Chapter 12
Methods to Solve the Equations of Motion

Abstract This chapter presents several methods to solve the equations of motion of spatial multibody systems. In particular, the standard approach, the Baumgarte method, the penalty method and the augmented Lagrangian formulation are revised here. In this process, a general procedure for dynamic analysis of multibody systems based on the standard Lagrange multipliers method is described. Moreover, the implications in terms of the resolution of the equations of motion, accuracy and efficiency are also discussed in this chapter.

Keywords Dynamic analysis · Baumgarte method · Penalty method

As it was presented previously, the Newton-Euler equations of motion for a constrained multibody system of rigid bodies are written as

$$\mathbf{M}\dot{\mathbf{v}} - \mathbf{D}^T\boldsymbol{\lambda} = \mathbf{g} \tag{12.1}$$

In dynamic analysis, a unique solution is obtained when the algebraic constraint equations at the acceleration level are considered simultaneously with the differential equations of motion. Therefore, the second time derivative of the constraint equations are considered here and written as

$$\mathbf{D}\dot{\mathbf{v}} = \boldsymbol{\gamma} \tag{12.2}$$

Equation (12.2) can be appended to Eq. (12.1), yielding a system of differential algebraic equations (DAE). This system of equations is solved for accelerations vector, $\dot{\mathbf{v}}$, and Lagrange multipliers, $\boldsymbol{\lambda}$. Then, in each integration time step, the accelerations vector, $\dot{\mathbf{v}}$, together with velocities vector, $\mathbf{v}$, is integrated in order to obtain the system velocities and positions for the next time step. This procedure is repeated until the final analysis time is reached. A set of initial conditions, positions and velocities, is required to start the dynamic simulation. In the present work, the initial conditions are based on the results of kinematic simulation of the mechanical systems. The subsequent initial conditions for each time step in the simulation are obtained in the usual manner from the final conditions of the previous time step (Nikravesh 2007).

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Equations (12.1) and (12.2) can be rewritten in the matrix form as

$$\begin{bmatrix} \mathbf{M} & \mathbf{D}^T \\ \mathbf{D} & \mathbf{0} \end{bmatrix} \begin{Bmatrix} \dot{\mathbf{v}} \\ \boldsymbol{\lambda} \end{Bmatrix} = \begin{Bmatrix} \mathbf{g} \\ \boldsymbol{\gamma} \end{Bmatrix} \tag{12.3}$$

The linear system of Eq. (12.3) can be solved by applying any method suitable for the solution of linear algebraic equations. The existence of null elements in the main diagonal of the leading matrix and the possibility of ill-conditioned matrices suggest that methods using partial or full pivoting are preferred. However, none of these formulations help in the presence of redundant constraints. Alternatively, the equations of motion can be solved analytically. For this purpose, Eq. (12.1) is rearranged to put the accelerations vector in evidence, yielding

$$\dot{\mathbf{v}} = \mathbf{M}^{-1}(\mathbf{g} + \mathbf{D}^T\boldsymbol{\lambda}) \tag{12.4}$$

In this process, it is assumed that the multibody system under analysis does not include any body with null mass or inertia so that the inverse of the mass matrix **M** exists. Thus, introducing Eq. (12.4) into Eq. (12.2) and after basic mathematical manipulation results in

$$\boldsymbol{\lambda} = \left[\mathbf{D}\mathbf{M}^{-1}\mathbf{D}^T\right]^{-1}(\boldsymbol{\gamma} - \mathbf{D}\mathbf{M}^{-1}\mathbf{g}) \tag{12.5}$$

Substituting now Eq. (12.5) into Eq. (12.4) yields

$$\dot{\mathbf{v}} = \mathbf{M}^{-1}\mathbf{g} + \mathbf{M}^{-1}\mathbf{D}^T\left\{\left[\mathbf{D}\mathbf{M}^{-1}\mathbf{D}^T\right]^{-1}(\boldsymbol{\gamma} - \mathbf{D}\mathbf{M}^{-1}\mathbf{g})\right\} \tag{12.6}$$

Thus, Eq. (12.6) can be solved for $\dot{\mathbf{v}}$ then, the velocities and positions can be obtained by integration process in a similar manner as it was described above. This manner to solve the dynamic equations of motion is commonly referred to as the standard Lagrange multipliers method (Nikravesh 1988). Figure 12.1 presents a flowchart that shows the algorithm of the standard solution of the equations of motion. At $t = t^0$, the initial conditions on $\mathbf{q}^0$ and $\mathbf{v}^0$ are required to start the integration process. These values cannot be specified arbitrarily, but must satisfy the constraint equations defined by Eqs. (7.2) and (7.3). The algorithm presented in Fig. 12.1 can be summarized by the following steps:

1. Start at instant of time t^0 with given initial conditions for positions $\mathbf{q}^0$ and velocities $\mathbf{v}^0$.
2. Assemble the global mass matrix **M**, evaluate the Jacobian matrix **D**, construct the constraint equations $\boldsymbol{\Phi}$, determine the right-hand side of the accelerations $\boldsymbol{\gamma}$, and calculate the force vector **g**.
3. Solve the linear set of the equations of motion (12.3) for a constrained multibody system in order to obtain the accelerations $\dot{\mathbf{v}}$ at instant t and the Lagrange multipliers $\boldsymbol{\lambda}$.

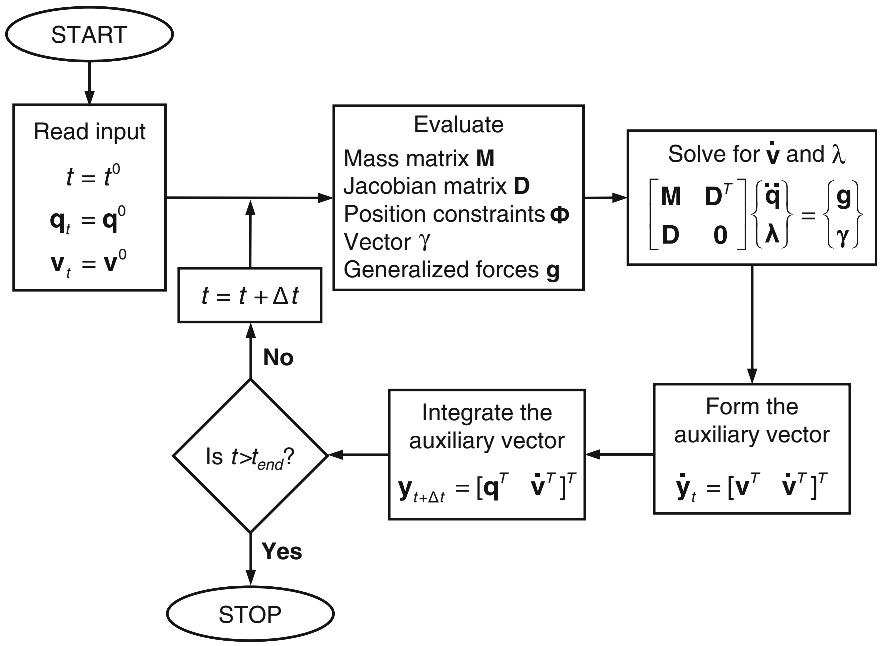

Fig. 12.1 Flowchart of computational procedure for dynamic analysis of multibody systems based on the standard Lagrange multipliers method

4. Assemble the vector $\dot{\mathbf{y}}_t$ containing the generalized velocities $\mathbf{v}$ and accelerations $\dot{\mathbf{v}}$ for instant of time t.
5. Integrate numerically the $\mathbf{v}$ and $\dot{\mathbf{v}}$ vectors for time step $t + \Delta t$ and obtain the new positions and velocities.
6. Update the time variable, go to step (2) and proceed with the process for a new time step, until the final time of analysis is reached.

The system of the motion Eq. (12.3) does not use explicitly the position and velocity equations associated with the kinematic constraints, that is, Eqs. (7.2) and (7.3). Consequently, for moderate or long simulations, the original constraint equations start to be violated due to the integration process and/or to inaccurate initial conditions. Therefore, methods able to eliminate errors in the position or velocity equations or, at least, to keep such errors under control, must be implemented. In order to keep the constraint violations under control, the Baumgarte stabilization method is considered here (Baumgarte 1972). This method allows constraints to be slightly violated before corrective actions can take place, in order to force the violation to vanish. The objective of Baumgarte method is to replace the differential Eq. (7.5) by the following equation

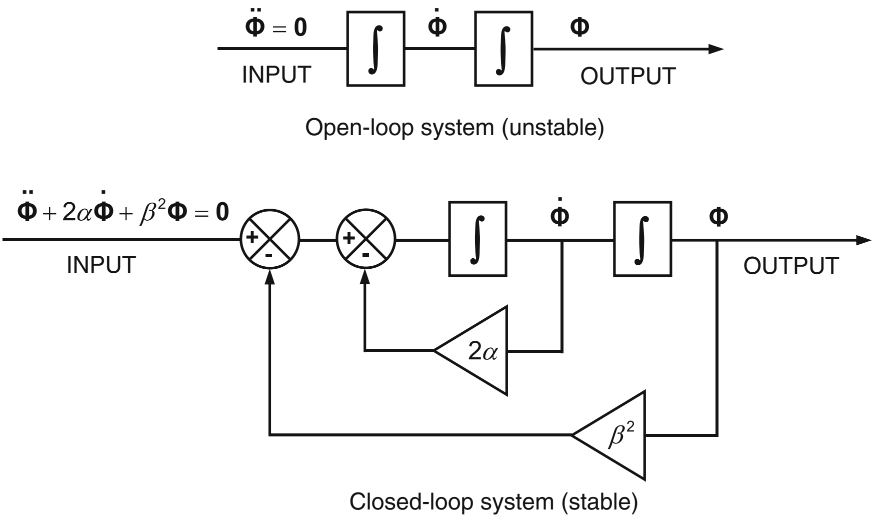

Fig. 12.2 Open loop and closed loop control systems

$$\ddot{\Phi} + 2\alpha\dot{\Phi} + \beta^2\mathbf{\Phi} = \mathbf{0} \tag{12.7}$$

Equation (12.7) is a differential equation for a closed-loop system in terms of kinematic constraint equations, in which the terms $2\alpha\dot{\Phi}$ and $\beta^2\mathbf{\Phi}$ play the role of control terms. The principle of the method is based on the damping of acceleration of constraint violation by feeding back the position and velocity of constraint violations, as illustrated in Fig. 12.2, which shows open-loop and closed-loop control systems. In the open-loop systems $\mathbf{\Phi}$ and $\dot{\Phi}$ do not converge to zero if any perturbation occurs and, therefore, the system is unstable. Thus, using the Baumgarte approach, the equations of motion for a system subjected to constraints are stated in the following form

$$\begin{bmatrix} \mathbf{M} & \mathbf{D}^T \\ \mathbf{D} & \mathbf{0} \end{bmatrix} \begin{Bmatrix} \dot{\mathbf{v}} \\ \lambda \end{Bmatrix} = \begin{Bmatrix} \mathbf{g} \\ \gamma - 2\alpha\dot{\Phi} - \beta^2\mathbf{\Phi} \end{Bmatrix} \tag{12.8}$$

If α and β are chosen as positive constants, the stability of the general solution of Eq. (12.8) is guaranteed. Baumgarte (1972) highlighted that the suitable choice of the parameters α and β is performed by numerical experiments. Hence, the Baumgarte method has some ambiguity in determining optimal feedback gains. Indeed, it seems that the value of the parameters is purely empiric, and there is no reliable method for selecting the coefficients α and β. The improper choice of these coefficients can lead to unacceptable results in the dynamic analysis of the multibody systems (Nikravesh 1984; Flores et al. 2011).

The penalty method presented by Jalón and Bayo (1994) constitutes an alternative way to solve the equations of motion. In this method, the equations of motion are modeled as a linear second-order differential equation that can be written in the form

$$m_c \ddot{\Phi} + d_c \dot{\Phi} + k_c \Phi = \mathbf{0} \tag{12.9}$$

Introducing Eq. (7.5) into Eq. (12.9) yields

$$m_c (\mathbf{D}\dot{\mathbf{v}} + \dot{\mathbf{D}}\mathbf{v}) + d_c \dot{\Phi} + k_c \mathbf{\Phi} = \mathbf{0} \tag{12.10}$$

Pre-multiplying Eq. (12.10) by the transpose of Jacobian matrix, $\mathbf{D}^T$, and after mathematical treatment, results in

$$m_c \mathbf{D}^T \mathbf{D}\dot{\mathbf{v}} = -\mathbf{D}^T (m_c \dot{\mathbf{D}}\mathbf{v} + d_c \dot{\Phi} + k_c \mathbf{\Phi}) \tag{12.11}$$

Let consider now the Newton-Euler equations of motion for a system of unconstrained system and written here as

$$\mathbf{M}\dot{\mathbf{v}} = \mathbf{g} \tag{12.12}$$

Adding Eqs. (12.12) and (12.11) yields

$$\mathbf{M}\dot{\mathbf{v}} + m_c \mathbf{D}^T \mathbf{D}\dot{\mathbf{v}} = \mathbf{g} - \mathbf{D}^T (-m_c \boldsymbol{\gamma} + d_c \dot{\Phi} + k_c \mathbf{\Phi}) \tag{12.13}$$

in which Eq. (7.6) has been employed. Finally, Eq. (12.13) can be written in the following form

$$(\mathbf{M} + \alpha \mathbf{D}^T \mathbf{D})\dot{\mathbf{v}} = \mathbf{g} - \alpha \mathbf{D}^T (-\boldsymbol{\gamma} + 2\mu\omega \dot{\Phi} + \omega^2 \mathbf{\Phi}) \tag{12.14}$$

where

$$\alpha = m_c, \quad \frac{d_c}{m_c} = 2\mu\omega \quad \text{and} \quad \frac{k_c}{m_c} = \omega^2 \tag{12.15}$$

Equation (12.14) can be solved for $\dot{\mathbf{v}}$. This method gives good results if α tends to infinity. Typical values of α, ω and μ are 1×10^7, 10 and 1, respectively (Jalón and Bayo 1994). It should be noted that with this penalty method, multibody systems with redundant constraints or kinematic singular configurations can be easily solved.

The augmented Lagrangian formulation is a methodology that penalizes the constraint violations, much in the same form as the Baumgarte stabilization method (Baumgarte 1972). This is an iterative procedure that presents a number of advantages relative to other methods because it involves the solution of a smaller set of equations, handles redundant constraints and still delivers accurate results in the vicinity of singular configurations. The augmented Lagrangian formulation consists of solving the system equations of motion by an iterative process. Let index i denote the i-th iteration. The evaluation of the system accelerations in a given time step starts as (Jalón and Bayo 1994)

$$\mathbf{M}\dot{\mathbf{v}}_i = \mathbf{g}, \quad (i = 0) \tag{12.16}$$

The iterative process to evaluate the system accelerations proceeds with the evaluation of

$$(\mathbf{M} + \alpha\mathbf{D}^T\mathbf{D})\dot{\mathbf{v}}_{i+1} = \mathbf{M}\dot{\mathbf{v}}_i - \alpha\mathbf{D}^T(-\gamma + 2\mu\omega\dot{\mathbf{\Phi}} + \omega^2\mathbf{\Phi}) \tag{12.17}$$

The iterative process continues until

$$\|\dot{\mathbf{v}}_{i+1} - \dot{\mathbf{v}}_i\| = \varepsilon \tag{12.18}$$

where ε is a specified tolerance. The augmented Lagrangian formulation involves the solution of a system of equations with a dimension equal to the number of coordinates of the multibody system. Though mass matrix $\mathbf{M}$ is generally positive semi-definite the leading matrix of Eq. (12.17) $\mathbf{M} + \alpha\mathbf{D}^T\mathbf{D}$ is always positive definite (Jalón and Bayo 1994). Even when the system is close to a singular position or when in presence of redundant constraints the system of equations can still be solved.

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