, where $\tau(\lambda) = \sqrt{\lambda^2 - 2ib\lambda - c}$. Since both ω and ω' are even functions with respect to τ, the representation is unambiguous. The next lemma gives an asymptotic representation of these two functions in terms of $\sin\lambda a$ and $\cos\lambda a$ for $x = a > 0$.

Lemma 12.3.1. *Let $b, c \in \mathbb{R}$ and $a > 0$. Then there are $R > 0$ and analytic functions $f_{j,k}$ on $\{z \in \mathbb{C} : |z| < R^{-1}\}$ for $j, k = 1, 2$, satisfying $f_{j,k}(-\overline{z}) = \overline{f_{j,k}(z)}$, $f_{1,1}(0) = f_{2,1}(0) = \cosh ba$ and $-f_{1,2}(0) = f_{2,2}(0) = \sinh ba$ such that for the solution $\omega(\cdot, a)$ of the initial value problem (12.3.1) at $x = a$ and its derivative $\omega'(\cdot, a)$ at $x = a$ we have the representations*

$$\omega(\lambda, a) = \frac{\sin \tau(\lambda)a}{\tau(\lambda)} = f_{1,1}(\lambda^{-1})\frac{\sin \lambda a}{\lambda} + i f_{1,2}(\lambda^{-1})\frac{\cos \lambda a}{\lambda}, \tag{12.3.2}$$

$$\omega'(\lambda, a) = \cos \tau(\lambda)a = f_{2,1}(\lambda^{-1}) \cos \lambda a + i f_{2,2}(\lambda^{-1}) \sin \lambda a, \tag{12.3.3}$$

for $|\lambda| > R$.

Proof. Let $r > 0$ such that $2|b|r + cr^2 < 1$. Let h_1 be the unique analytic branch of $z \mapsto \sqrt{1 - 2ibz - cz^2}$ on $\{z \in \mathbb{C} : |z| < r\}$ with $h_1(0) = 1$, i. e.,

$$h_1(z) = \sum_{j=0}^{\infty} \binom{\frac{1}{2}}{j} (-2ibz - cz^2)^j, \ |z| < r. \tag{12.3.4}$$

Note that $h_1(z) \neq 0$ for all $|z| < r$. For $|\lambda| > R = \frac{1}{r}$ we will now choose the branch of τ such that

$$\frac{\tau(\lambda)}{\lambda} = h_1(\lambda^{-1}).$$

For $|z| < r$ we define

$$h_2(z) = \frac{h_1(z) - 1}{z}. \tag{12.3.5}$$

Clearly, h_2 is analytic with $h_2(0) = -ib$. For $|\lambda| > R$ we conclude that

$$\tau(\lambda) - \lambda = \lambda \left(\frac{\tau(\lambda)}{\lambda} - 1 \right) = h_2(\lambda^{-1}).$$

It follows that

$$\begin{aligned} \frac{\sin \tau(\lambda)a}{\tau(\lambda)} &= \frac{\cos h_2(\lambda^{-1})a}{h_1(\lambda^{-1})} \frac{\sin \lambda a}{\lambda} + \frac{\sin h_2(\lambda^{-1})a}{h_1(\lambda^{-1})} \frac{\cos \lambda a}{\lambda}, \\ \cos \tau(\lambda)a &= \cos h_2(\lambda^{-1})a \cos \lambda a - \sin h_2(\lambda^{-1})a \sin \lambda a. \end{aligned}$$

For $|z| < r$ we define

$$f_{1,1}(z) = \frac{\cos h_2(z)a}{h_1(z)}, \ f_{1,2}(z) = -i\frac{\sin h_2(z)a}{h_1(z)}, \tag{12.3.6}$$

$$f_{2,1}(z) = \cos h_2(z)a, \ f_{2,2}(z) = i \sin h_2(z)a, \tag{12.3.7}$$

which proves the representations (12.3.2) and (12.3.3). We also obtain $f_{1,1}(0) = f_{2,1}(0) = \cos h_2(0)a = \cosh ba$ and $-f_{1,2}(0) = f_{2,2}(0) = i \sinh h_2(0)a = \sinh ba$. The symmetry of these functions follows from

$$h_1(-\overline{z}) = \sqrt{1 + 2ib\overline{z} - c\overline{z}^2} = \overline{h_1(z)}$$

and $h_2(-\overline{z}) = -\overline{h_2(z)}$ for $|z| < r$. ☐

Corollary 12.3.2. *Under the assumptions of Lemma* 12.3.1, *for each* $n \in \mathbb{N}$ *there are polynomials* $f_{j,k,n}$ *of degree* $\le n$ *and entire functions* $\psi_{j,n} \in \mathcal{L}^a$, $j,k = 1,2$, *such that* $f_{j,k,n}(-\overline{z}) = \overline{f_{j,k,n}(z)}$, $f_{j,1,n}(0) = \cosh ba$, $(-1)^j f_{j,2,n}(0) = \sinh ba$, $\psi_{j,n}(-\overline{z}) = (-1)^{n+j}\overline{\psi_{j,n}(z)}$ *and such that*

$$\frac{\sin\tau(\lambda)a}{\tau(\lambda)} = f_{1,1,n}(\lambda^{-1})\frac{\sin\lambda a}{\lambda} + i f_{1,2,n}(\lambda^{-1})\frac{\cos\lambda a}{\lambda} + \frac{\psi_{1,n}(\lambda)}{\lambda^{n+1}}, \tag{12.3.8}$$

$$\cos\tau(\lambda)a = f_{2,1,n}(\lambda^{-1})\cos\lambda a + i f_{2,2,n}(\lambda^{-1})\sin\lambda a + \frac{\psi_{2,n}(\lambda)}{\lambda^n}. \tag{12.3.9}$$

Proof. Let $f_{j,k,n}$, $j,k = 1,2$, be the Taylor polynomial about 0 of order n of the function $f_{j,k}$ from Lemma 12.3.1. Defining the function $\psi_{1,n}$ by (12.3.8), we have

$$\begin{aligned}\psi_{1,n}(\lambda) &= \lambda^{n+1}\frac{\sin\tau(\lambda)a}{\tau(\lambda)} - \lambda^n f_{1,1,n}(\lambda^{-1})\sin\lambda a - i\lambda^n f_{1,2,n}(\lambda^{-1})\cos\lambda a\\ &= \lambda^n\left(f_{1,1}(\lambda^{-1}) - f_{1,1,n}(\lambda^{-1})\right)\sin\lambda a + i\lambda^n\left(f_{1,2}(\lambda^{-1}) - f_{1,2,n}(\lambda^{-1})\right)\cos\lambda a,\end{aligned}$$

where the second identity follows from (12.3.2). The first of these representations shows that $\psi_{1,n}$ is an entire function with the stated symmetry. Finally, the second representation shows that $\psi_{1,n}$ is of the form $O(|\lambda|^{-1})\sin\lambda a + O(|\lambda|^{-1})\cos\lambda a$, and therefore $\psi_{1,n}$ is of exponential type $\le a$ and $O(|\lambda|^{-1})$ on the real axis. Hence $\psi_{1,n} \in \mathcal{L}^a$. The proof for $\psi_{2,n}$ is similar. □

Lemma 12.3.3 ([180, Lemma 3.4.2])**.** *For the functions u and v to have the representations*

$$u(\lambda) = \frac{\sin\lambda a}{\lambda} - \frac{4\pi^2 A a\cos\lambda a}{4\lambda^2a^2 - \pi^2} + \frac{f(\lambda)}{\lambda^2}, \tag{12.3.10}$$

$$v(\lambda) = \cos\lambda a + B\pi^2\frac{\sin\lambda a}{\lambda a} + \frac{g(\lambda)}{\lambda}, \tag{12.3.11}$$

where $A, B \in \mathbb{C}$, $f \in \mathcal{L}^a_e$, $f(0) = 0$, $g \in \mathcal{L}^a_o$, *it is necessary and sufficient that*

$$u(\lambda) = a\prod_{k=1}^{\infty}\left(\frac{\pi k}{a}\right)^{-2}(u_k^2 - \lambda^2), \quad u_k = \frac{\pi k}{a} + \frac{\pi A}{ak} + \frac{\alpha_k}{k}, \tag{12.3.12}$$

$$v(\lambda) = \prod_{k=1}^{\infty}\left(\frac{\pi}{a}\left(k - \frac{1}{2}\right)\right)^{-2}(v_k^2 - \lambda^2), \quad v_k = \frac{\pi}{a}\left(k - \frac{1}{2}\right) + \frac{\pi B}{ak} + \frac{\beta_k}{k}, \tag{12.3.13}$$

where $(\alpha_k)_{k=1}^\infty \in l_2$ *and* $(\beta_k)_{k=1}^\infty \in l_2$.

Proof. We define

$$\varphi_s(\lambda) = i\lambda^2 u(\lambda).$$

In view of

$$\frac{4\pi^2 a\lambda^2}{4\lambda^2a^2 - \pi^2} - \frac{\pi^2}{a} = \frac{\pi^4}{(4\lambda^2a^2 - \pi^2)a},$$

φ_s is an even entire function of the form φ as in (7.1.4) with $\sigma = 0$, $\alpha = 1$, $M = 0$ and $N = \frac{\pi^2 A}{a}$. Then the representation of u_k in (12.3.13) follows from part 1 in Lemma 7.1.3 if we observe that u is even and that we have to omit a double zeros at 0 from the sequence $(\tilde{\lambda}_k)_{k=-\infty,k\neq 0}^{\infty}$. Since $\lambda \mapsto \lambda \sin \lambda a$ is a sine type function, it follows from Lemma 11.2.29 that

$$u(\lambda) = a' \prod_{k=1}^{\infty} \left(\frac{\pi k}{a}\right)^{-2} (u_k^2 - \lambda^2)$$

with some $a' \neq 0$. From the product representation of the sine function we infer that

$$\begin{aligned} \frac{\lambda u(\lambda)}{\sin \lambda a} &= \frac{a'}{a} \prod_{k=1}^{\infty} \left(\frac{\pi k}{a}\right)^{-2} (u_k^2 - \lambda^2) \left(1 - \frac{\lambda^2 a^2}{\pi^2 k^2}\right)^{-1} \\ &= \frac{a'}{a} \prod_{k=1}^{\infty} \frac{u_k^2 - \lambda^2}{\frac{\pi^2 k^2}{a^2} - \lambda^2} \\ &= \frac{a'}{a} \prod_{k=1}^{\infty} \left(1 + \frac{u_k^2 - \frac{\pi^2 k^2}{a^2}}{\frac{\pi^2 k^2}{a^2} - \lambda^2}\right). \end{aligned}$$

Since $u_k^2 - \frac{\pi^2 k^2}{a^2}$ is bounded with respect to k and since $|\frac{\pi^2 k^2}{a^2} - \lambda^2| \geq \frac{\pi^2 k^2}{a^2}$ for λ on the imaginary axis, it follows that the right-hand side converges to $\frac{a'}{a}$ as $\lambda \to \infty$ along the imaginary axis, whereas the corresponding limit on the left-hand side is 1. Hence $a' = a$ and we have shown that the function given by (12.3.10) has the representation (12.3.12).

Conversely, assume that u is given by (12.3.12). Putting $\lambda_k = \frac{\pi k}{a}$, we have

$$u_k = \lambda_k + \frac{\pi^2 A}{a^2} \lambda_k^{-1} + \frac{\pi \alpha_k}{a} \lambda_k^{-1}.$$

Since $(\lambda_k)_{k\in\mathbb{Z}}$ is the sequence of the zeros of $\lambda \mapsto \sin \lambda a$, it follows from Remark 11.3.16 that

$$\lambda u(\lambda) = C_0 \sin \lambda a \left(1 + \frac{B_1}{\lambda}\right) - C_0 \frac{\pi^2 A}{a\lambda} \cos \lambda a + \frac{f_2(\lambda)}{\lambda}, \tag{12.3.14}$$

where $C_0 \neq 0$, B_1 is a constant and $f_2 \in \mathcal{L}^a$. Taking into account that u is an even entire function, we obtain $B_1 = 0$, $f_2 \in \mathcal{L}_e^a$ and $f_2(0) = 0$. From the first part of the proof we conclude that $C_0 = 1$. Hence u is of the form (12.3.10).

Similarly, the function defined by

$$\varphi_c(\lambda) = \lambda v(\lambda)$$

is an odd entire function of the form φ as in (7.1.4) with $\sigma = 1$, $\alpha = 0$, $M = \frac{\pi^2 B}{a}$ and $N = 0$. Arguing as for φ_s it follows that (12.3.11) and (12.3.13) are equivalent. ◻

Lemma 12.3.4. *For the entire functions u and v to admit the representations*

$$u(\lambda) = \frac{\sin \lambda a}{\lambda} - \frac{4\pi^2 Aa \cos \lambda a}{4\lambda^2 a^2 - \pi^2} + C\frac{\sin \lambda a}{\lambda^3} + \frac{f(\lambda)}{\lambda^3}, \tag{12.3.15}$$

$$v(\lambda) = \cos \lambda a + B\pi^2 \frac{\sin \lambda a}{\lambda a} + D\frac{\cos \lambda a}{4\lambda^2 a^2 - \pi^2} + \frac{g(\lambda)}{\lambda^2}, \tag{12.3.16}$$

where $A, B, C, D \in \mathbb{C}$, $f \in \mathcal{L}_o^a$, $g \in \mathcal{L}_e^a$, it is necessary and sufficient that

$$u(\lambda) = a \prod_{k=1}^{\infty} \left(\frac{\pi k}{a}\right)^{-2} (u_k^2 - \lambda^2), \quad u_k = \frac{\pi k}{a} + \frac{\pi A}{ak} + \frac{\alpha_k}{k^2}, \tag{12.3.17}$$

$$v(\lambda) = \prod_{k=1}^{\infty} \left(\frac{\pi}{a}\left(k - \frac{1}{2}\right)\right)^{-2} (v_k^2 - \lambda^2), \quad v_k = \frac{\pi}{a}\left(k - \frac{1}{2}\right) + \frac{\pi B}{ak} + \frac{\beta_k}{k^2}, \tag{12.3.18}$$

where $(\alpha_k)_{k=1}^{\infty} \in l_2$ and $(\beta_k)_1^{\infty} \in l_2$.

Proof. First assume that (12.3.15) or (12.3.16) hold. We are going to prove that the representations of the zeros u_k and v_k given in (12.3.12) and (12.3.13) can be written in the form (12.3.17) and (12.3.18). In case (12.3.15) we define

$$\tilde{\chi}(\mu) = \mu^2 u(\mu), \tag{12.3.19}$$

while in case (12.3.16) we define

$$\tilde{\chi}(\mu) = -\mu v\left(\mu + \frac{\pi}{2a}\right). \tag{12.3.20}$$

It is easy to see that in either case,

$$\tilde{\chi}(\mu) = (\mu + B_2\mu^{-1}) \sin \mu a + A_1 \cos \mu a + \Psi_1(\mu)\mu^{-1}$$

with $\Psi_1 \in \mathcal{L}^a$. Indeed, in case (12.3.15) we have $B_2 = C$ and $A_1 = -\pi^2 Aa^{-1}$, while in case (12.3.16) we have $B_2 = \frac{D}{4a^2}$ and $A_1 = -B\pi^2 a^{-1}$. Hence $\tilde{\chi}$ is of the form as considered in Lemma 7.1.5 with $n = 1$, $B_0 = 1$, $B_1 = 0$, $A_2 = 0$, except that we do not require that A_1 and B_1 are real and that Ψ_n is symmetric. But it is easy to see that these requirements are only used to guarantee that the zeros of $\tilde{\chi}$ can be indexed properly; the asymptotic representation (7.1.13) of the zeros holds without these requirements. We therefore conclude from Lemma 7.1.5 that the zeros of $\tilde{\chi}$ have the asymptotic representation

$$\mu_k = \frac{\pi k}{a} - \frac{A_1}{k\pi} + \frac{b_k^{(1)}}{k^2},$$

where $(b_k^{(1)})_{k=1}^{\infty} \in l_2$. Observing that $u_k = \mu_k$ and $v_k = \mu_k - \frac{\pi}{2a}$, the representations (12.3.17) and (12.3.18) follow.

Conversely, assume that (12.3.17) holds. By the first part of this proof, the zeros $(\lambda_k^{(0)})_{k\in\mathbb{Z}\setminus\{0\}}$ of the entire function u_0 defined by

$$u_0(\lambda) = \frac{\sin\lambda a}{\lambda} - \frac{4\pi^2 Aa\cos\lambda a}{4\lambda^2a^2-\pi^2} \tag{12.3.21}$$

have the asymptotic behaviour

$$\lambda_k^{(0)} = \frac{\pi k}{a} + \frac{\pi A}{ak} + \frac{\alpha_{0,k}}{k^2}, \tag{12.3.22}$$

where $(\alpha_{0,k})_{k=1}^\infty \in l_2$ and $\alpha_{0,-k} = -\alpha_{0,k}$ for all $k\in\mathbb{N}$. Comparing (12.3.22) with (12.3.17) we obtain

$$u_k = \lambda_k^{(0)} + \frac{\gamma_k}{\left(\lambda_k^{(0)}\right)^2},$$

where $(\gamma_k)_{k\in\mathbb{Z}\setminus\{0\}} \in l_2$. It is easy to see that $\lambda\mapsto\lambda u_0(\lambda)$ is a sine-type function of type a. In view of Remark 11.3.16 we obtain

$$\lambda u(\lambda) = C_0\lambda u_0(\lambda)\left(1+\frac{B_1}{\lambda}+\frac{C}{\lambda^2}\right)+\frac{f_2(\lambda)}{\lambda^2}, \tag{12.3.23}$$

where $C_0\neq 0$, B_1, C are constants and $f_2\in\mathcal{L}^a$. Taking into account that u and u_0 are even functions, we obtain $B_1=0$ and $f_2\in\mathcal{L}^a_o$. Since u satisfies the representation (12.3.10) in Lemma 12.3.3, it follows that $C_0=1$. Substituting (12.3.21) into (12.3.23) and observing $C_0=1$ and $B_1=0$, we obtain (12.3.15).

Finally assume that (12.3.18) holds. By the first part of this proof, the zeros $(\lambda_k^{(0)})_{k\in\mathbb{Z}\setminus\{0\}}$ of the entire function v_0 defined by

$$v_0(\lambda) = \cos\lambda a + B\pi^2\frac{\sin\lambda a}{\lambda a} \tag{12.3.24}$$

have the asymptotic behaviour

$$\lambda_k^{(0)} = \frac{\pi}{a}\left(|k|-\frac{1}{2}\right)\operatorname{sgn} k + \frac{\pi B}{ak} + \frac{\beta_{0,k}}{k^2}, \tag{12.3.25}$$

where $(\beta_{0,k})_{k=1}^\infty \in l_2$ and $\beta_{0,-k} = -\beta_{0,k}$ for all $k\in\mathbb{N}$. Comparing (12.3.25) with (12.3.18) we obtain

$$v_k = \lambda_k^{(0)} + \frac{\gamma_k}{\left(\lambda_k^{(0)}\right)^2},$$

where $(\gamma_k)_{k\in\mathbb{Z}\setminus\{0\}} \in l_2$. It is easy to see that v_0 is a sine-type function of type a. In view of Remark 11.3.16 we obtain

$$v(\lambda) = C_0 v_0(\lambda)\left(1+\frac{B_1}{\lambda}+\frac{D}{4a^2\lambda^2}\right)+\frac{f_2(\lambda)}{\lambda^2}, \tag{12.3.26}$$

where $C_0\neq 0$, B_1, D are constants and $f_2\in\mathcal{L}^a$. Taking into account that v and v_0 are even functions, we obtain $B_1=0$ and $f_2\in\mathcal{L}^a_e$. Since v satisfies the representation (12.3.11) in Lemma 12.3.3, it follows that $C_0=1$. Substituting (12.3.24) into (12.3.26) and observing $C_0=1$ and $B_1=0$, we obtain (12.3.16). ☐

12.4 The fundamental equation

Throughout this section let u and v be as in (12.3.10) and (12.3.11) with $A = B$ and such that the numbers u_k and v_k in (12.3.12) and (12.3.13) are real or pure imaginary for all $k \in \mathbb{N}$ and satisfy

$$v_1^2 < u_1^2 < v_2^2 < u_2^2 \cdots .$$

We consider the entire function χ defined by

$$\chi(\lambda) = v(\lambda) + i\lambda u(\lambda), \quad \lambda \in \mathbb{C}. \tag{12.4.1}$$

Lemma 12.4.1. *The function χ is of* SSHB *class.*

Proof. By definition, u and v are even functions, and we can write $u(\lambda) = Q(\lambda^2)$ and $v(\lambda) = P(\lambda^2)$ with entire functions Q and P, where the sets $\{u_k^2 : k \in \mathbb{N}\}$ and $\{v_k^2 : k \in \mathbb{N}\}$ are the sets of the zeros of Q and P, respectively, and all zeros of Q and P are simple and interlace. Since Q and P have the representations (12.3.12) and (12.3.13), it is easy to see that $Q(v_1^2)$ is an infinite product of positive numbers, whereas $P'(v_1^2)$ is the negative of an infinite product of positive numbers. It follows that $Q'(x)P(x) - Q(x)P'(x) > 0$ for all $x \in \mathbb{R}$ sufficiently close to v_1^2. Hence there are $x \in \mathbb{R}$ such that $P(x) \neq 0$ and such that $\theta = \frac{Q}{P}$ satisfies $\theta'(x) > 0$. Therefore θ is a Nevanlinna function by Theorem 11.1.6 and Remark 11.1.7. Then $\theta \in \mathcal{N}_+^{\mathrm{ep}}$ by Corollary 5.2.3, and thus χ is of SSHB class by Definition 5.2.6. ☐

We further define the function ψ by

$$\psi(\lambda) = e^{-i\lambda a}\chi(\lambda) \tag{12.4.2}$$

and the function S by

$$S(\lambda) = \frac{\psi(\lambda)}{\psi(-\lambda)}. \tag{12.4.3}$$

Proposition 12.4.2. *Let $\kappa := \#\{k \in \mathbb{N} : v_k^2 < 0\}$. Then S is meromorphic on $\mathbb{C}$ and analytic on $\mathbb{R}$, and S has exactly κ poles in the open upper half-plane. All poles in the open upper half-plane are simple and lie on the imaginary axis. Furthermore, for $\lambda \in \mathbb{C}$ such that $\psi(\lambda) \neq 0$ and $\psi(-\lambda) \neq 0$ we have*

$$S(-\overline{\lambda}) = \frac{\psi(-\overline{\lambda})}{\psi(\overline{\lambda})} = \overline{S(\lambda)}, \quad \textit{and} \quad \frac{1}{S(\lambda)} = \frac{\psi(-\lambda)}{\psi(\lambda)} = S(-\lambda). \tag{12.4.4}$$

In particular,

$$S(-\lambda) = \overline{S(\lambda)}, \ |S(\lambda)| = 1, \quad \lambda \in \mathbb{R}. \tag{12.4.5}$$

Proof. The function S is the quotient of two nonzeros entire functions and hence meromorphic on $\mathbb{C}$. Let $\check{\chi}(\lambda) = \chi(-\lambda)$, $\lambda \in \mathbb{C}$. Since χ and $\check{\chi}$ do not have common nonzero zeros, the statement on the poles and the analyticity of S on $\mathbb{R} \setminus \{0\}$ immediately follows from Theorem 5.2.9. Furthermore, the possible singularity of S at 0 is removable since $\lim\limits_{\lambda\to 0} S(\lambda) = -1$ if 0 is a (simple) zero of χ. The identities (12.4.4) and (12.4.5) are immediate consequences of the fact that χ is real on the imaginary axis. □

For $b \geq 0$ we define S_b by

$$S_b(\lambda) = 1 - S(ib + \lambda), \quad \lambda \in \mathbb{C},\ \psi(-ib - i\lambda) \neq 0. \tag{12.4.6}$$

Lemma 12.4.3.

1. *Let $b \geq 0$ such that ib is not a pole of S. Then $S_b \in L_2(\mathbb{R})$.*
2. *Let $\gamma \geq 0$ such that S is analytic on $\{\lambda \in \mathbb{C} : \operatorname{Im} \lambda \geq \gamma\}$ and define*

$$F(x) = \frac{e^{-\gamma x}}{2\pi} \int_{-\infty}^{\infty} S_\gamma(\lambda) e^{i\lambda x}\, d\lambda, \quad x \in \mathbb{R}. \tag{12.4.7}$$

Then the function F is real valued and independent of γ. Furthermore, the function $x \mapsto e^{\gamma x} F(x)$ is the inverse Fourier transform of S_γ and can be represented as $F_1 + F_2$, where $F_1 \in L_2(\mathbb{R})$, $F_2 \in W_2^1(\mathbb{R})$ are real-valued functions, $F_1(x) = 0$ for $x > 0$, and $F_2(x) = 0$ for $x > 2a$.

Proof. 1. We conclude from the representations (12.3.10) and (12.3.11) of u and v that

$$\chi(\lambda) = e^{i\lambda a} + \frac{A\pi^2}{a} \frac{\sin \lambda a}{\lambda} - \frac{4\pi^2 i\lambda A a \cos \lambda a}{4\lambda^2 a^2 - \pi^2} + \frac{g(\lambda) + if(\lambda)}{\lambda},$$

which gives

$$\psi(\lambda) = 1 + \frac{A\pi^2}{a} \frac{\sin \lambda a}{\lambda} e^{-i\lambda a} - \frac{4\pi^2 i\lambda A a \cos \lambda a}{4\lambda^2 a^2 - \pi^2} e^{-i\lambda a} + \frac{g(\lambda) + if(\lambda)}{\lambda} e^{-i\lambda a} \tag{12.4.8}$$

and

$$\frac{1}{\psi(-\lambda)} = 1 - \frac{A\pi^2}{a} \frac{\sin \lambda a}{\lambda} e^{i\lambda a} - \frac{4\pi^2 i\lambda A a \cos \lambda a}{4\lambda^2 a^2 - \pi^2} e^{i\lambda a} + \frac{g(\lambda) - if(\lambda)}{\lambda} e^{i\lambda a} + O(\lambda^{-2}) \tag{12.4.9}$$

for λ in the closed upper half-plane. Therefore, the function S satisfies

$$S(\lambda) = 1 - 2\frac{A\pi^2 i}{a} \frac{\sin^2 \lambda a}{\lambda} - \frac{8\pi^2 i\lambda A a \cos^2 \lambda a}{4\lambda^2 a^2 - \pi^2} + \frac{2}{\lambda}(g(\lambda) \cos \lambda a - f(\lambda) \sin \lambda a) \\ + O(\lambda^{-2}) e^{2 \operatorname{Im} \lambda a}.$$

Observing that

$$\frac{4a^2}{4\lambda^2 a - \pi^2} = \frac{1}{\lambda^2} + O(\lambda^{-4}),$$

this representation can be written in the form

$$1 - S(\lambda) = 2\frac{A\pi^2 i}{\lambda a} - \frac{2}{\lambda}(g(\lambda)\cos\lambda a - f(\lambda)\sin\lambda a) + O(\lambda^{-2})e^{2\,\mathrm{Im}\,\lambda a}. \quad (12.4.10)$$

By Proposition 12.4.2 we know that poles of S in the closed upper half-plane lie on the positive imaginary axis, and from (12.4.10) we thus infer that $1 - S$ is analytic and square integrable on each line $\mathrm{Im}\,\lambda = b$ with $b \geq 0$ for which ib is not a pole of S.

2. Using (12.4.4) we have

$$S_\gamma(-\lambda) = 1 - S(i\gamma - \lambda) = 1 - \overline{S(i\gamma + \lambda)} = \overline{S_\gamma(\lambda)}, \quad \lambda \in \mathbb{R},$$

so that

$$F(x) = \frac{e^{-\gamma x}}{\pi}\int_0^\infty \mathrm{Re}\left(S_\gamma(\lambda)e^{i\lambda x}\right) d\lambda, \quad x \in \mathbb{R},$$

which shows that F is real valued. We can write

$$\begin{aligned} F(x) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} (1 - S(i\gamma + \lambda))e^{i(i\gamma+\lambda)x}\, d\lambda \\ &= \frac{1}{2\pi}\int_{\mathrm{Im}\,\lambda=\gamma} (1 - S(\lambda))e^{i\lambda x}\, d\lambda =: F_\gamma(\lambda). \end{aligned}$$

If we now take $\gamma' > \gamma$, then it follows from Cauchy's theorem that

$$\begin{aligned} |F_{\gamma'}(x) - F_\gamma(x)| \leq \frac{1}{2\pi}\limsup_{R\to\infty}\Bigg(&\int_\gamma^{\gamma'} |(1 - S(R + it))e^{i(R+it)x}|dt \\ &+ \int_\gamma^{\gamma'} |(1 - S(-R + it))e^{i(-R+it)x}|dt\Bigg) \\ = \limsup_{R\to\infty} O(R^{-1}) = 0,& \end{aligned}$$

and therefore the function F is independent of the special choice of γ.

The first term of the representation (12.4.10) of $1 - S$ has a pole at $\lambda = 0$. But it is easy to see that (12.4.10) can be written as

$$1 - S(\lambda) = \frac{2A\pi^2 i}{a(\lambda + i)} + \frac{\omega(\lambda)}{\lambda + i},\ \lambda \in \mathbb{C},$$

where ω is analytic in the closed half-plane $\{\lambda \in \mathbb{C} : \mathrm{Im}\,\lambda \geq \gamma\}$, $\omega(i\gamma + \cdot) \in L_2(\mathbb{R})$, and $\lambda \mapsto \omega(\lambda)e^{-2\,\mathrm{Im}\,\lambda a}$ is bounded in that half-plane. We can therefore write $e^{\gamma x}F(x) = F_1(x) + F_2(x)$ with

$$F_1(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{2A\pi^2 i}{a(\lambda + i\gamma + i)}e^{i\lambda x}\, d\lambda, \quad F_2(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\omega(\lambda + i\gamma)}{\lambda + i\gamma + i}e^{i\lambda x}\, d\lambda.$$

For $x > 0$ and $R > 0$ it follows from Cauchy's integral theorem and Lebesgue's dominated convergence theorem that

$$\int_{-R}^{R} \frac{e^{i\lambda x}}{\lambda + i\gamma + i}\, d\lambda = -\int_0^{\pi} \frac{e^{Re^{i(\theta+\frac{\pi}{2})}x}}{Re^{i\theta} + i\gamma + i} Re^{i\theta}\, d\theta \to 0 \text{ as } R \to \infty.$$

Similarly, for $x < 0$ and $R > \gamma + 1$ we use the residue theorem and Lebesgue's dominated convergence theorem to conclude that

$$\int_{-R}^{R} \frac{e^{i\lambda x}}{\lambda + i\gamma + i}\, d\lambda = \int_{\pi}^{2\pi} \frac{e^{Re^{i(\theta+\frac{\pi}{2})}x}}{Re^{i\theta} + i\gamma + i} Re^{i\theta}\, d\theta - 2\pi i \operatorname{res}_{-i\gamma - i} \frac{e^{i\lambda x}}{\lambda + i\gamma + i}$$
$$\to -2\pi i e^{(\gamma+1)x} \text{ as } R \to \infty.$$

Therefore

$$F_1(x) = \begin{cases} 0 & \text{if } x > 0, \\ \frac{2A\pi^2}{a} e^{(\gamma+1)x} & \text{if } x < 0. \end{cases}$$

The function F_2 is differentiable with

$$F_2'(x) = \frac{i}{2\pi} \int_{-\infty}^{\infty} \frac{\lambda}{\lambda + i\gamma + i} \omega(\lambda + i\gamma) e^{i\lambda x}\, d\lambda, \quad x \in \mathbb{R}.$$

Hence F_2 and F_2' are inverse Fourier transforms of functions in $L_2(\mathbb{R})$, which shows that $F_2 \in W_2^1(\mathbb{R})$. For $x > 2a$ and $R > 0$ it follows from Cauchy's integral theorem, the boundedness of $\lambda \mapsto \omega(\lambda)e^{2i\lambda a}$ in the closed half-plane $\{\lambda \in \mathbb{C} : \operatorname{Im}\lambda \geq \gamma\}$, and Lebesgue's dominated convergence theorem that

$$\int_{-R}^{R} \frac{\omega(\lambda + i\gamma)}{\lambda + i\gamma + i} e^{i\lambda x}\, d\lambda = -\int_0^{\pi} \frac{\omega(Re^{i\theta} + i\gamma)}{Re^{i\theta} + i\gamma + i} e^{Re^{i(\theta+\frac{\pi}{2})}(2a-x)} Re^{i\theta}\, d\theta \to 0$$

as $R \to \infty$. Therefore, $F_2(x) = 0$ for $x > 2a$. □

The fundamental equation, see [180, (3,2,10), (3.3.7)] will formally be defined as

$$F(x+y) + H(x,y) + \int_x^{\infty} H(x,t)F(y+t)\, dt = 0, \quad 0 \leq x \leq y. \tag{12.4.11}$$

In order to have the limits of integration independent of x, we substitute $y + x$ for y and $t + x$ for t, see [180, (3.3.7′)], and the fundamental equation becomes

$$F(2x+y) + H(x,x+y) + \int_0^{\infty} H(x,x+t)F(2x+y+t)\, dt = 0, \quad x, y \geq 0. \tag{12.4.12}$$

We observe that in view of $F(x) = 0$ for $x > 2a$, the equation (12.4.12) can be written as

$$F(2x+y) + H(x,x+y) + \int_0^{2a-2x-y} H(x,x+t)F(2x+y+t)\, dt = 0, \quad (x,y) \in D, \tag{12.4.13}$$

where $D = \{(x,y) \in \mathbb{R}^2 : 0 \le x \le a, 0 \le y \le 2(a-x)\}$ since for any other values of $x, y \ge 0$, $H(x, x+y) = 0$ is necessary and sufficient for (12.4.12) to hold. However, it will be more convenient to use a rectangular region, and therefore we will consider the region $[0,a] \times [0,2a]$ instead of D.

For $0 \le x \le a$ define

$$(\mathbb{F}_x f)(y) = \int_0^{2a} F(2x+y+t)f(t)\,dt, \quad f \in L_2(0,2a),\ y \in [0,2a]. \tag{12.4.14}$$

The following result is a special case of [180, Lemmas 3.3.1, 3.3.2 and 3.3.3].

Lemma 12.4.4. *For $0 \le x \le a$, the operator $\mathbb{F}_x$ is a self-adjoint compact operator in the space $L_2(0,2a)$. If $v_1^2 > 0$, then $I + \mathbb{F}_x \gg 0$.*

Proof. Since $(y,t) \mapsto F(2x+y+t)$ is a real-valued and symmetric continuous function on $[0,2a] \times [0,2a]$, the operator $\mathbb{F}_x$ is self-adjoint and compact, see, e. g., [109, p. 240].

Now let $v_1^2 > 0$. Then S is analytic in the closed upper half-plane, and we can take $\gamma = 0$ in (12.4.7). In particular, $F \in L_2(\mathbb{R})$. Let $f \in L_2(0,2a)$. Putting $f(t) = (\mathbb{F}_x f)(t) = 0$ for $t \in \mathbb{R} \setminus [0,2a]$ and using the notations $\check{f}(t) = f(-t)$, $\tau_b(t) = y + b$ for $x, b \in \mathbb{R}$, the right-hand side of (12.4.14) can be written as a convolution, so that we arrive at

$$\mathbb{F}_x f = (F \circ \tau_{2x}) * \check{f} \text{ on } [0,\infty).$$

Hence there is a function g with support in $(-\infty, 0]$ such that

$$\mathbb{F}_x f + g = (F \circ \tau_{2x}) * \check{f} \text{ on } \mathbb{R}. \tag{12.4.15}$$

For $y < 0$ we have

$$|g(y)|^2 = \left| \int_0^{2a} F(2x+y+t)f(t)\,dt \right|^2 \le \int_0^{2a} |F(2x+y+t)|^2\,dt\,\|f\|^2,$$

where $\|\cdot\|$ denotes the norm in $L_2(\mathbb{R})$. Hence

$$\begin{aligned}\|g\| = \int_{-\infty}^0 |g(y)|^2\,dy &\le \int_{-\infty}^0 \int_0^{2a} |F(2x+y+t)|^2\,dt\,dy\,\|f\|^2 \\ &= \int_0^{2a} \int_{-\infty}^0 |F(2x+y+t)|^2\,dy\,dt\,\|f\|^2 \le 2a\|F\|^2\|f\|^2,\end{aligned}$$

which shows that $g \in L_2(-\infty, 0)$.

Taking the inner product with f in (12.4.15) and observing that $g\overline{f} = 0$, we arrive at

$$(\mathbb{F}_x f, f) = ((F \circ \tau_{2x}) * \check{f}, f).$$

Let $\hat{f}$ denote the Fourier transformation of f. Taking the Fourier transforms of the functions in the above inner product and observing Parseval's formula gives

$$((\mathbb{F}_x f)\hat{}, \hat{f}) = ((F \circ \tau_{2x})\hat{}\,\check{\hat{f}}, \hat{f}),$$

see, e.g., [108, (21.41)]. It is well known and easy to check that $\hat{f}$ is an entire function. We observe that for any function $h \in L_2(\mathbb{R})$ and $b \in \mathbb{R}$,

$$(h \circ \tau_b)\hat{}(\lambda) = \int_{-\infty}^{\infty} h(b+t)e^{-i\lambda t}\,dt = \int_{-\infty}^{\infty} h(t)e^{-i\lambda(t-b)}\,dt = e^{i\lambda b}\hat{h}(\lambda). \quad (12.4.16)$$

Since $\hat{F} = 1 - S$ by definition of F in (12.4.7), we conclude that

$$((\mathbb{F}_x f)\hat{}, \hat{f}) = \int_{-\infty}^{\infty} e^{2i\lambda x}(1 - S(\lambda))\hat{f}(-\lambda)\overline{\hat{f}(\lambda)}\,d\lambda.$$

Again from (12.4.16) and Parseval's identity we conclude that

$$\begin{aligned}\int_{-\infty}^{\infty} e^{2i\lambda x}\hat{f}(-\lambda)\overline{\hat{f}(\lambda)}\,d\lambda &= \int_{-\infty}^{\infty} e^{i\lambda x}\hat{f}(-\lambda)\overline{e^{-i\lambda x}\hat{f}(\lambda)}\,d\lambda \\ &= \int_{-\infty}^{\infty} (f \circ \tau_{-x})(-t)\overline{(f \circ \tau_{-x})(t)}\,dt.\end{aligned}$$

But $(f \circ \tau_{-x})(t) = f(t-x) = 0$ if $t < 0$ since f is zero outside $[0, 2a]$, and therefore the integrand on the right-hand side is zero. We conclude that

$$(\mathbb{F}_x f, f) = \frac{1}{2\pi}((\mathbb{F}_x f)\hat{}, \hat{f}) = -\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{2i\lambda x}S(\lambda)\hat{f}(-\lambda)\overline{\hat{f}(\lambda)}\,d\lambda. \quad (12.4.17)$$

Observing (12.4.5) it follows from the Cauchy–Schwarz–Bunyakovskii inequality that

$$(\mathbb{F}_x f, f) \geq -\frac{1}{2\pi}\int_{-\infty}^{\infty} |\hat{f}(-\lambda)|\,|\hat{f}(\lambda)|\,d\lambda \geq -\frac{1}{2\pi}\|\hat{f}\|^2 = -\|f\|^2. \quad (12.4.18)$$

Altogether, we conclude that

$$((I + \mathbb{F}_x)f, f) \geq 0 \quad (12.4.19)$$

for all $f \in L_2[0, 2a]$. Hence $I + \mathbb{F}_x \geq 0$.

Since $\mathbb{F}_x$ is compact, $I + \mathbb{F}_x$ is a Fredholm operator of index 0, and for $I + \mathbb{F}_x \gg 0$ it remains to show that $I + \mathbb{F}_x$ is injective. Hence let $f \in L_2(0, 2a)$ such that $(I + \mathbb{F}_x)f = 0$. For this f, (12.4.19) becomes and inequality, and hence also the Cauchy–Schwarz–Bunyakovskii inequality (12.4.18) is an equality. But this happens if and only if one function in the inner product is a nonnegative multiple of the other function, i.e., when there is $\alpha \geq 0$ such that

$$-e^{2i\lambda x}S(\lambda)\hat{f}(-\lambda) = \alpha\hat{f}(\lambda), \quad \lambda \in \mathbb{R}.$$

Observing that $S(0) = 1$ and that in case $f \neq 0$ the Taylor expansion of the entire function $\hat{f}$ about 0 leads to

$$\lim_{\lambda \to 0} \left| \frac{\hat{f}(\lambda)}{\hat{f}(-\lambda)} \right| = 1,$$

it follows that $\alpha = 1$, and the identity theorem gives

$$\hat{f}(\lambda) + e^{2i\lambda x} S(\lambda) \hat{f}(-\lambda) = 0, \quad \lambda \in \mathbb{C},\ S(\lambda) \neq 0, \tag{12.4.20}$$

which is also trivially true in case $f = 0$.

First let $x = 0$. Then (12.4.20) holds with $x = 0$. Hence all poles of S must be cancelled by zeros of $\check{\hat{f}}$, and since the poles of S are the zeros of $\check{\psi}$, there is an entire function ω such that

$$\hat{f} = \omega \psi.$$

Multiplying (12.4.20) by $\check{\psi}$ we arrive at

$$\omega \psi \check{\psi} = \hat{f} \check{\psi} = -\check{\hat{f}} \psi = -\check{\omega} \check{\psi} \psi,$$

which shows that ω is an odd entire function. We have seen at the beginning of this section that ψ has no zeros in the closed lower half-plane and satisfies the estimate (12.4.9). Hence $\frac{1}{\psi}$ is a bounded analytic function in the closed lower half-plane. Since $\hat{f}$ is the Fourier transform of a function in $L_2[0, 2a]$, also $\hat{f}$ is bounded in the closed lower half-plane. Indeed, it is easy to see that the integral over $|f|$ is such a bound. Hence $\omega = \frac{\hat{f}}{\psi}$ is bounded in the closed lower half-plane and then also bounded in the closed upper half-plane due to the symmetry of ω. By Liouville's theorem, ω is constant. But $\frac{1}{\psi}$ is bounded on $\mathbb{R}$, see (12.4.9), and $\hat{f}$ is an L_2 function on $\mathbb{R}$, so that this constant must be zero. We have shown that $\omega = 0$, and $\hat{f} = 0$ follows. Therefore $I + \mathbb{F}_0$ is injective, and $I + \mathbb{F}_0 \gg 0$ is proved.

Now let $x > 0$. Recall that we consider $(I + \mathbb{F}_x) f = 0$. Since $(\mathbb{F}_x f)(y) = 0$ for $y > 2(a - x)$, this implies $f(y) = 0$ for $y > 2(a - x)$. It follows for $0 < h < x$ that the function

$$f_h = \frac{1}{2}(f \circ \tau_{-x+h} + f \circ \tau_{-x-h})$$

has support in $[x - h, 2a - x + h]$, and therefore $f_h \in L_2[0, 2a]$. In view of (12.4.16) we conclude that

$$\hat{f}_h(\lambda) = \hat{f}(\lambda) e^{-i\lambda x} \cos \lambda h. \tag{12.4.21}$$

A substitution of (12.4.21) into (12.4.20) leads to

$$\hat{f}_h(\lambda) - S(\lambda) \hat{f}_h(-\lambda) = 0, \quad \lambda \in \mathbb{R},$$

which means that $((I + \mathbb{F}_0) f_h, f_h) = 0$ for $0 < h < a$. In view of

$$((I + \mathbb{F}_0) f_h, f_h) = ((I + \mathbb{F}_0)^{\frac{1}{2}} f_h, (I + \mathbb{F}_0)^{\frac{1}{2}} f_h)$$

we conclude that $(I + \mathbb{F}_0) f_h = 0$. But from the case $x = 0$ we already know that this implies $f_h = 0$. Then (12.4.21) gives $\hat{f} = 0$ and thus $f = 0$. $\square$

Lemma 12.4.5. *Let $v_1^2 > 0$. For $0 \le x \le a$, define the operator $\mathbb{F}_x^0$ as the restriction of $\mathbb{F}_x$ to $C[0,2a]$. Then $\mathbb{F}_x^0$ is a compact operator in the space $C[0,2a]$, and $I + \mathbb{F}_x^0$ is invertible.*

Proof. We begin by defining an auxiliary operator T by

$$((Tg)f)(y) = \int_0^{2a} g(y+t)f(t)\,dt, \quad g \in L_2(0,2a),\ f \in C[0,2a],\ y \in [0,2a].$$

Here we set $g(x) = 0$ for $x > 2a$. Clearly, $(Tg)f$ is a measurable function on $[0,2a]$ for all $g \in L_2(0,2a)$ and $f \in C[0,2a]$, and

$$|((Tg)f)(y)| \le \|g\|\,\|f\| \le \sqrt{2a}\|g\|\,\|f\|_0,$$

where $\|\cdot\|_0$ is the maximum norm in the Banach space $C[0,2a]$. This shows that $T \in L(L_2(0,2a), L(C[0,2a], L_\infty(0,2a)))$. Clearly, for continuous g with $g(2a) = 0$, also $(Tg)f$ is continuous. Observing that the set of such functions g is dense in $L_2(0,2a)$, that $C[0,2a]$ is closed in $L_\infty(0,2a)$ and that therefore $L(C[0,2a], C[0,2a])$ is closed in $L(C[0,2a], L_\infty(0,2a))$, it follows that $T \in L(L_2(0,2a), L(C[0,2a]))$. Furthermore, if $g \in W_2^1(0,2a)$ with $g(2a) = 0$, then we can write

$$g(x) = -\int_x^{2a} g'(\tau)\,d\tau, \quad x \ge 0,$$

see, e. g., [189, Proposition 2.1.5], and therefore

$$\begin{aligned} ((Tg)f)(y) &= -\int_0^{2a}\int_{y+t}^{2a} g'(\tau)\,d\tau\, f(t)\,dt \qquad = -\int_0^{2a}\int_y^{2a} g'(\tau+t)\,d\tau\, f(t)\,dt \\ &= -\int_y^{2a}\int_0^{2a} g'(\tau+t)\,dt\, f(t)\,d\tau = -\int_y^{2a} ((Tg')f)(\tau)\,d\tau. \end{aligned}$$

Since $(Tg')f$ is continuous by what we have already shown, it follows that $(Tg)f$ is differentiable with continuous derivative $((Tg)f)' = (Tg')f$. Then the norm of $(Tg)f$ in $C^1[0,2a]$ is

$$\|(Tg)f\|_0 + \|((Tg)f)'\|_0 = \|(Tg)f\|_0 + \|(Tg')f\|_0 \le \sqrt{2a}(\|g\| + \|g'\|)\|f\|_0,$$

which shows that $Tg \in L(C[0,2a], C^1[0,2a])$ if $g \in W_2^1[0,2a]$ with $g(2a) = 0$. But since the embedding from $C^1[0,2a]$ into $C[0,2a]$ is compact, see, e. g., [189, Proposition 2.1.7 and Lemma 2.4.1], it follows that Tg is a compact operator on $C[0,2a]$.

As we have seen in the proof of Lemma 12.4.4, we can take $\gamma = 0$ in (12.4.7). For $0 \le x \le a$ we now apply the above auxiliary result to the operators

$$\mathbb{F}_x^0 = T(F \circ \tau_{2x}),$$

which proves that $\mathbb{F}_x^0$ is a compact operator in $C[0,2a]$. Therefore $I + \mathbb{F}_x^0$ is a Fredholm operator with index 0. But $N(I + \mathbb{F}_x^0) \subseteq N(I + \mathbb{F}_x)$ and $I + \mathbb{F}_x$ is injective by Lemma 12.4.4. It follows that $I + \mathbb{F}_x$ is invertible. □

Lemma 12.4.6. *The operator functions* $x \mapsto \mathbb{F}_x$ *and* $x \mapsto \mathbb{F}_x^0$ *are differentiable on* $[0,a]$. *The derivative* $\mathbb{F}_x'$ *of* $x \mapsto \mathbb{F}_x$ *at* x *is the operator*

$$(\mathbb{F}_x' f)(y) = 2\int_0^{2a} F'(2x+y+t)f(t)\,dt, \quad f \in L_2(0,2a),\ y \in [0,2a], \tag{12.4.22}$$

and $(\mathbb{F}_x^0)'$ *is the restriction of* $\mathbb{F}_x'$ *to* $C[0,2a]$. *For each* $x \in [0,a]$ *and for all* $f \in L_2(a,b)$, *the function* $\mathbb{F}_x f$ *is differentiable, and* $(\mathbb{F}_x f)' = \frac{1}{2}\mathbb{F}_x' f$.

Proof. Arguing as at the beginning of the proof of Lemma 12.4.4 we see that $\mathbb{F}_x'$ is a bounded operator in $L_2(0,2a)$. For $x, x' \in [0,a]$ we define the auxiliary operator

$$\mathbb{F}_{x,x'} = \mathbb{F}_{x'} - \mathbb{F}_x - (x'-x)\mathbb{F}_x'.$$

Then it follows for $f \in L_2(0,2a)$ and $y \in [0,2a]$ that

$$\begin{aligned}(\mathbb{F}_{x,x'}f)(y)\\ &= \int_0^{2a} [F(2x'+y+t) - F(2x+y+t) - 2(x'-x)F'(2x+y+t)]f(t)\,dt.\end{aligned}$$

Since $F \in W_2^1(0,a)$ by Lemma 12.4.3, we can write

$$F(2x'+y+t) - F(2x+y+t) = \int_{2x}^{2x'} F'(\tau+y+t)\,d\tau,$$

see, e. g., [189, Proposition 2.1.5]. Therefore

$$(\mathbb{F}_{x,x'}f)(y) = \int_0^{2a}\int_{2x}^{2x'} [F'(\tau+y+t) - F'(2x+y+t)]f(t)\,d\tau\,dt.$$

Let $\varepsilon > 0$. Since the set of continuous functions on $[0,2a]$ is dense in $L_2(0,a)$, there is a continuous function g on $[0,2a]$ such that $\|F'-g\| < \varepsilon$. Then

$$\begin{aligned}&\left|\int_0^{2a}\int_{2x}^{2x'} [(F'-g)(\tau+y+t) - (F'-g)(2x+y+t)]f(t)\,d\tau\,dt\right|\\ &\qquad \le \left|\int_{2x}^{2x'}\int_0^{2a} |(F'-g)(\tau+y+t)|\,|f(t)|\,dt\,d\tau\right|\\ &\qquad\quad + \left|\int_{2x}^{2x'}\int_0^{2a} |(F'-g)(2x+y+t)|\,|f(t)|\,dt\,d\tau\right|\\ &\qquad \le 4|x'-x|\|F'-g\|\|f\| < 4\varepsilon|x'-x|\|f\|.\end{aligned}$$

Since g is continuous and therefore uniformly continuous, there exists $\delta > 0$ such that $|g(t)-g(t')| < \varepsilon$ for all $t, t' \in [0,2a]$ with $|t-t'| < 2\delta$. Hence it follows for

$|x' - x| < \delta$ that

$$\left| \int_0^{2a} \int_{2x}^{2x'} [g(\tau + y + t) - g(2x + y + t)] f(t)\, dt \right| \le \varepsilon \left| \int_{2x}^{2x'} \int_0^{2a} |f(t)|\, dt\, d\tau \right|$$
$$\le \varepsilon \sqrt{2a} |x' - x| \|f\|.$$

Altogether, we conclude that

$$\|\mathbb{F}_{x,x'} f\|_\infty \le \varepsilon (4 + \sqrt{2a}) |x' - x| \|f\|.$$

This shows that $\mathbb{F}_x$ is differentiable as an operator function from $L_2(0, 2a)$ to $L_\infty(0, 2a)$ with derivative $\mathbb{F}'_x$, see, e. g., [66, Section 8.1]. By the product rule, see, e. g., [66, 8.3.1], the same is clearly true if these operators are considered as operators into $L_2(0, 2a)$, that is, multiplied by the constant embedding from $L_\infty(0, 2a)$ to $L_2(0, 2a)$.

The same reasoning as above applies to the operator function $x \mapsto \mathbb{F}^0_x$. We only have to restrict f to functions in $C[0, 2a]$ and replace $\|f\|$ with $\sqrt{2a}\|f\|_0$.

In (12.4.14), we can interchange integration and differentiation with respect to y, and $(\mathbb{F}_x f)' = \frac{1}{2} \mathbb{F}'_x f$ is therefore an immediate consequence of (12.4.22). □

Proposition 12.4.7. *Let $v_1^2 > 0$ and let the function F be as defined in* (12.4.7)*. Then for every $x \in [0, a]$, $(I + \mathbb{F}_x) g = -F \circ \tau_{2x}$ has a unique solution $g = G(x, \cdot)$ in $L_2(0, 2a)$. The function G is continuous on $[0, a] \times [0, 2a]$ and $G(\cdot, y) \in W_2^1(0, a)$ for $y \in [0, 2a]$.*

Proof. As we have seen in the proof of Lemma 12.4.4, we can take $\gamma = 0$ in (12.4.7). The existence and uniqueness of G follows immediately from the invertibility of $I + \mathbb{F}_x$ for all $x \in [0, a]$, which was shown in Lemma 12.4.4, and we have

$$G(x, y) = -((I + \mathbb{F}_x)^{-1} (F \circ \tau_{2x}))(y)$$

for all $(x, y) \in [0, a] \times [0, 2a]$. Since $F \circ \tau_{2x}$ is continuous, Lemma 12.4.5 shows that we can also write

$$G(x, y) = -((I + \mathbb{F}^0_x)^{-1} (F \circ \tau_{2x}))(y),$$

and therefore $G(x, \cdot)$ is continuous. By Lemma 12.4.6, $x \mapsto \mathbb{F}^0_x$ is differentiable and therefore continuous on $[0, a]$, so that also $x \mapsto (I + \mathbb{F}^0_x)^{-1}$ is continuous. For $x, x' \in [0, a]$ we therefore conclude

$$\begin{aligned} |G(x', y) - G(x, y)| &\le |((I + \mathbb{F}^0_{x'})^{-1} (F \circ \tau_{2x'}))(y) - ((I + \mathbb{F}^0_x)^{-1} (F \circ \tau_{2x'}))(y)| \\ &\quad + |((I + \mathbb{F}^0_x)^{-1} (F \circ \tau_{2x'}))(y) - ((I + \mathbb{F}^0_x)^{-1} (F \circ \tau_{2x}))(y)| \\ &\le \|(I + \mathbb{F}^0_{x'})^{-1} - (I + \mathbb{F}^0_x)^{-1}\| \, \|(F \circ \tau_{2x'})\| \\ &\quad + \|(I + \mathbb{F}^0_x)^{-1}\| \, \|F \circ \tau_{2x'} - F \circ \tau_{2x}\| \\ &\to 0 \text{ as } x' \to x. \end{aligned}$$

In the last step we also have used that F is uniformly continuous. We have thus shown that $G(x,\cdot)$ and $G(\cdot,y)$ are continuous for all $x\in[0,a]$ and $y\in[0,2a]$, and a standard argument shows that G is continuous on $[0,a]\times[0,2a]$.

Since $x\mapsto \mathbb{F}_x$ is differentiable by Lemma 12.4.6 and since $I+\mathbb{F}_x$ is invertible for all $x\in[0,a]$ by Lemma 12.4.4, $x\mapsto (I+\mathbb{F}_x)^{-1}$ is differentiable on $[0,a]$ by the quotient rule, see [66, 8.3.2], and

$$\frac{d}{dx}(I+\mathbb{F}_x)^{-1} = -(I+\mathbb{F}_x)^{-1}\mathbb{F}_x'(I+\mathbb{F}_x)^{-1}.$$

Furthermore, also $x\mapsto F\circ\tau_{2x}$ is differentiable with derivative $2F'\circ\tau_{2x}$, and the product rule, see [66, 8.3.1], gives

$$\frac{\partial}{\partial x}G(x,\cdot) = (I+\mathbb{F}_x)^{-1}\mathbb{F}_x'(I+\mathbb{F}_x)^{-1}(F\circ\tau_{2x}) - 2(I+\mathbb{F}_x)^{-1}(F'\circ\tau_{2x}). \quad (12.4.23)$$

Since $F\circ\tau_{2x}$ is continuous, we know from Lemma 12.4.5 that the first summand can be written as

$$G_1(x,\cdot) := (I+\mathbb{F}_x^0)^{-1}(\mathbb{F}_x^0)'(I+\mathbb{F}_x^0)^{-1}(F\circ\tau_{2x}),$$

and since

$$x\mapsto (I+\mathbb{F}_x^0)^{-1}(\mathbb{F}_x^0)'(I+\mathbb{F}_x^0)^{-1}$$

is a continuous operator function in $L(C[0,2a])$, it follows like we have shown for G above that also G_1 is continuous on $[0,a]\times[0,2a]$. Next we consider the auxiliary operator G_0 defined by

$$(G_0g)(x,\cdot) := (I+\mathbb{F}_x)^{-1}(g\circ\tau_{2x}), \quad g\in L_2(0,2a), x\in[0,a].$$

Again for continuous g, we can replace $\mathbb{F}_x$ with $\mathbb{F}_x^0$, and the above considerations show that G_0g is continuous on $[0,2a]$ and therefore square integrable. We calculate

$$\begin{aligned}\left|\int_0^a\int_0^{2a}\left((I+\mathbb{F}_x)^{-1}(g\circ\tau_{2x})\right)(y)\,dy\,dx\right| &\le \int_0^a \|(I+\mathbb{F}_x)^{-1}\|\,\|g\|\,dx\\ &\le a\max_{x\in[0,a]}\|(I+\mathbb{F}_x)^{-1}\|\,\|g\|.\end{aligned}$$

By continuity, this extends to all $g\in L_2(0,2a)$, and we obtain that G_0 is a bounded operator from $L_2(0,a)$ to $L_2((0,a)\times(0,2a))$. Therefore, $G_2 := G_0F'$ belongs to $L_2((0,a)\times(0,2a))$. Finally, let $y\in[0,2a]$. Then

$$((I+\mathbb{F}_x)G_2)(x,\cdot) = F'\circ\tau_{2x}$$

gives

$$G_2(x,y) = -\int_0^a F(2x+y+t)G_2(x,t)\,dt + F'(2x+y).$$

Since $(x,t)\mapsto F(2x+y+t)G_2(x,t)$ is a square integrable kernel, it follows that the function $x\mapsto \int_0^a F(2x+y+t)G_2(x,t)\,dt$ is square integrable on $(0,a)$, see, e.g., [109, p. 240]. Altogether, $\frac{\partial}{\partial x}G(\cdot,y) = G_1(\cdot,y) - 2G_2(\cdot,y)\in L_2(0,a)$ follows. □

Proposition 12.4.8 ([180, Theorem 3.3.1]). *Let $v_1^2 > 0$ and consider the function F defined in* (12.4.7)*. Then the fundamental equation* (12.4.11) *has a unique continuous solution H on $D_0 = \{(x,y) \in \mathbb{R}^2 : 0 \le x \le y\}$. Furthermore, $q \in L_2(0,a)$, where*

$$q(x) := -2\frac{d}{dx}H(x,x), \quad x \in (0,a), \tag{12.4.24}$$

H and q are real valued, and H satisfies the integral equation

$$H(x,y) = \frac{1}{2}\int_{\frac{x+y}{2}}^{a+\frac{y-x}{2}} \int_{x+|y-\tau|}^{a-|a-\tau|} q(\sigma)H(\sigma,\tau)\,d\sigma\,d\tau + \frac{1}{2}\int_{\frac{x+y}{2}}^{2a} q(\sigma)\,d\sigma. \tag{12.4.25}$$

Proof. As we have seen in the proof of Lemma 12.4.4, we can take $\gamma = 0$ in (12.4.7). Recall that H satisfies the fundamental equation if and only if $(x,y) \mapsto H(x,x+y)$ satisfies (12.4.13). But from Proposition 12.4.7 we know that (12.4.13) has a unique solution G. Hence the fundamental equation has the unique continuous solution H given by $H(x,x+y) = G(x,y)$ for $(x,y) \in D_0$. Since F is real valued, Also H is real valued. In view of $H(x,x) = G(x,0)$, $q \in L_2(0,a)$ is a consequence of $G(\cdot,0) \in W_2^1(0,a)$, which was shown in Proposition 12.4.7.

To prove (12.4.25), we first consider the case that F is a twice continuously differentiable function on $(0,\infty)$ with support in $[0,2a]$ such that $I + \mathbb{F}_x^0$ has an inverse for all $x \in [0,a]$. We observe that $\mathbb{F}_x^0$ depends continuously on x and hence a compactness argument shows that the norm of $(I+\mathbb{F}_x^0)^{-1}$ is uniformly bounded for $x \in [0,a]$. In that case, the operator function $x \mapsto \mathbb{F}_x^0$ is twice differentiable and we can differentiate once more on both sides of (12.4.22). Adapting the proof of Proposition 12.4.7 to this case we see that G_x and G_{xx} exist and are continuous. It is also immediately clear from (12.4.13) that G_y, G_{xy} and G_{yy} exist and are continuous. Furthermore, for $x \in [0,a]$, $y \in [0,2a]$ and $h \in \mathbb{R}$ such that $x+h \in [0,a]$ and $y+2h \in [0,2a]$ we have

$$\begin{aligned}
&G(x,y+2h) - G(x,y)\\
&\quad = ((I+\mathbb{F}_x^0)^{-1}(F\circ\tau_{2x}))(y+2h) - ((I+\mathbb{F}_x^0)^{-1}(F\circ\tau_{2x}))(y)\\
&\quad = ((I+\mathbb{F}_x^0)^{-1}(F\circ\tau_{2(x+h)}))(y) - ((I+\mathbb{F}_x^0)^{-1}(F\circ\tau_{2x}))(y)\\
&\quad = ((I+\mathbb{F}_x^0)^{-1}[F\circ\tau_{2(x+h)} - F\circ\tau_{2x}])(y).
\end{aligned}$$

A reasoning as in the proof of Lemma 12.4.6 shows that

$$G_y(x,y) = \frac{1}{2}((I+\mathbb{F}_x^0)^{-1}(F'\circ\tau_{2x}))(y).$$

Arguing as in the proof of Proposition 12.4.7 and as above in this proof we see that also G_{yx} exists and is continuous. We have shown that G is twice continuously differentiable, and therefore also H is twice continuously differentiable.

Differentiating the fundamental equation (12.4.11) twice with respect to x and twice with respect to y we obtain

$$F''(x+y) + H_{xx}(x,y) - \frac{d}{dx}[H(x,x)F(x+y)] \\ - H_x(x,x)F(x+y) + \int_x^\infty H_{xx}(x,t)F(y+t)\,dt = 0$$

and

$$F''(x+y) + H_{yy}(x,y) + \int_x^\infty H(x,t)F''(y+t)\,dt = 0.$$

Integrating by parts twice we have

$$\int_x^\infty H(x,t)F''(y+t)\,dt = -H(x,x)F'(x+y) + H_y(x,x)F(x+y) \\ + \int_x^\infty H_{yy}(x,t)F(y+t)\,dt.$$

Taking now the difference of the two second-order partial derivatives we arrive at

$$H_{xx}(x,y) - H_{yy}(x,y) + q(x)F(x+y) \\ + \int_x^\infty [H_{xx}(x,t) - H_{yy}(x,t)]F(y+t)\,dt = 0.$$

With $y = x + \tilde{y}$ this equation becomes

$$H_{xx}(x,x+\tilde{y}) - H_{yy}(x,x+\tilde{y}) + q(x)F(2x+\tilde{y}) \\ + \int_0^\infty [H_{xx}(x,x+t) - H_{yy}(x,x+t)]F(2x+\tilde{y}+t)\,dt = 0. \quad (12.4.26)$$

Since the fundamental equation in the form (12.4.12) gives

$$q(x)(F \circ \tau_{2x}) = -q(x)(I + \mathbb{F}_x)H(x,x+\cdot) = -(I+\mathbb{F}_x)(q(x)H(x,x+\cdot)),$$

equation (12.4.26) can be written as

$$(I + \mathbb{F}_x)[H_{xx}(x,x+\cdot) - H_{yy}(x,x+\cdot)) - q(x)H(x,x+\cdot)] = 0.$$

Defining

$$\varphi(x,y) = H_{xx}(x,y) - H_{yy}(x,y) - q(x)H(x,y),$$

$\varphi(x,x+\cdot)$ is continuous for all $x \in [0,a]$ and satisfies

$$(I + \mathbb{F}_x)\varphi(x,x+\cdot) = 0.$$

In view of Lemma 12.4.4 we conclude that $\varphi = 0$.

Now put $\xi = 2a + x - y$ and $\eta = 2a - x - y$ and define

$$\tilde{u}(\xi, \eta) := H(x, y) = H\left(\frac{\xi - \eta}{2}, 2a - \frac{\xi + \eta}{2}\right). \tag{12.4.27}$$

We observe that $0 \le x \le y \le 2a - x$ gives the domain $0 \le \eta \le \xi \le 2a$ for ξ and η. As in the proof of Theorem 12.1.3 we obtain

$$\tilde{u}_{\xi\eta} = -\frac{1}{4}\left(H_{xx}\left(\frac{\xi - \eta}{2}, 2a - \frac{\xi + \eta}{2}\right) - H_{yy}\left(\frac{\xi - \eta}{2}, 2a - \frac{\xi + \eta}{2}\right)\right),$$

and $\varphi = 0$ leads to

$$\tilde{u}_{\xi\eta}(\xi, \eta) = -\frac{1}{4} q\left(\frac{\xi - \eta}{2}\right) \tilde{u}(\xi, \eta).$$

With

$$U(\xi, \eta) := \frac{1}{4} \int_{\xi}^{2a} \int_{0}^{\eta} q\left(\frac{\sigma - \tau}{2}\right) \tilde{u}(\sigma, \tau)\, d\tau\, d\sigma$$

we conclude that

$$\tilde{u}_{\xi\eta} - U_{\xi\eta} = 0.$$

Hence there are continuous functions f and g on $[0, 2a]$ such that

$$\tilde{u}(\xi, \eta) = U(\xi, \eta) + f(\xi) + g(\eta), \quad 0 \le \eta \le \xi \le 2a.$$

For definiteness, we may assume that $g(0) = 0$. For $\xi \in [0, 2a]$ we have

$$\tilde{u}(\xi, 0) = H\left(\frac{\xi}{2}, 2a - \frac{\xi}{2}\right) = 0 \quad \text{and} \quad U(\xi, 0) = 0,$$

which shows that $f = 0$. Similarly,

$$\tilde{u}(2a, \eta) = H\left(a - \frac{\eta}{2}, a - \frac{\eta}{2}\right) = \frac{1}{2} \int_{a - \frac{\eta}{2}}^{a} q(\sigma)\, d\sigma \quad \text{and} \quad U(2a, \eta) = 0,$$

which shows that

$$g(\eta) = \frac{1}{2} \int_{a - \frac{\eta}{2}}^{a} q(\sigma)\, d\sigma.$$

Altogether, we have shown that

$$\tilde{u}(\xi, \eta) = \frac{1}{4} \int_{\xi}^{2a} \int_{0}^{\eta} q\left(\frac{\sigma - \tau}{2}\right) \tilde{u}(\sigma, \tau)\, d\tau\, d\sigma + \frac{1}{2} \int_{a - \frac{\eta}{2}}^{a} q(\sigma)\, d\sigma. \tag{12.4.28}$$

Now let F be defined by (12.4.7). In $W_2^1(0, \infty)$ we can approximate the restriction of F to $[0, \infty)$ by a sequence $(F_n)_{n \in \mathbb{N}}$ of twice differentiable functions on $[0,\infty)$ with support in $[0,2a]$. Consider the corresponding operators $\mathbb{F}^0_{n,x}$, which converge as $n\to\infty$ and depend continuously on x. Then a compactness argument shows that we may assume without loss of generality that $I+\mathbb{F}^0_{n,x}$ is invertible for all $n\in\mathbb{N}$ and $x\in[0,a]$. Furthermore, $(I+\mathbb{F}^0_{n,x})^{-1}$ converges uniformly in x to $(I+\mathbb{F}^0_x)^{-1}$. Since also $F_n\to F$ uniformly as $n\to\infty$, a standard argument shows that $G_n\to G$ uniformly as $n\to\infty$, where the functions G_n are defined as

$$G_n(x,y) = -((I+\mathbb{F}^0_{n,x})^{-1}(F_n\circ\tau_x))(y).$$

Hence also $H_n\to H$ uniformly as $n\to\infty$ and the corresponding functions $\tilde u_n$ defined by (12.4.27) with H_n converge uniformly to $\tilde u$ as $n\to\infty$. Since the functions $\tilde u_n$ satisfy the integral equation (12.4.28), it follows that also $\tilde u$ satisfies (12.4.28). Substituting H for $\tilde u$ in (12.4.28) shows that H satisfies the integral equation (12.4.25). Here we have to observe that the region of integration is determined by

$$2a+x-y\le 2a+\sigma-\tau\le 2a \quad\text{and}\quad 0\le 2a-\sigma-\tau\le 2a-x-y,$$

which can be rewritten as

$$\frac{x+y}{2}\le\tau\le a+\frac{y-x}{2},\ \tau+x-y\le\sigma\le\tau,\quad\text{and}\quad x+y-\tau\le\sigma\le 2a-\tau. \qquad \square$$

Lemma 12.4.9. *Let $q\in L_2(0,a)$. Then the integral equation*

$$H(x,y) = \frac{1}{2}\int_{\frac{x+y}{2}}^{a+\frac{y-x}{2}}\int_{x+|y-\tau|}^{a-|a-\tau|} q(\sigma)H(\sigma,\tau)\,d\sigma\,d\tau + \frac{1}{2}\int_{\frac{x+y}{2}}^{2a} q(\sigma)\,d\sigma \tag{12.4.29}$$

has a unique solution H on $\{(x,y)\in\mathbb{R}^2 : 0\le x\le y\le 2a-x\}$.

Proof. Putting $\xi=2a+x-y$ and $\eta=2a-x-y$ and

$$\tilde u(\xi,\eta) := H(x,y) = H\left(\frac{\xi-\eta}{2}, 2a-\frac{\xi+\eta}{2}\right),$$

we have seen in the proof of Proposition 12.4.8 that the integral equation (12.4.29) becomes equivalent to the integral equation

$$\tilde u(\xi,\eta) = \frac{1}{4}\int_{\xi}^{2a}\int_0^{\eta} q\left(\frac{\sigma-\tau}{2}\right)\tilde u(\sigma,\tau)\,d\tau\,d\sigma + \frac{1}{2}\int_{a-\frac{\eta}{2}}^{a} q(\sigma)\,d\sigma$$

for $0\le\eta\le\xi\le 2a$. From the proof of Lemma 12.1.1 we see that its statement remains true for fixed ξ_0 and η_0. Thus the above integral equation can be written as $\tilde u = -\tilde T\tilde u + \tilde g$ with invertible operator $I+\tilde T$ on $C(\{(\xi,\eta)\in\mathbb{R}^2 : 0\le\eta\le\xi\le 2a\})$. Hence the integral equation for $\tilde u$ has a unique continuous solution, and it follows that (12.4.29) has a unique continuous solution H. $\square$

12.5 Two spectra and the fundamental equation

Lemma 12.5.1. *The spectrum of the Sturm–Liouville problem*

$$-y'' + q(x)y = \lambda y, \quad 0 \le x \le a, \tag{12.5.1}$$

$$y(0) = 0, \quad \cos\beta y(a) - \sin\beta y'(a) = 0, \tag{12.5.2}$$

with a real potential $q \in L_2(0,a)$ and $\beta \in [0,\pi]$ consists of an increasing sequence of simple real eigenvalues $(\lambda_k(\beta))_{k=1}^{\infty}$ which tend to ∞. For $0 < \beta' < \beta \le \pi$, the eigenvalues interlace as follows:

$$\lambda_1(\beta') < \lambda_1(\beta) < \lambda_2(\beta') < \lambda_2(\beta) < \cdots .$$

Proof. The result is well known. Indeed, it is easy to see from Theorems 10.3.5, 10.3.8 and the spectral theorem for compact operators that the spectrum consists of real eigenvalues which are bounded below and tend to ∞. Since the initial value problem (12.5.1), $y(0) = 0$ has a one-dimensional solution space, it follows that all eigenvalues are simple and that $\lambda_k(\beta) \ne \lambda_j(\beta')$ for all $k, j \in \mathbb{N}$ and $0 < \beta' < \beta \le \pi$. From [143, Theorem 4.2, (4.5)] we know that $\lambda_k(\beta)$ depends continuously on β and is strictly increasing as a function of β for all k, and the stated interlacing property of the eigenvalues follows. □

Corollary 12.5.2. *Let $q \in L_2(0,a)$ be real valued. Then the spectra of the Sturm–Liouville problem* (12.5.1) *subject to the boundary conditions $y(0) = y(a) = 0$ and $y(0) = y'(a) = 0$, respectively, consist of two sequences of real eigenvalues which interlace as follows:*

$$\zeta_1 < \xi_1 < \zeta_2 < \xi_2 < \cdots$$

and obey the asymptotic formulae

$$\xi_k = \frac{\pi^2 k^2}{a^2} - 2\frac{\pi^2 A}{a^2} + \alpha_k, \quad \zeta_k = \frac{\pi^2}{a^2}\left(k - \frac{1}{2}\right)^2 - 2\frac{\pi^2 A}{a^2} + \beta_k, \tag{12.5.3}$$

where $A \in \mathbb{R}$, $(\alpha_k)_{k=1}^{\infty} \in l_2$ and $(\beta_k)_{k=1}^{\infty} \in l_2$ are real-valued sequences.

Proof. The first part is an immediate consequence of Lemma 12.5.1 with $\lambda_k(\pi) = \xi_k$ and $\lambda_k(\frac{\pi}{2}) = \zeta_k$. For the asymptotic expansion of the eigenvalues we recall from Corollary 12.2.10 that

$$s(\lambda, a) = \frac{\sin\lambda a}{\lambda} - K(a,a)\frac{\cos\lambda a}{\lambda^2} + \frac{f(\lambda)}{\lambda^2},$$

$$s'(\lambda, a) = \cos\lambda a + K(a,a)\frac{\sin\lambda a}{\lambda} + \frac{g(\lambda)}{\lambda}$$

with $f, g \in \mathcal{L}^a$. An application of Lemma 12.3.3 and (12.2.19) proves (12.5.3) with

$$A = -\frac{a}{\pi^2}K(a,a) = -\frac{a}{2\pi^2}\int_0^a q(x)\,dx. \tag{12.5.4}$$

□

Let $q \in L_2(0,a)$ and define $q_a(x) = q(a-x)$. Clearly, $q_a \in L_2(0,a)$.

Proposition 12.5.3. *Let e_a be the solution of* (12.2.1), (12.2.2) *with respect to q_a. For $\lambda \in \mathbb{C}$ define*

$$e(\lambda, x) = \begin{cases} e^{-i\lambda a} e_a(-\lambda, a-x) & \text{if } 0 \le x \le a, \\ e^{-i\lambda x} & \text{if } a < x. \end{cases} \tag{12.5.5}$$

Then the function e is the Jost solution of (12.2.1) *as defined in Section* 2.1.

Proof. Clearly, $e(\lambda, \cdot)$ satisfies the differential equation (12.2.1) on $(0,a)$ and

$$\begin{aligned} e(\lambda, a) &= e^{-i\lambda a} e_a(-\lambda, 0) &&= e^{-i\lambda a}, \\ e'(\lambda, a) &= -e^{-i\lambda a} e_a'(-\lambda, 0) &&= -i\lambda e^{-i\lambda a} \end{aligned}$$

shows that e is indeed the Jost solution. □

Proposition 12.5.4. *Let K_a be the function $\tilde{K}$ from Theorem* 12.2.6 *with respect to the potential q_a and define*

$$K_\infty(x,t) = \begin{cases} K_a(a-x, a-t) & \text{if } 0 \le x \le t \le 2a - x, \\ 0 & \text{for all other } x, t \in \mathbb{R}. \end{cases} \tag{12.5.6}$$

Then K_∞ satisfies the integral equation (12.4.29),

$$e(\lambda, x) = e^{-i\lambda x} + \int_x^\infty K_\infty(x,t) e^{-i\lambda t}\, dt, \quad \lambda \in \mathbb{C},\ x \ge 0, \tag{12.5.7}$$

and

$$K_\infty(x,x) = \frac{1}{2}\int_x^a q(t)\, dt, \quad x \in [0,a]. \tag{12.5.8}$$

Proof. With the aid of (12.2.18) we calculate

$$\begin{aligned} K_\infty(x,y) = K_a(a-x, a-y) &= \frac{1}{2}\int_0^{a-\frac{x+y}{2}} q(a-s)\, ds \\ &+ \int_0^{a-\frac{x+t}{2}} \int_0^{\frac{y-x}{2}} q(a-s-p) K_\infty(a-s-p, a-s+p)\, dp\, ds \\ &= \frac{1}{2}\int_{\frac{x+y}{2}}^{a+\frac{y-x}{2}} \int_{x+|y-\tau|}^{a-|a-\tau|} q(\sigma) K_\infty(\sigma, \tau)\, d\sigma\, d\tau + \frac{1}{2}\int_{\frac{x+y}{2}}^{2a} q(\sigma)\, d\sigma. \end{aligned}$$

From Proposition 12.5.3, Theorem 12.2.6 and with the aid of the transformation $\tau = a - t$ we infer for $0 \le x \le a$ that

$$\begin{aligned} e(\lambda, x) &= e^{-i\lambda a} e_a(-\lambda, a - x) \\ &= e^{-i\lambda a} e^{i\lambda(a-x)} + \int_{x-a}^{a-x} K_a(a - x, t) e^{i\lambda t} e^{-i\lambda a}\, dt \\ &= e^{-i\lambda x} + \int_x^{2a-x} K_a(a - x, a - t) e^{-i\lambda t}\, dt, \end{aligned}$$

and (12.5.7) follows in view of (12.5.6). For $x > a$, (12.5.7) is obvious. Finally, we conclude from (12.2.19) that

$$\begin{aligned} K_\infty(x, x) = K_a(a - x, a - x) &= \frac{1}{2} \int_0^{a-x} q_a(\sigma)\, d\sigma \\ &= \frac{1}{2} \int_x^a q_a(a - \tau)\, d\tau = \frac{1}{2} \int_x^a q(t)\, dt. \end{aligned}$$ □

The function

$$v_a(\lambda, \cdot) := \frac{e(\lambda, \cdot)e^{i\lambda a} + e(-\lambda, \cdot)e^{-i\lambda a}}{2}$$

is the solution of (12.2.1) satisfying $v_a(\lambda, a) = 1$ and $v_a'(\lambda, a) = 0$, whereas the function

$$u_a(\lambda, \cdot) := \frac{e(-\lambda, \cdot)e^{-i\lambda a} - e(\lambda, \cdot)e^{i\lambda a}}{2i\lambda}$$

is the solution of (12.2.1) satisfying $u_a(\lambda, a) = 0$ and $u_a'(\lambda, a) = 1$. Observe that $e(\lambda, \cdot) = e^{-i\lambda a}(v_a(\lambda, \cdot) - i\lambda u_a(\lambda, \cdot))$. Since $x \mapsto q(a - x)$ belongs to $L_2(0, a)$, v_a and u_a have a representation like c and $-s$ in Theorem 12.2.9, respectively, with x replaced by $a - x$.

Clearly, the zeros of $u_a(\cdot, 0)$ are the eigenvalues of (12.5.1) with the boundary conditions $y(0) = y(a) = 0$, whereas the zeros of $v_a(\cdot, 0)$ are the eigenvalues of (12.2.1) with the boundary conditions $y(0) = y'(a) = 0$. With ζ_k and ξ_k according to Corollary 12.5.2 we put $v_k = \zeta_k^{\frac{1}{2}}$, $u_k = \xi_k^{\frac{1}{2}}$ for $k \in \mathbb{Z}$. Then it follows from Corollary 12.5.2 that these numbers satisfy the assumptions posed at the beginning of Section 12.4. Hence the entire function u defined there has exactly the same zeros as $u_a(\cdot, 0)$, and the entire function v defined there has exactly the same zeros as $v_a(\cdot, 0)$. The function v has the representation (12.3.11), and by the above discussion, also $v_a(\cdot, 0)$ has such a representation, with the same leading term as v. By Corollary 12.2.11 and its proof, both v and $v_a(\cdot, 0)$ are sine type functions. Hence they are multiples of each other by Lemma 11.2.29, and since they have the same leading terms, $v_a(\cdot, 0) = v$ follows. A corresponding argument for the function $\lambda \mapsto \lambda u(\lambda)$ and $\lambda \mapsto -\lambda u_a(\lambda, 0)$ gives $u_a(\cdot, 0) = -u$.

Hence the function ψ defined in (12.4.2) satisfies

$$\psi(\lambda) = e^{-i\lambda a}(v_a(\lambda, 0) - i\lambda u_a(\lambda, 0)) = e(\lambda, 0), \tag{12.5.9}$$

and it follows that the function S defined in (12.4.3) has the representation

$$S(\lambda) = \frac{e(\lambda, 0)}{e(-\lambda, 0)}, \quad \lambda \in \mathbb{R}. \tag{12.5.10}$$

Lemma 12.5.5 ([180, Lemma 3.1.5])**.** *For $\lambda \neq 0$ and $e(-\lambda, 0) \neq 0$,*

$$-\frac{2i\lambda s(\lambda, \cdot)}{e(-\lambda, 0)} = e(\lambda, \cdot) - S(\lambda)e(-\lambda, \cdot).$$

Proof. The function $s(\lambda, \cdot)$ is the solution of (12.2.1) which satisfies the initial conditions $s(\lambda, 0) = 0$, $s'(\lambda, 0) = 1$. On the other hand,

$$\omega(\lambda, \cdot) := e(-\lambda, 0)e(\lambda, \cdot) - e(\lambda, 0)e(-\lambda, \cdot)$$

is a solution of (12.2.1) with $\omega(\lambda, 0) = 0$. It is well known that the Wronskian

$$W(e(-\lambda, \cdot), e(\lambda, \cdot)) = e(-\lambda, \cdot)e'(\lambda, \cdot) - e'(-\lambda, \cdot)e(\lambda, \cdot)$$

is constant, and its value at a is $-2i\lambda$, so that $\omega'(\lambda, 0) = -2i\lambda$. Hence we have shown that $\omega(\lambda, \cdot) = -2i\lambda s(\lambda, \cdot)$. □

The proof of the following lemma is extracted from [180, pp. 204–206].

Lemma 12.5.6. *Let $\gamma \geq 0$ such that S is analytic on $\{\lambda \in \mathbb{C} : \operatorname{Im} \lambda \geq \gamma\}$ and define F by*

$$F(x) = \frac{e^{-\gamma x}}{2\pi} \int_{-\infty}^{\infty} S_\gamma(\lambda)e^{i\lambda x}\, d\lambda, \quad x \in \mathbb{R}, \tag{12.5.11}$$

where S_γ is defined by (12.4.6)*. Then the fundamental equation*

$$F(x + y) + K_\infty(x, y) + \int_x^\infty K_\infty(x, t)F(y + t)\, dt = 0, \quad 0 \leq x \leq y, \tag{12.5.12}$$

is satisfied.

Proof. In view of Lemma 12.5.5 and (12.5.7) we have

$$-\frac{2i\lambda s(\lambda, x)}{e(-\lambda, 0)} = e^{-i\lambda x} - e^{i\lambda x} + \int_x^\infty K_\infty(x, t)e^{-i\lambda t}\, dt - \int_x^\infty K_\infty(x, t)e^{i\lambda t}\, dt$$
$$+ (1 - S(\lambda))\left(e^{i\lambda x} + \int_x^\infty K_\infty(x, t)e^{i\lambda t}\, dt\right),$$

which can be rewritten as

$$
\begin{aligned}
-2i\lambda s(\lambda, x)\left(\frac{1}{e(-\lambda,0)} - 1\right) &+ 2i(\sin\lambda x - \lambda s(\lambda, x)) \\
&= \int_x^\infty K_\infty(x,t)e^{-i\lambda t}\,dt - \int_{-\infty}^{-x} K_\infty(x,-t)e^{-i\lambda t}\,dt \\
&\quad + (1 - S(\lambda))\left(e^{i\lambda x} + \int_x^\infty K_\infty(x,t)e^{i\lambda t}\,dt\right). \qquad (12.5.13)
\end{aligned}
$$

We multiply both sides of (12.5.13) by $\frac{1}{2\pi}e^{i\lambda y}$, $y \in \mathbb{R}$, and integrate along $\operatorname{Im}\lambda = \gamma$, resulting in an identity which we formally write as $I_l(x, y) = I_r(x, y)$. Then $I_r(x, y)$ is the sum of the 4 integrals

$$
\begin{aligned}
I_1(x,y) &= \frac{1}{2\pi}\int_{-\infty}^\infty \int_x^\infty K_\infty(x,t)e^{-i(i\gamma+\lambda)t}\,dt\,e^{i(i\gamma+\lambda)y}\,d\lambda, \\
I_2(x,y) &= -\frac{1}{2\pi}\int_{-\infty}^\infty \int_{-\infty}^{-x} K_\infty(x,-t)e^{-i(i\gamma+\lambda)t}\,dt\,e^{i(i\gamma+\lambda)y}\,d\lambda, \\
I_3(x,y) &= \frac{1}{2\pi}\int_{-\infty}^\infty S_\gamma(\lambda)e^{i(i\gamma+\lambda)x}e^{i(i\gamma+\lambda)y}\,d\lambda, \\
I_4(x,y) &= \frac{1}{2\pi}\int_{-\infty}^\infty S_\gamma(\lambda)\int_x^\infty K_\infty(x,t)e^{i(i\gamma+\lambda)t}e^{i(i\gamma+\lambda)y}\,dt\,d\lambda.
\end{aligned}
$$

The Fourier inversion formula gives

$$
I_1 = \frac{1}{2\pi}\int_{-\infty}^\infty \int_x^\infty e^{\gamma(t-y)}K_\infty(x,t)e^{-i\lambda t}\,dt\,e^{i\lambda y}\,d\lambda = K_\infty(x,y),
$$

and similarly

$$
I_2 = -K_\infty(x,-y) = 0 \quad \text{for } y > x.
$$

We further calculate

$$
I_3 = \frac{e^{-\gamma(x+y)}}{2\pi}\int_{-\infty}^\infty S_\gamma(\lambda)e^{i\lambda(x+y)}\,d\lambda = F(x+y)
$$

and

$$
\begin{aligned}
I_4(x,y) &= \frac{1}{2\pi}\int_x^\infty K_\infty(x,t)e^{-\gamma(y+t)}\int_{-\infty}^\infty S_\gamma(\lambda)e^{i\lambda(y+t)}\,d\lambda\,dt \\
&= \int_x^\infty K_\infty(x,t)F(y+t)\,dt.
\end{aligned}
$$

Hence we obtain

$$I_r(x,y) = K_\infty(x,y) + F(x+y) + \int_x^\infty K_\infty(x,t)F(y+t)\,dt, \quad 0 \le x < y. \tag{12.5.14}$$

To prove the fundamental equation, it remains to prove that $I_l(x,y) = 0$ for $0 \le x \le a$ and $y > x$.

Therefore, let $x \in [0,a]$ and $y > x$. We define the functions g_1 and g_2 by

$$g_1(\lambda) = \lambda s(\lambda,x)\left(\frac{1}{e(-\lambda,0)} - 1\right)e^{i\lambda y}, \; g_2(\lambda) = (\sin\lambda x - \lambda s(\lambda,x))e^{i\lambda y}. \tag{12.5.15}$$

The function g_2 is an entire function and the function g_1 is analytic on the set $\{\lambda \in \mathbb{C} : \operatorname{Im}\lambda \ge \gamma\}$ since the poles of S are the zeros of $\lambda \mapsto e(-\lambda,0)$. By Lemma 12.2.8 applied to K_a we conclude that $\frac{\partial}{\partial t}K_\infty(0,\cdot) \in L_2(-a,a)$. Therefore we have in view of (12.5.7) that

$$\begin{aligned}
e(-\lambda,0) - 1 &= \int_0^{2a} K_\infty(0,t)e^{i\lambda t}\,dt \\
&= \frac{1}{i\lambda}\left[K_\infty(0,2a)e^{2i\lambda a} - K_a(0,0)\right] - \frac{1}{i\lambda}\int_0^{2a}\frac{\partial}{\partial t}K_\infty(0,t)e^{i\lambda t}\,dt \\
&= O\left(\lambda^{-1}\right) \;\text{ for}\operatorname{Im}\lambda \ge 0.
\end{aligned}$$

Hence it follows that

$$\frac{1}{e(-\lambda,0)} - 1 = \frac{1}{1+O(\lambda^{-1})} - 1 = O(\lambda^{-1}) \text{ for}\operatorname{Im}\lambda \ge 0. \tag{12.5.16}$$

From (12.2.16) we conclude that $\lambda \mapsto \lambda s(\lambda,x)$ is bounded on each horizontal line and that $g_2(\lambda) = O(\lambda^{-1})$ for λ on any horizontal line. Together with (12.5.16) we conclude that g_1 and g_2 are square integrable on the line $\operatorname{Im}\lambda = \gamma$.

Up to a constant factor, $I_l(x,y)$ is the difference of the integrals $I_5(x,y)$ and $I_6(x,y)$ given by

$$I_{4+j}(x,y) = \int_{-\infty}^{\infty} g_j(\lambda)\,d\lambda, \; j = 1,2.$$

Clearly, Corollary 12.2.11 holds with a there replaced by any $x > 0$, and we can conclude in view of Lemma 11.2.6 and (12.5.16) that there are constants $M_1 > 0$ and $M_2 > 0$ such that

$$|g_j(i\gamma + Re^{i\theta})| \le M_j R^{-1} e^{-R(y-x)\sin\theta}, \quad R > 0, \; 0 \le \theta \le \pi, \; j = 1,2.$$

Hence it follows from Cauchy's theorem and Lebesgue's dominated convergence theorem that

$$I_j = -i\lim_{R\to\infty}\int_0^\pi g_j(i\gamma + Re^{i\theta})Re^{i\theta}\,d\theta = 0, \quad j = 5,6. \qquad \square$$

12.6 The potential and two spectra

The following result is well known, but for convenience we will present its proof.

Lemma 12.6.1. *Let h be a bounded measurable function on $[0,2a]\times[0,2a]$ and define*

$$(\mathbb{H}f)(x) = \int_x^{2a} h(x,t)f(t)\,dt, \quad f\in L_2(0,2a),\ 0\le x\le 2a. \tag{12.6.1}$$

Then the operator $\mathbb{H}$ is a Volterra operator on $L_2(0,2a)$, i. e., $\mathbb{H}$ is compact and its spectral radius is 0.

Proof. With

$$M = \sup\{|h(x,t)| : 0\le x,t\le 2a\},$$

the estimate

$$\int_0^{2a}\int_0^{2a} |h(x,t)|^2\,dt\,dx \le 4aM < \infty$$

shows that $\mathbb{H}$ is an integral operator with L_2 kernel and therefore compact, see, e. g., [109, p. 240].

Since the spectral radius of the adjoint $\mathbb{H}^*$ equals the spectral radius of $\mathbb{H}$, it suffices to show that the spectral radius of $\mathbb{H}^*$ is 0. For $f,g\in L_2(0,2a)$ we calculate

$$\begin{aligned}(\mathbb{H}^*f,g) = (f,\mathbb{H}g) &= \int_0^{2a} f(x)\int_x^{2a}\overline{h(x,t)}\,\overline{g(t)}\,dt\,dx\\ &= \int_0^{2a}\int_0^t \overline{h(x,t)}f(x)\,dx\,\overline{g(t)}\,dt,\end{aligned}$$

which shows that the adjoint $\mathbb{H}^*$ of $\mathbb{H}$ has the representation

$$(\mathbb{H}^*f)(t) = \int_0^t \overline{h(x,t)}f(x)\,dx, \quad f\in L_2(0,2a),\ 0\le t\le 2a.$$

Let $m>0$ and define the norm $\|\cdot\|_m$ on $L_2(0,2a)$ by

$$\|f\|_m^2 = \int_0^{2a} |f(x)|^2 e^{-2mx}\,dx, \quad f\in L_2(0,2a),$$

which is clearly equivalent to the standard L_2-norm. Then we obtain for each $f\in L_2(0,2a)$ and $0\le t\le 2a$ that

$$\begin{aligned}|(\mathbb{H}^*f)(t)|^2e^{-2mt} &= \left|\int_0^t \overline{h(x,t)}e^{-m(t-x)}f(x)e^{-mx}\,dx\right|^2\\ &\le \int_0^t |h(x,t)|^2e^{-2m(t-x)}\,dx\int_0^t |f(x)|^2e^{-2mx}\,dx\\ &\le \frac{M^2}{2m}\|f\|_m^2,\end{aligned}$$

which gives

$$\|\mathbb{H}^* f\|_m^2 \leq \frac{aM^2}{m}\|f\|_m^2.$$

Since the spectral radius of a bounded operator is bounded by the norm of the operator, it follows that the spectral radius of $\mathbb{H}^*$ is less or equal to $M\sqrt{am^{-1}}$. But $m > 0$ was arbitrary, and it follows that the spectral radius of $\mathbb{H}^*$ is 0. □

Theorem 12.6.2 ([180, Theorem 3.4.1, p. 248]). *For two sequences $(\xi_k)_{k=1}^\infty$ and $(\zeta_k)_{k=1}^\infty$ of real numbers to be the spectra of the boundary value problems generated by the Sturm–Liouville equation*

$$-y'' + q(x)y = \lambda y \text{ on } [0, a], \tag{12.6.2}$$

with a real potential $q \in L_2(0, a)$ and the boundary conditions $y(0) = y(a) = 0$ and $y(0) = y'(a) = 0$, respectively, it is necessary and sufficient that the sequences interlace:

$$\zeta_1 < \xi_1 < \zeta_2 < \xi_2 < \cdots$$

and obey the asymptotic formulae

$$\xi_k = \frac{\pi^2 k^2}{a^2} - 2\frac{\pi^2 A}{a^2} + \alpha_k, \quad \zeta_k = \frac{\pi^2}{a^2}\left(k - \frac{1}{2}\right)^2 - 2\frac{\pi^2 A}{a^2} + \beta_k,$$

where $A \in \mathbb{R}$, $(\alpha_k)_{k=1}^\infty \in l_2$ and $(\beta_k)_{k=1}^\infty \in l_2$. The potential q is uniquely determined by the sequences $(\xi_k)_{k=1}^\infty$ and $(\zeta_k)_{k=1}^\infty$.

Proof. The necessity of the interlacing property and the asymptotic distribution of the eigenvalues was shown in Corollary 12.5.2.

Next we are going to show that the potential is uniquely determined by the two spectra. Let $q \in L_2(a, b)$ be real valued and let $(\xi_k)_{k=1}^\infty$ and $(\zeta_k)_{k=1}^\infty$ be the corresponding spectra. Without loss of generality we may assume that $\zeta_1 > 0$, which can be achieved by a shift of the eigenvalue parameter λ in (12.6.2). Putting $u_k = (\xi_k)^{\frac{1}{2}}$ and $v_k = (\zeta_k)^{\frac{1}{2}}$ for $k \in \mathbb{N}$, we consider the two functions S and F defined by (12.4.3) and (12.4.7). It remains to show that q is uniquely determined by F. Indeed, in view of (12.5.8), q is uniquely determined by K_∞. From Lemma 12.5.6 we know that the function K_∞ associated with the potential q satisfies the fundamental equation (12.5.12), which is the same as (12.4.11). Then it follows from Proposition 12.4.8 that K_∞ is uniquely determined by F. Altogether, we have shown that the potential is uniquely determined by the two sequences.

Now let two sequences $(\xi_k)_{k=1}^\infty$ and $(\zeta_k)_{k=1}^\infty$ with the required properties be given. Shifting all elements in these two sequences by the same real number $b_0 > -\zeta_1$ if $\zeta_1 \leq 0$, we obtain that $\zeta_1 > 0$. With the two sequences $(\xi_k)_{k=1}^\infty$ and $(\zeta_k)_{k=1}^\infty$ we associate the functions S and F defined by (12.4.3) and (12.4.7). By Proposition 12.4.8, there is a unique solution H of the fundamental equation and a potential q defined by (12.4.24). With this q we associate the two sequences $(\tilde{\xi}_k)_{k=1}^\infty$ and

$(\tilde{\zeta}_k)_{k=1}^\infty$ representing the Dirichlet spectrum and the Dirichlet–Neumann spectrum, respectively, of (12.6.2). With these two sequences we can now associate functions $\tilde{S}$ and $\tilde{F}$ defined by (12.4.3) and (12.4.7). Let K_∞ be the function defined in (12.5.6) with respect to q. By Propositions 12.4.8 and 12.5.4, both H and K_∞ satisfy (12.4.29) with the same q. But since the solution of the integral equation (12.4.29) is unique by Proposition 12.4.8, $K_\infty = H$ follows. Let

$$(\mathbb{H}f)(x) = \int_x^{2a} H(x,t)f(t)\,dt, \quad f \in L_2(0,2a),\ x \in [0,2a].$$

Since $I + \mathbb{H}$ is invertible by Lemma 12.6.1, we obtain from the fundamental equations (12.4.11) and (12.5.12) for $y \in [0,2a]$ that

$$F \circ \tau_y = -(I+\mathbb{H})^{-1}H(\cdot,y) = -(I+\mathbb{H})^{-1}K_\infty(\cdot,y) = \tilde{F} \circ \tau_y.$$

Hence $F = \tilde{F}$, and since the definition of F in (12.4.7) is independent of $\gamma \geq 0$, we can take the same γ in F and $\tilde{F}$. But S_γ is the Fourier transform of $x \mapsto e^{\gamma x}F(x)$, which show that S_γ and therefore S is uniquely determined by F. Thus we have that $S = \tilde{S}$.

Next we will show that the sequences $(\xi_k)_{k=1}^\infty$ and $(\zeta)_{k=1}^\infty$ are uniquely determined by S. Indeed, it follows from (12.4.1), (12.4.2), (12.4.3) and the proof of Lemma 12.4.1 that

$$\frac{P(\lambda^2) + i\lambda Q(\lambda^2)}{P(\lambda^2) - i\lambda Q(\lambda^2)} = S(\lambda)e^{-2i\lambda a}.$$

The sequences $(\xi_k)_{k=1}^\infty$ and $(\zeta_k)_{k=1}^\infty$ interlace and are the zeros of the entire functions P and Q, respectively, with a corresponding result for the sequences $(\tilde{\xi}_k)_{k=1}^\infty$ and $(\tilde{\zeta}_k)_{k=1}^\infty$ and entire functions $\tilde{Q}$ and $\tilde{P}$. Hence it follows for $\lambda \neq 0$ that $Q(\lambda^2) = 0$ if and only if $S(\lambda)e^{-2i\lambda a} = 1$, that $P(\lambda^2) = 0$ if and only if $S(\lambda)e^{-2i\lambda a} = -1$, that $\tilde{Q}(\lambda^2) = 0$ if and only if $S(\lambda)e^{-2i\lambda a} = 1$, and that $P(\lambda^2) = 0$ if and only if $S(\lambda)e^{-2i\lambda a} = -1$. Since $\zeta_1 > 0$, it follows that the two sequences $(\xi_k)_{k=1}^\infty$ and $(\zeta_k)_{k=1}^\infty$ are indeed uniquely determined by S. Furthermore, the nonzero zeros of $\tilde{P}$ and $\tilde{Q}$ coincide with the nonzero zeros of P and Q, respectively. Hence $0 < \zeta_1 < \xi_1$ and $0 \leq \tilde{\zeta}_1 < \tilde{\xi}_1$, so that $\tilde{\xi}_1 = \xi_1$ since they are positive and the smallest zeros of $\tilde{Q}$ and Q, respectively. Now ζ_1 is a positive zero of P, and therefore also a positive zero of $\tilde{P}$, and $\tilde{\zeta}_1 = \zeta_1$ follows. Hence also the two sequences $(\tilde{\xi}_k)_{k=1}^\infty$ and $(\tilde{\zeta}_k)_{k=1}^\infty$ are uniquely determined by S. We have shown that the two spectra of the differential equation (12.6.2) with the q given by Proposition 12.4.8, after a possible backshift by b_0, are indeed the two sequences $(\xi_k)_{k=1}^\infty$ and $(\zeta_k)_{k=1}^\infty$. □