Chapter 4
Open Problems

The unsolved problems form an incentive for the development of any area of mathematics. Since this book has an ambitious purpose to embrace all or almost all modern C_p-theory, it was impossible to avoid dealing with open questions.

In this book, we have a wide selection of unsolved problems of C_p-theory. Of course, "unsolved" means "unsolved to the best of the knowledge of the author". I give a classification by topics but there is no mention whatsoever of whether the given problem is difficult or not. One good parameter is the year of publication but sometimes the problem is not solved for many years because of lack of interest or effort and not because it is too difficult.

I believe that almost all unsolved problems of importance in C_p-theory are present in the volumes of this book taken together. The reader understands, of course, that there is a big difference between the textbook material of the first three chapters and open questions to which an author must be assigned. I decided that it was my obligation to make this assignment and did my best to be frowned at (or hated!) by the least possible number of potential authors of open problems.

This volume contains 100 unsolved problems which are classified by topics presented in 8 sections the names of which outline what the given group of problems is about. At the beginning of each subsection we define the notions *which are not defined in the main text*. Each published problem has a reference to the respective paper or book. If it is unpublished, then my opinion on who is the author is expressed. The last part of each problem is a very brief explanation of its motivation and/or comments referring to the problems of the main text or some papers for additional information. If the paper is published and the background material is presented in the main text, we mention the respective exercises. If the main text contains no background we refer the reader to the original paper. If no paper is mentioned in the motivation part, then the reader must consult the paper/book in which the unsolved problem was published.

V.V. Tkachuk, *A Cp-Theory Problem Book*, Problem Books in Mathematics, DOI 10.1007/978-3-319-16092-4_4

To do my best to assign the right author to every problem I implemented the following simple principles:

1. If the unsolved problem is published, then I cite the publication and consider myself not to be involved in the decision about who is the author. Some problems are published many times and I have generally preferred to cite the articles in journals/books which are more available for the Western reader. Thus it may happen that I do not cite the earliest paper where the problem was formulated. Of course, I mention it explicitly, if the author of the publication attributes the problem to someone else.
2. If, to the best of my knowledge, the problem is unpublished then I mention the author according to my personal records. The information I have is based upon my personal acquaintance and communication with practically all specialists in C_p-theory. I am aware that it is a weak point and it might happen that the problem I attributed to someone, was published (or invented) by another person. However, I did an extensive work ploughing through the literature to make sure that this does not happen.

4.1 Sokolov Spaces and Corson Compact Spaces

A space X is *splittable over a class* $\mathcal{P}$ if for any set $A \subset X$ there exists a continuous map $f : X \to Y$ such that $Y \in \mathcal{P}$ and $f^{-1}f(A) = A$. This notion generalizes the class of spaces condensable into a space from $\mathcal{P}$. Observe that a space X is splittable in the sense of Section 1.5 if and only if it is splittable over $\mathbb{R}^\omega$ (see Problem 416). If X is a space, then the ω-*modification of* X is the set X with the topology generated by all G_δ-subsets of the space X.

4.1.1 Suppose that X is a Sokolov compact space and $p(C_p(X)) = \omega$. Must X be metrizable?

Published in Tkachuk (2005c)
Motivated by the fact that this is true if X is Corson compact (Problem 287)

4.1.2 Suppose that X is a Sokolov compact space. Does $C_p(X)$ condense into a Σ-product of real lines?

Published in Tkachuk (2005c)
Motivated by the fact that this is true if X is Corson compact (Problem 286)

4.1.3 Suppose that K is a scattered compact space and let X be the ω-modification of K. Must the space X be Sokolov?

Author V.V. Tkachuk

4.1.4 Suppose that K is a scattered compact space of finite dispersion index and let X be the ω-modification of K. Must the space X be Sokolov?

Author V.V. Tkachuk

4.1.5 Suppose that X is a Corson compact space such that $C_p(X)$ is hereditarily σ-metacompact. Is it true that X is metrizable?

Author V.V. Tkachuk
Motivated by the fact that if $C_p(X)$ is hereditarily Lindelöf then X is metrizable

4.1.6 Let $\mathcal{P}$ be a class of all Corson compacta. Suppose that X is a space such that each compact subspace of $C_p(X)$ belongs to $\mathcal{P}$. Is it true that, for any compact K, all compact subspaces of $C_p(X \times K)$ belong to $\mathcal{P}$?

Published in Arhangel'skii (1998b)
Related to some questions on Grothendieck spaces

4.1.7 Recall that $L(\kappa)$ is the one-point Lindelöfication of the discrete space of cardinality κ. Suppose that X is compact and $C_p(X)$ is a continuous image of a closed subspace of $L(\kappa)^\omega \times K$ for some uncountable cardinal κ and a compact space K. Is it true that X is Corson compact?

Published in Arhangel'skii (1989a)
Motivated by the fact that it is true if K is the one-point space.

4.1.8 Suppose that a compact space X is splittable over the class of Corson compact spaces. Is it true in ZFC that X is Corson compact?

Authors D. Jardón, V.V. Tkachuk

4.1.9 Suppose that a compact space X is splittable over the class of Corson compact spaces. Is it true in ZFC that $w(X) = d(X)$?

Authors D. Jardón, V.V. Tkachuk

4.1.10 Suppose that a compact space X is splittable over the class of Corson compact spaces. Is it true in ZFC that any continuous image of X is also splittable over the class of Corson compacta?

Authors D. Jardón, V.V. Tkachuk

4.1.11 Suppose that a compact space X is splittable over the class of Corson compact spaces. Is it true in ZFC that $X \times X$ is also splittable over the class of Corson compacta?

Authors D. Jardón, V.V. Tkachuk

4.1.12 Suppose that X is a compact space such that $\overline{A}$ is Corson compact for any $A \subset X$ with $|A| \leq \mathfrak{c}$. Must X be Corson compact?

Authors D. Jardón, V.V. Tkachuk

4.2 Gul'ko Compact Spaces

A space X is *splittable over a class* $\mathcal{P}$ if for any set $A \subset X$ there exists a continuous map $f : X \to Y$ such that $Y \in \mathcal{P}$ and $f^{-1}f(A) = A$. This notion generalizes the class of spaces condensable into a space from $\mathcal{P}$. Observe that a space X is splittable in the sense of Section 1.5 if and only if it is splittable over $\mathbb{R}^\omega$ (see Problem 416).

4.2.1 Suppose that a compact space X is splittable over the class of Gul'ko compact spaces. Is it true in ZFC that X is Gul'ko compact?

Authors D. Jardón, V.V. Tkachuk

4.2.2 Suppose that a compact space X is splittable over the class of Gul'ko compact spaces. Is it true in ZFC that any continuous image of X is also splittable over the class of Gul'ko compacta?

Authors D. Jardón, V.V. Tkachuk

4.2.3 Suppose that a compact space X is splittable over the class of Gul'ko compact spaces. Is it true in ZFC that $X \times X$ is also splittable over the class of Gul'ko compacta?

Authors D. Jardón, V.V. Tkachuk

4.2.4 Suppose that a compact space X is splittable over the class of Gul'ko compact spaces. Is it true in ZFC that X has a dense metrizable subspace?

Authors D. Jardón, V.V. Tkachuk

4.2.5 Suppose that X is a compact space such that $\overline{A}$ is Gul'ko compact for any $A \subset X$ with $|A| \leq \mathfrak{c}$. Must X be Gul'ko compact?

Authors D. Jardón, V.V. Tkachuk

4.2.6 Suppose that X is a compact space such that $\overline{A}$ is a Lindelöf Σ-space for any $A \subset C_p(X)$ with $|A| \leq \mathfrak{c}$. Must X be Gul'ko compact?

Authors D. Jardón, V.V. Tkachuk

4.3 Eberlein Compact Spaces

A space X is *splittable over a class* $\mathcal{P}$ if for any set $A \subset X$ there exists a continuous map $f : X \to Y$ such that $Y \in \mathcal{P}$ and $f^{-1}f(A) = A$. This notion generalizes the class of spaces condensable into a space from $\mathcal{P}$. Observe that a space X is splittable in the sense of Section 1.5 if and only if it is splittable over $\mathbb{R}^\omega$ (see Problem 416).

4.3.1 Suppose that X has a dense σ-bounded subspace. Is it true that every countably compact subspace of $C_p(X)$ is an Eberlein compact?

Published in Arhangel'skii (1989a)
Comment this is true if X has a dense σ-pseudocompact subspace.

4.3.2 Given a cardinal κ, does there exist an Eberlein compact X such that every Eberlein compact of weight $\leq \kappa$ is a continuous image of X?

Published in Benyamini et al. (1977)

4.3.3 Given a cardinal κ, does there exist a uniform Eberlein compact X such that every uniform Eberlein compact of weight $\leq \kappa$ is a continuous image of X?

Published in Benyamini et al. (1977)

4.3.4 Suppose that a compact space X is splittable over the class of Eberlein compact spaces. Is it true in ZFC that X is Eberlein compact?

Authors D. Jardón, V.V. Tkachuk

4.3.5 Suppose that a compact space X is splittable over the class of Eberlein compact spaces. Is it true in ZFC that X has a dense metrizable subspace?

Authors D. Jardón, V.V. Tkachuk

4.3.6 Suppose that a compact space X is splittable over the class of Eberlein compact spaces. Is it true in ZFC that any continuous image of X is also splittable over the class of Eberlein compacta?

Authors D. Jardón, V.V. Tkachuk

4.3.7 Suppose that a compact space X is splittable over the class of Eberlein compact spaces. Is it true in ZFC that $X \times X$ is also splittable over the class of Eberlein compacta?

Authors D. Jardón, V.V. Tkachuk

4.3.8 Suppose that X is a compact space such that $\overline{A}$ is Eberlein compact for any $A \subset X$ with $|A| \leq \mathfrak{c}$. Must X be Eberlein compact?

Authors D. Jardón, V.V. Tkachuk

4.3.9 Suppose that a pseudocompact space X is splittable over the class of Eberlein compact spaces. Must X be Eberlein compact?

Authors D. Jardón, V.V. Tkachuk

4.3.10 Suppose that a countably compact space X is splittable over the class of Eberlein compact spaces. Must X be Eberlein compact?

Authors D. Jardón, V.V. Tkachuk

4.3.11 Suppose that X is an Eberlein compact space such that $C_p(X)$ is hereditarily σ-metacompact. Is it true that X is metrizable?

Author V.V. Tkachuk
Motivated by the fact that if $C_p(X)$ is hereditarily Lindelöf then X is metrizable

4.3.12 Let $\mathcal{P}$ be a class of Eberlein compacta. Suppose that X is a space such that each compact subspace of $C_p(X)$ belongs to $\mathcal{P}$. Is it true that, for any compact K, all compact subspaces of $C_p(X \times K)$ belong to $\mathcal{P}$?

Published in Arhangel'skii (1998b)
Related to some questions on Grothendieck spaces

4.4 The Lindelöf Σ-Property in $C_p(X)$

There is no need to explain why the Lindelöf Σ-property is of vital importance in general topology and C_p-theory. However, after decades of very hard work on the subject, there is still no acceptable characterization of the Lindelöf Σ-property in $C_p(X)$. As a consequence, many basic questions solved in topology remain unanswered for the spaces $C_p(X)$.

4.4.1 Suppose that $\upsilon(C_p(X))$ is a Lindelöf Σ-space. Is it true that the space $\upsilon(C_pC_p(X))$ is Lindelöf Σ?

Published in Arhangel'skii (1992b)
Comment the answer is "yes" if $C_p(X)$ is a Lindelöf Σ-space or X is normal (Problems 246–247)

4.4.2 Suppose that $C_p(X, \mathbb{I})$ is a Lindelöf Σ-space. Let Y be the set of non-isolated points of X. Is it true that the closure of Y in υX is a Lindelöf Σ-space?

Published in Okunev (1993a)
Motivated by the fact that υX is a Lindelöf Σ-space if $C_p(X)$ is a Lindelöf Σ-space (Problem 206) and $C_p(X, \mathbb{I})$ is compact (and hence Lindelöf Σ) if X is discrete.

4.4.3 Let X be a space of countable spread such that $C_p(C_p(X))$ is a Lindelöf Σ- space. Is it true that X has a countable network?

Published in Tkachuk (2001)
Motivated by the positive answer if $C_p(X)$ is a Lindelöf Σ- space (Problem 275).

4.4.4 Is it true that, for any compact space X, the space $C_p(X)$ has a T_0-separating weakly σ-point-finite family of cozero sets? What happens in case when X is a Lindelöf Σ- space?

Published in Tkachuk (2001)
Motivated by the fact that this is true for any Lindelöf Σ- space with a unique non-isolated point (Problem 274).

4.4.5 Suppose that X is a product of separable metric spaces. Is it true that every Lindelöf subspace of $C_p(X)$ has a countable network?

Published in Tkachuk (2001)
Comment this is true for Lindelöf Σ- subspaces of $C_p(X)$ (Problem 254).

4.4.6 Is it consistent with ZFC that for any hereditarily Lindelöf space X every Lindelöf Σ- subspace of $C_p(X)$ has a countable network?

Published in Tkachuk (2001)
Motivated by the fact that if $C_p(X)$ is a Lindelöf Σ- space then it has a countable network (Problem 275).

4.4.7 Let $C_p(X)$ be a Lindelöf Σ-space and assume that $\psi(X) \leq \omega$ (or even $\chi(X) \leq \omega$). Is it true that $|X| \leq \mathfrak{c}$?

Published in Tkachuk (2005c)
Motivated by the fact that this is true if X is a closed subspace of a Σ-product of real lines (Problem 285)

4.5 The Lindelöf Property in X and $C_p(X)$

One of the basic results here is the fact that $\omega_1 + 1$ does not embed in $C_p(Y)$ for any Lindelöf space Y. A natural step forward would be to prove the same for any compact space of uncountable tightness. However, this is only known under PFA and one of the most important questions of the topic is whether this statement is true without any additional axioms.

4.5.1 Suppose that X is a compact space embeddable in $C_p(Y)$ for a Lindelöf space Y. Is it true in ZFC that $t(X) = \omega$?

Published in Arhangel'skii (1989a)
Related to Problems 085–089

4.5.2 Suppose that X is a compact space embeddable in $C_p(Y)$ for a Lindelöf space Y. Is it true that X is sequential under MA+$\neg$CH?

Published in Arhangel'skii (1989a)
Related to Problems 085–089
Comment the answer is "yes" under PFA (Arhangel'skii (1989a))

4.5.3 Suppose that X is a compact space embeddable in $C_p(Y)$ for a Lindelöf space Y. Let X_1 be a continuous image of X. Is it true that X_1 is embeddable in $C_p(Z)$ for some Lindelöf Z?

Published in Arhangel'skii (1989a)
Comment if the answer is positive then every compact space embeddable in $C_p(Y)$ for a Lindelöf space Y, has countable tightness (see Problems 085–089)

4.5.4 Assuming MA+$\neg$CH suppose that X is a separable compact subspace of $C_p(Y)$ for a Lindelöf space Y. Must X be metrizable?

Published in Arhangel'skii (1989a)
Related to Problems 080–081, 097–098
Motivated by the fact that the answer is "yes" if every finite power of Y is Lindelöf

4.5.5 Suppose that X is a separable compact subspace of $C_p(Y)$ for a Lindelöf space Y. Must X be hereditarily separable?

Published in Arhangel'skii (1998b)
Related to Problems 097–098

4.5.6 Suppose that X is a homogeneous compact space and $X \subset C_p(Y)$ for some Lindelöf space Y. Is it true that $|X| \leq \mathfrak{c}$?

Published in Arhangel'skii (1989a)
Motivated by the fact that it is true under PFA because $|X| \leq \mathfrak{c}$ for any sequential homogeneous compact X (Arhangel'skii (1978b))
Related to Problems 085–089

4.5.7 Let X be a space of countable extent. Is it consistently true (say, under MA+¬CH) that every compact subspace of $C_p(X)$ is Fréchet–Urysohn?

Published in Arhangel'skii (1997)
Related to Problems 085–089

4.5.8 Let X be a Lindelöf space. Is it consistent with ZFC that every compact subspace of $C_p(X)$ is Fréchet–Urysohn?

Published in Arhangel'skii (1998b)
Related to Problems 085–089

4.5.9 Let X be a Lindelöf k-space. Is it true in ZFC that every compact subspace of $C_p(X)$ is Fréchet–Urysohn? What happens if X is a Lindelöf first countable space?

Published in Arhangel'skii (1997)
Related to Problems 085–089

4.5.10 Let X be a Lindelöf space. Is it true that every countably compact subspace of $C_p(X)$ has countable tightness?

Published in Arhangel'skii (1997)
Related to Problems 085–089

4.5.11 Suppose that $C_p(X)$ is Lindelöf. Is it true that $C_p(X) \times C_p(X)$ is Lindelöf?

Published in Arhangel'skii (1988a), (1989a), (1990a), (1992b)
Related to Problem 032

4.5.12 Suppose that X is a compact space and $C_p(X)$ is Lindelöf. Is it true that $C_p(X) \times C_p(X)$ is Lindelöf?

Published in Arhangel'skii (1988a), [1989a], [1990a], [1992b]
Related to Problem 032

4.5.13 Is every Lindelöf space embeddable in a Lindelöf $C_p(X)$ (as a closed subspace)?

Published in Arhangel'skii (1992b)

4.5.14 Let X be a (compact) space such that $C_p(X)$ is Lindelöf. Is it true that there exists a (compact) zero-dimensional space Y such that X is a continuous image of Y and $C_p(Y)$ is Lindelöf?

Published in Arhangel'skii (1989a)
Related to Problem 032

4.5.15 Suppose that $C_p(X)$ is Lindelöf. Is it true that $t(L_p(X)) = \omega$?

Published in Arhangel'skii (1989a)

4.5.16 Suppose that $t(L_p(X)) = \omega$. Is it true that $C_p(X)$ is Lindelöf?

Author A.V. Arhangel'skii
Comment In many occasions Arhangel'skii formulated this problem personally.

4.5.17 Let X be a compact space such that $C_p(C_p(X))$ is Lindelöf. Is it true that $C_p(X)$ is Lindelöf? Must X be ω-monolithic?

Published in Arhangel'skii (1990a)
Comment if $C_p(C_p(X))$ is a Lindelöf Σ-space then so is $C_p(X)$ (Problem 221)

4.5.18 Suppose that $C_p(X)$ is Lindelöf and K is a compact subspace of $C_p(X)$. Is it true that $t(K) = \omega$?

Published in Arhangel'skii (1990a) (attributed to Okunev)
Motivated by the fact that it is true if $C_p(X)$ is a Lindelöf Σ-space
Comment this is true under PFA (Arhangel'skii (1998b))

4.5.20 Let X be a monolithic compact space of countable tightness. Suppose that every linearly orderable compact subspace of X is metrizable. Is it true in ZFC that $C_p(X)$ must be Lindelöf?

Published in Arhangel'skii (1992b)
Motivated by the fact that, in a Corson compact space, every closed linearly orderable subspace is metrizable (Problem 082)

4.6 Extral and Extendial Spaces

Every embedding of an absolute retract is an l-embedding; so the notion of an extral space is a generalization of the concept of an absolute retract. This generalization turned out to be quite ample, because the class of extral spaces contains all compact metrizable spaces. Another non-trivial result is that extendial spaces form a larger class than the class of metrizable spaces. Being extral or extendial imposes very strong restrictions on a space; since these restrictions often imply metrizability, a basic question about a given extral/extendial space is whether it is metrizable.

4.6.1 Suppose that X is a space which is t-extral and t-extendial. Must X be metrizable?

Published in Arhangel'skii (1998b)
Motivated by the theorem which says that every extral t-extendial space must be metrizable (Problem 488)

4.6.2 Is it true that $\mathbb{D}^{\omega_1}$ is t-embedded in $[0, 1]^{\omega_1}$?

Published in Arhangel'skii (1998b)
Motivated by the fact that $\mathbb{D}^{\kappa}$ is not l-embeddable in $[0, 1]^{\kappa}$ for any uncountable cardinal κ (Problems 483 and 493)

4.6.3 Is it true that $C_p(\mathbb{D}^{\omega_1})$ can be embedded in $C_p([0, 1]^{\omega_1})$?

Published in Arhangel'skii (1998b)
Motivated by the fact that there is no linear embedding of $C_p(\mathbb{D}^{\kappa})$ in $C_p([0, 1]^{\kappa})$ for any uncountable cardinal κ.

4.6.4 Suppose that $\varphi : C_p(\mathbb{D}^{\omega_1}) \to C_p([0, 1]^{\omega_1})$ is a continuous map. Is it true that $\varphi(C_p(\mathbb{D}^{\omega_1}))$ has a countable network?

Published in Arhangel'skii (1998a)
Motivated by the fact that if $\varphi : C_p(\mathbb{D}^{\omega_1}) \to C_p([0, 1]^{\omega_1})$ is a linear continuous map then $nw(\varphi(C_p(\mathbb{D}^{\omega_1}))) = \omega$.

4.6.5 Can every compact space be t-embedded (or l-embedded) in a homogeneous compact space? In particular, given an uncountable cardinal κ, does there exist a t-embedding of the space $A(\kappa)$ in a homogeneous compact space?

Published in Arhangel'skii (1998b)
Motivated by the fact that each metrizable compact space is l-embedded in any compact space containing it as a subspace.

4.6.6 Is it true that, for every compact X, the space $C_p(X)$ can be (linearly) embedded in $C_p(Y)$ for some homogeneous compact space Y?

Published in Arhangel'skii (1998b)

Motivated by the fact that each metrizable compact space is l-embedded in any compact space containing it as a subspace.

4.6.7 Let κ be an infinite cardinal number. Is it true that there exists a compact space $U(\kappa)$ of weight κ such that every compact space of weight not exceeding κ can be l-embedded (or t-embedded) in $U(\kappa)$?

Published in Arhangel'skii (1998b)
Motivated by the fact that this is true for $\kappa = \omega$.

4.6.8 Let κ be an infinite cardinal number. Is it true in ZFC that there exists a compact space $W(\kappa)$ of weight κ such that, for every compact space X with $w(X) \leq \kappa$, there exists a (linear) embedding of $C_p(X)$ in $C_p(W(\kappa))$?

Published in Arhangel'skii (1998b)
Motivated by the fact that this is true for $\kappa = \omega$.
Comment this is true under GCH (Esenin-Vol'pin (1949))

4.6.9 Is $\mathbb{D}^{\omega_1}$ a t-extral space?

Published in Arhangel'skii (1992b)
Motivated by the fact that $\mathbb{D}^{\omega_1}$ is not extral (Problem 493)

4.6.10 Let X be a non-metrizable t-extral space. Is it true that $\mathbb{D}^{\omega_1}$ (or $[0,1]^{\omega_1}$) embeds in X?

Published in Arhangel'skii (1992b)
Motivated by the analogous theorem for l-dyadic compact spaces (Problem 496).

4.6.11 Let X be a t-extral space of countable tightness. Is it true that X is metrizable?

Published in Arhangel'skii (1992b)
Motivated by the analogous theorem for l-dyadic compact spaces (Problem 496).

4.6.12 Let X be an ω-monolithic t-extral space. Is it true that X is metrizable?

Published in Arhangel'skii (1992b)
Motivated by the fact that any Corson t-extral space is metrizable (Problems 141 and 480).

4.6.13 Is it true that any continuous image of an extendial (t-extendial) compact space is extendial (t-extendial respectively)?

Published in Arhangel'skii and Choban (1992)
Motivated by the fact that any continuous image of a t-extendial compact space has some strong extension properties described in Arhangel'skii and Choban (1992)

4.6.14 Let X be a separable compact space such that $X \times X$ is t-extendial. Is it true in ZFC that X is metrizable?

Published in Arhangel'skii and Choban (1989)
Motivated by the fact that it is true under MA+¬CH (Problem 458).

4.7 Point-Finite Cellularity and Calibers

Every space $C_p(X)$ has the Souslin property; however ω_1 need not be a caliber of $C_p(X)$ and point-finite cellularity of $C_p(X)$ is not necessarily countable. This fact triggered an extensive study of calibers and point-finite cellularity in C_p-theory. Most of the open questions in this area are consequences of too little knowledge we have about the spaces X such that $C_p(X)$ has countable point-finite cellularity.

4.7.1 Is it true that $p(C_p(X)) = p((C_p(X))^\kappa)$ for any cardinal κ?

Published in Tkachuk (1984c)
Motivated by the fact that $p(C_p(X)) = p((C_p(X))^\mathfrak{c})$ and the positive answer for the case when $p(C_p(X)) = \omega$ (Tkachuk (1984c))

4.7.2 Is it consistent with ZFC that any perfectly normal compact space X with $p(C_p(X)) = \omega$ is metrizable?

Published in Okunev and Tkachuk (2001)
Related to Problem 287

4.7.3 Is there a ZFC example of a non-metrizable first countable compact space X such that $p(C_p(X)) = \omega$?

Published in Okunev and Tkachuk (2001)
Related to Problem 287

4.7.4 Let X be the Souslin continuum. Is it true that $p(C_p(X)) = \omega$?

Published in Okunev and Tkachuk (2001)
Related to Problem 287

4.7.5 Suppose that $p(C_p(X)) = \omega$. Is it true that $p(C_p(\upsilon X)) = \omega$?

Published in Okunev and Tkachuk (2001)
Related to Problems 228–232

4.7.6 Let X be a monolithic compact space with $p(C_p(X)) = \omega$. Must X be metrizable?

Published in Arhangel'skii (1992b)
Related to Problem 287

4.7.7 Suppose that ω_1 is a caliber of X. Is it true that all countably compact subspaces of $C_p(X)$ are metrizable?

Published in Tkachuk (1988)
Motivated by the fact that in some models of ZFC all compact subsets of $C_p(X)$ are metrizable (Problems SFFS-294–299) while this is not true (in ZFC) for pseudocompact subspaces of $C_p(X)$.

4.8 Grothendieck Spaces

A subset A of a space X is called *countably compact in* X if every infinite subset of A has an accumulation point in X. Call X *a* g*-space* if $\overline{A}$ is compact for every $A \subset X$, which is countably compact in X. A space X is called *Grothendieck* if $C_p(X)$ is a hereditarily g-space. If $C_p(X)$ is a g-space then X is called *weakly Grothendieck*.

The classes described above were introduced by Arhangel'skii to generalize the theorem of Grothendieck for compact spaces. Observe that, in the new terminology, every compact space is Grothendieck.

4.8.1 Assume that X is a k-space (a space of countable tightness) with $l^*(X) = \omega$. Is then X a Grothendieck space?

Published in Arhangel'skii (1998b)
Motivated by the fact that every Lindelöf Σ-space is Grothendieck.

4.8.2 Let X be a Lindelöf first countable space. Is then X a Grothendieck space?

Published in Arhangel'skii (1998b)
Motivated by the fact that every Lindelöf Σ-space is Grothendieck.

4.8.3 Is the product of two Grothendieck spaces a (weakly) Grothendieck space?

Published in Arhangel'skii (1998b)
Motivated by the fact that the product of two weakly Grothendieck spaces is not necessarily weakly Grothendieck.

4.8.4 Let X be a (weakly) Grothendieck space. Is it true that $X \times K$ is a (weakly) Grothendieck space for any compact space K? What happens if $K = \mathbb{I}$?

Published in Arhangel'skii (1998b)
Motivated by the fact that the product of two weakly Grothendieck spaces is not necessarily weakly Grothendieck.

4.8.5 Let X and Y be Grothendieck spaces. Is it true that $X \oplus Y$ is a Grothendieck space?

Published in Arhangel'skii (1998b)
Motivated by the fact that the discrete union of any family of weakly Grothendieck spaces is weakly Grothendieck.

4.8.6 Let X and Y be Grothendieck spaces. Suppose that K is a compact subspace of $C_p(X)$ and L is a compact subspace of $C_p(Y)$. Must the space $K \times L$ be Fréchet–Urysohn?

Published in Arhangel'skii (1998b)
Motivated by the fact that K and L are Fréchet–Urysohn and this property is not preserved by products even in compact spaces.

4.8.7 Is the product of a Grothendieck space and a countable space a (weakly) Grothendieck space?

Published in Arhangel'skii (1998b)
Motivated by the fact that any countable space is Grothendieck and it is not known whether the product of two Grothendieck spaces is Grothendieck.

4.8.8 Must a perfect preimage of a (weakly) Grothendieck space be a (weakly) Grothendieck space?

Published in Arhangel'skii (1998b)
Motivated by the fact that very little is known about the categorical behavior of (weakly) Grothendieck spaces.

4.8.9 Is every hereditarily Grothendieck space hereditarily separable?

Published in Arhangel'skii (1998b)
Motivated by the fact that any hereditarily separable space is hereditarily Grothendieck and any hereditarily Grothendieck space has countable spread.

4.8.10 Is any perfectly normal compact space hereditarily Grothendieck?

Published in Arhangel'skii (1998b)
Comment any hereditarily separable space is hereditarily Grothendieck.

4.8.11 Suppose that X is a Lindelöf ω-stable space of countable tightness. Is it true that every continuous image of X is a Grothendieck space?

Published in Arhangel'skii (1997)
Motivated by the fact that this is true under PFA.

4.8.12 A space Z is *isocompact* if any countably compact closed subspace of Z is compact. Is it true that $C_p(X)$ is isocompact if and only if it is a g-space?

Published in Arhangel'skii (1997)
Motivated by the fact that any g-space is isocompact

4.9 Raznoie (Unclassified Questions)

It is usually impossible to completely classify a complex data set such as the open problems in C_p-theory. This last group of problems contains the open questions which do not fit into any of the previous Sections.

4.9.1 Suppose that every compact subspace of $C_p(X)$ has countable tightness (is Fréchet–Urysohn). Is it true that, for any compact K, all compact subspaces of $C_p(X \times K)$ have countable tightness (are Fréchet–Urysohn)?

Published in Arhangel'skii (1998b)
Related to some questions on Grothendieck spaces

4.9.2 Is the countable Fréchet–Urysohn fan embeddable in $C_p(X)$ for a Hurewicz space X?

Published in Arhangel'skii (1989a)
Motivated by the fact that this fan is not embeddable in $C_p(X)$ if all finite powers of X are Hurewicz (Problem 057).

4.9.3 Let X be a hereditarily Lindelöf space. Is it true that every compact subspace of $C_p(X)$ has countable tightness?

Published in Arhangel'skii (1998b)
Motivated by the fact this is true under PFA even for Lindelöf X (Problem 089)

4.9.4 Let X be a space of countable spread. Is it true that every compact subspace of $C_p(X)$ has countable tightness?

Published in Arhangel'skii (1998b)
Motivated by the fact this is true under PFA if X is Lindelöf (Problem 089)

4.9.5 Let X be a dyadic compact space such that $C_p(X)$ is (hereditarily) subparacompact. Is it true that X is metrizable?

Author V.V. Tkachuk
Motivated by the fact that if $C_p(X)$ is Lindelöf then X is metrizable (Problems 086–087)

4.9.6 Let X be a zero-dimensional compact space such that $C_p(X, \mathbb{D})$ is normal. Must the space $C_p(X, \mathbb{D})$ be Lindelöf?

Author V.V. Tkachuk
Related to Problem 032

4.9.7 Let X be a zero-dimensional (compact) space such that $C_p(X, \mathbb{D})$ is normal. Must the space $C_p(X, \mathbb{D}) \times C_p(X, \mathbb{D})$ be normal?

Author V.V. Tkachuk
Related to Problem 032

4.9.8 Let X be a zero-dimensional (compact) space such that $C_p(X, \mathbb{D})$ is Lindelöf. Must the space $C_p(X, \mathbb{D}) \times C_p(X, \mathbb{D})$ be Lindelöf (or metacompact or subparacompact)?

Author V.V. Tkachuk
Related to Problem 032

4.9.9 Let X be a zero-dimensional (compact) space such that $C_p(X, \mathbb{D})$ is metacompact. Must the space $C_p(X, \mathbb{D}) \times C_p(X, \mathbb{D})$ be metacompact?

Author V.V. Tkachuk
Related to Problem 032

4.9.10 Let X be a compact monolithic space. Is it true that $C_p(X)$ can be condensed onto a subspace of a Σ-product of real lines?

Published in Arhangel'skii (1992b)
Related to Problems 285 and 286

4.9.11 Is it possible to embed $C_p(\mathbb{I})$ in a pseudoradial space?

Published in Arhangel'skii (1992b)
Motivated by the fact that $C_p(\mathbb{I})$ is not embeddable in any sequential space (Problem 062).