Chapter 53
New Constructions in the Theory of Elliptic Boundary Value Problems

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53.1 Introduction

How are potentials constructed for boundary value problems? One takes a fundamental solution of the corresponding differential operator *in whole space* $\mathbf{R}^m$, and with its help constructs the potentials according to boundary conditions. Further, one studies their boundary properties, and with the help of potentials reduces the boundary value problem to an equivalent integral equation on the boundary. The formulas for integral representation of solution of the boundary value problem were obtained for separate cases only (a ball, a half-space, such places, where one has explicit form for a Green function). Thus, an ideal result for a boundary value problem even with a smooth boundary is its reduction to an equivalent Fredholm equation and obtaining the existence and uniqueness theorem (without knowing how the solution looks) [Ag57, Fa88, Ke94, MiMiTa01, HsWe08]. We would like to show that potentials can arise from another point of view, without using fundamental solution, but using factorization idea and they obviously should take into account the boundary geometry. A smooth boundary is a hyper-plane locally (there is a Poisson formula for the Dirichlet problem, see also [Es81]), first type of non-smooth boundary is a conical surface.

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53.2 Operators, Equations, and Wave Factorization

We consider an elliptic pseudo-differential equation in a multi-dimensional cone and starting wave factorization concept we add some boundary conditions. For the simplest cases explicit formulas for solution are given like layer potentials for a classical case.

Let's go to studying solvability of pseudo-differential equations [Va00a, Va11, Va10]

$$(Au_+)(x) = f(x),\; x \in C_+^a, \tag{53.1}$$

in the space $H^s(C_+^a)$, where C_+^a is m-dimensional cone

$$C_+^a = \{x \in \mathbf{R}^m : x = (x_1,\ldots,x_{m-1},x_m), x_m > a|x'|, a > 0\},\; x' = (x_1,\ldots,x_{m-1}),$$

A is pseudo-differential operator ($\tilde{u}$ denotes the Fourier transform of u)

$$u(x) \longmapsto \int\limits_{\mathbf{R}^m} e^{ix\cdot\xi} A(\xi)\tilde{u}(\xi)d\xi,\;\; x \in \mathbf{R}^m,$$

with the symbol $A(\xi)$ satisfying the condition

$$c_1 \le |A(\xi)(1+|\xi|)^{-\alpha}| \le c_2.$$

(Such symbols are elliptic [Es81] and have the order $\alpha \in \mathbf{R}$ at infinity.)

By definition, the space $H^s(C_+^a)$ consists of distributions from $H^s(\mathbf{R}^m)$, whose support belongs to $\overline{C_+^a}$. The norm in the space $H^s(C_+^a)$ is induced by the norm from $H^s(\mathbf{R}^m)$. The right-hand side f is chosen from the space $H_0^{s-\alpha}(C_+^a)$, which is space of distributions $S'(C_+^a)$, admitting the continuation on $H^{s-\alpha}(\mathbf{R}^m)$. The norm in the space $H_0^{s-\alpha}(C_+^a)$ is defined by

$$||f||_{s-\alpha}^{+} = \inf ||lf||_{s-\alpha},$$

where *infimum* is chosen from all continuations l.

Further, we define a special multi-dimensional singular integral by the formula

$$(G_m u)(x) = \lim_{\tau\to 0+} \int\limits_{\mathbf{R^m}} \frac{u(y',y_m)dy'dy_m}{(|x'-y'|^2 - a^2(x_m - y_m + i\tau)^2)^{m/2}}$$

(we omit a certain constant, see [Va00a]). Let us recall, this operator is multi-dimensional analogue of the one-dimensional Cauchy type integral, or Hilbert transform.

We also need some notations before definition.

The symbol $\overset{*}{C^a_+}$ denotes a conjugate cone for C^a_+:

$$\overset{*}{C^a_+}= \{x \in \mathbf{R}^m : x = (x', x_m), ax_m > |x'|\},$$

$C^a_- \equiv -C^a_+$, $T(C^a_+)$ denotes radial tube domain over the cone C^a_+, i.e. domain in a complex space $\mathbf{C}^m$ of type $\mathbf{R}^m + iC^a_+$.

To describe the solvability picture for the equation (53.1) we will introduce the following definition.

Definition 1. Wave factorization for the symbol $A(\xi)$ is called its representation in the form

$$A(\xi) = A_{\neq}(\xi)A_{=}(\xi),$$

where the factors $A_{\neq}(\xi), A_{=}(\xi)$ must satisfy the following conditions:

1) $A_{\neq}(\xi), A_{=}(\xi)$ are defined for all admissible values $\xi \in \mathbf{R}^m$, without may be, the points $\{\xi \in \mathbf{R}^m : |\xi'|^2 = a^2\xi_m^2\}$;
2) $A_{\neq}(\xi), A_{=}(\xi)$ admit an analytical continuation into radial tube domains $T(\overset{*}{C^a_+}), T(\overset{*}{C^a_-})$, respectively, with estimates

$$|A^{\pm 1}_{\neq}(\xi + i\tau)| \leq c_1(1 + |\xi| + |\tau|)^{\pm\kappa},$$

$$|A^{\pm 1}_{=}(\xi - i\tau)| \leq c_2(1 + |\xi| + |\tau|)^{\pm(\alpha-\kappa)},\ \forall \tau \in \overset{*}{C^a_+}.$$

The number $\kappa \in \mathbf{R}$ is called index of wave factorization.

The class of elliptic symbols admitting the wave factorization is very large. There are the special chapter in the book [Va00a] and the paper [Va00b] devoted to this question, there are examples also for certain operators of mathematical physics.

Everywhere below we will suppose that the mentioned wave factorization does exist, and the sign $\sim$ will denote the Fourier transform, particularly $\tilde{H}(D)$ denotes the Fourier image of the space $H(D)$.

53.3 After the Wave Factorization

Now we will consider the equation (53.1) for the case $\kappa - s = n + \delta, n \in \mathbf{N}, |\delta| < 1/2$, only. A general solution can be constructed in the following way. We choose an arbitrary continuation lf of the right-hand side on a whole space $H^{s-\alpha}(\mathbf{R}^m)$ and introduce

$$u_-(x) = (lf)(x) - (Au_+)(x).$$

After wave factorization for the symbol $A(\xi)$ with preliminary Fourier transform we write

$$A_{\neq}(\xi)\tilde{u}_{+}(\xi)+A_{=}^{-1}(\xi)\tilde{u}_{-}(\xi)=A_{=}^{-1}(\xi)\tilde{lf}(\xi).$$

One can see that $A_{=}^{-1}(\xi)\tilde{lf}(\xi)$ belongs to the space $\tilde{H}^{s-\kappa}(\mathbf{R}^m)$, and if we choose the polynomial $Q(\xi)$, satisfying the condition

$$|Q(\xi)| \sim (1+|\xi|)^n,$$

then $Q^{-1}(\xi)A_{=}^{-1}(\xi)\tilde{lf}(\xi)$ will belong to the space $\tilde{H}^{-\delta}(\mathbf{R}^m)$.

Further, according to the theory of multi-dimensional Riemann problem [Va00a], we can decompose the last function on two summands (jump problem):

$$Q^{-1}A_{=}^{-1}\tilde{lf}=f_{+}+f_{-},$$

where $f_{+}\in\tilde{H}(C_{+}^{a})$,$f_{-}\in\tilde{H}(\mathbf{R}^m\setminus C_{+}^{a})$.

So, we have

$$Q^{-1}A_{\neq}\tilde{u}_{+}+Q^{-1}A_{=}^{-1}\tilde{u}_{-}=f_{+}+f_{-},$$

or

$$Q^{-1}A_{\neq}\tilde{u}_{+}-f_{+}=f_{-}-Q^{-1}A_{=}^{-1}\tilde{u}_{-}$$

In other words,

$$A_{\neq}\tilde{u}_{+}-Qf_{+}=Qf_{-}-A_{=}^{-1}\tilde{u}_{-}.$$

The left-hand side of the equality belongs to the space $\tilde{H}^{-n-\delta}(C_{+}^{a})$, and right-hand side is from $\tilde{H}^{-n-\delta}(\mathbf{R}^m\setminus C_{+}^{a})$, hence

$$F^{-1}(A_{\neq}\tilde{u}_{+}-Qf_{+})=F^{-1}(Qf_{-}-A_{=}^{-1}\tilde{u}_{-}),$$

where the left-hand side belongs to the space $H^{-n-\delta}(C_{+}^{a})$, and the right-hand side belongs to the space $H^{-n-\delta}(\mathbf{R}^m\setminus C_{+}^{a})$, that's why we conclude immediately that it is a distribution supported on ∂C_{+}^{a}.

The main tool now is to define the form of the distribution.

We denote T_a the bijection operator transferring ∂C_{+}^{a} into hyperplane $x_m=0$, more precisely, it is transformation $\mathbf{R}^m\longrightarrow\mathbf{R}^m$ of the following type

$$\begin{cases} t_1=x_1, \\ \cdots\cdots\cdots \\ t_{m-1}=x_{m-1}, \\ t_m=x_m-a|x'|. \end{cases}$$

Then the function

$$T_a F^{-1}(A_{\neq}\tilde{u}_+ - Qf_+)$$

will be supported on the hyperplane $t_m = 0$ and belongs to $H^{-n-\delta}(\mathbf{R}^m)$. Such distribution is a linear span of Dirac mass-function and its derivatives [GeSh59] and looks as the following sum

$$\sum_{k=0}^{n-1} c_k(t')\delta^{(k)}(t_m).$$

It is left to think, what is operator T_a in Fourier image. Explicit calculations give a simple answer:

$$FT_a u = V_a \tilde{u},$$

where V_a is something like a pseudo-differential operator with symbol $e^{-ia|\xi'|\xi_m}$, and, further, one can construct the general solution of our pseudo-differential equation (53.1).

We need some connections between the Fourier transform and the operator T_a:

$$(FT_a u)(\xi) = \int\limits_{\mathbf{R}^m} e^{-ix\cdot\xi} u(x_1,\dots,x_{m-1},x_m - a|x'|)dx =$$

$$= \int\limits_{\mathbf{R}^m} e^{-iy'\xi'} e^{-i(y_m + a|y'|)\xi_m} u(y_1,\dots,y_{m-1},y_m)dy =$$

$$= \int\limits_{\mathbf{R}^{m-1}} e^{-ia|y'|\xi_m} e^{-iy'\xi'} \hat{u}(y_1,\dots,y_{m-1},\xi_m)dy',$$

where $\hat{u}$ denotes the Fourier transform on the last variable, and the Jacobian is

$$\frac{D(x_1,x_2,\dots,x_m)}{D(y_1,y_2,\dots,y_m)} = \begin{vmatrix} 1 & 0 & \cdots 0 & 0 \\ 0 & 1 & \cdots 0 & 0 \\ 0 & 0 & \cdots 1 & 0 \\ \dots & \dots & \dots & \\ \frac{ay_1}{|y'|} & \frac{ay_2}{|y'|} & \dots \frac{ay_{m-1}}{|y'|} & 1 \end{vmatrix} = 1.$$

If we define a pseudo-differential operator by the formula

$$(Au)(x) = \int\limits_{\mathbf{R}^m} e^{ix\xi} A(\xi)\tilde{u}(\xi)d\xi,$$

and the direct Fourier transformation

$$\tilde{u}(\xi) = \int\limits_{\mathbf{R}^m} e^{-ix\xi} u(x)dx,$$

then we have the following relation formally at least

$$(FT_a u)(\xi) = \int\limits_{\mathbf{R}^{m-1}} e^{-ia|y'|\xi_m} e^{-iy'\xi'} \hat{u}(y_1, \ldots, y_{m-1}, \xi_m)dy. \tag{53.2}$$

In other words, if we denote the $(m-1)$-dimensional Fourier transform ($y' \to \xi'$ in distribution sense) of function $e^{-ia|y'|\xi_m}$ by $E_a(\xi', \xi_m)$, then the formula (53.2) will be the following

$$(FT_a u)(\xi) = (E_a * \tilde{u})(\xi),$$

where the sign $*$ denotes a convolution for the first $m-1$ variables, and the multiplier for the last variable ξ_m. Thus, V_a is a combination of a convolution operator and the multiplier with the kernel $E_a(\xi', \xi_m)$. It is very simple operator, and it is bounded in Sobolev–Slobodetski spaces $H^s(\mathbf{R}^m)$.

Notice that distributions supported on conical surface and their Fourier transforms were considered in [GeSh59], but the author did not find the multi-dimensional analogue of theorem on a distribution supported in a single point in all issues of this book.

Remark 1. One can wonder why we can't use this transform in the beginning to reduce the conical situation (53.1) to hyperplane one, and then to apply Eskin's technique [Es81]. Unfortunately, this is impossible because T_a is non-smooth transformation, but even for smooth transformation we obtain the same operator A with some additional compact operator. Obtaining the invertibility conditions for such operator is a very serious problem.

53.4 General Solution

The following result is valid (it follows from considerations of Section 53.3).

Theorem 1. *A general solution of the equation* (53.1) *in Fourier image is given by the formula*

$$\tilde{u}_+(\xi) = A_{\neq}^{-1}(\xi) Q(\xi) G_m Q^{-1}(\xi) A_{=}^{-1}(\xi) \widetilde{lf}(\xi) +$$

$$+ A_{\neq}^{-1}(\xi) V_{-a} F \left(\sum_{k=1}^{n} c_k(x') \delta^{(k-1)}(x_m) \right),$$

where $c_k(x') \in H^{s_k}(\mathbf{R}^{m-1})$ are arbitrary functions, $s_k = s - \kappa + k - 1/2$, $k = 1, 2, \ldots, n$, lf is an arbitrary continuation f on $H^{s-\alpha}(\mathbf{R}^m)$.

Starting from this representation one can suggest different statements of boundary value problems for the equation (53.1).

53.4.1 Another Singularity

It may be that the singularity point will be different from considered one. This matter will influence the structure of the operator V_a. So, if we consider an another m-dimensional cone, for example $C_+^{\vec{a}} = \{x \in \mathbf{R}^m : x = (x_1,...,x_{m-1},x_m), x_m > \sum_{k=1}^{m-1} a_k|x_k|,\ a_k > 0,\ k = 1,2,...,m-1\}$, $\vec{a} = (a_1,...,a_{m-1})$, then we need certain corrections for our studies, in general it will be the same. Namely, we need to define a special multi-dimensional singular integral by the formula

$$(G_m u)(x) = (2i)^{m-1} \lim_{\tau\to 0+} \int_{\mathbf{R^m}} \prod_{j=1}^{m-1} \frac{a_j(x_m - y_m + i\tau)^{m-2}}{(x_j - y_j)^2 - a_j^2(x_m + y_m + i\tau)^2} u(y)dy$$

(for details see also [Va00a]). Such operator corresponds to the Fourier multiplier (characteristic function, or indicator) of the pyramid $C_+^{\vec{a}}$.

The Jacobian for this transformation $T_{\vec{a}}$ is

$$\frac{D(x_1,x_2,...,x_m)}{D(y_1,y_2,...,y_m)} = \begin{vmatrix} 1 & 0 & \cdots 0 & 0 \\ 0 & 1 & \cdots 0 & 0 \\ 0 & 0 & \cdots 1 & 0 \\ & \ldots\ldots\ldots\ldots\ldots & & \\ -a\ sign(y_1) & -a\ sign(y_2) & \ldots. -a\ sign(y_{m-1}) & 1 \end{vmatrix} = 1,$$

and the argument continues as above. This kind of singularity is considered in a forthcoming paper by the author, to appear in *Adv. Dyn. Syst. Appl.*

53.5 Boundary Conditions: Simplest Version, the Dirichlet Condition

We consider a very simple case, when $f \equiv 0,\ a = 1,\ n = 1$. Then the formula from the theorem takes the form

$$\tilde{u}_+(\xi) = A_{\neq}^{-1}(\xi)V_{-1}\tilde{c}_0(\xi').$$

We consider the following construction separately. According to the Fourier transform our solution is

$$u_+(x) = F^{-1}\{A_{\neq}^{-1}(\xi)V_{-1}\tilde{c}_0(\xi')\}.$$

Let's suppose we choose the Dirichlet boundary condition on ∂C^1_+ for unique identification of an unknown function c_0, i.e.

$$(Pu)(y) = g(y),$$

where g is given function on ∂C^1_+, P is restriction operator on the boundary, so we know the solution on the boundary ∂C^1_+. Thus,

$$T_1 u(x) = T_1 F^{-1}\{A^{-1}_{\neq}(\xi)V_{-1}\tilde{c}_0(\xi')\},$$

so we have

$$FT_1 u(x) = FT_1 F^{-1}\{A^{-1}_{\neq}(\xi)V_{-1}\tilde{c}_0(\xi')\} = V_1\{A^{-1}_{\neq}(\xi)V_{-1}\tilde{c}_0(\xi')\}, \tag{53.3}$$

and we know $(P'T_1u)(x') \equiv v(x')$, where P' is the restriction operator on the hyperplane $x_m = 0$.

The relation between the operators P' and F is well-known [Es81]:

$$(FP'u)(\xi') = \int_{-\infty}^{+\infty} \tilde{u}(\xi', \xi_m)d\xi_m.$$

Returning to the formula (53.3) we obtain the following

$$\tilde{v}(\xi') = \int_{-\infty}^{+\infty} \{V_1\{A^{-1}_{\neq}(\xi)V_{-1}\tilde{c}_0(\xi')\}\}(\xi', \xi_m)d\xi_m, \tag{53.4}$$

where $\tilde{v}(\xi')$ is given function. Hence, the equation (53.4) is an integral equation for determining $c_0(x')$.

The Neumann boundary condition leads to analogous integral equation (see below).

53.6 Conical Potentials

We consider the particular case where $f \equiv 0, n = 1$. The formula for general solution of the equation (53.1) takes the form

$$\tilde{u}_+(\xi)) = A^{-1}_{\neq}(\xi)V_{-a}F\{c_0(x')\delta^{(0)}(x_m)\},$$

and after Fourier transform (for simplicity we write $\tilde{c}$ instead of $V_{-1}\tilde{c}_0$),

$$\tilde{u}_+(\xi) = A^{-1}_{\neq}(\xi)\tilde{c}(\xi'), \tag{53.5}$$

or equivalently the solution is

$$u_+(x) = F^{-1}\{A_{\neq}^{-1}(\xi)\tilde{c}(\xi')\}.$$

Then we apply the operator T_a to formula (53.5)

$$(T_a u_+)(t) = T_a F^{-1}\{A_{\neq}^{-1}(\xi)\tilde{c}(\xi')\}$$

and the Fourier transform

$$(FT_a u_+)(\xi) = FT_a F^{-1}\{A_{\neq}^{-1}(\xi)\tilde{c}(\xi')\}.$$

If the boundary values of our solution u_+ are known on ∂C_+^a, it means that the following function is given:

$$\int\limits_{-\infty}^{+\infty} (FT_a u_+)(\xi) d\xi_m.$$

So, if we denote

$$\int\limits_{-\infty}^{+\infty} (FT_a u_+)(\xi) d\xi_m \equiv \tilde{g}(\xi'),$$

then for determining $\tilde{c}(\xi')$ we have the following equation:

$$\int\limits_{-\infty}^{+\infty} (FT_a F^{-1})\{A_{\neq}^{-1}(\xi)\tilde{c}(\xi')\} d\xi_m = \tilde{g}(\xi'), \tag{53.6}$$

This is a convolution equation, and if evaluating the inverse Fourier transform $\xi' \to x'$, we'll obtain the conical analogue of layer potential.

53.6.1 Studying the Last Equation

Now we try to determine the form of the operator FT_aF^{-1} (see above Sec. 3). We write

$$(FT_a F^{-1}\tilde{u})(\xi) = (FT_a u)(\xi) = \int\limits_{\mathbf{R}^{m-1}} e^{-ia|y'|\xi_m} e^{-iy'\cdot\xi'} \hat{u}(y', \xi_m) dy', \tag{53.7}$$

where $y' = (y_1, ...y_{m-1}), \hat{u}$ is the Fourier transform of u on last variable y_m.

We denote the convolution operator with symbol $A_{\neq}^{-1}(\xi)$ by letter a, so that by definition

$$(a*u)(x) = \int_{\mathbf{R}^m} a(x-y)u(y)dy,$$

or, for Fourier images,

$$F(a*u)(\xi) = A_{\neq}^{-1}(\xi)\tilde{u}(\xi).$$

As above, we denote $\hat{a}(x', \xi_m)$ the Fourier transform of convolution kernel $a(x)$ on the last variable x_m. The integral in (53.6) takes the form (according to (53.7))

$$\int_{\mathbf{R}^{m-1}} e^{-ia|y'|\xi_m} e^{-iy'\cdot\xi'} (\hat{a}*c)(y', \xi_m)dy',$$

Taking into account the properties of the convolution operator and the Fourier transform we have the following representation (see Section 53.3)

$$E_a * (A_{\neq}^{-1}(\xi)\tilde{c}(\xi')),$$

or, in more detail,

$$\int_{\mathbf{R}^{m-1}} E_a(\xi'-\eta', \xi_m)A_{\neq}^{-1}(\eta', \xi_m)\tilde{c}(\eta')d\eta'.$$

Then the equation (53.6) takes the form

$$\int_{\mathbf{R}^{m-1}} K_a(\eta', \xi'-\eta')\tilde{c}(\eta')d\eta' = \tilde{g}(\xi'), \tag{53.8}$$

where

$$K_a(\eta', \xi') = \int_{-\infty}^{+\infty} \frac{E_a(\xi', \xi_m)}{A_{\neq}(\eta', \xi_m)} d\xi_m.$$

So, the integral equation (53.8) is an equation for determining $\tilde{c}(\xi')$. This is a conical analogue of the double-layer potential.

Suppose that we solved this equation and constructed the inverse operator L_a, so that $L_a\tilde{g} = \tilde{c}$. By the way, we'll note the unique solvability condition for the equation (53.8) (i.e. existence of bounded operator L_a) is necessary and sufficient for unique solvability for our Dirichlet boundary value problem. Using the formula (53.5) we obtain

$$\tilde{u}_+(\xi) = A_{\neq}^{-1}(\xi)(L_a\tilde{g})(\xi'),$$

or, relabeling,

$$\tilde{u}_+(\xi) = A_{\neq}^{-1}(\xi)\tilde{d}_a(\xi').$$

Then

$$u_+(x', x_m) = \int\limits_{\mathbf{R}^{m-1}} W(x' - y', x_m) d_a(y') dy', \tag{53.9}$$

where $W(x', x_m) = F_{\xi \to x}^{-1}(A_{\neq}^{-1}(\xi))$.

Formula (53.9) is the analogue of the Poisson integral for a half-space.

53.7 Comparison with the Half-Space Case for the Laplacian

For the half-space $x_m > 0$ we have the following (see Eskin's book [Es81]):

$$\tilde{u}_+(\xi) = \frac{\tilde{c}(\xi')}{\xi_m + i|\xi'|}.$$

If we have the Dirichlet condition on the boundary, then the function

$$\tilde{g}(\xi') = \int\limits_{-\infty}^{+\infty} \tilde{u}_+(\xi) d\xi_m$$

is given.

From the formula above we have

$$\tilde{g}(\xi') = \tilde{c}(\xi') \int\limits_{-\infty}^{+\infty} \frac{d\xi_m}{\xi_m + i|\xi'|},$$

and we need to calculate the last integral only.

For this case we can use the residue technique and find that the last integral is equal to $-\pi i$. Thus,

$$\tilde{u}_+(\xi) = -\frac{\tilde{g}(\xi')}{\pi i(\xi_m + i|\xi'|)}.$$

Consequently, our solution $u_+(x)$ is the convolution (for first $(m-1)$ variables) of the given function $g(x')$ and the kernel defined by inverse Fourier transform of function $(\xi_m + i|\xi'|)^{-1}$ (up to a constant). The inverse Fourier transform on variable ξ_m leads to the function $e^{-x_m|\xi'|}$, and further, the inverse Fourier transform $\xi' \to x'$ leads to Poisson kernel

$$P(x', x_m) = \frac{c_m x_m}{(|x'|^2 + x_m^2)^{m/2}},$$

c_m is certain constant defined by Euler Γ-function.

Thus, for the solution of the Dirichlet problem in half-space $\mathbf{R}^m_+$ for the Laplacian with given Dirichlet data $g(x')$ on the boundary $\mathbf{R}^{m-1}$ we have the integral representation

$$u_+(x', x_m) = \int\limits_{\mathbf{R}^{m-1}} P(x' - y', x_m) g(y') dy'.$$

53.8 Oblique Derivative Problem

We go back to formula (53.5). We can write

$$\xi_k \tilde{u}_+(\xi) = \xi_k A^{-1}_{\neq}(\xi) \tilde{c}(\xi),$$

or equivalently according to Fourier transform properties

$$\frac{\partial u_+}{\partial x_m} = F^{-1}\{\xi_k A^{-1}_{\neq}(\xi) \tilde{c}(\xi)\},$$

for arbitrary fixed $k = 1, 2, ..., m$.

Further, we apply the operator T_a and work as above. Our considerations will be the same, and in all places instead of $A^{-1}_{\neq}(\xi)$ will stand $\xi_k A^{-1}_{\neq}(\xi)$. We call this situation the oblique derivative problem, because $\frac{\partial}{\partial x_k}$ related to conical surface is not normal derivative exactly.

Remark 2. Some words on the Neumann problem. If we try to give normal derivative of our solution on conical surface different from origin, then we have the boundary value problem with variable coefficients because the boundary condition varies from one point to another one on conical surface. We need additional localization for such points to reduce it to the case of constant coefficients and consider corresponding model problem in $\mathbf{R}^m_+$. Roughly speaking, we would say that the solution looks locally different in dependence on the type of boundary point. In other words, local principle permits to work with symbols and boundary conditions independent of the space variable.

53.9 Conclusions

It seems that to solve explicitly the simplest boundary value problems in domains with conical point, we need to use another potentials different from classical single-layer and double-layer potentials. This fact will be shown for the Laplacian with Dirichlet condition on a conical surface by direct calculations in a future paper.

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