# Type-Reduction for Concave Type-2 Fuzzy Sets

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**Abstract.** An efficient centroid type-reduction strategy for general type-2 fuzzy set is proposed by Liu. In Liu's method, a type-2 fuzzy set will be decomposed into several interval type-2 fuzzy sets. However, if the membership function of the type-2 fuzzy set is concave, the primary membership of these interval type-2 fuzzy sets on some points may not have only one continuous interval. Existing type-reduction algorithms, such as Karnik-Mendel algorithm and Enhanced Karnik-Mendel algorithm, can not deal with this problem. We propose a method to decompose this problem into several subproblems which can then be solved by existing type-reduction algorithms. The union of the solutions to the subproblems is the final solution to the original problem.

**Keywords:** Karnik-Mendel (KM) algorithm, type-reduction, type-2 fuzzy set, interval type-2 fuzzy set.

## 1 Introduction

The type-2 fuzzy system has evolved from the type-1 fuzzy system [1], [4], [5], [7]. One of the major differences between these two systems lies on the defuzzification involved in the inference process. The defuzzification of a type-2 fuzzy set is composed of type-reduction and type-1 defuzzification. Therefore, type-reduction plays an important role in type-2 fuzzy systems [6].

Two kinds of type-reduction have been introduced. Liu proposed a method [2] to perform type-reduction for general type-2 fuzzy sets. A type-2 fuzzy set is decomposed into interval type-2 fuzzy sets, and one only needs to perform type-reduction for each resulting interval type-2 fuzzy set [2], [9], [10]. There are many type-reduction algorithms for interval type-2 fuzzy sets, such as Karnik-Mendel algorithm (KM) [3] and Enhanced Karnik-Mendel algorithm (EKM) [8]. EKM is an iterative algorithm which is a faster version of KM. The Enhanced Centroid-Flow algorithm [12] is a more accurate version of the Centroid-Flow algorithm [11]. This algorithm utilizes KM or EKM only at the central $\alpha$-level, and then lets its result flow upward to the maximum $\alpha$-level and downward to the minimum $\alpha$-level.

However, if the membership function of the type-2 fuzzy set is concave, the primary membership of these interval type-2 fuzzy sets on some points may

 
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not have only one continuous interval. The type-reduction methods mentioned above can not deal with this problem. We propose a method to decompose this problem into several subproblems which can then be solved by any of the above mentioned algorithm. The union of the solutions to the subproblems is the final solution to the original problem.

The rest of this paper is organized as follows. Section 2 provides a brief introduction to fuzzy sets. Our proposed method is described in Section 3. An illustrating example is shown in Section 4. Finally, a conclusion is given in Section 5.

## 2 Background

Some background about fuzzy sets is given here. For more details, please refer to [1] and the other cited literature.

### 2.1 Type-2 Fuzzy Set

A type-2 fuzzy set $\widetilde{A}$ can be expressed as

$$\widetilde{A} = \{(x,u), \mu_{\widetilde{A}}(x,u) | \forall x \in X, \forall u \in J_x\} \tag{1}$$

where $X$ is the universe for primary variable $x$,and $J_x$ is the primary membership of $\widetilde{A}$ at $x$. $\widetilde{A}$ can also be expressed as

$$\widetilde{A} = \int_{x\in X} \int_{u\in J_x} \mu_{\widetilde{A}}(x,u)/(x,u) \tag{2}$$

where $\int$ denotes the union of all admissible $x$ and $u$.

### 2.2 Interval Type-2 Fuzzy Set

An interval type-2 fuzzy set is a special case of type-2 fuzzy set. It's membership function degree is 1. An interval type-2 fuzzy set $\widetilde{A}$ can be expressed as

$$\widetilde{A} = \{(x,u), \mu_{\widetilde{A}}(x,u) = 1 | \forall x \in X, \forall u \in J_x\} \tag{3}$$

or

$$\widetilde{A} = \int_{x\in X} \int_{u\in J_x} 1/(x,u) \tag{4}$$

The footprint of uncertainty of $\widetilde{A}$, denoted by $FOU(\widetilde{A})$, is defined by

$$FOU(\widetilde{A}) = \bigcup_{x\in X} J_x = \bigcup_{x\in X} [\underline{I}(x), \bar{I}(x)] \tag{5}$$

where $[\underline{I}(x), \bar{I}(x)]$ is an interval set, and $[\underline{I}(x), \bar{I}(x)] \subseteq [0,1]$.

### 2.3 Liu's Method

An efficient centroid type-reduction strategy for general type-2 fuzzy set is proposed by Liu [2]. The key idea of Liu's method is to decompose a type-2 fuzzy set into several interval type-2 fuzzy sets, called $\alpha$-plane representation, as illustrated in Fig. 1. The $\alpha$-plane for a type-2 fuzzy set $\widetilde{A}$, denoted by $\widetilde{A}_\alpha$, is defined

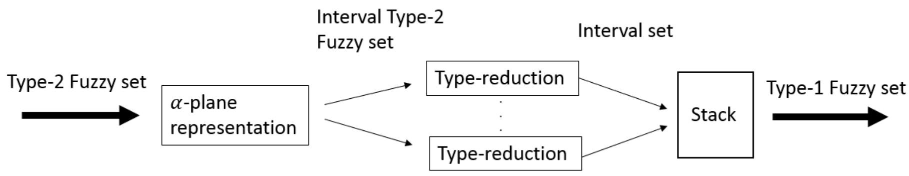


**Fig. 1.** Liu's method

to be

$$\widetilde{A}_\alpha = \bigcup_{x \in X} \{(x, u) | \mu_{\widetilde{A}}(x, u) \geq \alpha\} \tag{6}$$

The $\widetilde{A}_\alpha$ is an interval type-2 fuzzy set. For each interval type-2 fuzzy set, an interval set can be obtained by applying the centroid type-reduction. Finally, The union of the resulting interval sets form a type-1 fuzzy set. This type-1 fuzzy set is the centroid of the type-2 fuzzy set $\widetilde{A}$.

### 2.4 Enhanced Karnik-Mendel Algorithm

The Enhanced Karnik-Mendel (EKM) algorithm [8] is a type-reduction algorithm for interval type-2 fuzzy sets. It is an enhanced version of Karnik-Mendel algorithm (KM) [3]. An interval type-2 fuzzy set $\widetilde{A}$ can be regarded as composed of many type-1 fuzzy sets $A_i$ such as

$$\widetilde{A} = A_1 \cup A_2 ... \cup A_i \cup ... \tag{7}$$

where

$$A_i = \{(x, u) | \forall x \in X, u \in J_x\} \tag{8}$$

The centroid type-reduction of the interval type-2 fuzzy set $\widetilde{A}$ is composed of all the centers of the type-1 fuzzy sets $A_i$. As a result, one can get an interval set $[c_l, c_r]$. This interval set is the centroid of the interval type-2 fuzzy set $\widetilde{A}$. EKM is a fast algorithm to find such $c_l$ and $c_r$, and can be summarized in Table 1.

## 3 Proposed Method

The problem of Liu's method is the membership function of type-2 fuzzy set must be a convex function. If the membership function is concave function, $J_x$

**Table 1.** EKM to calculate the centroid of an interval type-2 fuzzy set

| Step | EKM for $c_l$ |
|---|---|
| 1 | Set $k = [N/2.4]$ (the nearest integer to $N/2.4$) for $N = \vert X\vert$. |
| 2 | Compute $c = \frac{\sum_{i=1}^{k} x_i \bar{\mu}_{\widetilde{A}}(x_i) + \sum_{i=k+1}^{N} x_i \underline{\mu}_{\widetilde{A}}(x_i)}{\sum_{i=1}^{k} \bar{\mu}_{\widetilde{A}}(x_i) + \sum_{i=k+1}^{N} \underline{\mu}_{\widetilde{A}}(x_i)}$ . |
| 3 | Find $k' \in [1, N-1]$ such that $x_{k'} \leq c \leq x_{k'+1}$. |
| 4 | Check $k = k'$: if yes, stop and set $c_l = c$, $L = k$; if no, set $k = k'$ and go to step 2. |

| Step | EKM for $c_r$ |
|---|---|
| 1 | Set $k = [N/2.4]$ (the nearest integer to $N/1.7$) for $N = \vert X\vert$. |
| 2 | Compute $c = \frac{\sum_{i=1}^{k} x_i \underline{\mu}_{\widetilde{A}}(x_i) + \sum_{i=k+1}^{N} x_i \bar{\mu}_{\widetilde{A}}(x_i)}{\sum_{i=1}^{k} \underline{\mu}_{\widetilde{A}}(x_i) + \sum_{i=k+1}^{N} \bar{\mu}_{\widetilde{A}}(x_i)}$ . |
| 3 | Find $k' \in [1, N-1]$ such that $x_{k'} \leq c \leq x_{k'+1}$. |
| 4 | Check if $k = k'$: if yes, stop and set $c_r = c$, $R = k$; if no, set $k = k'$ and go to step 2. |

may be a union which is composed of two or more intervals for the $\alpha$-plane representation $\widetilde{A}_\alpha$. The $FOU(\widetilde{A}_\alpha)$ may be

$$FOU(\widetilde{A}_\alpha) = \bigcup_{x \in X} J_x \tag{9}$$

where

$$J_x = [\underline{I_1}(x), \bar{I}_1(x)] \cup [\underline{I_2}(x), \bar{I}_2(x)] ... \cup [\underline{I_j}(x), \bar{I}_j(x)] \cup ...$$

If $J_x$ is composed of more than one interval, we can't use EKM algorithm to calculate the centroid.

We propose a method to perform the centroid type-reduction of a concave type-2 fuzzy set $\widetilde{A}$. The first step is to decompose $\widetilde{A}$ into several interval type-2 fuzzy sets $\widetilde{A}_{\alpha_i}$ for $\alpha_i \in [0, 1]$ by the $\alpha$-plane representation as shown in Fig. 2. Since $\widetilde{A}$ is a concave type-2 fuzzy set, the primary membership of $\widetilde{A}_{\alpha_i}$ may be the union of several interval sets, denoted by $J_x^{\alpha_i}$. The footprint of uncertainty of $\widetilde{A}_{\alpha_i}$ is defined by

$$FOU(\widetilde{A}_{\alpha_i}) = \bigcup_{x \in X} J_x^{\alpha_i}$$

and

$$J_x^{\alpha_i} = I_{x1}^{\alpha_i} \cup I_{x2}^{\alpha_i} ... \cup I_{xj}^{\alpha_i} \cup ... \tag{10}$$

where $I_{xj}^{\alpha_i}$ is an interval set, and $I_{xj}^{\alpha_i} \subseteq [0, 1]$.

For each interval type-2 fuzzy set $\widetilde{A}_{\alpha_i}$, we decompose it into several interval type-2 fuzzy sets $\widetilde{A}_{\alpha_i}^l$ as shown in Fig. 3. The footprint of uncertainty of $\widetilde{A}_{\alpha_i}^l$ is

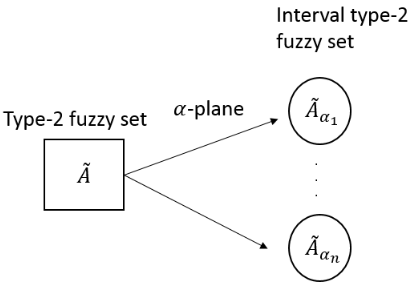


**Fig. 2.** The $\alpha$-plane representation

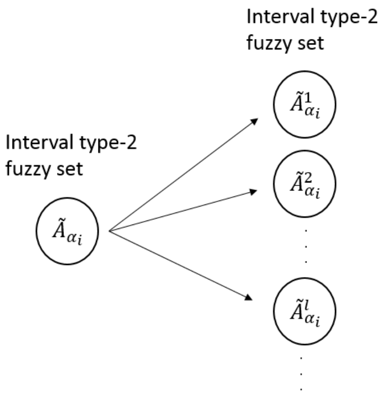


**Fig. 3.** The decomposition of an interval type-2 fuzzy set

defined by

$$FOU(\widetilde{A}^l_{\alpha_i}) = \bigcup_{x \in X} I_x^{\alpha_i} \tag{11}$$

where $I_x^{\alpha_i}$ is an interval set, and $I_x^{\alpha_i} \subseteq J_x^{\alpha_i}$. Then we can have $S$ interval type-2 fuzzy sets from $\widetilde{A}^l_{\alpha_i}$, i.e.,

$$S = \prod_{x \in X} s_x \tag{12}$$

where $s_x$ is the number of intervals contained in $J_x^{\alpha_i}$. If $\widetilde{A}$ is a convex type-2 fuzzy set, $s_x$ is 1 for all $x \in X$.

The primary membership of each interval type-2 fuzzy set $\widetilde{A}^l_{\alpha_i}$ is an interval set for all $x \in X$, so we can perform the centroid type-reduction by EKM for each interval type-2 fuzzy set $\widetilde{A}^l_{\alpha_i}$. Then, we collect all the results from the application of EKM to each interval type-2 fuzzy set $\widetilde{A}^l_{\alpha_i}$ and get a set $W_{\alpha_i}$

$$W_{\alpha_i} = \bigcup_{l=1}^{S} w^l_{\alpha_i} \tag{13}$$

where $w^l_{\alpha_i}$ is the result of applying EKM on $\widetilde{A}^l_{\alpha_i}$. Note that $w^l_{\alpha_i}$ is an interval set, but $W_{\alpha_i}$ is not necessarily an interval set. The $W_{\alpha_i}$ we obtain is the centroid of the interval type-2 fuzzy set $\widetilde{A}_{\alpha_i}$.

Finally, we can obtain a type-1 fuzzy set by doing a union of $W_{\alpha_i}$ for all $i$. This type-1 fuzzy set is then the centroid of the concave type-2 fuzzy set $\widetilde{A}$. The whole process of our proposed method for computing the centroid of a concave type-2 fuzzy set can be summarized in Fig. 4.

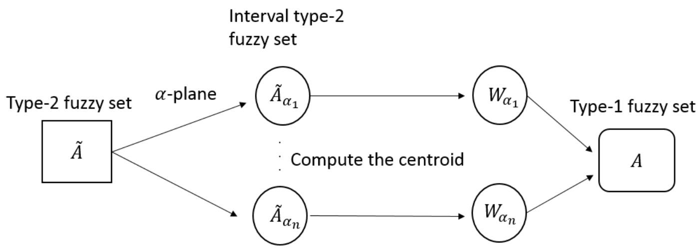


**Fig. 4.** Our method for computing the centroid of a concave type-2 fuzzy set

# 4 Numerical Results

In this section, we show the result of applying our method on a type-2 fuzzy set. Liu's method is not applicable for this case. The universal set $X$ involved is $[0.0, 0.2, 0.4, ..., 4]$ and the $\alpha$ set adopted is $[0.000, 0.025, 0.050, 0.075, ..., 1]$ for this example.

This example is a concave type-2 fuzzy set $\widetilde{A}$ which is shown in Fig. 5. The primary membership function $f_{\widetilde{A}}(x)$ and secondary membership function $\mu_{\widetilde{A}}(x, u)$ of $\widetilde{A}$ are defined by

$$f_{\widetilde{A}}(x) = exp\left[-\frac{(x-2)^2}{2*(0.3)^2}\right] \tag{14}$$

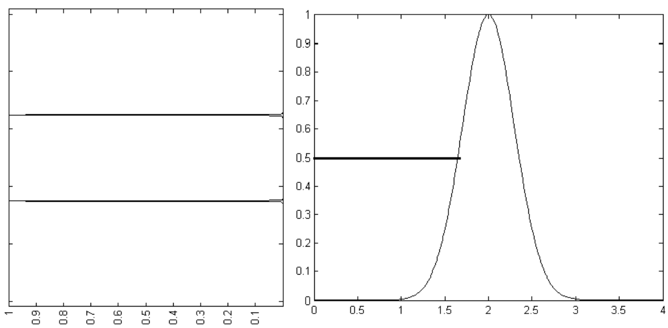

**Fig. 5.** $\widetilde{A}$ for illustration

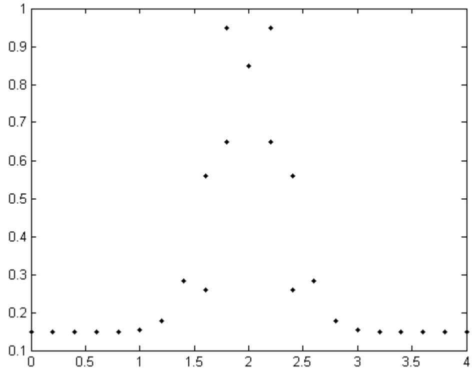


**Fig. 6.** The $\alpha$-plane with $\alpha = 0.5$

and

$$\mu_{\widetilde{A}}(x,u) = \begin{cases} exp\left[-\frac{(u-(f_{\widetilde{A}}(x)-0.15))^2}{2*(0.001)^2}\right], u \le f_{\widetilde{A}}(x) \\ exp\left[-\frac{(u-(f_{\widetilde{A}}(x)+0.15))^2}{2*(0.001)^2}\right], u > f_{\widetilde{A}}(x) \end{cases} \tag{15}$$

Firstly, we get $\alpha$-planes from $\widetilde{A}$. For example, the $\alpha$-plane with $\alpha = 0.5$ is shown in Fig. 6. Note that each point in this figure signifies an interval. Secondly, each $\alpha$-plane is further decomposed into several interval type-2 fuzzy sets. For the $\alpha$-plane with $\alpha = 0.5$, it is decomposed into 16 interval type-2 fuzzy sets. One of the decomposed interval type-2 fuzzy set is shown in Fig. 7, where each point signifies an interval. Finally, we do type-reduction for each decomposed interval type-2 fuzzy set using EKM. Since the secondary membership function of $\widetilde{A}$ is a concave function, $W_{\alpha_i}$ may contain several intervals. It means that the result of $\widetilde{A}$ can be a concave type-1 fuzzy set. The centroid obtained for $\widetilde{A}$ is shown in Fig. 8.

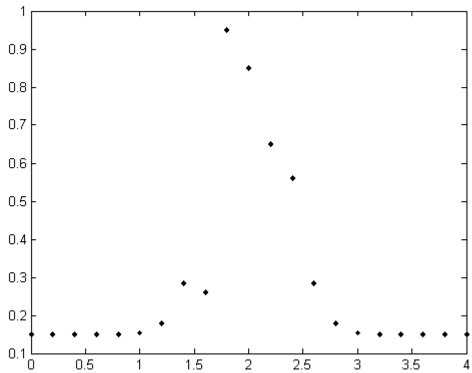


**Fig. 7.** A decomposed interval type-2 fuzzy set of Fig. 6

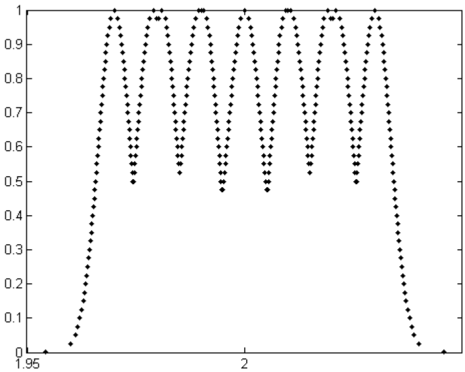


**Fig. 8.** The type-reduction result of $\widetilde{A}$

## 5 Conclusion

We have presented a method to perform type-reduction for type-2 fuzzy sets. Our method is based on Liu's method which can only handle the type-2 fuzzy sets with convex membership functions. By our method, an underlying type-2 fuzzy set is decomposed into several interval type-2 fuzzy sets by the $\alpha$-plane representation. Then, we decompose each interval type-2 fuzzy set into several new interval type-2 fuzzy sets. These new interval type-2 fuzzy sets are then handled by existing type-reduction algorithms, e.g., EKM, and we collect the type-reduction results from the new interval type-2 fuzzy sets to form the type-reduction result of each interval type-2 fuzzy set of the $\alpha$-plane representation. The union of the type-reduction results of all the $\alpha$ planes is the centroid of the original type-2 fuzzy set. In this way, type-reduction of both convex and concave type-2 fuzzy sets can be done properly.

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