Chapter 5
Angular Velocity and Acceleration

Abstract In this chapter, a complete characterization of the angular velocity and angular acceleration for rigid bodies in spatial multibody systems are presented. For both cases, local and global formulations are described taking into account the advantages of using Euler parameters. In this process, the transformation between global and local components of the angular velocity and time derivative of the Euler parameters are analyzed and discussed in this chapter.

Keywords Angular velocity · Angular acceleration · Spatial motion

In order to keep the present analysis simple, let concentrate on the rotation of a body and neglect its translational motion. For this desideratum, let consider that the $\xi\eta\zeta$ coordinate system is rotating and has its origin coincident with the origin of the nonrotating xyz coordinate system, as shown in Fig. 5.1. The angular velocity $\vec{\omega}$ describes the axis and the magnitude of the rotation of the $\xi\eta\zeta$ frame. This axis is called the instantaneous axis of rotation and should not be mistaken with the orientational axis of rotation. Thus, at this instant, if the rotation of the body is frozen, the axis around which the body must rotate in order for the two coordinate systems become parallel is the orientational axis of rotation (Shabana 1989; Schiehlen 1990).

The angular velocity vector can be projected onto either the $\xi\eta\zeta$ frame or xyz frame resulting into algebraic vectors expressed as

$$\boldsymbol{\omega}' = \{\, \omega_\xi \quad \omega_\eta \quad \omega_\zeta \,\}^T, \quad \boldsymbol{\omega} = \{\, \omega_x \quad \omega_y \quad \omega_z \,\}^T \tag{5.1}$$

Nikravesh (1988) demonstrated that the angular velocity and the time derivative of the transformation matrix $\mathbf{A}$ have the following relations

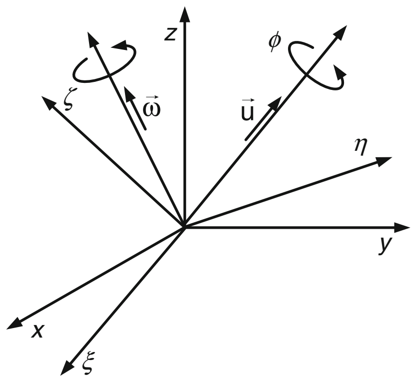
$$\dot{\mathbf{A}} = \mathbf{A}\tilde{\boldsymbol{\omega}}', \quad \dot{\mathbf{A}} = \tilde{\boldsymbol{\omega}}\mathbf{A} \tag{5.2}$$

or alternatively,

$$\mathbf{A}^T\dot{\mathbf{A}} = \tilde{\boldsymbol{\omega}}', \quad \dot{\mathbf{A}}\mathbf{A}^T = \tilde{\boldsymbol{\omega}} \tag{5.3}$$

P. Flores, *Concepts and Formulations for Spatial Multibody Dynamics*,
SpringerBriefs in Applied Sciences and Technology,
DOI 10.1007/978-3-319-16190-7_5

Fig. 5.1 Rotating $\xi\eta\zeta$ coordinates system



It should be noted that the angular velocity vector does not have an integral, i.e., there is no array of three rotational coordinates that its first time derivative is defined as the vector of angular velocity.

It is known that the global position of a point P that is fixed in the $\xi\eta\zeta$ coordinate system is given by the equation

$$\mathbf{s}^P = \mathbf{A}\mathbf{s}'^P \tag{5.4}$$

Differentiating this equation with respect to time yields

$$\dot{\mathbf{s}}^P = \dot{\mathbf{A}}\mathbf{s}'^P \tag{5.5}$$

Substituting Eq. (5.2) into Eq. (5.5) results in

$$\dot{\mathbf{s}}^P = \tilde{\boldsymbol{\omega}}\mathbf{A}\mathbf{s}'^P = \tilde{\boldsymbol{\omega}}\mathbf{s}^P \tag{5.6}$$

Thus, for any vector $\vec{s}$ attached to the $\xi\eta\zeta$ coordinate system, such as the one in Fig. 3.1, Eq. (5.6) can be written as (Nikravesh 1988)

$$\dot{\mathbf{s}} = \tilde{\boldsymbol{\omega}}\mathbf{s} = -\tilde{\mathbf{s}}\tilde{\boldsymbol{\omega}} \tag{5.7}$$

For a $\xi\eta\zeta$ frame that rotates and translates relative to the nonmoving xyz frame, the velocity of a point P_i that is fixed in the $\xi\eta\zeta$ frame can be determined. Thus, a point P_i can be located in the xyz frame by the relation

$$\mathbf{r}^P = \mathbf{r} + \mathbf{s}^P \tag{5.8}$$

The time derivative of this equation gives the velocity of point P as

$$\dot{\mathbf{r}}^P = \dot{\mathbf{r}} + \dot{\mathbf{s}}^P = \dot{\mathbf{r}} + \tilde{\boldsymbol{\omega}}\mathbf{s}^P \tag{5.9}$$

The transformation between the xyz components of the angular velocity vector and time derivative of Euler parameters is given by (Nikravesh 1988)

$$\boldsymbol{\omega} = 2\mathbf{G}\dot{\mathbf{p}} \tag{5.10}$$

In expanded form, Eq. (5.10) is

$$\begin{Bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{Bmatrix} = 2\begin{bmatrix} -e_1 & e_0 & -e_3 & e_2 \\ -e_2 & e_3 & e_0 & -e_1 \\ -e_3 & -e_2 & e_1 & e_0 \end{bmatrix}\begin{Bmatrix} \dot{e}_0 \\ \dot{e}_1 \\ \dot{e}_2 \\ \dot{e}_3 \end{Bmatrix} \tag{5.11}$$

The inverse transformation is found to be

$$\dot{\mathbf{p}} = \frac{1}{2}\mathbf{G}^T\boldsymbol{\omega} \tag{5.12}$$

In expanded form, Eq. (5.12) is

$$\begin{Bmatrix} \dot{e}_0 \\ \dot{e}_1 \\ \dot{e}_2 \\ \dot{e}_3 \end{Bmatrix} = \frac{1}{2}\begin{bmatrix} -e_1 & -e_2 & -e_3 \\ e_0 & e_3 & -e_2 \\ -e_3 & e_0 & e_1 \\ e_2 & -e_1 & e_0 \end{bmatrix}\begin{Bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{Bmatrix} \tag{5.13}$$

The transformation between the $\xi\eta\zeta$ components of the angular velocity vector and the time derivative of Euler parameters is given by (Nikravesh 1988)

$$\boldsymbol{\omega}' = 2\mathbf{L}\dot{\mathbf{p}} \tag{5.14}$$

In expanded form, Eq. (5.14) is

$$\begin{Bmatrix} \omega_\xi \\ \omega_\eta \\ \omega_\zeta \end{Bmatrix} = 2\begin{bmatrix} -e_1 & e_0 & e_3 & -e_2 \\ -e_2 & -e_3 & e_0 & e_1 \\ -e_3 & e_2 & -e_1 & e_0 \end{bmatrix}\begin{Bmatrix} \dot{e}_0 \\ \dot{e}_1 \\ \dot{e}_2 \\ \dot{e}_3 \end{Bmatrix} \tag{5.15}$$

The inverse transformation is found to be

$$\dot{\mathbf{p}} = \frac{1}{2}\mathbf{L}^T\boldsymbol{\omega}' \tag{5.16}$$

In expanded form, Eq. (5.16) is

$$\begin{Bmatrix} \dot{e}_0 \\ \dot{e}_1 \\ \dot{e}_2 \\ \dot{e}_3 \end{Bmatrix} = \frac{1}{2}\begin{bmatrix} -e_1 & -e_2 & -e_3 \\ e_0 & -e_3 & e_2 \\ e_3 & e_0 & -e_1 \\ -e_2 & e_1 & e_0 \end{bmatrix}\begin{Bmatrix} \omega_\xi \\ \omega_\eta \\ \omega_\zeta \end{Bmatrix} \tag{5.17}$$

Differentiating Eq. (5.10) with respect to time yields

$$\dot{\boldsymbol{\omega}} = 2\mathbf{G}\ddot{\mathbf{p}} + 2\dot{\mathbf{G}}\dot{\mathbf{p}} \tag{5.18}$$

Nikravesh (1988) showed that the product $\dot{\mathbf{G}}\dot{\mathbf{p}}$ is null, and hence, Eq. (5.17) can be simplified as

$$\dot{\boldsymbol{\omega}} = 2\mathbf{G}\ddot{\mathbf{p}} \tag{5.19}$$

In a similar manner, differentiating Eq. (5.14) with respect to time yields

$$\dot{\boldsymbol{\omega}}' = 2\mathbf{L}\ddot{\mathbf{p}} \tag{5.20}$$

Vectors $\dot{\boldsymbol{\omega}}$ and $\dot{\boldsymbol{\omega}}'$ are the global and local components of vector $\vec{\dot{\omega}}$ defined as the angular acceleration of the $\xi\eta\zeta$ frame. Finally, it can be shown that the inverses of Eqs. (5.19) and (5.20) are given by Nikravesh (1988)

$$\ddot{\mathbf{p}} = \frac{1}{2}\mathbf{G}^T\dot{\boldsymbol{\omega}} - \frac{1}{4}(\boldsymbol{\omega}^T\boldsymbol{\omega})\mathbf{p} \tag{5.21}$$

and

$$\ddot{\mathbf{p}} = \frac{1}{2}\mathbf{L}^T\dot{\boldsymbol{\omega}}' - \frac{1}{4}(\boldsymbol{\omega}'^T\boldsymbol{\omega}')\mathbf{p} \tag{5.22}$$

It is clear that $\boldsymbol{\omega}^T\boldsymbol{\omega} = \boldsymbol{\omega}'^T\boldsymbol{\omega}' = \omega^2$, where ω is the magnitude of $\vec{\omega}$.

References

Nikravesh PE (1988) Computer-aided analysis of mechanical systems. Prentice Hall, Englewood Cliffs

Schiehlen W (1990) Multibody systems handbook. Springer, Berlin

Shabana AA (1989) Dynamics of multibody systems. Wiley, New York