ESTIMATION OF A MULTIPLICATIVE MODEL FOR LOCATING LARGE RETAIL STORES

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Abstract

A Multiplicative Competitive Interaction Model may be used for predicting the sales volumes for new locations of large scale stores or shopping centers. The paper reviews the least squares technique for evaluating the parameters of the model. Teekens' approach is applied to obtain the maximum likelihood estimators for the multiplicative competitive interaction model. The minimum variance unbiased estimators and limitations of the various techniques are discussed.

Introduction

The problem of forecasting sales volumes for new locations of large scale stores or shopping centers has been addressed by several researchers. These can be classified into the Analog Models and the Gravitational Approaches. Nakanishi and Cooper (1974) have proposed a method for estimating the parameters of a general form of a Multiplicative Competitive Interaction Model (MCI) using the least squares approach. We use Teekens' (1972) technique for multiplicative models to develop the maximum likelihood estimators for the Nakanishi and Cooper model.

The Generalized Least Squares Estimators Of The Gravitational Model

Masao Nakanishi and Lee G. Cooper (1974) have proposed a method for estimating the parameters of Huff's model and in general for the parameters of a general form of a Multiplicative Competitive Interaction Model (MCI Model) using the least squares approach. The general MCI Model is:

$$p_{ij} = \frac{\left(\prod_{k=1}^{q} x_{kij}^{b_k}\right) t^*_{ij}}{\sum_{j=1}^{m} \left(\prod_{k=1}^{q} x_{kij}^{b_k}\right) t^*_{ij}} \quad \text{---(1)}$$

where t^*_{ij} = the specification error term such that $\log t^*_{ij}$ is independently and normally distributed with mean 0 and variance σ_t. Hence, t^*_{ij} will have a lognormal distribution.

p_{ij} = the probability that a consumer in choice situation i will choose object j.

x_{kij} = the k^{th} variable describing object j in choice situation i. b_k = the parameter for the sensitivity of p_{ij} with respect to variable k.

To develop the least squares estimators it is necessary to obtain linear transform of the original multiplicative model. Nakanishi and Cooper succeed in obtaining the linear transformation, which is the major contribution of their paper. After taking the logarithm of equation (1), simplifying and rearranging the equation they obtain the following:

$$\log\left(\frac{p_{ij}}{\tilde{p}_i}\right) = \sum_{k=1}^{q} b_k \log\left(\frac{x_{kij}}{\tilde{x}_{ki}}\right) + \log\left(\frac{t^*_{ij}}{\tilde{t}^*_i}\right) \quad \text{---(2)}$$

where $\tilde{p}_i$, $\tilde{t}^*_i$ and $\tilde{x}_{ki}$ are the geometric means of p_{ij}, t^*_{ij} and x_{kij}. The p_{ij}'s are true probabilities and cannot be observed and instead we observe proportions p'_{ij}.

Let $y_{ij} = \log\left(\frac{p'_{ij}}{\tilde{p}'_i}\right)$ and $z_{kij} = \log\left(\frac{x_{kij}}{\tilde{x}_{ki}}\right)$. Then, using equation (2) we can write:

$$y_{ij} = \sum_{k=1}^{q} b_k z_{kij} + c_{ij} \quad \text{---(3)}$$

where i = 1, 2, ...M and j = 1, 2, 3, ...m and C_{ij} is the stochastic disturbance term which consists of two parts: (i) the specification errors prepresented by t^*_{ij} and (ii) sampling errors (due to the use of proportions p'_{ij} instead of the actual probabilities p_{ij}) represented by r_{ij}.

i.e., $c_{ij} = y_{ij} - \sum_{k=1}^{q} b_k z_{kij}$

$$= y_{ij} - \log\left(\frac{p_{ij}}{\tilde{p}_i}\right) + \left\{\log\left(\frac{p_{ij}}{\tilde{p}_i}\right) - \sum_{k=1}^{q} b_k z_{kij}\right\}$$

$= r_{ij} + t_{ij}$ Having obtained the linear transform of the MCI Model, Nakanishi and Cooper then develop expressions for the GLS estimators for the following three cases: (i) When there is no sampling error (for very large samples) i.e., $r_{ij} = 0$, (ii) When there is no specification error, i.e., $t_{ij} = 0$, and (iii) When both sampling as well as specification errors are present and hence $c_{ij} = r_{ij} + t_{ij}$.

Teekens' Approach For Multiplicative Models

R. Teekens (1972) in his book, "Prediction Methods In Multiplicative Models" has indicated procedures for obtaining GLS, Maximum Likelihood and Minimum Variance Unbiased (MVU) estimators for multiplicative models of the form:

$$y_i = \prod_{k=1}^{K} \{f_{ki}(x_{ki})\}^{\delta_k}(v_i) = \prod_{k=1}^{K} (z_{ki})^{\delta_k}(v_i)$$

where v_i are the stochastic error terms, z_{ki} are the non-stochastic variables and δ_k are unknown parameters to be estimated for the model. To apply Teekens' procedure to the MCI model we have to use another modified transformation technique of the type used by Nakanishi and Cooper to obtain the linear form. The application of Teekens' results have yet another obstacle for our MCI model: consider the log transform of the MCI model as given in equation (2).

Teekens assumes the specification of the original error term v_i of his model (equivalent to our t_{ij} in the MCI model) and assumes v_i to have a lognormal distribution with mean = $E(v_i) = 1$, variance $(v_i) = w^2$ (a bounded variance); and v_i are distributed independently and identically. Hence, covariance $(v_i, v_j) = 0$ for $i \neq j$.

This leads to the result that $\log v_i$ has a normal distribution with mean = $-\frac{1}{2}\sigma^2$ and variance σ^2.

i.e., $\log v_i \sim N\{-\frac{1}{2}\sigma^2, \sigma^2\}$. This is different from Nakanishi and Cooper's error term specification where the properties of the transformed error term are specified and we have $\log t^*_{ij}$ distributed independently and normally with mean = 0 and variance = σ_t^2.

The error specification is the key decision in obtaining the parameter estimators. In examining the original equation (1) of our MCI model and the transformed equation (2), we recognize that the distribution of $\log p_{ij}$ is not necessarily normal since, although

$\log t^*_{ij}$ is normal, we have the term

$$\log\left\{\sum_{j=1}^{m}\left(\prod_{k=1}^{q} x_{kij}^{b_k}\right) t^*_{ij}\right\}$$ containing t^*_{ij}

summed over j and hence not necessarily normally distributed. Thus the distribution of $\log p_{ij}$ is not readily known. This does not allow us to use Teekens' approach for obtaining the Maximum Likelihood or MVU estimators.

The Maximum Likelihood Estimators

In order to develop the Maximum Likelihood (ML) estimators we must first examine the error distribution. We shall consider the MCI model given in equation (1) with the same notation used earlier, and $t^*_{ij} \sim N(0,\sigma_t^2)$ as before. Using equation (2) let $t_{ij} = \log\left(\frac{t^*_{ij}}{\tilde{t}_{ij}}\right)$, summing over j

and dividing by m we obtain: $\frac{1}{m}\sum_{j=1}^{m} t_{ij} = 0$

This implies that $\sum_{j=1}^{m} t_{ij} = 0$ and hence the $t_{ij} = \log\left(\frac{t^*_{ij}}{\tilde{t}^*_{ij}}\right)$ are not all independent. They are contemporaneously correlated. So, if we assume that there is correlation only across the j's and not with respect to the i's, then let $C = E(cc')$ denote the covariance matrix, which will be block diagonal under our preceding assumption. C will have the form:

$$C = \begin{pmatrix} \Sigma_{c1} & 0 & \cdots & 0 \\ 0 & \Sigma_{c2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \Sigma_{cm} \end{pmatrix}$$

The equation (3) can be written as:

$$\{ y = Zb + c \} \quad \text{---(4)}$$

where

$$y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{pmatrix}_{Mm\times 1} \quad z = \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_m \end{pmatrix}_{Mm\times 1} \quad b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_q \end{pmatrix}_{q\times 1} \quad c = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_m \end{pmatrix}_{Mm\times 1}$$

$$y_i = (y_{i1}, \; y_{i2} \; \text{---} y_{im})_{1\times m}$$

$$c_i' = (c_{i1}, \; c_{i2}, \; \text{---} c_{im})_{1\times m}$$

$$Z_i = \begin{pmatrix} Z_{1i1} & Z_{2i1} & Z_{3i1} & \text{---} & Z_{qi1} \\ Z_{1i2} & Z_{2i2} & Z_{3i2} & \text{---} & Z_{qi2} \\ \vdots & & & & \vdots \\ Z_{1im} & Z_{2im} & Z_{3im} & \text{---} & Z_{qim} \end{pmatrix}_{m\times q}$$

Further if $X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \sim N(\mu, \Sigma_{n\times n})$ and $\mu = \begin{pmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_n \end{pmatrix}$

Then the multivariate normal distributions' density function is:

$$f(x) = \frac{1}{(2p)^{n/2}}\left(|\Sigma_{n\times n}|^{-1/2}\right) e^{-1/2\{(X-\mu)'\,\Sigma^{-1}\,(x-\mu)\}}$$

using this result we can write the likelihood function for obtaining maximum likelihood estimates of the parameters b_k for our MCI model as:

$$L\{b_k\} = \frac{1}{(2p)^{Mm/2}}|C|^{-1/2}\, e^{-1/2\,(c'\,C^{-1}\,c)} \quad \text{since } c \sim N(0,C)$$

$$\therefore \log L = \frac{-Mm}{2}\log 2p + 1/2\log|C^{-1}|$$
$$-1/2\,\{y' C^{-1} y - y' C^{-1} Zb$$
$$-b'Z'C^{-1}y + b'Z'C^{-1}Zb\}$$

For maximum: $\frac{\partial \log L}{\partial b} = 0 = -2Z'\hat{C}^{-1}_{ML}\,y + 2(Z'\hat{C}^{-1}_{ML}Z)\,b_{ML}$

$$\{\hat{b}_{ML} = (Z'\hat{C}^{-1}_{ML}Z)^{-1}Z'\hat{C}^{-1}_{ML}y\}$$

So, to obtain $\hat{b}_{ML}$ we need $\hat{C}_{ML}$. Consider then the first order condition for maximum taking now the derivation with respect to the element W^{ij} of the Matrix C. First the log likelihood function is written as:

$\log L = \frac{-Mm}{2}\log 2p + 1/2\log|C^{-1}| - 1/2\,\text{trace}\, C^{-1}S$

where $S_{Mm\times Mm} = (y-Z\hat{b}_{ML})(y-Z\hat{b}_{ML})'$ = the matrix of squared residuals. $\therefore \log L = \frac{-Mm}{2}\log 2p + 1/2\log|C^{-1}|$

$$-1/2\sum_{j=1}^{Mm}\sum_{i=1}^{Mm} W^{ij}\,S_{ji}$$

Then for maximum $\frac{\partial \log L}{\partial W^{ij}} = 1/2\hat{C}^{ij}_{ML} - 1/2S_{ji} = 0$ where

$\hat{C}^{ij}_{ML}$ = the estimated co-factor of elements W_{ij} of matrix C. Therefore, $\hat{C}^{ij}_{ML} = S_{ji}$ is the likelihood estimate of C^{ij}_{ML} and $(\hat{C}_{ML} = S')$.

Further, since S is symmetric, $S' = S$. Hence, $\{\hat{C}_{ML} = S\}$ and $\{S = (y-Z\hat{b}_{ML})(y-Z\hat{b}_{ML})'\}$ is the M.L. estimator of C. So in order to obtain the matrix $\hat{C}_{ML}$ we need $\hat{b}$ and vice versa.

Hence the procedure to obtain the estimators can be split into two phases. In the first phase, we estimate $\hat{b}$ the least squares estimator of b and use this estimated b to get the estimator of $\hat{C}_{ML}$ which can then be used to obtain a revised maximum likelihood estimator of $\hat{b}_{ML}$ of b. This process can be repeated recursively if desired to refine the values of the estimators until we have convergence. We can note at this point that the extimator $\hat{C}_{ML}$ of C is S, the matrix of the squares of the residuals. Hence each element of C is

estimated by only one 'square of residual' term. This can be considered as a weakness of the procedure since the estimator of C is based on only one respective observation.
Haines, Simon and Alexis (1972) have obtained the Maximum likelihood estimators of the restricted form of the MCI model:

$$P_{ij} = \frac{\dfrac{S_j}{T_{ij}^{\lambda}}}{\sum_{j=1}^{m} \dfrac{S_j}{T_{ij}^{\lambda}}}$$

They obtain the likelihood function and the first order conditions of the log likelihood function for the maximum likelihood estimator of b. They used a search procedure defined by the GROPE program to obtain the λ which maximizes the log likelihood function. But this procedure does not necessarily identify the global maximum nor is it a minimum variance estimator.

The Minimum Variance Unbiased Estimators

The least squares estimators and the maximum likelihood estimators obtained are consistent and also linear unbiased with respect to the transformed model (the linear form). The Maximum Likelihood estimators are not unbiased with respect to the original MCI model. This is due to the property of the lognormal distribution. If $\overset{*}{b}$ has a normal distribution with mean = b and variance σ_b^2 and is its least squares estimator, then, $E(\hat{b}) = b$. Also, $e^{\overset{*}{b}}$ has a lognormal distribution, but $E(e^{\overset{*}{b}}) \neq e^b$ since e^b gives the median of the lognormal distribution, not the mean. The mean of the lognormal distribution of $e^{\overset{*}{b}}$ is $E(e^{\overset{*}{b}}) = e^{(b + 1/2\,\sigma_b^2)}$ In addition, the least squares and the maximum likelihood estimators are not the minimum variance estimators of the original MCI model.
In practice, the purpose of the estimation is to apply the model to prediction problems. So we need a Minimum Variance Unbiased (MVU) estimator (which may not always exist) for the original model. As we have shown, the estimator $e^{\hat{b}}$ is a 'median-unbiased' estimator of e^b when e^b has lognormal distribution and $\hat{b}$ is the least squares estimator of b. So $e^{\hat{b}}$ is not an unbiased estimator of e^b. Finney (1951) showed how the least squares estimator can be adjusted to an unbiased one for a particular case. Bradu and Mundlak (1970) extended Finney's approach to accommodate the general loglinear model. Let us now investigate the possibilities of obtaining the MVU estimators for our MCI model.
The model investigated by Bradu and Mundlak is of the form:

$$y = \prod_{i=1}^{k_1} X_i^{\alpha_i} e^{\varepsilon} = e^{\left(\sum_{i=1}^{k} \alpha_i x_i + \varepsilon\right)} \text{---(4) where } x_i = \log X_i$$

where $x_i > 0$ and $i = 1, 2, 3, \ldots k$ are the number of explanatory variables. The number of obsverations is n. The error term is ε and $\varepsilon' = (\varepsilon_1\ \varepsilon_2 \text{---} \varepsilon_n)'$ is assumed to have a normal distribution, i.e., $\varepsilon \sim N(0, \sigma^2 I_n)$. I_n = identity matrix and $E(x_{ij}\ \varepsilon_i) = 0$. Taking $y = (\ln Y_1\ \ln Y_2 \text{---} \ln Y_n)$; $\alpha = (\alpha_1, \alpha_2 \text{---} \alpha_k)$ and $X = (x_{ij})_{nxk}$ we have the transformed log model in the matrix form: $y = X\alpha + \varepsilon$.
Now the ordinary least squares estimators (OLS) can be obtained, but they will not be mean variance unbiased estimators. To develop the MVU estimators, Bradu and Mundlak introduce a special function:

$$g_{m_1}(t) = \sum_{p=0}^{\infty} W_p(t) = \sum_{p=0}^{\infty} \frac{m_1^b(m_1+2p)}{m_1(m_1+2)\text{--}(m_1+2p)} \cdot \left(\frac{m_1}{m_1+1}\right)^p \frac{t^p}{(p)!} \quad \text{---(6)}$$

Further, if we consider two independent random variables, a vector z and S^2 where $z \sim N\{\mu, c\sigma^2\}$ with c = a known constant and $S^2 = \frac{\sigma^2}{m_1}\chi^2_{m_1}$, i.e., $\frac{MS^2}{\sigma^2}$ has a $\chi^2_{m_1}$ distribution with m_1 = degrees of freedom, then:

$$E(S^{2p}) = \frac{m_1(m_1+2)\text{---}(m_1+2p)}{m_1^p\,(m_1+2p)}\sigma^{2p} \quad \text{---(7)} \quad p=0,1,2\text{---}$$

combining (6) and (7) we have:

$$E\{g_{m_1}(AS^2)\} = \sum_{p=0}^{\infty} \frac{1}{(p)!}\left(\frac{m_1}{m_1+1} A\sigma^2\right)^p = e^{\frac{m_1 A\sigma^2}{m_1+1}}$$

Hence, $g_m(AS^2)$ is an unbiased estimator of $e^{\frac{m_1 A\sigma^2}{m_1+1}}$ Also, since e^{bz} is also lognormally distributed (as is e^z), so

$$E(e^{Bz}) = e^{B\mu + 1/2B^2 c\sigma^2} = e^{B\mu} e^{1/2B^2 c\sigma^2}$$

Hence, e^{Bz} is an unbiased estimator of $e^{B\mu + 1/2B^2 c\sigma^2}$ and from the specifications of z and S^2 it follows that $e^{Bz}g_{m_1}(AS^2)$ is an unbiased estimator of $\{e^{B\mu + \left(1/2B^2c + \frac{m_1 A}{m_1+1}\right)\sigma^2}\}$.
Therefore, to obtain an unbiased estimator of the form $e^{\gamma\mu + \delta\sigma^2}$ we adapt the above unbiased estimator by putting $B=\gamma$ and $A = \frac{(m_1+1)(\delta - 1/2\,\gamma^2 c)}{m_1}$

Therefore, $g_{m_1}(AS^2) = g_{m_1}\frac{m_1+1}{m_1}(\delta - 1/2\,\gamma^2 c)S^2$ and

$T(e^{\gamma z} + \delta\sigma^2) = e^{\gamma z} g_{m_1}\{ \frac{m_1+1}{m_1} (\delta - 1/2\gamma^2 c)S^2\}$ is an unbiased estimator of $e^{\gamma z} + \delta\sigma^2$. Finney (7) uses S^2 as the least squares variance estimator with $z \sim N(\mu, c\sigma^2)$ and $\frac{m_1 S^2}{\sigma^2} \sim \chi^2_{m_1}$; and z and S^2 being stochastically independent.

The computation of the minimum variance unbiased estimators requires numerical values of the function $g_{m_1}(t)$. This function has been tabulated by various researchers in econometrics. The $g_{m_1}(t)$ function values for positive t are given by Aitchison and Brown('66)and the log $g_{m_1}(t)$ values for positive t are given by Thoni(1969). Bradu and Mundlak have given values of $g_{m_1}(t)$ for negative t values.

To turn to our MCI model, we can rewrite our equation (3) as follows:

$$y_{ij} = \sum_{k=1}^{q} b_k Z_{kij} + c_{ij} \qquad \text{---(3)}$$

Let y_{ij} =log Y_{ij} log $\frac{p_{ij}}{\tilde{p}_i}$ i.e., $Y_{ij} = \frac{p_{ij}}{\tilde{p}_i}$

$$\text{Then} \quad y_{ij} = e^{\sum_{k=1}^{q} b_k Z_{kij} + c_{ij}} \qquad \text{---(5)}$$

This is a similar form to that of Bradu and Mundlak's model (equation (4)). The big difference, however, is the error term is $\varepsilon \sim N(0, \sigma^2 I_n)$ and our error term is $\varepsilon \sim N(0,C)$. In other words we have a variance covariance matrix which is block-diagonal and hence cannot be written in the form $\sigma^2 I_n$. This again arises due to the covariance across the j subscript in the MCI model.

This implies that we cannot use the Finney-Bradu-Mundlak (1951, 1970) procedure and we cannot utilize their g_{m_1} (t) function form to obtain minimum variance unbiased estimators for the MCI model. We need a further generalization of their procedure, i.e., a multivariate generalization of the g_{m_1} (t) function which is not available at present in the econometrics literature.

If we do assume that we have the variance covariance matrix C of our MCI model of the form ($\sigma^2 I$), then we can apply the Bradu-Mundlak approach. However, that assumption would still not yield MVU estimators since by that assumption we have essentially thrown out some information, the correlation across the number of store (choices) and this would reduce the GLS to the OLS. The efficiency of the estimators would be reduced (with decreased information) and they would not have minimum variance. The assumption could also introduce a bias due to improper specification of the model.

Discussion

In our procedures used for evaluating the potential of new store locations we have in all cases used proportions (p_{ij}) obtained by aggregating data for individuals from the same area i in a sample, instead of the true probability values. This itself introduces error and the substitution can only be justified when the samples are large.

We throw away a lot of data in our aggregation process for proportion estimates. It would be perhaps worthwhile from this viewpoint to use an analysis on the lines suggested by Daniel McFadden (1974) for human choice behavior (which is essentially the problem we are addressing in our MCI model). McFadden uses the conditional logit distribution to analyze the choice behavior and the functional form is very close to the MCI form (but not the same).

In our aggregation we are assuming population homogeniety over our areas as well as over the entire population (all the i = 1, 2, 3...M areas) when we take the values to be the same for all these. We are further assuming the same pro= bability distribution for all the population and areas. This must be borne in mind as there could well be econometric problems of aggregation involved.

Finally, we have the specification problem involved when we use distance and size of the store as the only two variables determining the underlying utility function and hence the MCI model. Huff gives several arguments for the inclusion of these two variables and there is general agreement for their inclusion. However, other factors such as visibility, accessibility, promotional policies and the locality of the store would also influence the utility and hence the choice probability. These other factors can be included in the general MCI form. However, some of these quantities would be difficult to quantify and include in the model.

Apart from the transformed linear form considered in this paper, researchers have developed various numerical estimation techniques. Urban (1969) used an on-line iterative search program which used decision maker's judgments as one of the inputs. Alfred Kuehn, Timothy McGuire and Doyle Weiss (1966) used a non-linear least squares method which minimized the sum of the squared residuals by a direct search technique. Hlavac and Little (1966) and Haines, Simon and Alexis (1972) attempted to find the maximum likelihood estimators by Newton's method or a direct search technique.

All these techniques suffer from some common drawbacks: (i) none of them guarantee a global mixima or minima, (ii) they are costly in terms of computational efforts needed, (iii) the statistical properties of the estimators are not known except for the asymptotic properties of the maximum likelihood estimators. Haines has shown that assuming that the global maxima can be found , the maximum likelihood estimators are consistent, but they are not minimum variance estimators.

Following Nakanishi and Cooper's linear transformation of the model allows us to use generalized squares and the maximum likelihood methods on the transformed model, and in this case the statistics of the estimators are known. Further work is needed for obtaining the mean variance unbiased estimators for the model, and generalizing the Finney-Bradu-Mundlak $g_{m_1}(t)$ function to the multivariate case would be an important step towards obtaining the mean variance unbiased estimators. Obtaining a model developed using conditional logit approach would be another interesting area to push into.

The Multiplicative Competitive Interaction Model need not be restricted to a retail store location. It is useful over a number of cases, wherever there is a choice behavior to be analyzed and the underlying utility function is the Cobb Douglus type. For example, it can be extended into the field of production; for warehouse and factory location models. The Multiplicative Competitive Interaction Model as considered by Nakanishi and Cooper becomes important due to its generalized form which can be applied to a variety of different problems.

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