A Hierarchical WENO Reconstructed Discontinuous Galerkin Method for Computing Shock Waves

Y. Xia, M. Frisbey, and H. Luo

1 Introduction

The discontinuous Galerkin (DG) methods[1] have recently become popular for the solution of systems of conservation laws because of their several attractive features such as easy extension to and compact stencil for higher-order ($> 2nd$) approximation, flexibility in handling arbitrary types of grids for complex geometries, and amenability to parallelization and hp-adaptation. However, the DG Methods have their own share weaknesses. In particular, how to effectively control spurious oscillations in the presence of strong discontinuities, and how to reduce the computing costs and storage requirements for the DGM remain the two most challenging and unresolved issues in the DGM.

In order to reduce high costs associated with the DGM, Dumbser et al[2] have introduced a new family of reconstructed DGM, termed PnPm schemes and referred to as RDG(PnPm) in this paper, where Pn indicates that a piecewise polynomial of degree of n is used to represent a DG solution, and Pm represents a reconstructed polynomial solution of degree of m ($m \geq n$) that is used to compute the fluxes. The RDG(PnPm) schemes are designed to enhance the accuracy of the discontinuous Galerkin method by increasing the order of the underlying polynomial solution. The beauty of RDG(PnPm) schemes is that they provide a unified formulation for both finite volume and DGM, and contain both classical finite volume and standard DG methods as two special cases of RDG(PnPm) schemes, and thus allow for a direct efficiency comparison. When n=0, i.e. a piecewise constant polynomial is used to represent a numerical solution, RDG(P0Pm) is nothing but classical high order finite volume schemes, where a polynomial solution of degree m ($m > 1$) is reconstructed from a piecewise constant solution. When m=n, the reconstruction reduces to the identity operator, and RDG(PnPn) scheme yields a standard DG method.

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R. Bonazza and D. Ranjan (eds.), *29th International Symposium on Shock Waves 2*,
DOI: 10.1007/978-3-319-16838-8_25

The objective of the effort discussed in this paper is to develop a reconstructed discontinuous Galerkin method based on a Hierarchical WENO reconstruction, termed HWENO(P1P2)[3] [4] for computing shock waves on 3D hybrid grids. The HWENO(P1P2) method is designed not only to reduce the high computing costs of the DGM, but also to avoid spurious oscillations in the vicinity of strong discontinuities, thus effectively addressing the two shortcomings of the DGM. The developed HWENO(P1P2) method is used to compute a variety of shock wave problems on hybrid grids to demonstrate its accuracy, robustness, and non-oscillatory performance. The numerical experiments indicate that this HWENO(P1P2) method is able to capture shock waves sharply without any spurious oscillations, and achieve the designed third-order of accuracy for smooth flows.

2 Numerical Method

The Euler equations governing unsteady compressible inviscid flows can be expressed in conservative form as

$$\frac{\partial \mathbf{U}(\mathbf{x},t)}{\partial t} + \frac{\partial \mathbf{F}_j(\mathbf{U}(\mathbf{x},t))}{\partial x_j} = 0, \qquad \text{in } \Omega \tag{1}$$

where Ω is a bounded connected domain in $\mathbf{R}^d$, d is the number of spatial dimension, $\mathbf{U}$ is the conservative state vector, and $\mathbf{F}$ is the flux function vector.

The governing equation (1) is discretized using a discontinuous Galerkin finite element formulation. We assume that the domain Ω is subdivided into a collection of non-overlapping elements Ω_e, which can be tetrahedral, prism, pyramid, and hexahedral or their combinations. We introduce the following broken Sobolev space V_h^p

$$V_h^p = \{v_h \in [L_2(\Omega)]^m : v_h \mid_{\Omega_e} \in [V_p^m] \forall \Omega_e \in \Omega\}, \tag{2}$$

which consists of discontinuous vector-valued polynomial functions of degree p, and where m is the dimension of conservative state vector and V_p is the space of all polynomials of degree $\leq p$. To formulate the discontinuous Galerkin method, we introduce the following weak formulation, which is obtained by multiplying the above conservation law (1) by a test function W_h, integrating over an element Ω_e, and then performing an integration by parts,

$$\begin{cases} \text{find } \mathbf{U}_h \in V_h^p \text{ such as} \\ \frac{d}{dt}\int_{\Omega_e} \mathbf{U}_h \mathbf{W}_h d\Omega + \int_{\Gamma_e} \mathbf{F}_j(\mathbf{U}_h)\mathbf{n}_j \mathbf{W}_h d\Gamma - \int_{\Omega_e} \mathbf{F}_j(\mathbf{U}_h) \frac{\partial \mathbf{W}_h}{\partial x_j} d\Omega = 0 \qquad \forall \mathbf{W}_h \in V_h^p, \end{cases} \tag{3}$$

where $\mathbf{U}_h$ and $\mathbf{W}_h$ are represented by piecewise-polynomial functions of degree p, which are discontinuous between element interfaces and $\mathbf{n}_k$ denotes the unit outward normal vector to Γ_e: the boundary of Ω_e. Assume that B_i is the basis of polynomial function of degrees p, this is then equivalent to the following system of N equations,

$$\frac{d}{dt}\int_{\Omega_e}\mathbf{U}_h B_i d\Omega + \int_{\Gamma_e}\mathbf{F}_j(\mathbf{U}_h)\mathbf{n}_j B_i d\Gamma - \int_{\Omega_e}\mathbf{F}_j(\mathbf{U}_h)\frac{\partial B_i}{\partial x_j}d\Omega = 0 \qquad 1 \le i \le \mathrm{N}, \quad (4)$$

where N is the dimension of the polynomial space.

In our work, the numerical polynomial solutions are represented using a Taylor series expansion at the centroid of the cell. The unknowns to be solved in this formulation are the cell-averaged variables and their derivatives at the center of the cells, regardless of element shapes. This formulation belongs to the so-called modal discontinuous Galerkin method and has a number of attractive, distinct, and useful features. For example, cell-averaged variables and their derivatives are handily available in this formulation, making the implementation of both in-cell and inter-cell reconstruction schemes straightforward and simple. The use of the same basis functions for any shapes of elements: tetrahedron, pyramid, prism, and hexahedron makes the implementation of DGM on arbitrary meshes straightforward.

In comparison with reconstructed FV methods, the DGM have a significant drawback in that they require more degrees of freedom, additional domain integration, and more Gauss quadrature points for the boundary integration, and therefore more computational costs and storage requirements. On the one hand, the reconstruction methods that FV methods use to achieve higher-order accuracy are relatively inexpensive but less accurate and robust. On the other hand, the DGM that can be viewed as a different way to extend a FV method to higher orders are accurate and robust but costly. It is only natural and tempting to combine the efficiency of the reconstruction methods and the accuracy of the DG methods. This idea was originally introduced by Dumbser et al[2] in the frame of PnPm scheme, where Pn indicates that a piecewise polynomial of degree of n is used to represent a DG solution, and Pm represents a reconstructed polynomial solution of degree of m ($m \ge n$) that is used to compute the fluxes and source terms.

Our reconstructed DG method [3] [4] is based on a Hierarchical WENO reconstruction, termed HWENO(P1P2), and designed not only to reduce the high computing costs of the DGM, but also to avoid spurious oscillations in the vicinity of strong discontinuities, thus ensuring the nonlinear stability of the RDG method. In this HWENO(P1P2) method, a quadratic solution is first reconstructed to enhance the accuracy of the underlying DG method in two steps: (1) all second derivatives on each cell are first reconstructed using the solution variables and their first derivatives from adjacent face-neighboring cells via a strong interpolation; (2) the final second derivatives on each cell are then obtained using a WENO strategy based on the reconstructed second derivatives on the cell itself and its adjacent face-neighboring cells. This reconstruction scheme, by taking advantage of handily available and yet valuable information namely the gradients in the context of the DG methods, only involves von Neumann neighborhood and thus is compact, simple, robust, and flexible. As the underlying DG method is second-order, and the basis functions are at most linear functions, fewer quadrature points are then required for both domain and face integrals, and the number of unknowns (the number of degrees of freedom)

remains the same as for the DG(P1). Consequently, this RDG method is more efficient than its third order DG(P2) counterpart. The gradients of the quadratic polynomial solutions are then modified using a WENO reconstruction in order to eliminate non-physical oscillations in the vicinity of strong discontinuities, thus ensuring the non-linear stability of the RDG method.

3 Numerical Examples

As illustrative examples, a few numerical results are presented to demonstrate the accuracy and robustness of this DG method. One of advantages of the present RDG method is its ability to compute 1D, 2D, and 3D problems using the very same code, which greatly alleviates the need and pain for code maintenance and upgrade. Results for one-dimensional problems can be readily obtained by using hexedral cells and by setting the number of cells in both y- and z-directions to be 2. For two-dimensional problems, the number of cells in the z-direction is simply set to be 1. All computations use an explicit three-stage third-order TVD Runge-Kutta scheme to advance the solution in time.

Example 1: Sod Shock Tube Problem. The first example is the well-known Sod shock tube problem, which is chosen to demonstrate the ability of the HWENO(P1P2) method for sharply resolving shock waves without any spurious oscillations. Figure 1 compares the density and velocity profiles between computed and exact solutions, respectively.

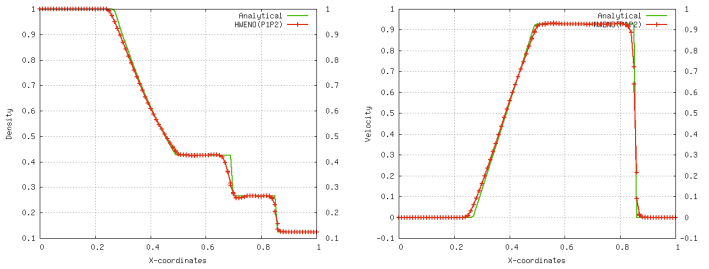

Fig. 1 Comparison of computed density and velocity profile for Sod shock tube problem by HWENO(P1P2) solution with the analytical solution

Example 2: Woodward-Collela Blast Wave Problem. The second example is Woodward-Collela blast wave problem, which is chosen to demonstrate that the HWENO(P1P2) can produce an essentially non-oscillatory solution for strong blast waves. Four hundred cells are used in this case, and the computed results are shown in Figure 2 for the density distribution at t=0.038, where the solid lines are obtained from the same computation with 4000 cells.

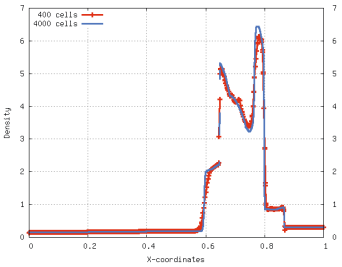

Fig. 2 Comparison of the computed RDG solutions between coarse and fine meshes for the Woodward-Collela blast wave problem

Example 3: Shu-Osher Shock-Entropy Wave Interaction Problem. The Shu-Osher shock-entropy wave interaction problem consists of not only large scales waves but also shocklets and fine scales structures, thus providing a good test case for the accuracy and robustness of the HWENO(P1P2) method at different scales. The computed density distribution at time t = 1.8 obtained by RDG solution is shown in Figure 3 using 400 cells, where the solution obtained using 2000 grid cells is also presented for comparison.

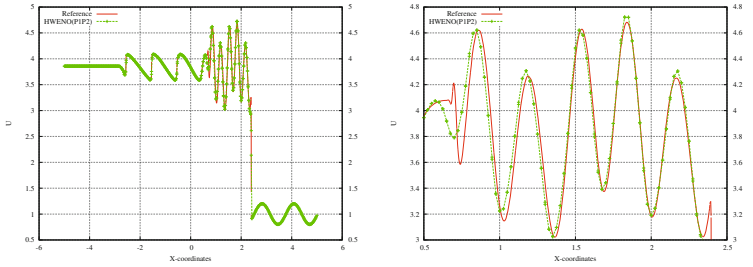

Fig. 3 Comparison of computed density with reference solution for Shu-Osher shock-entropy wave interaction problem

Example 4: An Incident Shock Past a Wedge. This example is presented to demonstrate the ability of the HWENO(P1P2) method for computing shock waves in a two-dimensional setting. The computed density contours for an incident shock past a wedge using the developed HWENO(P1P2) method is shown in Figure 4, where one can see that the HWENO(P1P2) method is able to capture wave structures accurately.

Example 5: A Blast Wave Past a Wall. Finally, a 3D blast simulation past a wall is presented in Figure 5 to demonstrate that the HWENO(P1P2) method can be used for solving problems of practical interests for realistic, engineering-type configurations.

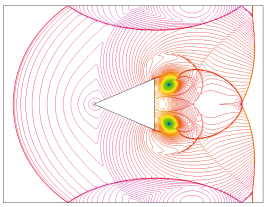

Fig. 4 Computed density contours for an incident shock M_s=1.3 past a wedge using the HWENO(P1P2) method.

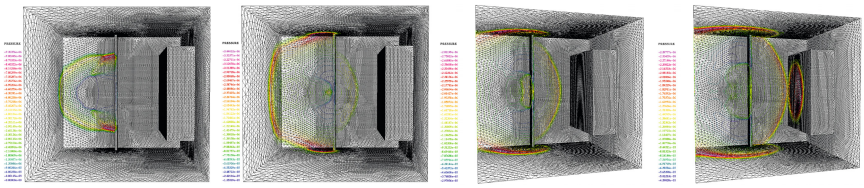

Fig. 5 Computed pressure contours at different times for an air blast wave past a wall

4 Conclusion

A hierarchical WENO reconstruction-based DG method, HWENO(P1P2), has been presented for computing shock waves on 3D hybrid grids. A number of numerical experiments for a variety of shock wave problems have been conducted to demonstrate the accuracy, robustness, and non-oscillatory performance of the HWENO (P1P2) method. The numerical results obtained indicate that the HWENO(P1P2) method is able to provide sharp resolution of shock waves without over- and under-shoots and achieve the designed third-order of accuracy for smooth flows, without significant increase in computing costs and memory requirements.

References

1. Cockburn, B., Shu, C.W.: The Runge-Kutta Discontinuous Galerkin Method for conservation laws V: Multidimensional System. Journal of Computational Physics 141, 199–224 (1998)
2. Dumbser, D., Balsara, D.S., Toro, E.F., Munz, C.D.: A unified framework for the construction of one-step finite volume and discontinuous Galerkin schemes on unstructured meshes. Journal of Computational Physics 227, 8209–8253 (2008)
3. Luo, H., Xia, Y., Li, S., Nourgaliev, R., Cai, C.: A Hermite WENO Reconstruction-based Discontinuous Galerkin Method for the Euler Equations on Tetrahedral Grids. Journal of Computational Physics 231, 5489–5503 (2012)
4. Luo, H., Xia, Y., Spiegel, S., Nourgaliev, R., Jiang, Z.: A Reconstructed Discontinuous Galerkin Method Based on a Hierarchical WENO Reconstruction for Compressible Flows on Tetrahedral Grids. Journal of Computational Physics 236, 477–492 (2013)