Diffusion Coefficient for the Disordered Harmonic Chain Perturbed by an Energy Conserving Noise

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Abstract We investigate the macroscopic behavior of the disordered harmonic chain of oscillators, through energy diffusion. The hamiltonian dynamics of the disordered system is perturbed by a degenerate conservative noise. After rescaling space and time diffusively, energy fluctuations in equilibrium evolve according to a linear heat equation (Simon, Equilibrium fluctuations for the disordered harmonic chain perturbed by an energy conserving noise, 2013). Here we concentrate on the diffusion coefficient, given by the non-gradient Varadhan's approach, and equivalently defined through the Green-Kubo formula. We compare the two approaches and investigate the convergence of the diffusion coefficient in a vanishing noise limit.

Keywords Disordered harmonic chain · Diffusion coefficient · Green-Kubo

1 Introduction

This work is based on [1], and addresses diffusion problems for harmonic chains of oscillators with random defects. The purely deterministic disordered harmonic chain of N oscillators was introduced in [2] and since then has attracted a lot of interest. After the first analyses of [2, 3], Ajanki and Huveneers [4] study this disordered chain when coupled at the boundaries to Langevin heat baths, with respective temperatures T_R and T_L. They prove an anomalous heat transport in the following sense: if J_N denotes the total energy current across the chain, then

$$\mathbb{E}\left[\int J_N \, \mathrm{d}\mu_{ss}^N\right] \sim (T_R - T_L)N^{-3/2}$$

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in the limit $N \to \infty$, where $\mathbb{E}$ states for the expectation w.r.t. the random environment and μ_{ss}^N is the non-equilibrium stationary state for the dynamical system.

Here we study the diffusive behavior for the disordered harmonic chain, but perturbed by an energy conserving noise. Thanks to the stochastic perturbation, the conductivity of the one-dimensional chain should become finite and positive. Besides, some homogenization effect occurs and the conductivity does not depend on the statistics of the disorder in the thermodynamic limit.

The disorder effect has already been investigated for lattice gas dynamics, for example in [5–8]. These papers share one feature: the models are non-gradient due to the presence of the environment. Non-gradient systems are usually solved by establishing a microscopic Fourier law up to a small fluctuating term, following the method initially developed by Varadhan in [9], and then generalized to non-reversible dynamics [10]. The study of disordered chains of oscillators perturbed by a conservative noise has appeared more recently [11–13]. In these papers, the thermal conductivity is studied via the Green-Kubo formula. Here, the diffusion coefficient is furthermore defined through fluctuating hydrodynamics.

In [14], we have obtained the diffusive scaling limit for a homogeneous chain of coupled harmonic oscillators perturbed by a noise, which randomly flips the sign of the velocities, so that the energy is conserved but not the momentum. Our first motivation was to investigate the same chain of harmonic oscillators, still perturbed by the velocity-flip noise, but now provided with i.i.d. random masses. In [14], a system of non-linear homogeneous hydrodynamic equations has been derived thanks to the relative entropy method, and one of the major ingredient for the proof was an exact *fluctuation-dissipation equation* (see for example [15]).

The disorder assumption makes all previous computations pointless: in particular, the fluctuation-dissipation equations are not directly solvable any more. To overcome this difficulty, one replaces these exact equations by approximations: more precisely, there exists a sequence of local functions for which an approximate fluctuation-dissipation decomposition holds, in the sense that the difference has a small space-time variance with respect to the dynamics in equilibrium. The main ingredients of the usual non-reversible non-gradient method are: a *spectral gap* for the symmetric part of the dynamics, and a *sector condition* for the total generator.

Our model has special features that enforce the Varadhan's method to be considered with new perspectives. In particular, the symmetric part of the generator is poorly ergodic, and does not have a spectral gap when restricted to microcanonical manifolds. Moreover, due to the degeneracy of the noise, the asymmetric part of the generator is not controlled by the symmetric part (in technical terms, the sector condition does not hold), with the only velocity-flip noise. Besides, remark that the energy current depends on the disorder, and has to be approximated by a fluctuation-dissipation equation which takes into account the fluctuations of the disorder itself.

Because of the high degeneracy of the velocity-flip noise, we add a second stochastic perturbation, that exchanges velocities (divided by the square root of mass) and positions at random independent Poissonian times, so that a *weak sector condition* can be proved (see [1, Proposition 5.7]). However, the spectral gap estimate and the usual sector condition still do not hold when adding the exchange noise, meaning that the stochastic perturbation remains very degenerate. Due to the harmonicity of the chain, the generator of the dynamics preserves the degree of polynomials, and even a degenerate noise is sufficient to apply Varadhan's method. The sector condition and the non-gradient decomposition are only needed for a specific class of functions. Contrary to the standard approach, we do not need to prove any general result concerning the so-called *closed forms* (we refer to [16, 17] for the general theory). This is a clear advantage of our model: since some difficult technical parts are in some sense simplified, the usual approach to non-gradient problems becomes slightly neater.

We show in addition that the diffusion coefficient can be equivalently defined by the Green-Kubo formula. This space-time variance of the current at equilibrium is only formal in the sense that a double limit (in space and time) has to be taken. As in [11], we prove here that the limit is well-defined, and that the homogenization effect occurs for the Green-Kubo formula: for almost every realization of the disorder, the thermal conductivity exists, is independent of the disorder, is positive and finite.

Finally, let us introduce $\gamma > 0$ the intensity of the flip noise, and $\lambda > 0$ the intensity of the exchange noise. We denote the diffusion coefficient by $D(\lambda, \gamma)$ when obtained through the variational formula in the Varadhan's method, and by $\overline{D}(\lambda, \gamma)$ when defined through the Green-Kubo formula. We prove in [1] that the two conductivities are equal: $D(\lambda, \gamma) = \overline{D}(\lambda, \gamma)$, when the two intensities λ, γ are positive. Furthermore, the Green-Kubo formula remains well-defined when $\lambda = 0$, namely: $\overline{D}(0, \gamma)$ exists, is finite and positive. Finally, $D(\lambda, \gamma)$ tends to $\overline{D}(0, \gamma)$ as λ vanishes. The existence question for $D(0, \gamma)$, when defining through hydrodynamics (or even fluctuating hydrodynamics) remains open. We start in the following section by introducing the model together with notations and definitions.

2 The Harmonic Chain Perturbed by Stochastic Jump Noises

We introduce the harmonic hamiltonian system described by the sequence $\{p_x, r_x\}$, where p_x stands for the momentum of the oscillator at site x, and r_x represents the distance between oscillator x and oscillator $x+1$. Each atom $x \in \mathbb{Z}$ has a mass M_x, the velocity of atom x is given by p_x/M_x. We assume the disorder $\mathbf{M} := \{M_x\}_{x\in\mathbb{Z}}$ to be a collection of real i.i.d. positive random variables that are bounded from above and below by positive constants. The equations of motions are given by

$$\frac{\mathrm{d}p_x}{\mathrm{d}t} = r_x - r_{x-1}, \qquad \frac{\mathrm{d}r_x}{\mathrm{d}t} = \frac{p_{x+1}}{M_{x+1}} - \frac{p_x}{M_x}, \qquad x \in \mathbb{Z}.$$

The dynamics conserves the total energy

$$\mathscr{E} := \sum_{x\in\mathbb{Z}} \left\{ \frac{p_x^2}{2M_x} + \frac{r_x^2}{2} \right\}.$$

To overcome the lack of ergodicity of deterministic chains, we add a stochastic perturbation to this new dynamics, so that the diffusion coefficient can be defined through Varadhan's approach (Theorem 3.3). The noise can be easily described: at independently distributed random Poissonian times, the quantity $p_x/\sqrt{M_x}$ and the interdistance r_x are exchanged, or the momentum p_x is flipped into $-p_x$.

Even if Theorem 3.3 could be proved *mutatis mutandis* for this harmonic chain, for pedagogical reasons we now focus on a simplified model (as in [18]), which has exactly the same features and involves less painful computations. From now on, we study the dynamics on the new configurations $\{\eta_x\}_{x\in\mathbb{Z}}$ written as

$$m_x \mathrm{d}\eta_x = (\eta_{x+1} - \eta_{x-1})\mathrm{d}t, \tag{1}$$

where $\mathbf{m} := \{m_x\}_{x\in\mathbb{Z}}$ is the new disorder with the same characteristics as before. It is convenient to change the variable η_x into $\omega_x := \sqrt{m_x}\eta_x$, and the total energy reads

$$\mathscr{E} = \sum_{x\in\mathbb{Z}} \omega_x^2.$$

Let us now introduce the corresponding stochastic energy conserving dynamics: the evolution is described by (1) between random exponential times, and at each ring one of the following interactions can happen:

a. Exchange noise—two nearest neighbour variables ω_x and ω_{x+1} are exchanged;
b. Flip noise—the variable ω_x at site x is flipped into $-\omega_x$.

We now describe the dynamics on the finite torus $\mathbb{T}_N := \{0, ..., N\}$, meaning that boundary conditions are periodic. The configuration $\{\omega_x\}_{x\in\mathbb{T}_N}$ evolves according to a dynamics which can be divided into two parts, a deterministic one and a stochastic one. The space of configurations of our system is given by $\Omega_N = \mathbb{R}^N$. The product and translation invariant measure that describes the disorder $\mathbf{m}$ on the space $\Omega_{\mathscr{D}} = [C^{-1}, C]^{\mathbb{Z}}$ is denoted by $\mathbb{P}$ and its expectation is denoted by $\mathbb{E}$. For a fixed disorder field $\mathbf{m} = \{m_x\}_{x\in\mathbb{Z}}$, the dynamics can be entirely defined by the generator of the Markov process $\{\omega_x(t)\,;\ x \in \mathbb{T}_N\}_{t\geqslant 0}$, that is

$$\mathscr{L}_N^{\mathbf{m}} = \mathscr{A}_N^{\mathbf{m}} + \gamma\mathscr{S}_N^{\text{flip}} + \lambda\mathscr{S}_N^{\text{exch}} \tag{2}$$

where,

$$\mathscr{A}_N^{\mathbf{m}} = \sum_{x\in\mathbb{T}_N} \left\{ \left(\frac{\omega_{x+1}}{\sqrt{m_x m_{x+1}}} - \frac{\omega_{x-1}}{\sqrt{m_{x-1} m_x}} \right) \frac{\partial}{\partial \omega_x} \right\},$$

and, for all functions $f : \Omega_{\mathscr{D}} \times \Omega_N \to \mathbb{R}$,

$$\mathscr{S}_N^{\text{flip}} f(\mathbf{m}, \omega) = \sum_{x\in\mathbb{T}_N} f(\mathbf{m}, \omega^x) - f(\mathbf{m}, \omega),$$
$$\mathscr{S}_N^{\text{exch}} f(\mathbf{m}, \omega) = \sum_{x\in\mathbb{T}_N} f(\mathbf{m}, \omega^{x,x+1}) - f(\mathbf{m}, \omega).$$

Here, the configuration ω^x is the configuration obtained from ω by flipping the momentum of particle x: $(\omega^x)_z = \omega_z$ if $z \neq x$ and $\omega_x^x = -\omega_x$. The configuration $\omega^{x,x+1}$ is obtained from ω by exchanging the momenta of particles x and $x+1$: $(\omega^{x,x+1})_z = \omega_z$ if $z \neq x, x+1$ while $\omega_x^{x,x+1} = \omega_{x+1}$, and $\omega_{x+1}^{x,x+1} = \omega_x$. We denote the total generator of the noise by $\mathscr{S}_N := \gamma \mathscr{S}_N^{\text{flip}} + \lambda \mathscr{S}_N^{\text{exch}}$, where $\gamma, \lambda > 0$ are two positive parameters which regulate the respective strengths of noises.

One quantity is conserved: the total energy $\sum \omega_x^2$. The following translation invariant product Gibbs measures μ_β^N on Ω_N are invariant for the process:

$$\mathrm{d}\mu_\beta^N(\omega) := \prod_{x\in\mathbb{T}_N} \sqrt{\frac{2\pi}{\beta}} \exp\left(-\frac{\beta}{2}\omega_x^2\right) \mathrm{d}\omega_x.$$

In the following, the expectation of f with respect to μ_β^N is denoted by $\langle f \rangle_\beta$. The index β stands for the inverse temperature: $\langle \omega_0^2 \rangle_\beta = 1/\beta$. From the definition, our model is not reversible with respect to the measure μ_β^N. Precisely, $\mathscr{A}_N^{\mathbf{m}}$ is an antisymmetric operator in $\mathbf{L}^2(\mu_\beta^N)$, whereas $\mathscr{S}_N$ is symmetric.

We denote by Ω the space of configurations in the infinite line, that is $\Omega := \mathbb{R}^{\mathbb{Z}}$, and by μ_β the product Gibbs measure on $\mathbb{R}^{\mathbb{Z}}$. Hereafter, for every $\beta > 0$, we denote by $\mathbb{P}_\beta^\star$ the probability measure on $\Omega_{\mathscr{D}} \times \Omega$ defined by $\mathbb{P}_\beta^\star := \mathbb{P} \otimes \mu_\beta$. We notice that $\mathbb{P}_\beta^\star$ is translation invariant and we write $\mathbb{E}_\beta^\star$ for the corresponding expectation.

Since the dynamics conserves the total energy, there exist instantaneous currents of energy $j_{x,x+1}$ such that $\mathscr{L}_N^{\mathbf{m}}(\omega_x^2) = j_{x,x+1}(\mathbf{m}, \omega) - j_{x-1,x}(\mathbf{m}, \omega)$. The quantity $j_{x,x+1}$ is the amount of energy flowing between the particles x and $x+1$, and is equal to

$$j_{x,x+1}(\mathbf{m}, \omega) = \frac{2\omega_x \omega_{x+1}}{\sqrt{m_x m_{x+1}}} + \lambda(\omega_{x+1}^2 - \omega_x^2).$$

We write $j_{x,x+1} = j_{x,x+1}^A + j_{x,x+1}^S$ where $j_{x,x+1}^A$ (resp. $j_{x,x+1}^S$) is the current associated to the antisymmetric (resp. symmetric) part of the generator:

$$j^A_{x,x+1}(\mathbf{m},\omega) = \frac{2\omega_x\omega_{x+1}}{\sqrt{m_x m_{x+1}}}, \qquad j^S_{x,x+1}(\mathbf{m},\omega) = \lambda(\omega^2_{x+1} - \omega^2_x).$$

Unfortunately the current cannot be directly written as the gradient of a local function, neither by an exact fluctuation-dissipation equation involving local functions (except if masses are equal). We also define the *static compressibility* that is equal to $\chi_\beta := \langle\omega_0^4\rangle_\beta - \langle\omega_0^2\rangle_\beta^2 = 2\beta^{-2}$.

2.1 Cylinder Functions and Dirichlet Form

For every $x \in \mathbb{Z}$ and f a measurable function on $\Omega_{\mathscr{D}} \times \Omega$, we consider the translated function $\tau_x f$, which is the function on $\Omega_{\mathscr{D}} \times \Omega$ defined by: $\tau_x f(\mathbf{m},\omega) := f(\tau_x\mathbf{m}, \tau_x\omega)$, where $\tau_x\mathbf{m}$ and $\tau_x\omega$ are the disorder and particle configurations translated by $x \in \mathbb{Z}$, respectively: $(\tau_x\mathbf{m})_z := m_{x+z}$, and $(\tau_x\omega)_z = \omega_{x+z}$. For a fixed positive integer ℓ, we define $\Lambda_\ell := \{-\ell, ..., \ell\}$. If the box is centered at site $x \in \mathbb{Z}$, we denote it by $\Lambda_\ell(x) := \{-\ell + x, ..., \ell + x\}$. If f is a measurable function on $\Omega_{\mathscr{D}} \times \Omega$, the *support* of f, denoted by Λ_f, is the smallest subset of $\mathbb{Z}$ such that $f(\mathbf{m},\omega)$ only depends on $\{m_x, \omega_x \ ; \ x \in \Lambda_f\}$ and f is called a *cylinder (or local) function* if Λ_f is finite. In that case, we denote by s_f the smallest positive integer s such that Λ_s contains the support of f and then $\Lambda_f = \Lambda_{s_f}$. For every cylinder function $f : \Omega_{\mathscr{D}} \times \Omega \to \mathbb{R}$, consider the formal sum

$$\Gamma_f := \sum_{x\in\mathbb{Z}} \tau_x f$$

which does not make sense but for which

$$(\nabla_x f)(\mathbf{m},\omega) := f(\mathbf{m},\omega^x) - f(\mathbf{m},\omega),$$
$$(\nabla_{x,x+1} f)(\mathbf{m},\omega) := f(\mathbf{m},\omega^{x,x+1}) - f(\mathbf{m},\omega)$$

are well-defined.

Definition 2.1 We denote by $\mathscr{C}$ the set of measurable cylinder functions φ on $\Omega_{\mathscr{D}} \times \Omega$, such that

1. for all $\omega \in \Omega$, the random variable $\mathbf{m} \mapsto \varphi(\mathbf{m},\omega)$ is continuous on $\Omega_{\mathscr{D}}$;
2. for all $\mathbf{m} \in \Omega_{\mathscr{D}}$, the function $\omega \mapsto \varphi(\mathbf{m},\omega)$ belongs to $\mathbf{L}^2(\mu_\beta)$ and has null average with respect to μ_β.

Definition 2.2 We introduce the set of *quadratic* cylinder functions on $\Omega_{\mathscr{D}} \times \Omega$, denoted by $\mathscr{Q} \subset \mathscr{C}$, and defined as follows: $f \in \mathscr{Q}$ if there exists a sequence $\{\psi_{i,j}(\mathbf{m})\}_{i,j\in\mathbb{Z}}$ of real cylinder measurable functions on $\Omega_{\mathscr{D}}$ such that

1. for all $i,j \in \mathbb{Z}$, $\omega \in \Omega$, the random variable $\mathbf{m} \mapsto \psi_{i,j}(\mathbf{m},\omega)$ is continuous;
2. $\psi_{i,j}$ vanishes for all but a finite number of pairs (i,j),

3. f is written as

$$f(\mathbf{m},\omega) = \sum_{i\in\mathbb{Z}} \psi_{i,i}(\mathbf{m})(\omega_{i+1}^2 - \omega_i^2) + \sum_{\substack{i,j\in\mathbb{Z}\\ i\neq j}} \psi_{i,j}(\mathbf{m})\omega_i\omega_j, \tag{3}$$

In other words, quadratic functions are homogeneous polynomials of degree two in the variable ω, that have null average with respect to μ_β for every $\mathbf{m} \in \Omega_{\mathscr{D}}$. An other definition through *Hermite polynomials* is given in [1]. We are now ready to define two sets of functions that will play further a crucial role.

Definition 2.3 Let $\mathscr{C}_0$ be the set of cylinder functions φ on $\Omega_{\mathscr{D}} \times \Omega$ such that there exists a finite subset Λ of $\mathbb{Z}$, and cylinder, measurable functions $\{F_x, G_x\}_{x\in\Lambda}$ defined on $\Omega_{\mathscr{D}} \times \Omega$, that verify

$$\varphi = \sum_{x\in\Lambda} \Big\{ \nabla_x(F_x) + \nabla_{x,x+1}(G_x) \Big\},$$

and such that, for all $x \in \Lambda$,

1. for all $\omega \in \Omega$, $\mathbf{m} \mapsto F_x(\mathbf{m},\omega)$ and $\mathbf{m} \mapsto G_x(\mathbf{m},\omega)$ are continuous on $\Omega_{\mathscr{D}}$;
2. for all $\mathbf{m} \in \Omega_{\mathscr{D}}$, $\omega \mapsto F_x(\mathbf{m},\omega)$ and $\omega \mapsto G_x(\mathbf{m},\omega)$ belong to $\mathbf{L}^2(\mu_\beta)$.

Let $\mathscr{Q}_0 \subset \mathscr{C}_0$ be the set of such functions φ, with the additional assumption that the cylinder functions F_x, G_x are homogeneous polynomials of degree two in ω.

Before giving a few properties of these two spaces, let us now consider operators $\mathscr{L}^{\mathbf{m}}$, $\mathscr{A}^{\mathbf{m}}$ and $\mathscr{S}$ acting on functions $f \in \mathscr{C}$ in the same way as (2), except that the sums now run on the whole line $\mathbb{Z}$. For a finite subset Λ_ℓ of $\mathbb{Z}$ defined as above, we denote by $\mathscr{L}^{\mathbf{m}}_{\Lambda_\ell}$, resp. $\mathscr{S}_{\Lambda_\ell}$, the restriction of the generator $\mathscr{L}^{\mathbf{m}}$, resp. $\mathscr{S}$, to the box Λ_ℓ, assuming periodic boundary conditions.

Definition 2.4 Let $\mathscr{C}_0$ (respectively $\mathscr{Q}_0$) be the set of cylinder (respectively quadratic cylinder) functions φ on $\Omega_{\mathscr{D}} \times \Omega$ such that there exists a finite subset $\Lambda \Subset \mathbb{Z}$, and cylinder functions $\{F_x, G_x\}_{x\in\Lambda}$ satisfying

$$\varphi = \sum_{x\in\Lambda} \nabla_x(F_x) + \nabla_{x,x+1}(G_x).$$

If φ belongs to $\mathscr{Q}_0$, we assume the cylinder functions F_x, G_x to be quadratic.

Finally we introduce the *Dirichlet form* associated to the generator: for any $x \in \mathbb{Z}$ and $f, g \in \mathscr{C}$, let us define $\mathscr{D}_\ell(\mu_\beta; f) := \big\langle (-\mathscr{L}^{\mathbf{m}}_{\Lambda_\ell}) f, f \big\rangle_\beta = \big\langle (-\mathscr{S}_{\Lambda_\ell}) f, f \big\rangle_\beta$.

2.2 Semi-inner Products and Diffusion Coefficient

For cylinder functions $g, h \in \mathscr{C}$, let us introduce

$$\langle\langle g, h\rangle\rangle_{\beta,\star} := \sum_{x\in\mathbb{Z}} \mathbb{E}^\star_\beta[g\,\tau_x h] \quad\text{and}\quad \langle\langle g\rangle\rangle_{\beta,\star\star} := \sum_{x\in\mathbb{Z}} x\,\mathbb{E}^\star_\beta[g\,\omega_x^2] \tag{4}$$

which are well-defined because g and h belong to $\mathscr{C}$ and therefore all but a finite number of terms vanish. Notice that $\langle\langle\cdot,\cdot\rangle\rangle_{\beta,\star}$ is a semi inner product.

Definition 2.5 We define the *diffusion coefficient* $D(\beta)$ for $\beta > 0$ as equal to

$$\lambda + \frac{1}{\chi_\beta}\inf_{f\in\mathscr{Q}}\sup_{g\in\mathscr{Q}}\left\{\langle\langle f, -\mathscr{S}f\rangle\rangle_{\beta,\star} + 2\langle\langle j^A_{0,1} - \mathscr{A}^{\mathbf{m}}f, g\rangle\rangle_{\beta,\star} - \langle\langle g, -\mathscr{S}g\rangle\rangle_{\beta,\star}\right\}.$$

The first term in the sum is only due to the exchange noise, whereas the second one comes from the hamiltonian part of the dynamics. Formally, this formula reads

$$D(\beta) = \lambda + \frac{1}{\chi_\beta}\langle\langle j^A_{0,1}, (-\mathscr{L}^{\mathbf{m}})^{-1} j^A_{0,1}\rangle\rangle_{\beta,\star}, \tag{5}$$

but the last term is ill-defined because $j^A_{0,1}$ is not in the range of $\mathscr{L}^{\mathbf{m}}$. More rigorously, we should define $\langle\langle j^A_{0,1}, (-\mathscr{L}^{\mathbf{m}})^{-1} j^A_{0,1}\rangle\rangle_{\beta,\star}$ as

$$\limsup_{z\to 0}\langle\langle j^A_{0,1}, (z-\mathscr{L}^{\mathbf{m}})^{-1} j^A_{0,1}\rangle\rangle_{\beta,\star}.$$

The scalar product above is now well-defined, and the problem is reduced to prove convergence as $z \to 0$. From Hille-Yosida Theorem (see [19, Proposition 2.1] for instance) (5) is equal to the infinite volume Green-Kubo formula:

$$\overline{D}(\beta) = \lambda + \frac{1}{\chi_\beta}\lim_{\substack{z\to 0\\ z>0}} \mathbb{E}\left[\int_0^{+\infty} e^{-zt}\left\langle \sum_{x\in\mathbb{Z}} j^A_{x,x+1}(t), j^A_{0,1}(0)\right\rangle_\beta \mathrm{d}t\right]. \tag{6}$$

In Sect. 4, we prove that (6) converges, inspired by [11]. Assuming the convergence in the Green-Kubo formula, one can easily see that $\overline{D}(\beta)$ does not depend on β. We denote

$$L(z) := \frac{1}{\chi_\beta}\int_0^{+\infty} e^{-zt}\,\langle\langle j^A_{0,1}(t), j^A_{0,1}(0)\rangle\rangle_{\beta,\star}\mathrm{d}t.$$

The function L is smooth on $(0,+\infty)$. The Hilbert space generated by the set of local functions and the inner product $\langle\langle\cdot,\cdot\rangle\rangle_{\beta,\star}$ is denoted by $\mathbf{L}^2_\star$. We define $h_z := h_z(\mathbf{m},\omega;\beta)$ as the solution of the resolvent equation (in $\mathbf{L}^2_\star$) $(z-\mathscr{L}^{\mathbf{m}})h_z = j^A_{0,1}$. Then,

$$L(z) = \frac{1}{\chi_\beta}\langle\langle h_z, j^A_{0,1}\rangle\rangle_{\beta,\star} = \frac{\beta^2}{2}\langle\langle h_z, j^A_{0,1}\rangle\rangle_{\beta,\star}. \tag{7}$$

Observe that if ω is distributed according to μ_β then $\beta^{1/2}\omega$ is distributed according to μ_1. Since $h_z(\mathbf{m}, \omega; 1) = h_z(\mathbf{m}, \omega; \beta)$ and $j^A_{x,x+1}$ is a homogeneous function of degree two in ω, it follows that the diffusion coefficient does not depend on β.

From now on, we assume $\beta = 1$. This assumption is justified since we are going to deal only with quadratic functions (as defined before). For instance, when one result is stated for the scalar product $\langle\langle\cdot\rangle\rangle_{1,\star}$, the same argument in the proof can be rewritten for any $\beta > 0$, after multiplying the process $\{\omega_x(t)\}$ by $\beta^{-1/2}$.

3 Non-gradient Varadhan Approach

In this section we are going to identify the diffusion coefficient D given in Definition 2.5. Roughly speaking, D is the asymptotic component of the energy current $j_{x,x+1}$ in the direction of the gradient $\omega^2_{x+1} - \omega^2_x$, and makes the expression below vanish:

$$\inf_{f\in\mathcal{Q}} \limsup_{N\to\infty} \limsup_{t\to\infty} \frac{1}{tN}\mathbb{E}^\star_1\Bigg[\bigg(\int_0^t \sum_{x\in\mathbb{T}_N} \big[j_{x,x+1} - D(\omega^2_{x+1} - \omega^2_x) - \mathcal{L}^{\mathbf{m}}(\tau_x f)\big]\mathrm{d}s\bigg)^2\Bigg].$$

3.1 *An Insight Through Additive Functionals of Markov Processes*

Consider a continuous time Markov process $\{Y_s\}_{s\geqslant 0}$ on a complete and separable metric space E, with an invariant measure π. We denote by $\langle\cdot\rangle_\pi$ the inner product in $\mathbf{L}^2(\pi)$ and by $\mathcal{L}$ the infinitesimal generator of the process. The adjoint of $\mathcal{L}$ in $\mathbf{L}^2(\pi)$ is denoted by $\mathcal{L}^\star$. Fix a function $V : E \to \mathbb{R}$ in $\mathbf{L}^2(\pi)$ such that $\langle V\rangle_\pi = 0$. Theorem 2.7 in [20] gives conditions which guarantee a CLT for

$$\frac{1}{\sqrt{t}}\int_0^t V(Y_s)\mathrm{d}s$$

and shows that the limiting variance equals

$$\sigma^2(V,\pi) = 2\lim_{z\to 0^+}\big\langle V, (z-\mathcal{L})^{-1}V\big\rangle_\pi.$$

Let the generator $\mathcal{L}$ be decomposed as $\mathcal{L} = \mathcal{S} + \mathcal{A}$, where $\mathcal{S} = (\mathcal{L} + \mathcal{L}^\star)/2$ and $\mathcal{A} = (\mathcal{L} - \mathcal{L}^\star)/2$ denote, respectively, the symmetric and antisymmetric parts of $\mathcal{L}$. Let $\mathcal{H}_1$ be the completion of $\mathbf{L}^2(\pi)$ with respect to the semi-norm $\|\cdot\|_1$:

$$\|f\|^2_1 := \big\langle f, (-\mathcal{L})f\big\rangle_\pi = \big\langle f, (-\mathcal{S})f\big\rangle_\pi.$$

Let $\mathscr{H}_{-1}$ be the dual space of $\mathscr{H}_1$ w.r.t. $\mathbf{L}^2(\pi)$, in other words, the Hilbert space generated by suitably regular functions and the norm $\|\cdot\|_{-1}$ defined by

$$\|f\|_{-1}^2 := \sup_g \left\{ 2\langle f, g\rangle_\pi - \|g\|_1^2 \right\},$$

where the supremum is carried over all local functions g. Formally, $\|f\|_{-1}$ can also be thought as $\langle f, (-\mathscr{S})^{-1} f\rangle_\pi$. The following result is a rigorous estimate of the time variance in terms of the $\mathscr{H}_{-1}$ norm, which is proved in [20, Lemma 2.4].

Lemma 3.1 *Given $T > 0$ and a mean zero function V in $\mathbf{L}^2(\pi) \cap \mathscr{H}_{-1}$,*

$$\mathbb{E}_\pi \left[\sup_{0 \leqslant t \leqslant T} \left(\int_0^t V(s) \mathrm{d}s \right)^2 \right] \leqslant 24T \|V\|_{-1}^2. \tag{8}$$

Then, we should take V proportional to

$$\sum_{x \in \mathbb{T}_N} \left[j_{x,x+1} - D(\omega_{x+1}^2 - \omega_x^2) - \mathscr{L}^{\mathbf{m}}(\tau_x f) \right]$$

and then take the limit as $N \to \infty$. In the right-hand side of (8) we obtain a variance that depends on N, and the main task is to show that this variance converges. Precisely, we can prove that the limit of the variance results in a semi-norm, which is denoted by $|||\cdot|||_1$ and defined in (9). The final step consists in minimizing this semi-norm on a well-chosen subspace, through orthogonal projections in Hilbert spaces. The hard point is that $|||\cdot|||_1$ only depends on the symmetric part of the generator $\mathscr{S}$, and the latter is really degenerate (it does not have a spectral gap).

In [1], we prove that the variance $\langle f, (-\mathscr{S})^{-1} f\rangle_1$ is well defined for every function f in $\mathscr{Q}_0$. In Sect. 3.2, we relate the previous limiting variance (taking the limit as N goes to infinity) to the suitable semi-norm. Finally, Sect. 3.3 focuses on the diffusion coefficient and its different expressions.

3.2 Limiting Variance and Semi-norm

We return to the case $\beta = 1$. We look for a variational formula for the variance

$$(2\ell)^{-1} \mathbb{E} \left\langle \left(-\mathscr{S}_{\Lambda_\ell}\right)^{-1} \sum_{|x| \leqslant \ell_\varphi} \tau_x \varphi, \sum_{|x| \leqslant \ell_\varphi} \tau_x \varphi \right\rangle_1$$

where $\varphi \in \mathscr{Q}_0$ and $\ell_\varphi = \ell - s_\varphi - 1$. We first introduce a semi-norm on $\mathscr{Q}_0$. For any cylinder function φ in $\mathscr{Q}_0$, let us define

$$|||\varphi|||_1^2 = 2 \sup_{g\in\mathscr{Q}} \left\{ \langle\langle \varphi, g\rangle\rangle_{1,\star} + \frac{\langle\langle\varphi\rangle\rangle_{1,\star\star}^2}{\lambda\chi_1} - \frac{\lambda}{4}\mathbb{E}_1^\star\left[\left(\nabla_{0,1}\Gamma_g\right)^2\right] - \frac{\gamma}{4}\mathbb{E}_1^\star\left[\left(\nabla_0\Gamma_g\right)^2\right]\right\}$$

$$= \sup_{\substack{g\in\mathscr{Q}\\ a\in\mathbb{R}}} \left\{2\langle\langle\varphi, g\rangle\rangle_{1,\star} + 2a\langle\langle\varphi\rangle\rangle_{1,\star\star} - \mathbb{E}\left[\mathscr{D}_0\left(\mu_1; a\omega_0^2 + \Gamma_g\right)\right]\right\}. \quad (9)$$

This formula can be formally restated as

$$|||\varphi|||_1^2 = \langle\langle\varphi, (-\mathscr{S})^{-1}\varphi\rangle\rangle_{1,\star} + \frac{2}{\lambda\,\chi_1}\langle\langle\varphi\rangle\rangle_{1,\star\star}^2. \quad (10)$$

Since φ belongs to $\mathscr{Q}_0$, one can prove that the first term in the right-hand side of (10) is well-defined (Proposition 4.4 in [1]). We are now in position to state the key result of the non-gradient Varadhan approach.

Theorem 3.2 *Consider a quadratic cylinder function $\varphi \in \mathscr{Q}_0$. Then*

$$\lim_{\ell\to\infty} (2\ell)^{-1}\mathbb{E}\left\langle\left(-\mathscr{S}_{\Lambda_\ell}\right)^{-1}\sum_{|x|\leqslant\ell_\varphi}\tau_x\varphi, \sum_{|x|\leqslant\ell_\varphi}\tau_x\varphi\right\rangle_1 = |||\varphi|||_1^2.$$

3.3 Hilbert Space and Projections

We can easily define from $|||\cdot|||_1$ a semi-inner product on $\mathscr{C}_0$ through polarization. Denote by $\mathscr{N}$ the kernel of the semi-norm $|||\cdot|||_1$ on $\mathscr{C}_0$. Then, the completion of $\mathscr{Q}_0|_{\mathscr{N}}$ denoted by $\mathscr{H}_1$ is a Hilbert space. Let us explain how the well-known Varadhan's approach is modified. Usually, the Hilbert space on which orthogonal projections are performed is the completion of $\mathscr{C}_0|_{\mathscr{N}}$, in other words it involves all local functions. Then, the standard procedure aims at proving that each element of that Hilbert space can be approximated by a sequence of functions in the range of the generator plus an additional term which is proportional to the current. The crucial steps for obtaining this decomposition consist in: first, controlling the antisymmetric part of the generator by the symmetric one for every cylinder function, and second, proving a strong result on germs of closed forms. These two key points are not valid in our model, but they can be proved when restricted to quadratic functions. It turns out that these weak versions are sufficient, since we are looking for a fluctuation-dissipation approximation that involves quadratic functions only.

In [1], we show that $\mathscr{H}_1$ is the completion of $\mathscr{S}\mathscr{Q}|_{\mathscr{N}} + \{j_{0,1}^S\}$. In other words, all elements of $\mathscr{H}_1$ can be approximated by $aj_{0,1}^S + \mathscr{S}g$ for some $a \in \mathbb{R}$ and $g \in \mathscr{Q}$. This is not irrelevant since the symmetric part of the generator preserves the degree of polynomial functions. The sum of the two subspaces $\{j_{0,1}^S\}$ and $\overline{\mathscr{S}\mathscr{Q}|_{\mathscr{N}}}$ is orthogonal. Nevertheless, this decomposition is not satisfactory, because we want the fluctuating term to be on the form $\mathscr{L}^{\mathbf{m}}(f)$, and not $\mathscr{S}(f)$. In order to make this replacement, we need to prove the weak sector condition, that gives a control of $|||\mathscr{A}^{\mathbf{m}}g|||_1$ by $|||\mathscr{S}g|||_1$, when g is a quadratic function. The only trouble is that this new decomposition is not orthogonal any more, so that we can not express the diffusion coefficient as a variational formula. This problem is solved by clever projections into a suitable Hilbert space. The main theorem given in [1] is:

Theorem 3.3 *For every $g \in \mathscr{Q}_0$, there exists a unique constant $a \in \mathbb{R}$, such that*

$$g + a(\omega_1^2 - \omega_0^2) \in \overline{\mathscr{L}^{\mathbf{m}}\mathscr{Q}} \quad \textit{in } \mathscr{H}_1.$$

In particular, there exists a unique number D, and a sequence $\{f_k\} \in \mathscr{Q}$ such that

$$|||j_{0,1} - D(\omega_1^2 - \omega_0^2) - \mathscr{L}^{\mathbf{m}}(f_k)|||_1 \xrightarrow[k\to\infty]{} 0.$$

Finally, one can prove more formulas for coefficient D defined in Theorem 3.3, and relate it to Definition 2.5, by following the argument given by instance in [21].

4 Convergence of Green-Kubo Formula

Remind that the *Green-Kubo formula* predicted by linear response theory is

$$\overline{\kappa}(z) := \lambda + \frac{1}{2}\langle\langle j_{0,1}^A, (z - \mathscr{L}^{\mathbf{m}})^{-1} j_{0,1}^A\rangle\rangle_{1,\star}. \tag{11}$$

Hereafter, we extend the inner-product $\langle\langle\cdot\rangle\rangle_{1,\star}$ (originally defined on $\mathscr{C}$) to the Hilbert space generated by the set of square integrable functions and denoted by $\mathbf{L}^2_\star$.

4.1 Existence of the Green-Kubo Formula

In this paragraph we prove the existence and finiteness of the Green-Kubo formula. The argument is based on the paper [11], where the author generalizes [22, 23].

Theorem 4.1 *The z-vanishing limit $\overline{D} := \lim \overline{\kappa}(z)$ exists, is finite and positive.*

Proof Recall (7). We have to prove that $\langle\langle h_z, j_{0,1}^A\rangle\rangle_{1,\star}$ converges as z vanishes, and that the limit is finite and non-negative. Then, from (11) it will follow that $\overline{D} \geqslant \lambda > 0$ and $\overline{D}$ is positive. We denote by $\|\cdot\|_1$ the semi-norm corresponding to the symmetric part of the generator due to the flip noise

$$\|f\|_1^2 = \langle\langle f, (-\gamma\mathscr{S}^{\text{flip}})f\rangle\rangle_{1,\star}$$

and $\mathscr{H}_\star$ is the Hilbert space obtained by the completion of $\mathbf{L}^2_\star$ w.r.t. that semi-norm. We multiply the resolvent equation by h_z and integrate with respect to $\langle\langle\cdot\rangle\rangle_{1,\star}$:

$$z\langle\langle h_z, h_z\rangle\rangle_{1,\star} + \|h_z\|_1^2 + \langle\langle h_z, (-\lambda\mathscr{S}^{\text{exch}})h_z\rangle\rangle_{1,\star} = \langle\langle h_z, j^A_{0,1}\rangle\rangle_{1,\star}.$$

Let us notice that $(-\gamma\mathscr{S}^{\text{flip}})(j^A_{0,1}) = 2\gamma j^A_{0,1}$. As a consequence, the Cauchy-Schwarz inequality for the scalar product $\langle\langle\cdot, (-\gamma\mathscr{S}^{\text{flip}})\cdot\rangle\rangle_{1,\star}$ on the right-hand side gives $\|h_z\|_1^2 \leqslant C$ for some positive constant C. Since $\{h_z\}_z$ is bounded in $\mathscr{H}_\star$, we can extract a weakly converging subsequence in $\mathscr{H}_\star$. We continue to denote this subsequence by $\{h_z\}_z$ and we denote by h_0 the limit.

Now we are going to show that the convergence is stronger (see (4) in Lemma 4.2 below) and that the limit is independent of the subsequence. Since the generator $\mathscr{L}^{\mathbf{m}}$ conserves the degree of homogeneous polynomial functions, we know that the solution of the resolvent equation is expected to be on the form

$$h_z(\omega) = \sum_{x,y\in\mathbb{Z}^2} \varphi_z(x, y)\omega_x\omega_y,$$

where $\varphi_z : \mathbb{Z}^2 \to \mathbb{R}$ is a square-summable symmetric function. Let $h_z = h_z^= + h_z^{\neq}$ be the decomposition of h_z according to the two subspaces $\mathscr{Q}^=$ and $\mathscr{Q}^{\neq}$, where $\mathscr{Q}^=$ is generated by $\{\omega_x^2, x \in \mathbb{Z}\}$ and $\mathscr{Q}^{\neq}$ is generated by $\{\omega_x\omega_y, x \neq y\}$. The main point in the following argument is that all gradient terms vanish in $\mathbf{L}^2_\star$.

First, one can see how the spaces $\mathscr{Q}^=$ and $\mathscr{Q}^{\neq}$ are mapped by the generators:

$$\begin{array}{ll} \mathscr{A}^{\mathbf{m}} : \mathscr{Q}^= \to \mathscr{Q}^{\neq} & \mathscr{A}^{\mathbf{m}} : \mathscr{Q}^{\neq} \to \mathscr{Q} \\ \mathscr{S}^{\text{flip}} : \mathscr{Q}^= \to \{0\} & \mathscr{S}^{\text{flip}} : \mathscr{Q}^{\neq} \to \mathscr{Q}^{\neq} \\ \mathscr{S}^{\text{exch}} : \mathscr{Q}^= \to \mathscr{Q}^= & \mathscr{S}^{\text{exch}} : \mathscr{Q}^{\neq} \to \mathscr{Q}^{\neq} \end{array}$$

Moreover, if $f \in \mathscr{Q}^=$, then $\mathscr{A}^{\mathbf{m}}(f)$ is a gradient in $\mathscr{Q}^{\neq}$, and $\mathscr{S}^{\text{exch}}(f)$ is a gradient in $\mathscr{Q}^=$. With all these considerations, the resolvent equation rewrites in $\mathbf{L}^2_\star$ as

$$\begin{cases} zh_z^= - \lambda\mathscr{S}^{\text{exch}}(h_z^=) = 0 \\ zh_z^{\neq} - \lambda\mathscr{S}^{\text{exch}}(h_z^{\neq}) - \gamma\mathscr{S}^{\text{flip}}(h_z^{\neq}) - \mathscr{A}^{\mathbf{m}}(h_z^{\neq}) = j^A_{0,1}. \end{cases}$$

The first equation means that $h_z^= = 0$ in $\mathbf{L}^2_\star$ and therefore the solution h_z of the resolvent equation is an element of $\mathscr{Q}^{\neq}$. As a consequence, we can write $(-\gamma\mathscr{S}^{\text{flip}})(h_z) = 2\gamma h_z$, and this remark is one of the key points in the following argument.

Lemma 4.2 *All the properties below are satisfied:*

1. $\lim_{z\to 0} z\langle\langle h_z, h_z\rangle\rangle_{1,\star} = 0$
2. $\{h_z\}$ *weakly converges as* z *goes to 0 towards* h_0 *in* $\mathbf{L}^2_\star$
3. $\langle\langle j^A_{0,1}, h_0\rangle\rangle_{1,\star} = \langle\langle h_0, (-\mathscr{S})h_0\rangle\rangle_{1,\star}$
4. $\langle\langle (h_z - h_0), (-\mathscr{S})(h_z - h_0)\rangle\rangle_{1,\star}$ *vanishes as* z *goes to 0*
5. *the weak limit of* $\{h_z\}$ *does not depend on the subsequence.*

We briefly prove the five points: (1) and (2) come from the fact that $(-\gamma\mathscr{S}^{\text{flip}})(h_z) = 2\gamma h_z$. To get (3), we multiply the resolvent equation by $h_{z'}$ and integrate:

$$z\langle\langle h_{z'}, h_z\rangle\rangle_{1,\star} + \langle\langle h_{z'}, (-\mathscr{S})h_z\rangle\rangle_{1,\star} + \langle\langle h_{z'}, (-\mathscr{A}^{\mathbf{m}})h_z\rangle\rangle_{1,\star} = \langle\langle h_{z'}, j^A_{0,1}\rangle\rangle_{1,\star}.$$

We first take the limit as $z' \to 0$ and then as $z \to 0$, and we use (1) and (2) to obtain (3). In the same way, multiplying the resolvent equation by h_z gives

$$z\langle\langle h_z, h_z\rangle\rangle_{1,\star} + \langle\langle h_z, (-\mathscr{S})h_z\rangle\rangle_{1,\star} = \langle\langle h_z, j^A_{0,1}\rangle\rangle_{1,\star}.$$

The first term of the left-hand side vanishes as z goes to 0, and the right-hand side converges to $\langle\langle h_0, (-\mathscr{S})h_0\rangle\rangle_{1,\star}$. This implies (4), that is

$$\langle\langle (h_z - h_0), (-\mathscr{S})(h_z - h_0)\rangle\rangle_{1,\star} \xrightarrow[z\to 0]{} 0.$$

The uniqueness of the limit follows by a standard argument with same ideas as before. We have proved the first part: the limit exists. To obtain its finiteness, we are going to give an upper bound, using the following variational formula:

$$\langle\langle j^A_{0,1}, (z - \mathscr{L}^{\mathbf{m}})^{-1} j^A_{0,1}\rangle\rangle_{1,\star} = \sup_f \left\{ 2\langle\langle f, j^A_{0,1}\rangle\rangle_{1,\star} - \|f\|^2_{1,z} - \|\mathscr{A}^{\mathbf{m}} f\|^2_{-1,z} \right\},$$

where the supremum is carried over local functions and the two norms $\|\cdot\|_{\pm 1,z}$ are

$$\|f\|^2_{\pm 1,z} = \langle\langle f, (z - \mathscr{S})^{\pm 1} f\rangle\rangle_{1,\star}.$$

For the upper bound, we neglect the term coming from the antisymmetric part $\mathscr{A}^{\mathbf{m}} f$:

$$\langle\langle j^A_{0,1}, (z - \mathscr{L}^{\mathbf{m}})^{-1} j^A_{0,1}\rangle\rangle_{1,\star} \leqslant \langle\langle j^A_{0,1}, (z - \mathscr{S})^{-1} j^A_{0,1}\rangle\rangle_{1,\star}.$$

In the right-hand side we can also neglect the part coming from the exchange symmetric part $\mathscr{S}^{\text{exch}}$, and remind that $\mathscr{S}^{\text{flip}}(j^A_{0,1}) = -2 j^A_{0,1}$. This gives an explicit finite upper bound. Then, we have from Lemma 4.2, Property (3) that

$$\lim_{z\to 0} \langle\langle j^A_{0,1}, (z - \mathscr{L}^{\mathbf{m}})^{-1} j^A_{0,1}\rangle\rangle_{1,\star} = \langle\langle j^A_{0,1}, h_0\rangle\rangle_{1,\star} = \langle\langle h_0, (-\mathscr{S})h_0\rangle\rangle_{1,\star} \geqslant 0.$$

4.2 Equivalence of the Definitions and Vanishing Noise Limit

Finally, we can rigorously prove [1] the equality between the variational formula for the diffusion coefficient and the Green-Kubo formula, precisely:

Theorem 4.3 *For every* $\lambda > 0$ *and* $\gamma > 0$*,*

$$\overline{D} := \lambda + \frac{1}{2} \lim_{\substack{z \to 0 \\ z > 0}} \langle\langle j^A_{0,1}, (z - \mathscr{L}^{\mathbf{m}})^{-1} j^A_{0,1} \rangle\rangle_{1,\star},$$

coincides with the diffusion coefficient D defined in Theorem 3.3.

Moreover, it can be easily shown that the homogenized Green-Kubo formula also converges if the strength λ of the exchange noise vanishes. First, we turn (11) into a new definition that highlights the dependence on $\lambda > 0$. For that purpose we introduce new notations: we define $\mathscr{S}_0 := \gamma \mathscr{S}^{\text{flip}}$, $\mathscr{S}_\lambda := \mathscr{S}_0 + \lambda \mathscr{S}^{\text{exch}}$, and then

$$\begin{cases} \mathscr{L}^{\mathbf{m}}_0 := \mathscr{A}^{\mathbf{m}} + \mathscr{S}_0 \\ \mathscr{L}^{\mathbf{m}}_\lambda := \mathscr{A}^{\mathbf{m}} + \mathscr{S}_\lambda = \mathscr{L}^{\mathbf{m}}_0 + \lambda \mathscr{S}^{\text{exch}} \end{cases} \text{and } J_0(\mathbf{m})(\omega) := \frac{\omega_0 \omega_1}{\sqrt{m_0 m_1}} = j^A_{0,1}(\mathbf{m}, \omega).$$

Let us introduce the homogenized Green-Kubo formula for both noises:

$$\overline{\kappa}(\lambda, z) := \langle\langle J_0(\mathbf{m}), (z - \mathscr{L}^{\mathbf{m}}_\lambda)^{-1} J_0(\mathbf{m}) \rangle\rangle_{1,\star} \quad (12)$$

and the homogenized Green-Kubo formula for flip noise only:

$$\overline{\kappa}_0(z) := \langle\langle J_0(\mathbf{m}), (z - \mathscr{L}^{\mathbf{m}}_0)^{-1} J_0(\mathbf{m}) \rangle\rangle_{1,\star}. \quad (13)$$

According to the previous paragraph, we already know that the Green-Kubo formulas (12) and (13) converge as z goes to 0. Then, the following diffusion coefficients are well defined, for all $\lambda > 0$,

$$\overline{D}(\lambda) := \lambda + \lim_{z \to 0} \overline{\kappa}(\lambda, z), \qquad \overline{D}_0 := \lim_{z \to 0} \overline{\kappa}_0(z).$$

The main result of this subsection is stated in the following theorem, proved in [1].

Theorem 4.4 *The function* $\lambda \mapsto \overline{D}(\lambda)$ *is continuous at 0.*

Let us remark that the theorem above does not imply the existence of the hydrodynamics diffusion coefficient $D(0, \gamma)$. This question remains open.

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