Chapter 11
Synchronization and Consensus

Synchronization is one of the fundamental aspects of self-organization in networks of systems. More generally, the emergence of macroscopic states is frequently encountered in dynamical systems when one starts to study coupling effects. Well-known examples include synchronization of oscillators, the emergence of consensus states in models that describe the opinion dynamics of social networks or multiagent systems, or flocking phenomena in biological networks such as swarms of birds or a school of fish. In all these different network models the dynamics of the individual states may "cluster" together or "synchronize" toward a common state that exhibits the system with a unique characteristic identity. The analysis and control of such synchronized states thus becomes an interesting new task for the design of networks. The phenomenon of synchrony was apparently noticed first by Huygens, who was, alongside his scientific activity, also a clock maker. Huygens noticed that two pendulum clocks hanging on a wall tend to synchronize. With time, a multitude of synchronization phenomena were observed in different fields, including, for example, ciliary beating in biology, laser physics, and the firing of neurons as in Parkinson disease. Although most realistic models are of course nonlinear, it appears to be of fundamental interest to explore these issues first in the simplified context of linear systems theory.

The synchronization of interconnected dynamical systems refers to the task of studying and designing coupling laws that force the state vectors of the node systems to converge to each other. In a classical model by Vicsek et. al. (1995), a simple local averaging rule was introduced as a basis for studying the cooperative behavior of animals. The closely related concept of consensus has emerged through studies of multiagent systems and social and computer networks. Obviously, the process of achieving a consensus is a fundamental step that leads to coherent social network structures. A fundamental model for opinion dynamics is due to Krause (1997) and exhibits interesting clustering and consensus effects. The Hegselmann–Krause model is a simplification of the Vicsek model that makes it more attractive for engineering implementations.

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The coupling structure of a multiagent network is defined by a weighted graph whose vertices correspond to the agent dynamics, while the weights of the edges relate to the coupling strengths. Thus the underlying graph structure plays an important role in investigating synchronization phenomena. In the systems engineering literature, diffusive coupling models have been popular for studying synchronization, but more general coupling topologies are of obvious interest, too. Nondiffusive couplings arise, for instance, in systems biology; an example is a model of gene-regulatory networks with a cyclic graph structure, as proposed in Hori, Kim and Hara (2011). In the sequel, we will allow for more general interconnection structures than diffusive coupling, with the consequence that the interconnection matrix is no longer a Laplacian matrix. We present a fairly general approach to problems of synchronization and clustering for networks of identical linear systems. Since the mathematical problems of consensus are somewhat easier to describe, we first present an overview of the main models used in this field. We describe simple autonomous models for mean value consensus and then examine synchronization, for both first-order and higher-order system dynamics. For the latter system class, synchronization is often referred to as both partial state and output synchronization.

11.1 Consensus and Clustering in Opinion Dynamics

Dynamical systems that achieve consensus or clustering may serve as models for opinion dynamics in social networks. Consensus in a network occurs when the participants agree upon their opinions, while clustering refers to the formation of different opinion spectra. We describe simple models for social dynamics using systems of differential or difference equations that exhibit cooperative dynamics. Consider a finite number of agents that are allowed to communicate their opinion on a certain topic. We identify the opinion of the $i-th$ agent with a real variable $x_i \in \mathbb{R}$, while communication among the agents is modeled by an interconnection graph. The topology of the graph is determined by the pattern describing the extent to which the opinions of each agent are influenced by the opinions of the other agents. Thus the opinions of N agents in the network are described by the evolution of a vector $x \in \mathbb{R}^N$, while the communication pattern is specified by a real $N \times N$ matrix $A(x)$. The simplest class of models for consensus that have been described in the literature is of the form

$$\dot{x} = A(x)x, \tag{11.1}$$

where the equilibrium states of (11.1) correspond to the consensus states with equal components $x_1 = \cdots = x_N$. The task is to specify realistic interconnection matrices $A(x)$ such that all solutions of (11.1) converge to a consensus state. Thus consensus, or synchronization, is essentially a stabilization problem in a subspace of desired states. Monotone maps $x \mapsto A(x)x$ are one method of choice here. Another phenomenon of interest is clustering, i.e., the effect that only certain classes of agents become asymptotically equal to each other. Further related topics of interest are synchronization of higher-order systems, distributed computing, formation control and Markov chains on a graph.

In this section we analyze a linear model for mean value consensus and then turn to a brief description of nonlinear models for consensus. In the subsequent sections we will then completely focus on the synchronization of identical node systems.

1. Linear Models for Consensus. We describe a simple linear model for consensus that is already useful for a number of applications. Let $\Gamma = (V,E)$ denote a weighted digraph without loops on $V = \{1,\dots,N\}$ with adjacency matrix $A = (a_{ij})$. Every vertex i is thought of as representing a dynamic agent $x_i \in \mathbb{R}^n$ described by a linear system

$$\begin{aligned} \dot{x}_i &= \alpha x_i + \beta v_i, \\ w_i &= \gamma x_i, \end{aligned} \tag{11.2}$$

with input v_i. A dynamical network with protocol v is then a feedback system

$$\begin{aligned} \dot{x}_i &= \alpha x_i + \beta v_i, \\ v_i &= \sum_{j=1}^{N} a_{ij} w_j. \end{aligned} \tag{11.3}$$

Definition 11.1. System (11.3) solves the **consensus problem** for a function $y = k(x)$ if there exists an interconnection matrix A and an asymptotically stable equilibrium point $x^* \in \mathbb{R}^{nN}$ of

$$\dot{x} = \Big(I_N \otimes \alpha + (I_N \otimes \beta)A(I_N \otimes \gamma)\Big)x$$

such that $x_1^* = \cdots = x_N^* = k(x(0))$. If k is chosen as the mean value $k(x) = \frac{1}{N}\sum_{i=1}^{N} x_i$, then this is called the solution to the **mean value consensus (MVC) problem**.

In its simplest possible form, the mean value consensus system for a system of integrators (where $\alpha = 0$ and $\beta = \gamma = I_n$)

$$\dot{x}_i = v_i, \quad i = 1,\dots,N$$

is of the form

$$\dot{x}_i = \sum_{j=1}^{N} A_{ij}(x_j - x_i), \quad i = 1,\dots,N,$$

where $A \in \mathbb{R}^{N\times N}$ is the adjacency matrix of a digraph Γ. Let

$$L = \operatorname{diag}(A\mathbf{e}) - A$$

denote the associated Laplacian matrix and $\mathbf{e} := (1,\dots,1)^\top \in \mathbb{R}^N$. Thus, for interconnection matrices of the form $A = L\otimes I_n$, one obtains the **standard MVC system** as

$$\dot{x} = -(L\otimes I_n)x. \tag{11.4}$$

For the general control system (11.2) and interconnection matrix $A = L\otimes K$ one obtains the **dynamic MVC system**

$$\dot{x} = \big(I_N \otimes \alpha - L\otimes \beta K\gamma\big)x.$$

This model will be studied in more detail in Section 11.2 on synchronization.

We begin with a discussion of (11.4). A directed weighted graph Γ with adjacency matrix A and Laplacian L is called **balanced** if

$$\sum_{j=1}^N a_{ij} = \sum_{j=1}^N a_{ji}$$

for all i. This is equivalent to $\mathbf{e}^\top L = 0$. Note that $L\mathbf{e} = 0$ is always satisfied for Laplacian matrices. Obviously, an undirected graph is balanced. The following characterization of systems that achieve mean value consensus is due to Olfati-Saber, Fax and Murray (2004).

Theorem 11.2. *Let Γ be a strongly connected weighted digraph and L the associated Laplacian. Then $L\mathbf{e} = 0$, and there exists a unique row vector $\mathbf{c} \in \mathbb{R}^{1\times N}$, with $\mathbf{c}L = 0$ and $\mathbf{c}\mathbf{e} = 1$. Moreover, the following properties are satisfied:*

1. *Every solution of (11.4) converges to a scalar multiple of $\mathbf{e}$, i.e., for each initial condition $x(0)$,*

$$\lim_{t\to\infty} x(t) = (\mathbf{e}\mathbf{c}\otimes I_n)x(0). \tag{11.5}$$

   In particular,

$$\lim_{t\to\infty} e^{-tL\otimes I_n} = \mathbf{e}\mathbf{c}\otimes I_n. \tag{11.6}$$

2. *System (11.4) achieves mean value consensus if and only if Γ is balanced, i.e., if and only if $\mathbf{e}^\top L = 0$. In that case,*

$$\lim_{t\to\infty} e^{-tL\otimes I_n} = \frac{1}{N}\mathbf{e}\mathbf{e}^\top \otimes I_n.$$

Proof. Since Γ is strongly connected, Theorem 8.36 shows that the Laplacian matrix L has a simple eigenvalue $\lambda_1 = 0$, with eigenvector $\mathbf{e}$ and $N-1$ eigenvalues $\lambda_2, \ldots, \lambda_N$ with positive real part. In particular, the left kernel of L is one-dimensional. After a similarity transformation S, one can assume that

$$-SLS^{-1} = \begin{pmatrix} 0 & L_{12} \\ 0 & L_{22} \end{pmatrix},$$

with L_{22} Hurwitz. Using the block-diagonal structure of SLS^{-1}, the matrix exponential of SLS^{-1} is readily computed. This shows that

$$\lim_{t\to\infty} e^{-tSLS^{-1}} = \begin{pmatrix} 1 & -L_{12}L_{22}^{-1} \\ 0 & 0 \end{pmatrix},$$

and the row vector $\lambda^\top = (1, -L_{12}L_{22}^{-1})$ satisfies $\lambda^\top SLS^{-1} = 0$ and $\lambda^\top e_1 = 1$. Thus $\mathbf{c} := \lambda^\top S$ satisfies $\mathbf{c}L = 0$ and $\mathbf{ce} = \mathbf{c}S^{-1}e_1 = \lambda^\top e_1 = 1$. Since the kernel of L is one-dimensional, $\mathbf{c}$ is uniquely determined.

One concludes that $\lim_{t\to\infty} e^{-tL}$ exists and is of the form $\lim_{t\to\infty} e^{-tL} = \mathbf{ec}$. The identity

$$e^{-tL\otimes I_n} = e^{-tL} \otimes I_n$$

then implies

$$\lim_{t\to\infty} e^{-tL\otimes I_n} = \mathbf{ec} \otimes I_n.$$

This proves (11.6), which in turn implies (11.5). The stronger mean value consensus property is satisfied if and only if

$$\lim_{t\to\infty} e^{-tL\otimes I_n} x(0) = \frac{1}{N}(\mathbf{ee}^\top \otimes I_n)x(0)$$

for all initial conditions $x(0)$. By (11.6), this is equivalent to $\mathbf{ec} \otimes I_n = \frac{1}{N}\mathbf{ee}^\top \otimes I_n$, i.e., to $c = \frac{1}{N}\mathbf{e}^\top$. This completes the proof. ∎

This result is easily extended as follows to graphs with time-varying interconnections. We present here only one such extension. Let $\mathscr{L}$ denote a compact set of real $N \times N$ matrices such that every element $L \in \mathscr{L}$ has nonnegative nondiagonal entries, satisfies $L\mathbf{e} = 0$, $\mathbf{e}^\top L = 0$, and is irreducible.

Theorem 11.3. *Let $L : [0,\infty) \longrightarrow \mathscr{L}$ be piecewise continuous. The solutions $x(t) \in \mathbb{R}^{nN}$ of*

$$\dot{x}(t) = -(L(t) \otimes I_n)x(t) \tag{11.7}$$

satisfy

$$\lim_{t\to\infty} x(t) = \frac{1}{N}(\boldsymbol{ee}^\top \otimes I_n)x(0),$$

with error bounds

$$\|x(t) - \frac{1}{N}(\boldsymbol{ee}^\top \otimes I_n)x(0)\|^2 \le e^{-2t\kappa}\|(I_{nN} - \frac{1}{N}\boldsymbol{ee}^\top \otimes I_n)x(0)\|^2 \le e^{-2t\kappa}\|x(0)\|^2,$$

where $\kappa := min_{L\in\mathscr{L}}\lambda_2(L+L^\top) > 0$.

Proof. Since $\mathscr{L}$ is a compact set of irreducible Laplacians, 0 is a simple eigenvalue of each matrix $L \in \mathscr{L}$. Moreover, by Theorem 8.36, the assumption $\mathbf{e}^\top L = 0$ implies that the symmetric matrix $P = L + L^\top$ is positive semidefinite with simple eigenvalue 0. Let $\lambda_1(P) = 0 \le \lambda_2(P) \le \cdots \le \lambda_N(P)$ denote the eigenvalues of P. Thus $\lambda_k(L + L^\top) > 0$ for all $L \in \mathscr{L}$ and $2 \le k \le N$. Since $\mathscr{L}$ is compact, κ exists and is strictly positive. Since $\lambda_2(L+L^\top)$ is the smallest eigenvalue of the restriction of $L+L^\top$ on the invariant subspace $(\mathbf{e}\otimes\mathbb{R}^n)^\perp$, we see that

$$2\delta^\top L\otimes I_n\delta = \delta^\top\Big((L+L^\top)\otimes I_n\Big)\delta \ge \lambda_2(L+L^\top)\|\delta\|^2 \ge \kappa\|\delta\|^2$$

for all $\delta \in (\mathbf{e}\otimes\mathbb{R}^n)^\perp$. Then every solution $x(t)$ of (11.7) satisfies $\mathbf{ee}^\top x(t) = \mathbf{ee}^\top x(0)$ since $(\mathbf{ee}^\top \otimes I_n)(L\otimes I_n) = 0$. Thus

$$\delta(t) := (I_{nn} - \frac{1}{N}\mathbf{ee}^\top \otimes I_n)x(t) = x(t) - \frac{1}{N}(\mathbf{ee}^\top \otimes I_n)x(0)$$

is a solution of (11.7). This implies

$$\frac{d\delta^\top\delta}{dt} = 2\delta^\top\frac{d\delta}{dt} = -2\delta^\top(L\otimes I_n)\delta \le -2\kappa\|\delta\|^2.$$

Therefore,

$$\|\delta(t)\| \le e^{-t\kappa}\|\delta(0)\|,$$

and the result follows. ■

Similar results are valid in the discrete-time case when the Laplacian matrix L is replaced by the **normalized Laplacian** matrix

$$F = D^{-1}A.$$

Here $D = \operatorname{diag}(A\mathbf{e})$ and A denotes the adjacency matrix of a weighted graph. If A is nonnegative, then the normalized Laplacian is a stochastic matrix. The following result gives a simple sufficient condition for consensus, which can be generalized in several directions (see Section 11.7, "Notes and References").

Theorem 11.4. *Let $x(0) \in \mathbb{R}^n$ and F be a primitive stochastic matrix. Then there exists a real number x_* such that the solution*

$$x(t+1) = Fx(t)$$

satisfies

$$\lim_{t\to\infty} x(t) = x_* \boldsymbol{e}.$$

In particular, this is satisfied for normalized Laplacians of connected graphs Γ with at least one loop around a vertex.

Proof. The primitivity of F implies (Theorem 8.18) that $\lim_{t\to\infty} F^t = \mathbf{e}c$ for some $c \in \mathbb{R}^{1\times n}$. This implies the result. ■

2. Distributed Algorithm for Solving Linear Equations. We illustrate the preceding analysis by showing how consensus algorithms can be used to design distributed methods for solving linear systems of equations. Distributed computing is of course a vast topic, and we leave it to the reader to explore further applications in this field.

Let us start with a classical problem from linear algebra, i.e., the computation of intersection points of a finite number of affine linear subspaces $L_i = a_i + V_i, i = 1,\ldots,N$, of a d-dimensional Hilbert space $\mathbb{H}$. We are interested in computing points $x^* \in E$ in the intersection $L_1 \cap \cdots \cap L_N$. For simplicity, let us assume that the vector spaces V_i are in general position in the sense that the direct sum decomposition

$$V_1^\perp \oplus \cdots \oplus V_N^\perp = \mathbb{H}$$

is satisfied. This implies both $V_1 \cap \cdots \cap V_N = \{0\}$ and $\sum_{i=1}^N \dim V_i^\perp = d$. In particular, whenever $L_1,\ldots,L_N$ intersect, they intersect at a unique point

$$L_1 \cap \cdots \cap L_N = \{x^*\},$$

and our goal is to compute the unique intersection point x^*. Of course, there are several methods available to do this, but our focus is on demonstrating how consensus algorithms can help in computing x^*. To this end, we introduce the selfadjoint projection operators $P_i : \mathbb{H} \longrightarrow V_i \subset \mathbb{H}$ onto V_i, with kernel $V_i^\perp$. Then the direct sum $P = \operatorname{diag}(P_1,\ldots,P_N) : \mathbb{H}^N \longrightarrow \mathbb{H}^N$ is a selfadjoint projection operator onto $V_1 \oplus \cdots \oplus V_N \subset \mathbb{H}^N$.

We next present graph-theoretic ideas relevant for distributed computing. Fix an undirected and connected graph Γ whose set of vertices $\mathscr{V} = \{1,\ldots,N\}$ is labeled by the N linear subspaces V_i, together with a set of M edges $\mathscr{E}$ that define which vertices can interact with each other during the course of running the algorithm. Let $\mathfrak{A}$ denotes the associated $0,1$-adjacency matrix of the graph. For technical reasons we assign to each vertex a single self-loop of the graph, so that the diagonal entries of the adjacency matrix are all equal to 1. Let $d_i \geq 2$ denote the degree of the ith vertex, and set $D := \operatorname{diag}(d_1,\ldots,d_N)$. Let $B = (b_{ij}) \in \mathbb{R}^{N\times M}$ denote the oriented incidence matrix of the graph defined by (8.13). Since Γ is connected, the incidence matrix B has full row rank $N-1$, and therefore the kernel of B has dimension $M-N+1$. The normalized graph Laplacian is the stochastic matrix

$$\mathscr{L} = D^{-1}\mathfrak{A} = I_N - D^{-1}BB^\top.$$

We need the following lemma.

Lemma 11.5. *Assume that* $V_1^\perp \oplus \cdots \oplus V_N^\perp = \mathbb{H}$. *Then*

$$\operatorname{Ker} P(B\otimes I_d) = \operatorname{Ker}(B\otimes I_d).$$

Proof. We first prove

$$\operatorname{Ker}(B^\top \otimes I_d) \cap \operatorname{Im} P = \{0\}. \tag{11.8}$$

In fact, $x \in \operatorname{Ker}(B^\top \otimes I_d)$ implies that $x = \mathbf{e}\otimes v$ for a suitable element $v \in \mathbb{H}$. Therefore, $x \in \operatorname{Ker}(B^\top \otimes I_d) \cap \operatorname{Im} P$ if and only if $x = \mathbf{e}\otimes v$, and there exist elements $\xi_1,\ldots,\xi_N$ with $v = P_1\xi_1 = \cdots = P_N\xi_N$. Equivalently,

$$v \in \bigcap_{j=1}^{N} \operatorname{Im} P_j = \bigcap_{j=1}^{N} V_j = \{0\}.$$

This proves (11.8). By taking orthogonal complements,

$$\operatorname{Im}(B\otimes I_d) + \operatorname{Ker} P = \mathbb{H}^N.$$

Since $\operatorname{rk}(B\otimes I_d) = d\operatorname{rk} B = d(N-1)$ and $\dim \operatorname{Ker} P = \sum_{i=1}^N \dim V_i^\perp = \dim \mathbb{H} = d$, one concludes that $\operatorname{Im}(B\otimes I_d)\cap \operatorname{Ker} P = \{0\}$. This implies the result. ■

Proposition 11.6. *Assume that* $V_1^\perp \oplus \cdots \oplus V_N^\perp = \mathbb{H}$. *Then each eigenvalue* λ *of* $P(\mathscr{L}\otimes I_d)P$ *is real and satisfies* $-1 < \lambda < 1$.

Proof. The normalized Laplacian $D^{-1}\mathfrak{A}$ has nonzero entries on the diagonal. Thus Theorem 8.42 applies and implies that all eigenvalues of $\mathscr{L}\otimes I_d$ are real and are contained in the interval $(-1,1]$. Applying a similarity transformation, we see that the same property is true for the symmetric matrix $D^{-\frac{1}{2}}\mathfrak{A}D^{-\frac{1}{2}} \otimes I_d$. Since P is a projection operator, the spectrum of $P(D^{-\frac{1}{2}}\mathfrak{A}D^{-\frac{1}{2}}\otimes I_d)P$ is contained in the convex hull of the spectrum of $D^{-\frac{1}{2}}\mathfrak{A}D^{-\frac{1}{2}}\otimes I_d$, i.e., it is contained in $(-1,1]$. Using

$D^{\frac{1}{2}}P(\mathscr{L}\otimes I_d)PD^{-\frac{1}{2}} = P(D^{-\frac{1}{2}}\mathfrak{A}D^{-\frac{1}{2}}\otimes I_d)P$, one concludes that $P(\mathscr{L}\otimes I_d)P$ has all its eigenvalues in $(-1,1]$. It remains to show that 1 is not an eigenvalue of $P(\mathscr{L}\otimes I_d)P$. In fact, otherwise there exists $v\in \operatorname{Im} P$ with $P(\mathscr{L}\otimes I_d)v = v$. Since $D^{-1}\mathfrak{A} = I - D^{-1}L = I - D^{-1}BB^\top$, this is equivalent to $P(D^{-1}BB^\top\otimes I_d)v = 0$. Since P and $D\otimes I_d$ commute, this is equivalent to $P(BB^\top\otimes I_d)v = 0$. Thus, using Lemma 11.5, $(BB^\top\otimes I_d)v = 0$ or, equivalently, $(B^\top\otimes I_d)v = 0$. This shows that $v\in \operatorname{Ker}(B^\top\otimes I_d)\cap \operatorname{Im} P$. By (11.8), thus $v = 0$, and we are done. ■

After these preparatory remarks, we are now ready to introduce and study the distributed algorithm for subspace intersections. The key idea is very simple to describe. Suppose one has computed for each $t\in\mathbb{N}$ and $i = 1,\dots,N$ an element $x_i(t)\in L_i$. Then for each $u_i(t)\in\mathbb{H}$ the linear control system

$$x_i(t+1) = x_i(t) + P_i u_i(t)$$

evolves in the affine subspace L_i. In fact, the right-hand side describes all elements of L_i. Choose the input vector $u_i(t)$ such that the difference

$$\|x_i(t+1) - \frac{1}{d_i}\sum_{j\in\mathscr{N}_i} x_j(t)\|^2$$

is minimized. By a straightforward computation, this leads to the recursion

$$x_i(t+1) = (I_d - P_i)x_i(t) + P_i\left(\frac{1}{d_i}\sum_{j\in\mathscr{N}_i} x_j(t)\right), \quad x_i(0)\in L_i, \tag{11.9}$$

which is clearly in distributed form. Using the vector notation $x(t) = \operatorname{col}(x_1(t), \dots, x_N(t))$, this is equivalent to

$$x(t+1) = (I_{dN} - P)x(t) + P(\mathscr{L}\otimes I_d)x(t), \quad x(0)\in L_1\times\cdots\times L_N.$$

Theorem 11.7. *Assume that $V_1^\perp\oplus\cdots\oplus V_N^\perp = \mathbb{H}$. Then the distributed algorithm (11.9) converges exponentially fast from every initial point $x(0)\in L_1\times\cdots\times L_N$ to $\mathbf{e}\otimes x^*$, where $x^*\in\mathbb{H}$ denotes the unique intersection point of $L_1\cap\cdots\cap L_N$.*

Proof. Let $z(t) := x(t) - \mathbf{e}\otimes x^*$. Then $z(t)$ satisfies the recursion

$$z(t+1) = P(\mathscr{L}\otimes I_d)z(t).$$

By Lemma 11.6, the eigenvalues of $P(\mathscr{L}\otimes I_d)$ are in the open unit interval $(-1,1)$. Thus $z(t)$ converges exponentially fast to 0. The result follows. ■

Of course, it is trivial to apply the preceding ideas to solving linear equations $Ax = b$. Let $A\in\mathbb{R}^{n\times n}$, with nonzero row vectors $A_1,\dots,A_n\in\mathbb{R}^{1\times n}$ and $b = \operatorname{col}(b_1,\dots,b_n)\in\mathbb{R}^n$. Defining the affine hyperplanes

$$L_i = \{x_i \in \mathbb{R}^n \mid A_i x_i = b_i\}$$

with subspaces $V_i = \operatorname{Ker} A_i$ we see that the solutions of $Ax = b$ are just the intersection points in $L_1 \cap \cdots \cap L_n$. Moreover, A is invertible if and only if $V_1^\perp \oplus \cdots \oplus V_n^\perp = \mathbb{R}^n$. The projection operators are

$$P_i = I - \frac{A_i^\top A_i}{\|A_i\|^2}.$$

The next consequence of Theorem 11.7 is obvious.

Theorem 11.8. *Let A be invertible. Then the distributed algorithm*

$$x_i(t+1) = \frac{A_i^\top b_i}{\|A_i\|^2} + (I - \frac{A_i^\top A_i}{\|A_i\|^2})\left(\frac{1}{d_i}\sum_{j\in\mathcal{N}_i} x_j(t)\right), \quad A_i x_i(0) = b_i,$$

converges from each solution of $A_i x_i(0) = b_i, i = 1,\ldots,n$, exponentially fast to $\operatorname{col}(A^{-1}b,\ldots,A^{-1}b)$.

3. Nonlinear Models for Consensus. Of course, the preceding, rather brief, discussion of linear consensus models can be extended in several directions, including to a study of the effects of dynamic or stochastic interconnections, allowing for nonlinear models and analyzing robustness issues. Here we focus on nonlinear models. We begin with a rather straightforward extension of Theorem 11.3 to nonlinear coupling models of the form

$$\dot{x} = -(L(x) \otimes I_n)x. \tag{11.10}$$

Of course, discrete-time models can be considered as well. Here $x \mapsto L(x)$ denotes a smooth function of Laplacian $N \times N-$ matrices

$$L(x) = D(x) - A(x),$$

where $D(x) = \operatorname{diag} A(x)\mathbf{e}$. To define $A(x)$, we introduce an **influence function** as a smooth strictly positive function $\chi : \mathbb{R} \longrightarrow [0,\infty)$ that is monotonically decreasing on $[0,\infty)$. The function χ is regarded as a measure of how strongly mutual agents influence each other. Thus, in applications to opinion dynamics, two agents, x_i and x_j, are thought of as influencing each other's opinions if the value of the influence function $\chi(x_i - x_j)$ is large, and small otherwise. Possible choices for such influence functions are, for example,

(a) Constant functions;
(b) The indicator function $\chi_r = \chi_{[-r,r]}$ on a compact interval $[-r,r]$;
(c) The potential $\chi(x) = k(1+x^2)^{-\beta} \quad$ for $\beta > 0$;
(d) The Gaussian distribution $\chi(x) = e^{-x^2}$.

Let $M = (m_{ij})$ denote a nonnegative matrix, for example, the weighted adjacency matrix of a graph. Thus M defines the relevant interconnections that are allowed between the various agents. Consider the adjacency matrix of a time-varying neighborhood graph defined as

$$A(x) = (m_{ij}\chi(\|x_i - x_j\|) \in \mathbb{R}^{N\times N}.$$

Note that $A(x)$ is symmetric for all x whenever the scaling matrix M is symmetric.

Theorem 11.9. *Assume that M is irreducible, symmetric, and nonnegative. For each initial condition $x(0)$ the unique solution $x(t)$ of (11.10) exists for all $t \geq 0$. Moreover, $\|x_i(t) - x_j(t)\|$ converges to 0 as $t \to \infty$.*

Proof. Let $\Delta = \mathbf{e} \otimes \mathbb{R}^n$ denote the diagonal in $\mathbb{R}^{nN} = \mathbb{R}^n \otimes \cdots \otimes \mathbb{R}^n$, and let

$$\phi(x) = \min_{x\in\Delta^\perp} \frac{x^\top (L(x) \otimes I_n)x}{\|x\|^2}$$

denote the Fiedler number, i.e., $\phi(x) = n\lambda_2(x)$, where $\lambda_2(x)$ denotes the second smallest eigenvalue of $L(x)$. Thus

$$x^\top (L(x) \otimes I_n)x = \frac{1}{2}\sum_{i,j=1}^{N} a_{ij}\|x_i - x_j\|^2 \geq \phi(x)\|x\|^2.$$

For each solution of (11.10),

$$\frac{d}{dt}\|x\|^2 = -2x^\top (L(x) \otimes I_n)x \leq -\phi(x)\|x\|^2 \leq 0.$$

Thus the norm $\|x(t)\|$ decreases monotonically, and therefore $x(t)$ is positively bounded. This shows the existence of solutions for all $t \geq 0$. To proceed with the analysis, we need a lower bound on the Fiedler number. This is achieved as follows. Let $L_M = D_M - M$ denote the associated Laplacian of M. Since M is assumed to be irreducible, the Fiedler number μ of L_M is strictly positive. Moreover, by the monotonicity of χ,

$$\chi(\|x_i - x_j\|^2) \geq \chi(\sum_{i,j=1}^{N}\|x_i - x_j\|^2) = \chi((2N-1)\|x\|^2).$$

By the symmetry of L, we have that $(\mathbf{e}^\top \otimes x^\top)(L \otimes I_n) = 0$. Therefore, the orthogonal complement $\Delta^\perp$ is invariant under the flow of (11.10). Thus, for all $0 \neq x \in \Delta^\perp$,

$$\begin{aligned}
\frac{d}{dt}\|x\|^2 &= -\sum_{i,j=1}^{N} a_{ij}\|x_i - x_j\|^2 = -\sum_{i,j=1}^{N} m_{ij}\chi(\|x_i - x_j\|^2)\|x_i - x_j\|^2 \\
&\leq -\chi((2N-1)\|x\|^2)\sum_{i,j=1}^{N} m_{ij}\|x_i - x_j\|^2 \\
&= -2x^\top (L_M \otimes I_n)\, x\chi((2N-1)\|x\|^2) \\
&\leq -2\mu\chi((2N-1)\|x\|^2)\|x\|^2 < 0.
\end{aligned}$$

This shows that each solution $x(t)$ of (11.10) satisfies $\lim_{t\to\infty} \mathrm{dist}(x(t),\Delta) = 0$. The result follows. ■

The **Vicsek model** is one of the first nonlinear models for swarm formation in a multiagent dynamical system. The system studied by Vicsek et. al. (1995) is described as follows. Consider N agents $x_1,\ldots,x_N \in \mathbb{R}^2$ in Euclidean space, modeled as simple integrators

$$\dot{x}_i = u_i, \quad i = 1,\ldots N.$$

For positive real numbers $r > 0$, define as follows a time-varying undirected neighborhood graph $\Gamma(t) = (V, E(t))$, with the set of vertices $V = \{1,\ldots,N\}$. The edges are characterized by the property that $\{i,j\} \in E(t)$ if and only if $\|x_i(t) - x_j(t)\| \leq r$. Let $\mathcal{N}_i(t) = \{j \mid \|x_i(t) - x_j(t)\| \leq r\}$ denote the set of $n_i(t)$ neighbors of agent i at time t. The agents are allowed to move with constant velocity but varying directions in the plane. The goal is to create a distributed feedback law that enables agents to synchronize the directions in which they move.

For discrete-time systems, a simple idea about how to achieve this is that each agent averages over the directions of all other agents within his/her neighborhood region. Following Jadbabaie, Lin and Morse (2003), this can be formalized in the following simple mean value consensus model. Fix a desired speed as $v > 0$ and

$$\begin{aligned}
\theta_i(t+1) &= \frac{1}{n_i(t)}\sum_{j\in\mathcal{N}_i(t)} \theta_j(t) \\
x_i(t+1) &= x_i(t) + ve^{\sqrt{-1}\theta_i(t+1)}.
\end{aligned} \tag{11.11}$$

Here $0 \leq \theta_i < 2\pi$ denotes the polar angle of x_i. This can be conveniently rewritten as follows. Let $\theta = (\theta_1,\ldots,\theta_N)^\top \in [0,2\pi]^N$, and let

$$\chi_r(t) = \begin{cases} 1 & 0 \leq t \leq r, \\ 0 & t > r \end{cases}$$

denote the characteristic function. The graph adjacency matrix of $\Gamma(t)$ is

$$A(t) := (\chi_r(\|x_i(t) - x_j(t)\|).$$

Let $D(t) = \operatorname{diag}(A(t)\mathbf{e})$, and let

$$F(t) = D(t)^{-1}A(t)$$

denote the normalized Laplacian. $F(t)$ is also called a **flocking matrix**. The discrete-time Vicsek model is then

$$\theta(t+1) = F(t)\theta(t). \tag{11.12}$$

Consensus in the Vicsek system occurs, provided for each initial condition $\theta(0)$ there exists a constant $\theta_* \in [0, 2\pi]$ such that the solution of (11.12) satisfies

$$\lim_{t\to\infty} \theta_i(t) = \theta_*$$

for $i = 1, \ldots, N$. Sufficient conditions for consensus depend on certain connectivity properties of the time-varying graphs $\Gamma(t)$, such as, for example, that the flocking matrix is primitive for all t. Note that the flocking matrix $F(t)$ is a nonnegative matrix that is congruent to the adjacency matrix $A(t)$. Therefore, $F(t)$ is irreducible if and only if $A(t)$ is irreducible, or, equivalently, if and only if the graph $\Gamma(t)$ is strongly connected. Moreover, $F(t)\mathbf{e} = \mathbf{e}$. Therefore, $F(t)$ is a stochastic matrix with positive entries on the diagonal. Thus, Theorem 8.23 implies that $F(t)$ is primitive if and only if $\Gamma(t)$ is strongly connected. Note further that $F(t)$ being a stochastic matrix for all t implies that the solutions $\theta(t)$ of the Vicsek model leave $[0, 2\pi]^N$ invariant.

Although the results on linear consensus provide explicit sufficient conditions for a consensus of general time-varying systems (11.12), they cannot be directly applied to the Vicsek model (11.11). In fact, in the Vicsek model (11.11) the entries of the flocking matrix $F(t)$ depend on the relative distances $\|x_i(t) - x_j(t)\|$ of the agents and thus depend in turn on $\theta(t)$. This shows that connectivity assumptions on the time-varying neighborhood graph, such as, for example, the primitivity of $F(t)$, cannot be assumed a priori. What is needed are assumptions on, for example, the initial positions $x_1(0), \ldots, x_N(0)$ of the agents that guarantee that $F(t)$ will remain primitive for all t. This is difficult to verify for the Vicsek model because of the hard constraints defined by the characteristic function $\chi_r(t)$.

A simplification of the Vicsek model is due to Krause (1997). While the Vicsek swarm model is a second-order model that describes the evolution of points in the plane, the **Hegselmann–Krause model** Hegselmann and Krause (2002) works in the real line. Its main motivation is drawn from understanding the dynamics of social networks, in particular the formation and development of opinions in such networks. We regard an opinion of an agent as a real variable x. Thus the opinion dynamics in a network of N agents is described by the evolution of N real variables $x_1, \ldots, x_N \in \mathbb{R}$. In its simplest form, the Hegselmann–Krause model is

$$x_i(t+1) = \frac{\sum_{j:|x_i(t)-x_j(t)|\le r} x_j(t)}{|\{j : |x_i(t)-x_j(t)| \le r\}|}, \quad i = 1,\dots,N. \tag{11.13}$$

Here agents i and j are thought of as influencing each other if and only if the distance between their opinions is small, i.e., $|x_i - x_j| \le r$. This defines a state-dependent weighted adjacency matrix $A(x) = (a_{ij}(x))$, with

$$a_{ij}(x) = \begin{cases} 1 & \text{if} \quad |x_i - x_j| \le r \\ 0 & \text{else.} \end{cases}$$

More generally, using the notion of influence functions, one can define the adjacency matrix of a time-varying neighborhood graph of opinions as

$$A(x) = (\chi(\|x_i - x_j\|) \in \mathbb{R}^{N\times N}$$

and the flocking matrix as the normalized Laplacian

$$F(x) = D(x)^{-1}A(x), \quad D(x) = \operatorname{diag}(A(x)\mathbf{e}).$$

Thus the ij-entry of $A(x)$ is small whenever agents x_i and x_j are not influencing each other's opinions. The generalized Hegselmann–Krause model for opinion dynamics is then

$$x(t+1) = F(x(t))x(t), \quad t \ge 0.$$

Thus, while the Hegselmann–Krause model (11.13) looks similar to the Vicsek model (11.11), the dynamics of (11.13) are simpler than that of (11.11). Without going into details, we mention that there exists an elegant convergence theory for the Krause model that is based on the theory of monotone operators.

11.2 Synchronization of Linear Networks

Using the theory of interconnected systems developed in Chapter 9, we now proceed to a general synchronization analysis of networks of linear systems. We consider networks of N identical interconnected linear systems, where the dynamics of each node $i = 1,2,\dots,N$ are described in state-space form as

$$\begin{aligned} \dot{x}_i(t) &= \alpha x_i(t) + \beta v_i(t), \\ w_i(t) &= \gamma x_i(t). \end{aligned} \tag{11.14}$$

Here the node system $\alpha \in \mathbb{R}^{n\times n}$, $\beta \in \mathbb{R}^{n\times m}$, $\gamma \in \mathbb{R}^{p\times n}$ is assumed to be reachable and observable. To define a network of such identical linear systems, we fix a **state interconnection structure**, defined by a matrix $A \in \mathbb{R}^{mN\times pN}$, and **input/output interconnection matrices** $B = (B_1^\top,\dots,B_N^\top)^\top \in \mathbb{R}^{mN\times q}$ and $C = (C_1,\dots,C_N) \in \mathbb{R}^{l\times pN}$, with $B_i \in \mathbb{R}^{m\times q}$ and $C_i \in \mathbb{R}^{l\times p}$, respectively. In the sequel, we will consider A as an $N\times N$ block matrix $A = (A_{ij})$, with blocks $A_{ij} \in \mathbb{R}^{m\times p}$. In particular, A is a square matrix if and only if $p = m$. Let $u = \operatorname{col}(u_1,\dots,u_q) \in \mathbb{R}^q$ denote the external control input applied to the whole network. The input to node i is then

$$v_i(t) = \sum_{j=1}^{N} A_{ij} w_j(t) + B_i u(t). \tag{11.15}$$

Like the external input to the network, the output of the network is a linear combination of the individual node outputs w_i as $y(t) = Cw(t)$, with $w = \operatorname{col}(w_1,\dots,w_N)$ and $y \in \mathbb{R}^l$. Let $x = \operatorname{col}(x_1,\dots,x_N) \in \mathbb{R}^{nN}$ denote the global state of the network.

A directed weighted **state interconnection graph** $\Gamma = (V,E)$ is associated with the state interconnection structure of the system as follows. The set of vertices $V = \{1,\dots,N\}$ corresponds to node systems (11.14). An edge $(i,j) \in E$ from system i to system j is defined if and only if $A_{ij} \neq 0$. We emphasize that the weights A_{ij} of the graph are matrices, unless $p = m = 1$. Similarly, graphs are defined for the input/output interconnection, respectively. So-called **diffusive coupling** refers to the special situation where $p = m = 1$ and A is the Laplacian matrix of an undirected weighted graph. Thus $a_{ij} < 0$ if and only if nodes $i \neq j$ are connected. Otherwise, for $i \neq j$ we define $a_{ij} = 0$. The diagonal elements of the Laplacian matrix A are defined by $a_{ii} = -\sum_{j\neq i} a_{ij}$. Then the interconnection law (11.15), with $B_i = 0$ and diffusive coupling, becomes

$$v_i(t) = \sum_{j\neq i} a_{ij}(w_j(t) - w_i(t)).$$

In the sequel, unless stated otherwise, we will not make restrictive assumptions on the structure of A. In particular, we will not assume that A is a Laplacian matrix or assume that the off-diagonal entries have a specific sign.

Using the interconnection matrices A,B,C and node dynamics α,β,γ, the resulting linear network has the form

$$\begin{aligned} \dot{x}(t) &= \mathscr{A}x(t) + \mathscr{B}u(t),\\ y(t) &= \mathscr{C}x(t), \end{aligned} \tag{11.16}$$

where

$$\mathscr{A} = I_N \otimes \alpha + (I_N \otimes \beta)A(I_N \otimes \gamma) \in \mathbb{R}^{nN\times nN} \quad \mathscr{B} = (I_N \otimes \beta)B \in \mathbb{R}^{nN\times q},$$

$$\mathscr{C} = C(I_N \otimes \gamma) \in \mathbb{R}^{l\times nN}.$$

Stated in terms of transfer functions, one obtains the node transfer function

$$G(z) = \gamma(zI_n - \alpha)^{-1}\beta$$

and associated left and right coprime factorizations

$$G(z) = D_\ell(z)^{-1}N_\ell(z) = N_r(z)D_r(z)^{-1} = V(z)T(z)^{-1}U(z).$$

Note that in this special case of a homogeneous network, our notation differs slightly from the preceding one. The network transfer function is

$$\mathscr{N}_G(z) = \mathscr{C}(zI_{nN} - \mathscr{A})^{-1}\mathscr{B}$$

and

$$\begin{aligned}\mathscr{N}_G(z) &= C\Big(I_N \otimes D_\ell(z) - (I_N \otimes N_\ell(z))A\Big)^{-1}(I_N \otimes N_\ell(z))B \\ &= C(I_N \otimes V(z))\Big(I_N \otimes T(z) - (I_N \otimes U(z))A(I_N \otimes V(z))\Big)^{-1}(I_N \otimes U(z))B.\end{aligned}$$

In principle, there exist two different approaches to the design of such networks. The first one, on which we will mainly focus in the sequel, is to consider the interconnection terms A, B, C as free design parameters. A natural question in this direction then concerns the design of networks, i.e., how one can change the system dynamics of the network (11.16) by a suitable choice of the coupling parameters A, B, C. This is closely related to feedback control problems, such as stabilization or self-organization. A second approach would consist in assuming the interconnection structure to be fixed and designing local controllers for the node system to change the dynamics of the network.

To treat synchronization issues more broadly, we recall some basic terminology from geometric control theory. Consider an invariant subspace $\mathscr{V} \subset \mathscr{X}$ of a linear operator $A : \mathscr{X} \longrightarrow \mathscr{X}$. Then there are two induced linear maps, the restriction operator $A|\mathscr{V} : \mathscr{V} \longrightarrow \mathscr{V}$ and the corestriction $A|_{\mathscr{X}/\mathscr{V}} : \mathscr{X}/\mathscr{V} \longrightarrow \mathscr{X}/\mathscr{V}$. The invariant subspace $\mathscr{V}$ is called **outer stable** if all eigenvalues of the corestriction $A|_{\mathscr{X}/\mathscr{V}}$ have negative real part. Consider a linear system with m inputs and p outputs,

$$\begin{aligned}\dot{x}(t) &= \mathscr{A}x(t) + \mathscr{B}u(t),\\ y(t) &= \mathscr{C}x(t),\end{aligned}$$

on an n-dimensional state space $\mathscr{X} \simeq \mathbb{R}^n$. A linear subspace $\mathscr{V} \subset \mathscr{X}$ is called **controlled invariant**, or $(\mathscr{A}, \mathscr{B})-$ invariant, if

$$\mathscr{A}\mathscr{V} \subset \mathscr{V} + \operatorname{Im}\mathscr{B}.$$

Equivalently, $\mathscr{V}$ is controlled invariant if and only if there exists a state feedback matrix $F \in \mathbb{R}^{m\times n}$, with

$$(\mathscr{A} + \mathscr{B}F)\mathscr{V} \subset \mathscr{V}.$$

$\mathscr{V}$ is called an **outer stabilizable** controlled invariant subspace if $\mathscr{V}$ is an outer stable invariant subspace for $\mathscr{A} + \mathscr{B}F$. Similarly, $\mathscr{V}$ is called **conditioned invariant**, or $(\mathscr{C}, \mathscr{A})-$ invariant, if

$$\mathscr{A}(\mathscr{V} \cap \mathrm{Ker}\,\mathscr{C}) \subset \mathscr{V}$$

or, equivalently, if there exists an output injection transformation $J \in \mathbb{R}^{n\times p}$, with

$$(\mathscr{A} + J\mathscr{C})\mathscr{V} \subset \mathscr{V}.$$

If $\mathscr{V}$ is outer stable for $\mathscr{A} + J\mathscr{C}$, then $\mathscr{V}$ is called an **outer detectable** conditioned invariant subspace. A linear subspace $\mathscr{V}$ is called $(\mathscr{A}, \mathscr{B}, \mathscr{C})$ invariant if it is simultaneously controlled invariant and conditioned invariant.

The term **synchronization** is usually linked to a concept of stability requiring that the state trajectories of the coupled node systems converge asymptotically to each other. Thus, for the interconnected system (11.16) with input $u = 0$, we require

$$\lim_{t\to\infty} \|x_i(t) - x_j(t)\| = 0 \tag{11.17}$$

for all $i, j = 1, \ldots, N$. Here $\|\cdot\|$ denotes the Euclidean norm on $\mathbb{R}^n$. Let $\mathbf{e} = (1, \ldots, 1)^\top \in \mathbb{R}^N$, and let

$$\Delta_n = \mathbf{e} \otimes \mathbb{R}^n = \{\mathrm{col}\,(\xi, \ldots, \xi) \mid \xi \in \mathbb{R}^n\} \subset \mathbb{R}^{nN}$$

be the linear subspace spanned by the column vectors of the matrix $\mathbf{e} \otimes I_n$. Thus Δ_n defines the diagonal in the N-fold direct sum space $\mathbb{R}^n \oplus \ldots \oplus \mathbb{R}^n$. We refer to Δ_n as the **synchronization subspace**. Let

$$\mathrm{dist}(x, \Delta) = \min_{v\in\Delta} \|x - v\|$$

denote the distance of a point $x \in \mathbb{R}^{nN}$ to Δ_n. Then, for the global state of the network $x(t)$, the convergence property (11.17) is equivalent to

$$\lim_{t\to\infty} \mathrm{dist}(x(t), \Delta) = 0.$$

For our purposes this property is a bit too weak because it does not imply an invariance of Δ under the flow. We therefore give the following stricter definition. The spectrum of a matrix M, that is, the set of eigenvalues, is denoted by $\sigma(M)$. The set $\mathbb{C}_- = \{z \in \mathbb{C} \mid \mathrm{Re}(z) < 0\}$ denotes the left half-plane in the field of complex numbers.

Definition 11.10. The state interconnection matrix A **synchronizes** (11.16), or (11.14), is **synchronizable** by A provided the following conditions are satisfied:

(i) **Synchronization Preservation:** The subspace $\Delta = \mathbf{e} \otimes \mathbb{R}^n$ is invariant under $\mathscr{A} = I_N \otimes \alpha + (I_N \otimes \beta)A(I_N \otimes \gamma)$.
(ii) **Outer Stability:** For all initial values $x(0) \in \mathbb{R}^{nN}$ and input $u(t) = 0$, the solutions of (11.16) satisfy

$$\lim_{t\to\infty} \|x_i(t) - x_j(t)\| = 0 \quad \text{for all } i, j = 1, \ldots, N.$$

Similarly, system (11.14) is called synchronizable by state feedback, or output injection, provided conditions (i) and (ii) are true for $\gamma = I_n$ and $A \in \mathbb{R}^{nN\times pN}$ or $\beta = I_n$ and $A \in \mathbb{R}^{mN\times nN}$, respectively. In either case, the restriction of $I_N \otimes \alpha + (I_N \otimes \beta)A(I_N \otimes \gamma)$ on the invariant subspace Δ_n is called the **synchronization dynamics**.

Obviously, the attractivity condition (ii) for synchronization is equivalent to the condition that the corestriction $\mathscr{A}|_{\mathbb{R}^{nN}/\Delta}$ is Hurwitz. Thus one obtains the following proposition.

Proposition 11.11. *The state interconnection matrix A synchronizes (11.16) if and only if Δ_n is an **outer stable** invariant subspace for $\mathscr{A} = I_N \otimes \alpha + (I_N \otimes \beta)A(I_N \otimes \gamma)$.*

Note that $\mathscr{A}$ is, in output feedback form,

$$\mathscr{A} = I_N \otimes \alpha + (I_N \otimes \beta)A(I_N \otimes \gamma) \tag{11.18}$$

for the decoupled system $(I_N \otimes \alpha, I_N \otimes \beta, I_N \otimes \gamma)$. Moreover, for $\gamma = I_n$, (11.18) is in state feedback form, while for $\beta = I_n$ one obtains output injection transformations. Thus, in view of Exercise 1, the synchronizability of (11.14) implies that Δ is both an outer detectable conditioned invariant and outer stabilizable controlled invariant subspace of the decoupled system $(I_N \otimes \alpha, I_N \otimes \beta, I_N \otimes \gamma)$. However, this reformulation in terms of geometric control theory does not take into consideration the underlying graph structure that defines the network. The problem is that no characterization of conditioned invariant subspaces is known such that the associated output injection transformations have a prescribed pattern of unconstrained entries and zero entries.

Definition 11.10 imposes no restrictions on the synchronization dynamics. In particular, it is very well possible and allowable that the synchronization dynamics are asymptotically stable, which sounds counterintuitive. In many studies of the synchronization phenomena, therefore, additional assumptions, such as marginal stability of the synchronization dynamics, are imposed. In the sequel we will not require such additional assumptions because they are often easily handled in a second design step.

The outer stability condition can be replaced by the equivalent condition that there exists a solution trajectory of the form $\mathbf{e} \otimes \xi(t) \in \Delta$ of (11.16) such that

$$\lim_{t\to\infty} \|x_i(t) - \xi(t)\| = 0 \quad \text{for all} \quad i = 1,\dots,N.$$

The existence of $\xi(t)$ is easily established; see, for example, the proof of Theorem 11.13.

We now aim at characterizing networks that are synchronizable. For simplicity, we focus on a special class of interconnection matrices that have been treated mainly in the literature on synchronization and consensus.

Definition 11.12. A state interconnection matrix A is called **decomposable** if there exist real matrices $L \in \mathbb{R}^{N\times N}, K \in \mathbb{R}^{m\times p}$, with

$$A = L \otimes K.$$

The matrix L then carries the main information about the underlying graph structure of the network, while K presents a uniform connection structure between the individual inputs and outputs of the nodes.

Theorem 11.13. *Assume that* $(\alpha,\beta,\gamma) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m} \times \mathbb{R}^{p\times n}$ *satisfies* $\operatorname{rk}\beta = m, \operatorname{rk}\gamma = p$. *Let* $\mathbf{e} = (1,\dots,1)^\top \in \mathbb{R}^N$. *Then (11.16) is synchronized by* $A = L\otimes K$ *if and only if the following properties are satisfied:*

1. $L\mathbf{e} = \lambda\mathbf{e}$ *for some* $\lambda \in \mathbb{R}$. *Either* λ *is a simple eigenvalue of* L *or* $\alpha + \lambda\beta K\gamma$ *is Hurwitz.*
2. $\alpha + \mu\beta K\gamma$ *is Hurwitz for all other eigenvalues* $\mu \neq \lambda$ *of* L.

Proof. We first prove the sufficiency part. Let $SLS^{-1} = J$ be in Jordan canonical form, with eigenvalues $\lambda_1,\dots,\lambda_n$ (counted with multiplicities). Then

$$(S\otimes I)\mathscr{A}(S^{-1}\otimes I) = I_N \otimes \alpha + J \otimes \beta K\gamma.$$

Thus, without loss of generality, one can assume that L is in Jordan canonical form with a block upper triangular matrix

$$I_N \otimes \alpha + J\otimes\beta K\gamma = \begin{pmatrix} \alpha+\lambda_1\beta K\gamma & * & \dots & * \\ 0 & \alpha+\lambda_2\beta K\gamma & * & \vdots \\ \vdots & & \ddots & * \\ 0 & \dots & 0 & \alpha+\lambda_N\beta K\gamma \end{pmatrix}.$$

Without loss of generality, assume that $\lambda = \lambda_1$. Clearly, $\mathbf{e}$ is an eigenvector of L if and only $L\mathbb{R}\mathbf{e} = \mathbb{R}\mathbf{e}$. Equivalently, for all $v\in\mathbb{R}^n$, we have that $\mathscr{A}(\mathbf{e}\otimes v) = (I_N\otimes\alpha + L\otimes\beta K\gamma)(\mathbf{e}\otimes v) = \mathbf{e}\otimes(\alpha v + \lambda\beta K\gamma v) \in \Delta$, i.e., $\mathscr{A}\Delta \subset \Delta$. Thus $\mathbf{e}$ is an eigenvector of L if and only if Δ is $\mathscr{A}$-invariant. If $\alpha+\lambda\beta K\gamma$ is Hurwitz, then, by condition 2, all block matrices $\alpha+\lambda_i\beta K\gamma$ are Hurwitz. Thus $\mathscr{A}$ is Hurwitz. Moreover, Δ is $\mathscr{A}$-invariant. Thus $A = L\otimes K$ synchronizes. If $\lambda = \lambda_1$ is a simple eigenvalue of L, then $(S^{-1}\otimes I)\Delta$ coincides with the subspace spanned by the first n basis vectors for the matrix $\alpha + \lambda\beta K\gamma$. By condition 2, all solutions of the transformed system $(I_N \otimes \alpha + J \otimes \beta K\gamma)$ converge to $(S^{-1} \otimes I)\Delta$. This again proves synchronization.

For the converse direction, note that the $\mathscr{A}$-invariance of Δ implies that $\mathbf{e}$ is an eigenvector of L. Assume that λ is not a simple eigenvalue, and $\alpha + \lambda\beta K\gamma$ is not Hurwitz. For simplicity assume, for example, that L contains a Jordan block of the form

$$\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}.$$

The other cases are treated similarly. Then $I \otimes \alpha + J \otimes \beta K\gamma$ is upper triangular and contains a diagonal block of the form

$$\begin{pmatrix} \alpha + \lambda\beta K\gamma & \beta K\gamma \\ 0 & \alpha + \lambda\beta K\gamma \end{pmatrix}.$$

But the system

$$\begin{aligned} \dot{x}_1 &= (\alpha + \lambda\beta K\gamma)x_1 + \beta K\gamma x_2, \\ \dot{x}_2 &= (\alpha + \lambda\beta K\gamma)x_2 \end{aligned}$$

is not synchronizing, as can be seen by choosing $x_2 = 0$ and x_1 in the unstable eigenspace of $\alpha + \lambda\beta K\gamma$. This proves the first condition. The second condition follows by similar reasoning as above. ■

Theorem 11.13 shows that the synchronization task for an interconnection matrix $L \otimes K$ is equivalent to a robust output feedback stabilization task. Such problems are in general hard to solve. In the next section, we derive a sufficient condition in the SISO case. The problem becomes simpler if we restrict ourselves to state feedback transformations, i.e., for $\gamma = I_n$ and $A = L \otimes K$, with $K \in \mathbb{R}^{m\times n}$. In that case, it is possible to solve the synchronization task by a simple constructive procedure based on algebraic Riccati equations. This is well known if the network is defined by a weighted Laplacian; see, for instance, Tuna (2009), and is still a viable approach for more general interconnection structures. The next result is a simple improvement of standard linear regulator theory.

Lemma 11.14. *Let (α, β) be stabilizable and $P = P^\top > 0, P \in \mathbb{R}^{n\times n}$ be the unique symmetric and positive definite solution of the algebraic Riccati equation*

$$\alpha^\top P + P\alpha - P\beta\beta^\top P + I_n = 0. \tag{11.19}$$

Then for all $\lambda \in \mathbb{C}$, with $\mathrm{Re}(\lambda) \geq \frac{1}{2}$, one obtains

$$\sigma(\alpha - \lambda\beta\beta^\top P) \subset \mathbb{C}_-.$$

Proof. Since $(\alpha,\beta,\gamma=I_n)$ is stabilizable and detectable, there exists a unique symmetric positive definite solution P of the algebraic Riccati equation (11.19). Thus, for complex numbers λ, one obtains

$$\begin{aligned}
&P(\alpha-\lambda\beta\beta^\top P)+(\alpha^\top-\bar{\lambda}P\beta\beta^\top)P\\
&=P\alpha+\alpha^\top P-2\,\mathrm{Re}(\lambda)P\beta\beta^\top P\\
&=-I_n+(1-2\,\mathrm{Re}(\lambda))P\beta\beta^\top P.
\end{aligned}$$

Since by assumption $1-2\mathrm{Re}(\lambda)\leq 0$, the matrix on the right-hand side of the equality is negative definite. The assertion then follows from Theorem 5.44 provided the pair

$$(F,G):=\Big(\alpha^\top-\bar{\lambda}P\beta\beta^\top, I_n+(2\,\mathrm{Re}(\lambda)-1)\,P\beta\beta^\top P\Big)$$

is reachable. But this is obvious, because $G=I_n+(2\,\mathrm{Re}(\lambda)-1)\,P\beta\beta^\top P$ is invertible. The result follows. ■

Theorem 11.15. *Assume that (α,β) is stabilizable and $\gamma:=I_n$. Let $P=P^\top$ be the unique positive definite symmetric solution of the algebraic Riccati equation*

$$\alpha^\top P+P\alpha-P\beta\beta^\top P+I_n=0.$$

Let $L\in\mathbb{R}^{N\times N}$ be a matrix with $L\boldsymbol{e}=\lambda\boldsymbol{e}$ and simple eigenvalue λ. Assume that all other eigenvalues $\mu\neq\lambda$ of L satisfy Re$(\mu)>0$ [or satisfy $\mathrm{Re}(\mu)<0$ *for all eigenvalues $\mu\neq\lambda$]. Choose $\tau\in\mathbb{R}$ such that for all eigenvalues $\mu\neq\lambda$ the inequality $\tau\mathrm{Re}(\mu)\geq\frac{1}{2}$ is fulfilled, and set $K:=\tau\beta^\top P\in\mathbb{R}^{m\times n}$. Then the state interconnection matrix $A=L\otimes K$ synchronizes the network (11.16).*

Proof. $\mathrm{Re}(\tau\mu)\geq\frac{1}{2}$, and thus, by Lemma 11.14, we obtain

$$\sigma(\alpha-\mu\beta K)=\sigma(\alpha-\mu\tau\beta\beta^\top P)\subseteq\mathbb{C}_-\,.$$

The result follows from Theorem 11.13. ■

The preceding result leads to an explicit sufficient condition for synchronization.

Corollary 11.16. *Assume that (α,β) is stabilizable and $\gamma:=I_n$. Let $P=P^\top$ be the unique positive definite symmetric solution of the algebraic Riccati equation*

$$\alpha^\top P+P\alpha-P\beta\beta^\top P+I_n=0.$$

Let L be the graph Laplacian of a weighted strongly connected digraph Γ, and let λ_2 denote the eigenvalue of L with smallest positive real part. Then $A=\tau L\otimes\beta^\top P$ synchronizes (11.16) for

$$\tau \geq \frac{1}{2\mathrm{Re}\ \lambda_2} > 0.$$

Proof. By Theorem 8.36, $\lambda_1 = 1$ is simple and all other eigenvalues $\lambda_i \neq 1$ have positive real part. Thus the result follows from Theorem 11.15. ∎

Stronger results can be obtained for restricted classes of node transfer functions. Recall that a square $m \times m$ strictly proper rational transfer function $G(z) = \gamma(zI_n - \alpha)^{-1}\beta$ is **positive real** if G has only poles in the closed left half-plane and

$$G(z) + G(\bar{z})^\top \succeq 0 \tag{11.20}$$

is positive semidefinite for all complex numbers z, with $\mathrm{Re}\ z > 0$. Let (α, β, γ) be reachable and observable. The **Positive Real Lemma** then asserts that $G(z)$ is positive real if and only if there exists a positive definite symmetric matrix P such that

$$\begin{aligned} -P\alpha - \alpha^\top P &\succeq 0, \\ \gamma &= \beta^\top P. \end{aligned} \tag{11.21}$$

Moreover, $G(z)$ is strictly positive real, i.e., (11.20) is valid for all z, with $\mathrm{Re}\ z \geq 0$, if and only if (11.21) is satisfied with $-P\alpha - \alpha^\top P \succeq 0$ being replaced by $-P\alpha - \alpha^\top P \succ 0$.

The following lemma is proven next.

Lemma 11.17. *Assume that (α, β, γ) is reachable and observable, with $m = p$ such that $G(z) = \gamma(zI_n - \alpha)^{-1}\beta$ is positive real. Then for all complex numbers λ with* $\mathrm{Re}\ \lambda > 0$, *the spectrum of $\alpha - \lambda\beta\gamma$ satisfies*

$$\sigma(\alpha - \lambda\beta\gamma) \subset \mathbb{C}_-.$$

Proof. By the Positive Real Lemma, a positive definite symmetric matrix P exists, with

$$\begin{aligned} -(\alpha - \lambda\beta\gamma)^* P - P(\alpha - \lambda\beta\gamma) &= -\alpha^\top P - P\alpha + \bar{\lambda}\gamma^\top\beta^\top P + \lambda P\beta\gamma \\ &= -\alpha^\top P - P\alpha + 2\mathrm{Re}\,\lambda\gamma^\top\gamma \\ &\succeq 2\mathrm{Re}\,\lambda\gamma^\top\gamma. \end{aligned}$$

The pair $(\alpha - \lambda\beta\gamma, \gamma)$ is observable. Thus every complex eigenvector v of $\alpha - \lambda\beta\gamma$ with eigenvalue w satisfies $\gamma v \neq 0$ and

$$-2\mathrm{Re}(w) v^* P v \geq 2\mathrm{Re}\ \lambda \|\gamma v\|^2 > 0.$$ ∎

This leads to the following simple sufficient condition for synchronization.

Theorem 11.18. *Let (α, β, γ) be reachable and observable, with $m = p$ such that $G(z) = \gamma(zI_n - \alpha)^{-1}\beta$ is positive real. Let L be the graph Laplacian of a weighted strongly connected digraph Γ. Then the state interconnection matrix $A = L \otimes I_m$ synchronizes the network (11.16).*

Proof. The eigenvalues of L are $\lambda_1 = 0, \lambda_2, \dots, \lambda_N$, with $0 < \mathrm{Re}(\lambda_2) \leq \dots \leq \mathrm{Re}(\lambda_N)$. Applying Lemma 11.17 one concludes that the eigenvalues of $\alpha - \lambda_i \beta\gamma$ have negative real part for $i = 2, \dots, N$. Thus the result follows from Theorem 11.13. ∎

11.3 Synchronization of Homogeneous Networks

In this section, we consider networks of linear systems that are SISO, that is, the node systems (11.14) are defined by reachable and observable systems $\alpha \in \mathbb{R}^{n\times n}$, $\beta \in \mathbb{R}^{n}$, and $\gamma \in \mathbb{R}^{1\times n}$. Let $g(z) := \gamma(zI_n - \alpha)^{-1}\beta$ denote the scalar strictly proper transfer function of the node system. Let

$$h(z) = \frac{1}{g(z)}$$

be the reciprocal of the transfer function. We allow for arbitrary multivariable interconnection matrices $(A, B, C) \in \mathbb{R}^{N\times N} \times \mathbb{R}^{N\times m} \times \mathbb{R}^{p\times N}$, with interconnection transfer function $\mathscr{N}(z) = C(zI_N - A)^{-1}B$. Thus the network transfer function is

$$\mathscr{N}_g(z) = C(h(z)I_N - A)^{-1}B,$$

i.e., can be represented as the composition

$$\mathscr{N}_g(z) = \mathscr{N}(h(z))$$

of the interconnection transfer function $\mathscr{N}(z)$ with $h(z)$. It is shown in Theorem 9.15 that a homogeneous network $(\mathscr{A}, \mathscr{B}, \mathscr{C})$ is reachable and observable if and only if (A, B, C) is reachable and observable. In the sequel we will always assume this. We next prove a simple frequency-domain characterization of the synchronizability of SISO node systems.

Theorem 11.19. *Assume that A satisfies $A\mathbf{e} = \lambda\mathbf{e}$, with λ a simple eigenvalue. Then A synchronizes the homogeneous network (11.16) if and only if*

$$h(\overline{\mathbb{C}_+}) \cap \sigma(A) \setminus \{\lambda\} = \emptyset. \tag{11.22}$$

Proof. Let $g(z) = \frac{p(z)}{q(z)}$ be a coprime factorization. Suppose A synchronizes (11.16). Let $\lambda_1 := \lambda, \lambda_2, \dots, \lambda_N$ denote the eigenvalues of A. By Theorem 11.13, the characteristic polynomials

$$\det(zI_n - (\alpha + \lambda_i \beta \gamma)) = q(z) - \lambda_i p(z)$$

are Hurwitz for $i = 2, \ldots, N$. This shows condition (11.22). Conversely, assume that (11.22) is satisfied. Then, for all $z \in \overline{\mathbb{C}_+}$ and all $2 \leq i \leq N$, one obtains $h(z) \neq \lambda_i$, i.e., $q(z) - \lambda_i p(z) \neq 0$. Thus $q(z) - \lambda_i p(z)$ is a Hurwitz polynomial for all $2 \leq i \leq N$. This completes the proof. ■

This leads to the following explicit description of synchronizing homogeneous networks.

Theorem 11.20. *N identical SISO minimal systems*

$$\begin{aligned} \dot{x}_i(t) &= \alpha x_i(t) + \beta u_i(t), \\ y_i(t) &= \gamma x_i(t) \end{aligned} \tag{11.23}$$

are synchronizable if and only if there exists a scalar proper rational real transfer function $f(z) \in \mathbb{R}(z)$ of McMillan degree $N-1$ with

$$f^{-1}(\infty) \cap h(\overline{\mathbb{C}_+}) = \emptyset.$$

Proof. Suppose (11.23) is synchronizable through an interconnection matrix A, where λ is a simple eigenvalue of A and $A\mathbf{e} = \lambda \mathbf{e}$. Thus A is similar to an upper triangular matrix

$$\begin{pmatrix} \lambda & A_{12} \\ 0 & A_{22} \end{pmatrix}$$

such that $I_{N-1} \otimes \alpha + A_{22} \otimes \beta\gamma$ is Hurwitz. The set of such matrices L_{22} is open, and therefore one can assume, without loss of generality, that A_{22} has $N-1$ distinct eigenvalues $\lambda_2, \ldots, \lambda_N$ that are distinct from λ. Thus we constructed a self-conjugate set of complex numbers $\lambda_2, \ldots, \lambda_r$ such that $q(z) - \lambda_i p(z)$ is Hurwitz for $i = 2, \ldots, r$. Choose a real transfer function $f(z)$ of McMillan degree $N-1$ that has its poles exactly at $\lambda_2, \ldots, \lambda_N$. Then $f^{-1}(\infty) \cap h(\overline{\mathbb{C}_+}) = \emptyset$. Conversely, assume that $f(z)$ is a real rational transfer function whose poles are disjoint from $h(\overline{\mathbb{C}_+})$. Choose a minimal realization $f(z) = c(zI_{N-1} - M)^{-1}b$, and let λ be a real number disjoint from the poles of f. Let $S \in GL_N(\mathbb{R})$ be such that $Se_1 = \mathbf{e}$. Then

$$A = S \begin{pmatrix} \lambda & 0 \\ 0 & M \end{pmatrix} S^{-1}$$

is an interconnection matrix that synchronizes. ■

The proof shows that the existence of synchronizing interconnection matrices is equivalent to an inverse eigenvalue problem with spectral constraints. For instance, to construct an interconnection matrix A with nonnegative entries requires solving the inverse eigenvalue problem of finding a nonnegative matrix with spectrum in $h(\overline{\mathbb{C}_+})$.

11.4 Polynomial Model Approach to Synchronization

Synchronization is usually defined and studied in the state-space context. However, our example of coupled oscillators from the introduction shows that it might be preferable to perform the study in a functional context and use the concept of state maps to see the connection with the state-space analysis. In this section we recall the preceding chapters to aim at a polynomial approach to synchronization. Our starting point is a polynomial matrix representation of the node system by taking N identical higher-order systems of differential equations

$$D_\ell(\sigma)\xi_i = N_\ell(\sigma)v_i, \quad i = 1,\dots,N. \tag{11.24}$$

Here $\sigma = \frac{d}{dt}$ denotes the differentiation operator, and $N_\ell(z) \in \mathbb{R}[z]^{p\times m}$ and $D_\ell(z) \in \mathbb{R}[z]^{p\times p}$ are polynomial matrices, with $D_\ell(z)$ nonsingular. We assume that the associated transfer function

$$G(z) = D_\ell(z)^{-1}N_\ell(z)$$

is strictly proper. More generally, one could incorporate outputs by considering the general class of Rosenbrock systems

$$\begin{aligned} T(\sigma)\xi_i &= U(\sigma)v_i, \\ w &= V(\sigma)\xi_i + W(\sigma)v_i. \end{aligned}$$

We will consider these systems later on. Of course, one can always replace the differentiation operator σ with the backward shift, thereby studying higher-order systems of difference equations. We freely use the terminology and results from Chapter 4.

To simplify our discussion, we will initially assume that the couplings between these systems are defined by a decomposable state interconnection matrix $A = L \otimes K \in \mathbb{R}^{mN\times pN}$, leading to a state feedback transformation

$$v = (L\otimes K)\xi + Bu.$$

As previously, matrix L incorporates the underlying interconnection structure defined by the graph of the network. The resulting interconnected system is then of the form

$$\Big(I_N \otimes D_\ell(\sigma) - L\otimes N_\ell(\sigma)K\Big)\xi = \big(I_N\otimes N_\ell(\sigma)\big)Bu. \tag{11.25}$$

Defining

$$\mathscr{T}(z) = I_N \otimes D_\ell(z) - L \otimes N_\ell(z)K,$$
$$\mathscr{U}(z) = \big(I_N \otimes N_\ell(z)\big)B,$$

one obtains the associated transfer function of the network (11.25) as

$$\Phi(z) = \mathscr{T}(z)^{-1}\mathscr{U}(z) = \Big(I_N \otimes D_\ell(z) - L \otimes N_\ell(z)K\Big)^{-1}\big(I_N \otimes N_\ell(z)\big)B.$$

Note that $\mathscr{T}(z)$ is factored as

$$\mathscr{T}(z) = (I_N \otimes D_\ell(z))(I_{pN} - L \otimes G(z)K),$$

and $I_{pN} - L \otimes G(z)K$ is biproper. Thus $\mathscr{T}(z)$ is nonsingular and $\Phi(z) = \mathscr{T}(z)^{-1}U(z) = (I_{pN} - L \otimes G(z)K)(I_N \otimes G(z))B)$ is strictly proper.

To study the effects of couplings, it is crucial to compare the relevant polynomial model spaces. Let X_{D_ℓ} denotes the polynomial model of the individual node system (11.24), so that

$$X_{I_N \otimes D_\ell} = X_{D_\ell} \oplus \cdots \oplus X_{D_\ell}$$

denote the state space of the decoupled system. Similarly,

$$X_{\mathscr{T}} = X_{I_N \otimes D_\ell - L \otimes N_\ell K}$$

denotes the state space of the interconnected system. The connection between these two spaces is clarified by the following lemma.

Lemma 11.21. *1. Let $V(z) \in \mathbb{R}[z]^{r\times r}, U(z) \in \mathbb{R}[z]^{r\times m}$ be polynomial matrices, with $V(z)$ nonsingular, such that the transfer function $V(z)^{-1}U(z)$ is strictly proper. Then, for a matrix $A \in \mathbb{R}^{m\times r}$, the polynomial model spaces X_V and X_{V-UA} are equal as sets and the map*

$$\pi_{V-UA} : X_V \longrightarrow X_{V-UA},$$
$$\pi_{V-UA}(f) = (V - UA)\pi_-((V - UA)^{-1}f)$$

defines an isomorphism of vector spaces (but in general not of $\mathbb{R}[z]$-modules).

2. In particular, $X_{I_N \otimes D_\ell} = X_{\mathscr{T}}$ as sets and the map

$$\pi_{\mathscr{T}} : X_{I_N \otimes D_\ell} \longrightarrow X_{\mathscr{T}}, \quad \pi_{\mathscr{T}}(f) = \mathscr{T}(z)\pi_-(\mathscr{T}(z)^{-1}f(z))$$

is an isomorphism of vector spaces. Moreover, each $f \in X_{D_\ell}$ satisfies

$$\pi_{\mathscr{T}}(\boldsymbol{e} \otimes f) = \boldsymbol{e} \otimes f. \tag{11.26}$$

Proof. By the strict properness of $V(z)^{-1}U(z)$, we see that

$$(V(z) - U(z)A)^{-1} = (I_r - V(z)^{-1}U(z)A)^{-1}V(z)^{-1}$$

is fulfilled, with $I_r - V(z)^{-1}U(z)A$ biproper. Therefore, if f is a vector of polynomials, then $V(z)^{-1}f(z)$ is strictly proper if and only if $(V(z) - U(z)A)^{-1}f(z)$ is strictly proper. This shows the equality of the polynomial models $X_V = X_{V-UA}$ as sets. The linearity of π_{V-UA} is obvious. Suppose $f \in X_V$ satisfies $(V - UA)\pi_-((V - UA)^{-1}f) = 0$. Then $g(z) = (V(z) - U(z)A)^{-1}f(z) = (I_r - V(z)^{-1}U(z)A)^{-1}V(z)^{-1}f(z)$ is a polynomial. Since $V(z)^{-1}f(z)$ is strictly proper and $I_r - V(z)^{-1}U(z)A$ is biproper, this implies that the polynomial $g(z) = (V(z) - U(z)A)^{-1}f(z)$ is strictly proper. Therefore, $f = 0$. This shows the injectivity of π_{V-UA}. For surjectivity, let f be an element of $X_{V-UA} = X_V$. Since π_{V-UA} is a projection operator, $\pi_{V-UA}(f) = f$. This shows the surjectivity of π_{V-UA} and completes the proof of the first claim.

The second claim is a simple consequence of the first claim by setting $V = I_N \otimes D_\ell, U = I_N \otimes N_\ell, A = L \otimes K$. Finally, (11.26) follows from a simple calculation

$$\begin{aligned}
\pi_{\mathscr{T}}(\mathbf{e} \otimes f) &= \mathscr{T}\pi_-(\mathscr{T}^{-1}\mathbf{e} \otimes f) = \mathscr{T}\pi_-\Big((I_{pN} - L \otimes G(z)K)^{-1}(\mathbf{e} \otimes D_\ell^{-1}f)\Big) \\
&= \mathscr{T}((I_{pN} - L \otimes G(z)K)^{-1}\mathbf{e} \otimes D_\ell^{-1}f) \\
&= (I_N \otimes D_\ell)\mathbf{e} \otimes D_\ell^{-1}f \\
&= \mathbf{e} \otimes f.
\end{aligned}$$

Here the third equation is true since $D_\ell^{-1}f$ is strictly proper and $(I_{pN} - L \otimes G(z)K)^{-1}$ is biproper. ■

One can identify X_{D_ℓ} with the diagonal in the direct sum $X_{I_N \otimes D_\ell} = X_{D_\ell} \oplus \ldots \oplus X_{D_\ell}$, i.e.,

$$X_{D_\ell} \simeq \{\mathbf{e} \otimes f(z) \mid f \in X_{D_\ell}\} \subset X_{I_N \otimes D_\ell}.$$

Similarly, using the identity (11.26), we define the diagonal in $X_{\mathscr{T}}$ as

$$\Delta := \{\mathbf{e} \otimes f(z) \in X_{\mathscr{T}} \mid f \in X_{D_\ell}\}.$$

Δ is called the **synchronization space** of the network (11.25). Let $S_{I_N \otimes D_\ell}$ and $S_{\mathscr{T}}$ denote the shift operators on the polynomial models $X_{I_N \otimes D_\ell}$ and $X_{\mathscr{T}}$, respectively. By Lemma 11.21, we know that the vector space isomorphism $\pi_{\mathscr{T}}$ is not a module homomorphism, i.e., $\pi_{\mathscr{T}}$ is not commuting with the shifts. We now show that by restricting $\pi_{\mathscr{T}}$ to the respective diagonal spaces a module isomorphism is obtained.

Lemma 11.22. *Assume that* $\boldsymbol{e}$ *is an eigenvector of* L *with eigenvalue* λ.

1. *The restriction of* $\pi_{\mathscr{T}}$ *on the diagonal* X_{D_ℓ} *of* $X_{I_N\otimes D_\ell}$

$$\tau : X_{D_\ell} \longrightarrow \Delta,\ \tau(f) = \boldsymbol{e}\otimes f$$

is an isomorphism of $\mathbb{R}$*-vector spaces.*

2. Δ *is an invariant subspace of the shift, i.e.,* $S_{\mathscr{T}}(\Delta)\subset\Delta$.
3. *There is a conjugacy of the shift operators*

$$S_{\mathscr{T}}|\Delta \simeq S_{D_\ell-\lambda N_\ell K}.$$

In particular,

$$\det(zI - S_{\mathscr{T}}|\Delta) = \det(D_\ell(z)-\lambda N_\ell(z)).$$

Proof. That τ defines an isomorphism of vector spaces is an obvious consequence of Lemma 11.21. Each vector of polynomials $g(z)$ satisfies the identity $\mathscr{T}(z)\mathbf{e}\otimes g(z) = \mathbf{e}\otimes D_\ell(z)g(z) - L\mathbf{e}\otimes N_\ell(z)Kg(z) = \mathbf{e}\otimes(D_\ell(z)-\lambda N_\ell(z)K)g(z)$. Thus each vector of polynomials f satisfies $\mathscr{T}(z)\mathbf{e}\otimes(D_\ell-\lambda N_\ell K)^{-1}f = \mathbf{e}\otimes f$, and therefore

$$\mathscr{T}(z)^{-1}(\mathbf{e}\otimes f(z)) = \mathbf{e}\otimes(D_\ell(z)-\lambda N_\ell(z)K)^{-1}f(z).$$

For vector polynomials f, Lemma 11.21 implies that $f\in X_{D_\ell}$ if and only if $f\in X_{D_\ell-\lambda N_\ell K}$. Thus $f\in X_{D_\ell}$ satisfies

$$\begin{aligned}
S_{\mathscr{T}}(\mathbf{e}\otimes f) &= \mathscr{T}\pi_-(\mathscr{T}^{-1}(\mathbf{e}\otimes zf))\\
&= (I_N\otimes D_\ell - L\otimes N_\ell K)\pi_-(\mathbf{e}\otimes(D_\ell-\lambda N_\ell K)^{-1}zf)\\
&= \mathbf{e}\otimes D_\ell\pi_-((D_\ell-\lambda N_\ell K)^{-1}zf) - \mathbf{e}\otimes\lambda N_\ell K\pi_-((D_\ell-\lambda N_\ell K)^{-1}zf)\\
&= \mathbf{e}\otimes\pi_{D_\ell-\lambda N_\ell K}(zf)\in\Delta.
\end{aligned}$$

Here the second equation follows from identity (11.4). This shows the invariance of Δ under the shift $S_{\mathscr{T}}$ as well as the conjugacy of $S_{\mathscr{T}}|\Delta$, with SD_ℓ. This completes the proof. ∎

With all these facts at hand, we proceed to define synchronization for a higher-order system as follows.

Definition 11.23. Let $G(z) = D_\ell(z)^{-1}N_\ell(z)$ be a left coprime factorization of the strictly proper transfer function $G(z)$, and let $\mathscr{T}(z) = I_N\otimes D_\ell(z) - L\otimes N_\ell(z)K$. The network (11.25) of higher-order systems synchronizes with the interconnection matrix $L\otimes K$ if it synchronizes for the shift realization on $X_{\mathscr{T}}$. Equivalently, then $\Delta\subset X_{\mathscr{T}}$ is an outer stable invariant subspace for the shift operator $S_{\mathscr{T}}$ on $X_{\mathscr{T}}$.

We thus have defined the synchronization of polynomial models (11.25) by the synchronization of the associated shift realizations. For further clarification, let us consider the polynomial models and associated shift operators in more detail. Let (α,β,γ) denote the shift realization of the node identical transfer function $G(z) = D_\ell(z)^{-1}N_\ell(z)$. Thus the shift realization of the decoupled transfer function $I_N \otimes D_\ell(z)^{-1}N_\ell(z)$ on the state space $X_{I_N\otimes D_\ell}$ is

$$\begin{aligned}
(I_N\otimes\alpha)f &= S_{I_N\otimes D_\ell}f = \pi_{I_N\otimes D_\ell}(zf),\\
(I_N\otimes\beta)\xi &= \pi_{I_N\otimes D_\ell}(I_N\otimes N_\ell(z)\xi) = (I_N\otimes N_\ell(z))\xi,\\
(I_N\otimes\gamma)f &= ((I_N\otimes D_\ell)^{-1}f)_{-1}.
\end{aligned}$$

Similarly, the shift operator of the transfer function $\mathscr{T}(z)^{-1}\mathscr{U}(z) = (I_N\otimes D_\ell(z) - L\otimes N_\ell(z)K)^{-1}N_\ell(z)B$ on the state space $X_{\mathscr{T}}$ is

$$S_{\mathscr{T}}f = \pi_{\mathscr{T}}(zf).$$

Note that for $f\in X_{I_N\otimes D_\ell}$ one has $(\mathscr{T}^{-1}f)_{-1} = ((I_N\otimes D_\ell)^{-1}f)_{-1}$. Therefore,

$$\begin{aligned}
S_{\mathscr{T}}f &= \mathscr{T}\pi_-(z\mathscr{T}^{-1}f)\\
&= zf - \mathscr{T}\pi_+(z\mathscr{T}^{-1}f)\\
&= zf - \mathscr{T}(\mathscr{T}^{-1}f)_{-1}\\
&= zf - \mathscr{T}((I_N\otimes D_\ell)^{-1}f)_{-1}.
\end{aligned}$$

Similarly, for $f\in X_{I_N\otimes D_\ell}$,

$$S_{I_N\otimes D_\ell}f = I_N\otimes D_\ell\pi_-(z(I_N\otimes D_\ell)^{-1}f) = zf - (I_N\otimes D_\ell)(I_N\otimes D_\ell^{-1}f)_{-1}.$$

Using $\mathscr{T}(z) = I_N\otimes D_\ell(z) - L\otimes N_\ell(z)K$, this shows the identity of linear operators on $X_{I_N\otimes D_\ell} = X_{\mathscr{T}}$:

$$S_{\mathscr{T}}f = S_{I_N\otimes D_\ell}f + L\otimes N_\ell K(D_\ell^{-1}f)_{-1}.$$

Using the N-fold direct sum decomposition $X_{I_N\otimes D_\ell} = X_{D_\ell}\oplus\cdots\oplus X_{D_\ell}$, the shift operator $S_{\mathscr{T}}$ has the form

$$\begin{aligned}
S_{\mathscr{T}} &= I_N\otimes\alpha + (I_N\otimes\beta)(L\otimes K)(I_N\otimes\gamma)\\
&= I_N\otimes\alpha + L\otimes\beta K\gamma.
\end{aligned}$$

Therefore, Δ is an outer stable invariant subspace for the shift $S_{\mathscr{T}}$ if and only if $\Delta\subset X_{I_N\otimes D_\ell}$ is an outer stable invariant subspace for $I_N\otimes\alpha + L\otimes\beta K\gamma$. This leads us to the following characterization of synchronization in the polynomial model $X_{\mathscr{T}}$.

Theorem 11.24. *Let $G(z) = D_\ell(z)^{-1}N_\ell(z)$ be a left coprime factorization of the strictly proper transfer function, and let (α, β, γ) be a minimal realization of $G(z)$. Let $A = L \otimes K$ be a decomposable state interconnection matrix. Assume that $L\boldsymbol{e} = \lambda \boldsymbol{e}$ for a simple eigenvalue λ. The following statements are equivalent:*

(a) The network (11.16) of higher-order systems synchronizes.
(b) Δ is an outer stable subspace for $I_N \otimes \alpha + L \otimes \beta K \gamma$.
(c) The following two conditions are satisfied:

(c1) $S_{\mathscr{T}}(\Delta) \subset \Delta$.
(c2) For all eigenvalues $\mu \neq \lambda$ of L,

$$\det\,(D_\ell(z) - \mu N_\ell(z)K)$$

is a Hurwitz polynomial.

Proof. The shift operator $S_{\mathscr{T}}$ is isomorphic to $\mathscr{A} = I_N \otimes \alpha + L \otimes \beta K \gamma$. Thus $\dot{x} = \mathscr{A}x$ synchronizes if and only if $S_{\mathscr{T}}(\Delta) \subset \Delta$ and the corestriction of $S_{\mathscr{T}}$ on $X_{\mathscr{T}}/\Delta$ has only eigenvalues with negative real part. Thus (a) and (b) are equivalent. It was shown in Lemma 11.22 that the restriction operator $S_{\mathscr{T}}|\Delta$ is conjugate to the shift operator $S_{D_\ell(z)-\lambda N_\ell(z)K}$. Therefore, $\det \mathscr{T}(z) = \det(zI - S_{\mathscr{T}})$ and

$$\det(zI - S_{\mathscr{T}}|\Delta) = \det S_{D_\ell(z)-\lambda N_\ell(z)K} = \det(D_\ell(z) - \lambda N_\ell(z)K).$$

Let $\lambda = \lambda_1, \ldots, \lambda_N$ denote the not necessarily distinct eigenvalues of L. Following the proof of Theorem 11.13 we see that

$$\det \mathscr{T}(z) = \prod_{i=1}^{N} \det(D_\ell(z) + \lambda_i N_\ell(z)K).$$

Since $\det(S_{\mathscr{T}}|\Delta) = \det(D_\ell(z) - \lambda N_\ell(z)K)$, we conclude from

$$\det \mathscr{T} = \det(S_{\mathscr{T}}|\Delta)\det(S_{\mathscr{T}}|(X_{\mathscr{T}}/\Delta))$$

that

$$\prod_{i=2}^{N} \det(D_\ell(z) - \lambda_i N_\ell(z)K)$$

equals the characteristic polynomial of the corestriction $S_{\mathscr{T}}|(X_{\mathscr{T}}/\Delta)$. This completes the proof. ∎

In the preceding approach, synchronization of a higher-order system (11.24) was defined via synchronization of the associated shift realization. One wonders if a direct approach is possible. We consider the synchronization task for Rosenbrock node systems of the general form

$$\begin{aligned} T(\sigma)\xi_i &= U(\sigma)v_i, \\ w_i &= V(\sigma)\xi_i + W(\sigma)v_i, \quad i = 1,\ldots,N. \end{aligned} \tag{11.27}$$

We assume that $T(z)$ is a nonsingular $r \times r$ polynomial matrix and that the transfer function

$$G(z) = V(z)T(z)^{-1}U(z) + W(z)$$

is strictly proper. We use the following simplified notation for the decoupled system as

$$\mathscr{T} = I_N \otimes T, \quad \mathscr{U} = I_N \otimes U, \quad \mathscr{V} = I_N \otimes V.$$

We consider couplings among the partial state components defined by a coupling matrix $A \in \mathbb{R}^{mN\times pN}$. In the sequel, we will restrict ourselves to a special class of interconnection matrices.

Definition 11.25. An $N \times N$ block matrix $A = (A_{ij})$ with blocks $A_{ij} \in \mathbb{R}^{m\times p}$ is called **admissible** provided

$$< A >:= \sum_{j=1}^{N} A_{1j} = \cdots = \sum_{j=1}^{N} A_{Nj}.$$

This definition is general enough to cover all preceding cases of interest. Thus every decomposable matrix $A = L\otimes K$, with L a Laplacian matrix, is admissible, as is every finite sum $A = L_1 \otimes K_1 + \ldots \otimes + L_s \otimes K_s$, with $N \times N$ Laplacian matrices $L_1,\ldots,L_s$. Thus the interconnected closed-loop system on partial states is

$$\Big(\mathscr{T}(\sigma) - \mathscr{U}(\sigma)A\mathscr{V}(\sigma)\Big)\xi = 0. \tag{11.28}$$

Define the **partial state synchronization space** of (11.27) as

$$\Delta_{\mathrm{ps}} := \{(\xi,\ldots,\xi) \mid \xi \in \mathbb{R}^r\} \subset \mathbb{R}^{rN}.$$

Similarly, we define the **state synchronization space** as

$$\Delta := \{(f_1,\ldots,f_N) \in X_{\mathscr{T}-\mathscr{U}A\mathscr{V}} \mid f_1 = \cdots = f_N\}.$$

Here $X_{\mathscr{T}-\mathscr{U}A\mathscr{V}}$ denotes the polynomial model associated with the nonsingular polynomial matrix $\mathscr{T}(z) - \mathscr{U}(z)A\mathscr{V}(z)$. Note that, in general, the two spaces do not have the same dimension. For admissible interconnection matrics the polynomial matrix $\mathscr{T}(z)-\mathscr{U}(z)A\mathscr{V}(z)$ is nonsingular and maps polynomial vectors of the form $\mathbf{e}\otimes f(z)$ onto polynomial vectors $\mathbf{e}\otimes g(z)$. More precisely, for $f(z)\in\mathbb{R}[z]^r$, one has

$$\Big(\mathscr{T}(z)-\mathscr{U}(z)A\mathscr{V}(z)\Big)\mathbf{e}\otimes f(z)=\mathbf{e}\otimes(T-U<A>V)f. \tag{11.29}$$

We proceed by giving two definitions of synchronization, one for the partial states and the other one in the state space.

Definition 11.26. Let A be an admissible interconnection matrix.

(a) The **partial state** system (11.28) **synchronizes** provided all solutions $\xi(t)=(\xi_1(t),\ldots,\xi_N(t))$ of (11.28) satisfy

$$\lim_{t\to\infty}\|\xi_i(t)-\xi_j(t)\|=0. \tag{11.30}$$

(b) The **shift realization** of (11.28) **synchronizes** provided $\Delta\subset X_{\mathscr{T}-\mathscr{U}A\mathscr{V}}$ is an outer stable invariant subspace of the shift operator $S_{\mathscr{T}-\mathscr{U}A\mathscr{V}}:X_{\mathscr{T}-\mathscr{U}A\mathscr{V}}\longrightarrow X_{\mathscr{T}-\mathscr{U}A\mathscr{V}}$.

We next prove that these two definitions are actually equivalent and derive a polynomial matrix characterization. Our result is a natural extension of Theorem 11.24.

Theorem 11.27. *Partial state synchronization of the Rosenbrock system (11.28) is equivalent to the synchronization of the associated shift realization. In either case, synchronization is satisfied if and only if*

$$\frac{\det(\mathscr{T}(z)-\mathscr{U}(z)A\mathscr{V}(z))}{\det(T(z)-U(z)<A>V(z))}$$

is a Hurwitz polynomial.

Proof. By assumption on A, the higher-order system (11.28) induces a higher-order system on the quotient space $\mathbb{R}^{rN}/\Delta_{\mathrm{ps}}$. Thus the asymptotic stability condition (11.30) is equivalent to the asymptotic stability of the induced system on $\mathbb{R}^{rN}/\Delta_{\mathrm{ps}}$. In view of (11.29), this in turn is equivalent to the polynomial

$$\frac{\det(\mathscr{T}(z)-\mathscr{U}(z)A\mathscr{V}(z))}{\det(T(z)-U(z)<A>V(z))}$$

being Hurwitz. Similarly, (11.28) synchronizes for the shift realization if and only if Δ is an outer stable invariant subspace for the shift operator $S_{\mathscr{T}-\mathscr{U}A\mathscr{V}}$ on $X_{\mathscr{T}-\mathscr{U}A\mathscr{V}}$. The assumption on A implies that Δ is invariant under the shift. In fact, for $f=(f_1,\ldots,f_N)=\mathbf{e}\otimes g\in\Delta$, and writing $\hat{T}:=\mathscr{T}-\mathscr{U}A\mathscr{V}$ for short,

$$\begin{aligned} S_{\hat{T}}(f) &= \hat{T}\pi_-(\hat{T}^{-1}zf) \\ &= \hat{T}\pi_-(\mathbf{e}\otimes(T-U<A>V)^{-1}zg) \\ &= \mathbf{e}\otimes(T-U<A>V)\pi_-(\mathbf{e}\otimes(T-U<A>V)^{-1}zg) \\ &= \mathbf{e}\otimes S_{T-U<A>V}(f)\in\Delta. \end{aligned} \tag{11.31}$$

This proves the invariance of Δ. For a nonsingular polynomial matrix $D_\ell(z)$, the characteristic polynomial of the shift operator S_{D_ℓ} on X_{D_ℓ} coincides with $\det D_\ell(z)$. By (11.31), the restriction of the shift $S_{\hat{T}}$ on Δ is conjugate to the shift operator $S_{T-U<A>V}$. Therefore, the characteristic polynomial of the $S_{\hat{T}}|\Delta$ is equal to $\det(T(z)-U(z)<A>V(z))$. Thus the characteristic polynomial of the corestriction $S_{\hat{T}}$ on $X_{\hat{T}}/\Delta$ is equal to

$$\frac{\det(S_{\hat{T}})}{\det(S_{\hat{T}}|\Delta)}=\frac{\det(S_{\hat{T}})}{\det(T(z)-U(z)<A>V(z))}.$$

This completes the proof. ■

The following invariance principle states that partial state synchronization holds irrespective of the choice of coprime factorization for the node models.

Theorem 11.28. *Suppose that the Rosenbrock node systems $\Sigma_{VT^{-1}U+W}$ and $\Sigma_{\overline{VT}^{-1}\overline{U}+\overline{W}}$ are strictly system equivalent. Let A be an admissible interconnection matrix. Then partial state synchronization for $\mathscr{T}(z)-\mathscr{U}(z)A\mathscr{V}(z)$ is satisfied if and only if it is satisfied for $\overline{\mathscr{T}}(z)-\overline{\mathscr{U}}(z)A\overline{\mathscr{V}}(z)$.*

Proof. By Corollary 9.6, the two networks obtained from $\Sigma_{VT^{-1}U+W}$ and $\Sigma_{\overline{VT}^{-1}\overline{U}+\overline{W}}$ by coupling them with the same interconnection matrix A are strictly system equivalent. In fact, by a careful inspection of the proof of Corollary 9.6, the strict system equivalences can be seen to preserve the sets of synchronized states Δ and $\overline{\Delta}$, respectively. Thus

$$\det(\mathscr{T}(z)-\mathscr{U}(z)A\mathscr{V}(z))=\det(\overline{\mathscr{T}}(z)-\overline{\mathscr{U}}(z)A\overline{\mathscr{V}}(z)).$$

Moreover, the same is true of the determinants of the corestrictions. The result follows. ■

Output Synchronization The preceding results lead to a simple characterization of output synchronization. To simplify matters, we will work with discrete-time state-space systems. We begin with deriving a simple version of the internal model principle. Consider a linear discrete-time system

$$\begin{aligned} x(t+1) &= Ax(t), \\ y(t) &= Cx(t). \end{aligned}$$

Then the set of possible output sequences $(y(t)|t \in \mathbb{N})$ is

$$\mathscr{Y} = \{(CA^t x_0) \mid x_0 \in \mathbb{R}^n\}$$

or, equivalently,

$$\mathscr{Y} = CX^{zI-A}.$$

We refer to $\mathscr{Y}$ as the **output behavior** of (C,A). If (C,A) is observable, then clearly this defines an autonomous behavior. In fact, for a left coprime factorization

$$C(zI-A)^{-1} = D_\ell(z)^{-1}N_\ell(z)$$

and (C,A) observable, Proposition 4.36 implies that

$$CX^{zI-A} = X^{D_\ell}.$$

Next we consider a second discrete-time system in first-order form as

$$\begin{aligned} x_1(t+1) &= A_1 x_1(t), \\ y_1(t) &= C_1 x_1(t). \end{aligned}$$

Let $\mathscr{Y}_1 = C_1 X^{zI-A_1}$ denote the associated output behavior. We say that the output behavior $\mathscr{Y}$ is a subbehavior of $\mathscr{Y}_1$ whenever $\mathscr{Y} \subset \mathscr{Y}_1$ or, equivalently, whenever

$$CX^{zI-A} \subset C_1 X^{zI-A_1}.$$

We arrive at the following version of the **internal model principle**, i.e., the characterization of subbehaviors of an autonomous behavior.

Proposition 11.29. *Assume that the pairs (C,A) and (C_1,A_1) are observable. Then $\mathscr{Y}$ is a subbehavior of $\mathscr{Y}_1$ if and only if there exists an invertible transformation $T \in GL_{n_1}(\mathbb{R})$ such that*

$$TA_1T^{-1} = \begin{pmatrix} A & A_1'' \\ 0 & A_1' \end{pmatrix}, \quad C_1T^{-1} = \begin{pmatrix} C & C_1' \end{pmatrix}.$$

Proof. Using the observability of (C_1,A_1), one obtains the left coprime factorizations

$$C_1(zI-A_1)^{-1} = D_{\ell,1}(z)^{-1}N_{\ell,1}(z).$$

Moreover, (C_1,A_1) is isomorphic to the shift realization of $D_{\ell,1}^{-1}N_{\ell,1}$, and similarly for (C,A). Proposition 4.36 implies that $\mathscr{Y}_1 = X^{D_{\ell,1}}$. Therefore, $\mathscr{Y}$ is a subbehavior of $\mathscr{Y}_1$ if and only if the inclusion

$$X^{D_\ell} \subset X^{D_{\ell,1}}.$$

By Theorem 3.35, the subspace X^{D_ℓ} is a submodule of $X^{D_{\ell,1}}$, i.e., $S_{D_{\ell,1}}X^{D_\ell} \subset X^{D_\ell}$. In particular, from Theorem 4.26 one obtains the equivalence of shift realizations $A_1|X^{D_\ell} \simeq S_{D_{\ell,1}}|X^{D_\ell} \simeq S_{D_\ell} \simeq A$, and $C_1|X^{D_\ell} \simeq C$. This completes the proof. ∎

Now we apply these ideas to output synchronization. Consider an observable pair (γ,α) with left coprime factorization

$$\gamma(zI-\alpha)^{-1} = D_\ell(z)^{-1}N_\ell(z).$$

Then the interconnected system with coupling matrix $A = L\otimes K$ is

$$\begin{aligned} x(t+1) &= \mathscr{A}x(t), \\ y(t) &= \mathscr{C}x(t), \end{aligned} \tag{11.32}$$

where $\mathscr{A} = I_N\otimes\alpha + L\otimes K\gamma$ and $\mathscr{C} = I\otimes\gamma$. Note that this system is output injection equivalent to the direct sum system $(I_N\otimes\gamma, I_N\otimes\alpha)$, and therefore $(\mathscr{C},\mathscr{A})$ is observable. Let $\mathscr{Y}$ denote the output behavior of (11.32). From the left coprime factorization

$$\mathscr{C}(zI-\mathscr{A})^{-1} = \mathscr{T}(z)^{-1}\mathscr{U}(z),$$

with

$$\mathscr{T}(z) = I_N\otimes D_\ell(z) - L\otimes N_\ell(z)K, \quad \mathscr{U}(z) = I_N\otimes N_\ell(z),$$

we obtain

$$\mathscr{Y} = \mathscr{C}X^{zI-\mathscr{A}} = X^{\mathscr{T}(z)}.$$

Definition 11.30. The **synchronized output behavior** of (11.32) is defined as the intersection of the diagonal in $z^{-1}\mathbb{R}[[z^{-1}]]^{pN}$ with $\mathscr{Y}$, i.e.,

$$\mathscr{Y}_{\text{sync}} := \{(h_1(z),\dots,h_N(z)) \in X^{\mathscr{T}} \mid h_1(z) = \cdots = h_N(z)\}.$$

System (11.32) is called **output synchronized** if the following requirements are satisfied:

1. There exists an initial state x_0 with output $y(t) = \operatorname{col}(y_1(t),\dots,y_N(t))$ satisfying $y_{\text{sync}}(t) := y_1(t) = \cdots = y_N(t)$ for all $t\ge 0$.

2. For all initial conditions,

$$\lim_{t\to\infty} \|y(t) - \mathbf{e}\otimes y_{\text{sync}}(t)\| = 0.$$

Arguing as in the proof of Theorem 11.27, it is easily seen that $\mathscr{Y}_{\text{sync}}$ is a submodule of $\mathscr{Y}$. We obtain a very simple characterization of output synchronizability.

Theorem 11.31. *Assume $Le = \lambda e$. System (11.32) is output synchronizable if and only if*

1. *The synchronized output behavior is nonempty, i.e., $\mathscr{Y}_{sync} \neq \emptyset$. Moreover,*

$$X^{D_\ell - \lambda N_\ell K} \simeq \mathscr{Y}_{sync} \subset X^{\mathscr{T}}.$$

2. $\det(D_\ell(z) - \mu N_\ell(z)K)$ *is a Hurwitz polynomial for all eigenvalues $\mu \neq \lambda$ of L.*

Proof. Clearly, condition 1 is equivalent to the existence of an element in the output behavior $\mathscr{Y}$ of (11.32), with all components being equal. This proves the equivalence of condition 1 with $\mathscr{Y}_{\text{sync}} \neq \emptyset$. For the other points, note that output synchronization is equivalent to partial state synchronization of the system

$$\mathscr{T}(\sigma)\xi = 0.$$

Using $\operatorname{Ker}\mathscr{T}(\sigma) = X^{\mathscr{T}}$, Theorem 11.27 implies that output synchronization is equivalent to synchronization of the associated shift realization of $X^{\mathscr{T}}$, i.e., that

$$\frac{\det\left(I\otimes D_\ell(z) - L\otimes N_\ell(z)K\right)}{\det\left(D_\ell(z) - \lambda N_\ell(z)K\right)} = \prod_{\mu\in\sigma(L)\setminus\{\lambda\}} \det\left(D_\ell(z) - \mu N_\ell(z)K\right)$$

is a Hurwitz polynomial. This completes the proof. ∎

Clustering. Finally, let us briefly discuss the more difficult question of clustering partial state vectors. While synchronization deals with the issue of driving the states of all the node systems of a network to each other, clustering is concerned with the more general task of allowing for different states to which the system can be driven. For simplicity, we focus on just two clusters; the general case can be treated similarly at the expense of more involved notation. Thus, for $I = \{1,\dots,r\} \subset \{1,\dots,N\}, J := \{r+1,\dots,N\}$, let

$$\Delta_{\text{ps}}(IJ) = \{(\xi_1,\dots,\xi_N) \mid \xi_1 = \cdots = \xi_r,\ \xi_{r+1} = \cdots = \xi_N\}$$

denote the set of IJ**-clustered partial states**. Taking the union of all nontrivial subsets I one obtains the set of 2**-clustered partial states** as

$$\Delta_{\text{ps}}^{[2]} = \bigcup_{0<|I|<N} \Delta_{\text{ps}}(IJ).$$

Similarly, we define the set of I**-clustered states** as

$$\Delta(IJ) = \{(f_1,\dots,f_N) \mid f_1 = \cdots = f_r,\ f_{r+1} = \cdots = f_N\}$$

in the polynomial model $X_{\mathscr{T}-\mathscr{U}A\mathscr{V}}$. The same made be said of the subset of **2-clustered states**. The counterpart to the set of admissible interconnection matrices is defined as follows (for IJ clustering only).

Definition 11.32. A block matrix $A = (A_{ij})$ with $m \times p$ blocks A_{ij} is called IJ**-admissible** if there are $m \times p$ matrices $\overline{A}_{II}, \overline{A}_{IJ}, \overline{A}_{JI}, \overline{A}_{JJ}$, with

$$\sum_{j=1}^{r} A_{1j} = \cdots = \sum_{j=1}^{r} A_{rj} := \overline{A}_{II}, \quad \sum_{j=r+1}^{N} A_{1j} = \cdots = \sum_{j=r+1}^{N} A_{rj} := \overline{A}_{IJ},$$

$$\sum_{j=1}^{r} A_{r+1,j} = \cdots = \sum_{j=1}^{r} A_{Nj} := \overline{A}_{JI}, \quad \sum_{j=r+1}^{N} A_{r+1,j} = \cdots = \sum_{j=r+1}^{N} A_{r+1,j} := \overline{A}_{JJ}.$$

Define

$$< A >_{IJ} = \begin{pmatrix} \overline{A}_{II} & \overline{A}_{IJ} \\ \overline{A}_{JI} & \overline{A}_{JJ} \end{pmatrix}.$$

The definition of IJ clustering then reads as follows.

Definition 11.33. Let A be an IJ-admissable interconnection matrix.

(a) The **partial state** system (11.28) IJ**-clusters** provided all solutions $\xi(t) = (\xi_1(t),\dots,\xi_N(t))$ of (11.28) satisfy

$$\lim_{t\to\infty} \|\xi_i(t) - \xi_j(t)\| = 0 \text{ for all } i,j = 1,\dots,r,$$

$$\lim_{t\to\infty} \|\xi_i(t) - \xi_j(t)\| = 0 \text{ for all } i,j = r+1,\dots,N.$$

(b) The **shift realization** of (11.28) IJ**-clusters** provided $\Delta(IJ) \subset X_{\mathscr{T}-\mathscr{U}A\mathscr{V}}$ is an outer stable invariant subspace of the shift operator $S_{\mathscr{T}-\mathscr{U}A\mathscr{V}} : X_{\mathscr{T}-\mathscr{U}A\mathscr{V}} \longrightarrow X_{\mathscr{T}-\mathscr{U}A\mathscr{V}}$.

Following the preceding analysis, one can then easily prove the next result; we omit the straightforward details.

Theorem 11.34. *Let A be IJ-admissible. Partial state IJ-clustering of the Rosenbrock system (11.28) is equivalent to IJ-clustering of the associated shift realization. In either case, IJ-clustering occurs if and only if*

$$\frac{\det(\mathscr{T}(z) - \mathscr{U}(z)A\mathscr{V}(z))}{\det(T(z) - U(z) < A >_{IJ} V(z))}$$

is a Hurwitz polynomial. The eigenvalues of the interconnected system on the IJ-clustered states are the roots of the polynomial

$$\det(T(z) - U(z) < A >_{IJ} V(z)).$$

In the special case of a decomposable interconnection matrix $A = L \otimes K$, the result can be stated in a more convenient form as follows.

Theorem 11.35. *Let K be arbitrary, and assume there are real numbers $\lambda_{11}, \lambda_{12}, \lambda_{21}, \lambda_{22}$ such that*

$$L = \begin{pmatrix} L_{11} & L_{12}, \\ L_{21} & L_{22}, \end{pmatrix},$$

with $L_{11} \in \mathbb{R}^{r \times r}, L_{22} \in \mathbb{R}^{(N-r)\times(N-r)}$ and $L_{ij}\boldsymbol{E} = \lambda_{ij}\boldsymbol{E}$ for all ij. Here $\boldsymbol{E}$ denotes the matrix of appropriate size, with all entries equal to 1. *Let*

$$\Delta_{cl}(z) := \det \begin{pmatrix} T(z) - r\lambda_{11}U(z)KV(z) & -(N-r)\lambda_{12}U(z)KV(z) \\ -r\lambda_{21}U(z)KV(z) & T(z) - (N-r)\lambda_{22}U(z)KV(z) \end{pmatrix}.$$

Then IJ-clustering occurs in the Rosenbrock system (11.28) and $A = L \otimes K$ if and only if

$$\frac{\det(I_N \otimes T(z) - L \otimes U(z)KV(z))}{\Delta_{cl}(z)} \tag{11.33}$$

is a Hurwitz polynomial. The eigenvalues of the interconnected system on the IJ-clustered states are the roots of the polynomial

$$\det \begin{pmatrix} T(z) - r\lambda_{11}U(z)KV(z) & -(N-r)\lambda_{12}U(z)KV(z) \\ -r\lambda_{21}U(z)KV(z) & T(z) - (N-r)\lambda_{22}U(z)KV(z) \end{pmatrix}. \tag{11.34}$$

Example 11.36. We investigate clustering for the case of three symmetrically coupled oscillators ($\lambda \neq 0$):

$$\begin{aligned} \ddot{x}_1 + a\dot{x}_1 + bx_1 &= \lambda \dot{x}_2, \\ \ddot{x}_2 + a\dot{x}_2 + bx_2 &= \lambda(\dot{x}_1 + \dot{x}_3), \\ \ddot{x}_3 + a\dot{x}_3 + bx_3 &= \lambda \dot{x}_2. \end{aligned}$$

Thus $T(z) = z^2 + az + b, U(z) = 1, V(z) = z$. The interconnection matrix is $A = L \otimes K = L$, with $K = 1$ and

$$L = \lambda \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

having the characteristic polynomial $z(z^2 - 2\lambda^2)$. Note that $\mathbf{e}$ is not an eigenvector of $A = L$, and therefore the network does not synchronize independently of the parameter value. By permuting the second and third columns and rows of L, we see that L is permutation equivalent to

$$\begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix} = \lambda \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

Thus L is admissible for $I = \{1,3\}, J = \{2\}$ with $\lambda_{11} = 0, \lambda_{12} = \lambda, \lambda_{21} = \lambda, \lambda_{22} = 0$. The characteristic polynomial for the coupled system is

$$\det(T(z)I_3 - LU(z)V(z)) = T(z)(T(z)^2 - 2\lambda^2 z^2),$$

which is Hurwitz if and only if $a > \sqrt{2}|\lambda|$ and $b > 0$. Polynomial (11.34) is

$$T(z)^2 - 2\lambda^2 z^2.$$

Thus the quotient (11.33) is the polynomial $T(z)$. One concludes that asymptotic clustering occurs if and only if $T(z)$ is Hurwitz, i.e., $a > 0, b > 0$.

11.5 Examples: Arrays of Oscillators

Because synchrony is usually connected to periodic phenomena, the basic components for modeling are mostly taken to be nonlinear oscillators. However, simpler linear system models, such as the classical harmonic oscillator, can be used for the same purpose. Of course, because its failure is structurally stable, the harmonic oscillator is an unsuitable model for most periodic physical phenomena. Nevertheless, the analysis of synchronization phenomena for the harmonic oscillator provides important insights into the general theory of synchronization. Arrays of coupled damped oscillators are perhaps the simplest models for synchronization of linear systems. The theory of such networks can be developed quite generally using tools from spectral graph theory developed in Chapter 8. Here we focus on applying such an analysis to the case of a finite number of damped harmonic oscillators that are ordered either linearly or circularly. The exposition is largely based on Fuhrmann, Priel, Sussmann and Tsoi (1987).

I. Linear Arrays of Oscillators. We begin by discussing an extension of the example in the introduction to a linear chain of N coupled identical oscillators. Thus, consider the dynamic equations

$$\begin{aligned}\ddot{x}_1 + a\dot{x}_1 + bx_1 &= 0,\\ \ddot{x}_2 + a\dot{x}_2 + bx_2 &= \lambda \dot{x}_1\\ &\vdots\\ \ddot{x}_N + a\dot{x}_N + bx_N &= \lambda \dot{x}_{N-1}.\end{aligned}$$

Here $\lambda \neq 0$ is assumed to be constant. Each SISO node system is in first-order form as

$$\alpha = \begin{pmatrix} 0 & 1 \\ -b & -a \end{pmatrix}, \quad \beta = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \gamma = \begin{pmatrix} 0 & 1 \end{pmatrix}, \tag{11.35}$$

or, equivalently, via the strictly proper transfer function

$$g(z) = \gamma(zI_2 - \alpha)^{-1}\beta = \frac{z}{z^2 + az + b}.$$

Assuming $b \neq 0$, which we will assume from now on, ensures the coprimeness of the factors z and $z^2 + az + b$ (Figure 11.1). The state interconnection matrix for this system is decomposable as $A = L \otimes K = L$, with $K = 1$ and

$$L = \lambda \begin{pmatrix} 0 & & & \\ 1 & 0 & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{pmatrix}.$$

In particular, L is nilpotent and $\mathbf{e}$ is not an eigenvector of L. Thus synchronization for this chain of oscillators does not occur. We rewrite this vectorially as a first-order system $\dot{x} = \mathscr{A}x$, where $\mathscr{A} \in \mathbb{R}^{2n \times 2n}$ has the block lower triangular form

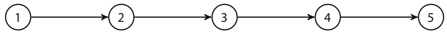

Fig. 11.1 Directed simple path

$$\begin{pmatrix} 0 & 1 & & & & & & \\ -b & -a & & & & & & \\ 0 & 0 & 0 & 1 & & & & \\ 0 & \lambda & -b & -a & & & & \\ & & \ddots & & \ddots & & & \\ & & & \ddots & & \ddots & & \\ & & & & 0 & 0 & 0 & 1 \\ & & & & 0 & \lambda & -b & -a \end{pmatrix}.$$

In this case, the spectral analysis becomes particularly simple because, by the lower triangular structure, one obtains

$$\det(zI - \mathscr{A}) = (z^2 + az + b)^N.$$

Clearly, this is also the minimal polynomial of our system. The system becomes a bit more interesting if we add inputs $u(t)$ to the system. For example, we might consider the case where only the first node is controlled, i.e., we consider

$$\begin{aligned} \ddot{x}_1 + a\dot{x}_1 + bx_1 &= u(t), \\ \ddot{x}_2 + a\dot{x}_2 + bx_2 &= \lambda \dot{x}_1 \\ &\vdots \\ \ddot{x}_N + a\dot{x}_N + bx_N &= \lambda \dot{x}_{N-1}. \end{aligned} \tag{11.36}$$

The network transfer function $\mathscr{N}(z)$ from the input to the states then becomes

$$\begin{pmatrix} z^2+az+b & & & \\ -\lambda z & \ddots & & \\ & \ddots & \ddots & \\ & & -\lambda z & z^2+az+b \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} (z^2+az+b)^{-1} \\ \lambda z(z^2+az+b)^{-2} \\ \vdots \\ (\lambda z)^{N-2}(z^2+az+b)^{-N+1} \\ (\lambda z)^{N-1}(z^2+az+b)^{-N} \end{pmatrix}.$$

Obviously, for $\lambda \neq 0$, this is a left coprime factorization, and thus (11.36) is reachable.

Next we move on to symmetrically coupled arrays of oscillators (Figure 11.2).

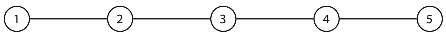

Fig. 11.2 Undirected simple path

Specifically, we consider the system

$$
\begin{aligned}
\ddot{x}_1 + a\dot{x}_1 + bx_1 &= \lambda \dot{x}_2, \\
\ddot{x}_2 + a\dot{x}_2 + bx_2 &= \lambda(\dot{x}_1 + \dot{x}_2) \\
&\vdots \\
\ddot{x}_{N-1} + a\dot{x}_{N-1} + bx_{N-1} &= \lambda(\dot{x}_{N-1} + \dot{x}_N), \\
\ddot{x}_N + a\dot{x}_N + bx_N &= \lambda \dot{x}_{N-1}.
\end{aligned}
\tag{11.37}
$$

We assume that $\lambda > 0$. The matrix $\mathscr{A}$ has again the tensor product structure $\mathscr{A} := I_n \otimes \alpha + L \otimes \beta\gamma$, but L has the symmetric matrix representation

$$
L = \lambda \begin{pmatrix} 0 & 1 & & & \\ 1 & 0 & 1 & & \\ & 1 & \ddots & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & 1 & 0 \end{pmatrix}.
$$

Matrix L has a tridiagonal symmetric Toeplitz structure. By Theorem 8.45, we know that the eigenvalues of L are distinct and equal to $2\cos\frac{k\pi}{N+1}$ for $k = 1, \ldots, N$, with an explicit formula for the eigenvectors, too. Thus the solutions of (11.37) can be written down in closed form. We will not do that here. Since $\mathbf{e}$ is not an eigenvalue of L, the system does not synchronize.

II. Circular arrays of oscillators. In the circularly oriented case, the interconnection structure is depicted as in Figure 11.3. Explicitly, we consider the system

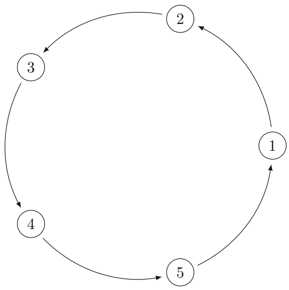

Fig. 11.3 Directed cycle graph

$$
\begin{aligned}
\ddot{x}_1 + a\dot{x}_1 + bx_1 &= \lambda \dot{x}_N, \\
\ddot{x}_2 + a\dot{x}_2 + bx_2 &= \lambda \dot{x}_1 \\
&\vdots \\
\ddot{x}_N + a\dot{x}_N + bx_N &= \lambda \dot{x}_{N-1}.
\end{aligned} \tag{11.38}
$$

Obviously, its state space is $2N$-dimensional and the dynamic equations can be written in first-order block circulant form, $\dot{x} = \mathscr{A}x$, as follows:

$$
\frac{d}{dt}\begin{pmatrix} x_1 \\ \dot{x}_1 \\ \vdots \\ \vdots \\ \vdots \\ x_N \\ \dot{x}_N \end{pmatrix} = \begin{pmatrix} 0 & 1 & & & & & 0 & 0 \\ -b & -a & & & & & 0 & \lambda \\ 0 & 0 & 0 & 1 & & & & \\ 0 & \lambda & -b & -a & & & & \\ & & \ddots & & \ddots & & & \\ & & & & \ddots & \ddots & & \\ & & & & 0 & 0 & 0 & 1 \\ & & & & 0 & \lambda & -b & -a \end{pmatrix} \begin{pmatrix} x_1 \\ \dot{x}_1 \\ \vdots \\ \vdots \\ \vdots \\ x_N \\ \dot{x}_N \end{pmatrix}.
$$

Using the special structure of matrix $\mathscr{A}$, its characteristic polynomial turns out to be

$$
d_{\mathscr{A}}(z) = (z^2 + az + b)^N - \lambda^N z^N.
$$

It is more convenient to analyze the system using the associated polynomial system matrices. We note that the coupling under consideration has a preferential direction. Later, we will also study more symmetric couplings. The interconnection matrix is $A = C$, where

$$
C = \begin{pmatrix} 0 & & & 1 \\ 1 & \ddots & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{pmatrix}.
$$

Note that C has the structure of a circulant matrix and thus can be diagonalized by the Fourier matrix. Its eigenvalues are exactly the $N-th$ roots of unity $1, \omega, \ldots, \omega^{N-1}$, where $\omega = e^{\frac{2\pi\sqrt{-1}}{N}}$ denotes the primitive root of unity. Note further that 1 is always an eigenvalue of C with associated eigenvector $\mathbf{e}$. To analyze the synchronization of (11.38), we consider the closed-loop polynomial system matrix $T(z) := (z^2 + az + b)I_N - z\lambda C$. The characteristic polynomial of $\mathscr{A}$ coincides with the determinant of $T(z)$. The zeros of $\det T(z)$ are equal to the roots of $z^2 + (a - \lambda\omega^k)z + b$ for $k = 0, \ldots, N-1$.

Theorem 11.37. *Let $a, \lambda > 0$. System (11.38) synchronizes if and only if*

$$a > \lambda \cos \frac{2\pi}{N}, \quad b > 0.$$

Proof. Using the stability test for complex polynomials via the positivity of the Hermite–Fujiwara matrix (Theorem 5.52), a complex polynomial $p(z) = z^2 + uz + v$ is Hurwitz if and only if the Hermite–Fujiwara matrix

$$\mathbf{H}_2(p) = 2 \begin{pmatrix} \mathrm{Re}(u\bar{v}) & -i\,\mathrm{Im}\,v \\ i\,\mathrm{Im}\,v & \mathrm{Re}\,u \end{pmatrix}$$

is positive definite, i.e., if and only if

$$\mathrm{Re}(u) > 0, \quad \mathrm{Re}(u)\mathrm{Re}(u\bar{v}) > \mathrm{Im}^2(v).$$

In our situation, $u = a - \lambda\omega^k, v = b$, with $a, b, \lambda > 0$ real. Thus $z^2 + (a - \lambda\omega^k)z + b$ is Hurwitz for $k = 1, \ldots, N-1$ if and only if $b > 0$ and $a - \lambda \cos \frac{2k\pi}{N} > 0$ for $k = 1, \ldots, N-1$. The result follows. ■

Note that, under our assumption that $a, b, \lambda > 0$, the inequality $|a - \lambda\omega^k| \leq a + \lambda$ is valid for all $1 \leq k \leq N$. Thus, if $(a+\lambda)^2 < 4b$, then there is oscillatory motion, with the different modes having damping terms of the form $e^{-\sigma_k t}$, with $\sigma_k = \mathrm{Re}\,\frac{a - \lambda\omega^k}{2}$. Obviously, the mode with the slowest rate of decay is the one where σ_k is (algebraically) the largest, and this occurs when $\omega^k = 1$, i.e., for $k = 0$. It is of interest to identify this mode. Indeed, if we look for eigenvectors of $\mathscr{A}$ of the form $(\xi, \ldots, \xi, \eta, \ldots, \eta)^\top$, then $\mathscr{A}x = \lambda x$ reduces to the pair of equations

$$\begin{aligned} \eta &= \lambda\xi, \\ -b - a\eta + \lambda\eta &= \lambda\eta, \end{aligned}$$

which leads to the equation

$$(\lambda^2 + (a - \lambda)\lambda + b)\xi = 0,$$

whose roots are eigenvalues of $\mathscr{A}$. Thus we see that the slowest rate of decay is in the synchronized mode.

If we apply symmetric nearest-neighbor coupling, we obtain

$$\ddot{x}_k + a\dot{x}_k + bx_k = \lambda(\dot{x}_{k-1} + \dot{x}_{k+1}) \qquad k = 1, \ldots, N, \tag{11.39}$$

with the understanding that $x_0 = x_N$ and $x_{N+1} = x_1$. The interconnection matrix A is the symmetric, circulant Toeplitz matrix

$$A = -\lambda \Gamma = -\lambda \begin{pmatrix} 0 & 1 & & & & 1 \\ 1 & 0 & 1 & & & \\ & 1 & \ddots & \ddots & & \\ & & \ddots & \ddots & 1 & \\ & & & 1 & 0 & 1 \\ 1 & & & & 1 & 0 \end{pmatrix},$$

which has $\mathbf{e}$ as an eigenvector. The associated eigenvalue -2λ is simple. By Theorem 8.48, the eigenvalues of A are $-2\lambda\cos(\frac{2k\pi}{N})$ for $k = 1,\dots,N$. The polynomial system matrix is $T(z) = (z^2 + az + b)I_N - \lambda z\Gamma$. System (11.39) can be written in state-space form as $\dot{x} = \mathscr{A}x$, where

$$\mathscr{A} = I_N \otimes \alpha - \lambda \Gamma \otimes \beta\gamma.$$

Here (α, β, γ) are as in (11.35). We observe that the eigenvalues of $\mathscr{A}$ are equal to the roots of the determinant $\det T(z)$, i.e., to the roots of

$$z^2 + (a - 2\lambda\cos(\frac{2k\pi}{N}))z + b = 0.$$

Note that $k = N$ corresponds to the synchronized mode $\mathbf{e}$. Note further that $a - 2\lambda\cos\frac{2k\pi}{N} < a - 2\lambda$ for all $1 \le k < N$. We arrive at the following theorem.

Theorem 11.38. *The symmetrically connected cycle (11.39) synchronizes if and only if $z^2 + (a - 2\lambda\cos(\frac{2k\pi}{N}))z + b$ is Hurwitz for $k = 1,\dots,N-1$. Equivalently, synchronization occurs if and only if $b > 0$ and $a > 2\lambda\cos\frac{2\pi}{N}$. The eigenvalues for the synchronized dynamics are equal to the roots of $z^2 + (a - 2\lambda)z + b$. All other eigenvalues of the system matrix $\mathscr{A}$ have real part $< a - 2\lambda$.*

11.6 Exercises

1. Prove the following result from Trentelmann, Stoorvogel and Hautus (2001):
   A linear subspace $\mathscr{V} \subset \mathscr{X}$ is (A, B, C)-invariant if and only if there exists an output feedback transformation K such that $(A + BKC)\mathscr{V} \subset \mathscr{V}$.
2. (a) Show that a homogeneous synchronizing network for the real rational node transfer function

$$g(z) = \frac{\alpha z + \beta}{z^2 + az + b}$$

exists if and only if there is a complex number $\lambda \in \mathbb{C}$ satisfying the inequalities

$$a|\lambda|^2 - \alpha \operatorname{Re}\lambda > 0,$$
$$\left(a|\lambda|^2 - \alpha \operatorname{Re}\lambda\right)\left(ab|\lambda|^2 + \alpha\beta - (a\beta + b\alpha)\operatorname{Re}\lambda\right) > \beta^2(\operatorname{Im}\lambda)^2.$$

(b) Deduce that a homogeneous synchronizing network for

$$g(z) = \frac{\varepsilon z + 1}{z^2 + 1}$$

exists if and only if $\varepsilon \neq 0$.

3. Give necessary and sufficient conditions when the system of four coupled second-order systems

$$\begin{aligned}
\ddot{x}_1 + a\dot{x}_1 + bx_1 &= \lambda \dot{x}_2,\\
\ddot{x}_2 + a\dot{x}_2 + bx_2 &= \lambda(\dot{x}_1 + \dot{x}_3),\\
\ddot{x}_3 + a\dot{x}_3 + bx_3 &= \lambda(\dot{x}_2 + \dot{x}_4),\\
\ddot{x}_4 + a\dot{x}_4 + bx_4 &= \lambda \dot{x}_3
\end{aligned}$$

clusters at x_1, x_4 and x_2, x_3, respectively.

11.7 Notes and References

There exists a huge literature from physics and systems engineering on synchronization, clustering, and consensus; we refer the reader to the survey paper by Doerfler and Bullo (2014) and the references therein. A new idea was recently proposed by R.W. Brockett in his 2014 Bernoulli lecture at the International Symposium on the Mathematical Theory of Networks and Systems (MTNS 2014) in Groningen, the Netherlands. Brockett asks a fundamental question concerning the potential mechanisms for synchronization: Given a symmetric matrix Q with distinct eigenvalues and a second-order system of the form

$$\begin{aligned}
\ddot{x} + \eta(x,\dot{x}) + Qx &= f(x,\dot{x},z),\\
\dot{z} &= g(x,\dot{x},z),
\end{aligned}$$

what are the simplest, physically plausible choices of f and g that result in synchronization? He argues that the system

$$\ddot{x} + \eta(x,\dot{x}) + (Q+Z)x = 0,$$

$$\dot{Z} = -\alpha Z + x\dot{x}^\top - \dot{x}x^\top$$

should be an interesting candidate. See Brockett (2003) and Brockett (2013) for a study of closely related equations that underpin this belief.

Theorem 11.4 can be extended to normalized Laplacians of time-varying graphs $\Gamma(t)$ under weak connectivity assumptions. Let $\Gamma_i = (V, E_i), i = 1, \ldots, m$, denote finitely many weighted directed graphs on the same vertex set V with associated adjacency matrices $A_1, \ldots, A_m$. The union $\Gamma = \Gamma_1 \cup \cdots \cup \Gamma_m$ is the graph $\Gamma = (V, E)$ with edge set $E = E_1 \cup \cdots \cup E_m$. Consensus results for time-varying graphs were established by Jadbabaie, Lin and Morse (2003) under the assumption that there exists $T \in \mathbb{N}$ such that the union of graphs $\Gamma(kT) \cup \Gamma(kT+1) \cup \cdots \cup \Gamma(kT+T)$ is connected for all $k \in \mathbb{N}_0$. Cao, Morse and Anderson (2008) have derived more generally sufficient conditions for consensus if the time-varying graph of each flocking matrix $F(t)$ has a root for all t.

There exists by now a rich and rapidly growing literature on distributed control and distributed optimization. We refer the reader to Tsitsiklis, Bertsekas and Athans (1986) and the book by Bertsekas and Tsitsiklis (1989) for early contributions and further references. The distributed algorithm (11.9) for finding intersection points of affine subspaces has appeared several times in more general contexts; see, for example, Nedic, Ozdaglar and Parrilo (2010) for intersections of convex sets and for an explanation of the connection with the classical alternating projection method by von Neumann. Theorem 11.8 is due to Mou and Morse (2013).

Simple simulation experiments – see, for example, Blondel, Hendrickx and Tsitsiklis (2009) – show that trajectories in the Hegselmann–Krause model (11.13) do not converge to a common consensus state; instead, they **cluster** around certain limit points. Thus the Hegselmann–Krause model is really more a model for clustering rather than for consensus. The fine structure of the cluster states is quite interesting and requires further mathematical analysis. For example, it has been experimentally observed, and is conjectured to be true in general, that the solutions of (11.13) actually cluster in distances of $|x_i^* - x_j^*| \geq 2r$. Moreover, the distribution of the cluster points is not exactly evenly distributed, even for uniform distributions of the initial conditions. For extensions of the Krause model to continuous-time models, see Blondel, Hendrickx and Tsitsiklis (2009, 2010).

Consensus problems for second-order systems, including consensus among velocities, have been treated by, for example, Anderson, Lin and Deghat (2012) and Ren (2008). Cucker and Smale (2007) proposed a new consensus algorithm for the velocities of N second-order agents in $\mathbb{R}^3$ using the state-dependent graph adjacency matrix and Laplacian

$$A(x) = \left((1 + \|x_i - x_j\|^2)^{-\beta}\right)_{i,j} \quad \text{and} \quad L(x) = \operatorname{diag}(A(x)\mathbf{e}) - A(x),$$

respectively. They established asymptotic convergence results for the velocities in the network of second-order systems

$$\ddot{x} + (I_{3N} - L(x) \otimes I_3)\dot{x} = 0,$$

depending on whether $\beta < \frac{1}{2}$ or $\beta \geq \frac{1}{2}$. Extensions to general interconnection graphs are in Cucker and Smale (2007a).

From a systems engineering point of view, synchronization is a property that is desirable for the purpose of using feedback strategies. A well-known decentralized control approach to synchronization assumes a fixed diffusive coupling, together with N local feedback controllers around the node systems. The synchronization task then becomes to tune the local feedback controllers so that the network synchronizes. This design strategy is quite different from the approach taken here, where one aims to achieve synchronization via appropriate selections of the coupling terms. Whatever approach one prefers, there is plenty of room for further improvements. For example, one might replace the local feedback controllers by adaptive ones. Such an approach has been proposed by Helmke, Prätzel-Wolters and Schmidt (1991) and Ilchmann (2013), where synchronization is modeled as an adaptive tracking problem for networks of systems. This leads to synchronization results that are robust with respect to variations in both the interconnection and system parameters.

Synchronization problems for homogeneous networks of linear systems using state feedback transformations have been explored by Scardovi and Sepulchre (2009) and Tuna (2008), for example. Robust synchronization tasks using state feedback and output injection are studied in Trentelmann, Takaba and Monshizadeh (2013). Variants of Theorem 11.13 for diffusive coupling have been shown by several researchers, including Ma and Zhang (2010) and Lunze (2011). For networks with diffusive couplings and using state feedback with $\gamma = I_n$, Ma and Zhang (2010) have shown that synchronizability is equivalent to (α, β) being stabilizable and the graph being connected. Other versions of synchronizability via output injection were studied by Tuna (2009). Versions of the internal model principle for synchronization have been considered by Wieland, Sepulchre and Allgöwer (2011) and Lunze (2012), who proved a special case of Proposition 11.29. Using state-space methods, Lunze (2012) proved an extension of Theorem 11.31 for heterogenous networks, however under the strong additional assumption that the system matrices of the agents are diagonalizable. We believe that the results for higher-order systems introduced in Section 11.4 lead to a more natural approach to synchronization than standard state-space methods.