Chapter 6
State Feedback and Output Injection

Our attention turns now to the study of the fundamental question: How does one use input variables in the actual control of a system? Naturally, the use of control functions depends on the desired objectives of performance. Moreover, there are various ways in which the control can be applied. One way is to determine, a priori, a control sequence in the discrete-time case, or a control function in the continuous-time case, and apply it. Thus the control is applied without regard to its lasting effects or to the actual system performance, except insofar as the design goals have been taken into account. This is referred to as **open-loop control**. Obviously, this kind of control is often far from being satisfactory. The reasons for this may be manifold, in that there is no exact modeling of systems or that there are no errorless determinations of state vectors or precisely known control mechanisms. Thus open-loop control does not take into account noise in the system or random variations occurring from external influences. However, one advantage of open-loop control lies in the computational ease of determining such controls, for instance using optimization techniques. This aspect becomes particularly important when dealing with the control of large networks of systems.

An alternative to open-loop control is **feedback control**. Our standing assumption is that some knowledge of the state is available to the control mechanism, or to the controller, and the control device takes this information into account. There is a wide range of possibilities in designing feedback laws. If at each moment the controller has access to the full state of the system, then one refers to it as **state feedback**. If only a function, linear in the case of interest to us, of the state variables is available, then this will be referred to as **output feedback**. A controller can be memoryless, that is, the control is determined only by the currently available information. In this case, it is called **constant gain feedback**. Alternatively, the controller itself may be a dynamic system, in which case one speaks of **dynamic feedback control**. Whatever the specific control strategy may be, feedback control has the well-known advantage of ensuring robustness, while its disadvantages lie in the computational burden that comes with either computing the feedback gains

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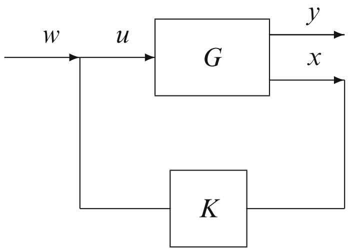
or estimating the full or partial state variables of the system. Thus feedback control can become an increasingly complex task for large-scale networks. The tasks of controlling large-scale interconnected systems, such as swarms or ensembles of systems, therefore rather ask for a hybrid approach where both open-loop and closed-loop control strategies are employed.

Schematically one considers the following feedback configuration:



Here G is the plant, or the original system, and K denotes the control device. Thus, in the linear, discrete-time case, the equations of the plant are

$$x_{t+1} = Ax_t + Bu_t,$$
$$y_t = Cx_t + Du_t,$$

and if the controller is memoryless, we consider static output feedback as

$$u_t = -Ky_t + w_t.$$

State feedback is a special case of this, and we set $p = n$ and $C = I_n$. A dynamic situation occurs when K is itself a linear system and hence is given internally by

$$z_{t+1} = Fz_t + Gy_t,$$
$$v_t = Hz_t + Jy_t,$$

while the coupling equation, which allows for another control apart from the feedback already incorporated, is

$$u_t = w_t + v_t.$$

In this chapter we focus on analyzing the effects of state feedback control, while the design of open-loop controls was addressed already in Chapter 4.

For a deeper analysis of state feedback and output injection problems for linear systems (A,B,C), it turns out to be useful to study special classes of linear subspaces in the state space that capture the dynamics of the subsystems of (A,B,C). Such subspaces are the controlled and conditioned invariant subspaces and were first introduced and extensively studied in the early 1970s by Basile and Marro, as well as Wonham and Morse in the context of geometric control theory. The textbooks by Wonham (1979) and Basile and Marro (1992) give comprehensive accounts of the geometric theory. This chapter will be confined to the derivation of basic characterizations of controlled and conditioned invariant subspaces, both in the state space and using functional model representations. In the subsequent Chapter 7, characterizations of functional observers will be given using conditioned invariant subspaces.

6.1 State Feedback Equivalence

This section is devoted to the study of the effects of state feedback transformations that act on a fixed input-to-state pair (A,B). Let $\mathscr{U}$ and $\mathscr{X}$ be finite-dimensional vector spaces over the field $\mathbb{F}$, and let (A,B) be a reachable pair with $A : \mathscr{X} \longrightarrow \mathscr{X}$ and $B : \mathscr{U} \longrightarrow \mathscr{X}$ linear transformations. Our assumption will be that $\dim \mathscr{U} = m$ and $\dim \mathscr{X} = n$. Through a choice of bases one can identify $\mathscr{U}$ and $\mathscr{X}$ with $\mathbb{F}^m$ and $\mathbb{F}^n$, respectively. In that case, A and B are represented by $n \times n$ and $n \times m$ matrices, respectively. The pair (A,B) stands for the linear system

$$x_{t+1} = Ax_t + Bu_t. \tag{6.1}$$

If the system is augmented by the identity readout map

$$y_t = x_t,$$

then the transfer function of the combined system is

$$G(z) = (zI - A)^{-1}B.$$

A **state feedback law** is given by

$$u_t = -Kx_t + w_t, \tag{6.2}$$

where w_t denotes the external input applied at time t. Substituting (6.2) back into (6.1) amounts to transforming the pair (A,B) into the pair $(A-BK,B)$. In this case, one says that $(A-BK,B)$ has been obtained from (A,B) by state feedback. Clearly, the applications of state feedback transformations form a commutative group. If the group is enlarged to the one generated by invertible transformations in the input space $\mathscr{U}$, state-space similarity transformations in $\mathscr{X}$, and state feedback transformations, then the full **state feedback group** $\mathscr{F}$ is obtained. Thus an element of $\mathscr{F}$ is a triple of linear maps (S,K,R), with $S : \mathscr{X} \longrightarrow \mathscr{X}$ and $R : \mathscr{U} \longrightarrow \mathscr{U}$ nonsingular and $K : \mathscr{X} \longrightarrow \mathscr{U}$. The feedback group acts on a pair (A,B) by the following rule:

$$(A,B) \overset{(S,K,R)}{\mapsto} \left(S(A-BR^{-1}K)S^{-1}, SBR^{-1}\right).$$

This implies the group composition law

$$(S,K,R)\circ(S_1,K_1,R_1) = (SS_1, RK_1 + KS_1, RR_1).$$

This composition law is clearly associative since it can be expressed in terms of matrix multiplication as follows:

$$\begin{pmatrix} S & 0 \\ K & R \end{pmatrix} \begin{pmatrix} S_1 & 0 \\ K_1 & R_1 \end{pmatrix} = \begin{pmatrix} SS_1 & 0 \\ KS_1 + RK_1 & RR_1 \end{pmatrix} \in GL_{n+m}(\mathbb{F}).$$

The **state feedback action** can be identified with

$$(A,B) \mapsto S\,(A,B) \begin{pmatrix} S & 0 \\ K & R \end{pmatrix}^{-1} = \left(S(A - BR^{-1}K)S^{-1}, SBR^{-1}\right).$$

This clearly shows that

$$(S,K,R)^{-1} = (S^{-1}, -R^{-1}KS^{-1}, R^{-1})$$

and, hence, that $\mathscr{F}$ is a bona fide group. It is clear from the matrix representation of the feedback group that every element of $\mathscr{F}$ is the product of three elementary types:

1. Similarity or change of basis in the state space, i.e., elements of the form $(S,0,I)$, with S invertible;
2. Similarity or change of basis in the input space, i.e., elements of the form $(I,0,R)$, with R invertible;
3. Pure feedbacks, i.e., elements of the form (I,K,I).

Indeed, one has the composition law

$$(S,K,R) = (S,0,I)(I,K,I)(I,0,R).$$

The feedback group $\mathscr{F}$ induces a natural equivalence relation in the set of reachable pairs (A,B) with state space $\mathscr{X}$ and input space $\mathscr{U}$. Let (A_i,B_i), $i = 1,2$, be input pairs with state spaces $\mathscr{X}_i$ and input spaces $\mathscr{U}_i$, respectively. The pair (A_2,B_2) is said to be **state feedback equivalent** to (A_1,B_1) if there exist invertible maps $Z : \mathscr{X}_1 \longrightarrow \mathscr{X}_2$ and $R : \mathscr{U}_1 \longrightarrow \mathscr{U}_2$ and a map $K : \mathscr{X}_1 \longrightarrow \mathscr{U}_2$ that satisfy

$$\begin{aligned} ZA_1 - A_2Z &= B_2K, \\ ZB_1 &= B_2R. \end{aligned}$$

It is trivial to check that this is indeed an equivalence relation. The equivalence classes are called the **orbits** of the feedback group, and one would like to obtain the orbit invariants as well as to isolate a single element in each orbit, a canonical form, that exhibits these invariants.

The dual concept the state feedback transformations is that of output injection. Here it is formulated for matrix representations of the system maps $A : \mathscr{X} \longrightarrow \mathscr{X}$, $B : \mathscr{U} \longrightarrow \mathscr{X}$, and $C : \mathscr{X} \longrightarrow \mathscr{Y}$, although a coordinate-free description is of course also possible. Thus two output pairs $(C_i, A_i) \in \mathbb{F}^{p\times n} \times \mathbb{F}^{n\times n}$, $i = 1,2$, are called **output injection equivalent** if there exist invertible matrices $R \in GL_n(\mathbb{F})$ and $S \in GL_p(\mathbb{F})$ and a matrix $L \in \mathbb{F}^{n\times p}$ such that

$$\begin{pmatrix} A_2 \\ C_2 \end{pmatrix} = \begin{pmatrix} R & L \\ 0 & S \end{pmatrix} \begin{pmatrix} A_1 \\ C_1 \end{pmatrix} R^{-1} = \begin{pmatrix} (RA_1 - LC_1)R^{-1} \\ SC_1R^{-1} \end{pmatrix}. \tag{6.3}$$

The relation of output injection equivalence defines a bona fide equivalence relation on the matrix space $\mathbb{F}^{p\times n} \times \mathbb{F}^{n\times n}$. The equivalence classes are given by the orbits

$$\left\{ (RCS^{-1}, S(A - LC)S^{-1}) \mid S \in GL_n(\mathbb{F}), R \in GL_p(\mathbb{F}), L \in \mathbb{F}^{n\times p} \right\}$$

of the **output injection group** $\mathscr{G}$, where

$$\mathscr{G} = \left\{ \begin{pmatrix} R & L \\ 0 & S \end{pmatrix} \;\middle|\; R \in GL_n(\mathbb{F}), S \in GL_p(\mathbb{F}), L \in \mathbb{F}^{n\times p} \right\} \subset GL_{n+p}(\mathbb{F}). \tag{6.4}$$

Conceptually, output injection seems much harder to grasp than state feedback. A clarification of its importance comes from a deeper study of observer theory, and this will be taken up in Chapter 7. There is therefore a measure of poetic justice in the fact that the analysis of the output injection case is, from a technical point of view, often significantly easier than that of the feedback case. The notion of output injection bears a natural duality with state feedback. In fact, a pair (C_1, A_1) is output injection equivalent to a pair (C_2, A_2) if and only if the dual pair $(A_1^\top, C_1^\top)$ is state feedback equivalent to $(A_2^\top, C_2^\top)$. This simple fact allows us to translate results for state feedback into corresponding results for output injection, and vice versa. However, while a natural approach is to dualize the feedback result, one can often develop an independent analysis of the output injection case, with the option of deriving results on the feedback group by duality considerations. Sometimes the proofs obtained along such lines become easier than those derived from state feedback analysis using duality arguments.

6.2 Polynomial Characterizations

The feedback group is introduced through a state-space formalism. However, as is seen in several instances, various aspects of linear systems theory are easier to handle if one operates with polynomial data, and this approach is our choice. Henceforth, by a choice of bases, $\mathscr{X}$ will be identified with $\mathbb{F}^n$ and $\mathscr{U}$ with $\mathbb{F}^m$.

Thus, for a reachable pair (A,B), the polynomial matrices $zI-A$ and B are left coprime. Since each factorization of a rational matrix function is associated with a right coprime factorization, one can write

$$(zI-A)^{-1}B = N(z)D(z)^{-1}, \tag{6.5}$$

with $N(z)$ and $D(z)$ right coprime. Furthermore, $N(z)$ and $D(z)$ are uniquely determined up to a common right unimodular factor. Thus, each reachable pair (A,B) is associated with the unique shift realization, defined in (4.23) as

$$S_D : X_D \longrightarrow X_D, \quad \pi_D : \mathbb{F}^m \longrightarrow X_D.$$

By Theorem 4.21, the pairs (A,B) and (S_D,π_D) are similar. Moreover, (6.5) defines a bijective correspondence between the similarity equivalence classes

$$\{(SAS^{-1},SB) \mid S \in GL_n(\mathbb{F})\}$$

of reachable pairs and the equivalence classes

$$\{(D(z)U(z)) \mid U(z) \in GL_n(\mathbb{F}[z])\},$$

with respect to right multiplication by unimodular polynomial matrices, of nonsingular polynomial matrices $D(z)$. The next theorem characterizes feedback equivalence in terms of the factorizations (6.5).

Theorem 6.1. *Let (A,B) be a reachable pair, with $A \in \mathbb{F}^{n\times n}$ and $B \in \mathbb{F}^{n\times m}$. Let $N(z)D(z)^{-1}$ be a right coprime factorization of $(zI-A)^{-1}B$. Then a necessary and sufficient condition for a reachable pair $(\overline{A},\overline{B})$ to be feedback equivalent to (A,B) is the existence of $R \in GL_m(\mathbb{F})$, $S \in GL_n(\mathbb{F})$, and $Q(z) \in \mathbb{F}[z]^{m\times m}$, for which $Q(z)D(z)^{-1}$ is strictly proper, such that*

$$(zI-\overline{A})^{-1}\overline{B} = SN(z)(D(z)+Q(z))^{-1}R^{-1}. \tag{6.6}$$

Proof. Assume $G(z) = (zI-A)^{-1}B = N(z)D(z)^{-1}$ are coprime factorizations, and let $(\overline{A},\overline{B})$ be feedback equivalent to (A,B). Thus, there exist invertible maps S and R such that $\overline{A} = S(A-BK)S^{-1}$ and $\overline{B} = SBR^{-1}$. Hence,

$$(zI-\overline{A})^{-1}\overline{B} = (S(zI-A+BK)^{-1}S^{-1})^{-1}SBR = S(zI-A+BK)^{-1}BR^{-1}\ .$$

Now, computing

$$\begin{aligned}(zI-A+BK)^{-1}B &= [(zI-A)(I+(zI-A)^{-1}BK)]^{-1}B \\ &= (I+(zI-A)^{-1}BK)^{-1}(zI-A)^{-1}B \\ &= (I+G(z)K)^{-1}G(z),\end{aligned}$$

and using the equality $G(z)(I+KG(z)) = (I+G(z)K)G(z)$, it follows that $(I+G(z)K)^{-1}G(z) = G(z)(I+KG(z))^{-1}$. Consequently,

$$\begin{aligned} G_f(z) &:= (zI-\overline{A})^{-1}\overline{B} = SG(z)(I+KG(z))^{-1}R^{-1} \\ &= SN(z)D(z)^{-1}(I+KN(z)D(z)^{-1})^{-1}R^{-1} \\ &= SN(z)(D(z)+KN(z))^{-1}R^{-1}. \end{aligned}$$

If one defines $Q(z) = KN(z)$, then clearly $G_f(z) = SN(z)(D(z)+Q(z))^{-1}R^{-1}$, and $Q(z)D(z)^{-1} = KN(z)D(z)^{-1}$ is strictly proper. This proves the necessity part of the theorem.

Conversely, assume that (6.6) is satisfied. Without loss of generality, it suffices to show that if, with $\overline{D}(z) = D(z)+Q(z)$ and $Q(z)D(z)^{-1}$ being strictly proper, the equality $N(z)\overline{D}(z)^{-1} = (zI-\overline{A})^{-1}\overline{B}$ is satisfied for a reachable pair $(\overline{A},\overline{B})$, then $(\overline{A},\overline{B})$ is feedback equivalent to (A,B). Thus it suffices to show that the pairs (S_D,π_D) and $(S_{\overline{D}},\pi_{\overline{D}})$ are feedback equivalent. Alternatively, it must be shown that, for some invertible $\mathbb{F}$-linear map $Y: X_{\overline{D}} \longrightarrow X_D$ and a linear map $K: X_D \longrightarrow \mathbb{F}^n$, the equality $S_D - YS_{\overline{D}}Y^{-1} = BK$ is valid, where $B: \mathbb{F}^n \longrightarrow X_D$ is defined by $Bu = \pi_{Du}$ for $u \in \mathbb{F}^n$. Clearly, the previous equation is equivalent to $S_{DY} - YS_{\overline{D}} = BKY = BK_1$. Hence, it suffices to show that

$$\operatorname{Im}(S_{DY} - YS_{\overline{D}}) \subset \operatorname{Im}\pi_D,$$

and this we proceed to do. We define the Toeplitz induced map $Y: X_{\overline{D}} \longrightarrow X_D$ by

$$Yf = \pi_D\pi_+ D\overline{D}^{-1}f, \qquad f \in X_{\overline{D}}.$$

Showing that Y is an invertible map follows, using Theorem 3.49, from the fact that $D(z)\overline{D}(z)^{-1}$ is biproper. For $f \in X_{\overline{D}}$ one computes

$$\begin{aligned} (YS_{\overline{D}} - S_{DY})f &= \pi_D\pi_+ D\overline{D}^{-1}\pi_{\overline{D}}zf - \pi_D z\pi_D\pi_+ D\overline{D}^{-1}f \\ &= \pi_D\pi_+ D\overline{D}^{-1}\overline{D}\pi_-\overline{D}^{-1}zf - \pi_D z\pi_+ D\overline{D}^{-1}f \\ &= \pi_D\pi_+ D\pi_-\overline{D}^{-1}zf - \pi_D z\pi_+ D\overline{D}^{-1}f \\ &= \pi_D\pi_+ D\overline{D}^{-1}zf - \pi_D z\pi_+ D\overline{D}^{-1}f \\ &= \pi_D(D\overline{D}^{-1}f)_{-1} \in \operatorname{Im}\pi_D. \end{aligned}$$

This completes the proof. ∎

The following theorem allows us to characterize state feedback equivalence in purely module-theoretic terms.

Theorem 6.2. *Let $D(z), \overline{D}(z) \in \mathbb{F}[z]^{m\times m}$ be nonsingular. Then the reachable pairs (S_D, π_D) and $(S_{\overline{D}}, \pi_{\overline{D}})$ obtained by the shift realizations $\Sigma_{\bullet D^{-1}}$ and $\Sigma_{\bullet \overline{D}^{-1}}$, respectively, are state feedback equivalent if and only if there exist a unimodular matrix $U(z) \in GL_m(\mathbb{F}[z])$ and a biproper rational matrix $\Gamma(z) \in \mathbb{F}[[z^{-1}]]^{m\times m}$, with*

$$D(z) = \Gamma(z)\overline{D}(z)U(z).$$

Equivalently, this is satisfied if and only if the left Wiener–Hopf indices of $D(z)$ and $\overline{D}(z)$ are equal.

Proof. To prove the sufficiency direction, assume that $D(z)$ is of the form $D(z) = \Gamma(z)\overline{D}(z)U(z)$ for a unimodular matrix $U(z)$ and $\Gamma(z)$ is biproper. Since the shift realizations of $\overline{D}(z)$ and $\overline{D}(z)U(z)$ are similar, we can assume without loss of generality that $U(z) = I_m$. Then the assumption is equivalent to $D(z)\overline{D}(z)^{-1}$ being biproper, i.e., the left Wiener–Hopf indices of $D(z)\overline{D}(z)^{-1}$ being zero. Then, by Theorem 3.47, the Toeplitz operator $T_{D\overline{D}^{-1}} : \mathbb{F}[z]^m \longrightarrow \mathbb{F}[z]^m$ being invertible, and so is the induced Toeplitz operator $\pi_D T_{D\overline{D}^{-1}} : X_{\overline{D}} \longrightarrow X_D$. Computing now

$$\begin{aligned}
(\pi_D T_{D\overline{D}^{-1}} S_{\overline{D}} - S_D \pi_D T_{D\overline{D}^{-1}})f &= \pi_D \pi_+ D\overline{D}^{-1} \pi_{\overline{D}} zf - \pi_D z \pi_D \pi_+ D\overline{D}^{-1} f \\
&= \pi_D \pi_+ D\overline{D}^{-1} \overline{D} \pi_- \overline{D}^{-1} zf - \pi_D z \pi_+ D\overline{D}^{-1} f \\
&= \pi_D \pi_+ D \pi_- \overline{D}^{-1} zf - \pi_D z \pi_+ D\overline{D}^{-1} f \\
&= \pi_D \pi_+ D\overline{D}^{-1} zf - \pi_D z \pi_+ D\overline{D}^{-1} f \\
&= \pi_D (D\overline{D}^{-1} f)_{-1} \in \operatorname{Im} \pi_D
\end{aligned}$$

proves the state feedback equivalence of $\Sigma_{\bullet D^{-1}}$ and $\Sigma_{\bullet \overline{D}^{-1}}$.

Conversely, assume that the pairs (S_D, π_D) and $(S_{\overline{D}}, \pi_{\overline{D}})$ are state feedback equivalent. Choose basis matrices $N(z), \overline{N}(z)$ for X_D and $X_{\overline{D}}$, and let (A,B) and $(\overline{A}, \overline{B})$ be the uniquely determined reachable pairs satisfying $(zI - A)^{-1}B = N(z)D(z)^{-1}$, $(zI - \overline{A})^{-1}\overline{B} = \overline{N}(z)\overline{D}(z)^{-1}$. By the transitivity of state feedback equivalence, the pairs (A,B) and $(\overline{A}, \overline{B})$ are state feedback equivalent. By Theorem 6.1, we obtain

$$\overline{N}(z)\overline{D}(z)^{-1} = (zI - \overline{A})^{-1}\overline{B} = SN(z)(D(z) + Q(z))^{-1}R^{-1},$$

with $Q(z)D(z)^{-1}$ strictly proper and $Q(z) = KN(z)$. Since $N(z)$ and $D(z)$ are assumed to be right coprime, $SN(z), D(z) + KN(z)$ are right coprime, too. Therefore, both pairs $(\overline{N}(z), \overline{D}(z))$ and $SN(z), D(z) + KN(z))$ are right coprime. Thus, there exists a unimodular polynomial matrix $U(z)$ with $\overline{D}(z) = R(D(z) + Q(z))U(z)$. Since QD^{-1} is strictly proper, the matrix

$$\Gamma(z) = D(z)(D(z) + Q(z))^{-1}R^{-1} = \left(I_m + Q(z)D(z)^{-1}\right)^{-1} R^{-1}$$

is biproper, and hence

$$\Gamma(z)\overline{D}(z) = D(z)U(z).$$

This completes the proof. ∎

The next theorem summarizes the preceding results.

Theorem 6.3. *For $i = 1,2$, let $(A_i, B_i) \in \mathbb{F}^{(n\times(n+m)}$, $\operatorname{rank} B_i = m$, be reachable pairs, with input-to-state transfer functions $G_i(z)$, having the coprime factorizations*

$$G_i(z) = (zI - A_i)^{-1}B_i = N_i(z)D_i(z)^{-1}.$$

The following assertions are equivalent:

(a) (A_2, B_2) is state feedback equivalent to (A_1, B_1).
(b) There exist state feedback transformation matrices $L \in GL_m(\mathbb{F}), K \in \mathbb{F}^{m\times n}$, $S \in GL_n(\mathbb{F})$ with

$$G_2(z) = SG_1(z)(I + KG_1(z))^{-1}L^{-1}.$$

(c) $D_1(z)$ and $D_2(z)$ have the same left Wiener–Hopf indices.
(d) $G_1(z)$ and $G_2(z)$ have the same left Wiener–Hopf indices.

Proof. The equivalence $(a) \iff (b) \iff (c)$ was shown already in Theorem 6.2. The implication $(b) \implies (d)$ is trivial. We prove $(d) \implies (c)$. Thus, there exist a biproper rational function $\Gamma(z) \in \mathbb{F}[[z^{-1}]]^{m\times m}$ and a unimodular polynomial matrix $U(z) \in GL_n(\mathbb{F}[z])$ such that

$$N_2(z)D_2(z)^{-1} = U(z)N_1(z)D_1(z)^{-1}\Gamma(z). \tag{6.7}$$

By the reachability of (A_1, B_1), the matrix $N_1(z)$ is right prime, and therefore $U(z)N_1(z)$ is right prime, too. Thus there exists a polynomial matrix $M(z)$ that satisfies $M(z)U(z)N_1(z) = I$. Multiplying the identity (6.7) by $M(z)$ on both sides, it follows that

$$M(z)N_2(z) = D_1(z)^{-1}\Gamma(z)D_2(z).$$

Taking determinants, one obtains

$$\det(M(z)N_2(z)) = \frac{\det D_2(z)}{\det D_1(z)}\det\Gamma(z).$$

Since $\Gamma(z)$ is biproper, the determinant $\det\Gamma(z)$ is biproper, too. Moreover, both $\det D_1(z) = \det(zI - A_1)$ and $\det D_2(z) = \det(zI - A_2)$ have degree n. Thus

$$\frac{\det D_2(z)}{\det D_1(z)}\det\Gamma(z)$$

is biproper. Hence, the polynomial $\det(M(z)N_2(z))$ is also biproper, which implies that $M(z)N_2(z)$ is unimodular. Thus

$$\Gamma(z)^{-1}D_1(z)M(z)N_2(z) = D_2(z)$$

implies that $D_1(z)$ and $D_2(z)$ have the same left Wiener–Hopf indices. This shows $(d) \implies (c)$, and the proof is complete. ■

The importance of the preceding result lies in showing that the classification of reachable pairs $(A,B) \in \mathbb{F}^{n\times(n+m)}$, up to state feedback equivalence, is equivalent to the classification of nonsingular $m \times m$ polynomial matrices $D(z)$ with identical left Wiener–Hopf indices. The bijective correspondence between orbits of the state feedback group and polynomial matrices with fixed Wiener–Hopf indices will be taken up in the next section.

Duality is used to derive the following two counterparts to Theorems 6.1 and 6.2.

Theorem 6.4. *Let $D(z), \overline{D}(z) \in \mathbb{F}[z]^{p\times p}$ be nonsingular. Then the observable pairs $((D\cdot)_{-1}, S_D)$ and $((\overline{D}\cdot)_{-1}, S_{\overline{D}})$, obtained by the shift realization in the state spaces $X_D, X_{\overline{D}}$, are output injection equivalent if and only if there exist a unimodular matrix $U(z) \in GL_p(\mathbb{F}[z])$ and a biproper rational matrix $\Gamma(z) \in \mathbb{F}[[z^{-1}]]^{p\times p}$ with*

$$D(z) = U(z)\overline{D}(z)\Gamma(z).$$

Equivalently, this is true if and only if the right Wiener–Hopf indices of $D(z)$ and $\overline{D}(z)$ are equal.

Proof. Using Proposition 3.48, this follows from Theorem 6.2 by duality considerations. ■

Theorem 6.5. *Let $(C_1,A_1),(C_2,A_2) \in \mathbb{F}^{(p+n)\times n}, \operatorname{rk} C_i = p$, be observable pairs with state-to-output transfer functions*

$$G_1(z) = C_1(zI - A_1)^{-1} = D_{\ell,1}(z)^{-1}N_{\ell,1}(z),$$

$$G_2(z) = C_2(zI - A_2)^{-1} = D_{\ell,2}(z)^{-1}N_{\ell,2}(z)$$

and left coprime factorizations $D_{\ell,1}(z), N_{\ell,1}(z)$ and $D_{\ell,2}(z), N_{\ell,2}(z)$, respectively. The following statements are equivalent:

(a) (C_1,A_1) is output injection equivalent to (C_2,A_2).
(b) There exists an output injection transformation matrix $P \in GL_p(\mathbb{F})$, $L \in \mathbb{F}^{n\times p}, S \in GL_n(\mathbb{F})$ with

$$G_2(z) = P(I_p + G_1(z)L)^{-1}G_1(z)S^{-1}.$$

(c) $D_{\ell,1}(z)$ and $D_{\ell,2}(z)$ have the same right Wiener–Hopf indices.
(d) $G_1(z)$ and $G_2(z)$ have the same right Wiener–Hopf indices.

Proof. A change of basis in the output space changes the transfer function by a left nonsingular factor. Similarly, a similarity transformation in the state space can be easily coped with. Thus, without loss of generality, one can assume that $A_2 = A_1 - LC_1$ and $C_2 = C_1$. Rewriting the coprime factorization as $N_{\ell,1}(z)(zI - A_1) = D_{\ell,1}(z)C_1$ and adding $N_{\ell,1}(z)LC_1$ to both sides, one obtains the intertwining relation $N_{\ell,1}(z)(zI - A_1 + LC_1) = (D_{\ell,1}(z) + N_{\ell,1}(z)L)C_1$, which can be written as

$$C_1(zI - A_1 + LC_1)^{-1} = (D_{\ell,1}(z) + N_{\ell,1}(z)L)^{-1}N_{\ell,1}(z) = D_{\ell,2}(z)^{-1}N_{\ell,2}(z).$$

It is easily checked that the factorization $\overline{G}_2(z) = (D_{\ell,1}(z) + N_{\ell,1}(z)L)^{-1}N_{\ell,1}(z)$ is left coprime. Thus, there exists a unimodular polynomial matrix $M(z)$ such that $D_{\ell,2}(z) = M(z)(D_{\ell,1}(z) + N_{\ell,1}(z)L)$ and $N_{\ell,2}(z) = M(z)N_{\ell,1}(z)$ are fulfilled, thereby obtaining the right Wiener–Hopf factorization

$$D_{\ell,2}(z) = M(z)D_{\ell,1}(z)\Gamma(z),$$

with $\Gamma(z) = D_{\ell,1}(z)^{-1}(D_{\ell,1}(z) + N_{\ell,1}(z)L) = I_p + G_1(z)L$ biproper. In particular, $D_{\ell,1}(z)$ and $D_{\ell,2}(z)$ have the same right Wiener–Hopf indices. This shows the implications $(a) \implies (b) \implies (c)$. The reverse directions follow as for the proofs of Theorems 6.1 and 6.2. The implication $(b) \implies (d)$ is trivial. The proof that $(d) \implies (c)$ runs parallel to the proof in Theorem 6.3 and is thus omitted. ∎

6.3 Reachability Indices and the Brunovsky Form

For discrete-time systems

$$\begin{aligned} x_{t+1} &= Ax_t + Bu_t \\ y_t &= Cx_t, \end{aligned}$$

with state space $\mathscr{X}$, input space $\mathscr{U}$, and output space $\mathscr{Y}$, there exists a fine structure in the state space according to how fast the various states are reached. Dually, one can ask how fast one can observe the state from the output. It turns out that this structure, manifested through the so-called reachability and observability indices, is all important for the study of the fundamental problems of systems theory, namely, a description of the inherent limitations of controllers to change the dynamics of the system and for state estimation purposes.

Consider the sequence of subspaces $\mathscr{V}_i(A,B) \subset \mathscr{X}$, defined by

$$\mathscr{V}_i(A,B) = \mathscr{B} + A\mathscr{B} + \cdots + A^i\mathscr{B}, \tag{6.8}$$

where $\mathscr{B} = \operatorname{Im} B$. Thus, in discrete time, the linear subspace $\mathscr{V}_i$ consists of all states that can be reached from zero in at most $i+1$ steps. Obviously, $\mathscr{V}_i \subset \mathscr{V}_{i+1}$. Applying the Cayley–Hamilton theorem and the assumption of reachability, $\mathscr{V}_{n-1+j} = \mathscr{R}$ is also valid for $j \geq 0$, where $\mathscr{R}$ denotes the reachable space of (A,B). Define a sequence of indices by

$$\nu_i(A,B) = \begin{cases} \dim \mathscr{B} & i = 0, \\ \dim \mathscr{V}_i - \dim \mathscr{V}_{i-1} & i \geq 1. \end{cases} \tag{6.9}$$

Thus

$$m \geq \nu_0(A,B) \geq \nu_1(A,B) \geq \ldots \geq \nu_n(A,B) = 0.$$

Define the dual set of indices by

$$\kappa_i(A,B) = \#\{\nu_j(A,B) \mid \nu_j(A,B) \geq i\}. \tag{6.10}$$

Thus $\kappa_1 \geq \cdots \geq \kappa_m$ and $\sum_{i=1}^m \kappa_i = \sum_{j=0}^n \nu_j$. Thus $\kappa = (\kappa_1, \ldots, \kappa_m)$ and $\nu = (\nu_0, \ldots, \nu_n)$ form dual partitions of $r = \dim \mathscr{V}_n(A,B)$. The indices $\kappa_1 \geq \cdots \geq \kappa_m$ are usually called the **controllability indices** of the pair (A,B). In the discrete-time case, it is more appropriate to call them, as we shall, the **reachability indices**. If the pair (A,B) is fixed, then one writes κ_i for $\kappa_i(A,B)$, and so forth. It follows trivially from (6.9) that $\kappa_1 + \cdots + \kappa_m = \nu_0 + \cdots + \nu_n = \dim \mathscr{R}$. Therefore, the reachability indices of a reachable pair on an n-dimensional state space form a partition of n, that is, a representation $\kappa_1 + \cdots + \kappa_m = n$. It is easily seen, by examples, that in fact all partitions of n into at most m parts arise as reachability indices of a suitable reachable pair (A,B).

Similarly, the observability indices of a pair $(C,A) \in \mathbb{F}^{p \times n} \times \mathbb{F}^{n \times n}$ will be introduced. To this end, define, for each $i = 1, \ldots, n$, the ranks of the ith partial observability matrix as

$$r_i(C,A) = \operatorname{rank} \begin{pmatrix} C \\ \vdots \\ CA^{i-1} \end{pmatrix}.$$

Thus the differences ($s_0 := 0$)

$$s_i = r_i - r_{i-1}, \quad i = 1, \ldots, n$$

measure the increase in the ranks of the partial observability matrices.

Definition 6.6. The **observability indices** of $(C,A) \in \mathbb{F}^{p \times n} \times \mathbb{F}^{n \times n}$ are the nonnegative integers $\lambda_1(C,A) \geq \ldots \geq \lambda_p(C,A)$ defined by

$$\lambda_i(C,A) = \#\{s_j(C,A) \mid s_j(C,A) \geq i\}.$$

In complete analogy with the reachability indices, the equality

$$\lambda_1 + \cdots + \lambda_p = n$$

is true if and only if (C,A) is observable. The following lemma will be needed.

Lemma 6.7. *The reachability indices are state feedback invariants, i.e.,*

$$\kappa_i(S(A+BK)S^{-1}, SBR^{-1}) = \kappa_i(A,B),\ i = 1,\dots,m,$$

is true for all matrices $R \in GL_m(\mathbb{F}), S \in GL_n(\mathbb{F}), K \in \mathbb{F}^{m\times n}$. Similarly, the observability indices are output injection invariants, i.e., for all $i = 1,\dots,p$ and for all matrices $R \in GL_p(\mathbb{F}), S \in GL_n(\mathbb{F}), L \in \mathbb{F}^{n\times p}$, the equality

$$\lambda_i(RCS^{-1}, S(A+LC)S^{-1}) = \lambda_i(C,A)$$

is valid.

Proof. It is obvious, with S,R invertible maps in the state space and input space, respectively, that

$$\mathscr{V}_i(S(A+BK)S^{-1}, SBR^{-1}) = S\mathscr{V}_i(A,B)$$

for all $i \geq 0$. While the spaces $\mathscr{V}_i$ defined in (6.8) change under the action of an element of the feedback group, their dimensions do not, i.e., they are invariant. This shows that the $\nu_i(A,B)$ are state feedback invariant, as are, thus, the reachability indices $\kappa_i(A,B)$. The proof is similar for the observability indices. ∎

The preceding definition of reachability indices was introduced in state-space terms. We now show the connection to invariants defined in terms of coprime factorizations of $(zI-A)^{-1}B$.

Theorem 6.8. *Let $(A,B) \in \mathbb{F}^{n\times n} \times \mathbb{F}^{n\times m}$ be a reachable pair, with $\operatorname{rank} B = m$. Let $N(z)D(z)^{-1}$ be a right coprime factorization of $(zI-A)^{-1}B$, and let $\kappa_1 \geq \cdots \geq \kappa_m > 0$ be the reachability indices of the pair (A,B) as defined in (6.10). Then:*

1. *The reachability indices of the pair (A,B) are equal to the minimal column indices of the submodule $D(z)\mathbb{F}[z]^m \subset \mathbb{F}[z]^m$;*
2. *The reachability indices of the pair (A,B) are equal to the left Wiener–Hopf factorization indices of $D(z)$;*
3. *The reachability indices of the pair (A,B) are equal to the minimal column indices of the submodule $\operatorname{Ker}(zI-A, -B) \subset \mathbb{F}[z]^{n+m}$, defined by the linear multiplication operator*

$$(zI-A, -B) : \mathbb{F}[z]^{n+m} \longrightarrow \mathbb{F}[z]^n.$$

Proof. Assume $\eta_1 \geq \cdots \geq \eta_m$ are the minimal column indices of $D(z)$. Let $\Delta(z) = \mathrm{diag}\,(z^{\eta_1}, \ldots, z^{\eta_m})$. Then there exist a unimodular polynomial matrix $U(z)$ and a biproper matrix $\Gamma(z)$ such that

$$D(z)U(z) = \Gamma(z)\Delta(z).$$

By Theorem 6.2, the pairs (S_D, π_D) and (S_Δ, π_Δ) are feedback equivalent and, hence, have the same reachability indices. Thus, it suffices to prove the theorem for $A = S_\Delta, B = \pi_\Delta$. The reachability indices of (S_Δ, π_Δ) are easily computed as follows. With $e_1, \ldots, e_m$ the standard basis elements of $\mathbb{F}^m$, clearly, by our assumption that B has full column rank, we get $\mathrm{Im}\,\pi_\Delta = \mathrm{span}\,\{e_1, \ldots, e_m\}$. On the other hand, the equality between the coprime factorizations $(zI - A)^{-1}B = N(z)D(z)^{-1}$ implies, using the shift realization, that (A, B) is similar to (S_D, π_D), so it is feedback equivalent to (S_Δ, π_Δ). Consider now the subspaces $\mathscr{V}_i$, defined in (6.8), that correspond to the pair (S_Δ, π_Δ). Clearly, $\dim \mathscr{V}_1 = \#\{\eta_i > 0\} = m$ and $\dim \mathscr{V}_k = \dim \mathscr{V}_{k-1} + \#\{\eta_i \geq k\}$. So $\nu_k = \dim \mathscr{V}_k - \dim \mathscr{V}_{k-1} = \#\{\eta_i \geq k\}$. Thus $\eta_1 \geq \cdots \geq \eta_m$ are the dual indices to the ν_i, but so are the reachability indices $\kappa_1, \ldots, \kappa_m$. Hence, necessarily, $\eta_i = \kappa_i$.

By part 1, the column indices of $D(z)$ are equal to the reachability indices of (S_D, π_D), i.e., to $\kappa_1, \ldots, \kappa_m$. Therefore, there exists a unimodular polynomial matrix $V(z)$ for which $D(z)V(z)$ is column proper with column indices $\kappa_1, \ldots, \kappa_m$. Writing $D(z)V(z) = \Gamma(z)\Delta(z)$, where $\Delta(z) = \mathrm{diag}\,(z^{\kappa_1}, \ldots, z^{\kappa_m})$, $\Gamma(z)$ is necessarily biproper because the leading term of $\Gamma(z)$ is $[DV]_{hc}$, which is nonsingular. This implies that, with $U(z) = V(z)^{-1}$, the left Wiener–Hopf factorization $D(z) = \Gamma(z)\Delta(z)U(z)$.

The equality $(zI - A)N(z) = BD(z)$ can be rewritten as

$$(zI - A, -B)\begin{pmatrix} N(z) \\ D(z) \end{pmatrix} = 0.$$

Using the coprimeness assumption (Theorem 2.27), it follows that

$$\mathrm{Ker}\,(zI - A, -B) = \begin{pmatrix} N(z) \\ D(z) \end{pmatrix} \mathbb{F}[z]^m.$$

Now $N(z)D(z)^{-1}$ is strictly proper, and thus the minimal column indices of $\begin{pmatrix} N(z) \\ D(z) \end{pmatrix}$ are equal to those of $D(z)$. By part 2, they coincide with the reachability indices of (A, B). ■

The next result, which is a straightforward consequence of Theorem 6.8, characterizes the reachability and observability indices of an observable pair in terms of Wiener–Hopf factorization indices.

Corollary 6.9. *1. Let (A,B) be a reachable pair, and let $N(z)D(z)^{-1} = (zI-A)^{-1}B$ be a right coprime factorization. Then the reachability indices of (A,B) are equal to the left Wiener–Hopf indices of $D(z)$.*
2. Let (C,A) be an observable pair, and let $D_\ell(z)^{-1}N_\ell(z) = C(zI-A)^{-1}$ be a left coprime factorization. Then the observability indices of (C,A) are equal to the right Wiener–Hopf indices of $D_\ell(z)$.

If $G(z)$ is a proper, rational transfer function, then its Wiener–Hopf factorization indices must have a system-theoretic interpretation. This is indeed the case, and a system-theoretic interpretation of the factorization indices of the denominators in coprime matrix fraction representations of $G(z)$ can be derived.

Theorem 6.10. *Let $G(z) \in \mathbb{F}[z]^{p\times m}$ be a proper rational function admitting the coprime matrix fraction representations*

$$G(z) = N_r(z)D_r(z)^{-1} = D_\ell(z)^{-1}N_\ell(z), \tag{6.11}$$

and let (A,B,C,D) be a reachable and observable realization of $G(z)$. Then the reachability indices of the realization are equal to the left Wiener–Hopf indices of $D_r(z)$ and the observability indices are equal to the right Wiener–Hopf indices of $D_\ell(z)$.

Proof. By the state-space isomorphism theorem, the pair (A,B) is isomorphic to the pair (S_{D_r}, π_{D_r}). By Proposition 2.19, there exists a unimodular matrix $U(z)$ such that $D_r(z)U(z)$ is column proper with column indices $\kappa_1 \geq \kappa_2 \geq \cdots \geq \kappa_m$. Clearly,

$$D_r(z)U(z) = \Gamma(z)\Delta(z), \tag{6.12}$$

with $\Delta(z) = \operatorname{diag}(z^{\kappa_1}, \ldots, z^{\kappa_m})$ and Γ biproper. By Theorem 6.2, this implies that (S_{D_r}, π_{D_r}) and (S_Δ, π_Δ) are feedback equivalent pairs. However, the reachability indices of (S_Δ, π_Δ) are easily seen to be equal to $\kappa_1, \ldots, \kappa_m$ (see the proof of the subsequently stated Theorem 6.14). Finally, (6.12) can be rewritten as

$$D_r(z) = G_-(z)\Delta(z)G_+(z),$$

with $G_-(z) = \Gamma(z)$ and $G_+(z) = U(z)^{-1}$. This is a left Wiener–Hopf factorization of $D_r(z)$. The statement concerning observability indices follows by duality. ■

In Corollary 6.9 and Theorem 6.10, it was shown that the reachability indices of a pair (A,B) coincide with the left Wiener–Hopf indices of the nonsingular polynomial matrix $D(z)$ appearing in a coprime factorization

$$(zI-A)^{-1}B = N(z)D(z)^{-1}.$$

One attempts to extend this analysis to Wiener–Hopf factorizations of strictly proper transfer functions $G(z) = C(zI-A)^{-1}B$. For simplicity, our focus will be on strictly proper transfer functions, although an extension to proper transfer functions is possible.

Definition 6.11. A reachable and observable system (A,B,C) is called **state feedback irreducible** if and only if $(S(A+BK)S^{-1}, SBR^{-1}, CS^{-1})$ is reachable and observable for all state feedback matrices $(S,K,R) \in \mathscr{F}_{n,m}$.

Of course, while the reachability of a system is always preserved by state feedback, this is no longer true of observability. It is a simple observation that transfer functions of the form

$$(zI-A)^{-1}B \quad \text{or} \quad C(zI-A)^{-1},$$

with (A,B) reachable or (C,A) observable, are feedback irreducible. Thus feedback irreducibility is an extension of the situation discussed previously.

To begin with the analysis of feedback irreducibility, one considers the single-input single-output case. Let

$$g(z) = \frac{p(z)}{q(z)} \in \mathbb{F}[z]$$

denote a scalar strictly proper transfer function of degree n, given by a coprime factorization, with $q(z)$ monic and $\deg p(z) < \deg q(z) = n$. Let (A,b,c) denote a minimal realization of $g(z)$. Without loss of generality, one can assume that (A,b) is in Brunovsky canonical form, i.e.,

$$\begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \\ 0 & \dots & \dots & 0 \end{pmatrix}, \quad b = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}, \quad c = \begin{pmatrix} c_0 & c_1 & \cdots & c_{n-1} \end{pmatrix},$$

with transfer function

$$g(z) = \frac{p(z)}{q(z)} = \frac{c_0 + \ldots + c_{n-1}z^{n-1}}{z^n}.$$

Thus the system (A,b,c) is feedback irreducible if and only if the pair

$$A+bk = \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \\ k_0 - q_0 & \dots & \dots & k_{n-1} - q_{n-1} \end{pmatrix}, \quad c = \begin{pmatrix} c_0 & c_1 & \cdots & c_{n-1} \end{pmatrix}$$

is observable for all state feedback matrices $k = (k_0, \ldots, k_{n-1})$. This in turn is equivalent to $p(z) = \sum_{j=0}^{n-1} c^j z^j$ being coprime to all monic polynomials of the form $q_k(z) = z^n + k_{n-1}z^{n-1} + \cdots + k_0$, i.e., that $p(z) = c_0 \neq 0$ is a nonzero constant polynomial. Thus a scalar strictly proper transfer function $g(z)$ is feedback irreducible if and only if it has no finite zeros, i.e., if and only if the relative degree $\deg q - \deg p$ of $g(z)$ is equal to n. This analysis is now extended to the matrix case, beginning with the following lemma.

Lemma 6.12. *Let $G(z) \in \mathbb{F}(z)^{m\times m}$ be proper with the right Wiener–Hopf factorization $G(z) = U(z)\Delta(z)\Gamma(z)$, with $U(z) \in \mathbb{F}[z]^{p\times p}$ unimodular, and $\Gamma(z)$ biproper. Let $N(z) \in \mathbb{F}[z]^{p\times m}$ be a right prime polynomial matrix, with $p \geq m$. Then*

$$N(z)G(z) = U_1(z)\begin{pmatrix}\Delta(z)\\0\end{pmatrix}\Gamma(z).$$

In particular, $G(z)$ and $N(z)G(z)$ have the same right factorization indices.

Proof. Since $N(z)$ is right prime, there exist unimodular matrices $V(z), W(z)$, with

$$N(z) = V(z)\begin{pmatrix}I_m\\0\end{pmatrix}W(z).$$

Thus,

$$N(z)G(z) = V(z)\begin{pmatrix}I_m\\0\end{pmatrix}W(z)U(z)\Delta(z)\Gamma(z) = U_1(z)\begin{pmatrix}\Delta(z)\\0\end{pmatrix}\Gamma(z),$$

where $U_1(z) = V(z)\mathrm{diag}\,(W(z)U(z), I)$. ■

Theorem 6.13. *Let (A,B,C) be a reachable and observable realization of a strictly proper transfer function $G(z)$ with right coprime factorization $G(z) = N(z)D(z)^{-1}$. Assume that $G(z)$ has full column rank. Then:*

1. *(A,B,C) is feedback irreducible if and only if $N(z)$ is right prime (i.e., left invertible);*
2. *Assume that (A,B,C) is feedback irreducible. Then the negatives of the reachability indices of (A,B) coincide with the right Wiener–Hopf indices and the negatives of the observability indices of (C,A)) coincide with the left Wiener–Hopf indices of the transfer function $G(z) = C(zI-A)^{-1}B$.*

Proof. Recall that every state feedback transformation $(A,B,C) \mapsto (A+BK,B,C)$ acts on transfer functions by right multiplication with a biproper rational function, that is,

$$C(zI-A+BK)^{-1}B = C(zI-A)^{-1}B(I_m+K(zI-A)^{-1}B)^{-1}.$$

Moreover, each right coprime factorization of $(zI-A)^{-1}B = H(z)D(z)^{-1}$ implies the intertwining relation $BD(z) = (zI-A)H(z)$. This induces the factorization

$$G(z) = C(zI-A)^{-1}B = N(z)D(z)^{-1},$$

with $N(z) = CH(z)$. This is summarized in the system equivalence relation

$$\begin{pmatrix} B & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} D(z) & -I \\ N(z) & 0 \end{pmatrix} = \begin{pmatrix} zI - A & -B \\ C & 0 \end{pmatrix} \begin{pmatrix} H(z) & 0 \\ 0 & I \end{pmatrix},$$

with $B, zI - A$ left coprime and $D(z), H(z)$ right coprime, which implies the following equivalence:

$$\begin{pmatrix} D(z) & -I \\ N(z) & 0 \end{pmatrix} \simeq_{FSE} \begin{pmatrix} zI - A & -B \\ C & 0 \end{pmatrix}.$$

Using the Shift Realization Theorem 4.26, it follows that the minimality of (A,B,C) implies the right coprimeness of $N(z), D(z)$. Similarly, from the factorization $(zI - A - BK)^{-1}B = H(z)(D(z) - KH(z))^{-1}$ follows the intertwining relation

$$\begin{pmatrix} B & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} D(z) + KH(z) & -I \\ N(z) & 0 \end{pmatrix} = \begin{pmatrix} zI - A + BK & -B \\ C & 0 \end{pmatrix} \begin{pmatrix} H(z) & 0 \\ 0 & I \end{pmatrix}.$$

Here $B, zI - A + BK$ and $D(z) + KH(z), H(z)$ are left coprime and right coprime, respectively. In particular,

$$\begin{pmatrix} D(z) + KH(z) & -I \\ N(z) & 0 \end{pmatrix} \simeq_{FSE} \begin{pmatrix} zI - A + BK & -B \\ C & 0 \end{pmatrix}$$

for each K. This shows that (A,B,C) is feedback irreducible if and only if the polynomial matrices $N(z)$ and $D(z) + KH(z)$ are right coprime for each K.

Next, it will be shown that this condition is equivalent to the right primeness of $N(z)$. Clearly, the right primeness of $N(z)$ implies for each state feedback matrix K the right coprimeness of $N(z)$ and $D(z) - KH(z)$. Thus $N(z)$ right prime implies feedback irreducibility. To prove the converse implication, let us assume that $N(z)$ is not right prime, i.e., there exists a polynomial factorization $N(z) = N'(z)F(z)$ with $N'(z)$ right prime and $F(z) \in \mathbb{F}[z]^{m \times m}$ nonsingular and nonunimodular. Applying Lemma 6.12, it follows that $G(z) = N(z)D(z)^{-1}$ and $F(z)D(z)^{-1}$ have the same right Wiener–Hopf indices. Let

$$F(z)D(z)^{-1} = U(z)\Delta(z)^{-1}\Gamma(z)$$

be the right Wiener–Hopf factorization, with $\Delta(z) = \operatorname{diag}(z^{\kappa_1}, \cdots, z^{\kappa_m})$. Then $E(z) := \Delta(z)U(z)^{-1}$ is a nonsingular polynomial matrix and

$$D_1(z) := E(z)F(z) = \Gamma(z)D(z)$$

is a nonsingular polynomial matrix with $\deg\det D_1(z) = \deg\det D(z)$. Computing

$$\begin{aligned} G(z)\Gamma(z)^{-1} &= N(z)D(z)^{-1}\Gamma(z)^{-1} = N(z)D_1(z)^{-1} \\ &= N'(z)E(z)^{-1} \end{aligned}$$

yields a nontrivial factorization. Thus, the McMillan degrees of $G(z)\Gamma(z)^{-1}$ and $G(z)$ are related as

$$\delta(G\Gamma^{-1}) \leq \deg\det E(z) < \deg\det D_1(z) = \deg\det D(z) = \delta(G).$$

This shows that $G(z)$ is feedback reducible and completes the proof of the first claim of the theorem.

By the first part, a full column rank coprime factorization $G(z) = N(z)D(z)^{-1}$ is feedback irreducible if and only if $N(z)$ is right prime. But then Lemma 6.12 implies that $G(z)$ and $D(z)^{-1}$ have the same right Wiener–Hopf indices. Thus the right Wiener–Hopf indices of $G(z)$ are equal to the negative of the left Wiener–Hopf indices of $D(z)$, which by Theorem 6.10 coincide with the reachability indices of (A,B). This completes the proof of the second claim 2. ■

Our attention turns now to the question of constructing a canonical form for reachable pairs under the action of the state feedback group.

Theorem 6.14. *1. Let $(A,B) \in \mathbb{F}^{n\times n} \times \mathbb{F}^{n\times m}$ be a reachable pair with reachability indices $\kappa_1 \geq \ldots \geq \kappa_m$. Then (A,B) is feedback equivalent to the block matrix representation*

$$\left(\begin{pmatrix} A_1 & & \\ & \ddots & \\ & & A_m \end{pmatrix}, \begin{pmatrix} B_1 & & \\ & \ddots & \\ & & B_m \end{pmatrix} \right), \tag{6.13}$$

with the matrices $A_j \in \mathbb{F}^{\kappa_j\times\kappa_j}$ and $B_j \in \mathbb{F}^{\kappa_j\times 1}$ defined by

$$A_j = \begin{pmatrix} 0 & & & \\ 1 & \ddots & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{pmatrix}, \quad B_j = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \tag{6.14}$$

*We will refer to $D(z) = diag\,(z^{\kappa_1}, \ldots, z^{\kappa_m})$ as the **polynomial Brunovsky form** and to (6.13) and (6.14) as the **Brunovsky canonical form**.*

2. Two reachable pairs $(A,B), (\overline{A},\overline{B}) \in \mathbb{F}^{n\times(n+m)}$ are state feedback equivalent if and only if they have the same reachability indices

$$\kappa_1(A,B) = \kappa_1(\overline{A},\overline{B}),\ \ldots,\ \kappa_m(A,B) = \kappa_m(\overline{A},\overline{B}).$$

Proof. The direct sum $X_\Delta = X_{z^{\kappa_1}} \oplus \cdots \oplus X_{z^{\kappa_m}}$ is a consequence of $\Delta(z)$ being diagonal. Let $\{e_1,\dots,e_m\}$ be the standard basis in $\mathbb{F}^m$; then the vectors

$$\{z^i e_j \mid 1 \le j \le m, 0 \le i < \kappa_j - 1\}$$

form a basis for X_Δ. Relative to these bases in $\mathbb{F}^m$ and X_Δ, the pair (S_Δ, π_Δ) has the matrix representation (6.13)–(6.14).

It is a trivial consequence of the Brunovsky canonical form that the reachability indices define a complete set of invariants for state feedback of reachable pairs. ■

For a reachable pair (A,B), the group

$$\mathrm{Stab}(A,B) = \left\{ \begin{pmatrix} S & 0 \\ K & R \end{pmatrix} \mid (S(A+BK)S^{-1}, SBR^{-1}) = (A,B) \right\} \subset GL_{n+m}(\mathbb{F})$$

of all elements of the feedback group that leave (A,B) invariant is called the **state feedback stabilizer group of** (A,B). Clearly, the stabilizers of feedback equivalent pairs are isomorphic. As a consequence, it suffices to study the stabilizer group for systems in Brunovsky canonical form. It follows that the structure of the stabilizer depends only on the reachability indices of the reachable pair (A,B). The relation between the state feedback stabilizer subgroup and the left factorization group introduced in Theorem 2.37 can be stated as follows.

Theorem 6.15. *Let (A,B) be a reachable pair, and let $N(z)D(z)^{-1}$ be a right coprime factorization of $(zI-A)^{-1}B$. Then the state feedback stabilizer group of (A,B) is isomorphic to the left factorization group of $D(z)$.*

Proof. The pair (A,B) is isomorphic to (S_D, π_D) and, in turn, state feedback equivalent to the polynomial Brunovsky form (S_Δ, π_Δ), with $\Delta(z) = \mathrm{diag}\,(z^{\kappa_1},\dots,z^{\kappa_m})$. It suffices, therefore, to study the state feedback stabilizer at (S_Δ, π_Δ). However, by Theorem 6.2, this is equivalent to finding all solutions of the equation

$$\Gamma(z)\Delta(z) = \Delta(z)U(z), \tag{6.15}$$

with $U(z) \in \mathbb{F}[z]^{m\times m}$ unimodular and $\Gamma(z) \in \mathbb{F}[[z^{-1}]]^{m\times m}$ biproper. Equation (6.15) is equivalent to $\gamma_{ij} z^{\kappa_j} = z^{\kappa_i} u_{ij}$, which in turn implies

$$\deg u_{ij} = \begin{cases} 0 & \kappa_i > \kappa_j \\ \le \kappa_j - \kappa_i & \kappa_j \ge \kappa_i \,. \end{cases} \tag{6.16}$$

Conversely, if $U(z)$ is unimodular and satisfies (6.16), then it is easily seen that equation (6.15) is solvable with a biproper $\Gamma(z)$. Thus the unimodular matrices $U(z)$ of (6.15) have a block triangular structure. By Theorem 2.37, the set of such unimodular matrices $U(z)$ coincides with the left factorization group of $\Delta(z)$. This structure is reflected in $\Gamma(z)$, which is uniquely determined by $U(z)$ and $\Delta(z)$. This completes the proof. ■

From the preceding discussion it is clear that the existence of a Wiener–Hopf factorization of a nonsingular polynomial matrix is equivalent to the existence of Brunovsky's canonical form for a reachable pair. Next, in a purely state-space-oriented manner, a refinement of the Brunovsky canonical form is derived. Recall that the Kronecker indices of a state space pair $(A, B = (b_1, \ldots, b_m))$ are defined by the following deletion process on the columns of the reachability matrix. Let $\leq$ denote the lexicographical ordering on $\{0, \ldots, n-1\} \times \{1, \ldots, m\}$ defined as

$$(i,j) \leq (k,\ell) \iff \begin{cases} i < k \\ \text{or} \\ i = k, j \leq \ell. \end{cases}$$

While going from left to right in the list

$$\left(b_1, \ldots, b_m, Ab_1, \ldots, Ab_m, \ldots, A^{n-1}b_1, \ldots, A^{n-1}b_m\right)$$

of mn vectors in $\mathbb{F}^n$*, delete all vectors* $A_k b_\ell$ *that are linearly dependent on the set of preceding vectors* $\{A_i b_j | (i,j) \leq (k,\ell)\}$.

It is easily seen that the remaining vectors constitute a list of the form

$$(b_1, Ab_1, \ldots, A^{k_1 - 1}b_1, \ldots, b_m, \ldots, A^{k_m - 1}b_m), \tag{6.17}$$

for unique nonnegative integers $k_1, \ldots, k_m$, called the **Kronecker indices**. Note that the Kronecker indices define an m-tuple of integers and not a set of numbers. By construction, the vectors in (6.17) form a basis of the reachable set $\mathscr{R}$ of (A,B). Thus (A,B) is reachable if and only if the Kronecker indices satisfy $k_1 + \cdots + k_m = n$. An important difference that distinguishes the Kronecker indices $\mathbf{k} = (k_1, \ldots, k_m)$ from the reachability indices $\kappa = (\kappa_1, \ldots, \kappa_m)$ is that the Kronecker indices are not ordered by magnitude. Thus $(2,0,3)$ and $(3,2,0)$ are Kronecker indices of different systems (A,B). It is easily seen that $\mathbf{k} = (k_1, \ldots, k_m)$ are Kronecker indices of a system (A,B), with $k_i \geq 1$ for all i; then the reachability indices of (A,B) arise by reordering the Kronecker indices in decreasing form. However, this is not true if one of the Kronecker indices is zero.

Let $\mathscr{U}_m$ denote the subgroup of $GL_m(\mathbb{F})$ consisting of all $m \times m$ upper triangular matrices U with identical entries $u_{11} = \cdots = u_{mm} = 1$ on the diagonal. The **restricted state feedback group** is then defined by all state feedback transformations

$$\begin{pmatrix} S & 0 \\ K & U \end{pmatrix},$$

with $S \in GL_n(\mathbb{F})$, $K \in \mathbb{F}^{m \times n}$, and $U \in \mathscr{U}_m$. Two linear systems $(A,B), (\overline{A}, \overline{B})$ are called **restricted state feedback equivalent** if $(\overline{A}, \overline{B}) = (S(A - BK)S^{-1}, SBU^{-1})$ is satisfied for a restricted state feedback transformation (S,K,U). We proceed to show that the Kronecker indices are feedback invariants.

Lemma 6.16. *Let $(A,B) \in \mathbb{F}^{n\times(n+m)}$ be reachable with Kronecker indices $\mathbf{k} = (k_1,\dots,k_m)$. For each $S \in GL_n(\mathbb{F})$, $U \in \mathcal{U}_m$, and $K \in \mathbb{F}^{m\times n}$, the pairs (A,B) and $(\overline{A},\overline{B}) = (S(A-BK)S^{-1}, SBU^{-1})$ have the same Kronecker indices $\mathbf{k} = (k_1,\dots,k_m)$.*

Proof. It is easily seen that the reachability matrix

$$R(A,B) = (B,\dots,A^{n-1}B)$$

satisfies

$$R(\overline{A},\overline{B}) = SR(A-BK,BU) = SR(A,B)V$$

for a suitable invertible upper triangular matrix $V \in GL_{nm}(\mathbb{F})$ with diagonal blocks $V_{11} = \cdots = V_{nn} = I_m$. This implies that the Kronecker indices of (A,B) and $(\overline{A},\overline{B})$ coincide. This completes the proof. ■

The following result will be needed.

Lemma 6.17. *Let $(A,B) \in \mathbb{F}^{n\times(n+m)}$ be reachable with Kronecker indices $\mathbf{k} = (k_1,\dots,k_m)$. Then there exists a unipotent matrix $U \in \mathcal{U}_m$ such that $\overline{B} = (\overline{b}_1,\dots,\overline{b}_m) = BU$ satisfies for each $j = 1,\dots,m$*

$$A^{k_j}\overline{b}_j \in ImB + \cdots + A^{k_j-1}ImB,$$
$$A^{k_j}\overline{b}_j \notin ImB + \cdots + A^{k_j-2}ImB.$$

Proof. By construction of the Kronecker indices there exist $c_{ij} \in \mathbb{F}, i < j$, and $z_j \in \mathrm{Im}B + \cdots + A^{k_j-1}\mathrm{Im}B$ such that

$$\begin{aligned} A^{k_j}b_j &= z_j + \sum_{i=1}^{j-1} c_{ij}A^{k_j}b_i, \\ A^{k_j-1}b_j &\notin \mathrm{Im}B + \cdots + A^{k_j-2}\mathrm{Im}B + A^{k_j-1}\,\mathrm{span}\{b_1,\dots,b_{j-1}\} \end{aligned} \tag{6.18}$$

holds. Define $\overline{b}_j = b_j - \sum_{i=1}^{j-1} c_{ij}b_i$ and

$$U = \begin{pmatrix} 1 & -c_{12} & \cdots & -c_{1m} \\ & \ddots & \ddots & \vdots \\ & & \ddots & -c_{m-1,m} \\ & & & 1 \end{pmatrix} \in \mathcal{U}_m.$$

Then $A^{k_j}\overline{b}_j = z_j \in \operatorname{Im} B + \cdots + A^{k_j-1}\operatorname{Im} B$. Suppose

$$A^{k_j-1}\overline{b}_j \in \operatorname{Im} B + \cdots + A^{k_j-2}\operatorname{Im} B.$$

Then $A^{k_j-1}b_j = A^{k_j-1}\overline{b}_j + \sum_{i=1}^{j-1} c_{ij}A^{k_j-1}b_i$, in contradiction to (6.18). This completes the proof. ■

Using the preceding lemmas, it will be shown next that the Kronecker indices define a complete set of invariants for the restricted state feedback equivalence of reachable pairs.

Theorem 6.18. *1. Let $(A,B) \in \mathbb{F}^{n\times(n+m)}$ be a reachable pair with Kronecker indices $\mathbf{k} = (k_1,\ldots,k_m)$. Then (A,B) is restricted state feedback equivalent to the block matrix*

$$\left(\begin{pmatrix} A_1 & & \\ & \ddots & \\ & & A_m \end{pmatrix}, \begin{pmatrix} B_1 & & \\ & \ddots & \\ & & B_m \end{pmatrix}\right) \tag{6.19}$$

$$A_j = \begin{pmatrix} 0 & & & \\ 1 & \ddots & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{pmatrix}, \qquad B_j = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \tag{6.20}$$

if $k_j \geq 1$. If $k_j = 0$, then the block A_j is absent and $B_j = 0$.

2. Two reachable pairs $(A,B),(\overline{A},\overline{B}) \in \mathbb{F}^{n\times(n+m)}$ are restricted state feedback equivalent if and only if their Kronecker indices coincide.

Proof. Choose U as in Lemma 6.17. Then there exist elements $\beta_{1,j},\ldots,\beta_{k_j,j} \in \operatorname{Im} B$ such that

$$A^{k_j}\overline{b}_j - A^{k_j-1}\beta_{1,j} - \cdots - \beta_{k_j,j} = 0. \tag{6.21}$$

For $j = 1,\ldots,m$ define the state vectors

$$x_{1,j}{=}\overline{b}_j, \quad x_{2,j}{=}A\overline{b}_j{-}\beta_{1,j}, \quad \ldots, \quad x_{k_j,j} = A^{k_j-1}\overline{b}_j - A^{k_j-2}\beta_{1j} - \cdots - \beta_{k_j-1,j}.$$

Let $\mathscr{X} \subset \mathbb{F}^n$ denote the span of the vectors $\{x_{i,j} \mid 1 \leq i \leq k_j, j = 1,\ldots,m\}$. Clearly, $\operatorname{Im} B \subset \mathscr{X}$. Using (6.21), it follows that $Ax_{k_j,j} = \beta_{k_j,j} \in \operatorname{Im} B \subset \mathscr{X}$. Thus $\mathscr{X}$ is A-invariant. Thus the reachability of (A,B) implies that $\{x_{i,j} \mid 1 \leq i \leq k_j, j = 1,\ldots,m\}$ defines a basis of $\mathbb{F}^n$. Choose $u_{i,j} \in \mathbb{F}^m$ with $\overline{B}u_{i,j} = \beta_{i,j}$, $i = 1,\ldots,k_j$, $j = 1,\ldots,m$. Then the feedback transformation $K : \mathbb{F}^n \longrightarrow \mathbb{F}^m$ defined by

$$Kx_{i,j} = -u_{i,j}, \quad i = 1,\ldots,k_j, j = 1,\ldots,m,$$

satisfies, for each $j = 1,\dots,m$,

$$(A-\overline{B}K)x_{i,j} = Ax_{i,j} - \beta_{i,j} = x_{i+1,j}, \quad 1 \le i \le k_j - 1,$$
$$(A-\overline{B}K)x_{K_j,j} = Ax_{K_j,j} - \beta_{k_j,j} = 0.$$

By choosing $S = (x_{1,1},\dots,x_{k_1,1},\dots,x_{1,m},\dots,x_{k_m,m})$, K, and U as above, one sees that S is invertible and $(S(A-BK)S^{-1}, SBU)$ has the form (6.19), (6.20). This completes the proof of the first part. The second follows easily from the first part, together with Lemma 6.16. ■

6.4 Pole Assignment

The study of the effects of state feedback on closed-loop dynamics begins with an analysis of the simple case of a single-input reachable system. It will be shown how, by the use of state feedback, the dynamics of the system, determined by its characteristic polynomial, can be arbitrarily assigned. This indicates the tremendous importance of feedback. In fact, as long as reachability is fulfilled, the original system can be flexibly modified by the use of feedback. In particular, every reachable system can be stabilized through feedback. The subsequent results are presented in an unashamedly matrix-oriented manner, beginning with the single-input case, where the analysis becomes particularly simple.

Theorem 6.19. *Let $(A,b) \in \mathbb{F}^{n\times n} \times \mathbb{F}^n$ be a reachable system with the n-dimensional state space $\mathbb{F}^n$. Let $f(z) = f_0 + \cdots + f_{n-1}z^{n-1} + z^n$ be a monic polynomial of degree n. Then there exists a unique feedback transformation $K \in \mathbb{F}^{1\times n}$ such that $A - bK$ has $f(z)$ as its characteristic polynomial.*

Proof. Let $q(z) = q_0 + \cdots + q_{n-1}z^{n-1} + z^n$ denote the characteristic polynomial of A. Since (A,b) is reachable, the pair (A,b) is state space equivalent to the reachable shift realization (S_q, π_q) on X_q. Thus, without loss of generality, one can identify (A,b) with the pair (S_q,π_q) and, by the choice of basis in X_q, one can assume that the pair (A,b) has the **control canonical form**

$$A = \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \\ -q_0 & -q_1 & \dots & -q_{n-1} \end{pmatrix}, \quad b = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}. \tag{6.22}$$

This can be done by taking a right coprime factorization $N(z)q(z)^{-1}$ of $(zI-A)^{-1}b$, with $q(z)$ the characteristic polynomial of A, and choosing in X_q the **control basis** $\mathscr{B}_{co} := \{\gamma_1(z),\dots,\gamma_n(z)\}$, where, for $i = 0,\dots,n$,

$$\gamma_i(z) = z^{n-i} + q_{n-1}z^{n-i-1} + \cdots + q_i.$$

A straightforward computation shows that the shift operator S_q acts on these basis vectors via

$$S_q(\gamma_i) = \gamma_{i-1}(z) - q_{i-1}.$$

Therefore, (A,b) in (6.22) is just the basis representation of (S_q,π_q) with respect to the control basis. Let $K = (k_0,\dots,k_{n-1})$ be the feedback map, i.e.,

$$Kx = k_0x_1 + \cdots + k_{n-1}x_n;$$

then

$$A - bK = \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \\ -k_0 - q_0 & \dots & \dots & -k_{n-1} - q_{n-1} \end{pmatrix}.$$

The unique choice $k_i = -q_i + f_i$ then yields $\det(zI - A + bK) = f(z)$. ■

The following two results present explicit formulas for the feedback gain K.

Theorem 6.20 (Ackermann Formula). *Let $(A,b) \in \mathbb{F}^{n\times n} \times \mathbb{F}^n$ be reachable, and let $f(z) = \sum_{j=0}^n f^j z^j, f_n = 1$, be a monic polynomial of degree n. Let $R(A,b) = (b,\dots,A^{n-1}b) \in GL_n(\mathbb{F})$ denote the reachability matrix. Then*

$$K = (0,\dots,0,1)R(A,b)^{-1}f(A)$$

is the unique element $K \in \mathbb{F}^{1\times n}$, with $\det(zI - A + bK) = f(z)$.

Proof. By Theorem 6.19, there exists a unique $K \in \mathbb{F}^{1\times n}$ that satisfies $\det(zI - A + bK) = f(z)$. Applying the Cayley–Hamilton theorem, one obtains

$$\sum_{j=0}^n f_j(A-bK)^j = f(A-bK) = 0,$$

and therefore

$$f(A) = -\sum_{j=0}^n f_j\left((A-bK)^j - A^j\right).$$

There exist row vectors $k_{j,\ell} \in \mathbb{F}^{1\times n}$, $k_{j,j-1} = -K$, with

$$(A-bK)^j - A^j = \sum_{\ell=0}^{j-1} A^\ell b k_{j,\ell}.$$

Thus

$$f(A) = -\sum_{j=0}^{n}\sum_{\ell=0}^{j-1} A^{\ell} b f_j k_{j,\ell} = -\sum_{i=0}^{n-1} A^{ib}\xi_i,$$

with $\xi_i = \sum_{j>i} f_j k_{j,i}$ and $\xi_{n-1} = f_n k_{n,n-1} = -K$. Defining the matrix

$$\xi = \begin{pmatrix} \xi_0 \\ \vdots \\ \xi_{n-1} \end{pmatrix} \in \mathbb{F}^{n\times n}$$

we obtain

$$f(A) = -R(A,b)\xi,$$

and hence $K = -e_n^{\top}\xi = e_n^{\top} R(A,b)^{-1} f(A)$. ■

We now turn to the analysis of state feedback in the general case $m \geq 1$.

Lemma 6.21. *Assume (A,B) is reachable and $b = Bv \neq 0$. Then there exist $u_0 = v, u_1, \ldots, u_{n-1} \in \mathbb{F}^m$ such that $(x_1, \ldots, x_n)$, recursively defined as*

$$x_1 = b, \quad x_k = Ax_{k-1} + Bu_k, \quad k = 2, \ldots, n,$$

is a basis of $\mathbb{F}^n$.

Proof. One constructs the input vectors u_k recursively, starting from $u_0 = v$. Suppose that $u_1, \ldots, u_{k-1}$ are such that $x_1, \ldots, x_k$ are linearly independent, satisfying ($x_0 := 0$) $x_j = Ax_{j-1} + Bu_{j-1}$ for $j = 1, \ldots, k$ and $k < n$. Let $L \subset \mathbb{F}^n$ denote the k-dimensional linear subspace spanned by $x_1, \ldots, x_k$. Then one chooses $u_k \in \mathbb{F}^m$ such that $x_{k+1} := Ax_k + Bu_k \notin L$. Such a vector u_k always exists, thereby proving the induction step that $\{x_1, \ldots, x_{k+1}\}$ is linearly independent. In fact, otherwise

$$Ax_k + Bu \in L$$

is true for all $u \in \mathbb{F}^m$. This implies $Ax_k \in L$, and therefore also $\operatorname{Im} B \subset L$, and, in turn, $Ax_j = x_{j+1} - Bu_j \in L$ for $j = 1, \ldots, k-1$. This shows that L is an A-invariant linear subspace that contains $\operatorname{Im} B$. The reachability of (A,B) thus implies $L = \mathbb{F}^n$, in contradiction to $\dim L = k < n$. ■

The preceding result has an interesting consequence for state feedback control.

Lemma 6.22 (Heymann). *Let $(A,B) \in \mathbb{F}^{n\times n} \times \mathbb{F}^{n\times m}$ and $b = Bv \neq 0$. Then there exists $K \in \mathbb{F}^{m\times n}$ such that $(A+BK, b)$ is reachable. In particular, for each reachable pair (A,B) there exists a feedback gain $K \in \mathbb{F}^{m\times n}$ such that $A+BK$ is cyclic.*

Proof. Choose $u_0,\dots,u_{n-1} \in \mathbb{F}^m$ and basis vectors $x_1,\dots,x_n$ of $\mathbb{F}^n$, as in Lemma 6.21. For each element $u_n \in \mathbb{F}^m$ there exists a unique $K \in \mathbb{F}^{m\times n}$, with

$$Kx_j = u_j, \quad j = 1,\dots,n.$$

This shows that

$$(A+BK)x_j = Ax_j + Bu_j = x_{j+1}$$

for $j = 1,\dots,n-1$. Since $b = x_1$, we obtain

$$(A+BK)^{jb} = x_{j+1}$$

for $j = 1,\dots,n-1$. Thus $(b,(A+BK)b,\dots,(A+BK)^{n-1}b)$ is a basis of $\mathbb{F}^n$, completing the proof. ∎

It is easy to see that, for nonzero $b \in \operatorname{Im} B$, the set of all such feedback gains K forms a Zariski-open subset of $\mathbb{F}^{m\times n}$. The celebrated pole-shifting theorem of M. Wonham is proved next. The reason for the name is due to the fact that poles of the rational function $(zI - A + BK)^{-1}B$ correspond to the eigenvalues of $A - BK$.

Theorem 6.23 (Pole-Shifting Theorem). *A linear system $(A,B) \in \mathbb{F}^{n\times n} \times \mathbb{F}^{n\times m}$ is reachable if and only if for every monic polynomial $f(z) \in \mathbb{F}[z]$ of degree n there exists $K \in \mathbb{F}^{m\times n}$, with*

$$\det(zI - A + BK) = f(z). \tag{6.23}$$

Proof. Suppose (A,B) is reachable. Choose a nonzero vector $b = Bv$ in the image space of B. By Heymann's Lemma 6.22, there exists $F \in \mathbb{F}^{m\times n}$ such that $(A+BF,b)$ is reachable. Thus, using Theorem 6.19, there exists a row vector $L \in \mathbb{F}^{1\times n}$ such that

$$\det(zI - A - BF + bL) = f(z).$$

This proves (6.23) for $K = -F + vL$.

To prove the converse, the Kalman decomposition is used. Thus, assume that (A,B) is a system with k-dimensional reachable subspace

$$\mathscr{R} = \operatorname{Im} B + A\operatorname{Im} B + \cdots + A^{n-1}\operatorname{Im} B.$$

Choose the basis vectors $v_1,\dots,v_k$ of $\mathscr{R}$ and extend them to a basis $v_1,\dots,v_n$ of the state space $\mathbb{F}^n$. Then the matrix $S = (v_1,\dots,v_n) \in \mathbb{F}^{n\times n}$ is invertible. Since $\mathscr{R}$ is an A-invariant linear subspace, this implies that the state-space equivalent system $(S^{-1}AS, S^{-1}B)$ has the structure

$$\begin{pmatrix} A_1 & A_2 \\ 0 & A_3 \end{pmatrix}, \quad \begin{pmatrix} B_1 \\ 0 \end{pmatrix}, \tag{6.24}$$

which is referred to as the **Kalman decomposition**; it exists for every $(A,B) \in \mathbb{F}^{n\times(n+m)}$. Note that (A_1,B_1) is uniquely determined up to a similarity transformation and is reachable. Moreover, the eigenvalues of A_3 are uniquely determined by the similarity orbit of (A,B). Thus, for the converse, one can assume, without loss of generality, that (A,B) is given by (6.24), with $k<n$. Hence, for each feedback matrix $K=(K_1,K_2)$,

$$A-BK = \begin{pmatrix} A_1-B_1K_1 & A_2-B_1K_2 \\ 0 & A_3 \end{pmatrix},$$

with the characteristic polynomial $\det(zI-A+BK) = \det(zI-A_1+B_1K_1)\det(zI-A_3)$. This implies that the characteristic polynomials $\det(zI-A+BK)$ of nonreachable pairs (A,B) all contain the same factor $\det(zI-A_3)$ and thus cannot be arbitrarily assigned. This completes the proof. ■

There is a simple, inductive proof of Wonham's theorem that works over an algebraically closed field $\mathbb{F}=\overline{\mathbb{F}}$. We learned the following argument from Carsten Scherer. Without loss of generality, assume that $\operatorname{rk} B = m$ and (A,B) is of the form

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \quad B = \begin{pmatrix} I_m \\ 0 \end{pmatrix}.$$

Then, using the Hautus test, one sees that the reachability of (A,B) implies that of (A_{22},A_{21}). Consider a monic polynomial $f(z)=f_1(z)f_2(z)$, with monic factors $f_1(z),f_2(z)$ of degrees $m,n-m$, respectively. Applying the induction hypothesis, there exists $K_2 \in \mathbb{F}^{m\times(n-m)}$ such that

$$\det(zI_{n-m}-A_{22}+A_{21}K_2) = f_2(z).$$

Let C be a matrix with the characteristic polynomial $f_1(z)$. With

$$S = \begin{pmatrix} I & K_2 \\ 0 & I \end{pmatrix}$$

and a suitable matrix X, one obtains

$$SAS^{-1} = \begin{pmatrix} A_{11}+K_2A_{21} & X \\ A_{21} & A_{22}-A_{21}K_2 \end{pmatrix}, \quad SB = B = \begin{pmatrix} I \\ 0 \end{pmatrix}.$$

Therefore, it follows that, with $F=(C-A_{11}-K_2A_{21},-X)$, one obtains

$$SAS^{-1}+SBF = \begin{pmatrix} C & 0 \\ A_{21} & A_{22}-A_{21}K_2 \end{pmatrix},$$

which has the characteristic polynomial $f_1(z)f_2(z)$. This completes the proof.

6.5 Rosenbrock's Theorem

We turn now to the question of finding the extent to which the dynamics of a system can be modified by state feedback. Of course, as far as eigenvalue assignment of $A-BK$ is concerned, it would suffice to know whether the eigenvalues of $A-BK$ could be freely assigned. A deeper question concerns the ability to alter the entire eigenstructure of $A-BK$, i.e., the Jordan canonical form. Rosenbrock showed, in a subtle analysis, that the invariant factors of $A-BK$ can be freely assigned subject only to a finite set of constraints arising from the reachability indices. Our aim in this section is to prove this fundamental result. In view of Theorem 6.14, the only invariants of a reachable pair (A,B) under the action of a feedback group are the reachability indices. On the other hand, the invariant factors of a nonsingular polynomial matrix $D(z)$ are invariant under left and right multiplication by unimodular polynomial matrices. Now if $N(z)D(z)^{-1}$ is a right coprime factorization of the input to state transfer function $(zI-A)^{-1}B$, then the column indices of $D(z)$, which are the minimal indices of the submodule $M = D(z)\mathbb{F}[z]^m$, are equal to the reachability indices of the pair (A,B). Thus it suffices to see how, starting with a polynomial matrix $\operatorname{diag}(\psi_1,\ldots,\psi_m)$, where the ψ_i satisfy $\psi_i|\psi_{i-1}$ for $i=2,\ldots,m$, that the minimal indices of M can be changed by left and right multiplication by unimodular matrices. Our starting point is the following.

Lemma 6.24. *Let $D(z)=(d_1(z),\ldots,d_m(z))\in\mathbb{F}[z]^{m\times m}$ be a nonsingular, column proper polynomial matrix with its columns $d_1(z),\ldots,d_m(z)$ of degrees $\lambda_1\geq\cdots\geq\lambda_m$. Assume, without loss of generality, that the highest column coefficient matrix is $[D]_{hc}=I_m$. If $1\leq j,k\leq m$, with $\deg d_j<\deg d_k$, then there exist elementary row and column operations transforming $D(z)$ into $D'(z)=(d_1'(z),\ldots,d_m'(z))$, with*

$$\deg d_i' = \begin{cases} \deg d_i & i\neq j,k, \\ \deg d_j+1 & i=j, \\ \deg d_k-1 & i=k, \end{cases}$$

and $[D']_{hc}=I_m$.

Proof. Adding the jth row, multiplied by z, to the kth row of $D(z)$, one gets a matrix $D^{(1)}(z)$ with columns $d_i^{(1)}(z)$, with

$$\deg d_i^{(1)} \begin{cases} =\deg d_i, & i\neq j,k, \\ =\deg d_j+1, & i=j, \\ \leq \deg d_k, & i=k. \end{cases}$$

Next, add a suitable multiple of the jth column to the kth column to obtain a $D^{(2)}(z)$, with

$$\deg d_i^{(2)} \begin{cases} = \deg d_i, & i \neq j,k, \\ = \deg d_j + 1, & i = j, \\ \leq \deg d_k, & i = k. \end{cases}$$

Since $\det D^{(2)}(z) = \det D(z)$, one necessarily has $\deg d_k^{(2)} = \deg d_k - 1$, and the highest column coefficient matrix satisfies $\det[D^{(2)}]_{hc} \neq 0$. Thus the matrix $D'(z) = [D^{(2)}]_{hc}^{-1} D^{(2)}(z)$ has the required properties. ■

As an example of the process, taken from Rosenbrock (1970), consider the nonsingular, column proper polynomial matrix

$$D(z) = \begin{pmatrix} z^2+2 & z^3 & z^5+z+1 \\ 2z+1 & z^4+3z+1 & 2z^2 \\ z+2 & 2z^2+1 & z^6-2z^4 \end{pmatrix}.$$

The column indices are $2,4,6$, and we will reduce the degree of the last column and increase the degree of the first. The successive stages are

$$D^{(1)}(z) = \begin{pmatrix} z^2+2 & z^3 & z^5+z+1 \\ 2z+1 & z^4+3z+1 & 2z^2 \\ z^3+3z+2 & z^4+2z^2+1 & 2z^6-2z^4+z^2+z \end{pmatrix},$$

$$D^{(2)}(z) = \begin{pmatrix} z^2+2 & z^3 & -z^5-4z^3+z+1 \\ 2z+1 & z^4+3z+1 & -4z^4-2z^3+2z^2 \\ z^3+3z+2 & z^4+2z^2+1 & -8z^4-4z^3+z^2+z \end{pmatrix},$$

$$[D^{(2)}]_{hc} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix}, \quad [D^{(2)}]_{hc}^{-1} = \begin{pmatrix} 0 & -1 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix},$$

and, finally,

$$D'(z) = \begin{pmatrix} z^3+z+1 & 2z^2-3z & -4z^4-2z^3-z^2+z \\ 2z+1 & z^4+3z+1 & -4z^4-2z^3+2z^2 \\ -z^2-2 & -z^3 & z^5+4z^3-z-1 \end{pmatrix}.$$

Proposition 6.25. *Let $M \subset \mathbb{F}[z]^m$ be a full submodule with minimal indices $\lambda_1 \geq \cdots \geq \lambda_m$, and let $\kappa_1 \geq \cdots \geq \kappa_m$ be a sequence of nonnegative integers. If the conditions*

$$\begin{array}{ll} \sum_{i=1}^{j} \lambda_i \geq \sum_{i=1}^{j} \kappa_i, & j = 1, \ldots, m-1, \\ \sum_{i=1}^{m} \lambda_i = \sum_{i=1}^{m} \kappa_i, & \end{array}$$

are satisfied, then there exists a submodule $N \subset \mathbb{F}[z]^m$, unimodularly equivalent to M, with minimal indices $\kappa_1 \geq \cdots \geq \kappa_m$.

Proof. The proof is by a purely combinatorial argument on partitions, applying Lemma 6.24. Recall that a partition of n is a decreasing sequence of integers $\kappa_1 \leq \cdots \leq \kappa_m$, with $\kappa_1 + \cdots + \kappa_m = n$. Define a partial order, the so-called **dominance order**, on partitions $\kappa = (\kappa_1, \ldots, \kappa_m)$ and $\lambda = (\lambda_1, \ldots, \lambda_m)$ of $\sum_{i=1}^m \kappa_i = n = \sum_{i=1}^m \lambda_i$ as

$$\kappa \preceq \lambda \iff \sum_{i=1}^{j} \kappa_i \leq \sum_{i=1}^{j} \lambda_i, \quad j = 1, \ldots, m-1.$$

A partition $\lambda \neq \kappa$ is called a cover of κ whenever λ is the smallest element in the dominance order that satisfies $\kappa \preceq \lambda$. The covers for the dominance order are characterized as follows. There exists $j < k$, with

$$\lambda_i = \begin{cases} \kappa_i & i, \neq j, k, \\ \kappa_j + 1, & i = j, \\ \kappa_k - 1, & i = k. \end{cases}$$

It is a simple and well-known combinatorial exercise to verify that two arbitrary partitions $\kappa \preceq \lambda$ are connected through a chain of covers, i.e.,

$$\kappa = \lambda^{(1)} \preceq \cdots \preceq \lambda^{(k)} = \lambda,$$

where $\lambda^{(i)}$ is a cover of $\lambda^{(i-1)}$, $i = 2, \ldots, k$. From this and Lemma 6.24 the result follows, as every product of elementary row and column operations is achieved by multiplying with appropriate unimodular matrices. ■

To prove Rosenbrock's theorem, one can start from the coprime factorization

$$(zI - A)^{-1} B = N(z) D(z)^{-1}$$

and try to modify the invariant factors of $D(z)$, keeping the reachability indices invariant. This is a difficult process, though conceptually more natural. Much easier, at the cost of being somewhat indirect, is to start from a polynomial matrix with the required invariant factors and modify the reachability indices, without changing the invariant factors.

Theorem 6.26 (Rosenbrock). *Let $(A, B) \in \mathbb{F}^{n \times n} \times \mathbb{F}^{n \times m}$ be a reachable pair with reachability indices $\kappa_1 \geq \cdots \geq \kappa_m$. Let $\psi_i \in \mathbb{F}[z]$ be such that $\psi_{i+1} \mid \psi_i$ for $i = 1, \ldots, m-1$. Then a necessary and sufficient condition for the existence of a matrix K such that the invariant factors of $A - BK$ are $\psi_1, \ldots, \psi_m$ is that*

$$\begin{aligned} \sum_{i=1}^{d} \deg \psi_i &\geq \sum_{i=1}^{d} \kappa_i, \qquad d=1,\ldots,m-1, \\ \sum_{i=1}^{m} \deg \psi_i &= \sum_{i=1}^{m} \kappa_i. \end{aligned} \tag{6.25}$$

Proof. Let $(zI-A)^{-1}B = N(z)D(z)^{-1}$ be a right coprime factorization such that $D(z)$ is column proper. Then $D(z)$ is a minimal-basis matrix for the full submodule $D(z)\mathbb{F}[z]^m \subset \mathbb{F}[z]^m$ with minimal indices $\kappa_1 \geq \cdots \geq \kappa_m$ and Smith form $\operatorname{diag}(\psi_1,\ldots,\psi_m)$. Let g_k denote the degree of the g.c.d. $\delta_k(D)$ of all $k\times k$ minors of $D(z)$. Since $D(z)$ is a minimal basis, a $k\times k$ principal minor of $D(z)$ has degree $\kappa_m+\cdots+\kappa_{m-k+1}$, and therefore $g_k \leq \kappa_m+\cdots+\kappa_{m-k+1}$ for $k=1,\ldots,m$. Similarly, since $\delta_k(D) = \psi_m\cdots\psi_{m-k+1}$, we obtain

$$g_k = \sum_{i=m-k+1}^{m} \deg \psi_i \leq \sum_{i=m-k+1}^{m} \kappa_i,$$

$$\sum_{i=1}^{m} \deg \psi_i = \sum_{i=1}^{m} \kappa_i,$$

which is equivalent to (6.25). This shows necessity.

Conversely, assume that conditions (6.25) are in force. By Proposition 6.25, the submodule M with the minimal-basis matrix $D_\psi := \operatorname{diag}(\psi_1,\ldots,\psi_m)$ is unimodularly equivalent to a full submodule $D(z)\mathbb{F}^m[z]$ with indices $\kappa_1 \geq \cdots \geq \kappa_m$. Thus there exist unimodular polynomial matrices $U(z), V(z)$ with $D(z) = U(z)D_\psi V(z)$, and $D(z)$ has invariant factors $\psi_1,\ldots,\psi_m$ and minimal indices $\kappa_1 \geq \cdots \geq \kappa_m$. Consider the shift realization (S_D,π_D). By Theorem 6.8, the reachability indices of (S_D,π_D) are $\kappa_1 \geq \cdots \geq \kappa_m$, and the invariant factors of S_D are equal to the Smith invariants of $D(z)$, i.e., they are $\psi_1,\ldots,\psi_m$. Now consider reachable pairs (A,B) with reachability indices $\kappa_1 \geq \cdots \geq \kappa_m$. By Theorem 6.1, the pair (A,B) is feedback equivalent to the pair (S_D,π_D), where S_D has invariant factors $\psi_1,\ldots,\psi_m$. This completes the proof. ∎

6.6 Stabilizability

Recall that a discrete-time linear dynamical system

$$x_{t+1} = Ax_t \tag{6.26}$$

on $\mathbb{F}^n$ is called **stable** whenever the sequence $x_t = A_t x_0$ converges to zero for all initial conditions $x_0 \in \mathbb{F}^n$. Similarly, a linear control system

$$x_{t+1} = Ax_t + Bu_t \tag{6.27}$$

in the state space $\mathbb{F}^n$ is called **open-loop stabilizable** if, for all initial conditions $x_0 \in \mathbb{F}^n$, there exists an input sequence $u_t \in \mathbb{F}^m$, with

$$\lim_{t\to\infty} x_t = 0.$$

Of course, these notions require specifying a topology on $\mathbb{F}^n$, and there are various ways to do that. This issue came up already in Chapter 5 in the discussion of the stability of linear systems. As in Chapter 5, we consider only two possibilities, depending on whether or not $\mathbb{F}$ is a subfield of the complex number field $\mathbb{C}$:

1. The Euclidean distance topology on subfields $\mathbb{F} \subset \mathbb{C}$;
2. The discrete topology on any other field $\mathbb{F}$.

Recall that **the discrete topology** is a unique topology on $\mathbb{F}$ whose open (and closed) subsets are subsets of $\mathbb{F}$. Thus every **finite field** is compact with respect to this topology. Moreover, if $\mathbb{F}$ is endowed with the discrete topology, then the stability of (6.26) means that the trajectories of (6.26) eventually become constant, i.e., $x_{t+T} = x_T, t \geq 0$, for a sufficiently large $T \in \mathbb{N}$. Equivalently, A is nilpotent. In contrast, if $\mathbb{F} \subset \mathbb{C}$ is endowed with the Euclidean topology, then the asymptotic stability of (6.26) is satisfied if and only if all eigenvalues λ of A have absolute value $|\lambda| < 1$. In this case, one says that A is **Schur stable**. The stability properties of linear systems (6.26) are thus summarized as follows.

Proposition 6.27. *Let $\mathbb{F}$ denote a field. A discrete-time dynamical system (6.26) is asymptotically stable if and only if*

1. *A is Schur stable whenever $\mathbb{F} \subset \mathbb{C}$;*
2. *A is nilpotent for the discrete topology on $\mathbb{F}$.*

For the remaining parts of this section, let us assume that $\mathbb{F} \subset \mathbb{C}$ is satisfied, so that one is dealing with the standard notion of stability. The standard **stability domain** for the discrete-time system (6.26) is the open unit disc in the complex plane

$$\mathbb{D} := \{z \mid |z| < 1\}.$$

In contrast, for continuous-time systems $\dot{x} = Ax$ it will be the open left half-plane $\mathbb{C}_- = \{z \mid \mathrm{Re}(z) < 0\}$. In more generality, one may consider a subset Λ of the complex field $\mathbb{C}$ and refer to it as a region of stability.

Definition 6.28. Let $\mathbb{F} \subset \mathbb{C}$ be a subfield.

1. A nonsingular polynomial matrix $T(z) \in \mathbb{F}[z]^{r\times r}$ will be called **Λ-stable**, with respect to a region of stability Λ, if $\det T(z)$ has all its zeros in Λ. If $\Lambda = \mathbb{D}$, then the polynomial matrix $T(z)$ is called **stable**.
2. The pair $(A,B) \in \mathbb{F}^{n\times n} \times \mathbb{F}^{n\times m}$ is called **Λ-feedback stabilizable** if there exists a state feedback gain $K \in \mathbb{F}^{m\times n}$ such that $A - BK$ has all eigenvalues in Λ.

For discrete-time systems (6.27) and $\Lambda = \mathbb{D}$, one refers simply to **feedback stabilizable** rather than to $\mathbb{D}$-feedback stabilizable.

Next, it is shown that the notions of feedback stabilizability and open-loop stabilizability are equivalent.

Proposition 6.29. *Let $\mathbb{F} \subset \mathbb{C}$ be a subfield.*

1. *A linear system (6.27) is open-loop stabilizable if and only if it is feedback stabilizable.*
2. *Let $\mathbb{F} = \mathbb{C}$. A linear system (6.27) is reachable if and only if it is Λ-feedback stabilizable for all nonempty subsets $\Lambda \subset \mathbb{C}$.*

Proof. Clearly, feedback stabilizability implies stabilizability. Suppose (A,B) is stabilizable. As a result of the pole-shifting theorem, it follows that the reachability of (A,B) is sufficient for feedback stabilizability. If (A,B) is not reachable, then, after applying a suitable similarity transformation $(A,B) \mapsto (SAS^{-1}, SB)$ by an invertible matrix $S \in GL_n(\mathbb{F})$, one can assume without loss of generality that (A,B) is in the Kalman decomposition form

$$A = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix}, \quad B = \begin{pmatrix} B_1 \\ 0 \end{pmatrix},$$

where (A_{11}, B_1) is reachable. By the pole-shifting theorem, there exists a feedback matrix $K = (K_1, K_2) \in \mathbb{F}^{m \times n}$ such that $A_{11} - B_1 K_1$ is stable. Since (A,B) is stabilizable, the matrix A_{22} must be stable. This implies that $A - BK$ is stable, i.e., (A,B) is feedback stabilizable.

Again, using the pole-shifting theorem, the eigenvalues of $A - BK$ for reachable pairs (A,B) can be placed arbitrarily in the complex plane. If (A,B) is not reachable, then the unreachable modes, i.e., the eigenvalues of A_{22} in the Kalman decomposition, are in the spectrum of $A - BK$ for every feedback matrix K. This proves the converse. ∎

The same argument as in the preceding proof shows that systems (6.27) over the field of real numbers $\mathbb{R}$ are reachable if and only if they are Λ-feedback stabilizable for all nonempty self-conjugate subsets $\Lambda \subset \mathbb{C}$. However, for other subfields of $\mathbb{C}$ such as, for example, the algebraic number fields, such simple characterizations cannot be expected.

A polynomial characterization of stabilizability is our next objective.

Theorem 6.30. *Let $\mathbb{F} \subset \mathbb{C}$ be a subfield, and let $\Lambda \subset \mathbb{C}$ be a nonempty subset. Let $G(z) = V(z)T(z)^{-1}U(z) + W(z) \in \mathbb{F}(z)^{p \times m}$ be proper rational, and let (A,B,C,D) be the associated shift realization (4.20) defined over $\mathbb{F}$. Then the following conditions are equivalent:*

1. *The shift realization is Λ-feedback stabilizable.*
2. *The g.c.l.d. $E(z)$ of $T(z)$ and $U(z)$ is stable.*
3. $\begin{pmatrix} T(z) & U(z) \end{pmatrix}$ *has full row rank for every $z \notin \Lambda$.*
4. $\begin{pmatrix} zI - A & B \end{pmatrix}$ *has full row rank for every $z \notin \Lambda$.*

Proof. By Theorem 4.26, the reachability subspace of the shift realization is $\mathscr{R} = EX_{T'}$. Let $\mathscr{F}$ be a complementary subspace to $EX_{T'}$ in X_T, i.e., $X_T = EX_{T'} + \mathscr{F}$ and $\mathscr{F} \cap EX_{T'} = \{0\}$. Then $\mathscr{F} \simeq X_T/EX_{T'}$, and this in turn is isomorphic to X_E. To see this, consider the projection map $\pi_E : X_T \longrightarrow X_E$. From the intertwining relation $T(z) = E(z)T'(z)$ it follows, using Theorem 3.21, that this map is surjective and its kernel is $EX_{T'}$. Thus the isomorphism is proved. In terms of this direct sum, the pair (A,B) has the Kalman decomposition

$$\begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix}, \quad \begin{pmatrix} B_1 \\ 0 \end{pmatrix},$$

with (A_{11},B_1) reachable and $A_{22} \simeq S_E$. Thus the pair (A,B) is feedback stabilizable if and only if A_{22} is stable or, equivalently, if and only if $E(z)$ is a stable polynomial matrix. This proves the equivalence of the first two conditions. The last two conditions are equivalent to the assertion that the g.c.l.d. $E(z)$ is invertible for all $z \notin \Lambda$, i.e., to $\det E(z)$ being Λ-stable. In turn, this is equivalent to the matrix $(zI - A, B)$ being full row rank for all $z \notin \Lambda$. This completes the proof. ∎

A different way of stating this for the field $\mathbb{F} = \mathbb{C}$ is as follows. For $\Lambda \subset \mathbb{C}$ and $A \in \mathbb{C}^{n\times n}$ we let

$$X_\Lambda(A) = \bigoplus_{\lambda\in\Lambda} \operatorname{Ker}(\lambda I - A)^n$$

denote the direct sum of the generalized eigenspaces of A with respect to the eigenvalues $\lambda \in \Lambda$. If Λ_- is a stability region and Λ_+ is its complement in $\mathbb{C}$, then every polynomial $p(z)$ has a factorization $p(z) = p_+(z)p_-(z)$, with $p_-(z)$ stable and $p_+(z)$ antistable. Clearly,

$$X_{\Lambda_+}(A) = \operatorname{Ker} d_+(A).$$

For each stability region Λ_- and its complement Λ_+ we will also write

$$X_\pm = X_{\Lambda_\pm}(A).$$

The preceding result can also be stated in a state-space representation.

Theorem 6.31. *A pair $(A,B) \in \mathbb{C}^{n\times n} \times \mathbb{C}^{n\times m}$ is Λ_--stabilizable if and only if one of the following equivalent conditions is satisfied:*

1. *$X_+(A)$ is contained in the reachable set of (A,B).*
2. *$\operatorname{rk}(zI - A,\ B) = n$ is valid for all $z \in \Lambda_+$.*

Proof. The first condition is often expressed by saying that the unstable modes are reachable. The equivalence of stabilizability with the first condition thus follows from the Kalman decomposition of (A,B). The equivalence of stabilizability with the second condition follows at once from Theorem 6.30. ∎

This construction can be extended to a field $\mathbb{F}$ as follows. Let $\mathbb{F}_-[z] \subset \mathbb{F}[z]$ denote a multiplicatively closed subset of nonzero polynomials such that, with an element $f(z) \in \mathbb{F}_-[z]$, all its prime factors are also contained in $\mathbb{F}_-[z]$, and further assume that $1 \in \mathbb{F}_-[z]$. Such subsets are called **saturated**. Denote by $\mathbb{F}_+[z]$ the set of all polynomials that are coprime with all elements of $\mathbb{F}_-[z]$. Elements of $\mathbb{F}_-[z]$ will be called **stable** polynomials and those of $\mathbb{F}_+[z]$ **antistable**. It is a consequence of the primary decomposition theorem that every polynomial $p(z)$ has a factorization $p(z) = p_-(z)p_+(z)$, with $p_-(z)$ stable and $p_+(z) \in \mathbb{F}_+[z]$. As an example, take $\mathbb{F} = \mathbb{R}$ and $\mathbb{R}_-[z]$ the set of all monic Schur polynomials of arbitrary degree (including 1). Alternatively, one considers the set of all monic real Hurwitz polynomials, including 1. As another example, one may consider $\mathbb{F}_-[z] := \{z^n \mid n \in \mathbb{N}_0\}$.

Let $A : \mathbb{F}^n \longrightarrow \mathbb{F}^n$ be a linear transformation and $d(z) = \det(zI - A)$ its characteristic polynomial, and let $d(z) = d_-(z)d_+(z)$ be its factorization into stable and antistable factors. In Chapter 3, it was shown that such a factorization induces essentially unique factorizations

$$zI - A = \overline{S}_+(z)S_-(z) = \overline{S}_-(z)S_+(z),$$

with $\overline{S}_-(z), S_-(z)$ stable and $\overline{S}_+(z), S_+(z)$ antistable. This leads to the spectral decomposition

$$\mathbb{F}^n = X_{zI-A} = \overline{S}_+(z)X_{S_-} \oplus \overline{S}_-(z)X_{S_+} = X_-(A) \oplus X_+(A),$$

where the subspaces $X_-(A), X_+(A)$ are the generalized eigenspaces associated with the sets of stable and antistable eigenvalues, respectively. With these constructions in our hands, Theorem 6.31 generalizes as follows; the proof is by a straightforward modification of the arguments for Theorem 6.31 and is omitted.

Theorem 6.32. *Let $\mathbb{F}_-[z]$ be a saturated subset of nonzero polynomials. For $(A,B) \in \mathbb{F}^{n\times n} \times \mathbb{F}^{n\times m}$, the following conditions are equivalent:*

1. *There exists $K \in \mathbb{F}^{m\times n}$ with $\det(zI - A + BK) \in \mathbb{F}_-[z]$.*
2. *$X_+(A)$ is contained in the reachable set (A,B).*
3. *rank$(zI - A \;\; B) = n$ is satisfied for all roots $z \in \overline{\mathbb{F}}$ of all irreducible polynomials in $\mathbb{F}_+[z]$.*

The dual notion to open-loop stabilizability is that of detectability. While we will introduce this concept in Chapter 7 in a larger context, here we confine ourselves to a more specialized situation.

Definition 6.33. The system

$$\begin{aligned} x_{t+1} &= Ax_t + Bu_t \\ y_t &= Cx_t \end{aligned} \tag{6.28}$$

is called **detectable**, provided all pairs of trajectories $(x_t),(\bar{x}_t)$ with the same input sequence (u_t) and identical output sequences $(Cx_t) = (C\bar{x}_t)$ satisfy

$$\lim_{t\to\infty}(x_t - \bar{x}_t) = 0.$$

Using linearity, it is easily seen that the detectability of (6.28) is satisfied if and only if all state trajectories (x_t) of the input-free or, equivalently, **autonomous** system

$$\begin{aligned} x_{t+1} &= Ax_t, \\ y_t &= Cx_t, \end{aligned} \tag{6.29}$$

with $Cx_t = 0$ for $t \geq 0$, satisfy

$$\lim_{t\to\infty} x_t = 0.$$

Thus (6.28) is detectable if and only if (6.29) is detectable. The system-theoretic interpretation of detectability is clarified by the following result.

Proposition 6.34. *Let $\mathbb{F} \subset \mathbb{C}$. The following conditions are equivalent:*

1. *System (6.28) is detectable.*
2. *The unobservable states*

$$\mathscr{O}_* = \bigcap_{i=0}^{n-1} Ker\, CA^{i-1}$$

   satisfy

$$\mathscr{O}_* \subset \{x_0 \in \mathbb{F}^n \mid \lim_{t\to\infty} x_t = 0\}.$$

3. *The unobservable modes $\lambda \in \mathbb{C}$ of (C,A) are all stable, i.e., satisfy $|\lambda| < 1$.*
4. *The dual system $(A^\top, C^\top)$ is stabilizable.*
5. *There exists an output injection transformation $L \in \mathbb{F}^{n\times p}$ such that $A - LC$ is Schur stable.*

Proof. The equivalence of statements (1) and (2) follows directly from the definition. Equivalently, the dual Kalman decomposition of (C,A) is of the form

$$SAS^{-1} = \begin{pmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{pmatrix}, \quad CS^{-1} = \begin{pmatrix} C_1 & 0 \end{pmatrix},$$

with (C_1, A_{11}) observable and A_{22} stable. Equivalently, the Kalman decomposition of $(A^\top, C^\top)$ is

$$\begin{pmatrix} A_{11}^\top & A_{21}^\top \\ 0 & A_{22}^\top \end{pmatrix}, \quad \begin{pmatrix} C_1^\top \\ 0 \end{pmatrix},$$

with $(A_{11}^\top, C_1^\top)$ reachable and $A_{22}^\top$ stable. Thus statement (2) is both equivalent to statement (3) and to $(A^\top, C^\top)$ being stabilizable. By Proposition 6.29, this is equivalent to the existence of $K \in \mathbb{F}^{n\times p}$ such that $A^\top + C^\top K$ is stable. With $L = K^\top$ this shows the equivalence of statements (4) and (5). ∎

Using this simple proposition, all mentioned results on state feedback stabilizability dualize to corresponding results on detectability. For a subset $\Lambda \subset \mathbb{C}$, a realization (A,B,C,D) defined over a subfield $\mathbb{F} \subset \mathbb{C}$ is called **Λ-output injection stabilizable**, if there exists $L \in \mathbb{F}^{n\times p}$ such that $\det(zI - A + LC)$ has all its roots in Λ. With this notation, the dual result to Theorem 6.30 is stated as follows.

Theorem 6.35. *Let $\mathbb{F} \subset \mathbb{C}$ be a subfield, and let $\Lambda \subset \mathbb{C}$ be a nonempty subset. Let $G(z) = V(z)T(z)^{-1}U(z) + W(z) \in \mathbb{F}(z)^{p\times m}$ be a proper rational matrix, and let (A,B,C,D) be the associated shift realization (4.20), defined over $\mathbb{F}$. Then the following conditions are equivalent:*

1. *The shift realization is Λ-output injection stabilizable.*
2. *The g.c.l.d. $E(z)$ of $T(z)$ and $V(z)$ is stable.*
3. $\begin{pmatrix} V(z) \\ T(z) \end{pmatrix}$ *has full column rank for every $z \notin \Lambda$.*
4. $\begin{pmatrix} C \\ zI - A \end{pmatrix}$ *has full column rank for every $z \notin \Lambda$.*

Proof. By transposing the transfer function $G(z)$, one obtains the factorization $G(z)^\top = U(z)^\top T(z)^{-\top} V(z)^\top + W(z)^\top$, having the shift realization $(A^\top, C^\top, B^\top, D^\top)$. Moreover, (C,A) is Λ-output injection stabilizable if and only if $(A^\top, C^\top)$ is Λ-feedback stabilizable. Thus the result follows by applying Theorem 6.30 to $G(z)^\top$. ∎

The next result is valid over an arbitrary field. It is obtained by dualizing Theorem 6.32; the straightforward arguments are omitted.

Theorem 6.36. *Let $\mathbb{F}_-[z]$ be a saturated subset of nonzero polynomials, and let $(A,B) \in \mathbb{F}^{n\times n} \times \mathbb{F}^{n\times m}$; then the following conditions are equivalent:*

1. *There exists $L \in \mathbb{F}^{n\times p}$ with $\det(zI - A + LC) \in \mathbb{F}_-[z]$.*
2. *The unobservable states $\mathscr{O}_*$ of (C,A) satisfy*

$$\mathscr{O}_* \subset X_-(A).$$

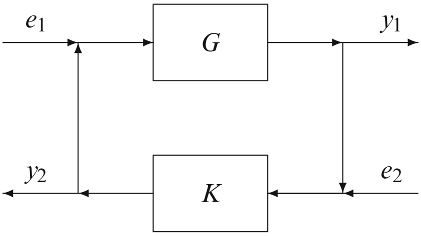
3. *The full rank condition*

$$\operatorname{rk}\begin{pmatrix} C \\ zI - A \end{pmatrix} = n$$

is fulfilled for all roots $z \in \overline{\mathbb{F}}$ of all irreducible polynomials in $\mathbb{F}_+[z]$.

6.7 Dynamic Output Feedback Stabilization

Next, let us consider the task of characterizing the notions of internal stability and stabilizability for the standard **output feedback connection** of a system with transfer function $G(z)$ and a controller with transfer function $K(z)$. This is a generalization of the situation discussed so far, insofar as it refers to **dynamic output feedback** rather than to static state feedback or output injection. For simplicity, we will focus on continuous-time systems described over the field of real numbers; the discrete-time case runs similarly. Schematically, the following diagram describes the feedback connection:



Our assumption is that both transfer functions $G(z) \in \mathbb{R}(z)^{p\times m}$ and $K(z) \in \mathbb{R}(z)^{m\times p}$ are proper rational. The full system equations are then given by

$$\begin{aligned} G(e_1+y_2) &= y_1, \\ K(e_2+y_1) &= y_2. \end{aligned} \tag{6.30}$$

Equivalently, the closed-loop feedback interconnection Σ_{cl} is described using state-space realizations

$$\begin{aligned} \dot{x} &= Ax+Bu_1, \\ y_1 &= Cx+Du_1, \end{aligned} \tag{6.31}$$

$$\begin{aligned} \dot{\xi} &= A_c\xi+B_cu_2, \\ y_2 &= C_c\xi+D_cu_2, \end{aligned} \tag{6.32}$$

for $G(z)$ and $K(z)$, respectively, together with the coupling equations

$$u_1 = e_1+y_2, \quad u_2 = e_2+y_1.$$

The feedback interconnection Σ_{cl} is called **well-posed**, provided $I-DD_c$ is invertible or, equivalently, if the transfer function $I-G(z)K(z)$ is properly invertible. This condition is easily seen to be equivalent to the $(m+p)\times(m+p)$-matrix

$$F = \begin{pmatrix} I_m & -D_c \\ -D & I_p \end{pmatrix}$$

being invertible. Thus the feedback interconnection of a strictly proper transfer function $G(z)$ with a proper controller $K(z)$ is well posed. The assumption of well-posedness allows one to eliminate the internal input variables u_1, u_2 via

$$F\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 & C_c \\ C & 0 \end{pmatrix}\begin{pmatrix} x \\ \xi \end{pmatrix} + \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}$$

from the state equations (6.31) and (6.32). The closed-loop state space representation Σ_{cl} of a well-posed feedback interconnection then takes the form

$$\begin{aligned} \dot{x}_{\mathrm{cl}} &= A_{\mathrm{cl}} x_{\mathrm{cl}} + B_{\mathrm{cl}} \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}, \\ \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} &= C_{\mathrm{cl}} x_{\mathrm{cl}} + D_{\mathrm{cl}} \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}, \end{aligned} \tag{6.33}$$

with system matrices

$$\begin{aligned} A_{\mathrm{cl}} &= \begin{pmatrix} A & 0 \\ 0 & A_c \end{pmatrix} + \begin{pmatrix} B & 0 \\ 0 & B_c \end{pmatrix} F^{-1} \begin{pmatrix} 0 & C_c \\ C & 0 \end{pmatrix}, \quad B_{\mathrm{cl}} = \begin{pmatrix} B & 0 \\ 0 & B_c \end{pmatrix} F^{-1} \\ C_{\mathrm{cl}} &= \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} F^{-1} \begin{pmatrix} 0 & C_c \\ C & 0 \end{pmatrix}, \quad D_{\mathrm{cl}} = \begin{pmatrix} D & 0 \\ 0 & D_c \end{pmatrix} F^{-1}. \end{aligned} \tag{6.34}$$

Thus $(A_{\mathrm{cl}}, B_{\mathrm{cl}}, C_{\mathrm{cl}})$ is static output feedback equivalent to the direct sum system

$$\left(\begin{pmatrix} A & 0 \\ 0 & A_c \end{pmatrix}, \begin{pmatrix} B & 0 \\ 0 & B_c \end{pmatrix}, \begin{pmatrix} C & 0 \\ 0 & C_c \end{pmatrix} \right),$$

which shows that (6.34) is reachable and observable if and only if both (A,B,C,D) and (A_c,B_c,C_c,D_c) are reachable and observable. Thus the minimality of the realizations (A,B,C,D), (A_c,B_c,C_c,D_c) of $G(z)$ and $K(z)$ implies the minimality of the closed-loop system $(A_{\mathrm{cl}}, B_{\mathrm{cl}}, C_{\mathrm{cl}}, D_{\mathrm{cl}})$.

Definition 6.37. The feedback interconnection Σ_{cl} is **internally stable** if and only if Σ_{cl} is well posed and the system matrix A_{cl} is stable, i.e., all eigenvalues of A_{cl} have negative real part.

Note that

$$\Phi(z) = D_{\mathrm{cl}} + C_{\mathrm{cl}}(zI - A_{\mathrm{cl}})^{-1} B_{\mathrm{cl}} \tag{6.35}$$

is the transfer function of the feedback interconnection (6.34).

Proposition 6.38. *$\Phi(z)$, the closed-loop transfer function in (6.35) from $\begin{pmatrix} e_1 \\ e_2 \end{pmatrix}$ to $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$, is given by*

$$\Phi = \begin{pmatrix} I & -G \\ -K & I \end{pmatrix}^{-1} \begin{pmatrix} G & 0 \\ 0 & K \end{pmatrix} = \begin{pmatrix} G & 0 \\ 0 & K \end{pmatrix} \begin{pmatrix} I & -K \\ -G & I \end{pmatrix}^{-1}. \tag{6.36}$$

In particular, the closed-loop transfer function G_f from e_1 to y_1 is given by

$$\begin{aligned} G_f(z) &= \begin{pmatrix} I & 0 \end{pmatrix} \Phi(z) \begin{pmatrix} I \\ 0 \end{pmatrix} \\ &= (I - G(z)K(z))^{-1}G(z) = G(z)(I - K(z)G(z))^{-1}. \end{aligned} \tag{6.37}$$

Proof. The system equations (6.30) can be written in matrix form as

$$\begin{pmatrix} I & -G \\ -K & I \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} G & 0 \\ 0 & K \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}.$$

It is easily calculated that

$$\begin{pmatrix} I & -G \\ -K & I \end{pmatrix}^{-1} = \begin{pmatrix} (I-GK)^{-1} & G(I-KG)^{-1} \\ (I-KG)^{-1}K & (I-KG)^{-1} \end{pmatrix},$$

and hence

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} (I-GK)^{-1}G & G(I-KG)^{-1}K \\ (I-KG)^{-1}KG & (I-KG)^{-1}K \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}. \tag{6.38}$$

The expression for the transfer function $G_f(z)$ from e_1 to y_1 follows easily from (6.38). ■

The definition of internal stability is stated solely in state-space terms. If the realizations (A,B,C,D) and (A_c,B_c,C_c,D_c) of the plant $G(z)$ and the controller $K(z)$ are stabilizable and detectable, then we can reformulate this condition in terms of the two transfer functions.

Proposition 6.39. *Assume that (A,B,C,D) and (A_c,B_c,C_c,D_c) are stabilizable and detectable. Then the feedback interconnection Σ_{cl} is internally stable if and only if $I - G(z)K(z)$ is properly invertible and the transfer function $\Phi(z)$ of Σ_{cl} is stable.*

Proof. By assumption, the realization $(A_{\mathrm{cl}},B_{\mathrm{cl}},C_{\mathrm{cl}},D_{\mathrm{cl}})$ is stabilizable and detectable. Thus the transfer function $\Phi(z)$ is stable if and only if A_{cl} is stable. Moreover, $I - G(\infty)K(\infty) = I - DD_c$. This completes the proof. ■

To characterize internally stabilizing controllers $K(z)$ for a transfer function $G(z)$, it is convenient to replace the ring of polynomials by the ring of stable proper rational functions. We proceed to formulate the relevant results on coprime factorizations in this context.

1. Coprime Factorizations over RH_∞.
Let RH_∞ denote the set of all proper rational transfer functions $g(z) \in \mathbb{R}(z)$ with poles only in the left half-plane $\mathbb{C}_-$. It is easily seen that RH_∞ is a ring and, indeed, a principal ideal domain; see Chapter 2 for details. Since any real rational function in $\mathbb{R}(z)$ can be expressed as the quotient of two elements from RH_∞, we conclude that $\mathbb{R}(z)$ is the field of fractions of RH_∞. Many important properties of rational matrix functions remain valid over RH_∞. One says that a rational matrix valued function $G(z) \in \mathbb{R}(z)^{p\times m}$ is in $RH_\infty^{p\times m}$ if and only if $G(z)$ is proper and stable. Two results for rational matrix functions – of importance for us, namely the existence of coprime and doubly coprime factorizations over the ring RH_∞ – are stated next. The proofs of these theorems run verbatim with those of Theorem 2.29, Theorem 2.33, and Corollary 2.34 in Chapter 2; they are therefore omitted.

Theorem 6.40. *For $G(z) \in \mathbb{F}(z)^{p\times m}$ the following assertions are valid:*

1. *There exist right coprime matrices $P(z) \in RH_\infty^{p\times m}, Q(z) \in RH_\infty^{m\times m}$ with* det $Q \neq 0$ *such that*
$$G(z) = P(z)Q(z)^{-1}. \tag{6.39}$$
If $P_1(z) \in RH_\infty^{p\times m}, Q_1(z) \in RH_\infty^{m\times m}$ are right coprime with det $Q_1 \neq 0$ *and*
$$P_1(z)Q_1(z)^{-1} = P(z)Q(z)^{-1} = G(z),$$
then there exists a unique unimodular matrix $U \in GL_m(RH_\infty)$ with $(P_1(z), Q_1(z)) = (P(z)U(z), Q(z)U(z))$.
2. *There exist left coprime matrices $P_\ell(z) \in RH_\infty^{p\times m}, Q_\ell(z) \in RH_\infty^{p\times p}$ with* det $Q_\ell \neq 0$ *such that*
$$G(z) = Q_\ell(z)^{-1}P_\ell(z). \tag{6.40}$$
If $P_{\ell,1}(z) \in RH_\infty^{p\times m}, Q_{\ell,1}(z) \in RH_\infty^{p\times p}$ are left coprime with det $Q_{\ell,1} \neq 0$ *and*
$$Q_{\ell,1}(z)^{-1}P_{\ell,1}(z) = Q_{\ell,2}(z)^{-1}P_{\ell,2}(z) = G(z),$$
then $(P_{\ell,1}(z), Q_{\ell,1}(z)) = (U(z)P_{\ell,2}(z), U(z)Q_{\ell,2}(z))$ for a uniquely determined unimodular matrix $U \in GL_m(RH_\infty)$.

Factorizations as in (6.39) and (6.40) are called ***right and left coprime factorizations of*** *G* ***over*** *RH_∞.*

We now relate the coprimeness of factorizations over RH_∞ to the solvability of Bezout equations, i.e., to unimodular embeddings. Let

$$G(z) = Q_\ell(z)^{-1}P_\ell(z) = P_r(z)Q_r(z)^{-1}$$

be right and left coprime factorizations of $G(z) \in \mathbb{F}(z)^{p\times m}$ over the ring RH_∞, respectively, which implies the intertwining relation

$$P_\ell(z)Q_r(z) = Q_\ell(z)P_r(z).$$

Theorem 6.41 (Doubly Coprime Factorization). *Let $P_\ell(z) \in RH_\infty^{p\times m}$ and $Q_\ell(z) \in RH_\infty^{p\times p}$ be right coprime and $P_r(z) \in RH_\infty^{p\times m}$ and $Q_r(z) \in RH_\infty^{m\times m}$ be left coprime, with*

$$Q_\ell(z)P_r(z) = P_\ell(z)Q_r(z).$$

Then there exist stable proper rational matrices $X(z) \in RH_\infty^{m\times p}, \overline{X}(z) \in RH_\infty^{m\times p}, Y(z) \in RH_\infty^{m\times m}, \overline{Y}(z) \in RH_\infty^{p\times p}$, with

$$\begin{pmatrix} Y(z) & X(z) \\ -P_\ell(z) & Q_\ell(z) \end{pmatrix} \begin{pmatrix} Q_r(z) & -\overline{X}(z) \\ P_r(z) & \overline{Y}(z) \end{pmatrix} = \begin{pmatrix} I_m & 0 \\ 0 & I_p \end{pmatrix}.$$

In particular, every right coprime factorization $G(z) = P(z)Q(z)^{-1}$ of a proper rational function $G(z) \in \mathbb{R}(z)^{p\times m}$ admits an extension to a unimodular matrix

$$\begin{pmatrix} Q(z) & -\overline{X}(z) \\ P(z) & \overline{Y}(z) \end{pmatrix} \in GL_{m+p}(RH_\infty).$$

2. Characterization of stabilizing controllers.
The characterization of controllers that internally stabilize a transfer function $G(z)$ is our next task. For this, the following abstract extension of Lemma 2.28 is needed.

Lemma 6.42. *Let R denote a principal ideal domain, and let $A \in R^{r\times r}, B \in R^{r\times m}, C \in R^{p\times r}$ be such that* $\det A \neq 0$*, A and B are left coprime and C and A are right coprime. Then $CA^{-1}B \in R^{p\times m}$ if and only if $A \in GL_r(R)$ is unimodular.*

Proof. The condition is obviously sufficient. For the necessity part consider a solution $X \in R^{r\times r}, Y \in R^{r\times p}$ of the Bezout equation

$$XA + YC = I_r.$$

The solution exists since A and C are right coprime. Thus $A^{-1}B = XB + YCA^{-1}B \in R^{r\times m}$. Using the dual Bezout equation

$$A\tilde{X} + B\tilde{Y} = I$$

we conclude that $A^{-1} = \tilde{X} + A^{-1}B\tilde{Y} \in R^{r\times r}$. Thus A is unimodular, i.e., $A \in GL_r(R)$. ∎

Theorem 6.43. *Assume that $G(z) \in \mathbb{F}(z)^{p\times m}$ is proper and that $K(z) \in \mathbb{F}(z)^{m\times p}$ is a proper controller, with the assumption that $I - G(\infty)K(\infty)$ is invertible. Assume that $G(z)$ and $K(z)$ have the following coprime factorizations over RH_∞:*

$$\begin{aligned} G(z) &= Q_\ell(z)^{-1}P_\ell(z) = P_r(z)Q_r(z)^{-1}, \\ K(z) &= S_\ell(z)^{-1}R_\ell(z) = R_r(z)S_r(z)^{-1}. \end{aligned} \tag{6.41}$$

Then:

1. *The transfer function $\Phi(z)$ of Σ_{cl} has the following coprime factorizations:*

$$\Phi(z) = \begin{pmatrix} Q_\ell & -P_\ell \\ -R_\ell & S_\ell \end{pmatrix}^{-1} \begin{pmatrix} P_\ell & 0 \\ 0 & R_\ell \end{pmatrix} = \begin{pmatrix} P_r & 0 \\ 0 & R_r \end{pmatrix} \begin{pmatrix} Q_r & -R_r \\ -P_r & S_r \end{pmatrix}^{-1}. \tag{6.42}$$

2. *For suitable units u_1, u_2, u_3 in RH_∞, the following equations are fulfilled:*

$$\begin{aligned} \det\begin{pmatrix} Q_\ell & -P_\ell \\ -R_\ell & S_\ell \end{pmatrix} &= u_1 \det(Q_\ell S_r - P_\ell R_r) = u_2 \det(S_\ell Q_r - R_\ell P_r) \\ &= u_3 \det\begin{pmatrix} Q_r & -R_r \\ -P_r & S_r \end{pmatrix}. \end{aligned}$$

3. *$\Phi(z)$ is proper and stable if and only if*

$$S_\ell Q_r - R_\ell P_r \in GL_m(RH_\infty) \quad \textit{or} \quad Q_\ell S_r - P_\ell R_r \in GL_p(RH_\infty).$$

4. *The closed-loop transfer function G_f from e_1 to y_1 has the following equivalent representations:*

$$\begin{aligned} G_f(z) &= S_r(z)(Q_\ell(z)S_r(z) - P_\ell(z)R_r(z))^{-1}P_\ell(z) \\ &= P_r(z)(S_\ell(z)Q_r(z) - R_\ell(z)P_r(z))^{-1}S_\ell(z) \\ &= \begin{pmatrix} P_r & 0 \end{pmatrix} \begin{pmatrix} Q_r & -R_r \\ -P_r & S_r \end{pmatrix}^{-1} \begin{pmatrix} I \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} I & 0 \end{pmatrix} \begin{pmatrix} Q_\ell & -P_\ell \\ -R_\ell & S_\ell \end{pmatrix}^{-1} \begin{pmatrix} P_\ell \\ 0 \end{pmatrix}. \end{aligned} \tag{6.43}$$

   More generally, the closed-loop transfer function is

$$\begin{aligned} \Phi(z) &= \begin{pmatrix} 0 & 0 \\ -I & 0 \end{pmatrix} + \begin{pmatrix} P_r \\ Q_r \end{pmatrix} (S_\ell Q_r - R_\ell P_r)^{-1} \begin{pmatrix} S_\ell & R_\ell \end{pmatrix} \\ &= \begin{pmatrix} 0 & -I \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} S_r \\ R_r \end{pmatrix} (Q_\ell S_r - P_\ell R_r)^{-1} \begin{pmatrix} P_\ell & Q_\ell \end{pmatrix}. \end{aligned}$$

Proof. The representations (6.42) follow by substituting the coprime factorizations (6.41) into equation (6.36). The right coprimeness of the factorization

$$\begin{pmatrix} P_r & 0 \\ 0 & R_r \end{pmatrix} \begin{pmatrix} Q_r & -R_r \\ -P_r & S_r \end{pmatrix}^{-1}$$

is equivalent to the right primeness of the matrix

$$\begin{pmatrix} P_r & 0 \\ 0 & R_r \\ Q_r & -R_r \\ -P_r & S_r \end{pmatrix},$$

i.e., after suitable elementary row operations, to the right primeness of

$$\begin{pmatrix} P_r & 0 \\ Q_r & 0 \\ 0 & R_r \\ 0 & S_r \end{pmatrix}.$$

In turn, this is equivalent to our assumption that P_r, Q_r and R_r, S_r are right coprime, respectively. A similar argument shows left coprimeness.
2. First, note that from the coprime factorizations (6.45) it follows that $\det Q_\ell = c \det Q_r$ and $\det S_\ell = d \det S_r$ for suitable units $c, d \in RH_\infty$. Next, computing

$$\begin{pmatrix} I & 0 \\ R_\ell Q_\ell^{-1} & I \end{pmatrix} \begin{pmatrix} Q_\ell & -P_\ell \\ -R_\ell & S_\ell \end{pmatrix} = \begin{pmatrix} Q_\ell & -P_\ell \\ 0 & S_\ell - R_\ell Q_\ell^{-1} P_\ell \end{pmatrix} = \begin{pmatrix} Q_\ell & -P_\ell \\ 0 & (S_\ell Q_r - R_\ell P_r) Q_r^{-1} \end{pmatrix},$$

and applying the multiplication rule of determinants, it follows that

$$\begin{aligned} \det \begin{pmatrix} Q_\ell & -P_\ell \\ -R_\ell & S_\ell \end{pmatrix} &= \det Q_\ell \cdot \det(S_\ell Q_r - R_\ell P_r) \cdot \det Q_r^{-1} \\ &= c \cdot \det(S_\ell Q_r - R_\ell P_r). \end{aligned}$$

The other equalities are derived analogously.
4. Substituting representations (6.41) into (6.37), the closed-loop transfer function $G_f(z)$ has the following representations:

$$\begin{aligned} G_f(z) &= S_r(z)(Q_\ell(z) S_r(z) - P_\ell(z) R_r(z))^{-1} P_\ell(z) \\ &= P_r(z)(S_\ell(z) Q_r(z) - R_\ell(z) P_r(z))^{-1} S_\ell(z). \end{aligned}$$

To obtain the third representation in (6.43), the coprime factorizations (6.42) are used to compute

$$G_f(z) = \begin{pmatrix} I & 0 \end{pmatrix} \begin{pmatrix} P_r & 0 \\ 0 & R_r \end{pmatrix} \begin{pmatrix} Q_r & -R_r \\ -P_r & S_r \end{pmatrix}^{-1} \begin{pmatrix} I \\ 0 \end{pmatrix}$$
$$= \begin{pmatrix} P_r & 0 \end{pmatrix} \begin{pmatrix} Q_r & -R_r \\ -P_r & S_r \end{pmatrix}^{-1} \begin{pmatrix} I \\ 0 \end{pmatrix}.$$

The last representation in (6.43) is similarly derived. The formulas for the closed-loop transfer function $\Phi(z)$ are similarly derived using formula (6.38). ■

The preceding result leads to the first main characterization of all stabilizing controllers.

Theorem 6.44. *The following assertions are equivalent for proper transfer functions $G(z) \in \mathbb{R}(z)^{p\times m}$ and $K(z) \in \mathbb{R}(z)^{m\times p}$ that define a well-posed feedback interconnection:*

1. *K internally stabilizes G.*
2. *The matrix*

$$\begin{pmatrix} Q_\ell & -P_\ell \\ -R_\ell & S_\ell \end{pmatrix} \in RH_\infty^{(p+m)\times(p+m)}$$

   is invertible over RH_∞.
3. *The matrix*

$$\begin{pmatrix} Q_r & -P_r \\ -R_r & S_r \end{pmatrix} \in RH_\infty^{(p+m)\times(p+m)}$$

   is invertible over RH_∞.
4. det $(Q_\ell S_r - P_\ell R_r)$ *is a biproper stable rational function with stable inverse.*
5. det $(S_\ell Q_r - R_\ell P_r)$ *is a biproper stable rational function with stable inverse.*

Proof. By Proposition 6.39, the controller K internally stabilizes K if and only if the transfer function $\Phi \in RH_\infty^{(p+m)\times(p+m)}$. By Lemma 6.42 this is true if and only if

$$\begin{pmatrix} Q_\ell & -P_\ell \\ -R_\ell & S_\ell \end{pmatrix}^{-1} \in RH_\infty^{(p+m)\times(p+m)}.$$

This in turn is equivalent to

$$\begin{pmatrix} Q_r & -P_r \\ -R_r & S_r \end{pmatrix}^{-1} \in RH_\infty^{(p+m)\times(p+m)}.$$

By Theorem 6.43, this is true if and only if det $(Q_\ell S_r - P_\ell R_r)$ [or det $(S_\ell Q_r - R_\ell P_r)$] is a biproper stable rational function with stable inverse. This proves the equivalence of parts (1)–(5). ■

3. The Youla–Kucera Parameterization.
The characterization of stabilizing controllers via unimodular embeddings is precise but has the disadvantage of not leading to an easily manageable parameterization of all such controllers. The Youla–Kucera parameterization resolves this issue by giving a complete parameterization of all stabilizing controllers via linear fractional transformations.

Theorem 6.45 (Youla-Kucera). *Let*

$$G(z) = Q_\ell(z)^{-1}P_\ell(z) = P_r(z)Q_r(z)^{-1} \in \mathbb{R}(z)^{p\times m}$$

be a proper rational, stable, and coprime factorization over RH_∞. Let

$$\begin{pmatrix} Y(z) & X(z) \\ P_\ell(z) & Q_\ell(z) \end{pmatrix} \begin{pmatrix} Q_r(z) & -\overline{X}(z) \\ -P_r(z) & \overline{Y}(z) \end{pmatrix} = \begin{pmatrix} I_m & 0 \\ 0 & I_p \end{pmatrix} \tag{6.44}$$

be a unimodular embedding for this doubly coprime factorization over RH_∞. Then all proper rational and internally stabilizing controllers for $G(z)$ are parameterized by

$$K = (\overline{X} - Q_r\Gamma)(\overline{Y} - P_r\Gamma)^{-1} = (Y - \Gamma P_\ell)^{-1}(X - \Gamma Q_\ell). \tag{6.45}$$

Here Γ denotes an arbitrary element of $(RH_\infty)^{m\times p}$ such that the matrices

$$\overline{Y}(\infty) - P_r(\infty)\Gamma(\infty) \quad \textit{and} \quad Y(\infty) - \Gamma(\infty)P_\ell(\infty)$$

are invertible.

Proof. By Theorem 6.44, all internally stabilizing controllers $K = R_rS_r^{-1}$ are such that the matrix

$$\begin{pmatrix} Q_r(z) & -R_r(z) \\ -P_r(z) & S_r(z) \end{pmatrix}$$

is unimodular over RH_∞. Thus there exists a unimodular matrix

$$U = \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix} \in GL_{m+p}(RH_\infty),$$

with

$$\begin{pmatrix} Q_r(z) & -R_r(z) \\ -P_r(z) & S_r(z) \end{pmatrix} = \begin{pmatrix} Q_r(z) & -\overline{X}(z) \\ -P_r(z) & \overline{Y}(z) \end{pmatrix} \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix} \in GL_{m+p}(RH_\infty).$$

This implies $U_{11} = I$, $U_{21} = 0$, and U_{22} is unimodular over RH_∞. Thus $\Gamma := U_{12}U_{22}^{-1}$ exists in RH_∞ and

$$\begin{pmatrix} -R_r(z) \\ S_r(z) \end{pmatrix} = \begin{pmatrix} Q_r(z) & -\overline{X}(z) \\ -P_r(z) & \overline{Y}(z) \end{pmatrix} \begin{pmatrix} \Gamma U_{22} \\ U_{22} \end{pmatrix}$$

follows, implying

$$K = R_r S_r^{-1} = (\overline{X} - Q_r \Gamma)(\overline{Y} - P_r \Gamma)^{-1}.$$

By the unimodular embedding (6.44), we see that

$$\begin{pmatrix} Y - \Gamma P_\ell & X - \Gamma Q_\ell \\ P_\ell & Q_\ell \end{pmatrix} \begin{pmatrix} Q_r & -R_r \\ -P_r & S_r \end{pmatrix} = \begin{pmatrix} I & -\Gamma \\ 0 & I \end{pmatrix} \begin{pmatrix} Y & X \\ P_\ell & Q_\ell \end{pmatrix} \begin{pmatrix} Q_r & -\overline{X} \\ -P_r & \overline{Y} \end{pmatrix} \begin{pmatrix} I & \Gamma \\ 0 & I \end{pmatrix}$$
$$= \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}.$$

Defining $R_\ell = X - \Gamma Q_\ell, S_\ell := Y - \Gamma P_\ell$, one obtains $S_\ell R_r = R_\ell S_r$, and therefore $S_\ell^{-1} R_\ell = R_r S_r^{-1}$, which completes the proof. ■

As a consequence of the Youla–Kucera parameterization, it is now shown that the closed-loop transfer function depends **affinely** on the stabilizing parameter Γ. This fact is very important in robust controller design and opens the way to applying convex optimization techniques to robust controller design.

Theorem 6.46. *Let*

$$G(z) = Q_\ell(z)^{-1} P_\ell(z) = P_r(z) Q_r(z)^{-1} \in \mathbb{R}(z)^{p\times m}$$

be a proper rational, stable, and coprime factorization over RH_∞. Let

$$K_0(z) = Y(z)^{-1} X(z) = \overline{X}(z) \overline{Y}(z)^{-1} \in \mathbb{R}(z)^{m\times p}$$

denote a proper rational, stable, and coprime factorization over RH_∞ of a stabilizing controller $K_0(z)$ of $G(z)$. Then the closed-loop transfer function $\Phi_K(z) =$

$$\begin{pmatrix} P_r(z) & 0 \\ 0 & \overline{X}(z) \end{pmatrix} \begin{pmatrix} Q_r(z) & -\overline{X}(z) \\ -P_r(z) & \overline{Y}(z) \end{pmatrix}^{-1} - \begin{pmatrix} P_r(z) \\ Q_r(z) \end{pmatrix} \begin{pmatrix} 0 & \Gamma(z) \end{pmatrix} \begin{pmatrix} Q_r(z) & -\overline{X}(z) \\ -P_r(z) & \overline{Y}(z) \end{pmatrix}^{-1}$$

of all proper stabilizing controllers $K(z)$ of $G(z)$ depends affinely on the stabilizing parameter Γ, where $\Gamma(z) \in RH_\infty$ is such that the matrices

$$\overline{Y}(\infty) - P_r(\infty)\Gamma(\infty) \quad \text{and} \quad Y(\infty) - \Gamma(\infty) P_\ell(\infty)$$

are invertible. In particular, the closed-loop transfer function from e_1 to y_1 has the affine parametric form

$$G_f(z) = (I - P_\ell \Gamma \overline{Y}^{-1}) G (I - K_0 G)^{-1} .$$

Proof. By Theorem 6.43, the transfer function for the feedback interconnection of G with $K = \overline{M}\overline{L}^{-1}$ is

$$G_f(z) = \begin{pmatrix} P_r & 0 \end{pmatrix} \begin{pmatrix} Q_r & -\overline{M} \\ -P_r & \overline{L} \end{pmatrix}^{-1} \begin{pmatrix} I \\ 0 \end{pmatrix}.$$

From the Youla–Kucera parameterization it follows that

$$\begin{pmatrix} Q_r & -\overline{M} \\ -P_r & \overline{L} \end{pmatrix} = \begin{pmatrix} Q_r & -\overline{X} \\ -P_r & \overline{Y} \end{pmatrix} \begin{pmatrix} I & \Gamma \\ 0 & I \end{pmatrix}.$$

Moreover,

$$\begin{pmatrix} Q_r & -\overline{X} \\ -P_r & \overline{Y} \end{pmatrix} = \begin{pmatrix} I & -K_0 \\ -G & I \end{pmatrix} \begin{pmatrix} Q_r & 0 \\ 0 & \overline{Y} \end{pmatrix}.$$

Using (6.38), we compute

$$\begin{aligned} G_f(z) &= \begin{pmatrix} P_r & 0 \end{pmatrix} \begin{pmatrix} I & -\Gamma \\ 0 & I \end{pmatrix} \begin{pmatrix} Q_r & -\overline{X} \\ -P_r & \overline{Y} \end{pmatrix}^{-1} \begin{pmatrix} I \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} P_r & -P_r \Gamma \end{pmatrix} \begin{pmatrix} Q_r^{-1} & 0 \\ 0 & \overline{Y}^{-1} \end{pmatrix} \begin{pmatrix} I & -K_0 \\ -G & I \end{pmatrix}^{-1} \begin{pmatrix} I \\ 0 \end{pmatrix} \\ &= G(I - K_0 G)^{-1} - P_r \Gamma \overline{Y}^{-1} G (I - K_0 G)^{-1}, \end{aligned}$$

which completes the proof. ■

To present a state-space representation of the Youla–Kucera parameterization, state-space formulas for the involved transfer functions are derived. Let (A,B,C,D) and (A_c,B_c,C_c,D_c) be realizations of the transfer functions $G(z)$ and $K(z)$, respectively. Choosing a state feedback F such that $A + BF$ is stable, the transfer functions

$$Q_r(z) = \left[\begin{array}{c|c} A+BF & B \\ \hline F & I \end{array}\right], \quad P_r(z) = \left[\begin{array}{c|c} A+BF & B \\ \hline C+DF & D \end{array}\right] \tag{6.46}$$

are in RH_∞ and define a right coprime factorization of $G(z) = P_r(z) Q_r(z)^{-1}$. Similarly, for J such that $A - JC$ is stable, the transfer functions

$$Q_\ell(z) = \left[\begin{array}{c|c} A-JC & J \\ \hline -C & I \end{array}\right], \quad P_\ell(z) = \left[\begin{array}{c|c} A-JC & B-JD \\ \hline C & D \end{array}\right]$$

are in RH_∞ and define a left coprime factorization of

$$G(z) = Q_\ell(z)^{-1} P_\ell(z).$$

A solution of the unimodular embedding (6.44) is constructed as follows. Define

$$\overline{Y}(z) = \left[\begin{array}{c|c} A+BF & J \\ \hline C+DF & I \end{array}\right], \quad \overline{X}(z) = \left[\begin{array}{c|c} A+BF & J \\ \hline F & 0 \end{array}\right].$$

Similarly,

$$Y(z) = \left[\begin{array}{c|c} A-JC & B-JD \\ \hline -F & I \end{array}\right], \quad X(z) = \left[\begin{array}{c|c} A-JC & J \\ \hline F & 0 \end{array}\right]. \tag{6.47}$$

The state-space representation of the stabilizing controller then has the form

$$\begin{aligned} \dot{\xi} &= A\xi + Bu + J(y - C\xi - Du) \\ u &= F\xi - \Gamma(\frac{d}{dt})(y - C\xi - Du), \end{aligned} \tag{6.48}$$

where $\Gamma(z)$ denotes a proper rational and stable matrix function such that the feedback system is well posed. One observes that the transfer function of (6.48) from y to u is exactly the controller transfer function $K(z)$ in Youla–Kucera form (6.45). By choosing a state-space realization of $\Gamma(z)$ as

$$\Gamma(z) = D_c + C_c(zI - A_c)B_c,$$

we obtain the (implicit) first-order representation of the stabilizing controller as

$$\begin{aligned} \dot{\xi} &= A\xi + Bu + J(y - C\xi - Du) \\ \dot{\xi}_c &= A_c\xi_c + B_c(y - C\xi - Du) \\ u &= F\xi - C_c\xi_c - D_c(y - C\xi - Du). \end{aligned}$$

For $D = 0$, this leads to the input/output representation of all stabilizing controllers:

$$\begin{aligned} \dot{\xi} &= (A + BF - (J - D_c)C)\xi - B_cC_c\xi_c + Jy \\ \dot{\xi}_c &= -B_{cC}\xi + A_c\xi_c + B_{cy} \\ u &= (F + D_{cC})\xi - C_c\xi_c - D_{cy}. \end{aligned}$$

6.8 Controlled Invariant Subspaces

The next two sections are devoted to a presentation of basic concepts of geometric control, i.e., the notions of controlled and conditioned invariant subspaces. Geometric control was developed in the early 1970s, by Francis, Wonham, and Morse on the one hand and by Basile and Marro on the other, as a tool for better understanding the structure of linear control systems within the context of state-space theory. Controlled and conditioned invariant subspaces generalize the class of invariant subspaces of a linear operator into a system-theoretic context; we refer the reader to Basile and Marro (1992) and Wonham (1979) for comprehensive accounts of the theory. The emphasis of geometric control is on clear, conceptual statements rather than a reliance on complex matrix manipulations. The term *geometry* refers to the basis-free study of classes of subspaces of the state space of a system realization. This development led to (sometimes iterative) design procedures based on elegant vector space geometric considerations.

However, it turned out that even the most fundamental problem of linear control – characterizing the limitations of pole placement by state feedback – was first solved by Rosenbrock using polynomial algebra. This brought about the need to find a bridge between the elegance of state-space problem formulations on the one hand and the computational effectiveness of polynomial algebra on the other.

The availability of the shift realizations allows us, given a system in a polynomial matrix description, to interpret the relation between the elements of the associated polynomial system matrix in state-space terms. Conversely, constructs coming from geometric control theory can, in the case of the shift realization, be completely characterized in polynomial terms. This leads to the most direct connection between abstract module theory, polynomial matrix descriptions, and state-space theory. In particular, the most basic objects of geometric control theory, namely, controlled and conditioned invariant subspaces, have very nice characterizations in terms of the zero structure of linear systems. Space limitations prevent us from delving deeper into the subject.

For a linear transformation A in $\mathscr{X}$ and an A-invariant subspace $\mathscr{V}$, denote by $A|_{\mathscr{V}}$ the restriction of A to $\mathscr{V}$. By a slight abuse of notation, an **induced map**, i.e., a map induced by A in the quotient space $\mathscr{X}/\mathscr{V}$, will be denoted by $A|_{\mathscr{X}/\mathscr{V}}$. Controlled invariant subspaces are introduced in state-space terms, and functional characterizations are derived.

Definition 6.47. 1. For an input pair (A,B), a subspace $\mathscr{V} \subset \mathscr{X}$ is called a **controlled invariant subspace**, or an (A,B)**-invariant subspace**, if

$$A\mathscr{V} \subset \mathscr{V} + \operatorname{Im} B. \tag{6.49}$$

2. Let $\mathscr{V}$ be a controlled invariant subspace for the pair (A,B). A feedback map $K : \mathscr{X} \longrightarrow \mathscr{U}$ that satisfies $(A - BK)\mathscr{V} \subset \mathscr{V}$ is called a **friend** of $\mathscr{V}$. For a controlled invariant subspace $\mathscr{V}$, denote by $\mathscr{F}(\mathscr{V})$ the set of all friends of $\mathscr{V}$.

3. One says that a family $\{\mathscr{V}_\alpha\}$ of controlled invariant subspaces is **compatible** if $\bigcap_\alpha \mathscr{F}(\mathscr{V}_\alpha) \neq \emptyset$, i.e., if there exists a single feedback map K such that

$$(A-BK)\mathscr{V}_\alpha \subset \mathscr{V}_\alpha$$

if fulfilled for all α.
4. A controlled invariant subspace $\mathscr{V}$ is called a **reachability subspace** if for each monic polynomial $q(z)$ of degree equal to $\dim \mathscr{V}$ there exists a friend $K \in \mathscr{F}(\mathscr{V})$ such that $q(z)$ is the characteristic polynomial of $(A-BK)|_{\mathscr{V}}$.

If $\mathscr{V} \subset \mathscr{X}$ is a controlled invariant subspace for the pair (A,B), and if $K \in \mathscr{F}(\mathscr{V})$, the notation $(A-BK)|_{\mathscr{V}}$ and $(A-BK)|_{\mathscr{X}/\mathscr{V}}$ will be used for restricted and induced maps, respectively. The next proposition is basic to all subsequent characterizations of controlled invariant subspaces.

Proposition 6.48. *For an input pair (A,B), the following statements are equivalent:*

1. *$\mathscr{V}$ is a controlled invariant subspace.*
2. *There exists a state feedback map $K : \mathscr{X} \longrightarrow \mathscr{U}$ such that the subspace $\mathscr{V}$ is $(A-BK)$-invariant.*
3. *For each $x_0 \in \mathscr{V}$, there exists an infinite sequence of inputs (u_t) such that the state trajectory (x_t) stays in $\mathscr{V}$.*

Proof. Assume $\mathscr{V}$ is controlled invariant. We choose a basis $\{v_1,\dots,v_m\}$ for $\mathscr{V}$. By our assumption, $Av_i = w_i + Bu_i$, with $w_i \in \mathscr{V}$ and $u_i \in U$. Define a linear map K on $\mathscr{V}$ by $Kv_i = u_i$, and arbitrarily extend it to all of $\mathscr{X}$. Thus there exists a linear map $K : \mathscr{X} \longrightarrow U$ such that $(A-BK)v_i = w_i$, i.e., $(A-BK)\mathscr{V} \subset \mathscr{V}$, and (1) implies (2).

Let $x_0 \in \mathscr{V}$. It suffices to show that there exists a control u such that $x_1 = Ax_0 + Bu \in \mathscr{V}$. Since $(A-BK)x_0 = x_1 \in \mathscr{V}$, we simply choose $u = -Kx_0$, and (2) implies (3). To show that (3) implies (1), consider $x_0 \in \mathscr{X}$. It suffices to show that there exists a u such that $Ax_0 = x_1 - Bu$. By our assumption, there exists a $u \in U$ such that $x_1 = Ax_0 + Bu \in \mathscr{V}$. ■

The following are simple, yet very useful, characterizations.

Proposition 6.49. *Let (A,B) be an input pair in the state space $\mathscr{X}$. Then:*

1. *A q-dimensional subspace $\mathscr{V} \subset \mathscr{X}$ is controlled invariant if and only if there exists a linear map $F : \mathscr{V}_0 \longrightarrow \mathscr{V}_0$ on a q-dimensional subspace $\mathscr{V}_0$ and an injective map $Z : \mathscr{V}_0 \longrightarrow \mathscr{X}$, with $\operatorname{Im} Z = \mathscr{V}$, so that for some K*

$$ZF = (A-BK)Z. \tag{6.50}$$

2. *A q-dimensional subspace $\mathscr{R} \subset \mathscr{X}$ is a reachability subspace if and only if there exists a reachable pair (F,G) in a q-dimensional state space $\mathscr{V}_0$ and an injective map $Z : \mathscr{V}_0 \longrightarrow \mathscr{X}$, with $\operatorname{Im} Z = \mathscr{R}$, so that for some K,L*

$$\begin{aligned} ZF &= (A-BK)Z, \\ ZG &= BL. \end{aligned} \tag{6.51}$$

Proof. If $\mathscr{V} \subset \mathscr{X}$ is controlled invariant, choose $\mathscr{V}_0 = \mathscr{V}$, $F = (A-BK)|\mathscr{V}$, and Z to be the embedding map of $\mathscr{V}$ into $\mathscr{X}$. Conversely, if F and Z exist and satisfy (6.50), then clearly $\mathscr{V} = \operatorname{Im} Z$ satisfies $(A+BK)\mathscr{V} \subset \mathscr{V}$, i.e., $\mathscr{V}$ is controlled invariant.

To prove the second claim, observe that $BL \subset \operatorname{Im} Z$ implies the existence of G. If $\mathscr{R}$ is a reachability subspace, it is in particular controlled invariant. So, with F and Z defined as previously, the first equation of (6.51) was derived. By the injectivity of Z, G is uniquely determined. Thus, the second equation in (6.51) follows. Conversely, if (6.51) holds, then, as previously, $\mathscr{R} = \operatorname{Im} Z$ is controlled invariant. Now, with $k = \dim \mathscr{X}_0$ and using the reachability of (F,G), we compute

$$\sum_{i=0}^{k-1} (A-BK)^i \operatorname{Im} BL = Z \sum_{i=0}^{k-1} F^i \operatorname{Im} G = Z\mathscr{X}_0 = \mathscr{R},$$

which shows that $\mathscr{R}$ is a reachability subspace. ■

Stated next are some elementary properties of controlled invariant subspaces.

Proposition 6.50. *Let (A,B) be an input pair in the state space $\mathscr{X}$. Then:*

1. *The set of controlled invariant subspaces is closed under sums;*
2. *For each subspace $\mathscr{K} \subset \mathscr{X}$ there exists a maximal controlled invariant subspace contained in $\mathscr{K}$ that is denoted by $\mathscr{V}^*(\mathscr{K})$.*

Proof. The first claim follows directly from (6.49). For the second claim note that the set of all controlled invariant subspaces contained in $\mathscr{K}$ is closed under sums and is nonempty because the zero subspace is included. $\mathscr{V}^*(\mathscr{K})$ is the sum of all these subspaces. ■

There exists a simple subspace algorithm to compute $\mathscr{V}^*$; see Wonham (1979).

Theorem 6.51 ($\mathscr{V}^*$-Algorithm). *Let $(A,B) \in \mathbb{F}^{n\times n} \times \mathbb{F}^{n\times m}$ and a linear subspace $W \subset \mathbb{F}^n$. Define a sequence of linear subspaces V_i, recursively constructed as follows:*

$$V_0 = W$$

$$V_{i+1} = W \cap A^{-1}(V_i + \operatorname{Im} B).$$

Then $V_0 \supset V_1 \supset V_2 \supset \cdots$ and, for a suitable $k \leq \dim W$, the equality $V_k = \mathscr{V}^*(W)$ is valid. In particular,

$$\mathscr{V}^*(W) = W \cap A^{-1}(W) \cap \cdots \cap A^{-k+1}(W) \cap A^{-k}(W + \operatorname{Im} B).$$

Proof. The construction of V_i implies $V_i \subset W$ for all i. Suppose that $V_i \subset V_{i-1}$ is satisfied for some i. Then

$$V_{i+1} = W \cap A^{-1}(V_i + \operatorname{Im} B) \subset W \cap A^{-1}(V_{i-1} + \operatorname{Im} B) = V_i.$$

Thus $V_j \subset V_{j-1}$ is satisfied for all $j \geq 1$. Moreover, by the same argument, $V_i = V_{i-1}$ implies $V_r = V_{i-1}$ for all $r \geq i$. Thus there exists $k \leq \dim W$ with $V_k = V_{k+r}$ for all $r \geq 1$. Since $V_{k+1} = W \cap A^{-1}(V_k + \operatorname{Im} B) = V_k$, the inclusions $V_k \subset W$ and $AV_k \subset V_k + \operatorname{Im} B$ follow. Thus V_k is a controlled invariant subspace contained in W. If $V \subset W$ is a controlled invariant subspace, then $V \subset V_0$. Let r be the largest nonnegative number with $V \subset V_r$. Then V being controlled invariant implies the inclusion $V \subset A^{-1}(V + \operatorname{Im} B)$, and therefore $V_{r+1} = W \cap A^{-1}(V_r + \operatorname{Im} B) \supset W \cap A^{-1}(V + \operatorname{Im} B) \supset V$. This proves that $V \subset V_i$ for all i, and therefore $V \subset V_k$. Thus V_k is the largest controlled invariant subspace contained in W. ∎

In applications, for example, to disturbance decoupling, it is often useful to extend the notion of controlled invariant subspaces by taking the output of the system under consideration. To this end, new geometric objects are introduced.

Definition 6.52. Let (A, B, C, D) be a state-space system in the state space $\mathscr{X}$.

1. A subspace $\mathscr{V} \subset \mathscr{X}$ is called **output nulling** if there exists a state feedback map K that satisfies

$$(A - BK)\mathscr{V} \subset \mathscr{V} \subset \operatorname{Ker}(C - DK). \tag{6.52}$$

2. Let $\mathscr{V}$ be a controlled invariant subspace for the pair (A, B). A feedback map K that satisfies (6.52) is called an **output nulling friend** of $\mathscr{V}$. Denote by $\mathscr{F}_{ON}(\mathscr{V})$ the set of all output nulling friends of $\mathscr{V}$.
3. A subspace $\mathscr{R}$ of the state space is called an **output nulling reachability subspace** if for each monic polynomial $q(z)$ of degree equal to $\dim \mathscr{R}$ there exists a friend $K \in \mathscr{F}_{ON}(\mathscr{V})$ such that $q(z)$ is the characteristic polynomial of $(A - BK)|_{\mathscr{V}}$.

Thus a subspace is output nulling if and only if, for each initial state in $\mathscr{V}$, one can find a state feedback controller that keeps the state in $\mathscr{V}$ while keeping the output zero. Usually, for an output nulling space $\mathscr{V}$, there exist also some external inputs that may be output nulled. Thus there exists a linear map L for which

$$\begin{aligned}(A - BK)\mathscr{V} &\subset \mathscr{V} \subset \operatorname{Ker}(C - DK)\\ \operatorname{Im} BL &\subset \mathscr{V}\\ DL &= 0.\end{aligned}$$

This is equivalent to nulling the output using a feedback law of the form

$$u = Kx + Lv.$$

Such an L, in fact a maximal one, can be constructed by considering the subspace

$$\mathscr{L} = \{\xi \mid B\xi \in \mathscr{V}, \quad D\xi = 0\},$$

and choosing L to be a basis matrix for this subspace.

In addition to the preceding basic definitions, a number of further useful classes of controlled invariant subspaces are linked with certain stability properties of the restriction and induced operators. Refer to Section 6.6 for the class of stable polynomials defined by a multiplicatively closed subset $\mathbb{F}_-[z] \subset \mathbb{F}[z]$.

Definition 6.53. Let (A,B,C,D) be a state-space system acting in the state space $\mathscr{X}$. A controlled invariant subspace $\mathscr{V}$ for the pair (A,B) is called stabilizable, or **inner stabilizable**, if there exists a friend $K \in \mathscr{F}(\mathscr{V})$ such that $(A-BK)|_{\mathscr{V}}$ is $\mathbb{F}_-[z]$-stable.

Analogously, a controlled invariant subspace $\mathscr{V}$ is called **outer stabilizable** if there exists a friend $K \in \mathscr{F}(\mathscr{V})$ such that $(A-BK)|_{\mathscr{X}/\mathscr{V}}$ is $\mathbb{F}_-[z]$-stable. One defines **inner antistabilizable** subspaces similarly.

If $\mathscr{V}$ is controlled invariant with respect to the reachable pair (A,B) and $K \in \mathscr{F}(\mathscr{V})$, then the pair induced by $(A-BK,B)$ in $\mathscr{X}/\mathscr{V}$ is also reachable; hence, $\mathscr{V}$ is both outer stabilizable and antistabilizable. From this point of view, it is more interesting to study inner stabilizability and antistabilizability.

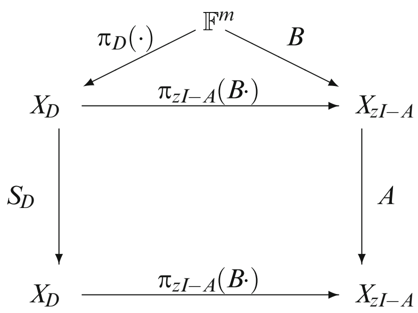
1. Polynomial Characterization of Controlled Invariant Subspaces

By Theorem 3.14, the study of invariant subspaces, in the polynomial model context, is directly related to factorization theory. So it is natural to try and extend this correspondence to the study of controlled and conditioned invariant subspaces as well as other classes of subspaces.

A consequence of the shift realization procedure is the compression of information. All information, up to state-space isomorphism, of a reachable pair (A,B) is encoded in one, nonsingular, polynomial matrix. To see this, recall that reachability is equivalent to the left coprimeness of $zI-A,B$. Taking coprime factorizations

$$(zI-A)^{-1}B = N(z)D(z)^{-1}, \tag{6.53}$$

the isomorphism of the input pairs (A,B) and (S_D,π_D) follows and is described by the following diagram:



Let (A_i, B_i), $i = 1, 2$, be state feedback equivalent pairs and $D_i(z)$ the respective denominators in the coprime factorizations (6.53). Then, by Theorem 6.2, the feedback equivalence is expressed by the fact that $D_2(z)D_1(z)^{-1}$ is biproper. Since a controlled invariant subspace is an invariant subspace for a feedback equivalent pair, we obtain the following theorem.

Theorem 6.54. *Let $D(z) \in \mathbb{F}[z]^{m\times m}$ be nonsingular. Then a subspace $\mathscr{V} \subset X_D$ is a controlled invariant subspace, i.e., an (S_D, π_D)-invariant subspace, if and only if there exist necessarily nonsingular, polynomial matrices $E_1(z)$, $F_1(z) \in \mathbb{F}[z]^{m\times m}$ such that*

1. $D(z)D_1(z)^{-1}$ is biproper for

$$D_1(z) := E_1(z)F_1(z). \tag{6.54}$$

2. The subspace $\mathscr{V}$ has the representation

$$\mathscr{V} = \pi_D T_{DD_1^{-1}}(E_1 X_{F_1}). \tag{6.55}$$

Proof. Assume there exist $D_1(z), E_1(z)$ and $F_1(z) \in \mathbb{F}[z]^{m\times m}$ such that (1) and (2) are satisfied. Then (6.54) implies that $E_1 X_{F_1}$ is an S_{D_1}-invariant subspace of X_{D_1}. From (1) it follows that the pairs (S_D, π_D) and (S_{D_1}, π_{D_1}) are feedback equivalent pairs, with the Toeplitz induced map $\pi_D T_{DD_1^{-1}} : X_{D_1} \longrightarrow X_D$ satisfying

$$S_D \pi_D T_{DD_1^{-1}} - \pi_D T_{DD_1^{-1}} S_{D_1} = \pi_{DK} \tag{6.56}$$

for some $K : X_{D_1} \longrightarrow \mathbb{F}^m$. This implies that $\mathscr{V}$ is a controlled invariant subspace.

Conversely, assume that $\mathscr{V} \subset X_D$ is a controlled invariant subspace of X_D. By the definition of controlled invariant subspaces, $\mathscr{V}$ is an invariant subspace of a feedback equivalent pair. This pair can be taken, without loss of generality, to be (S_{D_1}, π_{D_1}). This implies that $D(z)D_1(z)^{-1}$ is biproper. The map from X_{D_1} to X_D that exhibits the feedback, i.e., that satisfies (6.56), is simply an induced Toeplitz map. Since S_{D_1}-invariant subspaces of X_{D_1} are of the form $E_1 X_{F_1}$ for some factorization (6.54), it follows that $\mathscr{V}$ has the required representation (6.55). ■

We next aim at a characterization of controlled invariant subspaces of X_D in terms of rational models.

Theorem 6.55. *Let $D(z)$ be an $m \times m$ nonsingular polynomial matrix. Let the pair (A, B) be defined by the shift realization in the state space X_D. Then a subspace $\mathscr{V} \subset X_D$ is controlled invariant if and only if*

$$\mathscr{V} = \pi_D \pi_+ D X^{F_1}$$

for some nonsingular polynomial matrix $F_1(z)$ for which all left Wiener–Hopf factorization indices of $D(z)F_1(z)^{-1}$ are nonnegative.

Proof. Assume such an $F_1(z)$ exists. This implies the existence of a nonsingular polynomial matrix E_1 for which $DF_1^{-1}E_1^{-1}$ is biproper. We define

$$D_1(z) = E_1(z)F_1(z). \tag{6.57}$$

By Proposition 3.50, the map $Y : X_{D_1} \longrightarrow X_D$, defined, for $f(z) \in X_{D_1}$, by $Yf = \pi_D T_{DD_1^{-1}} f$, is invertible. Now, the factorization (6.57) implies that $E_1 X_{F_1}$ is an S_{D_1}-invariant subspace, so its image under Y is a controlled invariant subspace of X_D. Computing

$$\begin{aligned}\mathscr{V} &= Y(E_1X_{F_1}) = \pi_D T_{DD_1^{-1}} E_1 X_{F_1} = D\pi_- D^{-1}\pi_+ DF_1^{-1}E_1^{-1}E_1X_{F_1}\\ &= D\pi_- D^{-1}\pi_+ DX^{F_1} = \pi_D\pi_+ DX^{F_1} = D\pi^D X^{F_1},\end{aligned}$$

it is a consequence of Theorem 3.49 that the map $Z : X^F \longrightarrow X_D$, defined, for $h(z) \in X^F$, by

$$Zh = \pi_D \pi_+ Dh,$$

is injective. This shows that $\operatorname{Im} Z = \pi_D\pi_+ DX^F$ is indeed controlled invariant for the pair (A,B) defined by the shift realization.

Conversely, assume that $\mathscr{V} \subset X_D$ is a controlled invariant subspace. Every pair that is feedback equivalent to the pair $(S_D, \pi_D(\cdot))$ can be assumed, up to similarity, to be of the form $(S_{D_1}, \pi_{D_1}(\cdot))$, with $D(z)D_1(z)^{-1}$ biproper. An invariant subspace of X_{D_1} is of the form $\mathscr{V}_1 = E_1X_{F_1}$ for a factorization $D_1(z) = E_1(z)F_1(z)$ into nonsingular factors. The biproperness assumption on $D(z)D_1(z)^{-1}$ implies that all left Wiener–Hopf factorization indices of $D(z)F_1(z)^{-1}$ are nonnegative. The Toeplitz induced map $\overline{T}_{DD_1^{-1}} : X_{D_1} \longrightarrow X_D$, defined for $f \in X_{D_1}$ by $\overline{T}_{DD_1^{-1}} f = \pi_D\pi_+ DD_1^{-1} f$, is therefore, by Theorem 3.49, invertible, and hence its restriction to $E_1X_{F_1}$ is injective. Moreover, it satisfies $S_D\overline{T}_{DD_1^{-1}} - \overline{T}_{DD_1^{-1}} S_{D_1} = \pi_D K$ for some feedback map $K : X_{D_1} \longrightarrow \mathbb{F}^m$. Indeed, for $f \in X_{D_1}$, setting $g_+ = \pi_+ DD_1^{-1} f$ and $g_+ = \pi_+ DD_1^{-1} f$, we compute

$$\begin{aligned}(S_D\overline{T}_{DD_1^{-1}} - \overline{T}_{DD_1^{-1}} S_{D_1})f &= \pi_D z\pi_D\pi_+ DD_1^{-1} f - \pi_D\pi_+ DD_1^{-1}\pi_{D_1} zf\\ &= \pi_D z\pi_+ DD_1^{-1} f - \pi_D\pi_+ DD_1^{-1} D_1\pi_- D_1^{-1} zf\\ &= \pi_D\left\{z\pi_+ DD_1^{-1} f - \pi_+ D\pi_- D_1^{-1} zf\right\}\\ &= \pi_D\left\{z\pi_+ DD_1^{-1} f - \pi_+ D(I-\pi_+)D_1^{-1} zf\right\}\\ &= \pi_D\left\{z\pi_+ DD_1^{-1} f - \pi_+ zDD_1^{-1} f\right\}\\ &= \pi_D\left\{zg_+ - \pi_+ z(g_+ + g_-)\right\} = -\pi_D\pi_+ zg_- = -\pi_D\xi,\end{aligned}$$

with $\xi = \pi_+ z g_- \in \mathbb{F}^m$. Thus

$$\mathscr{V} = T\mathscr{V}_1 = \pi_D \pi_+ D D_1^{-1}(E_1 X_{F_1}) = \pi_D \pi_+ D F_1^{-1} E_1^{-1}(E_1 X_{F_1}) = \pi_D \pi_+ D X^{F_1}.$$

∎

Using Theorem 3.32, i.e., the isomorphism between polynomial and rational models, one can restate Theorem 6.54 in terms of the rational functional model. The trivial proof of the next result is omitted.

Theorem 6.56. *Let $D(z)$ be a nonsingular polynomial matrix. With the (A,B) defined by the shift realization (4.26), a subspace $\mathscr{V} \subset X^D$ is a controlled invariant subspace if and only if there exist nonsingular polynomial matrices $E_1(z)$, $F_1(z) \in \mathbb{F}[z]^{m\times m}$ such that*

1. *$D(z)D_1(z)^{-1}$ is biproper for $D_1(z) = E_1(z)F_1(z)$;*
2. *$\mathscr{V} = \pi^D X^{F_1}$.*

Proposition 6.57. *Let $D(z) \in \mathbb{F}[z]^{m\times m}$ be nonsingular. Then a subspace $\mathscr{V} \subset X^D$ is controlled invariant with respect to the shift realization $\Sigma^{\bullet D^{-1}}$ if and only if it has a (not necessarily unique) representation of the form*

$$\mathscr{V} = \pi^D X^F$$

for some nonsingular polynomial matrix $F(z) \in \mathbb{F}[z]^{m\times m}$.

Proof. Using the isomorphism of the shift realizations $\Sigma_{\bullet D^{-1}}$ and $\Sigma^{\bullet D^{-1}}$, it follows from Theorem 6.54 that the controlled invariant subspace $\mathscr{V}$ has the representation

$$\begin{aligned}\mathscr{V} &= D^{-1}\pi_D \pi_+ D X^F = D^{-1} D \pi_- D^{-1} \pi_+ D X^F \\ &= \pi_- D^{-1} \pi_+ D X^F = \pi^D X^F.\end{aligned}$$

Conversely, assume $\mathscr{V} = \pi^{D_\ell} X^F$ for some nonsingular $F(z) \in \mathbb{F}[z]^{m\times m}$. To prove that $\mathscr{V}$ is controlled invariant, one must show that for each $h \in \mathscr{V}$ there exist $h' \in X^F$ and $\xi \in \mathbb{F}^m$ such that $S^D \pi^D h = \pi^D h' + \pi_- D^{-1}\xi$. This is done by choosing $h' = S_- h$ and $\xi = (Dh)_{-1}$. To this end, one computes

$$\begin{aligned}S^D \pi^D h - \pi^D S_- h &= \pi_- z \pi_- D^{-1} \pi_+ D h - \pi_- D^{-1} \pi_+ D \pi_- z h \\ &= \pi_- z D^{-1} \pi_+ D h - \pi_- D^{-1} \pi_+ D \pi_- z h \\ &= \pi_- D^{-1} \{z \pi_+ D h - \pi_+ z D h\} = \pi_- D^{-1} \xi.\end{aligned}$$

∎

Theorem 6.54 can be strengthened to yield a particularly clean representation of controlled invariant subspaces, a representation that has no direct reference to specific factorizations or to particular representations of submodules.

Theorem 6.58. *Let $D(z) \in \mathbb{F}[z]^{m\times m}$ be nonsingular. With respect to the realization (4.26) in the state space X^D, a subspace $\mathscr{V} \subset X^D$ is controlled invariant if and only if*

$$\mathscr{V} = \pi^D \mathscr{L} \tag{6.58}$$

for some submodule $\mathscr{L} \subset z^{-1}\mathbb{F}[[z^{-1}]]^m$.

Proof. In view of Theorem 6.54, all that needs to be proven is that the image under the projection π^D of a submodule of $\mathscr{L} \subset z^{-1}\mathbb{F}[[z^{-1}]]^m$ is a controlled invariant subspace of X^D. Equivalently, one must show that if $h \in \mathscr{L}$, then there exist $h_1 \in \mathscr{L}$ and $\xi \in \mathbb{F}^m$ such that

$$S^D \pi^{Dh} = \pi_D h_1 + \pi_- D^{-1}\xi. \tag{6.59}$$

We will prove (6.59), with $h_1 = S_- h$ and $\xi = (Dh)_{-1}$. In this case,

$$\begin{aligned} &S^D \pi^{Dh} - \pi_D S_- h = \pi_- z \pi_- D^{-1} \pi_+ Dh - \pi_- D^{-1} \pi_+ D \pi_- zh \\ &= \pi_- z D^{-1} \pi_+ D zh - \pi_- D^{-1} \pi_+ D zh \\ &= \pi_- D^{-1} z \pi_+ Dh - \pi_+ z Dh = \pi_- D^{-1}\xi. \end{aligned}$$

∎

For the rational model characterization of controlled invariant subspaces as in Theorem 6.58, the shift realization $\Sigma^{\bullet D^{-1}}$ was used, with $D(z)$ as a right denominator. However, when analyzing output nulling subspaces, it turns out to be more convenient to work with the polynomial model shift realization $\Sigma_{T^{-1}V}$. To state the relevant characterization, one first extends the definition of a polynomial model from square nonsingular polynomial matrices to rectangular polynomial matrices $U(z) \in \mathbb{F}[z]^{p\times m}$ as

$$X_U := U(z)(z^{-1}\mathbb{F}[[z^{-1}]]^m) \cap \mathbb{F}[z]^p. \tag{6.60}$$

We refer to this space as the **rectangular polynomial model**. It is emphasized that X_U is certainly an $\mathbb{F}$-vector space of polynomials; however, unless U is nonsingular square, it is not a module over $\mathbb{F}[z]$.

Proposition 6.59. *Let $G(z)$ be a $p \times m$ proper rational matrix function with matrix fraction representation $G(z) = T(z)^{-1}U(z)$.*

1. *Assume $U(z) = E_1(z)U_1(z)$, with $E_1(z) \in \mathbb{F}[z]^{p\times p}$ a nonsingular polynomial matrix. Then*

$$\mathscr{V} = E_1 X_{U_1}$$

is an output nulling subspace of the shift realization $\Sigma_{T^{-1}U}$ in the state space X_T, and the following inclusions are valid:

$$E_1 X_{U_1} \subset X_U \subset X_T.$$

2. *A subspace $\mathscr{V} \subset X_T$ is output nulling if and only if $\mathscr{V} = E_1 X_{U_1}$, with $U(z) = E_1(z)U_1(z)$ and $E_1(z) \in \mathbb{F}[z]^{p\times p}$ nonsingular.*

Proof. Let $f \in E_1 X_{U_1}$, that is, $f = E_1 g$, with $g = U_1 h \in \mathbb{F}[z]^p$ for a strictly proper power series h. Since $T^{-1}U$ is proper and h is strictly proper, it follows that $T^{-1}f = (T^{-1}U)h$ is strictly proper. Thus $f \in X_T$, which shows $E_1 X_{U_1} \subset X_T$. Next we show that if h is strictly proper with $g = U_1 h \in \mathbb{F}[z]^p$, then also $U_1(S_- h) \in \mathbb{F}[z]^p$. Denoting by $\eta = h_{-1} \in \mathbb{F}^m$ the residue term of h, the equality $zh(z) = \eta + S_- h$ follows. This implies

$$U_1(S_- h) = U_1(zh) - U_1\eta = z(U_1 h) - U_1\eta = zg - U_1\eta \in \mathbb{F}[z]^p.$$

To show that $\mathscr{V} = E_1 X_{U_1}$ is controlled invariant, let (A,B,C,D) denote the shift realization $\Sigma_{T^{-1}U}$. For $f \in E_1 X_{U_1}$, i.e., $f = E_1 g$ and $g = U_1 h$, one computes

$$\begin{aligned} S_T f &= \pi_T(zf) = \pi_T(zE_1U_1h) = \pi_T(E_1U_1zh) = \pi_T(E_1U_1(\eta + S_- h)) \\ &= \pi_T(U\eta) + \pi_T(E_1U_1S_- h) = \pi_T(U\eta) + E_1U_1S_- h. \end{aligned}$$

Since we assume $T^{-1}U$ to be proper, there exists a representation $U(z) = T(z)D + V(z)$, with $T^{-1}V$ strictly proper. Hence,

$$\pi_T(U\eta) = \pi_T(TD + V)\eta = V\eta,$$

and therefore

$$Af = S_T f = E_1U_1(S_- h) + V\eta.$$

As $E_1U_1S_- h \in E_1 X_{U_1}$ and $V\eta \in \operatorname{Im} B$, this proves that $\mathscr{V}$ is controlled invariant. Next, compute

$$\begin{aligned} Cf &= (T^{-1}f)_{-1} = (T^{-1}Uh)_{-1} = (T^{-1}(TD + V)h)_{-1} \\ &= (Dh)_{-1} = Dh_{-1} = D\eta. \end{aligned}$$

Since η depends linearly on f, there exists a linear transformation K such that $\eta = Kf$. Thus $(A - BK)f \in \mathscr{V}$ and $(C - DK)f = 0$. This completes the proof of (1). The proof of the second claim is omitted. ■

The preceding result leads us to prove the following polynomial characterization of the maximal output nulling subspace.

Theorem 6.60. *With respect to the shift realization $\Sigma_{T^{-1}U}$ in the state space X_T, the maximal output nulling subspace is given as*

$$\mathscr{V}^* = X_U.$$

Proof. By Proposition 6.59 (1), it follows that X_U is output nulling. Moreover, by part (2) of the same proposition, each output nulling subspace $\mathscr{V}$ of X_T is of the form $\mathscr{V} = E_1 X_{U_1}$, with $U = E_1 U_1$ and E_1 nonsingular. Using Proposition 6.59 (1) once more, we obtain $\mathscr{V} \subset X_U$, which implies that X_U is the maximal output nulling subspace. ∎

A purely module-theoretic characterization of $\mathscr{R}^*$ is presented without proof.

Theorem 6.61. *Let $G(z)$ be a $p \times m$ proper rational matrix function with the left coprime matrix fraction representation $G(z) = T(z)^{-1}U(z)$. Then, with respect to the shift realization on X_T, the maximal output nulling reachability subspace is given by*

$$\mathscr{R}^* = X_U \cap U\mathbb{F}[z]^m.$$

6.9 Conditioned Invariant Subspaces

We begin by introducing basic concepts from geometric control theory that are relevant to observer design, i.e., conditioned invariant subspaces and related subspaces. The theory of such subspaces is dual to that of controlled invariant subspaces and thus can be developed in parallel.

Definition 6.62. 1. For an output pair (C,A), a subspace $\mathscr{V} \subset \mathscr{X}$ is called **conditioned invariant** if

$$A(\mathscr{V} \cap \operatorname{Ker} C) \subset \mathscr{V}.$$

2. For a conditioned invariant subspace $\mathscr{V}$ of the pair (C,A), an output injection map J for which $(A - JC)\mathscr{V} \subset \mathscr{V}$ is called a **friend** of $\mathscr{V}$. Denote by $\mathscr{G}(\mathscr{V})$ the set of all friends of $\mathscr{V}$.
3. For a pair (C,A), a conditioned invariant subspace $\mathscr{V} \subset \mathscr{X}$ is called **tight** if it satisfies

$$\mathscr{V} + \operatorname{Ker} C = \mathscr{X}.$$

4. A set of conditioned invariant subspaces $\mathscr{V}_\alpha$ is **compatible** if $\bigcap_\alpha \mathscr{F}(\mathscr{V}_\alpha) \neq \emptyset$.

5. A conditioned invariant subspace $\mathscr{V}$ will be called an **observability subspace** if, for each monic polynomial $q(z)$ of degree equal to codim $\mathscr{V}$, there exists a friend $J \in \mathscr{G}(\mathscr{V})$ such that $q(z)$ is the characteristic polynomial of $(A-JC)|_{\mathscr{X}/\mathscr{V}}$, the map induced on the quotient space $\mathscr{X}/\mathscr{V}$ by $A-JC$.

Some elementary properties of conditioned invariant subspaces are stated next.

Proposition 6.63. *Let (C,A) be an output pair acting in the state space $\mathscr{X}$. Then:*

1. *The set of conditioned invariant subspaces is closed under intersections;*
2. *For every subspace $\mathscr{L} \subset \mathscr{X}$, there exists a minimal conditioned invariant subspace containing $\mathscr{L}$; this subspace is denoted by $\mathscr{V}_*(\mathscr{L})$;*
3. *$\mathscr{V}$ is a conditioned invariant subspace if and only if there exists an output injection map $J : \mathbb{F}^p \longrightarrow \mathscr{X}$ such that $\mathscr{V}$ is $(A-JC)-$ invariant.*

Proof. The set of all conditioned invariant subspaces containing $\mathscr{L}$ is closed under intersections and is nonempty because $\mathscr{X}$ is included. $\mathscr{V}_*(\mathscr{L})$ is the intersection of all these subspaces. The proof of the last claim runs parallel to that of Proposition 6.48. Explicitly, choose a basis $\{v_1,\dots,v_r\}$ of $\mathscr{V}\cap \operatorname{Ker} C$, and extend it to a basis $\{v_1,\dots,v_q\}$ of $\mathscr{V}$. Then $Cv_{r+1},\dots,Cv_q \subset \mathbb{F}^p$ are linearly independent, with $q-r\le p$. Thus there exists a linear transformation $J:\mathbb{F}^p \longrightarrow \mathscr{X}$ that maps $Cv_{r+1},\dots,Cv_q$ to $Av_{r+1},\dots,Av_q \in \mathscr{V}$. Thus $(A-JC)v_i = Av_i$ for $1\le i\le r$ and $(A-JC)v_i = 0$ for $r+1\le i\le q$. Thus $(A-JC)\mathscr{V}\subset\mathscr{V}$. The converse is obvious. ∎

It may be instructive to see how the last claim can also be deduced from Proposition 6.48 by reasons of duality. In fact, assume for simplicity that $\mathscr{X}=\mathbb{F}^n$ and, thus, $A\in\mathbb{F}^{n\times n}, B\in\mathbb{F}^{n\times m}$ and $C\in\mathbb{F}^{p\times n}$. Consider a linear subspace $\mathscr{V}\subset\mathbb{F}^n$. Let $\mathscr{V}^\perp\subset\mathbb{F}^n$ denote the orthogonal complement with respect to the standard nondegenerate bilinear form $<x,y>=x^\top y$ on $\mathbb{F}^n$. Then

$$A\mathscr{V}\subset\mathscr{V}+\operatorname{Im} B$$

is satisfied if and only if

$$\mathscr{V}^\perp\cap\operatorname{Ker} B^\top=\mathscr{V}^\perp\cap(\operatorname{Im} B)^\perp=(\mathscr{V}+\operatorname{Im} B)^\perp\subset(A\mathscr{V})^\perp=(A^\top)^{-1}(\mathscr{V}^\perp),$$

i.e., if and only if

$$A^\top(\mathscr{V}^\perp\cap\operatorname{Ker} B^\top)\subset\mathscr{V}^\perp.$$

Thus $\mathscr{V}$ is (A,B)-invariant if and only if $\mathscr{V}^\perp$ is $(B^\top,A^\top)$-invariant. In this way, most results on controlled invariant subspaces for a linear system (A,B,C) can be equivalently reformulated as results on conditioned invariant subspaces for the dual system $(A^\top,C^\top,B^\top)$.

The dual notions to inner and outer stabilizable controlled invariant subspaces are introduced next.

Definition 6.64. Let (A,B,C,D) be a state-space system acting in the state space $\mathscr{X}$. A subspace $\mathscr{V} \subset \mathscr{X}$ is called **inner detectable** if there exists a friend $J \in \mathscr{G}(\mathscr{V})$ such that $(A-JC)|_{\mathscr{V}}$ is stable. A subspace $\mathscr{V}$ is **outer detectable** if there exists a $J \in \mathscr{G}(\mathscr{V})$ such that $(A-JC)|_{\mathscr{X}/\mathscr{V}}$ is stable. Again, the concepts of **inner and outer antidetectability** are naturally defined.

Let $\mathscr{V}$ be a conditioned invariant subspace for the observable pair (C,A), and let $J \in \mathscr{G}(\mathscr{V})$; then $(C, A-JC)$ has a restriction to $\mathscr{V}$, which is observable. Thus, if $\mathscr{V}$ is conditioned invariant, then it is both inner detectable and antidetectable. By standard duality considerations, one expects the notions of inner stabilizability to be related to outer detectability, and this indeed is the case.

1. Polynomial Characterizations of Conditioned Invariant Subspaces
We next derive polynomial characterizations of conditioned invariant subspaces.

Theorem 6.65. *Let $T(z) \in \mathbb{F}[z]^{p\times p}$ be nonsingular, and let (C,A) be the associated observable pair obtained via the shift realization $\Sigma_{T^{-1}\bullet}$. Then a subspace $\mathscr{V} \subset X_T$ is a conditioned invariant subspace, i.e., a $((T\cdot)_{-1}, S_T)$-invariant subspace, if and only if there exist nonsingular polynomial matrices $E_1(z)$, $F_1(z) \in \mathbb{F}[z]^{p\times p}$ such that:*

1. *$T_1(z)^{-1}T(z)$ is normalized biproper for*

$$T_1(z) = E_1(z)F_1(z); \tag{6.61}$$

2. *In terms of the factorization (6.61), the representation*

$$\mathscr{V} = E_1 X_{F_1}$$

is obtained.

Proof. Two proofs of this theorem are given.

Proof I:
$\mathscr{V}$ is a conditioned invariant subspace if and only if it is invariant for $A_1 = A - JC$. By Theorem 6.5, if the pair (C,A) is associated with the matrix fraction $T(z)^{-1}U(z)$, then the pair (C,A_1) is associated with the matrix fraction $T_1(z)^{-1}U(z)$, where $T_1(z)^{-1}T(z)$ is biproper. Since X_T and X_{T_1} are equal as sets, $\mathscr{V}$ is an S_{T_1}-invariant subspace of X_{T_1}. These subspaces are, by Theorem 3.11, of the form $\mathscr{V} = E_1 X_{F_1}$ with $T_1(z) = E_1(z)F_1(z)$.

Proof II:
In this proof, duality and the characterization of controlled invariant subspaces given in Theorem 6.54 will be used. The subspace $\mathscr{V} \subset X_T$ is conditioned invariant if and only if $\mathscr{V}^\perp \subset X_{T^\top}$ is controlled invariant, i.e., an $(S_{T^\top}, \pi_{T^\top})$-invariant subspace. By Theorem 6.54, there exists a polynomial matrix $T_1(z) \in \mathbb{F}[z]^{p\times p}$ such that $T^\top(z)T_1^\top(z)^{-1}$ is biproper and

$$\mathscr{V}^{\perp} = \pi_{T^\top} T_{T^\top T_1^{-\top}} (F_1^\top X_{E_1^\top}),$$

where $T_1(z) = E_1(z)F_1(z)$, and hence also $T_1^\top(z) = F_1^\top(z)E_1^\top(z)$. By the elementary properties of dual maps, $(\pi_{T^\top} T_{T^\top T_1^{-\top}})^* \mathscr{V} = \mathscr{V}_1 \subset X_{T_1}$ and $\mathscr{V}_1^{\perp} = F_1^\top X_{E_1^\top}$. Applying Theorem 3.11, one obtains $\mathscr{V}_1 = E_1 X_{F_1}$, and since $(\pi_{T^\top} T_{T^\top T_1^{-\top}})^* : X_T \longrightarrow X_{T_1}$ acts as the identity map, it follows that $\mathscr{V} = E_1 X_{F_1}$. ■

In view of Theorems 6.56 and 6.65, it is of considerable interest to characterize the factorizations appearing in these theorems. The key to this are Wiener–Hopf factorizations at infinity.

Proposition 6.66. *1. Let $D(z), F_1(z) \in \mathbb{F}[z]^{m\times m}$ be nonsingular. Then there exist $E_1(z) \in \mathbb{F}[z]^{m\times m}$, and $D_1(z) := E_1(z)F_1(z)$ such that $D(z)D_1(z)^{-1}$ is biproper if and only if all the left Wiener–Hopf factorization indices at infinity of $D(z)F_1(z)^{-1}$ are nonnegative.*
2. Let $T(z), E_1(z) \in F[z]^{p\times p}$ be nonsingular. Then there exist polynomial matrices $F_1(z) \in \mathbb{F}[z]^{p\times p}$, and $T_1(z) := E_1(z)F_1(z)$ such that $T_1(z)^{-1}T(z)$ is biproper if and only if the right Wiener–Hopf factorization indices at infinity of $E_1(z)^{-1}T(z)$ are nonnegative.

Proof. Define $D_1(z) = E_1(z)F_1(z)$. If $\Gamma(z) = D(z)D_1(z)^{-1}$ is biproper, then $D(z)F_1(z)^{-1} = \Gamma(z)E_1(z)$. Now let $E_1(z) = \Omega(z)\Delta(z)U(z)$ be a left Wiener–Hopf factorization of $E_1(z)$. Then necessarily the factorization indices of $E_1(z)$ are nonnegative, being the reachability indices of the input pair (S_{E_1}, π_{E_1}). It follows that

$$D(z)F_1(z)^{-1} = (\Gamma(z)\Omega(z))\Delta(z)U(z), \tag{6.62}$$

i.e., $D(z)F_1(z)^{-1}$ has nonnegative left factorization indices. Conversely, if (6.62) holds with $\Delta(z) = \mathrm{diag}\,(z^{\kappa_1}, \dots, z^{\kappa_m})$, and $\kappa_1 \geq \cdots \geq \kappa_m \geq 0$, then, defining $E_1(z) = \Delta(z)U(z)$, it follows that $D(z) = \Omega(z)D_1(z)$, with $D_1(z) = E_1(z)F_1(z)$ and $\Omega(z)$ biproper.

The proof of the second claim follows the lines of the proof of part 1 or can be derived from that theorem by duality. ■

The characterizations appearing in Theorems 6.54 and 6.65 are factorization-dependent. The following proposition makes this unnecessary.

Proposition 6.67. *Let $D_\ell(z) \in \mathbb{F}[z]^{p\times p}$ be nonsingular. A subspace $\mathscr{V} \subset X_{D_\ell}$ is conditioned invariant with respect to the shift realization $\Sigma_{D_\ell^{-1}\bullet}$ if and only if it has a (not necessarily unique) representation of the form*

$$\mathscr{V} = Ker\, \pi_T | X_{D_\ell} = X_{D_\ell} \cap T(z)\mathbb{F}[z]^p,$$

where $T(z) \in \mathbb{F}[z]^{p\times p}$ is a nonsingular polynomial matrix.

Proof. By Theorem 6.65, there exists a representation $\mathscr{V} = E_1X_{F_1}$, with $D_\ell^{-1}E_1F_1$ biproper. Assume $f \in \mathscr{V} = E_1X_{F_1}$; then $f \in X_{D_\ell}$ and $f = E_1g$, so $\mathscr{V} \subset X_{D_\ell} \cap E_1\mathbb{F}[z]^p$.

Conversely, if $f \in X_{D_\ell} \cap E_1\mathbb{F}[z]^p$, then $f = X_{D_\ell}$ and $f = E_1g$. Let $T_1 := E_1F_1$; then $T_1^{-1}D_\ell$ is biproper. Applying these facts, it follows that $D_\ell^{-1}f$ is strictly proper, and the biproperness of $T_1^{-1}D_\ell$ implies that also $T_1^{-1}f = T_1^{-1}E_1g = F_1^{-1}E_1^{-1}E_1g = F_1^{-1}g$ is strictly proper. But this shows that $g \in X_{F_1}$, and the inclusion $X_{D_\ell} \cap E_1\mathbb{F}[z]^p \subset \mathscr{V}$ is proved. ∎

Theorems 6.54 and 6.65 can be strengthened to yield a particularly clean representation of controlled and conditioned invariant subspaces that has no direct reference to specific factorizations or to particular representations of submodules.

Theorem 6.68. *Let $D_\ell(z) \in \mathbb{F}[z]^{p\times p}$ be nonsingular. With respect to the realization (4.24) in the state space X_{D_ℓ}, a subspace $\mathscr{V} \subset X_{D_\ell}$ is conditioned invariant if and only if*

$$\mathscr{V} = X_{D_\ell} \cap \mathscr{M} \tag{6.63}$$

for some submodule $\mathscr{M} \subset \mathbb{F}[z]^p$.

Proof. The *only if* part was proved in Theorem 6.65. To prove the *if* part, assume $\mathscr{V} = X_{D_\ell} \cap \mathscr{M}$, where $\mathscr{M}$ is a submodule of $\mathbb{F}[z]^p$. If $f \in \mathscr{V} \cap \operatorname{Ker} C$, then $(D_\ell^{-1}f)_{-1} = 0$, which implies that $S_{D_\ell}f = \pi_{D_\ell}zf = zf$. But $S_{D_\ell}f \in X_{D_\ell}$ and $S_{D_\ell}f = zf \in \mathscr{M}$. Therefore, $S_{D_\ell}f \in X_{D_\ell} \cap \mathscr{M}$ follows, which shows that $\mathscr{V}$ is a conditioned invariant subspace, thus proving the theorem. ∎

The availability of the representation (6.63) of conditioned invariant subspaces allows us to give different proofs of the closure of the sets of controlled/conditioned invariant subspaces under sums/intersections, respectively.

Proposition 6.69. *1. Let $D(z) \in \mathbb{F}[z]^{m\times m}$ be nonsingular. Let $\mathscr{V}_i$ be controlled invariant subspaces of X^D, with respect to the shift realization $\Sigma^{\bullet D^{-1}}$, and having the representations $\mathscr{V}_i = \pi^D\mathscr{L}_i$ for submodules $\mathscr{L}_i \subset z^{-1}\mathbb{F}[[z^{-1}]]^m$. Then $\mathscr{V} = \sum_i \mathscr{V}_i$ is controlled invariant with the representation $\mathscr{V} = \pi^D\sum_i \mathscr{L}_i$.*

2. Let $D_\ell(z) \in \mathbb{F}[z]^{p\times p}$ be nonsingular. Let $\mathscr{V}_i$ be conditioned invariant subspaces of X_{D_ℓ}, with respect to the shift realization $\Sigma_{D_\ell^{-1}\bullet}$, and having the representations $\mathscr{V}_i = X_{D_\ell} \cap \mathscr{M}_i$ for submodules $\mathscr{M}_i \subset \mathbb{F}[z]^p$. Then $\mathscr{V} = \cap_i\mathscr{V}_i$ is conditioned invariant and has the representation $\mathscr{V} = X_{D_\ell} \cap (\cap_i\mathscr{M}_i)$.

Proof. The proof is obvious. ∎

The representation formula (6.63) is at the heart of the analysis of conditioned invariant subspaces and opens up several interesting questions, some of which will be described later. Note first that, for unimodular polynomial matrices $U(z)$, one has $T\mathbb{F}[z]^p = TU\mathbb{F}[z]^p$ and $X^{UT} = X^T$. Thus it is not important to distinguish between representing polynomial matrices up to an appropriate, one-sided unimodular factor. The representations of controlled and conditioned invariant subspaces that appear in Theorem 6.65 have the advantage of using nonsingular polynomial matrices in the representations. The disadvantage is the nonuniqueness of the representing polynomial matrices. However, uniqueness modulo unimodular factors can be recovered by switching to the use of rectangular polynomial matrices. This is done next.

As noted already, the submodule $\mathscr{M}$ in a representation of the form $\mathscr{V} = X_{D_\ell} \cap \mathscr{M}$ is, in general, not unique. To get a unique representation, one needs to associate with a conditioned invariant subspace of X_{D_ℓ} a unique submodule, and none is more natural than the submodule of $\mathbb{F}[z]^p$ generated by $\mathscr{V}$.

Proposition 6.70. *Let $D_\ell(z) \in \mathbb{F}[z]^{p\times p}$ be nonsingular, and let $\mathscr{V} \subset X_{D_\ell}$ be a conditioned invariant subspace. Let $< \mathscr{V} >$ be the submodule of $\mathbb{F}[z]^p$ generated by $\mathscr{V}$, which is the smallest submodule of $\mathbb{F}[z]^p$ that contains $\mathscr{V}$. Then*

$$\mathscr{V} = X_{D_\ell} \cap < \mathscr{V} > .$$

Proof. Assume $\mathscr{V} = X_{D_\ell} \cap \mathscr{M}$ for some submodule of $\mathbb{F}[z]^p$. Clearly, $\mathscr{V} \subset \mathscr{M}$, and hence $< \mathscr{V} > \subset \mathscr{M}$, and so $\mathscr{V} \subset < \mathscr{V} > \subset \mathscr{M}$, which in turn implies

$$\mathscr{V} \quad \subset \quad X_{D_\ell} \cap < \mathscr{V} > \quad \subset \quad X_{D_\ell} \cap \mathscr{M} = \mathscr{V}.$$

∎

Corollary 6.71. *For each subset $E \subset X_{D_\ell}$, the intersection $X_{D_\ell} \cap < E >$ is the smallest conditioned invariant subspace of X_{D_ℓ} that contains E.*

Proof. $X_{D_\ell} \cap < E >$ is a conditioned invariant subspace and contains E. Let $\mathscr{W}$ be another conditioned invariant subspace containing E. Then $< E > \subset < \mathscr{W} >$, and hence

$$X_{D_\ell} \cap < E > \quad \subset \quad X_{D_\ell} \cap < \mathscr{W} > = \mathscr{W}.$$

∎

Finally, we arrive at a very useful characterization of conditioned invariant subspaces.

Theorem 6.72. *A subspace $\mathscr{V} \subset X_{D_\ell}$ is a conditioned invariant subspace if and only if it has a representation of the form*

$$\mathscr{V} = X_{D_\ell} \cap H(z)\mathbb{F}[z]^k,$$

where $H(z)$ is a full column rank $p \times k$ polynomial matrix whose columns are in $\mathscr{V}$. $H(z)$ is uniquely determined up to a right $k \times k$ unimodular factor.

Proof. Follows from Theorem 6.68 and the basis representation of submodules of $\mathbb{F}[z]^p$ by full column rank polynomial matrices. ∎

6.10 Zeros and Geometric Control

In this section we clarify the connection between the analysis of zeros based on module theory and that based on geometric control concepts. Recalling the definition of the zero module given in (4.43), we proceed with the following computational result. For any rectangular polynomial matrices we define the **rectangular polynomial model** X_U as

$$X_U := U(z)(z^{-1}\mathbb{F}[[z^{-1}]]^m) \cap \mathbb{F}[z]^m \tag{6.64}$$

and the **rectangular rational model** X^U as

$$X^U := \{h \in z^{-1}\mathbb{F}[[z^{-1}]]^m \mid \pi_-(Uh) = 0\}. \tag{6.65}$$

In particular, both identities $X^U = \operatorname{Ker} U(\sigma)$ and $UX^U = X_U$ are satisfied.

Proposition 6.73. *Let $G(z)$ be a strictly proper, $p \times m$ transfer function, with the left coprime factorization*

$$G(z) = T(z)^{-1}U(z). \tag{6.66}$$

Then:

1. *Viewed as linear multiplication maps from $\mathbb{F}(z)^m$ to $\mathbb{F}(z)^p$,*

$$Ker\, G = Ker\, U;$$

2.

$$\pi_- G^{-1}(\mathbb{F}[z]^p) = X^U = Ker\, U(\sigma) \tag{6.67}$$

and

$$U\pi_- G^{-1}(\mathbb{F}[z]^p) = X_U, \tag{6.68}$$

where X_U and X^U are defined by (6.64) and (6.65), respectively;

3.

$$U\pi_- Ker\, G = X_U \cap U\mathbb{F}[z]^m. \tag{6.69}$$

Proof. 1. Obvious.

2. Assume $h \in G^{-1}(\mathbb{F}[z]^p)$, i.e., $g = T^{-1}Uh \in \mathbb{F}[z]^p$. Defining $h_\pm = \pi_\pm h$, this implies $Tg = Uh = Uh_- + Uh_+$, or $Uh_- = Tg - Uh_+ \in \mathbb{F}[z]^p$ and, in turn, $h_- \in \operatorname{Ker} U(\sigma) = X^U$.

Conversely, if $h_- \in \operatorname{Ker} U(\sigma)$, then $Uh_- \in \mathbb{F}[z]^p$. By the left coprimeness of $T(z), U(z)$, there exist polynomial vectors $g(z), h_+(z)$ for which $Uh_- = Tg - Uh_+$. Therefore, with $h = h_- + h_+$ we have $g = T^{-1}Uh$, i.e., $h \in G^{-1}(\mathbb{F}[z]^p)$. From the equality $Uh_- = Tg - Uh_+$ it follows that $\pi_- Uh_- = 0$, i.e., (6.67) is proven.

3. Clearly, $G^{-1}(\{0\}) = \operatorname{Ker} G = \operatorname{Ker} U$. For $h = h_- + h_+ \in \operatorname{Ker} U$, we have $Uh_- = -Uh_+$, which implies $U(\sigma)h_- = \pi_- Uh_- = 0$, that is, $h_- \in X^U$ as well as $Uh_- = -Uh_+ \in U\mathbb{F}[z]^m$, and the inclusion $U\pi_- \operatorname{Ker} G \subset X_U \cap U\mathbb{F}[z]^m$ follows.

   Conversely, if $Uh \in X_U \cap U\mathbb{F}[z]^m$, then there exist $h_+ \in \mathbb{F}[z]^m$ and $h_- \in z^{-1}\mathbb{F}[[z^{-1}]]^m$ for which $Uh = Uh_- = -Uh_+$. From this it follows that $\pi_-(h_- + h_+) = h_-$ and $U(h_- + h_+) = 0$, or $(h_- + h_+) \in \operatorname{Ker} G$. This implies the inclusion $X_U \cap U\mathbb{F}[z]^m \subset U\pi_- \operatorname{Ker} G$. The two inclusions imply (6.69).

■

Following Wyman, Sain, Conte and Perdon (1989), we define

$$Z_\Gamma(G) = \frac{G(z)^{-1}(\mathbb{F}[z]^p)}{G(z)^{-1}(\mathbb{F}[z]^p) \cap \mathbb{F}[z]^m}, \tag{6.70}$$

$$Z_0(G) = \frac{\operatorname{Ker} G}{\operatorname{Ker} G \cap \mathbb{F}[z]^m}, \tag{6.71}$$

and recall the definition of the zero module, given in (4.43), namely,

$$Z(G) = \frac{G(z)^{-1}\mathbb{F}[z]^p + \mathbb{F}[z]^m}{\operatorname{Ker} G(z) + \mathbb{F}[z]^m}.$$

Theorem 6.74. *Let $G(z)$ be a (strictly) proper, $p \times m$ transfer function, with the left coprime factorization (6.66). Then:*

$$Z_\Gamma(G) \simeq \pi_- G^{-1}(\mathbb{F}[z]^p) = X^U, \tag{6.72}$$

$$Z_0(G) \simeq \pi_- Ker G. \tag{6.73}$$

For the zero module, defined by (4.43), the following isomorphism is true:

$$Z(G) \simeq \frac{X_U}{X_U \cap U\mathbb{F}[z]^m} = \frac{\mathscr{V}^*}{\mathscr{R}^*}. \tag{6.74}$$

Here $\mathscr{V}^*$ and $\mathscr{R}^*$ are defined by Theorems 6.60 and 6.61, respectively.

Proof. For the proof we will use the following standard module isomorphisms Lang (1965). Assuming M, N, M_i are submodules of a module X over a commutative ring R,

$$\frac{M+N}{N} \simeq \frac{M}{M \cap N}$$

and, assuming additionally the inclusions $M_0 \subset M_1 \subset M_2$,

$$\frac{M_2}{M_1} \simeq \frac{M_2}{M_0} \Big/ \frac{M_2}{M_1}.$$

Using these, we have the isomorphisms

$$Z_\Gamma(G) = \frac{G(z)^{-1}(\mathbb{F}[z]^p}{G(z)^{-1}(\mathbb{F}[z]^p) \,\cap\, \mathbb{F}[z]^m} \simeq \frac{G(z)^{-1}(\mathbb{F}[z]^p) + \mathbb{F}[z]^m}{\mathbb{F}[z]^m}$$

and

$$Z_0(G) = \frac{\operatorname{Ker} G}{\operatorname{Ker} G \,\cap\, \mathbb{F}[z]^m} \simeq \frac{\operatorname{Ker} G + \mathbb{F}[z]^m}{\mathbb{F}[z]^m}.$$

Clearly, the inclusion $\operatorname{Ker} G \subset G(z)^{-1}(\mathbb{F}[z]^p)$ implies the inclusions $\mathbb{F}[z]^m \subset \operatorname{Ker} G + \mathbb{F}[z]^m \subset G(z)^{-1}(\mathbb{F}[z]^p) + \mathbb{F}[z]^m$. Again, we obtain the isomorphism

$$Z(G) = \frac{G(z)^{-1}(\mathbb{F}[z]^p) + \mathbb{F}[z]^m}{\operatorname{Ker} G(z) \,+\, \mathbb{F}[z]^m} \simeq \frac{G(z)^{-1}(\mathbb{F}[z]^p) + \mathbb{F}[z]^m}{\mathbb{F}[z]^m} \Big/ \frac{\operatorname{Ker} G + \mathbb{F}[z]^m}{\mathbb{F}[z]^m}. \tag{6.75}$$

Note that (6.68) implies $U\pi_-(G(z)^{-1}(\mathbb{F}[z]^p) + \mathbb{F}[z]^m) = X_U$, and, similarly, (6.69) implies $U\pi_-(\operatorname{Ker} G + \mathbb{F}[z]^m) = X_U \cap U\mathbb{F}[z]^m$. Furthermore,

$$\operatorname{Ker} U\pi_- |(G(z)^{-1}(\mathbb{F}[z]^p) + \mathbb{F}[z]^m) = \operatorname{Ker} G + \mathbb{F}[z]^m,$$

and hence the isomorphism (6.74) follows. ■

The isomorphism (6.74) shows that the zero module is directly related to the transmission zeros Morse (1973). The modules $Z_\Gamma(G)$ and $Z_0(G)$ also have system-theoretic interpretations, but this is beyond the scope of the present monograph.

6.11 Exercises

1. Let $R(A,B) = (B, AB, \ldots, A^{n-1}B)$ denote the reachability matrix of $(A,B) \in \mathbb{F}^{n \times (n+m)}$. Prove that for a feedback $K \in \mathbb{F}^{m \times n}$ there exists an upper triangular block matrix $U \in \mathbb{F}^{nm \times nm}$ with diagonal blocks $U_{11} = \cdots = U_{nn} = I_m$ and

$$R(A+BK, B) = R(A,B)U.$$

2. Prove that the reachability indices of a pair (A,B) coincide with the Kronecker indices, arranged in decreasing order.
3. Assume that the reachable pair $(A,b) \in \mathbb{F}^{n\times n} \times \mathbb{F}^n$ is in Jordan canonical form,

$$A = \begin{pmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{pmatrix}, \quad A = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix},$$

with eigenvalues $\lambda_1,\dots,\lambda_n \in \mathbb{F}$. Let $\mu_1,\dots,\mu_n \in \mathbb{F}$. Prove the formula by Mayne-Murdoch, i.e., that the feedback gain $K = (k_1,\dots,k_n) \in \mathbb{F}^{1\times n}$,

$$k_i = \frac{\prod_j(\lambda_i - \mu_j)}{\prod_j(\lambda_i - \lambda_j)}, \quad i = 1,\dots,n,$$

satisfies

$$\det(zI - A + bK) = \prod_{j=1}^{n}(z - \mu_j).$$

4. (a) Let the pair $(A,B) \in \mathbb{F}^{n\times(n+m)}$ be reachable. For a monic polynomial $f(z) \in \mathbb{F}[z]$ of degree $m+n$, show the existence of matrices $X \in \mathbb{F}^{m\times n}$ and $Y \in \mathbb{F}^{m\times n}$ such that $f(z)$ is the characteristic polynomial of

$$M = \begin{pmatrix} A & B \\ X & Y \end{pmatrix}.$$

(b) Let

$$A = \begin{pmatrix} 1&0&0&0 \\ 0&0&1&0 \\ 0&0&0&0 \\ 1&0&0&0 \end{pmatrix}, \quad B = \begin{pmatrix} 1&0 \\ 1&0 \\ 0&1 \\ 0&0 \end{pmatrix}.$$

Determine the matrices X and Y such that the matrix M is nilpotent.
5. Let $\kappa = (\kappa_1,\dots,\kappa_m)$ denote a partition of n. Prove that the set of pairs $(A,B) \in \mathbb{F}^{n\times n} \times \mathbb{F}^{n\times m}$ whose reachability indices $\lambda = (\lambda_1,\dots,\lambda_m)$ satisfy $\sum_{j=1}^{r} \kappa_j \leq \sum_{j=1}^{r} \lambda_j$, $r = 1,\dots,m$, forms a Zariski-closed subset of $\mathbb{F}^{n\times n} \times \mathbb{F}^{n\times m}$. Let $n = km + \ell$ with $0 \leq \ell < m$. Deduce that the set of pairs $(A,B) \in \mathbb{F}^{n\times n} \times \mathbb{F}^{n\times m}$ with reachability indices $\kappa = (k+1,\dots,k+1,k,\dots,k)$ is a nonempty Zariski-open subset in $\mathbb{F}^{n\times n} \times \mathbb{F}^{n\times m}$.

6. Prove that a linear subspace $\mathscr{V} \subset \mathbb{F}^n$ is simultaneously (A,B)- and (C,A)-invariant if and only if there exists an output feedback gain $K \in \mathbb{F}^{m\times p}$ that satisfies $(A+BKC)\mathscr{V} \subset \mathscr{V}$.
7. Let (A,B,C) be reachable and observable. Then the following conditions are equivalent:

   (a) (A,B,C) is state feedback irreducible.
   (b) $(A+BK,B,C)$ is observable for all K.
   (c) $\mathscr{V}^*(\mathrm{Ker}\, C) = \{0\}$.

8. Consider the scalar real rational transfer function $G(s) = \frac{1}{(s-1)(s-2)}$ of a continuous-time linear system.

   (a) Show that $G(s) = P(s)Q(s)^{-1}$, with $P(s) = \frac{1}{(s+1)^2}$ and $Q(s) = \frac{(s-1)(s-2)}{(s+1)^2}$, is a coprime factorization over RH_∞.
   (b) Determine all stabilizing controllers of $G(s)$.

9. Let $G(s) \in RH_\infty$ be a $p \times m$ proper rational stable transfer function. Show that all stabilizing controllers of G are of the form

$$K(s) = \Gamma(s)(I - G(s)\Gamma(s))^{-1},$$

   with $\Gamma \in RH_\infty$ and $(I - G(s)\Gamma(s))^{-1}$ proper. Is K always in RH_∞?

6.12 Notes and References

A module-theoretic approach to the study of state feedback was initiated by Hautus and Heymann (1978); see also Fuhrmann (1979). The characterization in Theorem 6.13 of reachability indices via Wiener–Hopf indices for feedback irreducible systems can be extended to transfer functions that are not full column rank; see Fuhrmann and Willems (1979). A closely related characterization of feedback irreducibility is due to Heymann (1975).

The Brunovsky canonical form is due to Brunovsky (1970). The Kronecker indices appear first in the work by Popov (1972) and Wang and Davison (1976) on state-space canonical forms. Their characterization in Theorem 6.18 as complete invariants for restricted state feedback equivalence was shown by Helmke (1985). The dimension of the state feedback orbit of a reachable pair was expressed by Brockett (1977) via the reachability indices $\kappa_1 \geq \ldots \geq \kappa_m$ as

$$n^2 + nm + m^2 - \sum_{i,j=1}^{m} \max(\kappa_i - \kappa_j + 1, 0).$$

This formula is a straightforward consequence of Theorem 6.15. For a study of the feedback group, see also Tannenbaum (1981). For an analysis of the topological closure of each state feedback orbit in terms of the dominance order on partitions, see Hazewinkel and Martin (1983) and Helmke (1985).

The connection between reachability indices of a reachable pair and the Wiener–Hopf factorization indices is due to Gohberg, Lerer and Rodman (1978); see also Fuhrmann and Willems (1979). Rosenbrock's theorem is one important instance where polynomial arguments become much simpler than a pure state-space proof. The polynomial proof of Rosenbrock's theorem by Münzner and Prätzel-Wolters (1979) shows the elegancy of polynomial algebra arguments.

The Youla–Kucera parameterization of stabilizing controllers, together with a general theory of coprime factorizations, is nicely revealed in the book by Vidyasagar (1987). The state-space formulas (6.46)–(6.47) and (6.48) for the Youla–Kucera parameterization are taken from Kucera (2011). A more difficult problem is that of strong stabilizability that deals with the issue of finding a stable controller that stabilizes a plant. In Chapter 5.3 of Vidyasagar (1987), strong stabilizability is characterized in terms of parity interlacing conditions on the poles of plants with respect to unstable blocking zeros. A natural generalization of strong stabilizability is the simultaneous stabilization problem of N plants by a single controller, which has been studied by many researchers, including, for example, Blondel (1994); Ghosh and Byrnes (1983); Vidyasagar (1987).

For an early connection between state feedback pole placement and matrix extension problems, see Wimmer (1974b). The problem of pole placement and stabilization by static output feedback is considerably more difficult than that of state feedback control. Pole placement by constant output feedback is equivalent to an intersection problem in a Grassmann manifold, first solved, over the complex numbers, by Schubert in 1886. The connection to intersection theory in the Grassmann manifold was first revealed in the paper by Brockett and Byrnes (1981); see also Byrnes (1989). A striking result that generalized all of the preceding ones is due to Wang (1992), who showed that the pole-placement problem is generically solvable over the reals if $mp > n$. An interesting extension of the pole-placement problem to finite fields $\mathbb{F}$ is due to Gorla and Rosenthal (2010).

Geometric control was initiated by Francis and Wonham (1976) and Basile and Marro (1992); see also Wonham (1979). The mutual relations between the various subspaces are summarized by the so-called Morse relations and the Morse diamond; see Morse (1973). Readers who want to delve somewhat deeper into geometric control theory must master these important contributions. For a nice exposition and further results, we refer the reader to Aling and Schumacher (1984). Geometric control concepts can be applied, for example, to disturbance decoupling with measurement feedback Willems and Commault (1981) and to noninteracting control Falb and Wolovich (1967); Morse and Wonham (1971). A very important problem, which can be successfully dealt with in the framework of geometric control, is the so-called servomechanism or output regulation problem; see Francis (1977).

The polynomial model approach to the characterization of controlled and conditioned invariant subspaces yields clean module-theoretic representations. In particular, (6.58) is due to Fuhrmann and Willems (1980), whereas (6.63) was proved in Fuhrmann (1981). Closely related, and more general, characterizations of controlled and conditioned invariant subspaces in terms of transfer function representations are due to Hautus (1980) and Özgüler (1994). For the case of strictly proper rational transfer functions, the polynomial characterization of $\mathscr{V}^*$ in Theorem 6.60 is due to Emre and Hautus (1980) and Fuhrmann and Willems (1980). The characterization of the maximal output nulling reachability subspace $\mathscr{R}^*$ in Theorem 6.61 is due to Fuhrmann (1981) and Khargonekar and Emre (1982).