Chapter 9
The Simple Majority Rule in a Three-Valued Logic Framework

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9.1 Introduction

In the past decade a growing number of papers on different issues in social choice theory appealed to formal techniques originating in various branches of modern formal logic.[1] First-order (Grandi and Endriss 2013) and higher-order logic (Nipkow 2009), modal logic (Ågotnes et al. 2009; Pauly 2007), many-valued and fuzzy logics (Barrett and Salles 2011) helped provide more rigorous formalizations of social choice problems and, as a result, gain a deeper understanding of the field (Endriss 2011).

The study of judgment aggregation, initiated by (List and Pettit 2002) is an exemplar case. It focuses on the way in which the sets of judgments held by the members of a group can be aggregated to form a collective set of judgments. A judgment set is a subset of a given "agenda". The agenda is a set of propositions upon which a collective judgment is sought. An individual's judgment set contains exactly those propositions in the agenda that the individual believes to be true. The large literature on judgment aggregation usually assumes that the evaluations of the propositions allow for a proposition to be either true of false. However, an increasing

[1] Of course, logic had a much older role in modern social choice theory, going back to the early 1940s. In an interview published in Arrow et al. (2011). K.J. Arrow remembers an episode from his student years. In his last term he took a course on logic with the Polish logician A. Tarski. A testimony of the influence of Tarski on the young Arrow is to be found in the 1940 author preface to the English edition of *An Introduction to Logic* (Tarski 1959), where Tarski thanks K. J. Arrow for his help in reading proofs.

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number of papers focus on multi-valued evaluations: the truth values of a proposition range over a larger set, allowing for intermediate degrees of truth in between true and false. Dokow and Holzman (2010) work with non-binary aggregators that have multi-valued judgment aggregation as one natural interpretation. Pauly and van Hees (2006) and van Hees (2007) studied judgment aggregation in the framework of many-valued logic, drawing in particular on Post's many-valued systems. Duddy and Piggins (2013) used a Łukasiewicz-type multi-valued logic.

In this paper I shall follow a different approach to the use of many-valued logic in social choice. It was initiated nearly a half of century ago by Murakami (1966, 1968), who first considered logical mechanisms originating in many-valued logic in his pioneering study of representative decision-making.[2] On this approach, propositions do not describe issues, but attitudes of the individuals toward issues, like for example how to choose between two alternatives; thus, the atomic proposition p_1 is taken to describe the attitude of the individual A_1 with respect to the alternatives x and y, and p_2 to describe the attitude of the individual A_2 with respect to the same alternatives, etc. Attitudes can have two values: for example, A_1 either prefers x to y or prefers y to x; or can have multiple values, when intermediate cases are allowed (for example A_1 can be indifferent between x and y.)

The intuitive idea is that logical operators are similar to aggregation rules. An aggregation rule gives a social preference for each distribution of preferences of the members of a certain group. Similarly, logical operators like disjunction and conjunction give an aggregate truth-value for each distribution of truth-values of the propositions they connect. In general, if π is a binary logical operator, then $\pi(pq)$ gives the aggregate attitude of the group formed of the individuals A_1 and A_2 with respect to the two alternatives x and y.

Naturally, the question that immediately comes into one's mind is if logical operators corresponding to well-known aggregation rules like for example the simple majority rule or the absolute majority rule can be identified in a logical framework. Following this path, I rely on a Łukasiewiczian three-valued logic to define logical operators that can be easily compared with such aggregation rules like for example the simple and the absolute majority rules, the Jury rule or the extended Pareto rule.

On this account, compound propositions receive a quite different interpretation. Consider for example the proposition $\pi(\pi(p_1p_2)\pi(p_3p_4))$. Its meaning is this: by appealing to the aggregation rule π a collective decision of the group formed of the individuals A_1 and A_2 is reached. Similarly, a collective decision of the group formed of the individuals A_3 and A_4 is reached. Then these decisions are aggregated in a higher-order group formed of the two groups. It looks natural to try to interpret iterated applications of the decision rules in terms of representative systems or democracy. Murakami defined a representative system 'as a hierarchy of voting procedures, each of which may be called a council.Every individual casts a ballot

[2]Fine (1972) and Fishburn (1971) developed his approach.

or ballots in one council or councils. A decision in each council is represented in a higher council whose decision is, in turn, represented in a still higher council and so on' (Murakami 1966).

The paper is organized as follows. Section 9.2 presents the framework. The standard operators (negation, implication, disjunction and conjunction) in a Łukasiewicz-type three-valued logic are introduced. In Sect. 9.3 a new binary operator μ is also introduced; its intended meaning is that of the simple majority rule. Majority rule is usually studied in the general case, when the group of people who are to make a decision is large. The two-member groups are viewed as degenerate cases that do not deserve a special attention. Here I take the reverse path: I start with the binary rule and only then move to the general, n-ary case. The binary majority operator μ^2 has a very clear interpretation as a binary logical operator, just like the standard logical operators conjunction, disjunction and implication. In analogy with May's (1954) famous characterization of the majority rule, here μ^2 is characterized in terms of four properties: commutativity, self-duality, monotonicity and responsiveness. If the framework is slightly extended to allow for individual attitudes toward three alternatives then we can easily obtain a toy counterpart of Arrow's impossibility theorem. It shows that no binary operator can be both unanimous and transitive. The main argument in Sect. 9.4 is to prove that the binary operator μ^2 can be extended to the n-ary case, in the sense that all applications of the majority rule μ to a sequence consisting in n members can be defined in terms of the binary majority rule μ^2. Far from being a degenerate case, the binary majority rule operator is able to account for all n-ary applications. It is argued, however, that other binary operators corresponding to other voting rules cannot be so extended. Examples include the absolute majority rule operator and the Jury rule operator. Section 9.5 concludes.

9.2 The Framework

As usual (Urquhart et al. 2001), we start with a countable (infinite) set of propositions $\Sigma = \{p, q, r, \dots p_1. \ \dots \ p_n, \dots\}$, called a propositional signature. Θ is a finite set of operator names, sometimes also called (propositional) functions. A function ρ attaches to each operator in Θ a non-negative integer called its arity. A propositional language is a pair $\mathbf{L} = (\Theta, \ \rho)$. The set $\mathbf{L}_\Sigma$ of formulas of the language $\mathbf{L}$ given a signature Σ is defined inductively as follows:

(1) Members of Σ are $\mathbf{L}_\Sigma$ formulas:
(2) If $\theta \ \varepsilon \ \Theta$, $\rho(\theta) = m > 0$ and $\psi_1, \dots \psi_m$ are $\mathbf{L}_\Sigma$ formulas, then $\theta(\psi_1, \dots \psi_m)$ is also a $\mathbf{L}_\Sigma$ formula.

Say negation to the unary operator $\sim$; binary operators are, e.g. well-known logical operators like implication $\rightarrow$, conjunction $\wedge$, disjunction $\vee$, equivalence $\equiv$.

For a given signature, a propositional letter can take one of the following three values: 1, 0, −1. In Łukasiewicz's three-valued logic,[3] they refer to truth, possible, and false. However, the interpretation I shall constantly have in mind is different. Suppose that $p_1, \ldots p_n$ express the attitudes of n individuals $A_1, \ldots A_n$ with respect to two alternatives x and y. Value 1 means that the individual prefers alternative x to y; value −1 means that the individual prefers alternative y to x, and value 0 carries the meaning that the individual is indifferent between the two alternatives. A n-ary operator π^n represents an aggregation of the attitudes of the n individuals. A matrix $\mathbf{M}_\theta$ for an n-ary operator $\theta\ \varepsilon\ \Theta$ is function which attaches to each n-tuple $(a_1, \ldots a_n)$, where $a_i \varepsilon \{1, 0, -1\}$, a member of $\{1, 0, -1\}$. A matrix $\mathbf{M}_\Theta$ is the collection of all $\mathbf{M}_\theta$, for $\theta\ \varepsilon\ \Theta$. Łukasiewicz's three-valued logic is a pair L3 = (**L**, **M**), where **L** is a propositional language and **M** is a matrix for Θ. For the most important logical operators, $\mathbf{M}_\Theta$ is given by:

p	*p*	*Fp*	*Vp*
1	−1	−1	1
0	0	−1	1
−1	1	−1	1

→	1	0	−1
1	1	0	−1
0	1	1	0
−1	1	1	1

∧	1	0	−1
1	1	0	−1
0	0	0	−1
−1	−1	−1	−1

∨	1	0	−1
1	1	1	1
0	1	0	0
−1	1	0	−1

With the above interpretation of the meanings of p and q in mind, the expression $p \vee q$ expresses the attitude of the group formed of the individuals A and B relative to the alternatives x and y. The group's attitude follows the most favorable attitude to x of its members: if A or at least B prefers x, then the group also prefers x; if the most favorable attitude to x of the members of the group is indifference, then the group is indifferent between x and y, but if both A and B prefer y, then the group {A, B} also prefers y. Analogously, in the case of the expression $p \wedge q$ the attitude of the group {A, B} follows the least favorable attitude of its members toward x.

[3] I write 1, 0 and −1 as usually done is social choice theory instead of the usual values: 1, ½ and 0 in order to emphasize that a social choice interpretation is here intended.

Now let $\Sigma = \{p_1, p_2 \ldots p_n, \ldots\}$ be a signature. A value assignment is a sequence $\mathbf{a}_\Sigma = a_1 a_2 \ldots a_{n-1} a_n \ldots$, with $a_i \varepsilon \{1, 0, -1\}$. In what follows, I shall also appeal to an initial segment $\mathbf{a}_{\Sigma n} = a_1 a_2 \ldots a_{n-1} a_n$ of a value assignment and call it the value assignment for the sequence $\mathbf{p} = p_1 p_2 \ldots p_{n-1} p_n$ of propositional variables.

In this paper the following notation is used: when focusing on an n-ary operator π^n $(n \geq 1)$, I shall write π^n $(p_1 p_2 \ldots p_{n-1} p_n)$ and appeal to parentheses to clearly distinguish the expressions it applies to. When the arity n of an operator π^n is not the main focus, I shall simply write π instead of π^n. This convention applies to logical operators too. But the usual notations will be also appealed to when the operators $\wedge$, $\vee$, $\rightarrow$, and $\equiv$ will be used in the definitions of the properties of the operators. The expression $p \Rightarrow q$ is short for: $(p \rightarrow q) \wedge \sim (q \rightarrow p)$.

9.3 The Binary Majority Rule

In this section I shall discuss one binary operator μ^2 called the (simple) majority rule. It is defined as follows[4]: $\mu^2(pq) = \text{sgn}\,(p + q)$. The sgn function is given by: (i) if $n > 0$, then $\text{sgn}(n) = 1$; (ii) if $n < 0$, then $\text{sgn}(n) = -1$; and (iii) if $n = 0$, then $\text{sgn}(n) = 0$.

μ^2	1	0	−1
1	1	1	0
0	1	0	−1
−1	0	−1	−1

The operator μ^2 has the following intuitive interpretation. Suppose again that p and q express the attitudes of two voters A and B with respect to the two alternatives x and y. So, intuitively μ^2 aggregates the preferences of the two voters as follows: if both prefer an alternative, then they prefer it collectively; if both are indifferent, then their joint preference is also indifference; if they have opposite preferences, then they jointly do not prefer any alternative. Finally, if only one voter is indifferent, then their common preference is the preference of the other voter.

Below I shall prove that μ^2 is the only binary operator that satisfies some attractive properties. For the beginning, let me introduce a battery of such properties.

- A binary operator is independent if

$$(q \equiv r) \rightarrow (\pi(pq) \equiv \pi(pr)); \text{ and}$$
$$(p \equiv r) \rightarrow (\pi(pq) \equiv \pi(rq))$$

[4]This is the restriction of the general definition of the simple majority rule to the two member groups case; see the definition of μ in the next section.

Independence is built in the very definition of logical operators: when they aggregate propositions only truth-value of the compound propositions is relevant. Therefore in the remainder of this paper we shall take independence as satisfied by default.

- A binary operator π^2 is commutative if $\pi^2(pq) \equiv \pi^2(qp)$.

The operators $\wedge$, $\vee$, $\equiv$ are commutative; e.g., $(p \equiv q) \equiv (q \equiv p)$ states that $\equiv$ is commutative. Commutativity requires that the individuals A and B who hold attitudes toward the two alternatives x and y must be equally treated: it does not matter if we first consider A's attitude as expressed by p and then B's attitude as expressed by q, or the other way round.[5]

- A binary operator is self-dual if $\pi^2(pq) \equiv\sim \pi^2(\sim p \sim q)$.

Self-duality entails that the alternatives x and y must be equally treated.[6] Dictatorial and anti-dictatorial operators are self-dual. A dictatorial operator always gives the value of one of its arguments; and anti-dictatorial operator always gives the opposite value of one of its arguments. Example of a dictatorial binary operator: $\pi^2(pq) \equiv p$; example of an anti-dictatorial binary operator: $\pi^2(pq) \equiv\sim p$. Operators $\wedge$ and $\vee$ are not self-dual.

Some binary operators are both commutative and self-dual, such as:

α^2	1	0	−1
1	1	0	0
0	0	0	0
−1	0	0	−1

The operator α^2 is the binary absolute majority rule. We have $\alpha^2(pq) = 1$ if both $p = 1$ and $q = 1$; $\alpha(pq) = -1$ if both $p = -1$ and $q = -1$, and $\alpha^2(pq) = 0$ in all the other cases. I shall return to the absolute majority operator in the next section.

- A binary operator is monotonic if:

$$(p \to r) \to \left(\pi^2(pq) \to \pi^2(rq)\right); \quad \text{and}$$
$$(q \to r) \to \left(\pi^2(pq) \to \pi^2(pr)\right)$$

Examples: α^2 is monotonic; $\wedge$ and $\vee$ are also monotonic; but $\to$ is not monotonic.

- A binary operator is responsive if $\left(\left(\pi^2(pq) \equiv\sim \pi^2(pq)\right) \wedge (p \Rightarrow r)\right) \to \pi^2(rq)$

By responsiveness, if π^2 gives indifference, and one of its arguments is replaced with another argument having a higher value (0 or 1 instead of −1, or 1 instead of 0), then the value of π^2 moves to 1. The operators μ^2 and

[5] A commutative logical operator corresponds then to an anonymous rule.

[6] Clearly, a self-dual logical operator corresponds to a neutral rule.

$\eta^2 = \max(p+q-1, -1)$ are responsive; conjunction $\wedge$, disjunction $\vee$ and implication $\rightarrow$ are not responsive. The matrix for η^2 is given below[7]:

η^2	1	0	−1
1	1	0	−1
0	0	−1	−1
−1	−1	−1	−1

Note: if π^2 is self-dual and monotonic, then responsiveness entails:

$$\big(\big(\pi^2(pq) \equiv\sim \pi^2(pq)\big) \wedge (r \Rightarrow p)\big) \rightarrow\sim \pi^2(rq)$$

(and similarly for q). So, by responsiveness if $\pi^2(pq) = 0$ and one of its arguments is replaced by a proposition with a higher value, then the value of the aggregate $\pi^2(pq)$ moves to 1 (while if an argument is replaced by a proposition with a lower value then the value of $\pi^2(pq)$ goes down to −1).

Another important property is unanimity. Suppose that both arguments p and q of π^2 have the same value a; then $\pi^2(pq)$ has the same value a:

- A binary operator is unanimous if $(p \equiv q) \rightarrow \big(\pi^2(pq) \equiv p\big)$

Conjunction $\wedge$, disjunction $\vee$, α^2 and μ^2 are examples of unanimous operators. Implication and η^2 are not unanimous.

The first result to be presented is Theorem 1. It states that the binary operator μ^2 can be characterized in terms of the properties of commutativity, self-duality, monotonicity and responsiveness: μ^2 satisfies each of them, and no other logical operator satisfies them all.

Theorem 1 π^2 is a commutative, self-dual, monotonic and responsive binary operator if and only if $\pi^2 = \mu^2$.

Proof One direction of the proof is straightforward: μ^2 satisfies the four properties. For the converse direction, suppose that the binary operator π^2 is commutative, self-dual, monotonic and responsive. We want to prove that $\pi^2(pq) = \text{sgn}\,(p+q)$. We have nine cases: (11), (10), (1−1), (01), (00), (0−1), (−11), (−10), and (−1−1). Since π^2 is commutative, cases (10) and (01); (1−1) and (–11); (0−1) and (–10) are similar. Since π^2 is self-dual, cases (10) and (–10); (01) and (0–1); and (11) and (−1−1) are also similar. So we need to analyze only cases (11), (10), (1−1), and (00).

Case 1: (00). Since π^2 is self-dual, we have $\pi^2(00) \equiv\sim \pi^2(\sim 0 \sim 0) \equiv\sim \pi^2(00)$. But $\pi^2(00) \equiv\sim \pi^2(00)$ only if $\pi^2(00) = 0 = \text{sgn}\,(0+0)$.

Case 2: (10). We have $\pi^2(00) = 0$; responsiveness entails that $\pi^2(10) = 1 = \text{sgn}\,(1+0)$.

[7] Duddy and Piggins (2013) call η^2 the Łukasiewicz triangular norm T_L and appeal to it to characterize the deductively closed and free from veto power collective judgments. An important property of η^n is that it is associative.

Case 3: (11). As shown in case 2, $\pi^2(10) = 1$. We also have that $0 \to 1$, and so by the monotonicity of π^2 we get $\pi^2(11) = 1 = \text{sgn}\,(1 + 1)$.
Case 4: (1−1). By self-duality, we have $\pi^2\,(1 - 1) =\sim \pi^2\,(-11)$. By commutativity $\pi^2\,(-11) = \pi^2\,(1 - 1)$ and so $\pi^2\,(1 - 1) =\sim \pi^2\,(1 - 1)$, which can only hold when $\pi^2\,(1 - 1) = 0 = \text{sgn}\,(1 + (-1))$.

Theorem 1 is the counterpart in this formalism of the well-known axiomatization of the simple majority rule given in (May 1954). I appealed to a weaker version of his responsiveness axiom, which required the addition of a special monotonicity axiom.

One may attempt to characterize other logical operators in an analogous way. Here are two examples: conjunction $\wedge^2$ and disjunction $\vee^2$. I first introduce two new properties, top and bottom group preferences:

Top Group preference (**TGP**). If $p \Rightarrow q$, then $\pi^2(pq) \equiv q$.
Bottom Group Preference (**BGP**). If $p \Rightarrow q$, then $\pi^2(pq) \equiv p$.

By **TGP** the group always prefers the option most favorable to the alternative *x*, while by **BGP** the group always prefers the option most favorable to the alternative *y*.

Notice that no operator can be commutative, self-dual and also satisfy one of the axioms **BGP** and **TGP**. To see this, take for example **BGP** and consider the case (01). We have $0 \Rightarrow 1$ and so $\pi^2(01) = 0$ by **BGP**. Now self-duality requires that $\sim \pi^2\,(0 - 1) = 0$ and so $\pi^2\,(0 - 1) = 0$. But if π is commutative then $\pi^2\,(-10) = 0$. Since $-1 \Rightarrow 0$ by applying again **BGP** we get that $\pi^2\,(0 - 1) = -1$ – contradiction. As observed above, self-duality entails that the alternatives *x* and *y* must be treated equally; but both **TGP** and **BGP** entail that one of the two alternatives enjoys a special status.

The following theorem characterizes the two logical binary operators conjunction and disjunction:

Theorem 2

(a) π^2 is commutative, unanimous and satisfies **BGP** if and only if $\pi^2 = \wedge^2$.
(b) π^2 is commutative, unanimous and satisfies **TGP** if and only if $\pi^2 = \vee^2$.

Proof I give the proof of part (a); the proof for part (b) is similar. Clearly $\wedge^2$ satisfies the three properties. For the converse direction of the proof let us assume that π^2 has the three properties. We consider all possible cases:

Cases (11), (00) and (−1−1): since π^2 is unanimous, we have $\pi^2(aa) = a = \wedge^2(aa)$.
Cases (0−1) and (−10): since $-1 \Rightarrow 0$, by **BGP** we get that $\pi^2\,(-10) = -1 = \wedge^2\,(-10)$; by commutativity we have $\pi^2\,(-10) = \pi^2\,(-10) = -1 = \wedge^2\,(-10)$.
Cases (01) and (10): $0 \Rightarrow 1$ entails by **BGP** that $\pi^2(01) = 0 = \wedge^2(10)$. Commutativity gives again $\pi^2(10) = \pi^2(01) = 0 = \wedge^2(10)$.

Cases (1−1) and (−11): in the same way as above we have $\pi^2(-11) = 1 = \wedge^2(-11)$ because $-1 \Rightarrow 1$. Since π^2 is commutative it follows that $\pi^2(1-1) = \pi^2(-11) = 1 = \wedge^2(1-1)$.

As mentioned above, in the present framework the analysis is restricted to only two alternatives *x* and *y*. But let for a moment try to broaden it. Suppose that p^1, q^1 etc. express the attitudes of the individuals A, B etc. on the relation between the alternatives *x* and *y*; p^2, q^2 etc. express the attitudes of the individuals A, B etc. on the relation between the alternatives *y* and *z*; and p^3, q^3 etc. express the attitudes of the individuals A, B etc. on the relation between the alternatives *x* and *z*. I assume that all the individuals A, B etc. have consistent attitudes, and so the following must hold:

$$\left(\left(p^1 \to p^2\right) \wedge \left(p^2 \to p^3\right)\right) \to \left(p^1 \to p^3\right) \tag{9.1}$$

$$\left(\left(q^1 \to q^2\right) \wedge \left(q^2 \to q^3\right)\right) \to \left(q^1 \to q^3\right) \tag{9.2}$$

Say that a binary operator π is transitive if:

$$\left(\pi\left(p^1q^1\right) \wedge \pi\left(p^2q^2\right)\right) \to \pi\left(p^3q^3\right)$$

The following result is the counterpart of Arrow's theorem in this framework:

Theorem 3 If π is unanimous and transitive, then it is dictatorial.

Proof Assume that $p^1 = 1$, $p^2 = 0$, and also $q^1 = 0$, $q^2 = 0$. Then by unanimity we get $\pi\left(p^1q^1\right) = \pi(11) = 1$ and also $\pi\left(p^2q^2\right) = \pi(00) = 1$. (Here the fact that π satisfies Independence was assumed.) By the transitivity of π if follows that $\pi\left(p^3q^3\right) = 1$. Since π is unanimous, $\pi\left(p^3q^3\right) = 1$ must hold whenever $p^3 \equiv q^3$. We have three cases. First, if $p^3 = 1$ and $q^3 = 1$, then $\pi(11) = 1$. Secondly, if $p^3 = 0$ and $q^3 = 0$, then $\pi(00) = 1$. Observe that in both cases the expressions (9.1) and (9.2) have value 1: the individuals A and B have consistent preferences. Third, put $p^3 = -1$ and $q^3 = -1$. We get $\pi(-1-1) = 1$. But in this case we face a contradiction, because as we can easily check the expressions (9.1) and (9.2) do not hold: both have the value 0, i.e. both individuals A and B fail to hold transitive attitudes toward the three alternatives *x*, *y* and *z*. However, the possibility to construct the case when $p^1 = 1$, $p^2 = 0$, $p^3 = -1$ and $q^1 = 1$, $q^2 = 0$, $q^3 = -1$ and so to produce a contradiction is ruled out if π is dictatorial. For if π is dictatorial, we must have for example $\pi\left(p^1q^1\right) = p^1$, $\pi\left(p^2q^2\right) = p^2$ and $\pi\left(p^3q^3\right) = p^3$. Given (9.1), this does not allow for $\pi\left(p^1q^1\right) = 1$, $\pi\left(p^2q^2\right) = 0$ and $\pi\left(p^3q^3\right) = -1$; and similarly if π takes always the value of *q*.

9.4 *n*-ary Operators

In this section I introduce some n-ary operators originating in social choice theory and study their relation to the corresponding binary operators: the simple majority rule μ^n, the absolute majority rule α^n, the extended Pareto rule ε^n and the jury rule λ^n. Their definitions are as follows. Let $\mathbf{p} = p_1 p_2 \ldots p_{n-1} p_n$ be a sequence of propositional variables (the propositional letters in the sequence need not be different). As suggested above, intuitively p_i expresses the attitude of some voter i with respect to two alternatives x and y, with value $p_i = 1$ meaning that i prefers alternative x to y, $p_i = -1$ meaning that i prefers alternative y to x, and $p_i = 0$ carrying the meaning that i is indifferent between the two alternatives. Then $\mu^n(\mathbf{p}) = 1$ if more members of the society prefer x to y; $\mu^n(\mathbf{p}) = -1$ if more voters prefer y to x; and $\mu^n(\mathbf{p}) = 0$ if the votes are equally distributed between x and y. By the absolute majority rule α^n, an alternative is preferred by the society if it is preferred by more than half of the total number of members of the society; and the society is indifferent between two alternatives if none is preferred by more than half of the total members of the society.[8] By the extended Pareto rule ε^n an alternative is preferred by a society if all its members prefer it, and is indifferent in all the other cases. By the jury rule λ^n an alternative is preferred by a society if none of its members opposes it and at least some person prefers it, and is indifferent in all the other cases.

More formally, we have:

- The n-ary simple majority rule μ^n is defined by: $\mu^n(\mathbf{p}) = \operatorname{sgn}\left(\sum_{i=1}^{n} p_i\right)$.
- The n-ary absolute majority α^n is given by: $\alpha^n(p_1 p_2 \ldots p_{n-1} p_n) = 1$ if more than $n/2$ of the p_i's have value 1; $\alpha^n(p_1 p_2 \ldots p_{n-1} p_n) = -1$ if more than $n/2$ of the p_i's have value -1; and $\alpha^n(p_1 p_2 \ldots p_{n-1} p_n) = 0$ in all the other cases.
- The n-ary extended Pareto rule ε^n is defined by: $\varepsilon^n(\mathbf{p}) = 1$ if $p_i = 1$ for all i; $\varepsilon^n(\mathbf{p}) = -1$ if $p_i = -1$ for all i and $\varepsilon^n(\mathbf{p}) = 0$ in all the other cases.
- The n-ary jury rule λ^n is defined by: $\lambda^n(\mathbf{p}) = 1$ if $p_i \geq 0$ for all i, and $p_i = 1$ for some i; $\lambda^n(\mathbf{p}) = -1$ if $p_i \leq 0$ for all i and $p_i = -1$ for some i; and $\lambda^n(\mathbf{p}) = 0$ in all the other cases.

The binary operators are special cases of the n-ary ones. Specifically, note that $\mu^2 = \lambda^2$ and $\alpha^2 = \varepsilon^2$. However, it is interesting to study the converse relation: is it possible to extend binary operators to the n-ary case? For some binary operators this operation can be done in a quite straightforward manner by appealing to the property of associativity. For example, it is usual to extend conjunction $\wedge$ and disjunction $\vee$ to n arguments as follows:

[8] The binary case is awkward: since for a society formed of exactly two members more than half equals two, by α^2 an alternative is collectively preferred if it is preferred by both individuals, while indifference occurs in all the other cases.

$$\left.\begin{array}{l}\wedge\left(p_1 p_2 \ldots p_{n-1} p_n\right)=\min \left(p_1, p_2 \ldots p_{n-1}, p_n\right) \\ \vee\left(p_1 p_2 \ldots p_{n-1} p_n\right)=\max \left(p_1, p_2 \ldots p_{n-1}, p_n\right)\end{array}\right) \begin{array}{l}=\wedge\left(p_1 \wedge\left(p_2 \ldots p_{n-1} p_n\right)\right); \\ =\vee\left(p_1 \vee\left(p_2 \ldots p_{n-1} p_n\right)\right).\end{array}$$

When $n = 3$, we have in the case of conjunction: $\wedge (p_1 p_2 p_3) = (p_1 \wedge (p_2 \wedge p_3))$. Other operators also have this property. One example is the operator η introduced in the above section; the social choice operator ε (the extended Pareto rule) can also be extended in the same way:

$$\varepsilon\left(p_1 p_2 \ldots p_{n-1} p_n\right)=\varepsilon\left(p_1 \varepsilon\left(p_2 \ldots p_{n-1} p_n\right)\right)$$

But operators like the simple majority rule μ are not associative. Consider three propositions, p_1, p_2 and p_3, taken to express the attitudes of three persons A, B and C concerning the alternatives x and y. Suppose that the individual A votes for y while B and C vote for x. Then $p_1 = -1$, $p_2 = p_3 = 1$. By definition, $\mu\,(p_1 p_2 p_3) = \operatorname{sgn}\ (p_1 + p_2 + p_3)$. Then we must get $\mu\,(p_1 p_2 p_3) = \operatorname{sgn}\ (-1 + 1 + 1) = 1$. However, $\mu\,(p_1 \mu\,(p_2 p_3)) = \mu\,(-1 \mu(11)) = \mu\,(-11) = 0$, which shows that the attempt to extend the binary majority rule operator to the ternary case fails if we want to appeal to the standard method based on the property of associativity.

In this section I describe an alternative method to extend μ to the n-ary case. Then I prove that α^2 and λ^2 behave quite differently: they resist all attempts to be extended. This means that, e.g. in the case of the absolute majority rule α^2 we cannot construct any logical expression σ_α with the property that it contains only occurrences of the binary operator α^2 and $\alpha^n\,(p_1 p_2 \ldots p_{n-1} p_n) \equiv \sigma_\alpha$ is true for all value-assignments.

Let $\mathbf{p} = p_1 p_2 \ldots p_{n-1} p_n$ be a sequence of propositional variables. By definition we must have $\mu^n\,(\mathbf{p}) = \mu^n\,(p_1 p_2 \ldots p_{n-1} p_n) = \operatorname{sgn}\left(\sum_{i=1}^{n} p_i\right)$. I shall denote by $\mathbf{p}^{-i}$ the sequence resulting from $\mathbf{p}$ by deleting p_i from it. We need to construct a logical expression σ_μ with the property that contains only occurrences of the binary operator μ^2 and $\mu^n\,(p_1 p_2 \ldots p_{n-1} p_n) \equiv \sigma_\mu$ is true for all value-assignments. I first give two helpful lemmas.

Lemma 1

(a) If $\mu^n\,(\mathbf{p}) = 1$ and $p_i < 1$ for some $i = 1, \ldots n$ then $\mu^{n-1}\left(\mathbf{p}^{-i}\right) = 1$.
(b) If $\mu^n\,(\mathbf{p}) = 1$, then $\mu^{n-1}\left(\mathbf{p}^{-i}\right) \geq 1$ for each $i = 1, \ldots n$ and $\mu^{n-1}\left(\mathbf{p}^{-i}\right) = 1$ for some i.

Proof For (a), the proof is immediate once we appeal to the definition of μ^n. Observe that $\mu\,(\mathbf{p}) = 1$ entails that $\sum_{i=1}^{n-1} p_i \geq 1 - p_n$. Then clearly $\sum_{i=1}^{n-1} p_i \geq 1$ if $p_n < 1$ and so $\operatorname{sgn}\left(\sum_{i=1}^{n-1} p_i\right) = 1 = \mu\,(\mathbf{p}^{-n})$. For (b) suppose, without loss of generality, that $i = n$. By definition, $\mu^n(\mathbf{p}) = \mathrm{sgn}\left(\sum_{i=1}^{n-1} p_i + p_n\right)$. If $\mu^n(\mathbf{p}) = 1$, it follows that $\sum_{i=1}^{n-1} p_i + p_n \geq 1$. Now we can only have $\mu(\mathbf{p}^{-n}) < 0$ if $\mathrm{sgn}\left(\sum_{i=1}^{n-1} p_i\right) = -1$, and thus $\left(\sum_{i=1}^{n-1} p_i\right) < 0$; but in this case $\sum_{i=1}^{n-1} p_i + p_n < 1$ for each value of p_n – contradiction. But we cannot have $\mu^{n-1}\left(\mathbf{p}^{ik}\right) = 0$ for each i. Consider the following cases.

Case 1: $p_i = 1$ for all i. Then by unanimity $\mu^{n-1}\left(\mathbf{p}^{-i}\right) = 1$ for each i.
Case 2: there is some i such that $p_i < 1$. But then by (a) we have $\mu^{n-1}\left(\mathbf{p}^{-i}\right) = 1$.

The following dual propositions can be proved in a similar way:

(a) if $\mu(\mathbf{p}) = -1$ and $p_i > -1$ for some $i = 1, \ldots n$, then $\mu^{n-1}\left(\mathbf{p}^{-i}\right) = -1$.
(b) if $\mu(\mathbf{p}) = -1$, then $\mu\left(\mathbf{p}^{-i}\right) \leq 0$ for each $i = 1, \ldots n$ and $\mu\left(\mathbf{p}^{-i}\right) = -1$ for some i.

Lemma 2 $\mu^n(\mathbf{p}) = \mu^n\left(\mu^{n-1}\left(\mathbf{p}^{-1}\right), \ldots \mu(\mathbf{p}^{-n})\right)$

Proof The proof of the lemma is by induction on the number of members of the sequence $\mathbf{p}$. First, let $n = 2$, and so $\mathbf{p} = (p_1 p_2)$. But $\mu^2(p_1 p_2) = \mu^2\left(\mu\left(\mathbf{p}^{-1}\right)\mu^2\left(\mathbf{p}^{-2}\right)\right)$. Since as noted above the operator μ satisfies the property of unanimity we get $\mu\left(\mathbf{p}^{-1}\right) = \mu(p_2) = p_2$ and $\mu\left(\mathbf{p}^{-2}\right) = \mu(p_1) = p_1$ and so $\mu^2(p_1 p_2) = \mu^2(p_2 p_1)$, which is true by commutativity.

Now let $n > 2$. Suppose that s of the members of $\mathbf{p}$ have value 1, z of its members have value 0, and m of its members have value -1, and $n = s + z + m$. Since μ is commutative, we can write $\mathbf{p}$ as follows: $p_1, \ldots p_s,\ p_{s+1} \ldots p_{s+z},\ p_{s+z+1} \ldots p_{s+z+m} = p_n$. We have three cases.

Case 1: $\mu^n(\mathbf{p}) = 1$. Then by Lemma 1 all $\mathbf{p}^{-i}$'s are such that $\mu^{n-1}\left(\mathbf{p}^{-i}\right) = 1$ or $\mu^{n-1}\left(\mathbf{p}^{-i}\right) = 0$, and there is some i such that $\mu^{n-1}\left(\mathbf{p}^{-i}\right) = 1$. Then clearly $\mu^n\left(\mu^{n-1}\left(\mathbf{p}^{-1}\right), \ldots \mu^{n-1}(\mathbf{p}^{-n})\right) = 1$ because $\sum_{i=1}^{n} \mathbf{p}^{-i} \geq 1$.
Case 2: $\mu^n(\mathbf{p}) = -1$. The proof is just like in case 1.
Case 3: $\mu^n(\mathbf{p}) = 0$. We have two subcases:
Subcase 3a: $s = m = 0$, and so $z = n$. Observe also that $\mu^n(\mathbf{p}) = \mu^{n-1}\left(\mathbf{p}^{-i}\right)$ $(i = 1, \ldots n)$, because deleting a p_i with value 0 does not change the value of μ. Therefore by unanimity the result is proved. Subcase 3b: $s = m \neq 0$. We have in z cases $\mu^{n-1}\left(\mathbf{p}^{-i}\right) = 0$, in s cases $\mu^{n-1}\left(\mathbf{p}^{-i}\right) = -1$ and in m cases $\mu^{n-1}\left(\mathbf{p}^{-i}\right) = 1$. Since $s = m$, we get $\mathrm{sgn}\left(\sum_{i=1}^{n} \mathbf{p}^{-i}\right) =$

$$\operatorname{sgn}\left(\sum_{i=1}^{s}\mathbf{p}^{-i}+\sum_{i=s+1}^{s+z}\mathbf{p}^{-i}+\sum_{i=s+z+1}^{s+z+m=n}\mathbf{p}^{-i}\right) = \operatorname{sgn}(s-m+0) = 0 \text{ and so}$$
$$\mu^{n}\left(\mu^{n-1}\left(\mathbf{p}^{-1}\right),\ldots\mu^{n-1}\left(\mathbf{p}^{-n}\right)\right)=\operatorname{sgn}(0)=0.$$

Theorem 4 below states that all applications of the majority rule μ to a sequence consisting in n members can be defined in terms of iteratively applying μ to sequences containing only two members, i.e. in terms of the binary majority rule μ^2.

Theorem 4 Let $\mathbf{p} = p_1 \ldots p_n$. Then $\mu(\mathbf{p})$ is equivalent with an expression σ_μ which contains only the binary majority rule.

The proof is by induction on the number n of members of sequence $\mathbf{p}$. First, let $n = 3$. Then $\mathbf{p} = p_1p_2p_3$. All sequences $\mathbf{p}^{-i}$ have exactly two members. We can easily check that $\mu^3(\mathbf{p}) \equiv \mu^2\left(\mu^2\left(\mu^2\left(\mu^2\left(\mathbf{p}^{-1}\right)\mu^2\left(\mathbf{p}^{-2}\right)\right)\right.\right.$ $\left.\left.\mu^2\left(\mu^2\left(\mathbf{p}^{-1}\right)\mu^2\left(\mathbf{p}^{-3}\right)\right)\right)\mu^2\left(\mu^2\left(\mathbf{p}^{-2}\right)\mu^2\left(\mathbf{p}^{-3}\right)\right)\right)$. The expression in the right part of the equivalence states that we first apply μ^2 to each of the three subsets $\{p_2,p_3\}$, $\{p_1,p_3\}$ and $\{p_1,p_2\}$ of $\{p_1,p_2,p_3\}$: $\mu^2(p_2p_3)$; $\mu^2(p_1p_3)$; $\mu^2(p_1p_2)$. Then we apply μ^2 to each pair of these expressions and get: $\mu^2(\mu^2(p_2p_3)\mu^2(p_1p_3))$; $\mu^2(\mu^2(p_2p_3)\mu^2(p_1p_2))$; $\mu^2(\mu^2(p_1p_3)\mu^2(p_1p_2))$. Third, apply μ^2 to the first two expression thus obtained: $\mu^2(\mu^2(\mu^2(p_2p_3)\mu^2(p_1p_3))\ \mu^2(\mu^2(p_2p_3)\mu^2(p_1p_2)))$. Finally, apply μ^2 to this expression and the remaining $\mu^2(\mu^2(p_1p_3)\mu^2(p_1p_2))$.

Now let $n > 3$. By induction the proposition holds for all the sequences with at most $n-1$ members. Since each of the sequences $\mathbf{p}^{-i}$ has $n-1$ members, $\mu^{n-1}\left(\mathbf{p}^{-i}\right)$ is equivalent by induction with an expression containing only μ^2. Write $\sigma\left(\mathbf{p}^{-i}\right)$ for each of these n expressions. Now form all n sets of $n-1$ such expressions. For example, $\left\{\sigma\left(\mathbf{p}^{-1}\right),\ldots\sigma\left(\mathbf{p}^{-(n-1)}\right)\right\}$ contains all but $\sigma\left(\mathbf{p}^{-n}\right)$. By induction, $\mu^{n-1}\left(\sigma\left(\mathbf{p}^{-1}\right)\ldots\sigma\left(\mathbf{p}^{-(n-1)}\right)\right)$ is an expression equivalent with some expression containing only μ^2. I write $\Sigma(\mathbf{p})^{-n}$ for it. Next, define $\mu^{n-1}\left(\Sigma(\mathbf{p})^{-2}\ldots\Sigma(\mathbf{p})^{n}\right)$; again by induction, it is equivalent with an expression containing only μ^2. Finally, put $\sigma_\mu = \mu^2\left(\mu^{n-1}\left(\Sigma(\mathbf{p})^{-2}\ldots\Sigma(\mathbf{p})^{n}\right),\ \Sigma(\mathbf{p})^{-1}\right)$. I prove that $\mu^n(\mathbf{p}) \equiv \sigma_\mu$.[9]

Case 1: $\mu^n(\mathbf{p}) = 1$. By Lemma 2 we also have $\mu^{n}\left(\mu^{n-1}\left(\mathbf{p}^{-1}\right),\ldots\mu^{n-1}\left(\mathbf{p}^{-n}\right)\right) = 1$ and so $\mu^n\left(\sigma\left(\mathbf{p}^{-1}\right),\ldots\sigma\left(\mathbf{p}^{-n}\right)\right) = 1$. But by Lemma 1 all $\sigma(\mathbf{p}^{-i})$'s are such that $\sigma\left(\mathbf{p}^{-i}\right) = 1$ or $\sigma\left(\mathbf{p}^{-i}\right) = 0$, and there is some $\sigma\left(\mathbf{p}^{-i}\right)$ such that $\sigma\left(\mathbf{p}^{-i}\right) = 1$. Therefore for each $\Sigma(\mathbf{p})^{-i}$ we have that $\Sigma(\mathbf{p})^{-i} \geq 0$ and $\Sigma(\mathbf{p})^{-i} = 1$ for some i. So for $\sigma_\mu = \mu^2\left(\mu^{n-1}\left(\Sigma(\mathbf{p})^{-2}\ldots\Sigma(\mathbf{p})^{n}\right),\ \Sigma(\mathbf{p})^{-1}\right))$ we have two possibilities: a) $\Sigma(\mathbf{p})^{-1} = 0$. Then we must have $\Sigma(\mathbf{p})^{-i} = 1$ for some $i \geq 2$, which entails that $\mu^{n-1}\left(\Sigma(\mathbf{p})^{-2}\ldots\Sigma(\mathbf{p})^{n}\right) = 1$ and thus $\sigma_\mu = \mu^2(1,0) = 1$; b) $\Sigma(\mathbf{p})^{-1} = 1$.

[9]Note that for $n = 3$, σ_μ is exactly the expression used in the first step of this proof.

Then $\mu^{n-1}\left(\Sigma(\mathbf{p})^{-2}\ldots\Sigma(\mathbf{p})^{n}\right) = a \geq 0$, which gives again $\sigma_\mu = \mu^2(a,1) = 1$.

Case 2: $\mu^n(\mathbf{p}) = -1$. The proof is just like in case 1.

Case 3: $\mu^n(\mathbf{p}) = 0$. Then by Lemma 2 we have $\mu^n\left(\sigma\left(\mathbf{p}^{-1}\right), \ldots\sigma\left(\mathbf{p}^{-n}\right)\right) = 0$. The definition of μ^n entails that the number of $\sigma\left(\mathbf{p}^{-i}\right)$'s with the property that $\sigma\left(\mathbf{p}^{-i}\right) = 1$ is equal with the number of $\sigma\left(\mathbf{p}^{-i}\right)$'s with the property that $\sigma\left(\mathbf{p}^{-i}\right) = -1$. A similar argument entails that the number of $\Sigma(\mathbf{p})^{-i}$'s with the property that $\Sigma(\mathbf{p})^{-i} = 1$ is equal with the number of $\Sigma(\mathbf{p})^{-i}$'s with the property that $\Sigma(\mathbf{p})^{-i} = -1$. We have three subcases:

(a) $\Sigma(\mathbf{p})^{-i} = 1$. Then in the sequence $\Sigma(\mathbf{p})^{-2}, \ldots\Sigma(\mathbf{p})^{n}$ formed of $n-1$ members the number of $\Sigma(\mathbf{p})^{-i}$'s such that $\Sigma(\mathbf{p})^{-i} = -1$ is larger than the number of $\Sigma(\mathbf{p})^{-i}$'s such that $\Sigma(\mathbf{p})^{-i} = 1$ and so $\mu^{n-1}\left(\Sigma(\mathbf{p})^{-2}\ldots\Sigma(\mathbf{p})^{n}\right) = -1$. Then clearly $\sigma_\mu = \mu^2\left(\mu^{n-1}\left(\Sigma(\mathbf{p})^{-2}\ldots\Sigma(\mathbf{p})^{n}\right), \Sigma(\mathbf{p})^{-1}\right)) = \mu^2(-1,1) = 0 = \mu^n(\mathbf{p})$.

(b) $\Sigma(\mathbf{p})^{-1} = -1$. By an analogous argument we conclude that $\sigma_\mu = \mu^2(1,-1) = 0 = \mu^n(\mathbf{p})$.

(c) $\Sigma(\mathbf{p})^{-1} = 0$. Then in the sequence $\Sigma(\mathbf{p})^{-2}, \ldots\Sigma(\mathbf{p})^{n}$ formed of $n-1$ members the number of $\Sigma(\mathbf{p})^{-i}$'s such that $\Sigma(\mathbf{p})^{-i} = -1$ is equal with the number of $\Sigma(\mathbf{p})^{-i}$'s such that $\Sigma(\mathbf{p})^{-i} = 1$ and so $\mu^{n-1}\left(\Sigma(\mathbf{p})^{-2}\ldots\Sigma(\mathbf{p})^{n}\right) = 0$. Then $\sigma_\mu = \mu^2\left(\mu^{n-1}\left(\Sigma(\mathbf{p})^{-2}\ldots\Sigma(\mathbf{p})^{n}\right), \Sigma(\mathbf{p})^{-1}\right)) = \mu^2(0,0) = 0$.

Remark We can easily see that if $\mathbf{p}$ consists in just one member p, we can put $\mu(\mathbf{p}) = \mu(pp)$, and by unanimity we get $\mu(\mathbf{p}) \equiv p$. Therefore the unary case is also covered.

Theorem 5 λ^2 cannot be extended to the n-ary case.

Proof The proof consists in showing that there is no expression σ_λ which contains only the binary function λ^2 and $\lambda^n(\mathbf{p}) = \sigma_\lambda(\mathbf{p})$ for all $\mathbf{p} = p_1\ldots p_n$. We only need to consider the simplest case when we have three propositional variables p_1, p_2 and p_3 and show that $\lambda^3(p_1p_2p_3)$ is not definable in terms of λ^2. This means that there is no expression σ_λ with the property that $\lambda^3(p_1p_2p_3) = \sigma_\lambda(p_1p_2p_3)$ for all propositional variables p_1, p_2 and p_3 and σ_λ contains only occurrences of λ^2. Since λ^2 and μ^2 are identical, we can replace in σ_λ all occurrences of λ^2 with occurrences of μ^2. The proof has three steps. In the first step I show that σ_λ satisfies self-duality; in the second step I show that it satisfies monotonicity. Finally, I prove that if $\lambda^3(p_1p_2p_3) = \sigma_\lambda(p_1p_2p_3)$ for all propositional variables p_1, p_2 and p_3 then we get a contradiction.

First, I show by induction on the complexity of σ_λ that self-duality is satisfied. Suppose first that $\sigma_\lambda = \mu^3(p_1p_2p_3)$. Then clearly self-duality is preserved, for μ^3 is self-dual. Let $\sigma_\lambda = \mu^3\left(\sigma_\lambda^1(p_1p_2p_3), \sigma_\lambda^2(p_1p_2p_3), \sigma_\lambda^3(p_1p_2p_3)\right)$.

By induction $\sigma_\lambda^i(\sim p_1 \sim p_2 \sim p_3) \;=\sim\; \sigma_\lambda^i(p_1 p_2 p_3)$ $(i = 1, 2, 3)$. Then $\sim\; \sigma_\lambda(\sim p_1 \sim p_2 \sim p_3) \;=\sim\; \mu^3\Big(\sigma_\lambda^1(\sim p_1 \sim p_2 \sim p_3),\ \sigma_\lambda^2(\sim p_1 \sim p_2 \sim p_3),$ $\sigma_\lambda^3(\sim p_1 \sim p_2 \sim p_3)\Big) =\sim \mu^3\big(\sim \sigma_\lambda^1(p_1 p_2 p_3), \sim \sigma_\lambda^2(p_1 p_2 p_3)$ $, \sim \sigma_\lambda^3(p_1 p_2 p_3)\big) = \mu^3\big(\sigma_\lambda^1(p_1 p_2 p_3), \sigma_\lambda^2(p_1 p_2 p_3), \sigma_\lambda^3(p_1 p_2 p_3) = \sigma_\lambda$. In the second step we can proceed in a similar way to show that monotonicity is also holds.

Now let us move to the final step of the proof.[10] First, notice the following property of λ^3.

Let $p_1^1 = 1$, $p_2^1 = 1$ and $p_3^1 = -1$. By definition, $\lambda^3 = \left(p_1^1 p_2^1 p_3^1\right) = 0$. Let $p_2^2 = -1$. Then $\lambda^3\left(p_1^1 p_2^2 p_3^1\right) = 0$. Similarly, if $p_3^3 = 1$, then $\lambda^3\left(p_1^1 p_2^2 p_3^3\right) = 0$; and if $p_1^4 = -1$, then $\lambda^3\left(p_1^4 p_2^2 p_3^3\right) = 0$. But suppose that λ^3 is the result of extending λ^2 to the ternary case, i.e. there is some σ_λ such that $\lambda^3(p_1 p_2 p_3) = \sigma_\lambda(p_1 p_2 p_3)$ for all propositional variables p_1, p_2 and p_3. Then λ^3 must be monotonic and self-dual. Moreover, note that $p_1^1 =\sim p_1^4$, $p_2^1 =\sim p_2^2$, and $p_3^1 =\sim p_3^3$; so $\lambda^3\left(p_1^1 p_2^1 p_3^1\right) =\sim \lambda^3\left(\sim p_1^4 \sim p_2^2 \sim p_3^3\right)$ by self-duality. I shall prove that if $\sigma_\lambda\left(p_1^1 p_2^1 p_3^1\right) = 0$, then $\sigma_\lambda\left(p_1^1 p_2^2 p_3^1\right) \neq 0$ or $\sigma_\lambda\left(p_1^1 p_2^2 p_3^3\right) \neq 0$ or $\sigma_\lambda\left(p_1^4 p_2^2 p_3^3\right) \neq 0$. The proof is on induction on the complexity of σ_λ.

Case 1. $\sigma_\lambda(p_1 p_2 p_3) = \mu^3(p_1 p_2 p_3)$. Since $\sigma_\lambda\left(p_1^1 p_2^1 p_3^1\right) = 0$, we have $\mu^3\left(p_1^1 p_2^1 p_3^1\right) = 0$. But μ^3 is responsive, and we have $\left(p_2^2 \to p_2^1\right) \wedge \sim \left(p_2^1 \to p_2^2\right)$; so $\mu^3\left(p_1^1 p_2^2 p_3^1\right) = -1$, which gives $\sigma_\lambda\left(p_1^1 p_2^2 p_3^1\right) \neq 0$.

Case 2. $\sigma_\lambda(p_1 p_2 p_3) = \mu^3\left(\sigma_\lambda^1(p_1 p_2 p_3), \sigma_\lambda^2(p_1 p_2 p_3), \sigma_\lambda^3(p_1 p_2 p_3)\right)$. By induction, the property holds for all σ_λ^i $(i = 1, 2, 3)$. Since all σ_λ^i are monotonic, given the values of p_j^k's we have:

(1) $\sigma_\lambda^i\left(p_1^1 p_2^2 p_3^1\right) \to \sigma_\lambda^i\left(p_1^1 p_2^1 p_3^1\right)$
(2) $\sigma_\lambda^i\left(p_1^1 p_2^1 p_3^1\right) \to \sigma_\lambda^i\left(p_1^1 p_2^2 p_3^3\right)$
(3) $\sigma_\lambda^i\left(p_1^4 p_2^2 p_3^3\right) \to \sigma_\lambda^i\left(p_1^1 p_2^2 p_3^3\right)$

But by the definition of λ^3,

(4) $$\mu^3\left(\sigma_\lambda^1\left(p_1^1 p_2^1 p_3^1\right), \sigma_\lambda^2\left(p_1^1 p_2^1 p_3^1\right), \sigma_\lambda^3\left(p_1^1 p_2^1 p_3^1\right)\right)$$
$$= \mu^3\left(\sigma_\lambda^1\left(p_1^1 p_2^2 p_3^1\right), \sigma_\lambda^2\left(p_1^1 p_2^2 p_3^1\right), \sigma_\lambda^3\left(p_1^1 p_2^2 p_3^1\right)\right),$$

and similarly for the other two cases. So

(5) $$\sigma_\lambda^i\left(p_1^1 p_2^2 p_3^1\right) = \sigma_\lambda^i\left(p_1^1 p_2^1 p_3^1\right),$$

because otherwise μ^3's being responsive would contradict the above equivalence. A similar argument applies for the other two cases. However, by the inductive hypothesis we must have

(6) $\sigma_\lambda^i\left(p_1^1 p_2^1 p_3^1\right) \neq 0$ or $\sigma_\lambda^i\left(p_1^1 p_2^2 p_3^1\right) \neq 0$ or $\sigma_\lambda^i\left(p_1^1 p_2^2 p_3^3\right) \neq 0$ or $\sigma_\lambda^i\left(p_1^4 p_2^2 p_3^3\right) \neq 0$.

[10] It is inspired by the necessity part of Fine's proof of his Theorem 3 in (Fine 1972); I appeal to a very simplified version of his property of zigzaggedness.

From (5) and (6) we get:

(7) $\sigma_\lambda^i\left(p_1^1 p_2^2 p_3^1\right) = \sigma_\lambda^i\left(p_1^4 p_2^2 p_3^3\right) \neq 0$

But notice that $p_1^1 =\sim p_1^4$, $p_2^1 =\sim p_2^2$, and $p_3^1 =\sim p_3^3$; so we must also have $\sigma_\lambda^i\left(p_1^1 p_2^1 p_3^1\right) =\sim \sigma_\lambda^i\left(\sim p_1^4 \sim p_2^2 \sim p_3^3\right)$ by the self-duality of σ_λ^i – which contradicts (7).

Theorem 6 α^2 cannot be extended to the *n*-ary case.

Proof Again, it suffices to show that α^2 cannot be extended to α^3. Suppose that there is some expression σ_α which contains only the binary absolute majority rule α^2 and $\alpha^n(\mathbf{p}) = \sigma_\alpha(\mathbf{p})$ for all $\mathbf{p} = p_1 p_2 p_3$. Suppose that at the profile $\mathbf{p}$ we have $p_1 = 0, p_2 = p_3 = 1$. If σ_α contains at least one occurrence of p_1, then α^2 gives value 0, and the definition of α^2 entails that all other subsequent applications must result in the same value, so $\sigma_\alpha(\mathbf{p}) = 0$. But clearly we must have $\alpha^3(\mathbf{p}) = 1$ – contradiction. Therefore, we must construct σ_α such that it contains no occurrence of p_1. If on the other hand we take into account a value assignment $\mathbf{p}'$ such that $p_2 = 0$, $p_1 = p_3 = 1$ (in this case we also have $\alpha^3(\mathbf{p}') = 1$) we must conclude that σ_α is such that it contains no occurrence of p_2; a similar argument shows that σ_α does not satisfy the required property if it includes an occurrence of p_3. Therefore no σ_α satisfies the property that $\sigma_\alpha(\mathbf{p}) = \alpha(\mathbf{p})$ for all assignments $\mathbf{p}$.

9.5 Conclusion

Supposing that the language we use consists in propositions that express the attitudes of individuals on an issue (how to choose between the alternatives *x* and *y*), a Łukasiewiczian three-valued logic framework can be shown to be rich enough to allow for the reconstruction of many aggregation rules, the simple and the absolute majority rules among them. I argued that, on this account, in its primary use simple majority rule applies to groups of people consisting in only two members and so it can be modeled as a binary logical operator. I characterized it by means of some simple properties, in analogy with the famous result presented by May (1954) and also showed that an impossibility theorem can be obtained in this framework.

One of the missing steps in the appeal to simple majority rule, as applied to groups of people formed of an arbitrary number n ($n > 2$) of members, is its relation to the binary rule. I proved that the *n*-ary logical operator corresponding to simple majority rule can be obtained by extending the binary operator.[11]

[11]The three-valued logic is functionally complete in the following sense (Słupecki 1972): let **F** be a set of logical operators that contains all the unary operators and at least one essential operator. A binary operator is essential if it takes on all the values from $\{1, 0, -1\}$. Then each binary operator is definable in terms of the logical operators in **F**. In fact, we need not consider all the unary operators as given. Słupecki (Słupecki 1967) showed that all the binary logical operators can be defined by adding to the classical → (implication) and (negation) operators a new unary operator

The paper relies on a logical framework in which the propositions of the language used are interpreted as expressing the attitudes of the individuals toward certain issues. The aggregation of attitudes is then expressed by means of logical operators. Compound propositions are taken to express the attitudes of complex groups: some are formed of individuals, while others are higher-order and have also groups as their members. The aggregation of attitudes is an attempt to describe such complex situations of group decisions.

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T. It has the property that $Tp = 0$ for all values of p. Murakami (1968) proved a similar result: any binary voting rule can be defined in terms of iterate applications of μ, a negation operator and a constant rule: $Vp = 1$ for all values of p. Clearly, it is sufficient to show that the operators T and $\rightarrow$ are definable. First, implication is defined by: $p \rightarrow q = \mu\,(\sim pqVp)$; second, Slupecki's unary operator T is simply: $Tp = \mu\,(p \sim p)$. However, note that the ternary μ^3 is used. It is also necessary to show that the result extends to the n-ary case.

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