

On Gradient Like Properties of Population Games, Learning Models and Self Reinforced Processes

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Abstract We consider ordinary differential equations on the unit simplex of \mathbb{R}^n that naturally occur in population games, models of learning and self reinforced random processes. Generalizing and relying on an idea introduced in Dupuis and Fisher (On the construction of Lyapunov functions for nonlinear Markov processes via relative entropy, 2011), we provide conditions ensuring that these dynamics are gradient like and satisfy a suitable “angle condition”. This is used to prove that omega limit sets and chain transitive sets (under certain smoothness assumptions) consist of equilibria; and that, in the real analytic case, every trajectory converges toward an equilibrium. In the reversible case, the dynamics are shown to be C^1 close to a gradient vector field. Properties of equilibria—with a special emphasis on potential games—and structural stability questions are also considered.

1 Introduction

Let S be a finite set, say $S = \{1, \dots, n\}$. A *rate matrix* over S is a $n \times n$ matrix L such that $L_{ij} \geq 0$ for $i \neq j$ and $\sum_j L_{ij} = 0$. We let $\mathbf{R}(S)$ denote the space of such matrices. For $x \in \mathbb{R}^n$ and $L \in \mathbf{R}(S)$ we let xL denote the vector defined by $(xL)_i = \sum_j x_j L_{ji}$.

Let

$$\Delta = \{x \in \mathbb{R}^n : x_i \geq 0, \sum_i x_i = 1\}$$

be the *unit simplex of probabilities* over S . In this paper we are interested in ordinary differential equations on Δ having the form

$$\frac{dx}{dt} = xL(x) := F(x) \tag{1}$$

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where $L : \Delta \mapsto \mathbf{R}(S)$ is a sufficiently smooth function. Such dynamics occur—through a natural averaging procedure— in models of games describing strategic interactions in a large population of players, as well as in certain models of learning and reinforcement. These models are usually derived from qualitative assumptions describing the “microscopic” behavior of anonymous agents, and it is usually believed or assumed that similar qualitative microscopic behaviors should lead to similar global dynamics. However there is no satisfactory general theory supporting this belief.

To be more precise, under the assumption that $L(x)$ is irreducible, there exists a unique “invariant probability” for $L(x)$, $\pi(x) \in \Delta$ characterized by

$$\pi(x)L(x) = 0. \quad (2)$$

Several models corresponding to different rate functions $x \mapsto L(x)$ have the same invariant probability function $x \mapsto \pi(x)$. For instance, to each population game (see Sect. 2.1) which average ODE is given by (1), there is a canonical way to define a learning process (see Sect. 2.2) which average ODE is given by

$$\frac{dx}{dt} = -x + \pi(x) := F_\pi(x), \quad (3)$$

but there is no evidence that the dynamics of (1) and (3) are related in general.

The purpose of this paper is to provide sufficient conditions on $\pi(x)$ ensuring that (1) has a gradient-like structure. This heavily relies on an idea introduced in [10] where it was shown that the relative entropy between x and $\pi(x)$ is a strict Lyapounov function for systems of Gibbs type. We extend this idea to other class of systems beyond systems of Gibbs type, including population games and reinforcement process with imitative dynamics, and investigate further dynamics properties.

To give the flavor of the results presented in this paper, let $\pi : \Delta \mapsto \dot{\Delta}$ be a smooth function mapping Δ into its relative interior. Let χ_π be the set of vector fields having the form given by (the right hand side of) (1), where for each x , $L(x)$ is irreducible and verifies (2). Note that χ_π is a convex set of vector fields on Δ and that $F_\pi \in \chi_\pi$.

Theorem A

For all $F \in \chi_\pi$.

- (i) Equilibria (respectively non degenerate equilibria) of F coincide with equilibria (respectively non degenerate equilibria) of F_π .
- (ii) In general, global dynamics of F and F_π are “unrelated.” We construct an example for which F_π is globally asymptotically stable (every trajectory

converge toward a linearly stable equilibrium) while every non equilibrium trajectory for F converge to a limit cycle.

- (iii) Assume that there exists a C^k , $k \geq 1$ strictly increasing function $s : \mathbb{R} \mapsto \mathbb{R}$ such that $x \in \dot{\Delta} \mapsto s((\frac{x_i}{\pi_i(x)})_{i \in S})$ is the gradient (or quasi gradient) of some function $V : \dot{\Delta} \mapsto \mathbb{R}$. Then
- (a) V is a strict Lyapounov function for F (F is gradient-like) and verifies an *angle condition*,
 - (b) Omega limit sets and chain-transitive sets of F are equilibria,
 - (c) In the real analytic case, every solution to (1) converge toward an equilibrium,
 - (d) In the reversible case, hyperbolic equilibria of F coincide with non degenerate critical points of V and, provided there are finitely many equilibria, F is C^1 close to a gradient vector field for a certain Riemannian metric,
 - (e) The set χ_π is not (in general) structurally stable.

Section 2 describes a few examples that motivate this work. Section 3 contains some preliminary results and the main assumptions. Section 4 is devoted to Theorem A, (ii); Sect. 5 to (iii), (a), (b), (c); Sects. 6 and 7 to (iii)(d) and Sect. 8 to (iii)(e). Other results and examples are also discussed in these sections. For instance, in Sect. 6.1, the local dynamics (dynamics in the neighborhood of equilibria) of mean field systems associated to potential games is precisely described in term of Nash equilibria.

2 Motivating Examples

Throughout this section we see S as set of *pure strategies*. A Markov matrix over S is a $n \times n$ matrix K such that $K_{ij} \geq 0$ and $\sum_j K_{ij} = 1$. We let $\mathbf{M}(S)$ denote the sets of such matrices and we assume given a Lipschitz map

$$K : \Delta \mapsto \mathbf{M}(S).$$

For further reference we may call such a map a *revision protocol*. This terminology is borrowed from [23].

2.1 Population Games

Good references on the subject include [22] and the survey paper [23] from which some of the examples here are borrowed.

Consider a population of N agents, each of whom chooses a strategy in S at discrete times $k = 1, 2, \dots$. Depending on the context, an agent can be a player, a set of players, a biological entity, a communication device, etc. The state of the system at time $k \in \mathbb{N}$ is the vector $X_k^N = (X_{k,1}^N, \dots, X_{k,n}^N) \in \Delta$ where $NX_{k,i}^N$ equals the number of agents having strategy i . The system evolves as follows. Assume that at time k the system is in state $X_k^N = x$. Then an agent is randomly chosen in the population. If the chosen agent is an i -strategist, he/she switches to strategy j with probability $K_{ij}(x)$. This makes $(X_k^N)_{k \geq 1}$ a discrete time Markov chain, which transition probabilities are

$$\mathbf{P}(X_{k+1}^N = x + \frac{1}{N}(e_j - e_i) | X_k^N = x) = x_i K_{ij}(x)$$

where (e_1, \dots, e_n) is the standard basis of \mathbb{R}^n . Let

$$L(x) = -Id + K(x). \quad (4)$$

By standard mean-field approximation (see [5, 15] for precise statements, and [22, 23] for discussions in the context of games), the process $\{(X_k^N) : kN \leq T\}$ can be approximated by the solution to (1) (with $L(x)$ given by (4)) with initial condition $x = X_0^N$, over the time interval $[0, T]$.

2.1.1 Revision Protocols

Assume, as it is often the case in economic or biological applications, that the population game is determined by a continuous *Payoff-function* $U : \Delta \mapsto \mathbb{R}^n$. The quantity $U_i(x)$ represents the payoff (utility, fitness) of an i -strategist when the population state is x .

An *attachment-function* is a continuous map $w : \Delta \mapsto \mathbb{R}_+^n$. The weight $w_{ij}(x)$ can be seen as an a priori *attachment* of an i -strategist for strategy j . It can also encompass certain constraints on the strategy sets. For instance $w_{ij}(x) = 0$ (respectively $w_{ij}(x) \ll 1$) if a move from i to j is forbidden (respectively costly). We call the attachment function *imitative* if

$$w_{ij}(x) = x_j \tilde{w}_{ij}(x) \quad (5)$$

Most, not to say all, revision protocols in population games fall into one of the two next categories:

(i) [Sampling]

$$K_{ij}(x) = \frac{w_{ij}(x)f(U_j(x))}{\sum_k w_{ik}(x)f(U_k(x))} \quad (6)$$

where f is a non negative increasing function.

(ii) [Comparison]

$$K_{ij}(x) = w_{ij}(x)g(U_i(x), U_j(x)) \text{ for } i \neq j \quad (7)$$

$$K_{ii}(x) = 1 - \sum_{j \neq i} K_{ij}(x)$$

where $g(u, v)$ is nonnegative, decreasing in u , increasing in v and such that $\sum_{j \neq i} K_{ij}(x) \leq 1$.

2.2 Processes with Reinforcement and Adaptive Learning

Suppose now there is only one single agent in the population. In the context of games, one can imagine that this agent consists of a finite set of players and that S is the cartesian product of the strategy sets of the players. let $X_k \in S$ denote the strategy of this agent at time k . Let $\mu_k \in \Delta$ denote the empirical occupation measure of (X_k) up to time k . That is

$$\mu_k = \frac{1}{k} \sum_{j=1}^k \delta_{X_j}$$

where $\delta : S \mapsto \Delta$ is defined by $\delta_i = e_i$. Suppose now that the agent revises her strategies as follows:

$$\mathbf{P}(X_{k+1} = j | X_0, \dots, X_{k-1}, X_k = i) = K_{ij}(\mu_k).$$

The process (X_k) is no longer a Markov process but a *process with reinforcement* (see [20] for a survey of the literature on the subject). Using tools from stochastic approximation theory, it can be shown (see [1]) that, under certain irreducibility assumptions, the long term behavior of (μ_k) can be precisely related (see [1–3] and the brief discussion preceding Corollary 3) to the long term behavior of the differential equation on Δ

$$\frac{dx}{dt} = -x + \pi(x) \quad (8)$$

where $\pi(x) \in \Delta$ is the invariant probability of $K(x)$. Note that (8) can be rewritten as (1) with $L_{ij}(x) = \delta_{ij} - \pi_j(x)$.

3 Hypotheses, Notation, and Preliminaries

Let L be a rate matrix, as defined in the introduction. Then L is the infinitesimal generator of a continuous time Markov chains on S . A probability $\pi \in \Delta$ is called *invariant* for L if it is invariant for the associated Markov chain, or equivalently

$$\pi L = 0.$$

A sufficient condition ensuring that $\pi \in \Delta$ is invariant is that L is *reversible* with respect to π , meaning that

$$\pi_i L_{ij} = \pi_j L_{ji}.$$

The matrix is said *irreducible* if for all $i, j \in \{1, \dots, n\}$ there exist some integer k and a sequence of indices $i = i_1, i_2, \dots, i_{k-1}, i_k = j$ such that $L_{i_l, i_{l+1}} > 0$ for $l = 1, \dots, k-1$.

An irreducible rate matrix admits a unique invariant probability which can be expressed as a rational function of the coefficients (L_{ij}) (see e.g. Chapter 6 of [11]).

The relative interior of Δ is the set

$$\dot{\Delta} = \{x \in \Delta : \forall i \in S, x_i > 0\}.$$

From now on we assume given a C^1 map¹ $L : \Delta \mapsto \mathbf{R}(S)$ satisfying the following assumption:

Hypothesis 1 (Standing Assumption) For all $x \in \dot{\Delta}$, $L(x)$ is irreducible.

We sometimes assume

Hypothesis 2 (Occasional Assumption) For all $x \in \Delta$, $L(x)$ is irreducible.

In view of the preceding discussion Hypotheses 1 and 2 imply the following

Lemma 1 *There exists a C^1 map $\pi : \dot{\Delta} \mapsto \dot{\Delta}$ such that for all $x \in \dot{\Delta}$, $y = \pi(x)$ is the unique solution to the equation*

$$yL(x) = 0, \quad y \in \Delta.$$

If L is C^k, C^∞ or real analytic, the same is true for π . Under Hypothesis 2, π is defined on all Δ and maps Δ into $\dot{\Delta}$.

We let F_π denote the map defined as

$$F_\pi(x) = -x + \pi(x). \tag{9}$$

¹By this we mean that L is the restriction to Δ of a C^1 map defined in a neighborhood of Δ in $\text{aff}(\Delta) = \{x \in \mathbb{R}^n : \sum_i x_i = 1\}$.

Throughout, it is implicitly assumed that the domain of F_π is $\dot{\Delta}$ under Hypothesis 1 and Δ under Hypothesis 2.

We now consider the dynamics induced by (1). Without loss of generality, we may assume that (1) is defined on all \mathbb{R}^n and induces a flow $\Phi = (\Phi_t)$ leaving Δ positively invariant. Indeed, by convexity of Δ , the retraction $r : \mathbb{R}^n \mapsto \Delta$ defined by $r(x) = \mathbf{argmin}_{y \in \Delta} \|x - y\|$ is Lipschitz so that the differential equation

$$\frac{dy}{dt} = yL(r(y)) \tag{10}$$

is Lipschitz and sub-linear on all \mathbb{R}^n . By standard results, it then induces a flow $\Phi : \mathbb{R} \times \mathbb{R}^n \mapsto \mathbb{R}^n$ where $t \mapsto \Phi(t, y) = \Phi_t(y)$ is the unique solution to (10) with initial condition y . For all $x \in \Delta$ and $t \geq 0$, $\Phi_t(x) \in \Delta$ and the map $t \in \mathbb{R}^+ \mapsto \Phi_t(x)$ is solution to (1).

In the following we sometime use the notation $\Phi_t(x) = x(t) = (x_1(t), \dots, x_n(t))$. The *tangent space* of Δ is the space

$$T\Delta = \{u \in \mathbb{R}^n : \sum_{i=1}^n u_i = 0\}.$$

- Lemma 2** (i) *There exists $\alpha \geq 0$ such that for all $x \in \Delta$ $x_i(t) \geq e^{-\alpha t}x_i(0)$. In particular, $\dot{\Delta}$ is positively invariant.*
(ii) *If for all $x \in \partial\Delta$ $L(x)$ is irreducible, then $\Phi_t(\Delta) \subset \dot{\Delta}$ for all $t > 0$ and the dynamics (1) admits a global attractor*

$$A = \bigcap_{t \geq 0} \Phi_t(\Delta) \subset \dot{\Delta}.$$

Proof (i) Let $\alpha = \sup_{x \in \Delta} -L_{ii}(x)$. For all $j \neq i$ and $x \in \Delta$

$$\dot{x}_i \geq -\alpha x_i + x_j L_{ji}(x).$$

Hence

$$x_i(t) \geq e^{-\alpha t} [x_i(0) + \int_0^t e^{\alpha s} x_j(s) L_{ji}(x(s)) ds] \geq e^{-\alpha t} x_i(0).$$

The second inequality is the first statement. From the first inequality and the continuity of $L(x(t))$ it follows that $x_i(t) > 0$ for all $t > 0$ whenever that $x_j L_{ji}(x) > 0$. Let now $x \in \partial\Delta$. Assume without loss of generality that $x_1 > 0$. By irreducibility there exists a sequence $1 = i_1, i_2, \dots, i_k = j$ such that $L_{i_l, i_{l+1}}(x) > 0$. Hence, by continuity, $L_{i_l, i_{l+1}}(x(t)) > 0$ for all t small enough. It then follows that $x_j(t) > 0$ for all $t > 0$. ■

Remark 1 Assumption 1 is not needed in Lemma 2.

Throughout we let

$$\mathbf{Eq}(F) = \{x \in \Delta : F(x) = 0\}$$

denote the *equilibria* set of F . Note that in view of the preceding Lemmas

$$\mathbf{Eq}(F) \cap \dot{\Delta} = \{x \in \dot{\Delta} : F_\pi(x) = 0\}$$

and, in case $L(x)$ is irreducible for all $x \in \Delta$, $\mathbf{Eq}(F) \subset \dot{\Delta}$.

An equilibrium p is called *non degenerate* for F provided the Jacobian matrix $DF(p) : T\Delta \mapsto T\Delta$ is invertible.

Lemma 3 *Let $p \in \mathbf{Eq}(F) \cap \dot{\Delta}$. Then p is non degenerate for F if and only if it is non degenerate for F_π .*

Proof Let $L^T(x) : T\Delta \mapsto T\Delta$ be defined by $L^T(x)h = hL(x)$. Then for all $x \in \dot{\Delta}$ $F(x) = xL(x) = (x - \pi(x))L(x) = L^T(x)(x - \pi(x))$. Hence at every equilibrium $p \in \dot{\Delta}$ $DF(p) = -L^T(p)(DF_\pi(p))$. By irreducibility, $L^T(p)$ is invertible (see Lemma 8 in the Appendix). Thus $DF(p)$ is invertible if and only if $DF_\pi(p)$ is invertible. ■

4 Dynamics of F and F_π are Generally Unrelated

While F and F_π have the same equilibria, they may have quite different dynamics as shown by the following example.

Suppose $n = 3$ so that Δ is the unit simplex in \mathbb{R}^3 . Let G be a smooth vector field on Δ such that:

- (i) G points inward $\dot{\Delta}$ on $\partial\Delta$,
- (ii) Every forward trajectory of G converge to $p = (1/3, 1/3, 1/3)$,
- (iii) $JDG(p)J^{-1} = \begin{pmatrix} -\eta & -1 \\ 1 & -\eta \end{pmatrix}$, $\eta > 0$,

where $J : T\Delta \mapsto \mathbb{R}^2$ is defined by $J(u_1, u_2, u_3) = (u_1, u_2)$.

It is easy to construct such a vector field.

Choose $\varepsilon > 0$ small enough so that $\varepsilon G(x) + x$ lies in $\dot{\Delta}$ for all $x \in \Delta$ and set $\pi(x) = \varepsilon G(x) + x$. Then, F_π and G have the same orbits.

Let W be a 3×3 symmetric irreducible matrix with positive off-diagonal entries. Set $L_{ij}(x) = W_{ij}\pi_j(x)$ for $i \neq j$ and $L_{ii}(x) = -\sum_{j \neq i} L_{ij}(x)$. The matrix $L(x)$ is an irreducible rate matrix, reversible with respect to $\pi(x)$. It follows from Lemmas 2 and 3 that $F(x) = xL(x)$ has a global attractor contained in $\dot{\Delta}$ and a unique equilibrium given by p .

Furthermore,

$$DF(p) = -L(p)^T DF_\pi(p) = -\varepsilon L(p)^T DG(p) = -\frac{\varepsilon}{3} WDG(p)$$

where the last equality follows from the definition of L and the fact that $\pi(p) = p$. To shorten notation, set $b = \frac{\varepsilon}{3} W_{12}$, $c = \frac{\varepsilon}{3} W_{13}$ and $d = \frac{\varepsilon}{3} W_{23}$. Then

$${}_J DF(p) J^{-1} = \begin{pmatrix} (b+2c) & c-b \\ d-b & (b+2d) \end{pmatrix} \begin{pmatrix} -\eta-1 \\ 1-\eta \end{pmatrix}$$

The determinant of this matrix is positive and its trace equals

$$(c-d) - 2\eta(b+c+d).$$

If one now choose $c > d$ and η small enough, the trace is positive. This makes p linearly unstable. By Poincaré-Bendixson theorem, it follows that every forward trajectory distinct from p converges toward a periodic orbit.

Remark 2 It was pointed out to me by Sylvain Sorin and Josef Hofbauer that this example is reminiscent of the following phenomenon. Consider a population game which revision protocol takes the form

$$K_{ij}(x) = \frac{x_j}{R} \max(0, U_j(x) - U_i(x)) \text{ for } i \neq j$$

(here R is chosen so that $\sum_{j \neq i} K_{ij}(x) \leq 1$). This is a particular case of imitative pairwise comparison protocol (see Eq. (7)).

Then, the mean field ode is the classical replicator dynamics (see Example 3.2 in [23]):

$$\dot{x}_i = x_i(U_i(x) - \sum_{j \in S} x_j U_j(x)) \quad (11)$$

Here the rate matrix $L(x)$ is not irreducible and its set of invariant probabilities is easily seen to be the *Best Reply* set

$$BR(x) = \text{conv}(\{e_i : U_i(x) = \max_{j \in S} U_j(x)\}).$$

where $\text{conv}(A)$ stands for the convex hull of A . The vector field (9) is not defined but can be replaced by the differential inclusion

$$\dot{x} \in -x + BR(x). \quad (12)$$

If one assume that $U_i(x) = \sum_j U_{ij}x_j$ with U the payoff matrix given by a Rock-Paper-Scissors game,

$$U = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix};$$

Then $p = (1/3, 1/3, 1/3)$ is the unique equilibrium of (11) in $\dot{\Delta}$ (corresponding to the unique Nash equilibrium of the game) and every solution to (11) with initial condition in $\dot{\Delta} \setminus \{p\}$ is a periodic orbit. On the other hands, solutions to (12) converge to p . Phase portraits of these dynamics can be found in ([23], Section 5) and a detailed comparison of the replicator and the best reply dynamics is provided in [14].

5 Gradient Like Structure

For $u, v \in \mathbb{R}^n$ we let $\langle u, v \rangle = \sum_i u_i v_i$.

A map $h : \dot{\Delta} \mapsto \mathbb{R}^n$, is called a *gradient* if there exists a C^1 map $V : \dot{\Delta} \mapsto \mathbb{R}$ such that for all $x \in \dot{\Delta}$ and $u \in T\Delta$

$$\langle h(x), u \rangle = DV(x).u := \langle \nabla V(x), u \rangle.$$

It is called a *quasigradient* or a α -*quasigradient* if $x \mapsto \alpha(x)h(x)$ is a gradient for some continuous map $\alpha : \dot{\Delta} \mapsto \mathbb{R}_+^*$. That is

$$\alpha(x)\langle h(x), u \rangle = \langle \nabla V(x), u \rangle \quad (13)$$

for all $x \in \dot{\Delta}$ and $u \in T\Delta$.

Remark 3 If V is the restriction to $\dot{\Delta}$ of a C^1 map $W : \mathbb{R}^n \mapsto \mathbb{R}$, then $\nabla V(x)$ is the orthogonal projection of $\nabla W(x)$ onto $T\Delta$. That is

$$\nabla V_i(x) = \frac{\partial W}{\partial x_i}(x) - \frac{1}{n} \sum_{j=1}^n \frac{\partial W}{\partial x_j}(x), i = 1, \dots, n.$$

Remark 4 A practical condition ensuring that h is a gradient is that

- (a) h is the restriction to $\dot{\Delta}$ of a C^1 map $h : \mathbb{R}^n \mapsto \mathbb{R}^n$,
- (b) For all $x \in \dot{\Delta}$ and $i, j, k \in \{1, \dots, n\}$

$$\frac{\partial h_i}{\partial x_j}(x) + \frac{\partial h_j}{\partial x_k}(x) + \frac{\partial h_k}{\partial x_i}(x) = \frac{\partial h_i}{\partial x_k}(x) + \frac{\partial h_k}{\partial x_j}(x) + \frac{\partial h_j}{\partial x_i}(x).$$

This follows from ([13], Theorem 19.5.5.)

Notation

We use the following convenient notation. If x, y are vectors in \mathbb{R}^n and $s : \mathbb{R} \mapsto \mathbb{R}$ we let $x.y \in \mathbb{R}^n$ (respectively $\frac{x}{y}$ and $s(x)$) be the vector defined by $(xy)_i = x_i y_i$ (respectively $(\frac{x}{y})_i = \frac{x_i}{y_i}, s_i(x) = s(x_i)$).

5.1 Gradient Like Structure

A C^1 map $V : \dot{\Delta} \mapsto \mathbb{R}$ is called a *strict Lyapounov function* for F (or Φ) if for all $x \in \dot{\Delta}$

$$F(x) \neq 0 \Rightarrow \langle F(x), \nabla V(x) \rangle < 0.$$

Theorem 3 *Let $s :]0, \infty[\mapsto \mathbb{R}$ be a C^1 function with positive derivative and let $h^s : \dot{\Delta} \mapsto \mathbb{R}^n$ be the map defined by*

$$h^s(x) = s\left(\frac{x}{\pi(x)}\right).$$

Assume that h^s is a α -quasigradient. Then

- (i) *The map V (given by (13)) is a strict Lyapounov function for F on $\dot{\Delta}$;*
- (ii) *The critical points of V coincide with $\mathbf{Eq}(F) \cap \dot{\Delta}$;*
- (iii) *V satisfies the following angle condition: For every compact set $K \subset \dot{\Delta}$ there exists $c > 0$ such that*

$$| \langle \nabla V(x), F(x) \rangle | \geq c \|\nabla V(x)\| \|F(x)\|$$

for all $x \in K$.

Remark 5 (Gibbs Systems) *If $\pi(x)$ is a Gibbs measure,*

$$\pi_{\beta,i}(x) = \frac{e^{-(U_i^0 + \beta \sum_j U_{ij} x_j)}}{Z(x)} \tag{14}$$

where $U = (U_{ij})$ is a symmetric matrix, $\beta \geq 0$, and

$$Z(x) = \sum_j e^{-(U_j^0 + \beta \sum_k U_{jk} x_k)},$$

parts (i) and (ii) of Theorem 3 have been proved in [10], Theorems 5.3 and 5.5. Here $s(t) = \log(t)$ and

$$V(x) = \sum_i x_i \log(x_i) + \sum_j U_j^0 x_j + \frac{\beta}{2} \sum_{ij} U_{ij} x_i x_j. \quad (15)$$

Proof of Theorem 3

Part (i) relies on the following Lemma.

Lemma 4 *Let L be an irreducible transition matrix with invariant probability π . Let $x \in \Delta$, $f_i = \frac{x_i}{\pi_i}$, $s(f)_i = s(f_i)$ and $c_f = \inf_i s'(f_i) > 0$. Then there exists $\lambda(L) > 0$ depending continuously on L such that*

$$\langle xL, s(f) \rangle \leq -c_f \lambda(L) \text{Var}_\pi(f)$$

$$\text{where } \text{Var}_\pi(f) = \sum_i (f_i - 1)^2 \pi_i = \sum_i \frac{(x_i - \pi_i)^2}{\pi_i}.$$

The proof of this lemma uses elementary convexity arguments and classical tools from Markov chain theory. It is proved in appendix. Applying this lemma with $L = L(x)$ and $\pi = \pi(x)$ gives

$$\langle F(x), \nabla V(x) \rangle < 0$$

unless $x = \pi(x)$.

- (ii) The set $\mathbf{Eq}(F) \cap \dot{\Delta}$ coincides with fixed points of π in $\dot{\Delta}$. Let $x \in \dot{\Delta}$. $\nabla V(x) = 0 \Leftrightarrow h^s(x) \in \mathbb{R}1$ where 1 is the vector which components are all equal to 1. The function s being injective this is equivalent to $\frac{x_i}{\pi_i(x)} = \frac{x_j}{\pi_j(x)}$ for all i, j . That is $x = \pi(x)$.
- (iii) Let $K \subset \dot{\Delta}$. By Lemma 4 (applied with $L = L(x)$ and $\pi = \pi(x)$) and continuity of the maps involved, there exists $c > 0$ depending on K such that

$$|\langle \nabla V(x), F(x) \rangle| \geq c \sum_i (x_i - \pi_i(x))^2 = c \|x - \pi(x)\|^2.$$

To prove the angle condition it then suffices to show that both $\|F(x)\|$ and $\|\nabla V(x)\|$ are bounded by some constant times $\|x - \pi(x)\|$. Now, $F(x) = xL(x) - \pi(x)L(x)$ so that

$$\|F(x)\| \leq c_1 \|x - \pi(x)\|$$

with $c_1 = \sup_{x \in \Delta} \|L(x)\|$.

By Lipschitz continuity of s and compactness, there exist $c_2, c_3 > 0$ depending on K such that

$$\left| s\left(\frac{x_i}{\pi_i(x)}\right) - s(1) \right| \leq c_2 \left| \frac{x_i}{\pi_i(x)} - 1 \right| \leq c_3 |x_i - \pi_i(x)|.$$

Thus, for all $u \in T\Delta$ such that $\|u\| = 1$

$$\langle h^s(x), u \rangle = \langle h^s(x) - s(1)\mathbf{1}, u \rangle \leq \|h^s(x) - s(1)\mathbf{1}\| \leq c_3 \|x - \pi(x)\|.$$

This implies that $\|\nabla V(x)\| \leq c_3 \|x - \pi(x)\|$ and concludes the proof. ■

The following result proves to be useful for certain dynamics leaving invariant the boundary of the simplex. Such dynamics occur in population games using imitative protocols (see Eq. (5)) as well as in certain models of vertex reinforcement (see Example 3 below).

For $x \in \Delta$ let $\text{Supp}(x) = \{x \in \Delta : x_i > 0\}$.

Proposition 1 *Assume that assumptions of Theorem 3 hold. Assume furthermore that*

(a) *For all $x \in \Delta$*

$$x_i = 0 \Rightarrow L_{ji}(x) = 0$$

and the reduced rate matrix $[L_{ij}(x)]_{i,j \in \text{Supp}(x)}$ is irreducible

(b) *The maps $V : \dot{\Delta} \mapsto \mathbb{R}^n$ and $\alpha : \dot{\Delta} \mapsto \mathbb{R}_+^*$ (given by Eq. (13)) extend to C^1 (respectively continuous) maps $V : \Delta \mapsto \mathbb{R}^n$ and $\alpha : \Delta \mapsto \mathbb{R}_+^*$.*

Then V is strict Lyapounov function for F on Δ .

Proof Let $T\Delta(x) = \{u \in T\Delta : u_i = 0 \text{ for } i \notin \text{Supp}(x)\}$. By assumption (a) the map $x \mapsto \pi(x)$ is defined for all $x \in \Delta$ continuous in x and $\pi_i(x) = 0 \Leftrightarrow x_i = 0$.

Therefore, using assumption (b), the equation

$$\forall x \in \dot{\Delta}, \forall u \in T\Delta \quad \alpha(x) \langle h^s(x), u \rangle = \langle \nabla V(x), u \rangle$$

extends to

$$\forall x \in \Delta, \forall u \in T\Delta(x) \quad \sum_{i \in \text{Supp}(x)} s\left(\frac{x_i}{\pi_i(x)}\right) u_i = \langle \nabla V(x), u \rangle$$

Thus

$$\sum_{i \in \text{Supp}(x)} s\left(\frac{x_i}{\pi_i(x)}\right) (xL(x))_i = \langle \nabla V(x), F(x) \rangle$$

for all $x \in \Delta$. By Lemma 4 the left hand side is nonpositive and zero if and only if $x_i = \pi_i(x)$ for all $i \in \text{Supp}(x)$. ■

Remark 6 Note that under the assumptions of Proposition 1, the angle inequality of Theorem 3 doesn't hold on the boundary of the simplex

Example 1 Let $W : \mathbb{R}^n \mapsto \mathbb{R}$ be a C^k map, $k \geq 1$. Suppose that for all $x \in \Delta$

$$\pi_i(x) = \frac{f_i(x_i) \psi\left(\frac{\partial W}{\partial x_i}(x)\right)}{\sum_{j=1}^n f_j(x_j) \psi\left(\frac{\partial W}{\partial x_j}(x)\right)}$$

Then, Theorem 3 applies in the following cases:

Case 1 $\psi(u) = e^{-\beta u}$ with $\beta \geq 0$, and $f_i(t) > 0$ for all $t > 0$. It suffices to choose $s(t) = \log(t)$ and

$$V(x) = \sum_{i=1}^n x_i \log(x_i) - \sum_{i=1}^n \int_1^{x_i} \log(f_i(u)) du + \beta W(x) \quad (16)$$

Then h^s is the gradient of V .

Case 2 $\psi(u) = u^\beta$, $\beta > 0$, $f_i(t) = t$ and $\frac{\partial W}{\partial x_i} > 0$ on $\{x \in \Delta : x_i > 0\}$. It suffices to choose $s(t) = -t^{-1/\beta}$ and

$$V(x) = -W(x).$$

Then h^s is the α -quasigradient of V with

$$\alpha(x) = \left[\sum_j x_j \left(\frac{\partial W}{\partial x_j} \right)^\beta \right]^{-1/\beta}.$$

Example 2 (Potential Games) Examples of applications of Example 1, case 1, are given by *Potential Games* (see [22] for an comprehensive presentation and motivating examples). We use the notation of Sect. 2. A *Potential Game* is a game for which the payoff function is such that for all $x \in \Delta$

$$U_i(x) = -\frac{\partial W}{\partial x_i}(x), i = 1 \dots n$$

Consider a population game with a revision protocol given by (7). Suppose that the attachment matrix takes the form

$$w_{ij}(x) = f_j(x_j) \tilde{w}_{ij}(x)$$

with \tilde{w} irreducible and symmetric. Let $\beta \geq 0$. Assume furthermore that $g(u, v)$ takes one of the following form:

Pairwise comparison

$$g(u, v) = \frac{e^{\beta(v-u)}}{1 + e^{\beta(v-u)}} \text{ or } g(u, v) = \min(1, e^{\beta(v-u)}),$$

Imitation driven by dissatisfaction

$$g(u, v) = e^{-\beta u},$$

Imitation of success

$$g(u, v) = e^{\beta v}.$$

In all these situations, $K(x)$, hence $L(x)$ is reversible with respect to $\pi_\beta(x)$ with

$$\pi_{\beta,i}(x) = \frac{f_i(x_i) e^{-\beta \frac{\partial W}{\partial x_i}(x)}}{\sum_j f_j(x_j) e^{-\beta \frac{\partial W}{\partial x_j}(x)}}.$$

Theorem 3 applies with V given by (16).

Remark 7 (Gibbs Systems, 2) A particular case of potential games is obtained with $W(x) = \frac{1}{2} \sum_{ij} U_{ij} x_i x_j$ with $U = (U_{ij})$ symmetric, and $f_i(x) = e^{-U_i^0}$. Here payoffs are linear in x :

$$U_i(x) = - \sum_j U_{ij} x_j$$

and we retrieve the situation considered in [10]. See Remark 5.

Example 3 (Vertex Reinforcement) Let K be the revision protocol defined by

$$K_{ij}(x) = \frac{A_{ij} x_j^\gamma}{\sum_k A_{ik} x_k^\gamma}$$

where A is a matrix with positive entries and $\gamma \geq 1$. For population games (see Sect. 2.1) this gives a simple model of imitation: an agent of type i , when chosen, switches to j with a probability proportional to the (number of agents of type j) $^\gamma$. For processes with reinforcement (as defined in Sect. 2.2) the probability to jump from i to j at time n is proportional to (the time spent in j up to time n) $^\gamma$. This later model called a *vertex reinforced random walks* was introduced by Diaconis and first analyzed in Pemantle [19] (see also [1] and [7] for more references on the subject).

When A is symmetric, $K(x)$ is reversible with respect to

$$\pi_i(x) = \frac{x_i^\gamma \sum_k A_{ik} x_k^\gamma}{\sum_{ij} A_{ij} x_i^\gamma x_j^\gamma} = \frac{x_i \frac{\partial W}{\partial x_i}}{\sum_j x_j \frac{\partial W}{\partial x_j}} \quad (17)$$

with

$$W(x) = \sum_{i,j} A_{ij} x_i^\gamma x_j^\gamma \quad (18)$$

We are then in the situation covered by Example 1, case 2, with $\psi(u) = u, f_i(t) = t, s(t) = -\frac{1}{t}$ and $V = -W$.

Both Theorem 3 and Proposition 1 apply.

Example 4 (Interacting Urn Processes) Closely related to vertex reinforced random walks are models of *interacting urns* (see [6, 8, 24]). For these models $\pi_i(x) = x_i \frac{\partial W}{\partial x_i}$ for some smooth function W . This is a particular case of Example 1, case 2.

5.2 Limit Sets and Chain Transitive Sets

Using Lasalle's invariance principle we deduce the following consequences from Theorem 3.

Corollary 1 *Assume that assumptions of Theorem 3 hold. Then every omega limit set of Φ contained in $\dot{\Delta}$ is a connected subset of $\mathbf{Eq}(F) \cap \dot{\Delta}$.*

Combining this results with Lemma 2 (ii) and Proposition 1 gives

Corollary 2 *Assume that one of the following condition hold:*

- (a) *Assumptions of Theorem 3 and Hypothesis 2 or;*
- (b) *Assumptions of Proposition 1.*

Then every omega limit set of Φ is a connected subset of $\mathbf{Eq}(F)$.

A set L is called *attractor free* or *internally chain transitive* provided it is compact, invariant and $\Phi|_L$ has no proper attractor. For reinforced random processes like the ones defined in Sect. 2.2, limit sets of (μ_n) are, under suitable assumptions, attractor free sets of the associated mean field Eq. (8) (see [1]). More generally attractor free sets are limit sets of *asymptotic pseudo trajectories* (see [3]). It is then useful to characterize such sets. Note however that the existence of a strict Lyapounov function, doesn't ensure in general, that internally chain transitive sets consist of equilibria (see e.g. Remark 6.3 in [2]).

Corollary 3 *Assume that assumptions of Theorem 3 hold and that h^s is C^k for some $k \geq n - 2 = \dim(T\Delta) - 1$. Then every internally chain transitive set of Φ contained in $\dot{\Delta}$ is a connected subset of $\mathbf{Eq}(F) \cap \dot{\Delta}$. If we furthermore assume that $L(x)$ is irreducible for all $x \in \Delta$, then every internally chain transitive set of Φ is a connected subset of $\mathbf{Eq}(F)$*

Proof Let $C = \mathbf{Eq}(F) \cap \dot{\Delta}$ and $A \subset \dot{\Delta}$ an attractor free set. By Theorem 3, C coincide with critical points of V . By the assumption V is C^{k+1} so that by Sard's theorem (see [12]), $V(C)$ has empty interior. It follows (see e.g. Proposition 6.4 in [2]) that $A \subset C$. ■

5.3 Convergence Toward One Equilibrium

In case equilibria are isolated, Corollary 1 implies that every trajectory bounded away from the boundary converge to an equilibrium and that every trajectory converges in case $L(x)$ is irreducible for all x . However, when equilibria are degenerate, the gradient-like property is not sufficient to ensures convergence. There are known examples of smooth gradient systems which omega limit sets are a continuum of equilibria (see [18]). However, in the real analytic case, gradient like systems which verify an angle condition are known to converge.

Theorem 4 *Suppose that assumptions of Theorem 3 hold and that V is real analytic. Then every omega limit set meeting $\dot{\Delta}$ reduces to a single point.*

Proof Let p be an omega limit point. If V is real analytic, it satisfies a *Lojasiewicz inequality* at p in the sense that there exist $0 < \eta \leq 1/2$, $\beta > 0$ and a neighborhood $U(p)$ of p such that

$$|V(x) - V(p)|^{1-\eta} \leq \beta \|\nabla V(x)\|$$

for all x in a $U(p)$. Such an inequality called a “gradient inequality” was proved by Lojasiewicz [16] and used (by Lojasiewicz again) to show that bounded solutions of real analytic gradient vector fields have finite length, hence converge. When the dynamics is not a gradient, but only gradient like with V as a strict Lyapounov function, the same results holds provided that V satisfies an angle condition:

$$\langle \nabla V(x), F(x) \rangle \geq c \|F(x)\| \|\nabla V(x)\|$$

for all $x \in U(p)$. This is proved in [9] (see also [17], Theorem 7). ■

Example 5 (Gibbs Systems, 3) If π is given by (14) with U symmetric, V given by (15) is real analytic so that every solution to (1) converges toward an equilibrium.

6 Equilibria

Recall that point $p \in \mathbf{Eq}(F)$ is called *non degenerate* if the jacobian matrix $DF(p) : T\Delta \mapsto T\Delta$ is invertible. It is called *hyperbolic* if eigenvalues of $DF(p)$ have non zero real parts. If p is hyperbolic, $T\Delta$ admits a splitting

$$T\Delta = E_p^u \oplus E_p^s$$

invariant under $DF(p)$ such that the eigenvalues of $DF(p)|_{E_p^s}$ (respectively $DF(p)|_{E_p^u}$) have negative (respectively positive) real parts.

Point $p \in \mathbf{Crit}(V) = \{x \in \dot{\Delta} : \nabla(V)(x) = 0\}$ is called *non-degenerate* if $Hess(V)(p)$ the *Hessian* or V at p has full rank. In a suitable coordinate systems $Hess(V)(p)(u, u) = \sum_{i=1}^{n_+} u_i^2 - \sum_{j=1}^{n_-} u_j^2$ with $n_+ + n_- = \dim(T\Delta) = n - 1$. The number n_- is called the *index* of p (with respect to V) and is written $\mathbf{Ind}(p, V)$.

Proposition 2 *Assume that assumptions of Theorem 3 hold. Let $p \in \mathbf{Eq}(F) \cap \dot{\Delta}$. Then*

- (i) *Point p is non degenerate if and only if it is a non degenerate critical point of V .*
- (ii) *If furthermore L is C^2 and p is hyperbolic,*

$$\dim(E_p^u) = \mathbf{Ind}(p, V).$$

Proof From Lemma 3, p is non degenerate if and only if $DF_\pi(p)$ is invertible and (see the proof of Lemma 3)

$$DF(p) = -L^T(p)DF_\pi(p) \quad (19)$$

Now, a direction computation (details are left to the reader) of the Hessian of V at x leads to

$$\langle Hess(V)(x)u, v \rangle = \alpha(x) \langle s'(\frac{x}{\pi(x)})(u - \frac{x}{\pi(x)}D\pi(x)u), v \rangle_{1/\pi(x)}$$

where $\langle u, v \rangle_{1/\pi}$ stands for $\sum_i u_i v_i \frac{1}{\pi_i}$. Since $p = \pi(p)$

$$\langle Hess(V)(p)u, v \rangle = \alpha(p) s'(1) \langle (I - D\pi(p))u, v \rangle_{1/p} \quad (20)$$

for all $u, v \in T\Delta$. This proves that $HessV(p)$ is non degenerate if and only if $(I - D\pi(p)) = -DF_\pi(p)$ is non degenerate and concludes the proof of the first part.

We now prove the second part. By the stable manifold theorem, there exists a (local) C^2 manifold W_p^s tangent to E_p^s at p positively invariant under Φ and such that for all $x \in W_p^s$ $\lim_{t \rightarrow \infty} \Phi_t(x) = p$. Clearly p is a global minimum of V restricted to W^s . For otherwise there would exist $x \in W_p^s$ such that

$$V(p) > V(x) > \lim_{t \rightarrow \infty} V(\Phi_t(x)) = V(p).$$

Since p is also a critical point $\nabla V(p) = 0$. Let $u \in E_p^s$ and let $\gamma :]-1, 1[\mapsto W_p^s$ be a C^2 path with $\gamma(0) = p$, $\dot{\gamma}(0) = u$. Set $h(t) = V(\gamma(t))$. Then $\dot{h}(0) = 0$ (because p is a critical point of V) and $h''(0) = \langle HessV(p)u, u \rangle$ is non negative because $h(t) \geq h(0)$.

On the other hand, by the spectral decomposition of $HessV(p)$ we can write $T\Delta = E_V^s \oplus E_V^u$ with $\langle HessV(p)u, u \rangle > 0$ (respectively < 0) for all $u \in E_V^s \setminus \{0\}$ (respectively $E_V^u \setminus \{0\}$). Thus, $E_p^s \cap E_V^u = \{0\}$ and, consequently, $\dim(E_p^s) + \dim(E_V^u) \leq \dim(T\Delta)$. Similarly $\dim(E_p^u) + \dim(E_V^s) \leq \dim(T\Delta)$. This proves that $\dim(E_p^u) = \dim(E_V^u) = \mathbf{Ind}(p, V)$. ■

Remark 8 This later proposition shows that in the neighborhood of an hyperbolic equilibrium p , $\dot{x} = F(x)$ and $\dot{x} = -\nabla V(x)$ are topologically conjugate. Indeed, part (ii) of the proposition implies that the linear flows $\{e^{tDF(p)}\}$ and $\{e^{tHess(V)(p)}\}$ are topologically conjugate (see e.g. Theorem 7.1 in [21]), and by Hartman-Grobman Theorem (see again [21]), nonlinear flows are locally conjugate to their linear parts in the neighborhood of hyperbolic equilibria. However, note that while eigenvalues of $Hess(V)(p)$ are reals there is no evidence that the same is true for $DF(p)$ in general. The next proposition proves that this is the case when $L(x)$ is reversible with respect to $\pi(x)$.

Proposition 3 *Let $p \in \mathbf{Eq}(F) \cap \dot{\Delta}$. Assume that assumptions of Theorem 3 hold and that $L(p)$ is reversible with respect to $\pi(p) = p$. Then there exists a positive definite bilinear form $g_0(p)$ on $T\Delta$ such that for all $u, v \in T\Delta$*

$$g_0(p)(DF(p)u, v) = -\langle Hess(V)(p)u, v \rangle$$

In particular

- (i) $DF(p)$ has real eigenvalues,
- (ii) p is hyperbolic for F if and only if it is a non degenerate critical point of V .

Proof Let $p \in \mathbf{Eq}(F) \cap \dot{\Delta}$. Set $L = L(p)$ and recall that $L^T : T\Delta \mapsto T\Delta$ is defined by $L^T h = hL$. Then, by Lemma 8 in the Appendix, $-L^T$ is a definite positive operator for the scalar product on $T\Delta$ defined by $\langle u, v \rangle_{1/p} = \sum_i u_i v_i \frac{1}{p_i}$. Define now $g_0(p)$ by

$$g_0(p)(u, v) = -\langle (L^T)^{-1}u, v \rangle_{\perp}. \tag{21}$$

Using (19) and (20) it comes that for all $u, v \in T\Delta$

$$\begin{aligned} g_0(p)(DF(p)u, v) &= -\langle (I - D\pi(p))u, v \rangle_{1/p} \\ &= -[\alpha(p)s'(1)]^{-1} \langle Hess(V)(p)u, v \rangle. \end{aligned}$$

Replacing $g_0(p)$ by $\alpha(p)s'(1)g_0(p)$ proves the result. ■

A useful consequence of this later proposition is that it is usually much easier to verify non degeneracy of equilibria rather than hyperbolicity. Here is an illustration:

Example 6 (Gibbs Systems, 4) Consider the symmetric Gibbs model analyzed in [10] (see Remark 5 and Example 7). We suppose that the symmetric matrix $U = (U_{ij})$ is given and we treat $U^0 = (U_i^0)_{i \in S}$ and β as parameters. Let $\mathcal{E}_{rev}(U^0)$ denote the set of maps

$$\mathbb{R}^+ \times \Delta \mapsto T\Delta,$$

$$(\beta, x) \mapsto F_\beta(x) = xL_\beta(x)$$

such that L_β verifies assumption 2, is C^1 in x , and $L_\beta(x)$ is reversible with respect to $\pi_\beta(x)$ where

$$\pi_{\beta,i}(x) = \frac{e^{-U_i^0 - \beta \sum_j U_{ij}x_j}}{Z(x)}.$$

Proposition 4 *There exists an open and dense set $\mathcal{G}^0 \subset \mathbb{R}^n$ such that for all $U^0 \in \mathcal{G}^0$ and $F \in \mathcal{E}_{rev}(U^0)$*

- (i) *The set $\{(x, \beta) \in \Delta \times \mathbb{R}_+^+ : F_\beta(x) = 0\}$ is a C^∞ one dimensional submanifold,*
- (ii) *There exists an open dense set $\mathcal{B}^0 \subset \mathbb{R}^+$ containing 0 such that for all $\beta \in \mathcal{B}^0$ equilibria of F_β are hyperbolic.*

Proof Let $H : \dot{\Delta} \times \mathbb{R}^n \times \mathbb{R}^+ \mapsto T\Delta$ be defined by $H(x, U^0, \beta) = \nabla V_{U^0, \beta}(x)$ where $V_{U^0, \beta}$ is given by (15). Since $\frac{\partial H}{\partial U^0}(x, U^0, \beta)$ is the identity map, H is a submersion. Hence, by Thom's parametrized transversality Theorem (see [12], Chapter 3), there exists an open and dense set $\mathcal{G}^0 \in \mathbb{R}^n$ such that for all $U^0 \in \mathcal{G}^0$, $(x, \beta) \mapsto H(x, U^0, \beta)$ is a submersion. This proves (i). By the same theorem, for all $\beta \in \mathcal{B}^0$ with \mathcal{B}^0 open and dense in \mathbb{R}^+ , $x \mapsto H(x, U^0, \beta)$ is a submersion, meaning that critical points of $V_{U^0, \beta}$ are nondegenerate. By Proposition 3, equilibria of F_β are hyperbolic. ■

Remark 9 Other genericity results can be proved, if one fix U^0 or β and treat U as a parameter. Compare to the proof of Theorem 2.10 in [4] in an infinite dimensional setting.

6.1 Equilibria of Potential Games

Consider a population game with C^1 payoff function $U : \Delta \mapsto \mathbb{R}^n$. Recall that the game is called a potential game, provided $U_i(x) = -\frac{\partial W}{\partial x_i}(x)$ for all $x \in \Delta$ and some potential $W : \mathbb{R}^n \mapsto \mathbb{R}$.

Point $x^* \in \Delta$ is called a *Nash equilibrium* of U if, given the population state x^* , every agent has interest to play the mixed strategy x^* . That is

$$\forall i \in \{1, \dots, n\} U_i(x^*) \leq \langle U(x^*), x^* \rangle \quad (22)$$

Let

$$\text{Supp}(x^*) = \{i \in \{1, \dots, n\} : x_i^* > 0\}.$$

It follows from (22) that

$$\forall i \in \text{Supp}(x^*) U_i(x^*) = \langle U(x^*), x^* \rangle.$$

We let $\mathbf{NE}(U)$ denote the set of Nash equilibria of U . For all $\beta \geq 0$ and $x \in \Delta$ we let $\pi_\beta(x) \in \Delta$ be defined as

$$\pi_{\beta,i}(x) = \frac{e^{\beta U_i(x)}}{\sum_j e^{\beta U_j(x)}}, \quad i = 1, \dots, n \quad (23)$$

and we let $\chi(\beta, U)$ (respectively, $\chi_{rev}(\beta, U)$) denote the set of all vector fields having the form given by (1) where $L(x)$ is C^1 in x , irreducible and admits $\pi_\beta(x)$ as invariant (respectively reversible) probability. Recall (see Eq. (9)) that

$$F_{\pi_\beta} = -Id + \pi_\beta.$$

Our aim here is to describe $\mathbf{Eq}(F)$ for $F \in \chi(\beta, U)$ in term of $\mathbf{NE}(U)$ for large β , with a particular emphasis on potential games. Some of the results here are similar to the results obtained in Benaïm and Hirsch ($n \times 2$ coordinative games, 2000, unpublished manuscript) for $n \times 2$ pseudo games.

Proposition 5 *Let \mathcal{N} be a neighborhood of $\mathbf{NE}(U)$. There exists $\beta_0 \geq 0$ such that for all $\beta \geq \beta_0$ and $F \in \chi(\beta, U)$*

$$\mathbf{Eq}(F) \subset \mathcal{N}$$

Proof Equilibria of F coincide with equilibria of F_{π_β} . Let $x(\beta)$ be such an equilibrium. Then for all i, j

$$\frac{\log(x_i(\beta)) - \log(x_j(\beta))}{\beta} = U_i(x(\beta)) - U_j(x(\beta)).$$

Thus for every limit point $x^* = \lim_{\beta_k \rightarrow \infty} x(\beta_k)$ it follows that

$$U_i(x^*) = U_j(x^*)$$

if $i, j \in \text{Supp}(x^*)$ and

$$U_i(x^*) \geq U_j(x^*)$$

if $i \notin \text{Supp}(x^*)$ and $j \in \text{Supp}(x^*)$. ■

Remark 10 Note that Proposition 5 only requires the continuity of U .

We shall now prove some converse results.

A Nash equilibrium x^* is called *pure* if $\text{Supp}(x^*)$ has cardinal 1 and *mixed* otherwise. It is called *strict* if inequality (22) is strict for all $i \notin \text{Supp}(x^*)$.

Theorem 5 *Let x^* be a pure strict Nash equilibrium and \mathcal{N} a (sufficiently small) neighborhood of x^* . Then, there exists $\beta_0 > 0$ such that for all $\beta \geq \beta_0$ and $F \in \chi(\beta, U)$*

- (i) $\mathbf{Eq}(F) \cap \mathcal{N} = \{x_\beta^*\}$
- (ii) Equilibrium x_β^* is linearly stable for F_{π_β} .
- (iii) Assume furthermore that the game is a potential game. Then x_β^* is linearly stable for F under one of the following conditions:
 - (a) L (hence F) is C^2 and x_β^* is hyperbolic for F , or
 - (b) $F \in \chi_{rev}(\beta, U)$.

Proof Suppose without loss of generality that $x_1^* = 1$ and $x_i^* = 0$ for $i \neq 1$.

Set $R_{ij} = U_j - U_i$. By assumption and continuity, there exists $\delta > 0, \alpha > 0$ such that for all $x \in B(x^*, \alpha) = \{x \in \Delta : \|x - x^*\| \leq \alpha\}$,

$$R_{i1}(x) \geq \delta \text{ for } i > 1;$$

$$\|R_{ij}(x)\| \geq \delta \text{ if } R_{ij}(x^*) \neq 0$$

and

$$\|R_{ij}(x)\| \leq \delta \text{ if } R_{ij}(x^*) = 0.$$

Thus

$$1 \geq \pi_{\beta,1}(x) = (1 + \sum_{i>1} e^{-\beta R_{i1}(x)})^{-1} \geq (1 + (n-1)e^{-\beta\delta})^{-1}.$$

This implies that π_β maps $B(x^*, \alpha)$ into itself for β large enough. By Brouwer's Theorem, it then admits a fixed point x_β^* . To prove uniqueness and assertion (ii) it suffices to prove that π_β restricted to $B(x^*, \alpha)$ is a contraction. From the expression $\pi_{\beta,i} = (\sum_j e^{\beta R_{ij}})^{-1}$, we get

$$\frac{\partial \pi_{\beta,i}}{\partial x_m} = - \sum_j [\beta e^{\beta R_{ij}} (\sum_k e^{\beta R_{ik}})^{-2} \frac{\partial R_{ij}}{\partial x_m}] := \sum_j D_{ij} = \sum_{j \neq i} D_{ij}.$$

Let $j \neq i$. If $R_{ij}(x^*) \neq 0$

$$|D_{ij}| \leq \beta e^{\beta R_{ij}} (1 + e^{\beta R_{ij}})^{-2} \leq \beta \min(e^{\beta R_{ij}}, e^{-\beta R_{ij}}) \leq \beta e^{-\beta \delta}.$$

If $R_{ij}(x^*) = 0$. Then $i \neq 1$ and

$$|D_{ij}| \leq \beta e^{\beta R_{ij}} (e^{\beta R_{i1}})^{-2} = \beta e^{\beta(R_{ij}-2R_{i1})} \leq \beta e^{-\beta \delta}$$

These inequalities show that $\|D\pi_{\beta}(x)\| < 1$ for all $x \in B(x^*, \alpha)$ and β large enough, proving uniqueness of the equilibrium as well as assertion (ii). The last assertion follows from Propositions 2 and 3. ■

A Nash equilibrium x^* is called *fully mixed* if $Supp(x^*) = \{1, \dots, n\}$ and *partially mixed* if $1 < card(Supp(x^*)) < n$.

A fully mixed Nash equilibrium is called *non degenerate* if for all $u \in T\Delta$

$$[\forall w \in T\Delta \langle DU(x^*)u, w \rangle = 0] \Rightarrow u = 0.$$

Let

$$T\Delta(x^*) = \{u \in T\Delta : u_i = 0 \text{ for } i \notin Supp(x^*)\}.$$

A partially mixed equilibrium x^* is called *non degenerate* if for all $u \in T\Delta(x^*)$

$$[\forall w \in T\Delta(x^*) \langle DU(x^*)u, w \rangle = 0] \Rightarrow u = 0,$$

Lemma 5 Let $x^* \in \Delta$ be a mixed equilibria. Assume that $Supp(x^*) = \{1, \dots, r\}$ for some $1 < r \leq n$ and set

$$x^* = (q_1, \dots, q_{r-1}, 1 - \sum_{i=1}^{r-1} q_i, 0, \dots, 0).$$

Let, for $i = 1, \dots, r - 1$,

$$\begin{aligned} h_i^r(x_1, \dots, x_{r-1}, y_1, \dots, y_{n-r}) &= U_i(x_1, \dots, x_{r-1}, 1 - \sum_{i=1}^{r-1} x_i - \sum_{i=1}^{n-r} y_i, y_1, \dots, y_{n-r}) \\ &\quad - U_r(x_1, \dots, x_{r-1}, 1 - \sum_{i=1}^{r-1} x_i - \sum_{i=1}^{n-r} y_i, y_1, \dots, y_{n-r}) \end{aligned}$$

Then x^* is non degenerate if and only if the $(r-1) \times (r-1)$ matrix

$$\left[\frac{\partial h_i^r}{\partial x_j}((q, 0)) \right]_{i,j=1,\dots,r-1}$$

is invertible.

Proof One has

$$\frac{\partial h_i^r}{\partial x_j}(q, 0) = \left(\frac{\partial U_i}{\partial x_j}(x^*) - \frac{\partial U_i}{\partial x_r}(x^*) \right) - \left(\frac{\partial U_r}{\partial x_j}(x^*) - \frac{\partial U_r}{\partial x_r}(x^*) \right).$$

Let

$$v = (v_1, \dots, v_{r-1}, -\sum_{i=1}^{r-1} v_i, 0, \dots, 0) \in T\Delta(x^*)$$

and

$$w = (w_1, \dots, w_{r-1}, -\sum_{i=1}^{r-1} w_i, 0, \dots, 0) \in T\Delta(x^*).$$

Then it is easily seen that

$$\sum_{i=1}^{r-1} \sum_{j=1}^{r-1} \frac{\partial h_i^r}{\partial x_j}(q) v_i w_j = \langle DU(x^*)v, w \rangle.$$

This proves that x^* is non degenerate if and only if $\left[\frac{\partial h_i^r}{\partial x_j}((q, 0)) \right]_{i,j=1,\dots,r-1}$ is invertible. ■

Theorem 6 *Let x^* be a non degenerate fully mixed Nash equilibrium for U and \mathcal{N} a (sufficiently small) neighborhood of x^* . Then, there exists $\beta_0 > 0$ such that for all $\beta \geq \beta_0$ and $F \in \chi(\beta, U)$*

$$\mathbf{Eq}(F) \cap \mathcal{N} = \{x_\beta^*\}.$$

Assume furthermore that the game is a potential game with potential W . Then x_β^ is hyperbolic for F and its unstable manifold (for F) has dimension $\mathbf{Ind}(x^*, W|_\Delta)$ under one of the following conditions:*

- (a) L (hence F) is C^2 and x_β^* is hyperbolic for F , or
- (b) $F \in \chi_{\text{rev}}(\beta, U)$.

Proof Set $T = 1/\beta$. Equilibria of F_{π_β} are given by the set of equations

$$T(\log(x_i) - \log(x_n)) = U_i(x) - U_n(x), i = 1, \dots, n - 1$$

or, with the notation of Lemma 5,

$$T(\log(x_i) - \log(1 - \sum_{i=1}^{n-1} x_i)) = h_i^n(x_1, \dots, x_{n-1}), i = 1, \dots, n - 1. \quad (24)$$

Write $x^* = (q_1, \dots, q_{n-1}, 1 - \sum_{i=1}^{n-1} q_i)$. For $T = 0$, $q = (q_1, \dots, q_{n-1})$ is solution to (24). Hence, by the implicit function theorem (which hypothesis is fulfilled by the non degeneracy of x^* and Lemma 5) there exists $\alpha_0 > 0$, a neighborhood O of q in $(\mathbb{R}_+^*)^{n-1}$ and a C^1 map $T \in]-\alpha_0, \alpha_0[\mapsto q(T) \in O$ such that $(T, q(T))$ is the unique solution to (24) in $]-\alpha_0, \alpha_0[\times O$. This proves the first assertion of the theorem with $\beta_0 > 1/\alpha_0$ and $x_\beta^* = (q(1/\beta), 1 - \sum_{i=1}^{n-1} q_i(1/\beta))$.

In case, the game is a potential game with potential W , F is gradient-like with Lyapounov function V_β given by (16). Since x^* is fully mixed, $\frac{1}{q_i} < \infty$ so that $\|\frac{1}{\beta} \text{Hess} V_\beta(x_\beta^*) - \text{Hess} W(x^*)\| \rightarrow 0$ as $\beta \rightarrow \infty$. In particular, for β large enough $\text{Hess} V_\beta(x_\beta^*)$ is non degenerate, because x^* is non degenerate. The last assertion then follows from Propositions 2 and 3. ■

Theorem 7 *Let x^* be a strict and non degenerate partially mixed Nash equilibrium for U which support has cardinal $1 < r < n$. Let \mathcal{N} be a (sufficiently small) neighborhood of x^* . Then, there exists $\beta_0 > 0$ such that for all $\beta \geq \beta_0$ and $F \in \chi(\beta, U)$*

$$\text{Eq}(F) \cap \mathcal{N} = \{x_\beta^*\}.$$

Assume furthermore that the game is a potential game with potential W and that one of the following conditions hold:

- (a) L (hence F) is C^2 and x_β^* is hyperbolic for F , or
- (b) $F \in \chi_{\text{rev}}(\beta, U)$.

Then x_β^ is hyperbolic and*

$$k \leq \dim(E_{x_\beta^*}^u) \leq \min(n - r + k, r - 1).$$

with $k = \mathbf{Ind}(x^, W|_{\Delta(x^*)})$ and $\dim(E_{x_\beta^*}^u)$ stands for the dimension of the unstable manifold (for F).*

Proof Assume without loss of generality that $\text{Supp}(x^*) = \{1, \dots, r\}$ and set $x^* = (q_1, \dots, q_{r-1}, 1 - \sum_{i=1}^{r-1} q_i, 0, \dots, 0)$. Write every element of Δ as $(x_1, \dots, x_{r-1}, 1 - \sum_{i=1}^{r-1} x_i - \sum_{i=1}^{n-r} y_i, y_1, \dots, y_{n-r})$ and set $x = (x_1, \dots, x_{r-1}), y = (y_1, \dots, y_{n-r})$. Thus,

with $\beta = 1/T$, equilibria of $F_{\pi\beta}$ are given by the following system of equations:

$$T(\log(x_i) - \log(1 - \sum_{i=1}^{r-1} x_i - \sum_{i=1}^{n-r} y_i)) = h_i^r(x, y), i = 1, \dots, r-1 \quad (25)$$

and

$$T(\log(y_i) - \log(1 - \sum_{i=1}^{r-1} x_i - \sum_{i=1}^{n-r} y_i)) = h_{i+r}^r(x, y), i = 1 \dots n-r \quad (26)$$

where h_i^r is defined in Lemma 5. The triplet $(T = 0, x = q, y = 0)$ is solution to (25). Thus by the non degeneracy hypothesis and the implicit function theorem, there exists a smooth map

$$\hat{x} : \mathcal{O} \mapsto \mathcal{V}, (T, y) \mapsto \hat{x}(T, y)$$

where \mathcal{O} is a neighborhood of $(0, 0)$ in $\mathbb{R} \times \mathbb{R}^{n-r}$ and \mathcal{V} a neighborhood of q in \mathbb{R}^{r-1} such that $(T, \hat{x}(T, y), y)$ is solution to (25). Recall that $0 < \sum_{i=1}^{r-1} q_i < 1$ and $h_{i+r}^r(q, 0) < 0$ for all $i = 1, \dots, n-r$ (because x^* is strict). Thus, by choosing \mathcal{O} small enough we can furthermore ensure that

$$0 < 1 - \sum_{i=1}^{r-1} \hat{x}_i(T, y) - \sum_{i=1}^{n-r} y_i < 1 \quad (27)$$

and

$$h_{i+r}^r(\hat{x}(T, y), y) \leq -\delta < 0, i = 1 \dots n-r \quad (28)$$

for all $(T, y) \in \mathcal{O}$.

Now replacing x by $\hat{x}(T, y)$ in (26) leads to

$$y_i = G_i(T, y), i = 1 \dots n-r$$

where

$$G_i(T, y) = (1 - \sum_{i=1}^{r-1} \hat{x}_i(T, y) - \sum_{i=1}^{n-r} y_i) \exp\left(\frac{1}{T} h_{i+r}^r(\hat{x}(T, y), y)\right).$$

Using (27) and (28) we see that α small enough and $T \leq \frac{\log(1/\alpha)}{\delta}$ $G(T, \cdot)$ maps $\{y \in \mathbb{R}^{n-r} : 0 \leq y_i \leq \alpha\}$ into itself. By Brouwer's fixed point theorem, $G(T, \cdot)$ admits a fixed point $\hat{y}(T)$. Furthermore, $\|D_y G(T, y)\| \leq \frac{C}{T} e^{-\delta/T}$ for some constant C making $G(T, \cdot)$ a contraction. This implies that $\hat{y}(T)$ is unique. Finally define x_β^* by $x_{\beta,i}^* = \hat{x}_i(T, \hat{y}(T))$ for $1 \leq i < r$ and $x_{\beta,i+r}^* = \hat{y}_i(T)$ for $1 \leq 1 \leq n-r$.

We now prove the last assertions. By assumption, $T\Delta(x^*)$ admits a decomposition $T\Delta(x^*) = E_+ \oplus E_-$ with $\langle \text{Hess}(W)(x^*)u, u \rangle > 0$ (respectively < 0) for all $u \in E_+$ (respectively E_-) and $u \neq 0$.

Set $T\Delta_s(x^*) = \{u \in T\Delta : u_1 = \dots = u_r = 0\}$. Then

$$T\Delta = E_+ \oplus E_- \oplus T\Delta_s(x^*).$$

Let now V_β be the Lyapounov function given (16). Then for all $u \in T\Delta$

$$Q_\beta(u) := \left\langle \frac{1}{\beta} \text{Hess}(V_\beta)(x_\beta^*)u, u \right\rangle = \langle \text{Hess}W(x_\beta^*)u, u \rangle + \frac{1}{\beta} \sum_i \frac{1}{x_{\beta,i}^*} u_i^2.$$

The construction of x_β^* shows that $\frac{1}{\beta} \frac{1}{x_{\beta,i}^*} \rightarrow 0$ for $i \leq r$ and $\frac{1}{\beta} \frac{1}{x_{\beta,i}^*} \rightarrow \infty$ for $i > r$ when $\beta \rightarrow \infty$. Thus, for β large enough, Q_β is non degenerate, definite positive on E_+ and $T\Delta_s(x^*)$, and definite negative on E_- .

This implies that its index is bounded below by $k = \dim(E_-)$ and above by $\min(r - 1, n - r - k)$. This index equals the dimension of the stable manifold by Proposition 2. Under the reversibility assumption hyperbolicity follows from Proposition 3. ■

7 Reversibility and Gradient Structure

Recall that an irreducible rate matrix L is called *reversible* with respect to $\pi \in \dot{\Delta}$ is $\pi_i L_{ij} = \pi_j L_{ji}$. In this case π is the (unique) invariant probability of L . Here we will consider gradient properties of (1) under the assumption that $L(x)$ is reversible.

A $C^k, k \geq 0$ (Riemannian) metric on $\dot{\Delta}$ (or Δ) is a C^k map g such that for each $x \in \Delta$ $g(x) : T\Delta \times T\Delta \mapsto \mathbb{R}$ is a definite positive bilinear form. Given a C^1 map $V : \dot{\Delta} \mapsto \mathbb{R}$ we let $\text{grad}_g V$ denote the gradient vector field of V with respect to g . That is

$$g(x)(\text{grad}_g V(x), u) = \langle \nabla V(x), u \rangle$$

for all $u \in T\Delta$.

Proposition 6 *Assume that for all $x \in \dot{\Delta}$ $L(x)$ is reversible with respect to $\pi(x)$ and assume that the map $h : \dot{\Delta} \mapsto \mathbb{R}^n$, defined by*

$$h(x) = \frac{x}{\pi(x)}$$

is a α -quasigradient. Then there exists a metric g on $\dot{\Delta}$ such that for all $x \in \dot{\Delta}$ $F(x) = -\text{grad}_g V(x)$. If L and α are C^k then g is C^k .

Proof The proof is similar to the proof of Proposition 3. Let $A(x) : T\Delta \mapsto T\Delta$ be defined by $A(x)h = -hL(x)$. Then $A(x)$ and $L(x)$ are conjugate by the relation $\pi(x)L(x)h = A(x)\pi(x)h$ and $A(x)$ is a definite positive operator for the scalar product on $T\Delta$ defined by $\langle u, v \rangle_{1/\pi(x)} = \sum_i u_i v_i \frac{1}{\pi_i(x)}$. Define now a Riemannian metric on $T\Delta$ by

$$g_0(x)(u, v) = \langle A(x)^{-1}u, v \rangle_{\frac{1}{\pi(x)}}. \quad (29)$$

Since $F(x) = xL(x) = (x - \pi(x))L(x) = A(x)(-x + \pi(x))$, we get

$$g_0(x)(F(x), u) = -\left\langle \frac{x}{\pi(x)} - 1, u \right\rangle = -\left\langle \frac{x}{\pi(x)}, u \right\rangle.$$

If $x \mapsto \frac{x}{\pi(x)}$ is a quasi gradient, this makes F a gradient for the metric $g(x) = \alpha(x)g_0(x)$. ■

Example 7 Suppose that $L(x)$ is reversible with respect to π , independent on x . Then $x \mapsto \frac{x}{\pi}$ is the gradient of the χ^2 function $V(x) = \sum_i (\frac{x_i}{\pi_i} - 1)^2 \pi_i$. Hence $F(x) = -\text{grad}_g V(x)$ for some metric g .

Under the weaker assumption that $x \mapsto s(\frac{x}{\pi(x)})$ is a quasi-gradient for some strictly increasing function s (see Theorem 3) it is no longer true that F is a gradient, but it can be approximated by a gradient. The next Lemma is the key tool. Its proof is identical to the proof of Proposition 3.

Lemma 6 *Assume that assumptions of Theorem 3 hold and that for all $x \in \dot{\Delta}$, $L(x)$ is reversible with respect to $\pi(x)$. Then there exists a metric g_0 on $\dot{\Delta}$ such that for $p \in \mathbf{Eq}(F) \cap \dot{\Delta}$ and $u, v \in T\Delta$*

$$g_0(p)(DF(p)u, v) = -\langle \text{Hess}(V)(p)u, v \rangle.$$

If, furthermore, L and α (in Eq. (13)) are C^k then g_0 is C^k .

Theorem 8 *Assume that*

- (a) *Assumptions of Theorem 3 hold with s, α and L C^k , $k \geq 2$,*
- (b) *For all $x \in \dot{\Delta}$ $L(x)$ is reversible with respect to $\pi(x)$,*
- (c) *$\mathbf{Eq}(F) \cap \dot{\Delta}$ is finite.*

Then for every neighborhood \mathcal{U} of $\mathbf{Eq}(F) \cap \dot{\Delta}$ and every $\varepsilon > 0$ there exists a C^k metric g on $\dot{\Delta}$ such that

- (i) *$-\text{grad}_g V = F$ on $\dot{\Delta} \setminus \mathcal{U}$.*
- (ii) *$\|-\text{grad}_g V - F\|_{C^1, \mathcal{U}} \leq \varepsilon$ where*

$$\|G\|_{C^1, \mathcal{U}} = \sup_{x \in \mathcal{U}} \|G(x)\| + \|DG(x)\|.$$

Proof Let $\mathcal{E} = \dot{\Delta} \cap \mathbf{Eq}(F)$, $v(x) = d(x, \mathcal{E}) = \min_{p \in \mathcal{E}} \|x - p\|$ and let $\psi : \mathbb{R}^+ \mapsto [0, 1]$ be a C^∞ function which is 0 on $[0, 1]$, 1 on $[3, \infty[$ and such that $0 \leq \psi' \leq 1$. Fix $\varepsilon > 0$ and let $\lambda(x) = \psi(\frac{v(x)}{\varepsilon})$, $G_0 = -\text{grad}_{g_0} V$ where g_0 is given by Lemma 6 and

$$G(x) = (1 - \lambda(x))G_0(x) + \lambda(x)F(x).$$

Since for all $p \in \mathcal{E}$, $F(p) - G_0(p) = DF(p) - DG_0(p) = 0$ there exists a constant $C > 0$ such that

$$\|G_0(x) - F(x)\| \leq Cv(x)^2, \quad \|DG_0(x) - DF(x)\| \leq Cv(x).$$

Thus

$$\|G(x) - F(x)\| = (1 - \lambda(x))\|G_0(x) - F(x)\| \leq C(1 - \lambda(x))v(x)^2 \leq C\varepsilon^2$$

and

$$\begin{aligned} \|DG(x) - DF(x)\| &= \|(1 - \lambda(x))(DG_0(x) - DF(x)) + \langle \nabla \lambda(x), G_0(x) - F(x) \rangle\| \\ &\leq C((1 - \lambda(x))v(x) + \frac{1}{\varepsilon}v(x)^2) \leq C\varepsilon. \end{aligned}$$

This shows that G is a C^1 approximation of F which coincides with F on $\{v(x) \geq 3\varepsilon\}$ and with G_0 on $\{v(x) \leq \varepsilon\}$. Furthermore,

$$\langle \nabla V(x), G(x) \rangle = -(1 - \lambda(x))g_0(x)(G_0(x), G_0(x)) + \lambda(x)\langle \nabla V(x), F(x) \rangle \leq 0$$

with equality if and only if $x \in \mathcal{E}$.

Now, for all $x \in \dot{\Delta} \setminus \mathcal{E}$

$$T\Delta = \nabla V(x)^\perp \oplus \mathbb{R}G(x)$$

and the splitting is smooth in x . Hence $u \in T\Delta$ can be uniquely written as $u = P_x(u) + t_x(u)G(x)$ with $t_x(u) \in \mathbb{R}$ and $P_x(u) \in \nabla V(x)^\perp$. Let g be the metric on $\dot{\Delta} \setminus \mathcal{E}$ defined by

$$g(x)(u, v) = g_0(P_x(u), P_x(v)) + t_x(u)t_x(v)g_0(x)(G_0(x), G(x)).$$

Then g coincides with g_0 on $\{0 < x < v(x) < \varepsilon\}$ so that g can be extended to a C^2 metric on $\dot{\Delta}$. By construction of G and g , $G = -\text{grad}_g V$. ■

8 Questions of Structural Stability

Let $C_{pos}^k(\Delta, T\Delta)$ denote the set of C^k vector fields $F : \Delta \mapsto T\Delta$ leaving Δ positively invariant.

Two elements $F, G \in C_{pos}^k(\Delta, T\Delta)$ are said *topologically equivalent* if there exists a homeomorphism $h : \Delta \mapsto \Delta$ which takes orbits of F to orbits of G preserving their orientation. A set $\chi \subset C_{pos}^k(\Delta, T\Delta)$ is said *structurally stable* if all its elements are topologically equivalents.

Let $\pi : \Delta \mapsto \tilde{\Delta}$ be a smooth function. Assume that π verifies the assumption of Theorem 3 and that F_π has non degenerate equilibria. Let $\chi_{\pi, rev}$ denote the convex set of vector fields having the form given by (the right hand side of) (1), where for each $x \in \Delta$, $L(x)$ is irreducible and reversible with respect to $\pi(x)$. By Theorem 3, Proposition 3 and Theorem 8 all the elements of $\chi_{\pi, rev}$ have the same strict Lyapounov function V , hyperbolic equilibria (given by the critical points of V) and are C^1 close to $-grad_g V$ for some metric g . We may then wonder wether $\chi_{\pi, rev}$ is structurally stable. The following construction shows that this is not the case.

8.1 Potential Games are not Structurally Stable

Here Δ stands for the two-dimensional simplex in \mathbb{R}^3 . Let

$$\tilde{\Delta} = \{(y_1, y_2) \in \mathbb{R}^2 : y_1, y_2 \geq 0, y_1 + y_2 \leq 1\}$$

and $J : \mathbb{R}^3 \mapsto \mathbb{R}^2$ be the projection defined by $J(x_1, x_2, x_3) = (x_1, x_2)$. Note that J maps Δ homeomorphically onto $\tilde{\Delta}$.

Let $\tilde{W} : \mathbb{R}^2 \mapsto \mathbb{R}$ be a smooth function. Assume that

- (a) $-\nabla \tilde{W}$ points inward $\tilde{\Delta}$ on $\partial \tilde{\Delta}$;
- (b) The critical set $crit(\tilde{W}) = \{y \in \tilde{\Delta} : \nabla \tilde{W}(y) = 0\}$ consist of (finitely many) non degenerate points,
- (c) For all $u \in \mathbb{R}$

$$\frac{\partial \tilde{W}}{\partial y_1}(u, u) = \frac{\partial \tilde{W}}{\partial y_2}(u, u).$$

In particular, the diagonal $D(\tilde{\Delta}) = \{(y_1, y_2) \in \tilde{\Delta} : y_1 = y_2\}$ is positively invariant under the dynamics

$$\dot{y} = -\nabla \tilde{W}(y) \tag{30}$$

- (c) There is a saddle connection contained in $D(\tilde{\Delta})$, meaning that there are two saddle points of \tilde{W} $s^1, s^2 \in D(\tilde{\Delta})$ and some (hence every) point $y \in]s^1, s^2[$ which α limit set under (30) is s^1 and omega limit set is s^2 .

It is not hard to construct such a map.

Let $W : \mathbb{R}^3 \mapsto \mathbb{R}$ be defined by $W = \tilde{W} \circ J$.

Consider now the 3-strategies potential game associated to W . Payoffs are then defined by

$$U_i(x) = -\frac{\partial \tilde{W}}{\partial x_i}(x_1, x_2), i = 1, 2 \text{ and } U_3(x) = 0.$$

Using the notation of Sect. 6.1, record that $F_{\pi_\beta} = -Id + \pi_\beta$ where π_β is defined by (23), and $\chi_{rev}(\beta, U)$ is the set of vector fields given by (1) with $L(x)$ irreducible and reversible with respect to $\pi_\beta(x)$.

Proposition 7 *For all $\beta > 0$ sufficiently large, there exists $F \in \chi_{rev}(\beta, U)$ (which can be chosen C^1 close to F_{π_β}) which is not equivalent to F_{π_β} .*

Proof of Proposition 7

By definition of Nash equilibria (see Sect. 6.1) and condition (a) above, Nash equilibria of U are fully mixed and coincide with critical points of \tilde{W} :

$$crit(\tilde{W}) = J(\mathbf{NE}(U)).$$

Lemma 7 *For all $\varepsilon > 0$ there exists $\beta_0 > 0$ such that for all $\beta \geq \beta_0$ and $F \in \chi_{rev}(\beta, U)$ there is a one to one map*

$$p \in crit(\tilde{W}) \mapsto p_\beta \in \mathbf{Eq}(F),$$

such that

- (i) $\|p - J(p_\beta)\| \leq \varepsilon,$
- (ii) *The unstable (respectively stable) manifold of p_β has dimension $\mathbf{Ind}(p, \tilde{W},)$ (resp. $2 - \mathbf{Ind}(p, \tilde{W})$). In particular, s_β^1 and s_β^2 are saddle points.*
- (iii) $p \in D(\tilde{\Delta}) \Leftrightarrow J(p_\beta) \in D(\tilde{\Delta})$
- (iv) *Under the dynamics induced by F_{π_β} , the interval $[s_\beta^1, s_\beta^2]$ is invariant and for some (hence all) $q \in]s_\beta^1, s_\beta^2[$ the alpha limit (respectively omega limit) set of q equals s_β^1 (respectively s_β^2).*

Proof Assertions (i) and (ii) this follows from Propositions 5 and 6.

On $J^{-1}(D(\tilde{\Delta})) = \{(x_1, x_1, 1 - 2x_1)\}$ equilibria of F_{π_β} are given by the implicit equation $T(\log(x_1) - \log(1 - 2x_1)) = U_1(x_1, x_1)$ where $T = 1/\beta$. Solutions for $T = 0$ coincide with $J^{-1}(D(\tilde{\Delta}) \cap crit(\tilde{W}))$. For $T > 0$ and small enough, assertion (iii) then follows from the implicit function theorem.

By condition (c), $\frac{\partial \tilde{W}}{\partial x_1} = \frac{\partial \tilde{W}}{\partial x_2}$ on $D(\tilde{\Delta})$. Thus $U_1(x) = U_2(x)$ (hence $F_{\pi_{\beta,1}}(x) = F_{\pi_{\beta,2}}(x)$) on $J^{-1}(D(\tilde{\Delta}))$ proving invariance of $[s_\beta^1, s_\beta^2] \subset J^{-1}(D(\tilde{\Delta}))$. Assertion (iv) follows since, by (iii), there are no equilibria in $]s_\beta^1, s_\beta^2[$. ■

We now construct $F \in \chi_{rev}(\beta, U)$. Let $L(x)$ be the rate matrix defined for $i \neq j$ by

$$L_{ij}(x) = \pi_{\beta,j}(x) \text{ if } i, j \notin \{1, 3\}$$

$$L_{13}(x) = (1 + a(x))\pi_{\beta,3}(x) \text{ and } L_{31}(x) = (1 + a(x))\pi_{\beta,1}(x)$$

where $a : \Delta \mapsto \mathbb{R}^+$ is a smooth function to be defined below. Then Eq. (1) reads

$$\begin{aligned} \dot{x}_1 &= (x_2\pi_{\beta,1}(x) - x_1\pi_{\beta,2}(x)) + (x_3\pi_{\beta,1}(x) - x_1\pi_{\beta,3}(x))(1 + a(x)), \\ \dot{x}_2 &= (x_1\pi_{\beta,2}(x) - x_2\pi_{\beta,1}(x)) + (x_3\pi_{\beta,2}(x) - x_2\pi_{\beta,3}(x)), \\ \dot{x}_3 &= -\dot{x}_1 - \dot{x}_2. \end{aligned}$$

Thus, on $x_1 = x_2$,

$$\begin{aligned} \dot{x}_1 - \dot{x}_2 &= [x_3\pi_{\beta,1}(x) - x_1\pi_{\beta,3}(x)]a(x) \\ &= \frac{a(x)}{Z(x)}(x_3e^{\beta U_1(x)} - x_1). \end{aligned}$$

The map $x \mapsto x_3e^{\beta U_1(x)} - x_1$ vanishes at points s_β^1, s_β^2 and has a constant sign over $[s_\beta^1, s_\beta^2]$ (for otherwise there would exist an equilibrium for F in $]s_\beta^1, s_\beta^2[$ contradicting Lemma 7). Let $p = (s_\beta^1 + s_\beta^2)/2$ and B_η be the Euclidean open ball with center p and radius η . Choose η small enough so that

- (i) $B_\eta \cap [s_\beta^1, s_\beta^2] =]q^1, q^2[$ with $s_\beta^1 < q^1 < q^2 < s_\beta^2$ where $<$ stands for the natural ordering on $[s_\beta^1, s_\beta^2]$.
- (ii) $x \mapsto x_3\pi_{\beta,1}(x) - x_1\pi_{\beta,3}(x)$ has constant sign on B_η .

Let $x \mapsto a(x)$ be such $a = 0$ on $\Delta \setminus B_\eta$, $a > 0$ on B_η and $0 \leq a \leq \eta$ on Δ . Then, the alpha limit set of q^1 equals s_β^1 , for both F and F_{π_β} but since $\dot{x}_1 - \dot{x}_2$ doesn't vanish on B_η the trajectory through q^1 exits B_η at a point $\neq q^2$ and, consequently, the omega limit set of q^1 for F is distinct from s_β^2 . This proves that F and F_{π_β} are not equivalent.

8.2 Open Question

The preceding construction shows that $\chi_{rev}(\beta, U)$ is not structurally stable for an arbitrary potential game but this might be the case for particular examples. Consider for example the Gibbs model described in Remark 5. For $U^0 \in \mathbb{R}^n$ and $U = (U_{ij})$ symmetric, let $\chi_{rev}(\beta, U^0, U)$ be the set of C^1 vector field given by (1) with $L(x)$ irreducible and reversible with respect to the Gibbs measure (14).

Question

For generic (U^0, U) and β large enough, is $\chi_{rev}(\beta, U^0, U)$ structurally stable ?

Appendix

Let L be an irreducible rate matrix and $\pi \in \dot{\Delta}$ denote the invariant probability of L . That is the unique solution (in Δ) of $\pi L = 0$. For all $f, g \in \mathbb{R}^n$ we let

$$\langle f, g \rangle = \sum_i f_i g_i, \langle f, g \rangle_\pi = \sum_i f_i g_i \pi_i \text{ and } \langle f, g \rangle_{1/\pi} = \sum_i f_i g_i \frac{1}{\pi_i}.$$

The *Dirichlet form* of L is the map $\mathcal{E} : \mathbb{R}^n \mapsto \mathbb{R}_+$ defined as

$$\mathcal{E}(f) = -\langle f, Lf \rangle_\pi = \frac{1}{2} \sum_{i,j} (f_i - f_j)^2 L_{ij} \pi_i.$$

By irreducibility, $\mathcal{E}(f) > 0$ unless f is constant, and the *spectral gap*

$$\lambda = \sup\{\mathcal{E}(f) : \langle f, 1 \rangle_\pi = 0, \langle f, f \rangle_\pi = 1\}$$

is positive. We let L^* be the irreducible rate matrix defined by

$$L_{ij}^* = \frac{\pi_j L_{ji}}{\pi_i}.$$

Note that L^* admits π as invariant probability and that L^* is the adjoint of L for $\langle \cdot, \cdot \rangle_\pi$.

We let $L^T : T\Delta \mapsto T\Delta$ be defined by

$$L^T h = hL.$$

Finally recall that for all $f \in \mathbb{R}^n$ $\frac{f}{\pi}$ stands for the vector defined by $(\frac{f}{\pi})_i = \frac{f_i}{\pi_i}$, $i = 1 \dots n$.

Lemma 8 For all $u, v \in T\Delta$

$$\langle L^T u, v \rangle_{1/\pi} = \langle L^* \left(\frac{u}{\pi}, \frac{v}{\pi} \right) \rangle_{\pi}$$

In particular L^T is invertible and L^T is a definite negative operator for $\langle \cdot, \cdot \rangle_{\frac{1}{\pi}}$ whenever L is reversible with respect to π .

Proof The first assertion follows from elementary algebra. For the second, note that $\langle L^T u, u \rangle_{1/\pi} = -\mathcal{E}(\frac{u}{\pi})$. Thus, by irreducibility,

$$\langle L^T u, u \rangle_{1/\pi} < 0$$

unless $u = 0$. ■

Proof of Lemma 4

Given $f \in \mathbb{R}^n$ we write $f \geq 0$ if $f_i \geq 0$ for all i . We let $1 \in \mathbb{R}^n$ denote the vector which components are all equal to 1. For all $t \geq 0$ we let $P_t = e^{tL}$. Since L is a rate matrix, (P_t) is a Markov semigroup meaning that $P_t f \geq 0$ for all $f \in \mathbb{R}^n$ with $f \geq 0$ and $P_t 1 = 1$.

Lemma 9 Let $I \subset \mathbb{R}$ be an open interval and $S : I \mapsto \mathbb{R}$ a C^2 function such that $S''(t) \geq \alpha > 0$. Let $f \in \mathbb{R}^n$ be such that $f_i \in I$ for all i . Then

$$\frac{d}{dt} \langle S(P_t f), 1 \rangle_{\pi} |_{t=0} \leq -\alpha \mathcal{E}(f).$$

Proof For all $u, v \in I$ $S(v) - S(u) - S'(u)(v - u) \geq \alpha/2(v - u)^2$. Hence for all i, j

$$S(f_j) - S((P_t f)_i) - S'((P_t f)_i)(f_j - (P_t f)_i) \geq \alpha/2(f_j - (P_t f)_i)^2.$$

Applying P_t to this inequality gives

$$P_t(Sf)_i - S((P_t f)_i) \geq \alpha/2 P_t(f_i - (P_t f)_i)^2 = \alpha/2(P_t f_i^2 - (P_t f)_i^2)$$

Hence

$$P_t(Sf) - S((P_t f)) \geq \alpha/2 P_t(f - (P_t f))^2 = \alpha/2(P_t f^2 - (P_t f)^2).$$

Therefore, using the fact that $\langle P_t g, 1 \rangle_\pi = \langle g, 1 \rangle_\pi$ leads to

$$\langle S f - S(P_t f), 1 \rangle_\pi \geq \alpha \langle f^2 - (P_t f)^2, 1 \rangle_\pi.$$

Dividing by t and letting $t \rightarrow 0$ leads to the desired inequality. ■

Let $S :]0, \infty[\rightarrow \mathbb{R}$ be a C^2 function with positive second derivative. Let $H_\pi^S : \Delta \mapsto \mathbb{R}$ be the map defined by

$$H_\pi^S(x) = \sum_i \pi_i S\left(\frac{x_i}{\pi_i}\right).$$

Corollary 4 For all $x \in \Delta$

$$\langle \nabla H_\pi^S(x), xL \rangle \leq -\alpha \lambda \text{Var}_\pi(f)$$

where $f_i = \frac{x_i}{\pi_i}$

Proof For $x \in \Delta$ let $x(t) = x e^{tL}$, $f_i = \frac{x_i}{\pi_i}$, $f_i(t) = \frac{x_i(t)}{\pi_i}$ and $P_t^* g = e^{tL^*} g$. Note that P_t^* is the adjoint of P_t with respect to $\langle \cdot, \cdot \rangle_\pi$.

For all $g \in \mathbb{R}^n$, $\langle x(t), g \rangle = \langle x, P_t g \rangle = \langle f, P_t g \rangle_\pi = \langle P_t^* f, g \rangle_\pi$ so that $f(t) = P_t^* f$. Hence by the preceding lemma applied to L^* it follows that

$$\langle \nabla H_\pi^S(x), xL \rangle = \frac{d}{dt} \langle S(P_t^* f), 1 \rangle_\pi |_{t=0} \leq -\alpha \mathcal{E}(f) \leq -\alpha \lambda \text{Var}_\pi(f)$$

where $\alpha = \min_i S''\left(\frac{x_i}{\pi_i}\right) > 0$. ■

We now prove the Lemma. Set $S(t) = \int_1^t s(u) du$. Then for all $u \in T \Delta$

$$\langle \nabla H_\pi^S(x), u \rangle = \sum_i u_i s\left(\frac{x_i}{\pi_i}\right)$$

and the results follows from Corollary 4.

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