Chapter 9
Interconnected Systems

The system-theoretic study of interconnected systems is not new. It started with the work by Gilbert (1963) on controllability and observability for generic classes of systems in parallel, series, and feedback interconnections. Complete characterizations for multivariable linear systems were obtained by Callier and Nahum (1975) for series and feedback interconnections and in a short note by Fuhrmann (1975) for parallel interconnections. We refer the reader to Chapter 10 for a proof of these classical characterizations using the techniques developed here. However, the interconnection structures of most complex systems are generally not of the series, parallel, or feedback type. Thus, one needs to pass from the standard interconnections to more complex ones, where the interconnection pattern between the node systems is described by a weighted directed graph. This will be done in the first part of this chapter. The main tool used is the classical concept of strict system equivalence. This concept was first introduced by Rosenbrock in the 1970s for the analysis of higher-order linear systems and was subsequently developed into a systematic tool for realization theory through the work of Fuhrmann. Rosenbrock and Pugh (1974) provided an extension of this notion toward a permanence principle for networks of linear systems. Section 9.2 contains a proof of a generalization of this permanence principle for dynamic interconnections. From this principle we then derive our main results on the reachability and observability of interconnected systems. This leads to very concise and explicit characterizations of reachability and observability for homogeneous networks consisting of identical SISO systems. Further characterizations of reachability are obtained for special interconnection structures, such as paths, cycles, and circulant structures.

Before we delve into the technical details of this chapter, let us discuss some simple examples on the reachability of interconnected systems (Fig. 9.1). The first example is defined by a continuous-time consensus algorithm on a path graph whose nodes are first-order scalar systems. Thus, for the path graph $\Gamma_N = (\mathscr{V}, \mathscr{E})$ with

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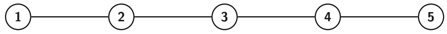


Fig. 9.1 Interconnection graph of (9.1) for $N = 5$

vertex set $\mathscr{V} = \{1,\ldots,N\}$ and set of edges $\mathscr{E} = \{\{1,2\},\{2,3\},\ldots,\{N-1,N\}\}$, consider the autonomous dynamical system

$$\begin{aligned}
\dot{z}_1(t) &= z_2(t) - z_1(t),\\
\dot{z}_i(t) &= z_{i+1}(t) - 2z_i(t) + z_{i-1}(t), \quad i = 2,\ldots,N-1, \\
\dot{z}_N(t) &= z_{N-1}(t) - z_N(t).
\end{aligned} \tag{9.1}$$

Equivalently, the system can be written in matrix form as $\dot{z} = -L_N z$, where

$$L_N = \begin{pmatrix} 1 & -1 & & & & \\ -1 & 2 & -1 & & & \\ & -1 & \ddots & \ddots & & \\ & & \ddots & \ddots & -1 & \\ & & & -1 & 2 & -1 \\ & & & & -1 & 1 \end{pmatrix}$$

denotes the Laplacian matrix of the graph Γ_N. We emphasize that system (9.1) achieves consensus in the sense that all solutions satisfy $\lim_{t\to\infty}(z_i(t) - z_j(t)) = 0$. More generally, consider an autonomous system of linear differential equations

$$\begin{aligned}
\dot{z}_1 &= a_{11}z_1 + \cdots + a_{1N}z_N\\
&\vdots \\
\dot{z}_N &= a_{N1}z_1 + \cdots + a_{NN}z_N
\end{aligned} \tag{9.2}$$

with a matrix of coefficients $\mathfrak{A} = (a_{ij}) \in \mathbb{R}^{N\times N}$. To study the influence of a node on the evolution of the remaining system variables, select, say, the last variable z_N and consider it a control variable. This leads to the linear control system

$$\dot{x}(t) = Ax(t) + bu(t) \tag{9.3}$$

in the variable $x = \operatorname{col}(z_1,\ldots,z_{N-1})$, where

$$A = \begin{pmatrix} a_{11} & \ldots & a_{1N-1} \\ \vdots & \ddots & \vdots \\ a_{N-11} & \ldots & a_{N-1N-1} \end{pmatrix}, \quad b = \begin{pmatrix} a_{1N} \\ \vdots \\ a_{N-1N} \end{pmatrix}.$$

In particular, from (9.1) one obtains the reachable system

$$A = \begin{pmatrix} 1 & -1 & & & & \\ -1 & 2 & -1 & & & \\ & -1 & \ddots & \ddots & & \\ & & \ddots & \ddots & -1 & \\ & & & -1 & 2 & -1 \\ & & & & -1 & 2 \end{pmatrix}, \quad b = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ -1 \end{pmatrix}.$$

More generally, one can select a finite number of state variables in an interconnected autonomous system and replace them by free input variables that act on the remaining system. Thus, starting from a system of ordinary differential equations, one obtains a control system, and one can study its reachability properties. This leads to the topic of **pinning control**. Now, suppose that certain of the entries a_{ij} are set to zero, i.e., assume that the coefficients a_{ij} are defined by the adjacency matrix of an undirected graph on N nodes. Then the reduced control system (9.3) has an induced graph structure and clearly defines an interconnected linear system. One can then ask about the extent to which the graph structure of the autonomous system (9.2) impacts the reachability properties of the network (9.3). Such pinning reachability questions have been considered in the past few years by a number of researchers, including Tanner (2004), Liu, Slotine and Barabasi (2011), and Parlangeli and Notarstefano (2012). The techniques that we will develop in this chapter can be applied to such problems.

To further illustrate the issue of pinning control, let us consider some examples of networks studied by Tanner (2004) and Parlangeli and Notarstefano (2012). The first example is perhaps a bit surprising because it shows that a complete graph can lead to unreachability.

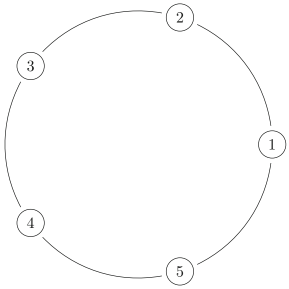
Example 9.1. Suppose that $\mathfrak{A}$ is an (unweighted) adjacency matrix of the complete graph K_N on N vertices. Then (9.3) is equal to

$$A = \begin{pmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}.$$

Obviously, the system (A,b) is unreachable for $N \geq 2$. The same conclusion is true if $\mathfrak{A}$ is the Laplacian of K_N.

Example 9.2. Here $\mathfrak{A}$ is the Laplacian matrix of the path graph Γ_N on N vertices. The pinned system is then

Fig. 9.2 Cycle graph



$$A = \begin{pmatrix} 1 & -1 & & & & \\ -1 & 2 & -1 & & & \\ & -1 & \ddots & \ddots & & \\ & & \ddots & \ddots & -1 & \\ & & & -1 & 2 & -1 \\ & & & & -1 & 2 \end{pmatrix}, \quad b = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ -1 \end{pmatrix},$$

which is reachable. One could extend this example by replacing the rth state variable z_r by an input. This situation has been analyzed by Parlangeli and Notarstefano (2012).

Example 9.3. Now assume that $\mathfrak{A}$ is the symmetric adjacency matrix of the cycle graph on N vertices and edges $\{1,2\},\{2,3\},\ldots,\{N-1,N\},\{N,1\}$ (Fig. 9.2).

Thus $\mathfrak{A}$ is the tridiagonal matrix

$$\mathfrak{A}_N = \begin{pmatrix} 0 & 1 & & & 1 \\ 1 & 0 & 1 & & \\ & 1 & \ddots & \ddots & \\ & & \ddots & \ddots & 1 \\ 1 & & & 1 & 0 \end{pmatrix},$$

and therefore one obtains the pinned system for z_N as

$$A = \begin{pmatrix} 0 & 1 & & & \\ 1 & 0 & 1 & & \\ & 1 & \ddots & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & 1 & 0 \end{pmatrix}, \quad b = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 1 \end{pmatrix},$$

which is not reachable. Likewise, by pinning the variable z_{N-1}, one obtains the unreachable system

$$A = \begin{pmatrix} 0 & 1 & & & & 1 \\ 1 & 0 & 1 & & & \\ 0 & 1 & \ddots & \ddots & & \\ & & \ddots & \ddots & 1 & 0 \\ & & & 1 & 0 & 0 \\ 1 & & & 0 & 0 & 0 \end{pmatrix}, \quad b = \begin{pmatrix} 0 \\ \vdots \\ \vdots \\ 0 \\ 1 \\ 1 \end{pmatrix}.$$

One can also consider pinning control problems where the adjacency matrix $\mathfrak{A}$ is replaced by the associated Laplacian matrix. In that situation, one encounters more interesting reachability phenomena, which have been studied by Parlangeli and Notarstefano (2012).

The preceding examples show that the reachability of the pinned system depends in a nontrivial way on the underlying graph structure of the system, as well as on the selection of the pinned control variables. In Section 9.5, we will study the reachability of such networks in greater generality. In fact, system (9.2) can be interpreted as a system of n integrators,

$$\begin{aligned} \dot{z}_1(t) &= u_1(t) \\ &\vdots \\ \dot{z}_N(t) &= u_N(t), \end{aligned}$$

with feedback terms $u_i = \sum_{j=1}^N a_{ij} z_j$. If one replaces the integrator dynamics with a general first-order systems $\dot{z}_i = \alpha_i z_i + \beta_i u_i$ with local state variables $z_i \in \mathbb{R}^{n_i}$, system matrices $\alpha_i, \beta_i \in \mathbb{F}^{n_i \times n_i}$, and using the same coupling terms $u_i = \sum_{j=1}^N a_{ij} z_j$, then the closed-loop system is

$$\begin{aligned} \dot{z}_1(t) &= \alpha_1 z_1(t) + \beta_1 \sum_{j=1}^N a_{1j} z_j(t) \\ &\vdots \\ \dot{z}_N(t) &= \alpha_N z_N(t) + \beta_N \sum_{j=1}^N a_{Nj} z_j(t) \,. \end{aligned}$$

By pinning the last variable z_N, one obtains the control system

$$\dot{z}_1(t) = \alpha_1 z_1(t) + \beta_1 \sum_{j=1}^{N-1} a_{1j} z_j(t) + \beta_1 a_{1N} u(t)$$

$$\vdots$$

$$\dot{z}_{N-1}(t) = \alpha_{N-1} z_{N-1}(t) + \beta_{N-1} \sum_{j=1}^{N-1} a_{N-1,j} z_j(t) + \beta_{N-1} a_{N-1,N} u(t) \; .$$

Thus one can ask when such a system is reachable and how one can relate reachability to the graph properties that define the structure of the matrix of coupling parameters $\mathfrak{A}$. We will now develop systematic tools for the reachability and observability analysis of such systems.

9.1 Interconnection Models

State-Space Representations. We present a state-space formulation of the situation we are interested in and introduce notation to be used subsequently. Consider N discrete-time linear systems, which we refer to as **node systems** $\Sigma_i, i = 1,\ldots,N$,

$$\begin{aligned} x_i(t+1) &= \alpha_i x_i(t) + \beta_i v_i(t), \\ w_i(t) &= \gamma_i x_i(t). \end{aligned} \tag{9.4}$$

Here $\alpha_i \in \mathbb{F}^{n_i \times n_i}$, $\beta_i \in \mathbb{F}^{n_i \times m_i}$, and $\gamma_i \in \mathbb{F}^{p_i \times n_i}$ are the associated system matrices, and $\mathbb{F}$ denotes a field. Assume that each system is reachable and observable. To interconnect the node systems, apply static coupling laws

$$v_i(t) = \sum_{j=1}^{N} A_{ij} w_j(t) + B_i u(t) \in \mathbb{F}^{m_i}$$

with constant matrices $A_{ij} \in \mathbb{F}^{m_i \times p_j}$ and $B_i \in \mathbb{F}^{m_i \times m}$, although more complex dynamic interconnections laws are possible, too, and will be considered later on. The interconnected output is

$$y(t) = \sum_{i=1}^{N} C_i w_i(t) + Du(t), \;\text{ with }\; C_i \in \mathbb{F}^{p \times p_i},\; i = 1,\ldots,N.$$

To express the closed-loop system in compact matrix form, define $\overline{n} := n_1 + \cdots + n_N$, $\overline{m} := m_1 + \cdots + m_N$, $\overline{p} := p_1 + \cdots + p_N$. Moreover,

$$A := (A_{ij})_{ij} \in \mathbb{F}^{\overline{m}\times\overline{p}}, \quad C := (C_1,\ldots,C_N) \in \mathbb{F}^{p\times\overline{p}}, \quad B := \begin{pmatrix} B_1 \\ \vdots \\ B_N \end{pmatrix} \in \mathbb{F}^{\overline{m}\times m}, \quad D \in \mathbb{F}^{p\times m}$$

and

$$\alpha := \begin{pmatrix} \alpha_1 & & \\ & \ddots & \\ & & \alpha_N \end{pmatrix} \in \mathbb{F}^{\overline{n}\times\overline{n}}, \qquad \beta := \begin{pmatrix} \beta_1 & & \\ & \ddots & \\ & & \beta_N \end{pmatrix} \in \mathbb{F}^{\overline{n}\times\overline{m}},$$

$$\gamma := \begin{pmatrix} \gamma_1 & & \\ & \ddots & \\ & & \gamma_N \end{pmatrix} \in \mathbb{F}^{\overline{p}\times\overline{n}}, \qquad x(t) := \begin{pmatrix} x_1(t) \\ \vdots \\ x_N(t) \end{pmatrix} \in \mathbb{F}^{\overline{n}}.$$

Thus, the global state-space representation of the node systems Σ_i is

$$\begin{aligned} x(t+1) &= \alpha x(t) + \beta v(t), \\ w(t) &= \gamma x(t), \end{aligned}$$

and the interconnection is

$$\begin{aligned} v(t) &= Aw(t) + Bu(t), \\ y(t) &= Cw(t) + Du(t)\ . \end{aligned}$$

Here $u(t)$ is the external input and $y(t)$ the external output of the network. The restriction to strictly proper rather than proper node systems is not crucial and is done here only to simplify some of the subsequent expressions. Thus the network dynamics has the state-space form

$$\begin{aligned} x(t+1) &= \mathscr{A}x(t) + \mathscr{B}u(t), \\ y(t) &= \mathscr{C}x(t) + Du(t), \end{aligned} \tag{9.5}$$

with

$$\mathscr{A} := \alpha + \beta A\gamma \in \mathbb{F}^{\overline{n}\times\overline{n}}, \quad \mathscr{B} := \beta B \in \mathbb{F}^{\overline{n}\times m}, \quad \mathscr{C} := C\gamma \in \mathbb{F}^{p\times\overline{n}}. \tag{9.6}$$

It is convenient to describe an interconnected system in terms of the transfer functions of the node systems. The ith node transfer function is defined as a strictly proper transfer function of McMillan degree n_i and is given in state-space form as

$$G_i(z) = \gamma_i(zI - \alpha_i)^{-1}\beta_i. \tag{9.7}$$

Define the **node transfer function** as

$$G(z) := \operatorname{diag}(G_1(z),\ldots,G_N(z)) = \gamma(zI-\alpha)^{-1}\beta.$$

In the case where $\overline{m} = \overline{p}$, the **interconnection transfer function** is defined as

$$\mathscr{N}(z) = C(zI-A)^{-1}B + D. \tag{9.8}$$

The global **network transfer function** is then defined as

$$\mathscr{N}_G(z) = \mathscr{C}(zI-\mathscr{A})^{-1}\mathscr{B} + D;$$

thus, explicitly,

$$\mathscr{N}_G(z) = C\gamma(zI-\alpha-\beta A\gamma)^{-1}\beta B + D. \tag{9.9}$$

A network of systems (9.5) is called **homogeneous** if the transfer functions of the node systems (9.7) are identical scalar rational functions that are strictly proper. The reachability and observability analysis of homogeneous networks is particularly easy, as is subsequently shown.

Polynomial Matrix Descriptions. A general class of higher-order system representations, the so-called **polynomial matrix descriptions** (PMD), was introduced by Rosenbrock (1970). In this case, Σ_i is defined in terms of systems of higher-order difference equations:

$$\begin{aligned} T_i(\sigma)\xi_i &= U_i(\sigma)v_i, \\ w_i &= V_i(\sigma)\xi_i + W_i(\sigma)v_i, \end{aligned} \tag{9.10}$$

with transfer functions

$$G_i(z) = V_i(z)T_i(z)^{-1}U_i(z) + W_i(z).$$

Here, as well as in other parts of this book, σ denotes the **backward shift operator** (4.8), defined for discrete-time systems. For continuous-time systems, σ denotes the differentiation operator. Let

$$T(z) := \operatorname{diag}(T_1(z),\ldots,T_N(z)) \in \mathbb{F}[z]^{\overline{r}\times\overline{r}},$$

and similarly define $V(z), U(z), W(z)$. Here $\overline{r} = \sum_{i=1}^N r_i$. Using this notation, (9.10) can be rewritten as the polynomial matrix representation

$$\begin{pmatrix} 0 \\ I \end{pmatrix} w = \begin{pmatrix} T(\sigma) & -U(\sigma) \\ V(\sigma) & W(\sigma) \end{pmatrix} \begin{pmatrix} \xi \\ v \end{pmatrix}.$$

The transfer function of the decoupled system is

$$V(z)T(z)^{-1}U(z)+W(z).$$

The interconnections are

$$\begin{aligned} v &= Aw+Bu, \\ y &= Cw+Du. \end{aligned}$$

The network transfer function (for $W=0$) is then

$$\mathcal{N}_G(z) = CV(z)(T(z)-U(z)AV(z))^{-1}U(z)B+D.$$

Matrix Fraction Systems. A special, but interesting, class of polynomial matrix representations is described as **matrix fraction descriptions** (MFD). Here, the node systems Σ_i, $i=1,\dots,N$, are given by polynomial matrix descriptions:

$$\begin{aligned} D_{\ell,i}(\sigma)\xi_i &= N_{\ell,i}(\sigma)v_i, \\ w_i &= \xi_i, \end{aligned}$$

with transfer function representations

$$G_i(z) = D_{\ell,i}(z)^{-1}N_{\ell,i}(z) = N_{r,i}(z)D_{r,i}(z)^{-1}.$$

Define the polynomial matrices $D_\ell(z) \in \mathbb{F}[z]^{\overline{p}\times\overline{p}}, N_\ell(z) \in \mathbb{F}[z]^{\overline{p}\times\overline{m}}$ by

$$D_\ell(z) = \operatorname{diag}\left(D_{\ell,1}(z),\dots,D_{\ell,N}(z)\right), \quad N_\ell(z) = \operatorname{diag}\left(N_{\ell,1}(z),\dots,N_{\ell,N}(z)\right),$$

and similarly for $D_r(z), N_r(z)$. If the interconnections are given by

$$\begin{aligned} v(t) &= Aw(t)+Bu(t), \\ y(t) &= Cw(t)+Du(t)\ , \end{aligned}$$

then the network transfer function is

$$\begin{aligned} \mathcal{N}_G(z) &= C(D_\ell(z)-N_\ell(z)A)^{-1}N_\ell(z)B+D \\ &= CN_r(z)(D_r(z)-AN_r(z))^{-1}B+D\ . \end{aligned} \tag{9.11}$$

9.2 Equivalence of Interconnected Systems

We next treat the mixed case, where several different models for decoupled node systems are possible, namely, state space, left and right matrix fractions, and polynomial system matrices. For each of the polynomial-based representations one can associate a state-space realization via the shift realization described in Theorem 4.26. The following theorem shows that the similarity of shift realizations associated with different representations of a decoupled system is preserved for interconnected systems. This is true despite the fact that, as a result of interconnection, the reachability and observability properties of the uncoupled node systems may have been lost. In view of Definition 4.30, proving a similarity of the realizations associated with polynomial system matrices is equivalent to showing that the polynomial system matrices are strictly system equivalent. This is the case even when the associated realizations are not minimal, and hence the state-space isomorphism theorem is not applicable. This constitutes a great simplification because strict system equivalence can be verified without computing the realizations.

Special emphasis will be placed on polynomial matrix descriptions because they cover all system representations of interest to us. Thus, we assume the node systems have the polynomial matrix descriptions

$$\begin{aligned} T_i(z)\xi_i(z) &= U_i(z)v_i, \\ w_i &= V_i(z)\xi_i + W_i(z)v_i\ , \end{aligned} \tag{9.12}$$

with transfer function $G_i(z) = V_i(z)T_i(z)^{-1}U_i(z) + W_i(z)$. The system interconnections are given by

$$\begin{aligned} v_i &= \sum_{j=1}^{N} A_{ij}w_j + B_i u, \\ y &= \sum_{j=1}^{N} C_j w_j + Du. \end{aligned} \tag{9.13}$$

Let

$$T(z) = \mathrm{diag}\,(T_1(z),\ldots,T_r(z)) \in \mathbb{F}[z]^{\bar{r}\times\bar{r}},$$

and similarly for $V(z), U(z), W(z)$. Using this notation, (9.12) can be rewritten in matrix form as

$$\begin{pmatrix} 0 \\ I \end{pmatrix} w = \begin{pmatrix} T(z) & -U(z) \\ V(z) & W(z) \end{pmatrix} \begin{pmatrix} \xi \\ v \end{pmatrix}. \tag{9.14}$$

Similarly, equation (9.13) can be rewritten as

$$\begin{aligned} v &= Aw + Bu, \\ y &= Cw + Du. \end{aligned}$$

More generally, we allow for dynamic interconnections described by first-order difference equations of the form

$$\begin{aligned} E(\sigma)v &= A(\sigma)w + B(\sigma)u, \\ y &= C(\sigma)w + Du. \end{aligned} \tag{9.15}$$

Here $E(z)$ is a square nonsingular polynomial matrix, $A(z), B(z), C(z)$ are appropriately sized polynomial matrices, and D is a constant feedthrough matrix. Assuming that the rational function $E(z)^{-1}(A(z), B(z))$ is proper, and by ignoring the output part, consider a first-order shift realization for

$$E(\sigma)v = A(\sigma)w + B(\sigma)u$$

as

$$\begin{aligned} \zeta(t+1) &= F\zeta(t) + G_1 w(t) + G_2 u(t), \\ v(t) &= H\zeta(t) + J_1 w(t) + J_2 u(t). \end{aligned} \tag{9.16}$$

Therefore, one obtains the strict system equivalence

$$\left(\begin{array}{c|cc} E(z) & -A(z) & -B(z) \\ \hline I & 0 & 0 \end{array}\right) \simeq_{FSE} \left(\begin{array}{c|cc} zI - F & -G_1 & -G_2 \\ \hline H & J_1 & J_2 \end{array}\right).$$

Clearly, ξ, v, and w are latent variables, whereas u and y are manifest variables. Thus equations (9.14) and (9.15) can be combined to yield a polynomial matrix description of the following closed-loop interconnected system:

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ I \end{pmatrix} y = \left(\begin{array}{ccc|c} T(z) & -U(z) & 0 & 0 \\ V(z) & W(z) & -I & 0 \\ 0 & E(z) & -A(z) & -B(z) \\ \hline 0 & 0 & C(z) & D \end{array}\right) \begin{pmatrix} \xi \\ v \\ w \\ u \end{pmatrix}.$$

Similarly, for $C(z) = C$ constant, the closed-loop interconnected system has the first-order representation

$$\begin{aligned} z(t+1) &= \mathscr{A}_c(t) + \mathscr{B}_c u(t), \\ y(t) &= \mathscr{C}_c z(t) + Du(t), \end{aligned} \tag{9.17}$$

with system matrices

$$\mathscr{A}_c = \left(\begin{array}{c|c} \alpha + \beta J_1 \gamma & \beta H \\ \hline G_1 \gamma & F \end{array}\right), \mathscr{B}_c = \begin{pmatrix} \beta J_2 \\ G_2 \end{pmatrix}, \mathscr{C}_c = \begin{pmatrix} C\gamma & 0 \end{pmatrix}.$$

Theorem 9.4. *Consider two pairs of N node systems with polynomial system matrices*

$$\begin{pmatrix} T_i^{(\nu)}(z) & -U_i^{(\nu)}(z) \\ V_i^{(\nu)}(z) & W_i^{(\nu)}(z) \end{pmatrix}, \nu = 1, 2,\ i = 1, \ldots, N.$$

Assume that, for all i,

$$\begin{pmatrix} T_i^{(1)}(z) & -U_i^{(1)}(z) \\ V_i^{(1)}(z) & W_i^{(1)}(z) \end{pmatrix} \simeq_{FSE} \begin{pmatrix} T_i^{(2)}(z) & -U_i^{(2)}(z) \\ V_i^{(2)}(z) & W_i^{(2)}(z) \end{pmatrix}.$$

Defining $T^{(\nu)}(z) = \operatorname{diag}(T_1^{(\nu)}(z), \ldots, T_N^{(\nu)}(z))$, *and similarly for the other matrices, then*

$$\begin{pmatrix} T^{(1)}(z) & -U^{(1)}(z) \\ V^{(1)}(z) & W^{(1)}(z) \end{pmatrix} \simeq_{FSE} \begin{pmatrix} T^{(2)}(z) & -U^{(2)}(z) \\ V^{(2)}(z) & W^{(2)}(z) \end{pmatrix}.$$

Assume that each of the two systems is connected by the same interconnection rule (9.15). Then

$$\left(\begin{array}{ccc|c} T^{(1)}(z) & -U^{(1)}(z) & 0 & 0 \\ V^{(1)}(z) & W^{(1)}(z) & -I & 0 \\ 0 & E(z) & -A(z) & -B(z) \\ \hline 0 & 0 & C(z) & D \end{array}\right) \simeq_{FSE} \left(\begin{array}{ccc|c} T^{(2)}(z) & -U^{(2)}(z) & 0 & 0 \\ V^{(2)}(z) & W^{(2)}(z) & -I & 0 \\ 0 & E(z) & -A(z) & -B(z) \\ \hline 0 & 0 & C(z) & D \end{array}\right). \tag{9.18}$$

Assume that C is constant, with

$$\begin{pmatrix} T(z) & -U(z) \\ V(z) & W(z) \end{pmatrix} \simeq_{FSE} \begin{pmatrix} zI - \alpha & -\beta \\ \gamma & 0 \end{pmatrix}$$

and

$$\left(\begin{array}{c|cc} E(z) & -A(z) & -B(z) \\ \hline I & 0 & 0 \end{array}\right) \simeq_{FSE} \left(\begin{array}{c|cc} zI - F & -G_1 & -G_2 \\ \hline H & J_1 & J_2 \end{array}\right).$$

Then

$$\left(\begin{array}{ccc|c} T(z) & -U(z) & 0 & 0 \\ V(z) & W(z) & -I & 0 \\ 0 & E(z) & -A(z) & -B(z) \\ \hline 0 & 0 & C & D \end{array}\right) \simeq_{FSE} \left(\begin{array}{cc|c} zI-\alpha-\beta J_1\gamma & -\beta H & -\beta J_2 \\ -G_1\gamma & zI-F & -G_2 \\ \hline C\gamma & 0 & D \end{array}\right).$$

Proof. By our assumption, there exist polynomial matrices $M(z),N(z),X(z),Y(z)$, with $M(z),T^{(2)}(z)$ left coprime and $T^{(1)}(z),N(z)$ right coprime, for which

$$\begin{pmatrix} M(z) & 0 \\ -X(z) & I \end{pmatrix}\begin{pmatrix} T^{(1)}(z) & -U^{(1)}(z) \\ V^{(1)}(z) & W^{(1)}(z) \end{pmatrix} = \begin{pmatrix} T^{(2)}(z) & -U^{(2)}(z) \\ V^{(2)}(z) & W^{(2)}(z) \end{pmatrix}\begin{pmatrix} N(z) & Y(z) \\ 0 & I \end{pmatrix}.$$

In turn, this implies

$$\begin{pmatrix} M(z) & 0 & 0 & 0 \\ -X(z) & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{pmatrix}\left(\begin{array}{ccc|c} T^{(1)}(z) & -U^{(1)}(z) & 0 & 0 \\ V^{(1)}(z) & W^{(1)}(z) & -I & 0 \\ 0 & E(z) & -A(z) & -B(z) \\ \hline 0 & 0 & C(z) & D \end{array}\right)$$

$$= \left(\begin{array}{ccc|c} T^{(2)}(z) & -U^{(2)}(z) & 0 & 0 \\ V^{(2)}(z) & W^{(2)}(z) & -I & 0 \\ 0 & E(z) & -A(z) & -B(z) \\ \hline 0 & 0 & C(z) & D \end{array}\right)\begin{pmatrix} N(z) & Y(z) & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{pmatrix}.$$

The left coprimeness of

$$\begin{pmatrix} M(z) & 0 & 0 & 0 \\ -X(z) & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{pmatrix} \quad \text{and} \quad \left(\begin{array}{ccc|c} T^{(2)}(z) & -U^{(2)}(z) & 0 & 0 \\ V^{(2)}(z) & W^{(2)}(z) & -I & 0 \\ 0 & E(z) & -A(z) & -B(z) \\ \hline 0 & 0 & C(z) & D \end{array}\right)$$

follows from the left coprimeness of $M(z)$ and $T^{(2)}(z)$, and similarly for right coprimeness. Thus (9.18) follows. For the remaining part, observe that the following strict equivalences are valid:

$$\left(\begin{array}{ccc|c} T(z) & -U(z) & 0 & 0 \\ V(z) & W(z) & -I & 0 \\ 0 & E(z) & -A(z) & -B(z) \\ \hline 0 & 0 & C & D \end{array}\right) \simeq_{FSE} \left(\begin{array}{cccc|c} T(z) & -U(z) & 0 & 0 & 0 \\ V(z) & W(z) & 0 & -I & 0 \\ 0 & E(z) & 0 & -A(z) & -B(z) \\ 0 & 0 & I & 0 & 0 \\ \hline 0 & 0 & C & 0 & D \end{array}\right) \simeq_{FSE}$$

$$\left(\begin{array}{cccc|c} T(z) & -U(z) & 0 & 0 & 0 \\ V(z) & W(z) & 0 & -I & 0 \\ 0 & E(z) & E(z) & -A(z) & -B(z) \\ 0 & 0 & I & 0 & 0 \\ \hline 0 & 0 & C & 0 & D \end{array}\right) \simeq_{FSE} \left(\begin{array}{cccc|c} T(z) & -U(z) & 0 & 0 & 0 \\ V(z) & W(z) & 0 & -I & 0 \\ 0 & 0 & E(z) & -A(z) & -B(z) \\ 0 & -I & I & 0 & 0 \\ \hline 0 & 0 & C & 0 & D \end{array}\right)$$

$$\simeq_{FSE} \left(\begin{array}{cccc|c} zI-\alpha & -\beta & 0 & 0 & 0 \\ \gamma & 0 & 0 & -I & 0 \\ 0 & 0 & zI-F & -G_1 & -G_2 \\ 0 & -I & H & J_1 & J_2 \\ \hline 0 & 0 & 0 & C & D \end{array}\right) \simeq_{FSE} \left(\begin{array}{cc|c} zI-\alpha-\beta J_1\gamma & -\beta H & -\beta J_2 \\ -G_1\gamma & zI-F & -G_2 \\ \hline C\gamma & 0 & D \end{array}\right).$$

∎

We emphasize that the formulation of this theorem for polynomial matrix descriptions covers several cases of interest, including nonminimal state-space descriptions and (not necessarily coprime) matrix fraction descriptions. The next result for constant interconnections is a straightforward consequence of Theorem 9.4; however, the proof is more specific in exhibiting the required isomorphisms. For ease of exposition we assume that the node transfer function $G(z) = V(z)T(z)^{-1}U(z) + W(z)$ is strictly proper, with $W(z) = 0$.

Theorem 9.5. *Assume the strictly proper transfer function $G(z)$ of decoupled node systems has the following, minimal, representations:*

$$\begin{aligned} G(z) &= \gamma(zI-\alpha)^{-1}\beta \\ &= D_\ell(z)^{-1}N_\ell(z) = N_r(z)D_r(z)^{-1} = V(z)T(z)^{-1}U(z). \end{aligned} \tag{9.19}$$

Let A, B, C, D be interconnection matrices.

1. *The shift realizations associated with the interconnected polynomial system matrices*

$$\begin{pmatrix} T(z)-U(z)AV(z) & -U(z)B \\ CV(z) & D \end{pmatrix}, \quad \begin{pmatrix} D_r(z)-AN_r(z) & -B \\ CN_r(z) & D \end{pmatrix}$$

are similar.

2. *The shift realizations associated with the interconnected polynomial system matrices*

$$\begin{pmatrix} T(z)-U(z)AV(z) & -U(z)B \\ CV(z) & D \end{pmatrix}, \quad \begin{pmatrix} D_\ell(z)-N_\ell(z)A & -N_\ell(z)B \\ C & D \end{pmatrix}$$

are similar.

3. *The shift realizations associated with the interconnected polynomial system matrices*

$$\begin{pmatrix} D_\ell(z)-N_\ell(z)A & -N_\ell(z)B \\ C & D \end{pmatrix}, \quad \begin{pmatrix} D_r(z)-AN_r(z) & -B \\ CN_r(z) & D \end{pmatrix}$$

are similar.

4. *The realizations associated with the interconnected polynomial system matrices*

$$\begin{pmatrix} T(z)-U(z)AV(z) & -U(z)B \\ CV(z) & D \end{pmatrix}, \quad \begin{pmatrix} zI-\alpha-\beta A\gamma & -\beta B \\ C\gamma & D \end{pmatrix}$$

are similar.

Proof. Without loss of generality, one can assume that $D=0$.

1. By our assumption of minimality, using the state-space isomorphism theorem and the definition of FSE, it follows that all polynomial system matrices

$$\begin{pmatrix} \alpha & -\beta \\ \gamma & 0 \end{pmatrix}, \begin{pmatrix} D_\ell(z) & -N_\ell(z) \\ I & 0 \end{pmatrix}, \begin{pmatrix} D_r(z) & -I \\ N_r(z) & 0 \end{pmatrix}, \text{and} \begin{pmatrix} T(z) & -U(z) \\ V(z) & 0 \end{pmatrix}$$

are system equivalent. Our plan is to show that all polynomial system matrices of the connected system, namely,

$$\begin{pmatrix} zI-\alpha-\beta A\gamma & -\beta \\ \gamma & 0 \end{pmatrix}, \begin{pmatrix} D_\ell(z)-N_\ell(z)A & -N_\ell(z) \\ I & 0 \end{pmatrix}, \begin{pmatrix} D_r(z)-AN_r(z) & -I \\ N_r(z) & 0 \end{pmatrix}, \text{ and}$$
$$\begin{pmatrix} T(z)-U(z)AV(z) & -U(z) \\ V(z) & 0 \end{pmatrix},$$

are also system equivalent. Noting that the transfer function of the unconnected system has the representation (9.19), it follows that there exists a polynomial matrix $S(z)$ for which $T(z)^{-1}U(z)=S(z)D_r(z)^{-1}$, and hence both the intertwining relation

$$U(z)D_r(z)=T(z)S(z)$$

and

$$N_r(z)=V(z)S(z)$$

hold. Since $UAN_r=UAVS$, the identity

$$\begin{pmatrix} U(z) & 0 \\ 0 & I \end{pmatrix}\begin{pmatrix} D_r(z) & -I \\ N_r(z) & 0 \end{pmatrix}=\begin{pmatrix} T(z) & -U(z) \\ V(z) & 0 \end{pmatrix}\begin{pmatrix} S(z) & 0 \\ 0 & I \end{pmatrix}$$

implies

$$\begin{pmatrix} U(z) & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} D_r(z) - AN_r(z) & -I \\ N_r(z) & 0 \end{pmatrix} = \begin{pmatrix} T(z) - U(z)AV(z) & -U(z) \\ V(z) & 0 \end{pmatrix} \begin{pmatrix} S(z) & 0 \\ 0 & I \end{pmatrix}.$$

In turn, this implies

$$\begin{pmatrix} D_r(z) - AN_r(z) & -I \\ N_r(z) & 0 \end{pmatrix} \simeq_{FSE} \begin{pmatrix} T(z) - U(z)AV(z) & -U(z) \\ V(z) & 0 \end{pmatrix}.$$

2. The proof is similar.
3. The equality $D_\ell(z)^{-1}N_\ell(z) = N_r(z)D_r(z)^{-1}$ leads to the intertwining relation

$$\begin{pmatrix} N_\ell(z) & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} D_r(z) - AN_r(z) & -I \\ N_r(z) & 0 \end{pmatrix} = \begin{pmatrix} D_\ell(z) - N_\ell(z)A & -N_\ell(z) \\ I & 0 \end{pmatrix} \begin{pmatrix} N_r(z) & 0 \\ 0 & I \end{pmatrix}.$$

In turn, this implies

$$\begin{pmatrix} D_r(z) - AN_r(z) & -I \\ N_r(z) & 0 \end{pmatrix} \simeq_{FSE} \begin{pmatrix} D_\ell(z) - N_\ell(z)A & -N_\ell(z) \\ I & 0 \end{pmatrix}, \tag{9.20}$$

which proves the similarity of the associated shift realizations.
4. By the state-space isomorphism theorem, the shift realizations associated with the polynomial system matrices

$$\begin{pmatrix} zI - \alpha & -\beta \\ \gamma & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} T(z) & -U(z) \\ V(z) & 0 \end{pmatrix}$$

are similar. Thus, applying Definition 4.30,

$$\begin{pmatrix} zI - \alpha & -\beta \\ \gamma & 0 \end{pmatrix} \simeq_{FSE} \begin{pmatrix} T(z) & -U(z) \\ V(z) & 0 \end{pmatrix}.$$

Our next step is to go from a state-space representation to a minimal right matrix fraction representation. To this end, let $H(z)D_r(z)^{-1}$ be a right coprime factorization of $(zI - \alpha)^{-1}\beta$. This implies the intertwining relation $\beta D_r(z) = (zI - \alpha)H(z)$. Moreover, define $N_r(z) = \gamma H(z)$. Similarity implies system equivalence, but one can actually write down the equivalence explicitly, namely,

$$\begin{pmatrix} \beta & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} D_r(z) & -I \\ N_r(z) & 0 \end{pmatrix} = \begin{pmatrix} zI - \alpha & -\beta \\ \gamma & 0 \end{pmatrix} \begin{pmatrix} H(z) & 0 \\ 0 & I \end{pmatrix}.$$

Thus

$$\begin{pmatrix} D_r(z) & -I \\ N_r(z) & 0 \end{pmatrix} \simeq_{FSE} \begin{pmatrix} zI-\alpha & -\beta \\ \gamma & 0 \end{pmatrix}.$$

Passing on to the polynomial system matrices associated with a coupled system, an easy computation shows that

$$\begin{pmatrix} \beta & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} D_r(z)-AN_r(z) & -I \\ N_r(z) & 0 \end{pmatrix} = \begin{pmatrix} zI-\alpha-\beta A\gamma & -\beta \\ \gamma & 0 \end{pmatrix} \begin{pmatrix} H(z) & 0 \\ 0 & I \end{pmatrix}.$$

This implies

$$\begin{pmatrix} zI-\alpha-\beta A\gamma & -\beta \\ \gamma & 0 \end{pmatrix} \simeq_{FSE} \begin{pmatrix} D_r(z)-AN_r(z) & -I \\ N_r(z) & 0 \end{pmatrix}. \tag{9.21}$$

Using equations (9.20) and (9.21) and the transitivity of FSE, one obtains

$$\begin{pmatrix} zI-\alpha-\beta A\gamma & -\beta \\ \gamma & 0 \end{pmatrix} \simeq_{FSE} \begin{pmatrix} T(z)-U(z)AV(z) & -U(z) \\ V(z) & 0 \end{pmatrix}.$$

So far, the required system equivalences were shown for $C = I$ and $B = I$. However, it is easily seen that if two polynomial system matrices

$$\begin{pmatrix} T_1(z) & -U_1(z) \\ V_1(z) & 0 \end{pmatrix}, \quad \begin{pmatrix} T_2(z) & -U_2(z) \\ V_2(z) & 0 \end{pmatrix}$$

are FSE, then for matrices B and C the polynomial matrices

$$\begin{pmatrix} T_1(z) & -U_1(z)B \\ CV_1(z) & 0 \end{pmatrix}, \quad \begin{pmatrix} T_2(z) & -U_2(z)B \\ CV_2(z) & 0 \end{pmatrix}.$$

are also system equivalent. This completes the proof. ■

One can reformulate this result as follows.

Corollary 9.6 (Permanence Principle). *Suppose strictly proper input-output systems $\Sigma_1,\dots,\Sigma_N$ are coupled by interconnection matrices A,B,C to define a network Σ. Assume that the systems Σ_i are system equivalent to systems $\hat{\Sigma}_i$, $i = 1,\dots,N$. Let $\hat{\Sigma}$ be a network obtained by interconnecting $\hat{\Sigma}_1,\dots,\hat{\Sigma}_N$ using the identical interconnections A,B,C. Then Σ is system equivalent to $\hat{\Sigma}$.*

We list an obvious consequence of this result for poles and zeros of systems that was introduced in Chapter 4.7. The proof is left to the reader. Recall that $\overline{\mathbb{F}}$ denotes the algebraic closure of $\mathbb{F}$.

Corollary 9.7. *Assume that the strictly proper transfer function $G(z)$ of the decoupled node systems has the following, minimal, representations:*

$$\begin{aligned} G(z) &= \gamma(zI-\alpha)^{-1}\beta, \\ &= D_\ell(z)^{-1}N_\ell(z) = N_r(z)D_r(z)^{-1}, \\ &= V(z)T(z)^{-1}U(z). \end{aligned}$$

Consider the interconnection matrices A,B,C,D. Then:

1. *The interconnected system $(\alpha+\beta A\gamma,\beta B,C\gamma,D)$ has a finite zero at $z\in\overline{\mathbb{F}}$ if and only if*

$$\mathrm{rk}\begin{pmatrix} T(z)-U(z)AV(z) & -U(z)B \\ CV(z) & D \end{pmatrix} < \bar{r}+grk;$$

   here grk denotes the generic rank of $\mathscr{N}_G(z)$;
2. *$(\alpha+\beta A\gamma,\beta B,C\gamma,D)$ has a pole at $z\in\overline{\mathbb{F}}$ if and only if*

$$\det(T(z)-U(z)AV(z))=0;$$

3. *Assume $\mathbb{F}=\mathbb{R}$. Then $(\alpha+\beta A\gamma,\beta B,C\gamma,D)$ is discrete-time (continuous-time) asymptotically stable if and only if* $\det(T(z)-U(z)AV(z))$ *is a Schur (Hurwitz) polynomial.*

9.3 Reachability and Observability of Networks of Systems

The permanence principle is our main tool for analyzing the reachability and observability properties of interconnected systems. The results will depend on the type of interconnections, i.e., whether they are static or dynamic.

1. Static Interconnections. The question we are interested in is to decide when an interconnected system in state-space form (9.5) is reachable or observable. Of course, if the input and output interconnection matrices B,C are identity matrices, then the effect of the interconnection matrix A is simply by the action of static output feedback on the decoupled, block-diagonal system α,β,γ. In particular, reachability and observability would be preserved. However, except for this trivial case, it is more difficult to characterize reachability and observability. A naive approach might be to compute the $\bar{n}\times\bar{n}m$-Kalman reachability matrix (or, equivalently, the Hautus test) for system (9.5) and check its rank. But this requires checking the rank of a potentially huge matrix. Moreover, owing to the additive perturbation structure of $\mathscr{A}$, the impact of interconnection parameters on the reachability properties is hard to assess. Therefore, one searches for an alternative reachability characterization that exhibits interconnection parameters and node dynamics in a more direct form.

The characterization of the reachability and observability of the shift realization associated with a polynomial system matrix, as in Theorem 4.26, allows us to easily derive such characterizations for interconnected systems. Since reachability and observability are preserved by FSE, Theorem 9.5 implies the following result.

Theorem 9.8. *With the same assumptions as in Corollary 9.7, let*

$$\mathscr{A} = \alpha + \beta A \gamma, \quad \mathscr{B} = \beta B, \quad \mathscr{C} = C\gamma.$$

1. *The transfer function $\mathscr{N}_G(z)$ of an interconnected system has the following representations:*

$$\begin{aligned}\mathscr{N}_G(z) &= \mathscr{C}(zI - \mathscr{A})^{-1}\mathscr{B} + D \\ &= C(D_\ell(z) - N_\ell(z)A)^{-1}N_\ell(z)B + D = CN_r(z)(D_r(z) - AN_r(z))^{-1}B + D \\ &= CV(z)(T(z) - U(z)AV(z))^{-1}U(z)B + D.\end{aligned}$$

2. *The following statements are equivalent:*
   (a) The system $(\mathscr{A}, \mathscr{B}, \mathscr{C})$ is reachable.
   (b) $D_\ell(z) - N_\ell(z)A$ and $N_\ell(z)B$ are left coprime.
   (c) $D_r(z) - AN_r(z)$ and B are left coprime.
   (d) $T(z) - U(z)AV(z)$ and $U(z)B$ are left coprime.
3. *The following statements are equivalent:*
   (a) The system $(\mathscr{A}, \mathscr{B}, \mathscr{C})$ is observable.
   (b) $D_\ell(z) - N_\ell(z)A$ and C are right coprime.
   (c) $D_r(z) - AN_r(z)$ and $CN_r(z)$ are right coprime.
   (d) $CV(z)$ and $T(z) - U(z)AV(z)$ are right coprime.
4. *The system $(\mathscr{A}, \mathscr{B}, \mathscr{C})$ is minimal if and only if $D_r(z) - AN_r(z)$ and B are left coprime and $D_r(z) - AN_r(z)$ and $CN_r(z)$ are right coprime.*

Proof. By Theorem 9.5, the triple $(\mathscr{A}, \mathscr{B}, \mathscr{C})$ is similar to the shift realizations of each of three polynomial matrix representations $(D_\ell(z) - N_\ell(z)A, N_\ell(z)B, C)$, $(D_r(z) - AN_r(z), B, CN_r(z))$, and $(T(z) - U(z)AV(z), U(z)B, CV(z))$. The Shift Realization Theorem 4.26 then implies that the reachability and observability of these realizations is equivalent to left coprimeness and right coprimeness, respectively. The result follows. ∎

Corollary 9.9. *A necessary condition for the reachability/observability of an interconnected system is the reachability/observability of all node systems.*

Proof. By Theorem 9.8, an interconnected system is reachable if and only if the polynomial matrices

$$\begin{pmatrix} T(z) & -U(z) & 0 \\ V(z) & 0 & -I \\ 0 & I & A \end{pmatrix} \text{ and } \begin{pmatrix} 0 \\ 0 \\ B \end{pmatrix}$$

are left coprime. Clearly, for this, the left coprimeness of $T(z)$ and $U(z)$ is necessary. Because of the diagonal nature of both polynomial matrices, this is equivalent to the left coprimeness of $T_i(z)$ and $U_i(z)$ for all i. One argues similarly for observability. ■

Let us point out that a similar result, phrased in terms of decoupling zeros, appeared in Rosenbrock and Pugh (1974). At that time, the connection between decoupling zeros and the properties of reachability and observability had not yet been clarified.

While Corollary 9.9 provides a simple necessary condition for reachability, the condition is in general not sufficient. As a simple consequence of Theorem 9.8 one obtains the following Hautus-type characterization of the reachability and observability of networks. Let $\overline{\mathbb{F}}$ denote the algebraic closure of the field $\mathbb{F}$.

Theorem 9.10. *(a)* $(\mathscr{A},\mathscr{B})$ *is reachable if and only if*

$$\operatorname{rk}\left(T(z) - U(z)AV(z)\;\; U(z)B\right) = \bar{r}, \quad \forall z \in \overline{\mathbb{F}}.$$

(b) $(\mathscr{C},\mathscr{A})$ *is observable if and only if*

$$\operatorname{rk}\begin{pmatrix} T(z) - U(z)AV(z) \\ CV(z) \end{pmatrix} = \bar{r}, \quad \forall z \in \overline{\mathbb{F}}.$$

The preceding result exhibits, in a clear way, how the different components of the network contribute to reachability and observability. In comparison with the Kalman reachability matrix, the size is reduced to $\bar{r} \times (\bar{r} + m)$. Note that for homogeneous networks with scalar node functions, the matrices $T(z), U(z), V(z)$ become scalar multiples of the identity matrix. Therefore, Theorem 9.10 implies that the reachability and observability properties of a homogeneous network are actually independent of the choice of the strictly proper node function. Thus, for homogeneous networks and scalar nodes, the network realization $(\mathscr{A},\mathscr{B})$ is reachable if and only if (A,B) is reachable. This greatly simplifies the analysis of scalar homogeneous networks; see Section 9.6 for further details and applications. For homogeneous networks with multivariable node transfer functions, the result is not true without further assumptions. We will now extend Theorem 9.8 to dynamical interconnection laws and analyze in detail some special interconnection schemes.

2. Dynamic Interconnections. We next consider more general dynamical coupling laws between the various node systems. This is important for network control applications, where one wants to allow for possible delays in interconnections, thus modeling potential communication delays between subsystems. Let

$$\begin{aligned} x(t+1) &= \alpha x(t) + \beta v(t) \\ w(t) &= \gamma x(t) \end{aligned} \tag{9.22}$$

denote the uncoupled array of node systems. As before, assume that (α, β, γ) is reachable and observable with right and left coprime factorizations of the block-diagonal transfer function

$$\gamma(zI-\alpha)^{-1}\beta = D_\ell(z)^{-1}N_\ell(z) = N_r(z)D_r(z)^{-1}.$$

Let $\gamma(zI-\alpha)^{-1}\beta = V(z)T(z)^{-1}U(z)+W(z)$ be a polynomial matrix fraction decomposition, with $V(z),T(z)$ right coprime and $T(z),U(z)$ left coprime. Consider the dynamic interconnection law (9.15) via

$$\begin{aligned} E(\sigma)v &= A(\sigma)w + B(\sigma)u, \\ y &= Cw + Du. \end{aligned}$$

Here $E(z)$ is a square nonsingular polynomial matrix, $A(z)$ and $B(z)$ are appropriately sized polynomial matrices, and C and D are constant matrices. Assuming that the rational function $E(z)^{-1}(A(z),B(z))$ is proper, there exists a proper, first-order shift realization as

$$\begin{aligned} \zeta(t+1) &= F\zeta(t) + G_1w(t) + G_2u(t), \\ v(t) &= H\zeta(t) + J_1w(t) + J_2u(t). \end{aligned} \tag{9.23}$$

Equations (9.22) and (9.23) can be combined to yield the first-order representation

$$\begin{aligned} z(t+1) &= \mathscr{A}_c z(t) + \mathscr{B}_c u(t), \\ y(t) &= \mathscr{C}_c z(t) + Du(t), \end{aligned} \tag{9.24}$$

with system matrices

$$\mathscr{A}_c = \left(\begin{array}{c|c} \alpha+\beta J_1\gamma & \beta H \\ \hline G_1\gamma & F \end{array}\right), \mathscr{B}_c = \begin{pmatrix} \beta J_2 \\ G_2 \end{pmatrix}, \mathscr{C}_c = \begin{pmatrix} C\gamma & 0 \end{pmatrix}.$$

The interconnected system then has the following PMD representation:

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ I \end{pmatrix} y = \left(\begin{array}{ccc|c} T(z) & -U(z) & 0 & 0 \\ V(z) & W(z) & -I & 0 \\ 0 & E(z) & -A(z) & -B(z) \\ \hline 0 & 0 & C & D \end{array}\right) \begin{pmatrix} \xi \\ v \\ w \\ u \end{pmatrix}.$$

The special choice $T(z) = D_r(z), U(z) = I, V(z) = N_r(z), W(z) = 0$ leads to the following result.

Theorem 9.11. *Given N node strictly proper systems, with right and left coprime matrix fraction representations $D_\ell(z)^{-1}N_\ell(z) = N_r(z)D_r(z)^{-1}$, respectively, the following strict system equivalences are fulfilled:*

$$\left(\begin{array}{cc|c} zI-\alpha-\beta J_1\gamma & -\beta H & -\beta J_2 \\ -G_1\gamma & zI-F & -G_2 \\ \hline C\gamma & 0 & D \end{array}\right) \simeq_{FSE}$$

$$\left(\begin{array}{ccc|c} D_r(z) & -I & 0 & 0 \\ N_r(z) & 0 & -I & 0 \\ 0 & E(z) & -A(z) & -B(z) \\ \hline 0 & 0 & C & D \end{array}\right) \simeq_{FSE} \left(\begin{array}{cc|c} I & 0 & 0 \\ 0 & -A(z)N_r(z)+E(z)D_r(z) & -B(z) \\ \hline 0 & CN_r(z) & D \end{array}\right).$$

Proof. The first equivalence follows from Theorem 9.4. The FSE representations

$$\left(\begin{array}{ccc|c} N_\ell(z) & -D_\ell(z) & 0 & 0 \\ 0 & 0 & I & 0 \\ \hline 0 & 0 & 0 & I \end{array}\right) \left(\begin{array}{ccc|c} D_r(z) & -I & 0 & 0 \\ N_r(z) & 0 & -I & 0 \\ 0 & E(z) & -A(z) & -B(z) \\ \hline 0 & 0 & C & D \end{array}\right) = \left(\begin{array}{ccc|c} 0 & -N_\ell(z) & D_\ell(z) & 0 \\ 0 & E(z) & -A(z) & -B(z) \\ \hline 0 & 0 & C & D \end{array}\right)$$

as well as

$$\left(\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & I & 0 \\ \hline 0 & 0 & I \end{array}\right) \left(\begin{array}{cc|c} I & 0 & 0 \\ 0 & -A(z)N_r(z)+E(z)D_r(z) & -B(z) \\ \hline 0 & CN_r(z) & D \end{array}\right)$$
$$= \left(\begin{array}{ccc|c} 0 & -N_\ell(z) & D_\ell(z) & 0 \\ 0 & E(z) & -A(z) & -B(z) \\ \hline 0 & 0 & C & D \end{array}\right) \left(\begin{array}{cc|c} I & 0 & 0 \\ 0 & D_r(z) & 0 \\ 0 & N_r(z) & 0 \\ \hline 0 & 0 & I \end{array}\right)$$

are satisfied. It is easily seen that these representations define FSE transformations. The result follows. ■

We conclude that the reachability of the node systems and left coprimeness of $(E(z), A(z), B(z))$ are necessary conditions for the reachability of a dynamically interconnected network. The next theorem characterizes the reachability and observability properties of interconnected systems. The proof is an obvious consequence of the preceding equivalence result and therefore omitted.

Theorem 9.12. *Assume that (α, β, γ) are reachable and observable and N_r and D_r are right coprime, with $N_r(z)D_r(z)^{-1} = \gamma(zI-\alpha)^{-1}\beta$. Assume further that*

$(F,(G_1,G_2),H)$ is reachable and observable with left coprime factorization $E(z)^{-1}(A(z),B(z)) = (J_1,J_2) + H(zI-F)^{-1}(G_1,G_2)$. Let

$$\mathscr{A}_c = \left(\begin{array}{c|c} \alpha+\beta J_1\gamma & \beta H \\ \hline G_1\gamma & F \end{array}\right), \mathscr{B}_c = \begin{pmatrix} \beta J_2 \\ G_2 \end{pmatrix}, \mathscr{C}_c = \begin{pmatrix} C\gamma & 0 \end{pmatrix}$$

denote the realization of the dynamically interconnected system (9.24).

1. $(\mathscr{A}_c,\mathscr{B}_c,\mathscr{C}_c)$ is reachable if and only if

$$\operatorname{rk}\begin{pmatrix} A(z)N_r(z)-E(z)D_r(z), B(z) \end{pmatrix} = \overline{n}, \quad \forall z \in \overline{\mathbb{F}}.$$

2. $(\mathscr{A}_c,\mathscr{B}_c,\mathscr{C}_c)$ is observable if and only if

$$\operatorname{rk}\begin{pmatrix} A(z)N_r(z)-E(z)D_r(z) \\ CN_r(z) \end{pmatrix} = \overline{n}, \quad \forall z \in \overline{\mathbb{F}}.$$

As a special case of the general dynamical coupling law one can characterize reachability and observability for delayed interconnection schemes of the form

$$v_i(t) = \sum_{j=1}^{N} A_{ij} w_j(t-L_{ij}) + B_{iu}(t)$$

for nonnegative integers L_{ij}. This network is described as follows using the interconnection law (9.15). We use the notations $L_i := \max_{j=1,\ldots,N} L_{ij}$, $i = 1,\ldots,N$, and $\overline{L}_{ij} = L_i - L_{ij} \geq 0$. Define the polynomial matrices $E_L(z) = \operatorname{diag}(z^{L_1},\cdots,z^{L_N})$, $A_L(z) = (A_{ij}z^{\overline{L}_{ij}})_{i,j=1,\ldots,N} \in \mathbb{F}^{\overline{p}\times\overline{m}}[z]$, $B(z) := E_L(z)B$, and $C(z) = C$. One obtains the following characterization of reachability and observability for delayed networks.

Theorem 9.13. *The shift realization of a delayed network is reachable if and only if*

$$\operatorname{rk}\begin{pmatrix} A_L(z)N_r(z)-E_L(z)D_r(z), E_L(z)B \end{pmatrix} = \overline{n}, \quad \forall z \in \overline{\mathbb{F}}.$$

The shift realization of a delayed network is observable if and only if

$$\operatorname{rk}\begin{pmatrix} A_L(z)N_r(z)-E_L(z)D_r(z) \\ CN_r(z) \end{pmatrix} = \overline{n}, \quad \forall z \in \overline{\mathbb{F}}.$$

As a special case, consider homogeneous networks of identical SISO systems with node transfer functions $G_i(z) = \frac{p(z)}{q(z)}$ satisfying $p(0) \neq 0$. In this situation the preceding result implies the following corollary.

Corollary 9.14. *Consider a network of identical node transfer functions $G_i(z) = \frac{p(z)}{q(z)}$ satisfying $p(0) \neq 0$. Assume that A is invertible and (A,B,C) is reachable (observable). Assume further that all delays L_{ij} are identical and* *equal to* $L \geq 1$. *Then the delayed network* $(\mathscr{A}_c, \mathscr{B}_c, \mathscr{C}_c)$ *is reachable (observable), independently of the value of L.*

Proof. Let $A = (A_{ij})$. Since $L_{ij} = L$, one obtains $\overline{L}_{ij} = 0$, and thus $A(z) = A$. From Theorem 9.12 we conclude that reachability is equivalent to $(p(z)A - z^{Lq}(z)I, z^L B)$ having full row rank. For $z = 0$ this is true since $p(0) \neq 0$ and A is invertible. For $z \neq 0$ this is equivalent to $(A - \frac{z^{Lq}(z)}{p(z)} I, B)$ having full row rank, which again follows from the reachability of (A, B). For observability one argues similarly. ■

9.4 Homogeneous Networks

Clearly, the simplest classes of networks are the homogeneous ones, defined by interconnections of identical linear systems with SISO node transfer function $g(z)$. Thus, assume that the dynamics of the identical node systems in a linear network are described by a single scalar strictly proper transfer function

$$g(z) = \gamma(zI_n - \alpha)^{-1}\beta,$$

with $\alpha \in \mathbb{F}^{n\times n}, \beta \in \mathbb{F}^n, \gamma \in \mathbb{F}^{1\times n}$ reachable and observable. This is a special case of (9.7). Define $h(z) = 1/g(z)$. Let $\mathscr{N}(z) = C(zI - A)^{-1}B$ denote the $p \times m$ interconnection transfer function. Then the network transfer function $\mathscr{N}_G(z)$ of the homogeneous network is $\mathscr{N}_G(z) = C(h(z)I - A)^{-1}B$, i.e., it is the composition of rational functions $\mathscr{N} \circ h$. Let $(\mathscr{A}, \mathscr{B}, \mathscr{C})$ be the shift realization of the network transfer function $\mathscr{N}_G(z)$ associated with a minimal factorization of $g(z) = p/q$. Then the matrices in (9.6) are represented in Kronecker product form as

$$\begin{aligned}
\mathscr{A} &= I_N \otimes \alpha + A \otimes \beta\gamma \in \mathbb{F}^{nN\otimes nN},\\
\mathscr{B} &= B \otimes \beta \in \mathbb{F}^{nN\times m},\\
\mathscr{C} &= C \otimes \gamma \in \mathbb{F}^{p\times nN}.
\end{aligned} \tag{9.25}$$

The following test for the reachability of homogeneous networks is a simple consequence of Theorem 9.10. It was first stated and proved by Hara, Hayakawa and Sugata (2009). Our proof is quite different and avoids complicated state-space canonical form arguments.

Theorem 9.15. *The shift realization* $(\mathscr{A}, \mathscr{B}, \mathscr{C})$ *of* $\mathscr{N}_G(z)$*, defined by (9.25), is reachable (observable) if and only if the realization* (A, B, C) *of the interconnection transfer function* $\mathscr{N}(z)$ *is reachable (observable). In particular, the reachability of* $(\mathscr{A}, \mathscr{B}, \mathscr{C})$ *is independent of the choice of the node transfer function* $g(z)$*, as long as* $g(z)$ *is scalar rational and strictly proper.*

Proof. By Theorem 9.8, $(\mathscr{A},\mathscr{B},\mathscr{C})$ is reachable if and only if $Q(z)-P(z)A, P(z)B$ are left coprime. The coprime factorization of the decoupled system $Q(z)^{-1}P(z) = g(z)I_N$ is $Q(z) = q(z)I_N$, $P(z) = p(z)I_N$, with $g(z) = p(z)/q(z)$ coprime. Thus $(\mathscr{A},\mathscr{B})$ is reachable if and only if

$$\mathrm{rk}(q(z)I_N - p(z)A, p(z)B) = N \tag{9.26}$$

for all $z \in \overline{\mathbb{F}}$. If $p(z) = 0$, then, by coprimeness, $q(z) \neq 0$, and (9.26) is satisfied. The fundamental theorem of algebra implies that for all $w \in \overline{\mathbb{F}}$ there exists $z \in \overline{\mathbb{F}}$, with $p(z) \neq 0$ and $w = \frac{q(z)}{p(z)}$. Dividing by $p(z)$, it follows that the left coprimeness condition is equivalent to $\mathrm{rk}(wI_N - A, B) = N$ for all $w \in \overline{\mathbb{F}}$. Thus the reachability of $(\mathscr{A},\mathscr{B})$ is equivalent to the reachability of (A,B), and we are done. One argues similarly for observability and minimality. ∎

The amazing consequence of the preceding theorem is that reachability can be analyzed completely independently of the choice of node function. We will now make this even more explicit by relating coprime factorizations of $\mathscr{N}_G$ and $\mathscr{N}$. Assume that $(A,B) \in \mathbb{F}^{N\times(N+m)}$ is reachable with reachability indices $\kappa_1 \geq \cdots \geq \kappa_m$. Choose a right coprime factorization

$$(zI - A)^{-1}B = N(z)D(z)^{-1}$$

by $N \times m$ and $m \times m$ polynomial matrices $N(z)$ and $D(z)$, respectively. Therefore, $\det D(z) = \det(zI - A)$. Without loss of generality, one can assume that $D(z)$ is in column proper form, i.e., the leading coefficient matrix of $D(z)$ is $D_0\Delta(z)$, with D_0 invertible and

$$\Delta(z) = \mathrm{diag}\,(z^{\kappa_1}, \cdots, z^{\kappa_m}).$$

Let $g(z)$ denote a strictly proper, scalar rational transfer function with coprime factorization $g(z) = p(z)/q(z)$ and McMillan degree n. Define the homogenizations

$$N_g(z) = N\Big(\frac{q(z)}{p(z)}\Big)\Delta(p(z)), \quad D_g(z) = D\Big(\frac{q(z)}{p(z)}\Big)\Delta(p(z)).$$

Proposition 9.16. *Under the preceding assumptions, the following assertions are true:*

1. *$N_g(z)$ and $D_g(z)$ are right coprime polynomial matrices and $D_g(z)$ is in column proper form.*
2. $\det D_g(z) = \det(q(z)I - p(z)A)$.
3. *The reachability indices of the shift realization of $N_g(z)D_g(z)^{-1}$ are equal to $(n\kappa_1, \ldots, n\kappa_m)$.*

Proof. That $N_g(z)$ and $D_g(z)$ are polynomials follows easily from $N(z)$ and $D(z)$ being in column proper form. For right coprimeness, one must show that

$$\operatorname{rk}\begin{pmatrix} N_g(z) \\ D_g(z) \end{pmatrix} = m \tag{9.27}$$

for all z in the algebraic closure of the field $\mathbb{F}$. If z is not a zero of p, then $\Delta(p(z))$ is invertible. Therefore, the rank condition (9.27) follows from the corresponding rank condition for $N(z)$ and $D(z)$ at the point $h(z) = \frac{q(z)}{p(z)}$. If $p(z) = 0$, then $D_g(z) = D_0\Delta(q(z))$ is invertible. Moreover, by the strict properness of $N(z)D(z)^{-1}$, at such a point z one has $N_g(z) = 0$. This proves right coprimeness. The column properness of $D_g(z)$ follows from the fact that the leading term of $D_g(z)$ is $D_0\Delta(q(z))$. This also implies part 3, i.e., that the reachability indices of the shift realization of $N_g(z)D_g(z)^{-1}$ are $n\kappa_i$ for $i = 1,\dots,m$.

2. It suffices to verify the formula for the transfer function for all z that are not zeros of $p(z)$. Note that

$$N_g(z)D_g(z)^{-1} = N(h(z))D(h(z))^{-1} = C(h(z)I - A)^{-1}B = C(q(z)I - p(z)A)^{-1}p(z)B.$$

Finally, for each z that is not a zero of p,

$$\det D_g(z) = \det D(h(z)) \det\Delta(p(z)) = \det D(h(z))p(z)^n = \det(q(z)I - p(z)A)).$$

This completes the proof. ■

9.5 Special Coupling Structures

Many coupling patterns in interconnected systems arise by specifying linear dependency relations among the coefficients of the coupling matrices A,B,C. Thus, for example, one may consider $0-*$ patterns in which the entries of A,B,C are either 0 or free independent variables. Other examples include block upper triangular matrices, symmetric matrices, and Toeplitz matrices A. In this section, we will explore the reachability task for some of these interconnection structures. A more systematic approach would require tools from graph theory. Here we pursue modest goals and focus on the analysis of special cases such as path graphs and circular structures.

1. Paths. Path graphs or, more generally, trees are among the simplest hierarchical interconnection patterns. Certainly, the easiest example of a coupling pattern that comes from a path is the nearest-neighbor interconnection scheme with controls at the first node. Thus, consider N node systems Σ_i with reachable and observable state-space representations $\alpha_i \in \mathbb{F}^{n\times n}$, $\beta_i \in \mathbb{F}^{n\times m}$, and $\gamma_i \in \mathbb{F}^{p\times n}$. For $i = 1,\dots,N$, let

$$\gamma_i(zI - \alpha_i)^{-1}\beta_i = D_{\ell,i}(z)^{-1}N_{\ell,i}(z) = N_{r,i}(z)D_{r,i}(z)^{-1} \tag{9.28}$$

denote left and right coprime factorizations of the associated transfer functions. For simplicity assume $m = p$. Consider the state interconnection matrices

$$A = \begin{pmatrix} 0 & \cdots & \cdots & 0 \\ I_m & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & I_m & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ \vdots \\ I_m \\ \vdots \\ 0 \end{pmatrix}, \tag{9.29}$$

the I_m-component of B being at position $1 \leq r \leq N$. Clearly, (9.29) represents a nearest-neighbor interaction of N systems, with the external controls entering at node r. The closed-loop system matrix then has the lower bidiagonal form

$$\mathscr{A} = \begin{pmatrix} \alpha_1 & & & & \\ \beta_2\gamma_1 & \alpha_2 & & & \\ & \beta_3\gamma_2 & \ddots & & \\ & & \ddots & \ddots & \\ & & & \beta_N\gamma_{N-1} & \alpha_N \end{pmatrix}, \quad \mathscr{B} = \begin{pmatrix} 0 \\ \vdots \\ \beta_r \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \tag{9.30}$$

Note that, for $r = 1$, the network (9.30) is simply the series connection $\Sigma_1 \wedge \ldots \wedge \Sigma_N$ of N systems and thus is reachable if and only if the $(N-1)m \times Nm$ polynomial Sylvester-type matrix

$$\begin{pmatrix} N_{r,1}(z) & D_{r,2}(z) & & & \\ & \ddots & \ddots & & \\ & & \ddots & \ddots & \\ & & & N_{r,N-1}(z) & D_{r,N}(z) \end{pmatrix}$$

is left prime. Applying Theorem 9.8, one observes that the system is not reachable for $r > 1$.

The situation becomes more interesting for symmetric couplings defined by the interconnection matrices $A = J \otimes I_m$ and $B = e_k \otimes I_m$, where

$$J = \begin{pmatrix} 0 & 1 & & & \\ 1 & 0 & 1 & & \\ & 1 & \ddots & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & 1 & 0 \end{pmatrix}. \tag{9.31}$$

The interconnected system is then

$$\mathscr{A} = \begin{pmatrix} \alpha_1 & \beta_1\gamma_2 & & \\ \beta_2\gamma_1 & \alpha_2 & \ddots & \\ \vdots & \ddots & \ddots & \beta_{N-1}\gamma_N \\ 0 & . & \beta_N\gamma_{N-1} & \alpha_N \end{pmatrix}, \quad \mathscr{B} = \begin{pmatrix} 0 \\ . \\ \beta_k \\ . \\ 0 \end{pmatrix}. \tag{9.32}$$

Again, applying Theorem 9.8 to the coprime factorization (9.28), we conclude that (9.32) is reachable if and only if the polynomial matrices

$$\begin{pmatrix} D_{r,1}(z) & N_{r,1}(z) & & \\ N_{r,2}(z) & D_{r,2}(z) & \ddots & \\ \vdots & \ddots & \ddots & N_{r,N-1}(z) \\ 0 & . & N_{r,N}(z) & D_{r,N}(z) \end{pmatrix}, \quad \begin{pmatrix} 0 \\ . \\ I_m \\ . \\ 0 \end{pmatrix}$$

are left prime. Equivalently, the polynomial matrix

$$\begin{pmatrix} D_{r,1}(z) & N_{r,1}(z) & & & & & & & \\ N_{r,2}(z) & D_{r,2}(z) & N_{r,2}(z) & & & & & & \\ & \ddots & \ddots & \ddots & & & & & \\ & & N_{r,k-1}(z) & D_{r,k-1}(z) & N_{r,k-1}(z) & & & & \\ & & & & N_{r,k+1}(z) & D_{r,k+1}(z) & N_{r,k+1}(z) & & \\ & & & & & \ddots & \ddots & \ddots & \\ & & & & & & N_{r,N-1}(z) & D_{r,N-1}(z) & N_{r,N-1}(z) \\ & & & & & & & N_{r,N}(z) & D_{r,N}(z) \end{pmatrix}$$

is left prime.

For identical node systems with $D(z) := D_{r,1}(z) = \ldots = D_{r,N}(z)$ and $N(z) := N_{r,1}(z) = \ldots = N_{r,N}(z)$, more explicit results can be obtained using the spectral information on (9.31). By Theorem 8.45, matrix J has N distinct real eigenvalues $2\cos\frac{k\pi}{N+1}, k = 1,\ldots,N$, with eigenvectors given by the columns of

$$T = \sqrt{\frac{2}{N+1}} \begin{pmatrix} \sin\frac{\pi}{N+1} & \cdots & \sin\frac{N\pi}{N+1} \\ \vdots & \ddots & \vdots \\ \sin\frac{N\pi}{N+1} & \cdots & \sin\frac{N^2\pi}{N+1} \end{pmatrix}.$$

Note that the column vectors $x^{(k)}$ of T are pairwise orthogonal with Euclidean norm

$$\frac{N+1}{2}\|x^{(k)}\|^2 = \sum_{j=1}^{N} \sin^2\left(\frac{kj\pi}{N+1}\right) = \frac{N}{2} - \frac{1}{2}\sum_{j=1}^{N} \cos\frac{2kj\pi}{N+1}$$
$$= \frac{N}{2} - \frac{1}{2}\sum_{j=1}^{N} \mathrm{Re}(\omega^{kj}) = \frac{N+1}{2},$$

where $\omega = e^{\frac{2\pi\sqrt{-1}}{N+1}}$. Thus $T = \sqrt{\frac{2}{N+1}}(\sin\frac{kl\pi}{N+1})_{k,l}$ is a real orthogonal matrix such that $T^{-1}JT$ is diagonal. Then $T \otimes I_N$ diagonalizes A with eigenvalues $2\cos\frac{k\pi}{N+1}, k = 1,\ldots,N$, each one occurring with multiplicity m. Moreover,

$$T^{-1}e_r = \sqrt{\frac{N+1}{2}}\begin{pmatrix} \sin\frac{r\pi}{N+1} \\ \vdots \\ \sin\frac{Nr\pi}{N+1} \end{pmatrix}$$

has a zero entry if and only if $N+1$ divides rk for some $1 \le k \le N$, i.e., if and only if $N+1$ and r are not coprime. This leads to an explicit characterization of reachability that is independent of the node system.

Theorem 9.17. *The interconnected system (9.32) with identical nodes is reachable if and only if $N+1$ and r are coprime.*

Proof. The matrix $T \otimes I_m\,(I_N \otimes D(z) - J \otimes N(z))\,T^{-1} \otimes I_m$ is block-diagonal with block-diagonal entries $D(z) - 2\cos\frac{k\pi}{N+1}N(z)$, $1 \le k \le N$. Therefore, the pair $(I_N \otimes D(z) - J \otimes N(z), T^{-1}e_r \otimes I_m)$ is left coprime if and only if $T^{-1}e_r$ has no zero entry, i.e., if and only if r and $N+1$ are coprime. ■

2. Simple Circulant Structures. Here the reachability problem for linear systems with special circulant interconnection structures are explored. We refer the reader to Brockett and Willems (1974) and Lunze (1986) for earlier work on circulant systems. Further motivation derives from the observation that such systems present the simplest kind of systems with symmetries; see, for example, Hazewinkel and Martin (1983). Symmetric systems also arise in recent studies on spatially invariant systems; see, for example, Bamieh, Paganini and Dahleh (2002).

Consider now N nodes with transfer functions $G_i(z)$ coupled circularly. Specifically, in terms of minimal state-space realizations (9.4), one has the following state-space equations describing the individual nodes ($i = 1,\ldots,N$):

$$\begin{aligned} \sigma x_i &= \alpha_i x_i(t) + \beta_i v_i(t), \\ w_i(t) &= \gamma_i x_i(t). \end{aligned}$$

This by itself is not sufficient to describe an interconnected system. We need to describe the cross influence between the nodes and the way in which the external input influences the nodes. There are many options. The cross influence between the nodes can be one- or two-sided nearest-neighbor interactions up to interactions between all nodes in the same way. Then one must consider how the external input influences the individual nodes. The two extreme cases are, first, that the external input is applied only to one node and, second, that it is applied directly to all nodes. Similar cases of interest exist for the global output of an interconnected system. We briefly review some of the options.

2a. Unidirectional Nearest-Neighbor Coupling and One-Node External Control. The coupling information is described by the matrices

$$A = \begin{pmatrix} 0 & \cdots & 0 & I \\ I & \ddots & & 0 \\ & \ddots & \ddots & \vdots \\ & & I & 0 \end{pmatrix}, \quad B = \begin{pmatrix} I \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad C = \begin{pmatrix} I & 0 & \cdots & 0 \end{pmatrix}. \tag{9.33}$$

The coupled system has the following representation:

$$\mathscr{A} = \begin{pmatrix} \alpha_1 & 0 & \cdots & 0 & \beta_1\gamma_N \\ \beta_2\gamma_1 & \alpha_2 & \ddots & & 0 \\ & \beta_3\gamma_2 & \ddots & \ddots & \vdots \\ & & \ddots & \ddots & 0 \\ & & & \beta_N\gamma_{N-1} & \alpha_N \end{pmatrix}, \quad \mathscr{B} = \begin{pmatrix} \beta_1 \\ 0 \\ \vdots \\ \vdots \\ 0 \end{pmatrix}, \quad \mathscr{C} = \begin{pmatrix} \gamma_1 & 0 & \cdots & 0 \end{pmatrix}. \tag{9.34}$$

2b. Bidirectional Nearest-Neighbor Coupling and One-Node External Control. The coupling in (9.33) is unidirectional. Alternatively, one can use the more symmetric, nearest-neighbor, coupling described by

$$A = \begin{pmatrix} 0 & I & & & I \\ I & \ddots & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & I \\ I & & & I & 0 \end{pmatrix}, \quad B = \begin{pmatrix} I \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad C = \begin{pmatrix} I & 0 & \cdots & 0 \end{pmatrix}.$$

In this case, the coupled system has the following representation:

$$\mathscr{A} = \begin{pmatrix} \alpha_1 & \beta_1\gamma_2 & 0 & \cdots & \beta_1\gamma_N \\ \beta_2\gamma_1 & \alpha_2 & \ddots & & 0 \\ 0 & \beta_3\gamma_2 & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \beta_{N-1}\gamma_N \\ \beta_N\gamma_1 & 0 & \cdots & \beta_N\gamma_{N-1} & \alpha_N \end{pmatrix}, \quad \mathscr{B} = \begin{pmatrix} \beta_1 \\ 0 \\ \vdots \\ \vdots \\ 0 \end{pmatrix}, \quad \mathscr{C} = \begin{pmatrix} \gamma_1 & 0 & \cdots & 0 \end{pmatrix}. \tag{9.35}$$

2c. Full Coupling and One-Node External Control. The coupling is described by

$$A = \begin{pmatrix} 0 & I & & & I \\ I & \ddots & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & I \\ I & & & I & 0 \end{pmatrix}, \quad B = \begin{pmatrix} I \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad C = \begin{pmatrix} I & 0 & \cdots & 0 \end{pmatrix}.$$

In this case, the coupled system has the following representation:

$$\mathscr{A} = \begin{pmatrix} \alpha_1 & \beta_1\gamma_2 & 0 & \cdots & \beta_1\gamma_N \\ \beta_2\gamma_1 & \alpha_2 & \ddots & & 0 \\ 0 & \beta_3\gamma_2 & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \beta_{N-1}\gamma_N \\ \beta_N\gamma_1 & 0 & \cdots & \beta_N\gamma_{N-1} & \alpha_N \end{pmatrix}, \quad \mathscr{B} = \begin{pmatrix} \beta_1 \\ 0 \\ \vdots \\ \vdots \\ 0 \end{pmatrix}, \quad \mathscr{C} = \begin{pmatrix} \gamma_1 & 0 & \cdots & 0 \end{pmatrix}. \tag{9.36}$$

If all nodes have the same state-space representation, that is, in the homogeneous case, then the matrices $\mathscr{A}$ in the representations (9.34), (9.35), and (9.36) will all have a block-circulant structure; see the next subsection for a discussion of general block-circulant structures.

For our purposes, it is advantageous to obtain matrix fraction representations of the various interconnections. However, because the computations are similar, we restrict ourselves to a single case, for which a characterization of reachability is obtained, which is summarized by the following theorem. This should be compared with the criteria for the reachability of series connections derived subsequently in (10.5).

Theorem 9.18. *Consider the node systems Σ_i, $i = 1, \dots, N$, with coprime matrix fraction representations as in (9.7). The circular interconnection system (9.34) is reachable if and only if the polynomial matrix*

$$\begin{pmatrix} N_{r,1}(z) & D_{r,1}(z) & & & \\ & N_{r,2}(z) & D_{r,2}(z) & & \\ & & \ddots & \ddots & \\ & & & N_{r,N-1}(z) & D_{r,N}(z) \end{pmatrix} \tag{9.37}$$

is left prime.

Proof. Applying Theorem 9.8, one sees that (9.34) is reachable if and only if the pair of polynomial matrices

$$(D_r(z) - AN_r(z), B) = \left(\begin{pmatrix} D_{r,1} & & & -N_{r,N} \\ -N_{r,1} & D_{r,2} & & \\ & \ddots & \ddots & \\ & & -N_{r,N-1} & D_{r,N} \end{pmatrix}, \begin{pmatrix} I_m \\ 0 \\ \vdots \\ 0 \end{pmatrix} \right)$$

is left coprime. After a simple column operation, this is equivalent to the left primeness of (9.37). ■

3. Block Circulant Structures. Following Brockett and Willems (1974), we begin by presenting a state-space formulation of the situation we are interested in and introduce our subsequent notation. Since Fourier transform techniques will be applied, we restrict ourselves to the field $\mathbb{C}$ of complex numbers. An $N \times N$ **block-circulant matrix** has the form

$$\mathbf{A} = \begin{pmatrix} A_0 & A_1 & \cdots & A_{N-2} & A_{N-1} \\ A_{N-1} & A_0 & \cdots & \cdots & A_{N-2} \\ A_{N-2} & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ A_1 & \cdots & A_{N-2} & A_{N-1} & A_0 \end{pmatrix},$$

where A_i denotes an $n \times n$ matrix with complex coefficients. Similarly, let $\mathbf{B}$ and $\mathbf{C}$ denote block-circulant matrices, where the block matrices are of the form $B_i \in \mathbb{C}^{n\times m}$ and $C_i \in \mathbb{C}^{p\times n}$. Consider the input and output matrices, respectively,

$$\beta = \begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_N \end{pmatrix}, \quad \gamma = \begin{pmatrix} \gamma_1 & \gamma_2 & \cdots & \gamma_N \end{pmatrix}.$$

Here, the submatrices satisfy $\beta_i \in \mathbb{C}^{m\times r}$ and $\gamma_i \in \mathbb{C}^{s\times p}$, respectively. Consider N interconnected discrete-time block-circulant linear systems

$$\begin{aligned} x(t+1) &= \mathbf{A}x(t) + \mathbf{B}\beta u(t), \\ y(t) &= \gamma \mathbf{C}x(t). \end{aligned} \tag{9.38}$$

We are interested in characterizing when such systems are reachable. Let $\omega := \exp(2\pi\sqrt{-1}/N)$ denote the primitive Nth root of unity, and let

$$\Phi = \frac{1}{\sqrt{N}}(\omega^{(k-1)(\ell-1)})_{k,\ell} \tag{9.39}$$

denote the $N \times N$ **Fourier matrix**. Note that the Fourier matrix Φ is the reachability matrix of the reachable pair

$$\Delta(\omega) = \begin{pmatrix} 1 & & & \\ & \omega & & \\ & & \ddots & \\ & & & \omega^{N-1} \end{pmatrix}, \quad \mathbf{1} = \frac{1}{\sqrt{N}} \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}.$$

It is easily seen that the block-circulant matrix is exactly of the form $\mathbf{A} = \sum_{i=0}^{N-1} S^i \otimes A_i$, where

$$S = \Phi \Delta(\omega) \Phi^* = \begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & \ddots & 1 \\ 1 & & & & 0 \end{pmatrix} \tag{9.40}$$

denotes the standard $N \times N$ circulant matrix. In particular, finite sums and products of block-circulant matrices with square blocks are block-circulant. Moreover, the Fourier matrix is unitary and $\Phi \otimes I_n$ block-diagonalizes all block-circulant matrices $\mathbf{A} = \sum_{i=0}^{N-1} S^i \otimes A_i$. Block-circulant matrices $\mathbf{A}$ are best analyzed in terms of the associated matrix polynomial

$$A(z) := \sum_{k=0}^{N-1} A_k z^k \in \mathbb{C}[z]^{n \times n}.$$

Thus

$$\mathbf{A} = (\Phi \otimes I_n) \begin{pmatrix} A(1) & & & \\ & A(\omega) & & \\ & & \ddots & \\ & & & A(\omega^{N-1}) \end{pmatrix} (\Phi^* \otimes I_n),$$

and similarly for $\mathbf{B}$ and $\mathbf{C}$. This shows that one has full knowledge on the eigenstructure of block-circulant matrices. Explicitly, the eigenvalues of block-circulant matrices are the eigenvalues of $A(1), \ldots, A(\omega^{N-1})$, respectively, while the eigenvectors of $\mathbf{A}$ are equal to $(\Phi \otimes I_N)v$ for the eigenvectors v of $\operatorname{diag}(A(1), \ldots, A(\omega^{N-1}))$.

Define $b \in \mathbb{C}^{Nm \times r}$, $c \in \mathbb{C}^{s \times Np}$ as

$$b = (\Phi \otimes I_m)\beta = \begin{pmatrix} b_N \\ b_1 \\ \vdots \\ b_{N-1} \end{pmatrix}, \quad c = \gamma(\Phi^* \otimes I_p) = \begin{pmatrix} c_N & c_1 \cdots c_{N-1} \end{pmatrix}.$$

Thus, in the discrete-time case (and similarly for continuous-time systems), the block-circulant system (9.38) is state-space equivalent to the parallel connected system

$$
\begin{aligned}
x_k(t+1) &= A(\omega^k)x_k(t) + B(\omega^k)b_k u(t),\\
y_k(t) &= c_k C(\omega^k)x_k(t), \quad k = 1,\ldots,N.
\end{aligned}
\tag{9.41}
$$

We emphasize that this is simply the parallel sum of N systems $(c_kC(\omega^k), A(\omega^k), B(\omega^k)b_k)$. The $s \times r$ transfer function of system (9.41) is

$$\mathscr{N}_G(z) = c\,\mathrm{diag}\,(G_1(z),\cdots,G_N(z))b,$$

where $G_k(z) = C(\omega^k)(zI - A(\omega^k))^{-1}B(\omega^k)$. Let

$$(zI - A(\omega^k))^{-1}B(\omega^k)b_k := N_k(z)D_k(z)^{-1}$$

denote a right coprime factorization into polynomial matrices $N_k(z), D_k(z)$, $k \in 1,\ldots,N$. Thus $N_k(z)$ and $D_k(z)$ are $n \times r$ and $r \times r$ polynomial matrices, respectively. From Theorem 10.4 one arrives at the following theorem.

Theorem 9.19. *Assume that $(A(\omega^k), B(\omega^k)b_k)$ are reachable for $k = 1,\cdots,N$. The block-circulant system (9.38) is reachable if and only if the N polynomial matrices $D_k(z) \in \mathbb{C}[z]^{r\times r}, k = 1,\ldots,N$ are mutually left coprime. In particular, for $r = 1$ and reachable pairs $(A(\omega^k), B(\omega^k)b_k), k = 1,\ldots,N$, system (9.38) is reachable if and only if the polynomials* $\det(zI - A(\omega^k))$ *are pairwise coprime for $k = 1,\cdots,N$.*

The preceding result generalizes previous results by Lunze (1986) and Brockett and Willems (1974). These authors considered block-circulant control systems in which each subsystem was controlled independently. Thus, they effectively assumed that b was the identity matrix. This excludes several interesting cases, such as leader–follower networks. The more general case treated here is motivated by the more recent work of Brockett (2010), in which the inputs are broadcasted to all nodes of a network. The next example has been studied by Lunze (1986).

Example 9.20. Consider the circulant system

$$x(t+1) = \mathbf{A}x(t) + \mathbf{B}u(t)$$

with independent controls. Let $\mathbf{A}$ and $\mathbf{B}$ be circulant matrices with $A(z) = A_0 - A_1 + A_1(1 + \cdots + z^{N-1})$ and $B(z) = B_0$. For $z = 1$ one obtains $A(1) = A_0 + (N-1)A_1$, and $A(\omega^i) = A_0 - A_1$ for $i = 1,\ldots,N-1$. Therefore, $(A(z), B(z))$ is reachable for all Nth roots of unity z if and only if the two systems $(A_0 - A_1, B_0)$ and $(A_0 + (N-1)A_1, B_0)$ are reachable. This coincides with the result by Lunze (1986). In contrast, if one replaces $\mathbf{B}$ with $\mathbf{B}\beta$, with β a column vector, then Theorem 9.19 implies that the system is not reachable for $N > 2$.

4. Periodic Interconnections. We proceed to discuss briefly an extension to periodic interconnection laws for discrete-time systems. We refer the reader to Bittanti and Colaneri (2010) for background material on periodic linear systems. Thus, consider N reachable and observable discrete-time decoupled node systems

$$\begin{aligned} x_k(t+1) &= \alpha_k x_k(t) + \beta_k v_k(t), \\ w_k(t) &= \gamma_k x_k(t), \quad k = 1,\ldots,N, \end{aligned}$$

with system matrices $\alpha_k \in \mathbb{F}^{n_k \times n_k}, \beta_k \in \mathbb{F}^{n_k \times m_k}, \gamma_k \in \mathbb{F}^{p_k \times n_k}$. Equivalently, introducing the global state vectors $x = \operatorname{col}(x_1,\ldots,x_N) \in \mathbb{F}^{\overline{n}}$, and similarly for the input and output vectors v, w, one obtains the global decoupled system

$$\begin{aligned} x(t+1) &= \alpha x(t) + \beta v_k(t), \\ w(t) &= \gamma x(t), \end{aligned}$$

where $\alpha = \operatorname{diag}(\alpha_1,\ldots,\alpha_N)$, and one argues similarly for β, γ. We emphasize at this point that (α, β, γ) is assumed to be a time-invariant reachable and observable system. One could also investigate periodic node systems, but we will not do so here. Let $G(z) = N_r(z)D_r(z)^{-1} = \gamma(zI - \alpha)^{-1}\beta$ be a right coprime factorization of the global node transfer function. Consider the periodic interconnection law

$$\begin{aligned} v(t) &= A_t w(t) + B_t u(t), \\ y(t) &= C_t w(t), \end{aligned}$$

with $A_t = A_{t+\tau}, B_t = B_{t+\tau}, C_t = C_{t+\tau}$ time-varying matrices of period $\tau \in \mathbb{N}$. The closed-loop, first-order system is then the τ-periodic system

$$\begin{aligned} x(t+1) &= \mathscr{A}_t x(t) + \mathscr{B}_t u(t), \\ y(t) &= \mathscr{C}_t x(t), \end{aligned} \tag{9.42}$$

with $\mathscr{A}_t = \alpha + \beta A_t \gamma, \mathscr{B}_t = \beta B_t, \mathscr{C}_t = C_t \gamma$. For simplicity, let us focus on reachability. Define

$$\overline{A} = \begin{pmatrix} 0 & & & A_\tau \\ A_1 & \ddots & & \\ & \ddots & \ddots & \\ & & A_{\tau-1} & 0 \end{pmatrix}, \quad \overline{B} = \begin{pmatrix} B_\tau & & & \\ & B_1 & & \\ & & \ddots & \\ & & & B_{\tau-1} \end{pmatrix}, \quad \overline{C} = \begin{pmatrix} 0 & & & C_\tau \\ C_1 & \ddots & & \\ & \ddots & \ddots & \\ & & C_{\tau-1} & 0 \end{pmatrix}.$$

The reachability properties of a periodic system are characterized by the reachability properties of the so-called lifted system. We refer the reader to Bittanti and Colaneri (2010) for a discussion on the reachability properties of periodic systems and a proof of the equivalence of the reachability of periodic systems and of the reachability of lifted systems. Let S denote the standard $\tau \times \tau$ circulant matrix defined in (9.40).

Proposition 9.21. *The closed-loop periodic system (9.42) is reachable (observable) if and only if the time-invariant* ***lifted system***

$$\mathscr{A}_e = S^\top \otimes \alpha + (I \otimes \beta)\overline{A}(I \otimes \gamma), \quad \mathscr{B}_e = (I \otimes \beta)\overline{B}, \quad \mathscr{C}_e = \overline{C}(I \otimes \gamma)$$

is reachable (observable).

Since we are using the Fourier transform, let us assume from now on that the systems are defined over the field $\mathbb{F} = \mathbb{R}$ of real numbers or over the field $\mathbb{F} = \mathbb{C}$ of complex numbers. Define $\omega = e^{2\pi\sqrt{-1}/\tau}, \overline{\omega} = e^{-2\pi\sqrt{-1}/\tau}$, and let Φ denote the $\tau \times \tau$ Fourier matrix. Then

$$(\Phi^* \otimes I)\mathscr{A}_e(\Phi \otimes I) = \Delta(\overline{\omega}) \otimes \alpha + (I \otimes \beta)\hat{A}(I \otimes \gamma), \quad (\Phi^* \otimes I)\mathscr{B}_e = (I \otimes \beta)\hat{B},$$

where $\Delta(\overline{\omega}) = \operatorname{diag}(1, \overline{\omega}, \ldots, \overline{\omega}^{\tau-1})$, $\hat{A} = (\Phi^* \otimes I)\overline{A}(\Phi \otimes I)$, and $\hat{B} = (\Phi^* \otimes I)\overline{B}$. Thus $(\mathscr{A}_e, \mathscr{B}_e)$ is reachable if and only if $(\Delta(\overline{\omega}) \otimes \alpha + (I \otimes \beta)\hat{A}(I \otimes \gamma), (I \otimes \beta)\hat{B})$ is reachable. The latter system is obtained by interconnecting the decoupled node system

$$\hat{\alpha} = \operatorname{diag}(\alpha, \overline{\omega}\alpha, \ldots, \overline{\omega}^{\tau-1}\alpha), \; \hat{\beta} = I \otimes \beta, \; \hat{\gamma} = I \otimes \gamma$$

with the interconnection matrices $\hat{A}$ and $\hat{B}$. The transfer function of the system $(\hat{\alpha}, \hat{\beta}, \hat{\gamma})$ is easily computed from the right coprime factorization $\overline{P}(z)\overline{Q}(z)^{-1}$ of $\gamma(zI - \alpha)^{-1}\beta$ as

$$\hat{G}(z) := (I \otimes \gamma)(zI - \hat{\alpha})^{-1}(I \otimes \beta) = \hat{P}(z)\hat{Q}(z)^{-1},$$

with right coprime factors

$$\hat{Q}(z) = \operatorname{diag}\left(D_r(z), D_r(\omega z), \ldots, D_r(\omega^{\tau-1}z)\right),$$
$$\hat{P}(z) = \operatorname{diag}\left(\omega N_r(\omega z), \omega N_r(\omega z), \ldots, \omega^{\tau-1}N_r(\omega^{\tau-1}z)\right).$$

Applying Theorem 9.10 we arrive at the following corollary.

Corollary 9.22. *A periodically interconnected network of linear systems (9.42) is reachable if and only if the matrix* $\left(\hat{Q}(z) - \hat{A}\hat{P}(z), \hat{B}\right)$ *has full row rank for all z.*

9.6 Exercises

1. For matrix fraction representations (9.11), prove that the map

$$Z : X_D \longrightarrow X_{D-NA}, \quad Zf = \pi_{D-NA}(f)$$

defines an isomorphism of $\mathbb{F}[z]$-modules.

2. Let $(\kappa_1,\ldots,\kappa_m)$ denote the reachability indices of (A,B), and let (α,β,γ) denote a minimal realization of a McMillan degree n, SISO transfer function. Show that the reachability indices of the homogeneous network $\mathscr{A} = I_N \otimes \alpha + A \otimes \beta\gamma$, $\mathscr{B} = B \otimes \beta$ are $(n\kappa_1,\ldots,n\kappa_m)$. Deduce that every state x of the network can be reached from 0 in at most $n\kappa_1$ steps.
3. Deduce from Proposition 9.16 the formula

$$\delta(\mathscr{N} \circ h) = \delta(\mathscr{N})\delta(g)$$

for the McMillan degree of the network transfer function of a homogeneous network. Apply this to obtain a new proof of Theorem 9.15.
4. Let $\lambda \in \mathbb{C}$ be nonzero and $p_C(z) := \sum_{j=0}^{N-1} c_{jz}^{j} \in \mathbb{C}[z]$. A complex **$\lambda$-circulant matrix** is a Toeplitz matrix of the form

$$C_\lambda = \begin{pmatrix} c_0 & c_1 & \cdots & c_{N-1} \\ \lambda c_{N-1} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & c_1 \\ \lambda c_1 & \cdots & \lambda c_{N-1} & c_0 \end{pmatrix}, \quad S_\lambda := \begin{pmatrix} 0 & 1 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ \lambda & \cdots & 0 & 0 \end{pmatrix}.$$

 a. Prove that a λ-circulant is equal to $p_C(S_\lambda)$. Conversely, each such matrix is a λ-circulant. Deduce that the set of λ-circulants is an Abelian algebra.
 b. Let γ denote an Nth root of λ, i.e., $\gamma^N = \lambda$. Prove that the eigenvectors of a λ-circulant matrix C_λ are the columns of the matrix $\operatorname{diag}(1,\gamma,\ldots,\gamma^{N-1})\Phi$. What are the eigenvalues?

5. Extend Theorem 9.19 to λ-circulant interconnection matrices $\mathbf{A},\mathbf{B},\mathbf{C}$.
6. Compute the eigenvalues and eigenvectors of the $N \times N$ circulant matrix

$$\mathbf{A} = \begin{pmatrix} a & & & b \\ b & \ddots & & \\ & \ddots & \ddots & \\ & & b & a \end{pmatrix}.$$

For which $\beta \in \mathbb{R}^N$ is $\dot{x}(t) = \mathbf{A}x(t) + \beta u(t)$ reachable?

9.7 Notes and References

A natural question in network analysis is that of structural controllability, i.e., the classification of all networks that are reachable for a generic choice of coupling parameters. We refer the reader to Dion, Commault and van der Woude (2013) for a survey on this topic. Liu, Slotine and Barabasi (2011) characterized all graphs such that (9.3) is structurally reachable, i.e., if the network is reachable for a generic choice of the nonzero coefficients in the coupling parameters A, b. Their work is based on the characterization by Lin (1974) on structural controllability and makes it possible to estimate the percentage of so-called driver nodes in a network, i.e., those state variables that, after pinning, lead to structurally reachable systems. We emphasize that the work by Liu, Slotine and Barabasi (2011) deals only with the very special situation in which the node systems are first-order integrators $\dot{x}_i = u_i$. For networks of a more general type of node system their conclusions on structural controllability need not hold. Theorem 9.15 enables one to take the first steps in that direction, i.e., to extend the graph-theoretic characterization of the structural controllability of linear systems to homogeneous networks. Characterizing structural controllability for heterogeneous networks is an open problem. Linear systems theory as developed over abstract fields is useful in several areas, including coding theory. This provides further motivation for the algebraic approach taken in this book. Observability and state estimation tasks for linear systems over finite fields are studied in Sundaram and Hadjicostis (2013).

Most of the present chapter is based on Fuhrmann and Helmke (2013). Our central tool for the structural analysis of networks is the equivalence theorem of Rosenbrock and Pugh (1974), extended in Theorem 9.4 to include dynamic couplings. All the subsequent results proven in this chapter follow directly from Theorem 9.4. Theorem 9.15 has been proven by Hara, Hayakawa and Sugata (2009) using complicated canonical form arguments. The early paper by Sontag (1979), which proves the same result in larger generality, has apparently been overlooked.

The key to a deeper understanding of homogeneous networks is the fact that the network transfer functions $\mathcal{N}_G = \mathcal{N} \circ h$, see (9.9), is the composition of the interconnection transfer function $\mathcal{N}(z)$, as defined by (9.8), with the reciprocal $h(z) = 1/g(z)$ of the scalar node transfer function $g(z)$. This simple observation in fact characterizes the transfer functions of homogeneous networks. The problem of characterizing the transfer functions of homogeneous networks is thus equivalent to the question, first raised by J.F. Ritt, of which rational functions can be written as a composition of two rational functions. Ritt (1922) proved that a complex scalar rational function f is the composition of two scalar rational functions if and only if the Galois group (or monodromy group) of f is imprimitive. This shows that a rational function f is the transfer function of a homogeneous network if and only if the Galois group of f is imprimitive. Ritt also solved the decomposition problem for complex polynomials; we refer the reader to Müller (1995) for a classification of the Galois groups of indecomposable polynomials. Of course, a full classification of Galois groups defined by rational functions is difficult and refers to the so-called inverse problem of Galois theory. Even more so, the characterization of imprimitive Galois groups of rational functions remains an open problem. We refer the reader to Brockett (1983) for related work on Galois groups attached to linear feedback systems. Algebraic-geometric characterizations of decomposable rational functions in terms of root loci or associated Bezoutian curves have been obtained by Pakovich (2011).

The necessary conditions for transfer functions of homogeneous networks are easily described in terms of fundamental system invariants. For example, the McMillan degree of rational functions is multiplicative, i.e., $\delta(f_1 \circ f_2) = \delta(f_1)\delta(f_2)$ is satisfied for rational functions f_1, f_2. A similar property holds for the Cauchy index of a rational function. If $F(z) = F(z)^\top$ is a real symmetric $m \times m$ proper rational function, then the **matrix Cauchy index** $CI(F)$, see Bitmead and Anderson (1977), of $F(z)$ is defined and the matrix Cauchy index of the composition $F \circ h$ with $h = 1/g$ satisfies

$$CI(F \circ h) = CI(F) \cdot CI(g) \,. \tag{9.43}$$

Consequently, this imposes a constraint on the Cauchy index of homogeneous networks. Formula (9.43) follows easily from the well-known interpretation of the Cauchy–Maslov index as the topological degree of the associated rational curve in the Lagrange–Grassmann manifold; see, for example, Byrnes and Duncan (1981). See Helmke (1989) for a generalization of formula (9.43) in terms of Bezoutian matrices.

The question of characterizing homogeneous networks is reminiscent of, but not equivalent to, the classical synthesis problem for electrical circuits, tackled by Brune (1931) and Bott and Duffin (1949). An amazing M.S. thesis by Ladenheim (1948) presents a catalog of 108 circuits that are claimed to realize all biquadratic positive real transfer functions. Despite these efforts, and those of many others, this fundamental circuit synthesis problem remains unsolved to date, but it has attracted some attention lately; see, for example, Kalman (2010), Smith (2002), Jiang and Smith (2011), and Hughes and Smith (2013).

Another interesting topic is the model reduction of networks of systems. There exist several options for doing this, either by reducing the number of nodes and coupling parameters or by order reduction of the node systems. For homogeneous networks, the situation becomes particularly nice. Mullis and Roberts (1976) have shown that if the discrete-time node transfer function $g(z)$ is allpass with respect to the unit circle, then the discrete-time reachability Gramian satisfies $W_c(\mathscr{A}, \mathscr{B}) = W_c(A,B) \otimes W_c(\alpha, \beta)$, and similarly for the observability Gramian. This has been generalized by Koshita, Abe and Kawamata (2007) for bounded real transfer functions $g(z)$ and leads to useful techniques for model reduction by balanced truncation.