Chapter 3
Bonus Results: Some Hidden Statements

The reader has, evidently, noticed that an essential percentage of the problems of the main text is formed by purely topological statements some of which are quite famous and difficult theorems. A common saying among C_p-theorists is that any result on C_p-theory contains only 20% of C_p-theory; the rest is general topology.

It is evident that the author could not foresee all topology which would be needed for the development of C_p-theory; so a lot of material had to be dealt with in the form of auxiliary assertions. After accumulating more than seven hundred such assertions, the author decided that some deserve to be formulated together to give a "big picture" of the additional material that can be found in solutions of problems.

This section presents 100 topological statements which were proved in the solutions of problems without being formulated in the main text. In these formulations the main principle is to make them clear for an average topologist. A student could lack the knowledge of some concepts of the formulation; so the index of this book can be used to find the definitions of the necessary notions.

After every statement we indicate the exact place (in this book) where it was proved. We did not include any facts from C_p-theory because more general statements are proved sooner or later in the main text.

The author considers that most of the results that follow are very useful and have many applications in topology. Some of them are folkloric statements and quite a few are published theorems, sometimes famous ones. For example, Fact 2 of U.086 is a famous result of Efimov (1977), Fact 1 of U.071 is a result of Arhangel'skii (1978b). Fact 1 of U.190 is a theorem of Yakovlev (1980) and Fact 4 of U.083 is a result of Shapirovsky (1974).

To help the reader find a result he/she might need, we have classified the material of this section according to the following topics: *standard spaces, metrizable spaces, compact spaces and their generalizations, properties of continuous maps, covering properties, normality and open families, completeness and convergence properties, ordered, zero-dimensional and product spaces, and cardinal invariants and set*

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theory. The author hopes that once we understand in which subsection a result should be, then it will be easy to find it.

3.1 Standard Spaces

By *standard spaces* we mean the real line, its subspaces and it powers, Tychonoff and Cantor cubes as well as ordinals together with the Alexandroff and Stone–Čech compactifications of discrete spaces.

U.074. Fact 1. *Given a regular uncountable cardinal κ suppose that $\lambda < \kappa$ and $C_\alpha \subset \kappa$ is a club for any $\alpha < \lambda$. Then $C = \bigcap\{C_\alpha : \alpha < \lambda\}$ is also a club.*

U.074. Fact 2. *Let κ be an uncountable regular cardinal. Then*

(i) if $A \subset \kappa$ is stationary then $|A| = \kappa$;
(ii) if $A \subset B \subset \kappa$ and A is stationary then B is also stationary;
(iii) if $A \subset \kappa$ is stationary and $C \subset \kappa$ is a club then $A \cap C$ is stationary;
(iv) given a cardinal $\lambda < \kappa$ suppose that $A_\alpha \subset \kappa$ for all $\alpha < \lambda$ and $\bigcup\{A_\alpha : \alpha < \lambda\}$ is stationary. Then A_α is stationary for some $\alpha < \lambda$.

U.074. Fact 3. *Suppose that κ is a regular uncountable cardinal and A is a stationary subset of κ. Assume that $f : A \to \kappa$ and $f(\alpha) < \alpha$ for any $\alpha \in A$. Then there is $\beta < \kappa$ such that the set $\{\alpha \in A : f(\alpha) = \beta\}$ is stationary.*

U.074. Fact 4. *For any ordinal ξ the space $\xi + 1$ is compact and scattered and hence $C_p(\xi + 1)$ is a Fréchet–Urysohn space.*

U.074. Fact 5. *If ξ is any ordinal then any closed non-empty $F \subset \xi$ is a retract of ξ, i.e., there exists a continuous map $r : \xi \to F$ such that $r(\alpha) = \alpha$ for any $\alpha \in F$.*

U.074. Fact 6. *If ξ is an ordinal such that cf(ξ) $> \omega$ then, for any second countable space M and a continuous map $f : \xi \to M$ there is $z \in M$ and $\eta < \xi$ such that $f(\alpha) = z$ for any $\alpha \in [\eta, \xi)$.*

U.074. Fact 8. *Given an ordinal ξ assume that $\kappa =$ cf(ξ) $\geq \omega$ and $\mu \in \xi$. Then there exists a map $f : \kappa \to [\mu, \xi)$ such that $\alpha < \beta < \kappa$ implies $f(\alpha) < f(\beta)$, the set $F = f[\kappa]$ is closed in ξ and $f : \kappa \to F$ is a homeomorphism. In particular, κ embeds in $[\mu, \xi)$ as a closed subspace.*

U.074. Fact 9. *For any ordinal α there exists a unique $n(\alpha) \in \omega$ and a unique limit ordinal $\mu(\alpha)$ such that $\alpha = \mu(\alpha) + n(\alpha)$.*

U.086. Fact 2. *If κ is a regular uncountable cardinal and $\varphi : \mathbb{D}^\kappa \to \mathbb{I}^\kappa$ is a continuous onto map then there is a closed $F \subset \mathbb{D}^\kappa$ such that $F \simeq \mathbb{D}^\kappa$ and $\varphi|F$ is injective.*

U.174. Fact 2. *Given an infinite set A and $n \in \omega$ let $\sigma_n(A) = \{x \in \mathbb{D}^A : |x^{-1}(1)| \leq n\}$. Then $\sigma_n(A)$ is a scattered compact space.*

U.176. Fact 1. *Suppose that $A \subset \omega_1$ is a stationary set such that $\omega_1 \backslash A$ is also stationary and let $T(A) = \{F \subset A : F$ is closed in $\omega_1\}$. Then all elements of $T(A)$ are countable and hence compact; given $F, G \in T(A)$ say that $F \leq G$ if F is an initial segment of G, i.e., for the ordinal $\alpha = \max(F)$, we have*

$G \cap (\alpha + 1) = F$. *Then* $(T(A), \leq)$ *is a tree which has no uncountable chains and no dense* σ*-antichains.*

U.176. Fact 2. *Given an infinite set* T *consider the set* $[P, Q] = \{x \in \mathbb{D}^T : x(P) \subset \{1\}$ *and* $x(Q) \subset \{0\}\}$ *for any disjoint finite sets* $P, Q \subset T$. *Suppose additionally that we have a family* $\mathcal{U} = \{[P_a, Q_a] : a \in A\}$ *such that the set* A *is infinite and* $\sup\{|P_a \cup Q_a| : a \in A\} < \omega$. *Then* $\mathcal{U}$ *is not disjoint.*

U.342. Fact 4. *Given an infinite cardinal* κ *let* $S_n = \{x \in \mathbb{D}^\kappa : |x^{-1}(1)| \leq n\}$ *for any* $n \in \omega$. *Then* S_n *is a continuous image of* $(A(\kappa))^n$ *for any* $n \in \omega$.

U.342. Fact 5. *If* $\kappa \leq \mathfrak{c}$ *is an infinite cardinal then* $A(\kappa)$ *is weakly metrizably fibered. As a consequence, if* A *is a set such that* $|A| \leq \mathfrak{c}$ *then any compact subspace of* $\sigma_0(A)$ *is weakly metrizably fibered.*

U.362. Fact 1. *Let* $u \in K = (A(\omega_1))^\omega$ *be the point with* $u(n) = a$ *for all* $n \in \omega$. *Then the space* $K \backslash \{u\}$ *is not metacompact.*

U.370. Fact 1. *If* $A \subset \omega_1$ *is a stationary set then* A *is not discrete as a subspace of* ω_1.

U.415. Fact 1. *For any space* Z *the set* $\mathbb{Q}^Z$ *is uniformly dense in* $\mathbb{R}^Z$, *i.e., for any* $f \in \mathbb{R}^Z$ *there is a sequence* $\{f_n : n \in \mathbb{N}\} \subset \mathbb{Q}^Z$ *which converges uniformly to* f.

U.426. Fact 1. *Suppose that* λ *is a cardinal with* $\lambda^\omega = \lambda$ *and* A *is a set such that* $|A| = \lambda$. *Then the space* $\mathbb{I}^A$ *has a strongly* σ*-discrete dense subspace of cardinality* λ.

U.463. Fact 1. *There exists a surjective map* $\xi : \mathbb{D}^\omega \to \omega^\omega$ *such that* $\xi^{-1}(U)$ *is an* F_σ*-set in* $\mathbb{D}^\omega$ *for any open* $U \subset \omega^\omega$.

U.471. Fact 1. *The topology of the double arrow space is zero-dimensional and generated by its lexicographical order. In particular, the double arrow space is a linearly ordered zero-dimensional perfectly normal, hereditarily separable non-metrizable compact space.*

U.481. Fact 3. *Suppose that* A *is a set and* $p \in \mathbb{I}^A$. *Assume that, for some* $n \in \mathbb{N}$, *we have subspaces* $K_0, \ldots, K_{n-1} \subset \mathbb{I}^A$ *such that every* K_i *is homeomorphic to* $A(\omega_1)$ *and* p *is the unique non-isolated point of* K_i; *let* $D_i = F_i \backslash \{p\}$. *Then, for any family* $\{U_0, \ldots, U_{n-1}\} \subset \tau(\mathbb{I}^A)$ *such that* $D_i \subset U_i$ *for all* $i < n$, *we have* $U_0 \cap \ldots \cap U_{n-1} \neq \emptyset$.

U.490. Fact 1. *The space* $A(\omega_1)$ *does not embed in a linearly ordered space.*

U.493. Fact 1. *For any set* A *both spaces* $\mathbb{R}^A$ *and* $\mathbb{I}^A$ *are connected.*

U.497. Fact 2. *The space* $\beta\mathbb{R}$ *is not metrizable and there are no non-trivial convergent sequences in* $\beta\mathbb{R} \backslash \mathbb{R}$.

3.2 Metrizable Spaces

The results of this Section deal with metrics, pseudometrics or metrizable spaces in some way. We almost always assume the Tychonoff separation axiom; so our second countable spaces are metrizable and hence present here too.

U.050. Fact 1. *Given a metric space (M, ρ) a family $\mathcal{C} \subset \tau(M)$ is a base in M if and only if, for any $\varepsilon > 0$ there is a collection $\mathcal{C}' \subset \mathcal{C}$ such that $\bigcup \mathcal{C}' = M$ and mesh$(\mathcal{C}') \leq \varepsilon$.*

U.050. Fact 2. *Suppose that Z is an arbitrary space. Then*

(1) for any pseudometrics d_1 and d_2 on the space Z, the function $d = d_1 + d_2$ is a pseudometric on the space Z;
(2) if d is a pseudometric on the space Z then $a \cdot d$ is a pseudometric on Z for any $a > 0$;
(3) for any pseudometrics d_1 and d_2 on the space Z, the function $d = \max\{d_1, d_2\}$ is a pseudometric on the space Z;
(4) if d_1 is a pseudometric on the space Z and $a > 0$ then the function $d : Z \times Z \to \mathbb{R}$ defined by $d(x, y) = \min\{d_1(x, y), a\}$ for all $x, y \in Z$ is a pseudometric on Z;
(5) if, for any $i \in \omega$, a function d_i is a pseudometric on the space Z and $d_i(x, y) \leq 1$ for any $x, y \in Z$ then $d = \sum_{i \in \omega} 2^{-i} \cdot d_i$ is a pseudometric on Z;
(6) if $f : Z \to \mathbb{R}$ is a continuous function then the function $d : Z \times Z \to \mathbb{R}$ defined by $d(x, y) = |f(x) - f(y)|$ for any $x, y \in Z$ is a pseudometric on the space Z;
(7) if d_1 is a metric and d_2 is a pseudometric on the space Z then $d = d_1 + d_2$ is a metric on the space Z.

U.062. Fact 1. *Let A be a non-empty closed subspace of a metrizable space M. Then there exists a continuous linear map $e : C_p(A) \to C_p(M)$ such that $e(f)|A = f$ for any $f \in C_p(A)$.*

U.062. Fact 2. *Suppose that M is a metrizable space and $A \subset M$ is a non-empty closed subset of M; let $I_A = \{f \in C_p(M) : f|A \equiv 0\}$. Then there exists a linear homeomorphism between $C_p(M)$ and $C_p(A) \times I_A$ and, in particular, $C_p(A)$ embeds in $C_p(M)$ as a closed linear subspace.*

U.094. Fact 1. *For any second countable space Z there is a countable space T such that Z embeds in $C_p(T)$ as a closed subspace.*

U.138. Fact 1. *If a space has a dense metrizable subspace then it has a σ-disjoint π-base. For first countable spaces the converse is also true, i.e., a first countable space Z has a dense metrizable subspace if and only if it has a σ-disjoint π-base.*

U.318. Fact 1. *A space Z can be condensed onto a metrizable space if and only if it has a G_δ-diagonal sequence $\{\mathcal{U}_n : n \in \omega\}$ such that $\mathcal{U}_{n+1}$ is a star refinement of $\mathcal{U}_n$ for any $n \in \omega$.*

U.347. Fact 1. *Suppose that* Z *is a space,* (M, ρ) *is a metric space and we have a map* $f : Z \to M$. *Then*

(a) for any $\varepsilon > 0$ *the set* $O_\varepsilon = \{z \in Z : osc(f, z) < \varepsilon\}$ *is open in* Z*;*
(b) the map f *is continuous at a point* $z \in Z$ *if and only if* $osc(f, z) = 0$.

3.3 Compact Spaces and Their Generalizations

This Section contains some statements on compact, countably compact and pseudocompact spaces.

U.039. Fact 1. *Any perfect preimage of a countably compact space is countably compact.*

U.071. Fact 1. *If MA+¬CH holds then any compact space X of weight at most ω_1 is pseudoradial. In particular, $\mathbb{D}^{\omega_1}$ is pseudoradial under MA+¬CH.*

U.072. Fact 1. *Let λ be an infinite cardinal. If X is a dyadic space such that the set $C = \{x \in X : \pi\chi(x, X) \le \lambda\}$ is dense in X then $w(X) \le \lambda$. In particular, if X has a dense set of points of countable π-character then X is metrizable.*

U.072. Fact 2. *Under CH every pseudoradial dyadic space is metrizable.*

U.077. Fact 1. *If K is an infinite compact space then $|\mathcal{C}(K)| \le w(K)$. In particular, for any metrizable compact K the family of all clopen subsets of K is countable.*

U.080. Fact 1. *If K is a compact ω-monolithic space of countable tightness then K is Fréchet–Urysohn and has a dense set of points of countable character. This is true in ZFC, i.e., no additional axioms are needed for the proof of this Fact.*

U.086. Fact 1. *Given a regular uncountable cardinal κ if K is a compact space such that $\pi\chi(x, K) \ge \kappa$ for any $x \in K$ then there is a closed $P \subset K$ which maps continuously onto $\mathbb{D}^\kappa$ and hence K maps continuously onto $\mathbb{I}^\kappa$.*

U.086. Fact 3. *If K is a dyadic space and $w(K) > \kappa$ for some infinite cardinal κ then $\mathbb{D}^{\kappa^+}$ embeds in K.*

U.104. Fact 1. *Suppose that K is a non-empty compact space with no points of countable character. Then K cannot be represented as a union of $\le \omega_1$-many cosmic subspaces.*

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U.174. Fact 1. *Let X be a countably compact σ-discrete space, i.e., $X = \bigcup_{n\in\omega} X_n$ where each X_n is a discrete subspace of X. Then X is scattered.*

U.185. Fact 1. *Suppose that Z is a compact space, F is non-empty and closed in Z and, additionally, there is a point-countable open cover $\mathcal{U}$ of the set $Z\backslash F$ such that $\overline{U} \subset Z\backslash F$ for any $U \in \mathcal{U}$. Then F is a W-set in Z.*

U.222. Fact 1. *Suppose that K is a compact space and some $x \in K$ is not a π-point. Then $K\backslash\{x\}$ is pseudocompact and K is canonically homeomorphic to $\beta(K\backslash\{x\})$, i.e., there exists a homeomorphism $\varphi : \beta(K\backslash\{x\}) \to K$ such that $\varphi(y) = y$ for any $y \in K\backslash\{x\}$.*

U.337. Fact 1. *Let K be an infinite compact space with $|K| = \kappa$. Suppose that U is an open subset of K and F is a closed subset of the subspace U. Then there exist families $\{S(\alpha, n) : \alpha < \kappa,\ n \in \omega\}$ and $\{V(\alpha, n) : \alpha < \kappa,\ n \in \omega\}$ with the following properties:*

(i) *the set $S(\alpha, n)$ is compact and $S(\alpha, n) \subset F$ for any $\alpha < \kappa$ and $n \in \omega$;*
(ii) *$V(\alpha, n)$ is open in K and $S(\alpha, n) \subset V(\alpha, n) \subset U$ for any $\alpha < \kappa$ and $n \in \omega$;*
(iii) *$\bigcup\{S(\alpha, n) : \alpha < \kappa,\ n \in \omega\} = F$;*
(iv) *if $\alpha < \beta < \kappa$ then $S(\beta, n) \cap V(\alpha, m) = \emptyset$ for any $m, n \in \omega$.*

U.342. Fact 1. *Suppose that K is compact and $K = \bigcup\{K_n : n \in \omega\}$ where K_n is weakly metrizably fibered and closed in K for any $n \in \omega$. Then K is also weakly metrizably fibered.*

U.342. Fact 2. *If K_n is a compact weakly metrizably fibered space for any $n \in \omega$ then $K = \prod_{n\in\omega} K_n$ is also weakly metrizably fibered.*

U.342. Fact 3. *Any closed subspace and any continuous image of a compact weakly metrizably fibered space is weakly metrizably fibered.*

U.417. Fact 1. *If Z is a pseudocompact space and $\psi(z, Z) \le \omega$ for some $z \in Z$ then $\chi(z, Z) \le \omega$.*

U.417. Fact 2. *If Z is a pseudocompact first countable space then, for any $Y \subset Z$ with $|Y| \le \mathfrak{c}$ there is $Y' \subset Z$ such that $|Y'| \le \mathfrak{c}$, the space Y' is pseudocompact and $Y \subset Y'$.*

U.481. Fact 1. *Given compact spaces X and Y suppose that $\varphi : C_p(X) \to C_p(Y)$ is a continuous linear map. Then the map $\varphi : C_u(X) \to C_u(Y)$ is continuous as well.*

U.497. Fact 1. *If Z is a space and Y is a dense locally compact subspace of Z then Y is open in Z.*

3.4 Properties of Continuous Maps

We consider the most common classes of continuous maps: open, closed, perfect and quotient. The respective results basically deal with preservation of topological properties by direct and inverse images.

U.074. Fact 11. *Given spaces Y and Z assume that $f : Y \to Z$ is a continuous map such that there is $P \subset Y$ for which $f(P) = Z$ and $f|P : P \to Z$ is a quotient map. Then f is quotient. In particular, any retraction is a quotient map.*

U.077. Fact 2. *Given spaces Z, T and a continuous map $f : Z \to T$, for any $B \subset T$, the set $G(f, B) = \{(z, f(z)) : z \in f^{-1}(B)\} \subset Z \times B$ is closed in $Z \times B$.*

U.093. Fact 2. *Suppose that $\varphi_t : E_t \to M_t$ is a compact-valued upper semicontinuous onto map for any $t \in T$. Let $E = \prod_{t\in T} E_t$, $M = \prod_{t\in T} M_t$ and define a multi-valued map $\varphi = \prod_{t\in T} \varphi_t : E \to M$ by $\varphi(x) = \prod_{t\in T} \varphi_t(x(t))$ for any $x \in E$. Then $\varphi : E \to M$ is a compact-valued upper semicontinuous onto map.*

U.481. Fact 4. *Given a space X suppose that $F \subset X$ is C^*-embedded in X and we have a continuous map $f : F \to \mathbb{I}^A$ for some A. Then there exists a continuous map $g : X \to \mathbb{I}^A$ such that $g|F = f$. In particular, if the space X is normal and F is closed in X then any continuous map from F to a Tychonoff cube can be continuously extended to the whole space X.*

U.481. Fact 6. *Let $\mathcal{U} = \{U_n : n \in \omega\}$ be disjoint family of non-empty open subsets of a space Z. Assume that $f_n \in C^*(Z)$ and $f_n(Z \setminus U_n) \subset \{0\}$ for all $n \in \omega$. If, additionally, the sequence $\{||f_n|| : n \in \omega\}$ converges to zero then the function $f = \sum_{n\in\omega} f_n$ is continuous on Z.*

U.493. Fact 2. *Any continuous image of a connected space is connected. As a consequence, if X is connected, Y is zero-dimensional and $f : X \to Y$ is a continuous map then $f(X)$ is a singleton.*

3.5 Covering Properties, Normality and Open Families

This section contains results on the covering properties which are traditionally considered not to be related to compactness, such as the Lindelöf property, paracompactness and its derivatives.

U.082. Fact 1. *If Z is a space and $l(Z) \le \kappa$ for some infinite cardinal κ then any indexed set $Y = \{y_\alpha : \alpha < \kappa^+\} \subset Z$ has a complete accumulation point, i.e., there is $z \in Z$ such that $|\{\alpha < \kappa^+ : y_\alpha \in U\}| = \kappa^+$ for any $U \in \tau(z, Z)$.*

U.093. Fact 3. *If Z^ω is Lindelöf then $Z^\omega \times T$ is also Lindelöf for any K-analytic space T.*

U.095. Fact 1. *Assume that we have an uncountable space Z such that $w(Z) \le \mathfrak{c}$ and there is a countable $Q \subset Z$ such that Z is concentrated around Q, i.e., $|Z \backslash U| \le \omega$ for any $U \in \tau(Q, Z)$. Then the Continuum Hypothesis (CH) implies that there is an uncountable $T \subset Z$ such that $Q \subset T$ and T^n is Lindelöf for any $n \in \mathbb{N}$.*

U.102. Fact 1. *Given a space Z, any σ-locally finite open cover of Z has a locally finite refinement.*

U.102. Fact 2. *Suppose that Z is a space and $\mathcal{F}$ is a discrete family of closed subsets of Z. If there exists a locally finite closed cover $\mathcal{C}$ of the space Z such that every $C \in \mathcal{C}$ meets at most one element of $\mathcal{F}$ then the family $\mathcal{F}$ is open-separated, i.e., for any $F \in \mathcal{F}$ we can choose $O_F \in \tau(F, Z)$ such that the family $\{O_F : F \in \mathcal{F}\}$ is disjoint.*

U.175. Fact 1. *If Z is an uncountable space which is a continuous image of a Lindelöf k-space then there is an infinite compact $K \subset Z$.*

U.177. Fact 1. *In a Lindelöf space Z every uncountable $A \subset Z$ has a condensation point, i.e., there is $z_0 \in Z$ for which $|A \cap U| > \omega$ for any $U \in \tau(z_0, Z)$. In addition, if Z is a space with $l(Z) \le \omega_1$ then Z is Lindelöf if and only if every uncountable $A \subset Z$ has a condensation point.*

U.177. Fact 2. *If Z is a space with strong condensation property and $l(Z^\omega) \le \omega_1$ then Z^ω is Lindelöf.*

U.188. Fact 1. *Given a space Z suppose that $\mathcal{U}$ is an open cover of Z such that $\overline{U}$ is Lindelöf for any $U \in \mathcal{U}$. Then $\mathcal{U}$ can be shrunk, i.e., for any $U \in \mathcal{U}$ there is a closed set $F_U \subset U$ such that $\{F_U : U \in \mathcal{U}\}$ is a cover of Z.*

U.193. Fact 3. *A space Z is hereditarily normal if and only if any pair of separated subsets of Z are open-separated.*

U.271. Fact 1. *Given a space Z and a family $\mathcal{U} \subset \tau^*(Z)$ there is a discrete $D \subset Z$ such that $\bigcup\{U \in \mathcal{U} : D \cap U \neq \emptyset\} = \bigcup \mathcal{U}$.*

U.271. Fact 2. *Suppose that λ is an infinite cardinal, Z is a space and $\mathcal{B} \subset \tau^*(Z)$ is a family with $ord(z, \mathcal{B}) < \lambda$ for any $z \in Z$. Then there exists a family $\{D_\alpha : \alpha < \lambda\}$ of discrete subspaces of Z such that, for the set $D = \bigcup\{D_\alpha : \alpha < \lambda\}$, we have $D \cap B \neq \emptyset$ for any $B \in \mathcal{B}$. In particular, if $\mathcal{B}$ is point-finite then there is a σ-discrete subset of Z which is dense in $\mathcal{B}$.*

U.284. Fact 1. *Suppose that Z is a Lindelöf Σ-space, $Y \subset Z$ and there is a countable family $\mathcal{A}$ of Lindelöf Σ-subspaces of Z that separates Y from $Z \backslash Y$. Then Y is a Lindelöf Σ-space.*

U.285. Fact 5. *Suppose that X is a Lindelöf Σ-space and $\mathcal{F}$ is a fixed countable network with respect to a compact cover $\mathcal{C}$ of the space X. Assume additionally that $\mathcal{F}$ is closed under finite intersections and $f : X \to Y$ is a continuous onto map. If $A \subset X$ is a set such that $f(A \cap F)$ is dense in $f(F)$ for any $F \in \mathcal{F}$ then $f(\overline{A}) = Y$.*

U.363. Fact 1. *Suppose that a space X has an open σ-point-finite cover $\mathcal{U}$ such that $\overline{U}$ is compact for any $U \in \mathcal{U}$. Then X is σ-metacompact.*

3.6 Completeness and Convergence Properties

This Section deals mainly with Čech-complete spaces. Some results on convergence properties are presented as well.

U.061. Fact 1. *If Z is a sequential space, $A \subset Z$ and $z \in \overline{A} \setminus A$ then z has a countable π-network in Z which consists of infinite subsets of A.*

U.074. Fact 10. *For any space Z, if $C_p(Z)$ is Fréchet–Urysohn and $Y \neq \emptyset$ is an F_σ-subset of Z then $C_p(Y)$ is also Fréchet–Urysohn.*

U.421. Fact 3. *If X is Čech-complete and $\psi(x, X) \leq \omega$ for some $x \in X$ then $\chi(x, X) \leq \omega$. In particular, any splittable Čech-complete space is first countable.*

U.421. Fact 7. *If X is a Čech-complete paracompact space with a G_δ-diagonal then X is metrizable.*

U.429. Fact 1. *For any Čech-complete space X there exists a dense paracompact subspace $Y \subset X$ such that Y is a G_δ-set in X (and hence Y is Čech-complete).*

3.7 Ordered, Zero-Dimensional and Product Spaces

The space $C_p(X)$ being dense in $\mathbb{R}^X$, the results on topological products form a fundamental part of C_p-theory. The main line here is to classify spaces which could be embedded in (or expressed as a continuous image of) a nice subspace of a product.

U.003. Fact 1. *If* $\dim Z_t = 0$ *for any* $t \in T$ *and* $Z = \bigoplus\{Z_t : t \in T\}$ *then* $\dim Z = 0$.

U.067. Fact 1. *For an infinite cardinal* κ *let* $\preceq$ *be the lexicographic order on the set* $D_\kappa = \kappa \times \mathbb{Z}$*, i.e., for any* $a, b \in D_\kappa$ *such that* $a = (\alpha, n),\ b = (\beta, m)$ *let* $a \prec b$ *if* $\alpha < \beta$*; if* $\beta < \alpha$ *then we let* $b \prec a$*. Now if* $\alpha = \beta$ *then* $a \preceq b$ *if* $n \leq m$ *and* $b \preceq a$ *if* $m \leq n$*. Then* $\preceq$ *is a linear order on* D_κ *and the space* $(D_\kappa, \tau(\preceq))$ *is discrete. Besides, if* $\kappa > \omega$ *then* $|\{a \in D_\kappa : a \preceq b\}| < \kappa$ *for any* $b \in D_\kappa$*. In particular, any discrete space* X *is linearly orderable.*

U.067. Fact 2. *Suppose that, for every* $t \in T$*, the topology of a space* X_t *can be generated by a linear order* $\preceq_t$ *which has a maximal and a minimal element. Then the space* $X = \bigoplus\{X_t : t \in T\}$ *is linearly orderable.*

U.067. Fact 3. *Suppose that* $X = \{x\} \cup \{x_\alpha : \alpha < \kappa\}$ *where* κ *is an infinite regular cardinal, the enumeration of* X *is faithful and* x *is the unique non-isolated point of* X*. For every* $\alpha < \kappa$ *let* $O_\alpha = \{x\} \cup \{x_\beta : \beta \geq \alpha\}$*. If the family* $\{O_\alpha : \alpha < \kappa\}$ *is a local base at* x *in* X *then there is a linear order* $\preceq$ *on* X *such that* $\tau(\preceq) = \tau(X)$*, the point* x *is the maximal element of* $(X, \preceq)$ *and* x_0 *is its minimal element.*

U.104. Fact 2. *Suppose that* N_t *is a cosmic space for each* $t \in T$ *and take any point* $u \in N = \prod\{N_t : t \in T\}$*. If* $|T| \leq \omega_1$ *then* $\Sigma(N, u)$ *is a union of* $\leq \omega_1$*-many cosmic spaces.*

U.190. Fact 1. *Any subspace of a* σ*-product of second countable spaces is metacompact.*

U.359. Fact 4. *Suppose that a zero-dimensional first countable* X *is strongly homogeneous, i.e., any non-empty clopen subset of* X *is homeomorphic to* X*. Then* X *is homogeneous.*

U.366. Fact 4. *Any* σ*-product of compact metrizable spaces must have a closure-preserving cover by metrizable compact subspaces.*

3.8 Cardinal Invariants and Set Theory

To classify function spaces using cardinal invariants often gives crucial information. This Section includes both basic, simple results on the topic as well as very difficult classical theorems.

U.003. Fact 2. *Given an infinite cardinal* κ *a space* Z *is zero-dimensional and* $w(Z) \le \kappa$ *if and only if* Z *is homeomorphic to a subspace of* $\mathbb{D}^\kappa$.

U.027. Fact 1. *For any space* T *and a closed* $F \subset T$ *we have* $\psi(F, T) \le l(T \setminus F)$. *In particular,* $\psi(t, T) \le l(T \setminus \{t\})$ *for any* $t \in T$.

U.074. Fact 7. *For any space* Z *we have* $|C_p(Z)| \le w(Z)^{l(Z)}$.

U.083. Fact 1. *Given a space* Z *and an infinite cardinal* κ *suppose that* $\psi(Z) \le 2^\kappa$ *and* $l(Z) \cdot t(Z) \le \kappa$. *Assume additionally that* $|\overline{A}| \le 2^\kappa$ *for any* $A \subset Z$ *with* $|A| \le \kappa$. *Then* $|Z| \le 2^\kappa$.

U.083. Fact 2. *Given a space* Z *let* $\mathcal{R}(Z)$ *be the family of all regular open subsets of* Z, *i.e.,* $\mathcal{R}(Z) = \{U \in \tau(Z) : U = \mathrm{Int}(\overline{U})\}$. *Then* $|\mathcal{R}(Z)| \le \pi w(Z)^{c(Z)}$.

U.083. Fact 3. *For any space* Z *we have* $\pi w(Z) \le \pi\chi(Z) \cdot d(Z)$.

U.083. Fact 4. *For any space* Z *we have* $w(Z) \le \pi\chi(Z)^{c(Z)}$.

U.127. Fact 1. *If a space* Z *is* κ*-monolithic and* $s(Z) \le \kappa$ *for some infinite cardinal* κ *then* $hl(Z) \le \kappa$.

U.274. Fact 1. *If* Z *is a space with a unique non-isolated point then* $Z \oplus \{t\} \simeq Z$ *for any* $t \notin Z$.

U.337. Fact 2. *If* n *is a natural number and* T *is an infinite set then for any family* $\mathcal{N} = \{N_t : t \in T\}$ *such that* $|N_t| \le n$ *for any* $t \in T$, *there is a set* D *and an infinite* $T' \subset T$ *such that* $N_s \cap N_t = D$ *for any distinct* $s, t \in T'$.

U.381. Fact 1. *For any* $n, l \in \omega$ *there exists a number* $g(n, l) \in \omega$ *such that, for any family* $\mathcal{N} = \{N_t : t \in T\}$ *with* $|N_t| \le n$ *for any* $t \in T$, *if* $|T| \ge g(n, l)$ *then there is* $S \subset T$ *such that* $|S| \ge l$ *and* $\{N_t : t \in S\}$ *is a* Δ*-system, i.e., there is a set* D *such that* $N_s \cap N_t = D$ *for any distinct* $s, t \in S$.

U.418. Fact 1. *If* Z *is a space then* $|Z| \le nw(Z)^{\psi(Z)}$.