# Chapter 10
# Equations of Motion for Constrained Systems

**Abstract** In this chapter, the formulation of motion's equations of multi-rigid body systems is described. The generalized coordinates are the centroidal Cartesian coordinates, being the system configuration restrained by constraint equations. The present formulation uses the Newton-Euler's equations of motion, which are augmented with the constraint equations that lead to a system of differential algebraic equations. This formulation is straightforward in terms of assembling the equations of motion and providing all reaction forces.

**Keywords** Equations of motion · Newton-Euler formulation · Spatial systems

The translational equations of motion for an unconstrained rigid body can be expressed as (Shabana 1989; Schiehlen 1990)

$$m\ddot{\mathbf{r}} = \mathbf{f} \tag{10.1}$$

in which $m$ represents the mass of the body, $\ddot{\mathbf{r}}$ denotes the acceleration of the center of mass and $\mathbf{f}$ represents the sum of all forces acting on the body (Jalón and Bayo 1994; Nikravesh 2008).

Nikravesh (1988) demonstrated that the rotational equations of motion for a rigid body can be written in the form

$$J\dot{\boldsymbol{\omega}} + \tilde{\boldsymbol{\omega}}J\omega = \mathbf{n} \tag{10.2}$$

where $\mathbf{J}$ is the global inertia tensor, $\dot{\boldsymbol{\omega}}$ denotes the global angular accelerations, $\boldsymbol{\omega}$ is global angular velocities and $\mathbf{n}$ represents the sum of all moments acting on the body. Thus, the translational and rotational equations of motion, also known as the Newton-Euler equations of motion, for an unconstrained rigid body can be obtained by combining Eqs. (10.1) and (10.2), which in the matrix form are written as

$$\begin{bmatrix} m\mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{J} \end{bmatrix} \begin{Bmatrix} \ddot{\mathbf{r}} \\ \dot{\boldsymbol{\omega}} \end{Bmatrix} + \begin{Bmatrix} \mathbf{0} \\ \tilde{\boldsymbol{\omega}}J\omega \end{Bmatrix} = \begin{Bmatrix} \mathbf{f} \\ \mathbf{n} \end{Bmatrix} \tag{10.3}$$

or, alternatively,



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$$\begin{bmatrix} m\mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{J} \end{bmatrix} \begin{Bmatrix} \ddot{\mathbf{r}} \\ \dot{\boldsymbol{\omega}} \end{Bmatrix} = \begin{Bmatrix} \mathbf{f} \\ \mathbf{n} - \tilde{\boldsymbol{\omega}} J \omega \end{Bmatrix} \tag{10.4}$$

The equations of motion can also be derived and expressed in terms of local components, namely the rotational equations of motion. However, the form how the equations of motion are presented here is consistent with the kinematic constraints offered in the previous sections. Thus, in a compact form, Eq. (10.4) can be expressed as

$$\mathbf{M}_i \dot{\mathbf{v}}_i = \mathbf{g}_i \tag{10.5}$$

where

$$\mathbf{M}_i = \begin{bmatrix} m_i\mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{J}_i \end{bmatrix}, \quad \dot{\mathbf{v}}_i = \begin{Bmatrix} \ddot{\mathbf{r}}_i \\ \dot{\boldsymbol{\omega}}_i \end{Bmatrix}, \quad \mathbf{g}_i = \begin{Bmatrix} \mathbf{f}_i \\ \mathbf{n}_i - \tilde{\boldsymbol{\omega}}_i \mathbf{J}_i \boldsymbol{\omega}_i \end{Bmatrix} \tag{10.6}$$

Hence, the Newton-Euler equations of motion of a multibody system composed by $n_b$ unconstrained bodies are written as

$$\mathbf{M}\dot{\mathbf{v}} = \mathbf{g} \tag{10.7}$$

in which

$$\mathbf{M} = \begin{bmatrix} \mathbf{M}_1 & & & \\ & \mathbf{M}_1 & & \\ & & \ddots & \\ & & & \mathbf{M}_{n_b} \end{bmatrix}, \quad \dot{\mathbf{v}} = \begin{Bmatrix} \dot{\mathbf{v}}_1 \\ \dot{\mathbf{v}}_2 \\ \vdots \\ \dot{\mathbf{v}}_{n_b} \end{Bmatrix}, \quad \mathbf{g} = \begin{Bmatrix} \mathbf{g}_1 \\ \mathbf{g}_2 \\ \vdots \\ \mathbf{g}_{n_b} \end{Bmatrix} \tag{10.8}$$

In turn, for a multibody system of constrained bodies, the Newton-Euler equations of motion are written as (Nikravesh 1988)

$$\mathbf{M}\dot{\mathbf{v}} = \mathbf{g} + \mathbf{g}^{(c)} \tag{10.9}$$

where $\mathbf{g}^{(c)}$ denotes the vector of reaction forces that can be expressed in terms of the Jacobian matrix and Lagrange multipliers as (Nikravesh 1988; Jalón and Bayo 1994)

$$\mathbf{g}^{(c)} = \mathbf{D}^T \boldsymbol{\lambda} \tag{10.10}$$

Finally, the dynamic equations of motion for a constrained multibody system can be written in its general form as

$$\mathbf{M}\dot{\mathbf{v}} - \mathbf{D}^T \boldsymbol{\lambda} = \mathbf{g} \tag{10.11}$$

Different methods of resolution of the equations of motion will be presented and discussed in the next sections of the present document.

It is known that physically, the Lagrange multipliers are related to the joint reaction forces. In what follows, the relation between the constrained reaction forces and the constraint equations is revisited. For this purpose, let first consider that $\mathbf{g}^{(c)}$ can be transformed to a coordinate system consistent with $\mathbf{q}$ and denoted as $\mathbf{g}^{(*)}$. Furthermore, it is assumed that there are $m$ independent constraint equations written as

$$\mathbf{\Phi} \equiv \mathbf{\Phi}(\mathbf{q}) = 0 \tag{10.12}$$

For frictionless kinematic joints, the work done by the constraint forces in a virtual displacement $\delta\mathbf{q}$ is zero, i.e.,

$$\mathbf{g}^{(*)T}\delta\mathbf{q} = 0 \tag{10.13}$$

Since the virtual displacement $\delta\mathbf{q}$ must be consistent with the constraint equations, then Eq. (10.12) yields

$$\mathbf{D}\delta\mathbf{q} = \mathbf{0} \tag{10.14}$$

The vector of $n$ coordinates $\mathbf{q}$ may be partitioned into a set of $m$ dependent coordinates $\mathbf{u}$, and a set of $n$-$m$ independent coordinates $\mathbf{v}$, as

$$\mathbf{q} \equiv \begin{bmatrix} \mathbf{u}^T & \mathbf{v}^T \end{bmatrix}^T \tag{10.15}$$

This yields a partitioned vector of virtual displacements and a partitioned Jacobian matrix as

$$\delta\mathbf{q} \equiv \begin{bmatrix} \delta\mathbf{u}^T & \delta\mathbf{v}^T \end{bmatrix}^T, \quad \mathbf{D} = \begin{bmatrix} \mathbf{D_u} & \mathbf{D_v} \end{bmatrix} \tag{10.16}$$

Hence, Eq. (10.13) can be rewritten as

$$\mathbf{g}_{\mathbf{u}}^{(*)T}\delta\mathbf{u} + \mathbf{g}_{\mathbf{v}}^{(*)T}\delta\mathbf{v} = 0 \tag{10.17}$$

or

$$\mathbf{g}_{\mathbf{u}}^{(*)T}\delta\mathbf{u} = -\mathbf{g}_{\mathbf{v}}^{(*)T}\delta\mathbf{v} \tag{10.18}$$

In a similar way, from Eq. (10.14) yields

$$\mathbf{D_u}\delta\mathbf{u} = -\mathbf{D_v}\delta\mathbf{v} \tag{10.19}$$

Appending now Eqs. (10.18) and (10.19) results in

$$\begin{bmatrix} \mathbf{g}_{\mathbf{u}}^{(*)T} \\ \mathbf{D}_{\mathbf{u}} \end{bmatrix} \delta\mathbf{u} = -\begin{bmatrix} \mathbf{g}_{\mathbf{v}}^{(*)T} \\ \mathbf{D}_{\mathbf{v}} \end{bmatrix} \delta\mathbf{v} \tag{10.20}$$

The matrix to the left in Eq. (10.20) is an $(m + 1) \times m$ matrix. Since $\mathbf{D}_{\mathbf{u}}$ is an $m \times m$ nonsingular matrix, the first row of the $(m + 1) \times m$ matrix can be expressed as a linear combination of the other rows of the matrix as

$$\mathbf{g}_{\mathbf{u}}^{(*)} = \mathbf{D}_{\mathbf{u}}^{T}\boldsymbol{\lambda} \tag{10.21}$$

where $\boldsymbol{\lambda}$ is an $m$-vector of multipliers known as Lagrange multipliers. Substituting now Eq. (10.21) into (10.18) yields

$$\boldsymbol{\lambda}^{T}\mathbf{D}_{\mathbf{u}}\delta\mathbf{u} = -\mathbf{g}_{\mathbf{v}}^{(*)T}\delta\mathbf{v} \tag{10.22}$$

or

$$-\boldsymbol{\lambda}^{T}\mathbf{D}_{\mathbf{v}}\delta\mathbf{v} = -\mathbf{g}_{\mathbf{v}}^{(*)T}\delta\mathbf{v} \tag{10.23}$$

in which Eq. (10.19) has been employed. Vector $\delta\mathbf{v}$ is an arbitrary independent vector. The consistency of the constraints for virtual displacements $\delta\mathbf{q}$ is guaranteed by solving Eq. (10.19) for $\delta\mathbf{u}$. Since Eq. (10.23) must hold for any arbitrary $\delta\mathbf{v}$, then

$$\boldsymbol{\lambda}^{T}\mathbf{D}_{\mathbf{v}} = \mathbf{g}_{\mathbf{v}}^{(*)T} \tag{10.24}$$

or

$$\mathbf{g}_{\mathbf{v}}^{(*)} = \mathbf{D}_{\mathbf{v}}^{T}\boldsymbol{\lambda} \tag{10.25}$$

Appending Eq. (10.21)–(10.25) yields

$$\mathbf{g}^{(*)} = \mathbf{D}^{T}\boldsymbol{\lambda} \tag{10.26}$$

which expresses the constraint reaction forces.

## References

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