Chapter 1
Behavior of Compactness in Function Spaces

The reader who has found his (or her) way through the first thousand problems of this book is fully prepared to enjoy working professionally in C_p-theory. Such a work implies choosing a topic, reading the papers with the most recent progress thereon and attacking the unsolved problems. Now, the first two steps are possible without doing heavy library work, because Chapter 1 provides information on the latest advances in all areas of C_p-theory, where compactness is concerned. Here, many ideas, results and constructions came from functional analysis giving a special flavor to this part of C_p-theory, but at the same time making it more difficult to master. I must warn the reader that most topics, outlined in the forthcoming bulk of 500 problems, constitute the material of important research papers—in many cases very difficult ones. The proofs and solutions, given in Chapter 2, are complete, but sometimes they require a very high level of understanding of the matter. The reader should not be discouraged if some proofs seem to be unfathomable. We still introduce new themes in general topology and formulate, after a due preparation, some non-trivial results which might be later used in C_p-theory.

Section 1.1 contains general facts on "nice" behavior of $C_p(X)$ and its subspaces when X has some compactness-like property. There are two statements which deserve to be called the principal ones: Arhangel'skii's theorem on compact subspaces of $C_p(X)$ for X Lindelöf, under PFA (Problem 089) and Okunev's theorem on Lindelöf subspaces of $C_p(X)$ for a compact separable space X, under MA+$\neg$CH (Problem 098).

Section 1.2 deals with Corson compact spaces and their applications. The most important results include a theorem of Gul'ko, Michael and Rudin which states that any continuous image of a Corson compact space is Corson compact (Problem 151), Sokolov's example of a Corson compact space which is not Gul'ko compact (Problem 175) and Sokolov's theorem on Lindelöf property in iterated function spaces of a Corson compact space (Problems 160 and 162).

In Section 1.3 we present the latest achievements in the exploration of Lindelöf Σ-property in X and $C_p(X)$. Many results of this section are outstanding. We would

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like to mention Gul'ko's theorem which says that every Gul'ko compact is Corson compact (Problem 285), Okunev's theorems on Lindelöf Σ-property in iterated function spaces (Problems 218 and 219), Leiderman's theorem on existence of dense metrizable subspaces of Gul'ko compact spaces (Problem 293) and Reznichenko's example of a Gul'ko compact space which is not Preiss–Simon (Problem 222).

Section 1.4 outlines the main topics in the theory of Eberlein compact spaces. Here, the bright results are also numerous. It is worth mentioning a theorem of Amir and Lindenstrauss on embeddings of Eberlein compact spaces in $\Sigma_*(A)$ (Problem 322), Rosenthal's characterization of Eberlein compact spaces (Problem 324), a theorem of Benyamini, Rudin, Wage and Gul'ko on continuous images of Eberlein compact spaces (Problem 337), Gruenhage's characterization of Eberlein compact spaces (Problem 364) and Grothendieck's theorem on equivalence of the original definition of Eberlein compact spaces to the topological one (Problem 400).

Section 1.5 is devoted to the study of splittable spaces and embeddings which admit nice extension operators. It has two main results: Arhangel'skii and Shakhmatov's theorem on pseudocompact splittable spaces (Problem 417) and a theorem of Arhangel'skii and Choban on compactness of every t-extral space (Problem 475).

1.1 The Spaces $C_p(X)$ for Compact and Compact-Like X

Given an infinite cardinal κ, call a space X *κ-monolithic* if, for any $Y \subset X$ with $|Y| \leq \kappa$, we have $nw(\overline{Y}) \leq \kappa$. A space is *monolithic* if it is κ-monolithic for any κ. A space X is zero-dimensional if it has a base whose elements are clopen. The space X is *strongly zero-dimensional (this is denoted by* $\dim X = 0$*)* if any finite open cover of X has a disjoint refinement. If we have topologies τ and μ on a set Z then τ is *stronger than* μ if $\tau \supset \mu$. If $\tau \subset \mu$ then τ is said to be *weaker than* μ.

Given a space X and $A \subset C(X)$, denote by $\mathrm{cl}_u(A)$ the set $\{f \in C(X) :$ there exists a sequence $\{f_n : n \in \omega\} \subset A$ such that $f_n \rightrightarrows f\}$. If X is a space, call a set $A \subset C(X)$ *an algebra*, if A contains all constant functions and $f + g \in A$, $f \cdot g \in A$ whenever $f, g \in A$. If we have spaces X and Y say that a set $A \subset C(X, Y)$ *separates the points of* X if, for any distinct $x, y \in X$, there is $f \in A$ with $f(x) \neq f(y)$. The set A *separates the points and the closed subsets of* X if, for any closed $F \subset X$ and any $x \in X \setminus F$ there is $f \in A$ such that $f(x) \notin \overline{f(F)}$.

The space $\mathbb{D}$ is the two-point set $\{0, 1\}$ endowed with the discrete topology. If T is a set and $S \subset T$ then $\chi_S^T : T \to \mathbb{D}$ is *the characteristic function of* S *in* T defined by $\chi_S^T(x) = 1$ for all $x \in S$ and $\chi_S^T(x) = 0$ whenever $x \in T \setminus S$. If the set T is clear we write χ_S instead of χ_S^T. A map $f : X \to Y$ is *finite-to-one* if $f^{-1}(y)$ is finite (maybe empty) for any $y \in Y$.

If X is a space and $A, B \subset C_p(X)$ let $\mathrm{MIN}(A, B) = \{\min(f, g) : f \in A,\ g \in B\}$ and $\mathrm{MAX}(A, B) = \{\max(f, g) : f \in A,\ g \in B\}$. For any $n \in \mathbb{N}$ consider the set $G_n(A) = \{af + bg : a, b \in [-n, n],\ f, g \in A\}$. Given $Y \subset C_p(X)$ we let $\mathcal{S}_1(Y) = \{Y\}$. If we have $\mathcal{S}_k(Y)$ for some $k \in \mathbb{N}$, let $\mathcal{S}_{k+1}(Y) = \{\mathrm{MIN}(A, B) : A, B \in \mathcal{S}_k(Y)\} \cup \{\mathrm{MAX}(A, B) : A, B \in \mathcal{S}_k(Y)\} \cup \{G_n(A) : A \in \mathcal{S}_k(Y),\ n \in \mathbb{N}\}$. This defines a family $\mathcal{S}_n(Y)$ for every $n \in \mathbb{N}$; let $\mathcal{S}(Y) = \bigcup\{\mathcal{S}_n(Y) : n \in \mathbb{N}\}$.

The expression $X \simeq Y$ says that the spaces X and Y are homeomorphic. If $\mathcal{P}$ is a topological property then $\vdash \mathcal{P}$ is to be read "has $\mathcal{P}$". For example, $X \vdash \mathcal{P}$ says that a space X has the property $\mathcal{P}$. For a space X, the class $\mathcal{E}(X)$ consists of all continuous images of products $X \times K$, where K is a compact space. A class $\mathcal{P}$ of topological spaces is called *k-directed* if it is finitely productive (i.e., $X, Y \in \mathcal{P} \Longrightarrow X \times Y \in \mathcal{P}$) and $X \in \mathcal{P}$ implies that $\mathcal{E}(X) \subset \mathcal{P}$. A k-directed class $\mathcal{P}$ is *sk-directed* if $\mathcal{P}$ is closed-hereditary, i.e., if $X \in \mathcal{P}$ then $Y \in \mathcal{P}$ for any closed $Y \subset X$. A property (or a class of spaces) $\mathcal{Q}$ is *weakly k-directed* if any metrizable compact space has $\mathcal{Q}$ (belongs to $\mathcal{Q}$) and $\mathcal{Q}$ is preserved (invariant) under continuous images and finite products.

Given a space X and an infinite cardinal κ, the space $o_\kappa(X) = (X \times D(\kappa))^\kappa$ is called *the κ-hull of* X. The space X is called *κ-invariant* if $X \simeq o_\kappa(X)$. A class $\mathcal{P}$ is called *κ-perfect* if, for every $X \in \mathcal{P}$, we have $o_\kappa(X) \in \mathcal{P}$, $\mathcal{E}(X) \subset \mathcal{P}$ and $Y \in \mathcal{P}$ for any closed $Y \subset X$. If $\mathcal{P}$ is a class of spaces, then $\mathcal{P}_\sigma$ consists of the spaces representable as a countable union of elements of $\mathcal{P}$. The class $\mathcal{P}_\delta$ contains the spaces which are countable intersections of elements of $\mathcal{P}$ in some larger space. More formally, $X \in \mathcal{P}_\sigma$ if $X = \bigcup\{X_n : n \in \omega\}$ where each $X_n \in \mathcal{P}$. Analogously, $X \in \mathcal{P}_\delta$ if there exists a space Y and $Y_n \subset Y$ such that $Y_n \in \mathcal{P}$ for all $n \in \omega$ and $\bigcap\{Y_n : n \in \omega\} \simeq X$. Then $\mathcal{P}_{\sigma\delta} = (\mathcal{P}_\sigma)_\delta$.

A space is called *k-separable* if it has a dense σ-compact subspace; the space X is *Hurewicz* if, for any sequence $\{\mathcal{U}_n : n \in \omega\}$ of open covers of X, we can choose, for each $n \in \omega$, a finite $\mathcal{V}_n \subset \mathcal{U}_n$ such that $\bigcup\{\mathcal{V}_n : n \in \omega\}$ is a cover of X. We say that X is *an Eberlein–Grothendieck space (or EG-space)* if X embeds into $C_p(Y)$ for some compact space Y. A space X is *radial* if, for any $A \subset X$ and any $x \in \overline{A}\backslash A$, there exists a regular cardinal κ and a transfinite sequence $S = \{x_\alpha : \alpha < \kappa\} \subset A$ such that $S \to x$ in the sense that, for any open $U \ni x$, there is $\alpha < \kappa$ such that, for each $\beta \geq \alpha$, we have $x_\beta \in U$. The space X is *pseudoradial* if $A \subset X$ and $A \neq \overline{A}$ implies that there is a regular cardinal κ and a transfinite sequence $S = \{x_\alpha : \alpha < \kappa\} \subset A$ such that $S \to x \notin A$. A subset A of a space X is called *bounded* if every $f \in C(X)$ is bounded on A, i.e., there exists $N \in \mathbb{R}$ such that $|f(x)| \leq N$ for all $x \in A$. A family $\mathcal{U}$ is an *ω-cover of a set A*, if, for any finite $B \subset A$, there is $U \in \mathcal{U}$ such that $B \subset U$. A *Luzin space* is any uncountable space without isolated points in which every nowhere dense subset is countable. *An analytic space* is a continuous image of the space $\mathbb{P}$ of irrational numbers. Given a space X, let $vet(X) \leq \kappa$ if, for any $x \in X$ and any family $\{A_\alpha : \alpha < \kappa\} \subset \exp X$ with $x \in \bigcap\{\overline{A_\alpha} : \alpha < \kappa\}$, we can choose, for each $\alpha < \kappa$, a finite $B_\alpha \subset A_\alpha$ such that $x \in \overline{\bigcup\{B_\alpha : \alpha < \kappa\}}$. The cardinal $vet(X) = \min\{\kappa \geq \omega : vet(X) \leq \kappa\}$ is called *fan tightness of the space X*.

It is also important to mention that the wizards of set theory invented an axiom which is called PFA (from Proper Forcing Axiom). Any reasonably comprehensive formulation of PFA is outside of the reach of this book. However, any professional topologist must know that it exists and that it is consistent with the usual system of axioms (referred to as ZFC) of set theory provided there exist some large cardinals. I am not going to give a rigorous definition of large cardinals because we won't need them here. However, we must be aware of the fact that it is absolutely evident for everybody (who knows what they are!) that their existence is consistent with ZFC notwithstanding that this consistence is not proved yet. So, it is already a common practice to use nice topological consequences of PFA. The one we will need is the following statement: "If X is a compact space of weight ω_1 and $t(X) > \omega$ then $\omega_1 + 1$ embeds in X".

Given a space X a continuous map $r : X \to X$ is called *a retraction* if $r \circ r = r$. If f is a function then $\text{dom}(f)$ is its domain; given a function g the expression $f \subset g$ says that $\text{dom}(f) \subset \text{dom}(g)$ and $g|\text{dom}(f) = f$. If we have a set of functions $\{f_i : i \in I\}$ such that $f_i|(\text{dom}(f_i) \cap \text{dom}(f_j)) = f_j|(\text{dom}(f_i) \cap \text{dom}(f_j))$ for any indices $i, j \in I$ then we can define a function f with $\text{dom}(f) = \bigcup_{i \in I} \text{dom}(f_i)$ as follows: given any $x \in \text{dom}(f)$, find any $i \in I$ with $x \in \text{dom}(f_i)$ and let $f(x) = f_i(x)$. It is easy to check that the value of f at x does not depend on the choice of i so we have consistently defined a function f which will be denoted by $\bigcup\{f_i : i \in I\}$.

001. Prove that, if X is a normal space and $\dim X = 0$ then $\dim \beta X = 0$ and hence βX is zero-dimensional.

002. Let X be a zero-dimensional compact space. Suppose that Y is second countable and $f : X \to Y$ is a continuous onto map. Prove that there exists a compact metrizable zero-dimensional space Z and continuous onto maps $g : X \to Z$ and $h : Z \to Y$ such that $f = h \circ g$.

003. Prove that there exists a continuous map $k : \mathbb{K} \to \mathbb{I}$ such that, for any compact zero-dimensional space X and any continuous map $f : X \to \mathbb{I}$, there exists a continuous map $g_f : X \to \mathbb{K}$ such that $f = k \circ g_f$.

004. Prove that, for any zero-dimensional compact X, the space $C_p(X, \mathbb{I})$ is a continuous image of $C_p(X, \mathbb{D}^\omega)$.

005. Given a countably infinite space X prove that the following conditions are equivalent:

(i) $C_p(X, \mathbb{D})$ is countable;
(ii) $C_p(X, \mathbb{D}) \simeq \mathbb{Q}$;
(iii) X is compact.

006. For an arbitrary space X prove that

(i) for any $P \subset C_p(X)$ there is an algebra $A(P) \subset C_p(X)$ such that $P \subset A(P)$ and $A(P)$ is *minimal* in the sense that, for any algebra $A \subset C_p(X)$, if $P \subset A$ then $A(P) \subset A$;
(ii) $A(P)$ is a countable union of continuous images of spaces which belong to $\mathcal{H}(P) = \{P^m \times K$ for some $m \in \mathbb{N}$ and metrizable compact $K\}$.
(iii) if $\mathcal{Q}$ is a weakly k-directed property and $P \vdash \mathcal{Q}$ then $A(P) \vdash \mathcal{Q}_\sigma$, i.e., $A(P)$ is a countable union of spaces with the property $\mathcal{Q}$;

007. Given a compact space X suppose that $A \subset C_p(X)$ is an algebra. Prove that both $\overline{A}$ and $\mathrm{cl}_u(A)$ are algebras in $C_p(X)$.

008. Let X be a compact space. Suppose that $A \subset C_p(X)$ separates the points of X, contains the constant functions and has the following property: for each $f, g \in A$ and $a, b \in \mathbb{R}$ we have $af + bg \in A$, $\max(f, g) \in A$, $\min(f, g) \in A$. Prove that every $f \in C_p(X)$ is a uniform limit of some sequence from A.

009. Let X be a compact space and suppose that $Y \subset C_p(X)$ separates the points of X. Prove that

(i) for any algebra $A \subset C_p(X)$ with $Y \subset A$, we have $\mathrm{cl}_u(A) = C_p(X)$;
(ii) if Y contains a non-zero constant function then $\mathrm{cl}_u(\bigcup \mathcal{S}(Y)) = C_p(X)$.

010. For a space X, suppose that $Y \subset C_p(X)$ and $\mathrm{cl}_u(Y) = C_p(X)$. Prove that $C_p(X) \in (\mathcal{E}(Y))_\delta$.

011. Prove that every k-directed non-empty class is weakly k-directed. Give an example of a weakly k-directed class which is not k-directed.

012. Prove that any class $\mathcal{K} \in \{$compact spaces, σ-compact spaces, k-separable spaces$\}$ is k-directed. How about the class of countably compact spaces?

013. Let $\mathcal{P}$ be a weakly k-directed class. Prove that, for any $Y \subset C_p(X)$ such that $Y \in \mathcal{P}$, we have $\mathcal{S}(Y) \subset \mathcal{P}$.

014. Given a k-directed class $\mathcal{Q}$ and a compact space X suppose that some set $Y \subset C_p(X)$ separates the points of X and $Y \in \mathcal{Q}$. Prove that $C_p(X) \in \mathcal{Q}_{\sigma\delta}$, i.e., there is a space Z such that $C_p(X) \subset Z$ and $C_p(X) = \bigcap\{C_n : n \in \omega\}$ where every $C_n \subset Z$ is a countable union of spaces with the property $\mathcal{Q}$.

015. For a compact space X suppose that $Y \subset C_p(X)$ separates the points of X. Prove that there exists a compact space K and a closed subspace $F \subset o_\omega(Y) \times K$ such that $C_p(X)$ is a continuous image of F.

016. Prove that, for any compact space X, there exists a compact space K and a closed subspace $F \subset (C_p(X))^\omega \times K$ such that $C_p(X^\omega)$ is a continuous image of F.

017. Let X be a compact space such that $(C_p(X))^\omega$ is Lindelöf. Show that $C_p(X^\omega)$ is Lindelöf. As a consequence, $C_p(X^n)$ is Lindelöf for each $n \in \mathbb{N}$.

018. Assume that X is compact and $\mathcal{P}$ is an ω-perfect class. Prove that it follows from $C_p(X) \in \mathcal{P}$ that $C_p(X^\omega) \in \mathcal{P}$.

019. Let $\mathcal{P}$ be an ω-perfect class of spaces. Prove that the following properties are equivalent for any compact X:

(i) the space $C_p(X)$ belongs to $\mathcal{P}$;
(ii) there exists $Y \subset C_p(X)$ such that Y is dense in $C_p(X)$ and $Y \in \mathcal{P}$;
(iii) there exists $Y \subset C_p(X)$ which separates the points of X and belongs to $\mathcal{P}$;
(iv) the space X embeds into $C_p(Z)$ for some $Z \in \mathcal{P}$.

020. Prove that the class $L(\Sigma)$ of Lindelöf Σ-spaces is ω-perfect. As a consequence, for any compact X, the following properties are equivalent:

(i) the space $C_p(X)$ is Lindelöf Σ;
(ii) there exists $Y \subset C_p(X)$ such that Y is dense in $C_p(X)$ and $Y \in L(\Sigma)$;
(iii) there exists $Y \subset C_p(X)$ which separates the points of X and belongs to $L(\Sigma)$;
(iv) the space X embeds into $C_p(Y)$ for some Lindelöf Σ-space Y.

021. Let X be a compact space such that $C_p(X)$ is Lindelöf Σ. Show that $C_p(X^\omega)$ is a Lindelöf Σ-space and so is $C_p(X^n)$ for each $n \in \mathbb{N}$.

022. Prove that the class $K(\mathcal{A})$ of K-analytic spaces is ω-perfect. Thus, for any compact X, the following properties are equivalent:

(i) the space $C_p(X)$ is K-analytic;
(ii) there exists $Y \subset C_p(X)$ such that Y is dense in $C_p(X)$ and $Y \in K(\mathcal{A})$;
(iii) there exists $Y \subset C_p(X)$ which separates the points of X and belongs to $K(\mathcal{A})$;
(iv) the space X embeds into $C_p(Y)$ for some K-analytic space Y.

023. Let X be a compact space such that $C_p(X)$ is K-analytic. Show that $C_p(X^\omega)$ is a K-analytic space and so is $C_p(X^n)$ for each $n \in \mathbb{N}$.

024. Observe that any K-analytic space is Lindelöf Σ. Give an example of a space X such that $C_p(X)$ is Lindelöf Σ but not K-analytic.

025. Give an example of X such that $C_p(X)$ is K-analytic but not $K_{\sigma\delta}$.

026. Let X be a Lindelöf Σ-space. Prove that $C_p(X)$ is normal if and only if $C_p(X)$ is Lindelöf. In particular, if X is compact then $C_p(X)$ is normal if and only if it is Lindelöf.

027. Suppose that X is a Lindelöf Σ-space such that $C_p(X)\backslash\{f\}$ is normal for some $f \in C_p(X)$. Prove that X is separable. In particular, if X is ω-monolithic and $C_p(X)\backslash\{f\}$ is normal for some $f \in C_p(X)$ then X has a countable network.

028. Let X and $C_p(X)$ be Lindelöf Σ-spaces and suppose that $C_p(X)\backslash\{f\}$ is normal for some $f \in C_p(X)$. Prove that X has a countable network.

029. Let M_t be a separable metrizable space for all $t \in T$. Suppose that Y is dense in $M = \prod\{M_t : t \in T\}$ and Z is a continuous image of Y. Prove that, if $Z \times Z$ is normal then $ext(Z) = \omega$ and hence Z is collectionwise normal.

030. Prove that, for any infinite zero-dimensional compact space X, there exists a closed $F \subset C_p(X, \mathbb{D}^\omega) \subset C_p(X)$ which maps continuously onto $(C_p(X))^\omega$.

031. Prove that, for any infinite zero-dimensional compact space X, there exists a closed $F \subset C_p(X, \mathbb{D}^\omega) \subset C_p(X)$ which maps continuously onto $C_p(X^\omega)$.

032. Prove that the following conditions are equivalent for an arbitrary zero-dimensional compact X:

(i) $C_p(X, \mathbb{D}^\omega)$ is normal;
(ii) $C_p(X, \mathbb{I})$ is normal;
(iii) $C_p(X)$ is normal;
(iv) $C_p(X)$ is Lindelöf;
(v) $(C_p(X))^\omega$ is Lindelöf;
(vi) $C_p(X^\omega)$ is Lindelöf.

033. Observe that $C_p(X)$ is monolithic for any compact X. Using this fact prove that, for any compact space X, each compact subspace $Y \subset C_p(X)$ is a Fréchet–Urysohn space.

034. Prove that, for any metrizable space M, there is a compact space K such that M embeds in $C_p(K)$.

035. Prove that the following conditions are equivalent for any compact X:

(i) there is a compact $K \subset C_p(X)$ which separates the points of X;
(ii) there is a σ-compact $Y \subset C_p(X)$ which separates the points of X;
(iii) there is a σ-compact $Z \subset C_p(X)$ which is dense in $C_p(X)$;
(iv) X embeds into $C_p(K)$ for some compact K;
(v) X embeds into $C_p(Y)$ for some σ-compact Y.

036. Suppose that X is compact and embeds into $C_p(Y)$ for some compact Y. Prove that it is possible to embed X into $C_p(Z)$ for some Fréchet–Urysohn compact space Z.

037. Give an example of a compact space X embeddable into $C_p(Y)$ for some compact Y but not embeddable into $C_p(Z)$ for any compact first countable space Z.

038. Suppose that X embeds into $C_p(Y)$ for some compact Y. Prove that it is possible to embed X into $C_p(Z)$ for some zero-dimensional compact space Z.

039. Suppose that X embeds into $C_p(Y)$ for some countably compact Y. Prove that it is possible to embed X into $C_p(Z)$ for some zero-dimensional countably compact space Z.

040. Give an example of a space Y which embeds in $C_p(X)$ for a pseudocompact space X but does not embed in $C_p(Z)$ for any countably compact Z.

041. Prove that a countable space Y embeds into $C_p(X)$ for some pseudocompact space X if and only if Y embeds into $C_p(Z)$ for some compact metrizable space Z.

042. Give an example of a space Y which embeds into $C_p(X)$ for a countably compact space X but does not embed into $C_p(Z)$ for a compact space Z.

043. Let $\xi \in \beta\omega\backslash\omega$. Prove that the countable space $\omega_\xi = \omega \cup \{\xi\}$, considered with the topology inherited from $\beta\omega$, does not embed into $C_p(X)$ for a pseudocompact X.

044. (Grothendieck's theorem). Suppose that X is a countably compact space and $B \subset C_p(X)$ is a bounded subset of $C_p(X)$. Prove that $\overline{B}$ is compact. In particular, the closure of any pseudocompact subspace of $C_p(X)$ is compact.

045. Prove that there exists a pseudocompact space X for which there is a closed pseudocompact $Y \subset C_p(X)$ which is not countably compact.

046. Let X be a σ-compact space. Prove that any countably compact subspace of $C_p(X)$ is compact.

047. Let X be a space and suppose that there is a point $x_0 \in X$ such that $\psi(x_0, X) = \omega$ and $x_0 \notin \overline{A}$ for any countable $A \subset X$. Prove that there is an infinite closed discrete $B \subset C_p(X)$ such that B is bounded in $C_p(X)$.

048. Prove that there exists a σ-compact space X such that $C_p(X)$ contains an infinite closed discrete subspace which is bounded in $C_p(X)$.

049. Prove that there exists a σ-compact space X such that $C_p(X)$ does not embed as a closed subspace into $C_p(Y)$ for any countably compact space Y.

050. Given a metric space (M, ρ) say that a family $\mathcal{U} \subset \exp M\backslash\{\emptyset\}$ is *ρ-vanishing* if $\operatorname{diam}_\rho(U) < \infty$ for any $U \in \mathcal{U}$ and the diameters of the elements of $\mathcal{U}$ converge to zero, i.e., the family $\{U \in \mathcal{U} : \operatorname{diam}_\rho(U) \geq \varepsilon\}$ is finite for any $\varepsilon > 0$. Prove that, for any separable metrizable X, the following conditions are equivalent:

(i) X is a Hurewicz space;
(ii) for any metric ρ which generates the topology of X, there is a ρ-vanishing family $\mathcal{U} \subset \tau(X)$ such that $\bigcup\mathcal{U} = X$;
(iii) for any metric ρ which generates the topology of X, there exists a ρ-vanishing base $\mathcal{B}$ of the space X;

(iv) there exists a metric ρ which generates the topology of X, such that, for any base $\mathcal{B}$ of the space X, there is a ρ-vanishing family $\mathcal{U} \subset \mathcal{B}$ for which $\bigcup \mathcal{U} = X$;
(v) for any metric ρ which generates the topology of X and any base $\mathcal{B}$ of the space X there is ρ-vanishing family $\mathcal{B}' \subset \mathcal{B}$ such that $\mathcal{B}'$ is also a base of X;
(vi) every base of X contains a family which is a locally finite cover of X.

051. Prove that X^ω is a Hurewicz space if and only if X is compact.
052. Prove that any separable Luzin space is a Hurewicz space.
053. Prove that any Hurewicz analytic space is σ-compact.
054. Give an example of a Hurewicz space which is not σ-compact.
055. Prove that, under CH, there exists a Hurewicz space whose square is not normal.
056. Prove that X^n is a Hurewicz space for every $n \in \mathbb{N}$, if and only if, for any sequence $\{\gamma_k : k \in \omega\}$ of open ω-covers of the space X, we can choose, for each $k \in \omega$, a finite $\mu_k \subset \gamma_k$ such that the family $\bigcup\{\mu_k : k \in \omega\}$ is an ω-cover of X.
057. Let X be any space. Prove that X^n is a Hurewicz space for all $n \in \mathbb{N}$ if and only if $vet(C_p(X)) \leq \omega$.
058. Prove that if $C_p(X)$ is Fréchet–Urysohn then $vet(C_p(X)) \leq \omega$.
059. Prove that, under MA+¬CH, there exists a second countable space X such that X^n is a Hurewicz space for each natural n, while X is not σ-compact.
060. Say that a space is *subsequential* if it embeds in a sequential space. Prove that every sequential space has countable tightness and hence each subsequential space also has countable tightness.
061. For any point $\xi \in \beta\omega\backslash\omega$ prove that the countable space $\omega \cup \{\xi\}$ is not subsequential.
062. Prove that $C_p(\mathbb{I})$ is not subsequential.
063. Prove that the following are equivalent for any pseudocompact X:

(i) $C_p(X)$ is a Fréchet–Urysohn space;
(ii) $C_p(X)$ embeds in a sequential space;
(iii) X is compact and scattered.

064. Prove that radiality is a hereditary property; show that pseudoradiality is closed-hereditary. Give an example showing that pseudoradiality is not hereditary.
065. Prove that any quotient (pseudo-open) image of a pseudoradial (radial) space is a pseudoradial (radial) space.
066. Prove that any radial space of countable tightness is Fréchet–Urysohn.
067. Prove that a space is radial (pseudoradial) if and only if it is a pseudo-open (quotient) image of a linearly ordered space.
068. Prove that any radial space of countable spread is Fréchet–Urysohn.
069. Prove that any radial dyadic space is metrizable.
070. Prove that $\beta\omega\backslash\omega$ is not pseudoradial.

071. Prove that $\mathbb{D}^{\omega_1}$ is not pseudoradial under CH and pseudoradial under MA+$\neg$CH.

072. Prove that it is independent of ZFC whether every dyadic pseudoradial space is metrizable.

073. Prove that, for any space X, the space $C_p(X)$ is radial if and only if it is Fréchet–Urysohn.

074. An uncountable cardinal κ is called *ω-inaccessible* if $\lambda^\omega < \kappa$ for any cardinal $\lambda < \kappa$. Recall that, if ξ is an ordinal then $\mathrm{cf}(\xi) = \min\{|A| : A$ is a cofinal subset of $\xi\}$. Prove that, for an infinite ordinal ξ, the space $C_p(\xi)$ is pseudoradial if and only if either $\mathrm{cf}(\xi) \leq \omega$ or ξ is an ω-inaccessible regular cardinal (here, as usual, ξ is considered with its interval topology). Observe that ω-inaccessible regular cardinals exist in ZFC and hence there exist spaces X such that $C_p(X)$ is pseudoradial but not radial.

075. Let X be a compact space. Prove that, if $C_p(X)$ is pseudoradial then it is Fréchet–Urysohn (and hence X is scattered).

076. Let X be any space such that $C_p(X, \mathbb{D}) \times \omega^\omega$ is not Lindelöf. Prove that the space $C_p(X, \mathbb{D}^\omega)$ is not Lindelöf.

077. Suppose that X is a compact space such that a countable set $M \subset X$ is open and dense in X. Assume also that the set of isolated points of $Y = X \setminus M$ is uncountable and dense in Y. Prove that $ext(C_p(X, \mathbb{D}) \times \omega^\omega) > \omega$.

078. Suppose that X is a compact space such that a countable set $M \subset X$ is open and dense in X. Assume also that the set I of isolated points of $Y = X \setminus M$ is uncountable and dense in Y; let $F = Y \setminus I$. Prove that, under MA+$\neg$CH, any uncountable subset of the set $E = \{f \in C_p(X, \mathbb{D}) : f(F) = \{0\}\}$ contains an uncountable set D which is closed and discrete in $C_p(X, \mathbb{D})$.

079. Let X be a compact space of weight ω_1 in which we have a countable dense set L and a nowhere dense closed non-empty set F. Assuming MA+$\neg$CH prove that there exists $M \subset L$ such that $\overline{M} \setminus M = F$ and all points of M are isolated in the space $M \cup F$.

080. Prove that, under MA+$\neg$CH, if X is a compact space such that $C_p(X)$ is normal, then X is Fréchet–Urysohn, ω-monolithic and has a dense set of points of countable character.

081. Assume MA+$\neg$CH. Show that, if a compact space X has the Souslin property and $C_p(X)$ is normal then X is metrizable.

082. Prove that $w(X) = l(C_p(X))$ for any linearly orderable compact space X. In particular, if $C_p(X)$ is Lindelöf then X is metrizable.

083. Given an infinite compact space X prove that we have $|\overline{Y}| \leq 2^{l(Y) \cdot c(X)}$ for any $Y \subset C_p(X)$.

084. Suppose that X is a compact space with the Souslin property and $C_p(X)$ has a dense Lindelöf subspace. Prove that $w(X) \leq |C_p(X)| \leq 2^\omega$.

085. Prove that, for any uncountable regular cardinal κ, if $Z \subset C_p(\kappa + 1)$ separates the points of $\kappa + 1$ then $l(Z) \geq \kappa$.

086. Prove that, if X is a dyadic space and $Y \subset C_p(X)$ then $nw(Y) = l(Y)$. In particular, any Lindelöf subspace of $C_p(X)$ has a countable network.

087. Prove that, if X is a dyadic space and $C_p(X)$ has a dense Lindelöf subspace then X is metrizable.

088. Given a space X suppose that $K \subset C_p(X)$ is a compact space of uncountable tightness. Show that there exists a closed $X_1 \subset X$ such that $C_p(X_1)$ contains a compact subspace of weight and tightness ω_1.

089. Prove that the axiom PFA implies that, for any Lindelöf space X and any compact $K \subset C_p(X)$, we have $t(K) \leq \omega$.

090. Given a space X and a set $A \subset X$ denote by τ_A the topology on X generated by the family $\tau(X) \cup \exp(X \backslash A)$ as a subbase; let $X[A] = (X, \tau_A)$. In other words, the space $X[A]$ is constructed by declaring isolated all points of $X \backslash A$ and keeping the same topology at the points of A. Prove that, for any uncountable Polish space M and $A \subset M$ the following conditions are equivalent:

(i) the space $(M[A])^\omega$ is Lindelöf;
(ii) if $\mathcal{F}$ is a countable family of finite-to-one continuous maps from the Cantor set $\mathbb{K}$ to M then $\bigcap\{f^{-1}(A) : f \in \mathcal{F}\} \neq \emptyset$;
(iii) if $\mathcal{F}$ is a countable family of injective continuous maps from the Cantor set $\mathbb{K}$ to M then $\bigcap\{f^{-1}(A) : f \in \mathcal{F}\} \neq \emptyset$.

Deduce from this fact that, for any uncountable Polish space M there is a disjoint family $\{A_\alpha : \alpha < \mathfrak{c}\}$ of subsets of M such that $(M[A_\alpha])^\omega$ is Lindelöf for any $\alpha < \mathfrak{c}$.

091. Given a space X and a set $A \subset X$ denote by τ_A the topology on X generated by the family $\tau(X) \cup \exp(X \backslash A)$ as a subbase; let $X[A] = (X, \tau_A)$. Prove that, if M is a Polish space, $A \subset M$ and $n \in \mathbb{N}$ then the following conditions are equivalent:

(i) the space $(M[A])^n$ is Lindelöf;
(ii) if $\mathcal{F}$ is a family of finite-to-one continuous maps from the Cantor set $\mathbb{K}$ to M and $|\mathcal{F}| \leq n$ then $\bigcap\{f^{-1}(A) : f \in \mathcal{F}\} \neq \emptyset$;
(iii) if $\mathcal{F}$ is a family of injective continuous maps from the Cantor set $\mathbb{K}$ to M and $|\mathcal{F}| \leq n$ then $\bigcap\{f^{-1}(A) : f \in \mathcal{F}\} \neq \emptyset$.

Deduce from this fact that, for any uncountable Polish space M there is a disjoint family $\{A_\alpha : \alpha < \mathfrak{c}\}$ of subsets of M such that for every $\alpha < \mathfrak{c}$ the space $(M[A_\alpha])^k$ is Lindelöf for any $k \in \mathbb{N}$ while $(M[A_\alpha])^\omega$ is not Lindelöf.

092. Suppose that $\mathcal{P}$ is an sk-directed class of spaces and $Y \in \mathcal{P}$. Prove that if $X \subset C_p(Y)$ and the set of non-isolated points of X is σ-compact then $C_p^*(X)$ belongs to the class $\mathcal{P}_{\sigma\delta}$.

093. Prove that there exist separable, scattered σ-compact spaces X and Y such that both $(C_p(X))^\omega$ and $(C_p(Y))^\omega$ are Lindelöf while $C_p(X) \times C_p(Y)$ is not normal and contains a closed discrete set of cardinality $\mathfrak{c}$.

094. Show that there is a separable scattered σ-compact space X and a countable space M such that the space $(C_p(X))^\omega$ is Lindelöf while we have the equality $ext(C_p(X) \times C_p(M)) = \mathfrak{c}$ and $C_p(X) \times C_p(M)$ is not normal.

095. Prove that, under CH, there exists a separable scattered compact space X such that $(C_p(X, \mathbb{D}))^n$ is Lindelöf for any natural n, while $(C_p(X, \mathbb{D}))^\omega$ is not Lindelöf.

096. Prove that there is a scattered, separable, zero-dimensional σ-compact space X with $(C_p(X, \mathbb{D}))^n$ Lindelöf for each natural n, while $(C_p(X, \mathbb{D}))^\omega$ is not Lindelöf.

097. Assume MA+$\neg$CH. Let X be a space with $l^*(X) = \omega$. Prove that any separable compact subspace of $C_p(X)$ is metrizable.

098. Assume MA+$\neg$CH. Let X be a separable compact space. Prove that, for any $Y \subset C_p(X)$ with $l^*(Y) = \omega$, we have $nw(Y) = \omega$.

099. Prove that there exists a separable σ-compact space X such that $(C_p(X))^\omega$ is Lindelöf and $s(X) > \omega$.

100. Assume MA+$\neg$CH. Prove that there is a separable σ-compact space X such that $C_p(X)$ does not embed into $C_p(Y)$ for a separable compact space Y.

1.2 Corson Compact Spaces

All spaces are assumed to be Tychonoff. For a product $X = \prod\{X_t : t \in T\}$ of the spaces X_t and $x \in X$, let $\Sigma(X, x) = \{y \in X : |\{t \in T : y(t) \neq x(t)\}| \leq \omega\}$. The space $\Sigma(X, x)$ is called the *Σ-product of spaces $\{X_t : t \in T\}$ with the center x*. Again, if $x \in X = \prod\{X_t : t \in T\}$, let $\sigma(X, x) = \{y \in X : |\{t \in T : y(t) \neq x(t)\}| < \omega\}$. The space $\sigma(X, x)$ is called the *σ-product of spaces $\{X_t : t \in T\}$ with the center x*. If some statement about Σ-products or σ-products is made with no center specified, then this statement holds (or must be proved) for an arbitrary center. The symbols $\Sigma(A)$ and $\sigma(A)$ are reserved for the respective Σ- and σ-products of real lines with the center zero, i.e., $\Sigma(A) = \{x \in \mathbb{R}^A : |\{a \in A : x(a) \neq 0\}| \leq \omega\}$ and $\sigma(A) = \{x \in \mathbb{R}^A : |\{a \in A : x(a) \neq 0\}| < \omega\}$. Now, $\Sigma_*(A) = \{x \in \mathbb{R}^A :$ for any $\varepsilon > 0$ the set $\{a \in A : |x(a)| \geq \varepsilon\}$ is finite$\}$. A family $\mathcal{U} \subset \exp A$ is called *ω-continuous* if $\bigcup\{U_n : n \in \omega\} \in \mathcal{U}$ whenever $U_n \in \mathcal{U}$ and $U_n \subset U_{n+1}$ for all $n \in \omega$. The family $\mathcal{U}$ is *ω-cofinal* if, for every countable $B \subset A$, there exist $U \in \mathcal{U}$ such that $B \subset U$. If $B \subset A$ and $x \in \Sigma(A)$, let $r_B(x)(a) = x(a)$ if $a \in B$ and $r_B(x)(a) = 0$ otherwise. Clearly, $r_B : \Sigma(A) \to \Sigma(A)$ is a continuous map. Call a set $Y \subset \Sigma(A)$ *invariant* if the family $\{B \subset A : r_B(Y) \subset Y\}$ is ω-continuous and ω-cofinal.

A compact space X is *Corson compact* if it embeds into $\Sigma(A)$ for some A. Given a space X and an infinite cardinal κ, the space $(X \times D(\kappa))^\kappa$ is called *the κ-hull of* X and is denoted by $o_\kappa(X)$. The space X is called *κ-invariant* if $X \simeq o_\kappa(X)$. For a space X, the class $\mathcal{E}(X)$ consists of all continuous images of products $X \times K$, where K is a compact space. Define a class $\mathcal{P}$ to be *κ-perfect* if, for any $X \in \mathcal{P}$, we have $o_\kappa(X) \in \mathcal{P}$, $\mathcal{E}(X) \in \mathcal{P}$ and $Y \in \mathcal{P}$ for any closed $Y \subset X$.

Let T be an infinite set. An arbitrary family $\mathcal{A} \subset \exp T$ is called *adequate* if $\bigcup \mathcal{A} = T$, $\exp A \subset \mathcal{A}$ for any $A \in \mathcal{A}$, and $A \in \mathcal{A}$ whenever all finite subsets of A belong to $\mathcal{A}$. Given $A \subset T$, let $\chi_A(t) = 1$ if $t \in A$ and $\chi_A(t) = 0$ if $t \notin A$. The map $\chi_A : T \to \{0, 1\}$ is called *the characteristic function of A in the set T*. The symbol $\mathbb{D}$ denotes the two-point discrete space $\{0, 1\}$. If we have a set T and an adequate family $\mathcal{A}$ on T let $K_{\mathcal{A}} = \{\chi_A \in \mathbb{D}^T : A \in \mathcal{A}\}$. Another object associated with $\mathcal{A}$ is the space $T^*_{\mathcal{A}}$ whose underlying set is $T \cup \{\xi\}$, where $\xi \notin T$, all points of T are isolated in $T^*_{\mathcal{A}}$ and the basic neighbourhoods of ξ are the complements of finite unions of elements of $\mathcal{A}$. A subspace $X \subset \mathbb{D}^T$ is called *adequate* if $X = K_{\mathcal{A}}$ for some adequate family $\mathcal{A}$ on T.

Given an uncountable cardinal κ, a space X belongs to $\mathcal{M}(\kappa)$ if there exists a compact K such that X is a continuous image of a closed subset of $L(\kappa)^\omega \times K$. The space X is called *primarily Lindelöf* if X is a continuous image of a closed subspace of $L(\kappa)^\omega$ for some uncountable cardinal κ.

Suppose that S_α is homeomorphic to $\omega + 1$ for each $\alpha < \kappa$. In the space $S = \bigoplus\{S_\alpha : \alpha < \kappa\}$ let F be the set of non-isolated points. Introduce a topology τ on the set $S/F = \{F\} \cup (S \backslash F)$ declaring the points of $S \backslash F$ isolated and defining the local base at F as the family of all sets $\{F\} \cup (U \backslash F)$ where U is an open set (in S) which contains F. The space $V(\kappa) = (S/F, \tau)$ is called the *Fréchet–Urysohn κ-fan*. A family $\mathcal{U} \subset \exp X$ is *T_0-separating in X* if, for any distinct $x, y \in X$, there exists $U \in \mathcal{U}$ such that $|U \cap \{x, y\}| = 1$. A continuous surjective map $f : X \to Y$ is *irreducible* if, for any closed $F \subset X$ with $F \neq X$, we have $f(F) \neq Y$.

Let X be a space. Denote by $AD(X)$ the set $X \times \{0, 1\}$. Given $x \in X$, let $u_0(x) = (x, 0)$ and $u_1(x) = (x, 1)$. Thus, $AD(X) = u_0(X) \cup u_1(X)$. Declare the points of $u_1(X)$ isolated. Now, if $z = (x, 0) \in AD(X)$ then the base at z is formed by the sets $u_0(V) \cup (u_1(V) \backslash \{u_1(x)\})$ where V runs over the open neighbourhoods of x. The space $AD(X)$, with the topology thus defined, is called *the Alexandroff double of the space X*. Recall that, if we have a map $f : X \to Y$ then the map $f^n : X^n \to Y^n$ is defined by $f^n(x) = (f(x_1), \ldots, f(x_n))$ for any $x = (x_1, \ldots, x_n) \in X^n$. A space X is called *Sokolov space*, if, for any family $\{F_n : n \in \mathbb{N}\}$ such that F_n is a closed subset of X^n for each $n \in \mathbb{N}$, there exists a continuous map $f : X \to X$ such that $nw(f(X)) \leq \omega$ and $f^n(F_n) \subset F_n$ for all $n \in \mathbb{N}$. Given a space X, we let $C_{p,0}(X) = X$ and $C_{p,n+1}(X) = C_p(C_{p,n}(X))$ for all $n \in \omega$. The spaces $C_{p,n}(X)$ are called *iterated function spaces of X*.

In this section, we make use of a two-player game with complete information introduced by G. Gruenhage. In this game (which we call the Gruenhage game or W-game), there are two players (the concept of "player" is considered axiomatic), who play a game of ω moves on a space X at a fixed set $H \subset X$. The first player is called OP (for "open") and the second one's name is PT (for "point"). The n-th move of OP consists in choosing an open set $U_n \supset H$. The player PT responds by choosing a point $x_n \in U_n$. After ω moves have been made, the sequence $\mathcal{P} = \{(U_n, x_n) : n \in \mathbb{N}\}$ is called a *play of the game*; for any $n \in \mathbb{N}$, the set $\{U_1, x_1, \ldots, U_n, x_n\}$ is called *an initial segment of the play $\mathcal{P}$*.

Now, if $\mathcal{P} = \{(U_n, x_n) : n \in \mathbb{N}\}$ is a play in the W-game at H in the space X then the set $\{x_n : n \in \mathbb{N}\}$ is taken into consideration to determine who won the game. If $x_n \to H$ in the sense that any open $U \supset H$ contains all but finitely many points x_n, then OP wins. If not, then PT is the winner.

A strategy for the player OP is any function s, whose domain is the family $\mathrm{dom}(s) = \{\emptyset\} \cup \{F : F = (U_1, x_1, \ldots U_n, x_n),\ n \in \mathbb{N},\ H \subset U_i \in \tau(X)$ and $x_i \in U_i$ for all $i \leq n\}$ and $s(F)$ is an open set containing H for any $F \in \mathrm{dom}(s)$. If $P = \{(U_n, x_n) : n \in \mathbb{N}\}$ is a play, we say that *OP applied the strategy s in P*, if $U_1 = s(\emptyset)$ and, for any $n \geq 2$, we have $U_n = s(U_1, x_1, \ldots, U_{n-1}, x_{n-1})$. The strategy s is called *winning* if any play in which s is applied, is favorable for the player OP, i.e., OP wins in every play where he/she applies the strategy s. A set $H \subset X$ is *a W-set (or has the W-property)* if OP has a winning strategy in the game on X with the fixed set H. If every point of X is a W-set, X is called *W-space (or a space with the W-property).* A family $\mathcal{U}$ of subsets of X is called *point-countable* if, for any $x \in X$, the family $\{U \in \mathcal{U} : x \in U\}$ is countable. A space X is *metalindelöf* if any open cover of X has a point-countable open refinement. A space X is *metacompact* if any open cover of X has a point-finite open refinement.

101. Let M_t be a metrizable space for each $t \in T$. For an arbitrary point $a \in M = \prod\{M_t : t \in T\}$, prove that $\Sigma(M, a)$ is a Fréchet–Urysohn space. In particular, $\Sigma(A)$ is a Fréchet–Urysohn space for any A.

102. Let M_t be a metrizable space for each $t \in T$. For an arbitrary point $a \in M = \prod\{M_t : t \in T\}$, prove that $\Sigma(M, a)$ is a collectionwise normal space. In particular, $\Sigma(A)$ is a collectionwise normal space for any A.

103. Let M_t be a second countable space for each $t \in T$. For an arbitrary point $a \in M = \prod\{M_t : t \in T\}$, prove that $ext(\Sigma(M, a)) \leq \omega$. In particular, $ext(\Sigma(A)) = \omega$ for any set A.

104. Let M_t be a second countable space for any $t \in T$. Take any point $a \in M = \prod\{M_t : t \in T\}$. Prove that, if a compact space X is a continuous image of a dense subspace of $\Sigma(M, a)$ then X is metrizable. In particular, if a compact X is a continuous image of $\sigma(M, a)$ or $\Sigma(M, a)$ then X is metrizable.

105. Prove that, if $|A| = \kappa \geq \omega$ then the space $\Sigma_*(A)$ is homeomorphic to $C_p(A(\kappa))$.

106. Prove that, if $|A| = \kappa > \omega$ then the space $\Sigma(A)$ is homeomorphic to $C_p(L(\kappa))$.

107. Prove that, for any κ, there is a compact subspace of $C_p(A(\kappa))$ which separates the points of $A(\kappa)$. As a consequence, $C_p(A(\kappa))$ and $\Sigma_*(\kappa)$ are $K_{\sigma\delta}$-spaces and hence Lindelöf Σ-spaces.

108. Prove that $\sigma(A)$ is a σ-compact space (and hence a Lindelöf Σ-space) for any A.

109. Prove that, for any uncountable set A, there is a closed countably compact non-compact subspace in $\Sigma(A)$ and hence $\Sigma(A)$ is not realcompact.

110. Prove that, for any infinite A, every pseudocompact subspace of $\Sigma_*(A)$ is compact.

111. Prove that any metrizable space M embeds in $\Sigma_*(A)$ for some A.

112. Observe that any pseudocompact continuous image of $\Sigma_*(A)$ is compact and metrizable for any infinite A. Give an example of a countably compact non-compact space which is a continuous image of $\Sigma(\omega_1)$.

113. Prove that, for any uncountable A, the space $\Sigma(A)$ is not embeddable into $\Sigma_*(B)$ for any set B.

114. Prove that, for any uncountable A, the space $\Sigma_*(A)$ is not embeddable into $\sigma(B)$ for any set B.

115. Prove that, for any uncountable A, neither of the spaces $\Sigma(A)$ and $\Sigma_*(A)$ maps continuously onto the other.

116. Prove that, for any A, the space $\Sigma(A)$ embeds in a countably compact Fréchet–Urysohn space.

117. Show that, if A is an uncountable set, then $\Sigma_*(A)$ cannot be embedded in a σ-compact space of countable tightness. In particular, neither $\Sigma(A)$ nor $\Sigma_*(A)$ are embeddable in a compact space of countable tightness if $|A| > \omega$.

118. Let X be a compact space. Prove that X is Corson compact if and only if X has a point-countable T_0-separating family of open F_σ-sets. Deduce from this fact that any metrizable compact space is Corson compact.

119. Let M_t be a second countable space for any $t \in T$. Prove that, for any point $a \in M = \prod\{M_t : t \in T\}$, every compact subset of $\Sigma(M, a)$ is Corson compact.

120. Prove that any Corson compact space is monolithic, Fréchet–Urysohn and has a dense set of points of countable character. As a consequence, $\omega_1 + 1$ is not Corson compact.

121. Prove that $d(X) = w(X)$ for any Corson compact space. Thus, the two arrows space is not Corson compact.

122. Let X be a Corson compact space such that $C_p(X)\backslash\{f\}$ is normal for some $f \in C_p(X)$. Prove that X is metrizable. In particular, if $C_p(X)$ is hereditarily normal, then X is metrizable.

123. Prove that any linearly ordered and any dyadic Corson compact space is metrizable.

124. Let X be a Corson compact space. Prove that the Alexandroff double $AD(X)$ is also Corson compact. In particular, $AD(X)$ is Corson compact for any metrizable compact X.

125. Let X_t be a Corson compact space for any $t \in T$. Prove that the one-point compactification of the space $\bigoplus\{X_t : t \in T\}$ is also Corson compact.

126. Prove that, under CH, there exists a compact space of countable spread which is not perfectly normal.

127. Let X be a Corson compact space such that $s(X) = \omega$. Prove that X is perfectly normal.

128. Let X be an ω-monolithic compact space such that $s(C_p(X)) = \omega$. Prove that X is metrizable. In particular, a Corson compact space X is metrizable whenever $s(C_p(X)) = \omega$.

129. Let X be a compact space of countable tightness. Prove that X maps irreducibly onto a Corson compact space.

130. Given spaces X and Y assume that there exists a closed continuous irreducible onto map $f : X \to Y$. Prove that $d(X) = d(Y)$ and $c(X) = c(Y)$.

131. Prove that, under the Jensen's axiom ($\diamondsuit$), there is a perfectly normal non-metrizable Corson compact space X. Therefore, under $\diamondsuit$, a Corson compact space X need not be metrizable if $c(X) = \omega$.

132. Prove that any Corson compact space X, with ω_1 precaliber of X, is metrizable.

133. Assuming MA+$\neg$CH, prove that any Corson compact space X, with $c(X) = \omega$, is metrizable.

134. Prove that a compact space X can fail to be Corson compact being a countable union of Corson compact spaces.

135. Prove that there exists a compact space X which is not Corson compact being a union of three metrizable subspaces.

136. Suppose that X is compact and X^ω is a countable union of Corson compact subspaces. Prove that X is Corson compact.

137. Prove that any countable product of Corson compact spaces is Corson compact. In particular, X^ω is Corson compact whenever X is Corson compact.

138. Let X be a Corson compact space. Prove that X has a dense metrizable subspace if and only if it has a σ-disjoint π-base.
139. Prove that $\mathcal{M}(\kappa)$ is an ω-perfect class for any κ.
140. Prove that for any Corson compact space X the space $C_p(X)$ belongs to $\mathcal{M}(\kappa)$ for some uncountable κ.
141. Prove that if κ is an uncountable cardinal and $Y \in \mathcal{M}(\kappa)$ then Y^ω is Lindelöf. In particular, $(C_p(X))^\omega$ is Lindelöf for any Corson compact space X.
142. Prove that any countable union of primarily Lindelöf spaces is a primarily Lindelöf space.
143. Prove that any countable product of primarily Lindelöf spaces is a primarily Lindelöf space.
144. Prove that any continuous image as well as any closed subspace of a primarily Lindelöf space is a primarily Lindelöf space.
145. Prove that any countable intersection of primarily Lindelöf spaces is a primarily Lindelöf space.
146. Prove that primarily Lindelöf spaces form a weakly k-directed class.
147. Given a space X let $r : X \to X$ be a retraction. For any $f \in C_p(X)$ let $r_1(f) = f \circ r$. Prove that $r_1 : C_p(X) \to C_p(X)$ is also a retraction.
148. Given an uncountable cardinal κ and a set $A \subset L(\kappa)$ define a map $p_A : L(\kappa) \to L(\kappa)$ by the rule $p_A(x) = a$ if $x \notin A$ and $p_A(x) = x$ for all $x \in A$ (recall that $L(\kappa) = \kappa \cup \{a\}$ and a is the unique non-isolated point of $L(\kappa)$). Prove that

(i) p_A is a retraction on $L(\kappa)$ onto $A \cup \{a\}$ for any $A \subset L(\kappa)$;
(ii) if $B \subset L(\kappa)$ and F is a closed subset of $(L(\kappa))^\omega$ then there exists $A \subset L(\kappa)$ such that $B \subset A$, $|A| \le |B| \cdot \omega$ and $(p_A)^\omega(F) \subset F$. Here, as usual, the map $q_A = (p_A)^\omega : (L(\kappa))^\omega \to (L(\kappa))^\omega$ is the countable power of the map p_A defined by $q_A(x)(n) = p_A(x(n))$ for any $x \in (L(\kappa))^\omega$ and $n \in \omega$.

149. Prove that, for any primarily Lindelöf space X, the space $C_p(X)$ condenses linearly into $\Sigma(A)$ for some A.
150. Prove that the following conditions are equivalent for any compact space X:

(i) X is Corson compact;
(ii) $C_p(X)$ is primarily Lindelöf;
(iii) there is a primarily Lindelöf $P \subset C_p(X)$ which separates the points of X;
(iv) X embeds in $C_p(Y)$ for some primarily Lindelöf space Y.

151. Prove that a continuous image of a Corson compact space is Corson compact.
152. Observe that $\Sigma_*(A)$ and $\sigma(A)$ are invariant subsets of $\Sigma(A)$; prove that, for any infinite cardinal κ and any closed $F \subset \Sigma(A)$ we have

(i) if $B_\alpha \subset A, r_{B_\alpha}(F) \subset F$ for any $\alpha < \kappa$ and $\alpha < \beta < \kappa$ implies $B_\alpha \subset B_\beta$ then $r_B(F) \subset F$ where $B = \bigcup_{\alpha<\kappa} B_\alpha$;
(ii) for any non-empty $D \subset A$ with $|D| \le \kappa$ there is a set $E \subset A$ such that $|E| \le \kappa$, $D \subset E$ and $r_E(F) \subset F$.

In particular, F is invariant in $\Sigma(A)$.

153. Prove that the following properties are equivalent for any X:

(i) X is a Sokolov space;
(ii) if, for any $n \in \mathbb{N}$, a set $B_n \subset X^n$ is chosen then there exists a continuous map $f : X \to X$ such that $nw(f(X)) \leq \omega$ and $f^n(B_n) \subset \overline{B_n}$ for each $n \in \mathbb{N}$;
(iii) if F_{nm} is a closed subset of X^n for all $n, m \in \mathbb{N}$, then there exists a continuous map $f : X \to X$ such that $nw(f(X)) \leq \omega$ and $f^n(F_{nm}) \subset F_{nm}$ for all $n, m \in \mathbb{N}$.

154. Prove that if X is a Sokolov space then $X \times \omega$ is a Sokolov space and every closed $F \subset X$ is also a Sokolov space.

155. Given a Sokolov space X and a second countable space E, prove that $C_p(X, E)$ is also a Sokolov space.

156. Prove that X is a Sokolov space if and only if $C_p(X)$ is a Sokolov space.

157. Let X be a Sokolov space with $t^*(X) \leq \omega$. Prove that $C_p(X, E)$ is Lindelöf for any second countable space E.

158. Prove that

(i) any $\mathbb{R}$-quotient image of a Sokolov space is a Sokolov space;
(ii) if X is a Sokolov space then X^ω is also a Sokolov space;
(iii) a space with a unique non-isolated point is Sokolov if and only if it is Lindelöf.

159. Let X be a space with a unique non-isolated point. Prove that the following properties are equivalent:

(i) $l(X) \leq \omega$ and $t^*(X) \leq \omega$;
(ii) X is a Sokolov space and $t^*(X) \leq \omega$;
(iii) $C_{p,n}(X)$ is Lindelöf for all $n \in \mathbb{N}$;
(iv) $C_p(X)$ is Lindelöf.

160. Let X be an invariant subspace of $\Sigma(A)$. Prove that X is a Sokolov space. Deduce from this fact that every Corson compact space is Sokolov.

161. Prove that every Sokolov space is collectionwise normal and has countable extent. Deduce from this fact that $ext(C_{p,n}(X)) \leq \omega$ for any Sokolov space X and $n \in \mathbb{N}$.

162. Let X be a Sokolov space. Prove that

(i) if $t^*(X) \leq \omega$ then $C_{p,2n+1}(X)$ is Lindelöf for any $n \in \omega$.
(ii) if $l^*(X) \leq \omega$ then $C_{p,2n}(X)$ is Lindelöf for any $n \in \mathbb{N}$;
(iii) if $l^*(X) \cdot t^*(X) \leq \omega$ then $C_{p,n}(X)$ is Lindelöf for any $n \in \mathbb{N}$.

163. Prove that every Sokolov space is ω-stable and ω-monolithic. Deduce from this fact that every Sokolov compact space is Fréchet–Urysohn and has a dense set of points of countable character.

164. Prove that a metrizable space is Sokolov if and only if it is second countable.

165. Let X be a Sokolov space with $l^*(X) \cdot t^*(X) = \omega$. Prove that

(i) if X has a small diagonal then $nw(X) = \omega$;
(ii) if ω_1 is a caliber of X then $nw(X) = \omega$.

166. Prove that if X is a Sokolov space with a G_δ-diagonal then $nw(X) = \omega$.

167. Let X be a Lindelöf Σ-space. Prove that if X is Sokolov then $t(X) \leq \omega$ and $C_{p,n}(X)$ is Lindelöf for any $n \in \mathbb{N}$. In particular, if K is Sokolov compact (or Corson compact) then $C_{p,n}(K)$ is Lindelöf for any $n \in \mathbb{N}$.

168. Let T be an infinite set. Prove that, if $\mathcal{A}$ is an adequate family on T then $K_\mathcal{A}$ is a compact space.

169. Let T be an infinite set. Suppose that $\mathcal{A}$ is an adequate family on T. Prove that $K_\mathcal{A}$ is a Corson compact space if and only if all elements of $\mathcal{A}$ are countable.

170. Let T be an infinite set; suppose that $\mathcal{A}$ is an adequate family on T and u is the function on $K_\mathcal{A}$ which is identically zero. For any $t \in T$ let $e_t(f) = f(t)$ for any $f \in \mathcal{K}_\mathcal{A}$. Observe that $Z = \{e_t : t \in T\} \cup \{u\} \subset C_p(K_\mathcal{A}, \mathbb{D})$; let $\varphi(\xi) = u$ and $\varphi(t) = e_t$ for any $t \in T$. Prove that $\varphi : T^*_\mathcal{A} \to Z$ is a homeomorphism and Z is closed in $C_p(K_\mathcal{A}, \mathbb{D})$. In particular, the space $T^*_\mathcal{A}$ is homeomorphic to a closed subspace of $C_p(K_\mathcal{A}, \mathbb{D})$.

171. Suppose that T is an infinite set and $\mathcal{A}$ is an adequate family on T. Prove that the spaces $C_p(K_\mathcal{A}, \mathbb{D})$ and $C_p(K_\mathcal{A})$ are both continuous images of the space $(T^*_\mathcal{A} \times \omega)^\omega$.

172. Let T be an infinite set. Suppose that $\mathcal{A}$ is an adequate family on T. Prove the space $C_p(K_\mathcal{A})$ is K-analytic if and only if $T^*_\mathcal{A}$ is K-analytic.

173. Let T be an infinite set. Suppose that $\mathcal{A}$ is an adequate family on T. Prove the space $C_p(K_\mathcal{A})$ is Lindelöf Σ if and only if $T^*_\mathcal{A}$ is Lindelöf Σ.

174. Observe that every adequate compact space is zero-dimensional. Give an example of a zero-dimensional Corson compact space which is not homeomorphic to any adequate compact space.

175. Let T be a subspace of $\mathbb{R}$ of cardinality ω_1. Consider some well-ordering $\prec$ on T and let $<$ be the order on T induced from the usual order on $\mathbb{R}$. Denote by $\mathcal{A}_1$ the family of all subsets of T on which the orders $<$ and $\prec$ coincide (i.e., $A \in \mathcal{A}_1$ if and only if, for any distinct $x, y \in A$, we have $x < y$ if and only if $x \prec y$). Let $\mathcal{A}_2$ be the family of all subsets of T on which the orders $<$ and $\prec$ are opposite (i.e., $A \in \mathcal{A}_2$ if and only if, for any distinct $x, y \in A$, we have $x < y$ if and only if $y \prec x$). Check that $\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2$ is an adequate family and that $X = K_\mathcal{A}$ is a Corson compact space for which $C_p(X)$ is not a continuous image of any Lindelöf k-space. In particular, $C_p(X)$ is not a Lindelöf Σ-space.

176. Give a ZFC example of a Corson compact space without a dense metrizable subspace.

177. Give an example of a compact space X for which $(C_p(X))^\omega$ is Lindelöf while X is not Corson compact.

178. Prove that any Corson compact space is a continuous image of a zero-dimensional Corson compact space.

179. Prove that every first countable space is a W-space and every W-space is Fréchet–Urysohn.

180. Suppose that $f : X \to Y$ is an open continuous onto map. Prove that if X is a W-space then so is Y.

181. Suppose that X is a separable space and a closed set $F \subset X$ has an outer base of closed neighbourhoods (i.e., for any $U \in \tau(F, X)$ there is $V \in \tau(F, X)$ such that $\overline{V} \subset U$). Prove that if F is a W-set in X then $\chi(F, X) \leq \omega$. In particular, if X is a separable W-space then $\chi(X) = \omega$.

182. Show that there exist W-spaces which are not first countable and Fréchet–Urysohn spaces which are not W-spaces.

183. Prove that any subspace of a W-space is a W-space and any countable product of W-spaces is a W-space.

184. Prove that any Σ-product of W-spaces is a W-space. Deduce from these facts that if X is a Corson compact space then every non-empty closed $F \subset X$ is a W-set; in particular, X is a W-space.

185. Prove that, if X is a compact space of countable tightness, then a non-empty closed $H \subset X$ is a W-set if and only if $X \backslash H$ is metalindelöf.

186. Let X be a compact scattered space. Prove that a non-empty closed $H \subset X$ is a W-set if and only if $X \backslash H$ is metacompact.

187. (Yakovlev's theorem) Prove that any Corson compact space is hereditarily metalindelöf.

188. Prove that the following are equivalent for any compact space X:

(i) X is Corson compact;
(ii) every closed subset of $X \times X$ is a W-set in $X \times X$;
(iii) the diagonal $\Delta = \{(x, x) : x \in X\}$ of the space X is a W-set in $X \times X$;
(iv) the space $(X \times X) \backslash \Delta$ is metalindelöf;
(v) the space $X \times X$ is hereditarily metalindelöf.

189. Give an example of a compact W-space X such that some continuous image of X is not a W-space.

190. Suppose that X is a compact space which embeds into a σ-product of second countable spaces. Prove that the space $X^2 \backslash \Delta$ is metacompact; here, as usual, the set $\Delta = \{(x, x) : x \in X\}$ is the diagonal of the space X.

191. Observe that any countably compact subspace of a Corson compact space is closed and hence compact. Deduce from this fact that there exists a countably compact space X which embeds into $\Sigma(A)$ for some A but is not embeddable into any Corson compact space.

192. Let M_α be a separable metrizable space for any $\alpha < \omega_1$. Prove that a dense subspace Y of the space $\prod\{M_\alpha : \alpha < \omega_1\}$ is normal if and only if Y is collectionwise normal.

193. Prove that if $2^{\omega_1} = \mathfrak{c}$ then there exists a dense hereditarily normal subspace Y in the space $\mathbb{D}^{\mathfrak{c}}$ such that $ext(Y) = \omega_1$. Deduce from this fact that it is independent of ZFC whether normality implies collectionwise normality in the class of dense subspaces of $\mathbb{D}^{\mathfrak{c}}$.

194. Let X be a monolithic compact space of countable tightness. Prove that any dense normal subspace of $C_p(X)$ is Lindelöf. In particular, if X is a Corson compact space and Y is a dense normal subspace of $C_p(X)$ then Y is Lindelöf.

195. Let X be a Corson compact space. Prove that there exists a σ-discrete set $Y \subset C_p(X)$ which separates the points of X.

196. Prove that, under CH, there exists a compact space X such that no σ-discrete $Y \subset C_p(X)$ separates the points of X.

197. Let X be a metrizable space. Prove that there is a discrete $Y \subset C_p(X)$ which separates the points of X.

198. Prove that, for each cardinal κ, there exists a discrete $Y \subset C_p(\mathbb{I}^\kappa)$ which separates the points of $\mathbb{I}^\kappa$.

199. Prove that $C_p(\beta\omega\backslash\omega)$ cannot be condensed into $\Sigma_*(A)$ for any A.

200. Prove that, for any Corson compact X and any $n \in \mathbb{N}$, the space $C_{p,n}(X)$ linearly condenses onto a subspace of $\Sigma(A)$ for some A.

1.3 More of Lindelöf Σ-Property. Gul'ko Compact Spaces

All spaces are assumed to Tychonoff. Given two families $\mathcal{A}$ and $\mathcal{B}$ of subsets of a space X, say that $\mathcal{A}$ *is a network with respect to* $\mathcal{B}$ if, for any $B \in \mathcal{B}$ and any open $U \supset B$, there is $A \in \mathcal{A}$ such that $B \subset A \subset U$. A space X is *a Lindelöf Σ-space,* if X has a cover $\mathcal{C}$ such that all elements of $\mathcal{C}$ are compact and there exists a countable family $\mathcal{F}$ which is a network with respect to $\mathcal{C}$. The class of Lindelöf Σ-spaces is denoted by $L(\Sigma)$.

Given a space X, denote by A the set $C(X, \mathbb{I})$ and, for each $f \in A$, let $\beta_x(f) = f(x)$. Then $\beta_x : A \to \mathbb{I}$ and the subspace $\tilde{X} = \{\beta_x : x \in X\} \subset \mathbb{I}^A$ is homeomorphic to X. Identifying the spaces X and $\tilde{X}$, we consider that $X \subset \mathbb{I}^A$. Denote by βX the closure of X in $\mathbb{I}^A$. The space βX is called the *Čech-Stone compactification of the space* X. Let $\upsilon X = \{y \in \beta X : H \cap X \neq \emptyset$ for any non-empty G_δ-set $H \subset \beta X$ such that $y \in H\}$. The space υX is called *the Hewitt realcompactification of the space* X. The space X is *realcompact* if $X = \upsilon X$. If $\varphi : X \to Y$ is a continuous mapping then its *dual map* $\varphi^* : C_p(Y) \to C_p(X)$ is defined by $\varphi^*(f) = f \circ \varphi$ for any $f \in C_p(Y)$.

Given a space X, we let $C_{p,0}(X) = X$ and $C_{p,n+1}(X) = C_p(C_{p,n}(X))$ for all $n \in \omega$. The spaces $C_{p,n}(X)$ are called *iterated function spaces of* X. A space X has *a small diagonal* if, for every uncountable $A \subset (X \times X) \backslash \Delta$, there is a neighbourhood U of the diagonal $\Delta = \{(x, x) : x \in X\}$ such that $A \backslash U$ is uncountable. A space Y is *Eberlein–Grothendieck* if it can be embedded into $C_p(K)$ for some compact space K. Say that a space X is $K_{\sigma\delta}$ if there exists a space Y such that $X \subset Y$ and $X = \bigcap\{Y_n : n \in \omega\}$, where each Y_n is a σ-compact subset of Y. A *K-analytic space* is a continuous image of a $K_{\sigma\delta}$-space.

A family $\mathcal{U} \subset \exp X$ is said to be *point-finite at* $x \in X$ if $\{U \in \mathcal{U} : x \in U\}$ is finite. The family $\mathcal{U}$ is *weakly σ-point-finite* if there exists a sequence $\{\mathcal{U}_n : n \in \omega\}$ of subfamilies of $\mathcal{U}$ such that, for every $x \in X$, we have $\mathcal{U} = \bigcup\{\mathcal{U}_n : n \in M_x\}$ where $M_x = \{n \in \omega : \mathcal{U}_n$ is point-finite at $x\}$. The family $\mathcal{U}$ is *T_0-separating* if, for any distinct $x, y \in X$, there exists $U \in \mathcal{U}$ such that $|U \cap \{x, y\}| = 1$. A set U is *a cozero set* in a space X if there is $f \in C_p(X)$ such that $U = X \backslash f^{-1}(0)$. The spaces which have a countable network are called *cosmic*.

Say that X is a *Gul'ko space* if $C_p(X)$ is a Lindelöf Σ-space. A compact Gul'ko space is called *Gul'ko compact*. The space X is *Talagrand* if $C_p(X)$ is K-analytic. A compact Talagrand space is called Talagrand compact.

A space X is called *d-separable,* if it has a dense σ-discrete subspace. If X is a space and $\mathcal{U}, \mathcal{V} \subset \tau^*(X)$, we say that $\mathcal{V}$ is *a π-base for $\mathcal{U}$* if, for every $U \in \mathcal{U}$, there is $V \in \mathcal{V}$ such that $V \subset U$. The *point-finite cellularity* $p(X)$ of a space X is the supremum of cardinalities of point-finite families of non-empty open subsets of X. A space X is *κ-stable* if, for any continuous onto map $f : X \to Y$, we have $nw(Y) \leq \kappa$ whenever $iw(Y) \leq \kappa$. Given a cardinal κ, a space X is *κ-monolithic* if $A \subset X$ and $|A| \leq \kappa$ implies $nw(\overline{A}) \leq \kappa$. Now, X is *a Preiss–Simon space* if, for any closed $F \subset X$ and any non-isolated $x \in F$, there exists a sequence $\{U_n : n \in \omega\}$ of open non-empty subsets of F such that $U_n \to x$, i.e., any neighbourhood of x contains all but finitely many of U_n's.

An uncountable regular cardinal κ is *a caliber* of a space X if, for any family $\mathcal{U} \subset \tau^*(X)$ of cardinality κ, there exists $\mathcal{U}' \subset \mathcal{U}$ such that $|\mathcal{U}'| = \kappa$ and $\bigcap \mathcal{U}' \neq \emptyset$. An uncountable regular cardinal κ is called *a precaliber* of a space X if, for any family $\mathcal{U} \subset \tau^*(X)$ of cardinality κ, there exists $\mathcal{U}' \subset \mathcal{U}$ such that $|\mathcal{U}'| = \kappa$ and $\mathcal{U}'$ is centered ($\equiv$has the finite intersection property, i.e., $\bigcap \mathcal{V} \neq \emptyset$ for any finite $\mathcal{V} \subset \mathcal{U}'$).

The symbol $\mathbb{P}$ stands for the space of the irrationals which is identified with ω^ω; in particular, if $p, q \in \mathbb{P}$, we say that $p \leq q$ if $p(n) \leq q(n)$ for any $n \in \omega$. Let $\omega^0 = \{\emptyset\}$ and $\omega^{<\omega} = \bigcup\{\omega^n : n \in \omega\}$. Usually, when considering $\omega^{<\omega}$, we identify $n = 0$ with the empty set and every $n \in \mathbb{N}$ with the set $\{0, \ldots, n-1\}$. If $s \in \omega^{<\omega}$ and $n \in \omega$ then $t = s^\frown n \in \omega^{<\omega}$ is defined as follows: there is a unique $k \in \omega$ with $s \in \omega^k$; let $t|k = s$ and $t(k) = n$.

Say that a space X is $\mathbb{P}$*-dominated* if there is a compact cover $\{K_p : p \in \mathbb{P}\}$ of the space X such that $p, q \in \mathbb{P}$ and $p \leq q$ imply $K_p \subset K_q$. In other words, the space X is $\mathbb{P}$-dominated if it has a $\mathbb{P}$-directed compact cover. A space X is said to be *strongly* $\mathbb{P}$*-dominated* if it has a $\mathbb{P}$-directed compact cover $\mathcal{C}$ such that, for any compact $K \subset X$ there is $C \in \mathcal{C}$ such that $K \subset C$, i.e., $\mathcal{C}$ "swallows" all compact subsets of X.

Given a space Z, the family of all compact subsets of Z is denoted by $\mathcal{K}(Z)$. A space X *is dominated by a space* Y if there is a compact cover $\{F_K : K \in \mathcal{K}(Y)\}$ of the space X such that $K, L \in \mathcal{K}(Y)$ and $K \subset L$ imply $F_K \subset F_L$.

Given a set A and a point $x \in \mathbb{R}^A$, let $\mathrm{supp}(x) = \{a \in A : x(a) \neq 0\}$. If we have a family $s = \{A_n : n \in \omega\} \subset \exp A$ and $\bigcup s = A$ then $N_x = \{n \in \omega : A_n \cap \mathrm{supp}(x)$ is finite$\}$ for any $x \in \mathbb{R}^A$. Let $\Sigma_s(A) = \{x \in \Sigma(A) : A = \bigcup\{A_n : n \in N_x\}\}$; here, as usual, $\Sigma(A) = \{x \in \mathbb{R}^A : |\mathrm{supp}(x)| \leq \omega\}$ and $\Sigma_*(A) = \{x \in \mathbb{R}^A :$ for any $\varepsilon > 0$, the set $\{a \in A : |x(a)| > \varepsilon\}$ is finite$\}$. The spaces $\Sigma(A)$ (or $\Sigma_*(A)$) will be called Σ-products (Σ_*-products) of real lines. If X is a space and $M \subset X$, let μ_M be the topology generated by $\tau(X) \cup \{\{x\} : x \in X \backslash M\}$. The space (X, μ_M) is usually denoted by X_M.

The statement CH (called *Continuum Hypothesis*) says that the first uncountable ordinal is equal to the continuum, i.e., $\omega_1 = \mathfrak{c}$. The statement "$\kappa^+ = 2^\kappa$ for any infinite cardinal κ" is called *Generalized Continuum Hypothesis (GCH).*

201. Suppose that $X = \upsilon Y$ and Z is a subspace of $\mathbb{R}^X$ such that $C_p(X) \subset Z$. Prove that there exists $Z' \subset \mathbb{R}^Y$ such that $C_p(Y) \subset Z'$ and Z' is a continuous image of Z.

202. Suppose that X is σ-compact. Prove that there exists a $K_{\sigma\delta}$-space Z such that $C_p(X) \subset Z \subset \mathbb{R}^X$.

203. Suppose that υX is σ-compact. Prove that there exists a K-analytic space Z such that $C_p(X) \subset Z \subset \mathbb{R}^X$.

204. Prove that X is pseudocompact if and only if there exists a σ-compact space Z such that $C_p(X) \subset Z \subset \mathbb{R}^X$.

205. Give an example of a Lindelöf space X for which there exists no Lindelöf space Z such that $C_p(X) \subset Z \subset \mathbb{R}^X$.

206. Prove that υX is a Lindelöf Σ-space if and only if $C_p(X) \subset Z \subset \mathbb{R}^X$ for some Lindelöf Σ-space Z. In particular,

(i) if $C_p(X)$ is a Lindelöf Σ-space, then υX is a Lindelöf Σ-space;
(ii) (Uspenskij's theorem) if X is a Lindelöf Σ-space then there exists a Lindelöf Σ-space Z such that $C_p(X) \subset Z \subset \mathbb{R}^X$;
(iii) if $\upsilon(C_p(X))$ is a Lindelöf Σ-space then υX is Lindelöf Σ.

207. Given a natural $n \geq 1$, suppose that there exists a Lindelöf Σ-space Z such that $C_{p,n}(X) \subset Z \subset \mathbb{R}^{C_{p,n-1}(X)}$. Prove that there exists a Lindelöf Σ-space Y such that $C_p(X) \subset Y \subset \mathbb{R}^X$.

208. Suppose that $C_p(X)$ is a Lindelöf Σ-space. Prove that $C_{p,n}(X)$ is ω-stable and ω-monolithic for any natural n.

209. Prove that a space X is dominated by a space homeomorphic to the irrationals if and only if X is $\mathbb{P}$-dominated.

210. Suppose that X is dominated by a second countable space. Prove that there is a countable family $\mathcal{F}$ of subsets of X which is a network with respect to a cover of X with countably compact subspaces of X.

211. Suppose that a space X has a countable family $\mathcal{F}$ which is a network with respect to a cover of X with countably compact subspaces of X. Prove that υX is a Lindelöf Σ-space.

212. Prove that the property of being dominated by a second countable space is preserved by countable unions, products and intersections as well as by closed subspaces and continuous images.

213. Show that every Lindelöf Σ-space is dominated by a second countable space. Prove that X is a Lindelöf Σ-space if and only if X is Dieudonné complete and dominated by a second countable space.

214. Prove that, for any space X, the space $C_p(X)$ is dominated by a second countable space if and only $C_p(X)$ is Lindelöf Σ.

215. Prove that, for any space X, the space $C_p(X)$ is $\mathbb{P}$-dominated if and only if $C_p(X)$ is K-analytic.

216. Prove that, for any space X, the space $C_p(X)$ is strongly $\mathbb{P}$-dominated if and only if X is countable and discrete.

217. Observe that there exist spaces X for which $C_p(X, \mathbb{I})$ is Lindelöf Σ while $C_p(X)$ is not Lindelöf. Supposing that υX and $C_p(X, \mathbb{I})$ are Lindelöf Σ-spaces prove that $C_p(X)$ is a Lindelöf Σ-space. In particular, if X is Lindelöf Σ then the space $C_p(X)$ is Lindelöf Σ if and only if $C_p(X, \mathbb{I})$ is a Lindelöf Σ-space.

218. (Okunev's theorem). Suppose that X and Y are Lindelöf Σ-spaces such that $Y \subset C_p(X)$. Prove that $C_p(Y)$ is a Lindelöf Σ-space.

219. Let X and $C_p(X)$ be Lindelöf Σ-spaces. Prove that, for every natural n, the space $C_{p,n}(X)$ is a Lindelöf Σ-space. In particular, if X is compact and $C_p(X)$ is Lindelöf Σ then all iterated function spaces of X are Lindelöf Σ-spaces.

220. For an arbitrary Lindelöf Σ-space X, prove that every countably compact subspace $Y \subset C_p(X)$ is Gul'ko compact.

221. Suppose that $C_p(X)$ is a Lindelöf Σ-space. Prove that every countably compact $Y \subset C_p(X)$ is Gul'ko compact.

222. (Reznichenko's compactum) Prove that there exists a compact space M with the following properties:

(i) $C_p(M)$ is a K-analytic space, i.e., M is Talagrand compact;
(ii) there is $x \in M$ such that $M \backslash \{x\}$ is pseudocompact and M is the Stone–Čech extension of $M \backslash \{x\}$.

As a consequence, there is an example of a K-analytic space X such that some closed pseudocompact subspace of $C_p(X)$ is not countably compact.

223. Suppose that, for a countably compact space X, there exists a condensation $f : X \to Z \subset C_p(Y)$, where $C_p(Y)$ is a Lindelöf Σ-space. Prove that f is a homeomorphism and X is Gul'ko compact.

224. Give an example of a pseudocompact non-countably compact space X which can be condensed onto a compact $K \subset C_p(Y)$, where $C_p(Y)$ is Lindelöf Σ.

225. Give an example of a space X such that $C_p(X)$ is Lindelöf Σ and some pseudocompact subspace of $C_p(X)$ is not countably compact.

226. Observe that if there exist spaces X such that $C_p(X)$ is a Lindelöf Σ-space while $t(C_p(X)) > \omega$. Prove that, if $C_p(X)$ is Lindelöf Σ and $Y \subset C_p(X)$ is pseudocompact then Y is Fréchet–Urysohn.

227. Show that there exists a space X such that $C_p(X)$ is a Lindelöf Σ-space and $t(Y) > \omega$ for some σ-compact subspace $Y \subset C_p(X)$.

228. Let X be an arbitrary space. Denote by $\pi : C_p(\upsilon X) \to C_p(X)$ the restriction map. Prove that, for any countably compact $Y \subset C_p(X)$, the space $\pi^{-1}(Y) \subset C_p(\upsilon X)$ is countably compact.

229. Give an example of a space X such that $\pi^{-1}(Y)$ is not pseudocompact for some pseudocompact $Y \subset C_p(X)$. Here $\pi : C_p(\upsilon X) \to C_p(X)$ is the restriction map.

230. Assume that υX is a Lindelöf Σ-space and $\pi : C_p(\upsilon X) \to C_p(X)$ is the restriction map. Prove that, for any compact $Y \subset C_p(X)$, the space $\pi^{-1}(Y) \subset C_p(\upsilon X)$ is also compact.

231. Assume that υX is a Lindelöf Σ-space and $\pi : C_p(\upsilon X) \to C_p(X)$ is the restriction map. Prove that, for any Lindelöf Σ-space Y contained in $C_p(X)$, the space $\pi^{-1}(Y) \subset C_p(\upsilon X)$ is Lindelöf Σ.

232. Let X be a pseudocompact space and denote by $\pi : C_p(\beta X) \to C_p(X)$ the restriction map. Prove that, for any Lindelöf Σ-space (compact space) $Y \subset C_p(X)$, the space $\pi^{-1}(Y) \subset C_p(\beta X)$ is Lindelöf Σ (or compact, respectively).

233. Give an example of a pseudocompact X such that $\pi^{-1}(Y) \subset C_p(\beta X)$ is not Lindelöf for some Lindelöf $Y \subset C_p(X)$. Here $\pi : C_p(\beta X) \to C_p(X)$ is the restriction map.

234. Observe that $C_p(X)$ is a Lindelöf Σ-space if and only if $C_p(\upsilon X)$ is Lindelöf Σ; prove that, for any X, the space $C_p(X)$ is K-analytic if and only if $C_p(\upsilon X)$ is K-analytic. In other words, X is a Talagrand space if and only if υX is Talagrand.

235. Suppose that $C_p(X)$ is a Lindelöf Σ-space. Prove that $C_{p,n}(\upsilon X)$ is a Lindelöf Σ-space for every $n \in \mathbb{N}$.

236. Given an arbitrary space X let $\pi : C_p(\upsilon X) \to C_p(X)$ be the restriction mapping. Let $\pi^*(\varphi) = \varphi \circ \pi$ for any function $\varphi \in \mathbb{R}^{C_p(X)}$ and observe that the map $\pi^* : \mathbb{R}^{C_p(X)} \to \mathbb{R}^{C_p(\upsilon X)}$ is an embedding. Identifying the space $\upsilon(C_p(C_p(X)))$ with the subspace $\{\varphi \in \mathbb{R}^{C_p(X)} : \varphi$ is strictly ω-continuous on $C_p(X)\}$ of the space $\mathbb{R}^{C_p(X)}$ (see TFS-438) prove that

(i) $\pi^*(C_p(C_p(X))) \subset \pi^*(\upsilon(C_p(C_p(X)))) \subset C_p(C_p(\upsilon X))$;
(ii) if $C_p(X)$ is normal then $\pi^*(\upsilon(C_p(C_p(X)))) = C_p(C_p(\upsilon X))$ and hence the spaces $\upsilon(C_p(C_p(X)))$ and $C_p(C_p(\upsilon X))$ are homeomorphic.

237. Suppose that $C_p(X)$ is a Lindelöf Σ-space. Prove that $C_{p,2n}(\upsilon X)$ is homeomorphic to $\upsilon(C_{p,2n}(X))$ for every $n \in \mathbb{N}$.

238. Suppose that $C_p(X)$ is a Lindelöf Σ-space. Prove that $C_{p,2n+1}(\upsilon X)$ can be condensed onto $C_{p,2n+1}(X)$ for every $n \in \omega$.

239. Suppose that $C_{p,2k+1}(X)$ is a Lindelöf Σ-space for some $k \in \omega$. Prove that $C_{p,2n+1}(X)$ is a Lindelöf Σ-space every $n \in \omega$.

240. Suppose that $C_{p,2k}(X)$ is a Lindelöf Σ-space for some $k \in \mathbb{N}$. Prove that $C_{p,2n}(X)$ is a Lindelöf Σ-space every $n \in \mathbb{N}$.

241. Give an example of a space X such that $C_p(X)$ is not Lindelöf while $C_{p,2n}(X)$ is a Lindelöf Σ-space for every $n \in \mathbb{N}$.

242. Give an example of a space X such that $C_pC_p(X)$ is not Lindelöf while $C_{p,2n+1}(X)$ is a Lindelöf Σ-space for every $n \in \omega$.

243. Prove that, for any space X, only the following distributions of the Lindelöf Σ-property in iterated function spaces are possible:

(i) $C_{p,n}(X)$ is not a Lindelöf Σ-space for any $n \in \mathbb{N}$;
(ii) $C_{p,n}(X)$ is a Lindelöf Σ-space for any $n \in \mathbb{N}$;
(iii) $C_{p,2n+1}(X)$ is a Lindelöf Σ-space and $C_{p,2n+2}(X)$ is not Lindelöf for any $n \in \omega$;
(iv) $C_{p,2n+2}(X)$ is a Lindelöf Σ-space and $C_{p,2n+1}(X)$ is not Lindelöf for any $n \in \omega$.

244. Suppose that $C_{p,2k+1}(X)$ is a Lindelöf Σ-space for some $k \in \omega$. Prove that, if $C_{p,2l+2}(X)$ is normal for some $l \in \omega$, then $C_{p,n}(X)$ is a Lindelöf Σ-space for any $n \in \mathbb{N}$.

245. Suppose that $C_{p,2k+2}(X)$ is a Lindelöf Σ-space for some $k \in \omega$. Prove that, if $C_{p,2l+1}(X)$ is normal for some $l \in \omega$, then $C_{p,n}(X)$ is a Lindelöf Σ-space for any $n \in \mathbb{N}$.

246. Prove that, if $C_p(X)$ is a Lindelöf Σ-space, then $\upsilon(C_pC_p(X))$ is a Lindelöf Σ-space.

247. Prove that, if X is normal and $\upsilon(C_p(X))$ is a Lindelöf Σ-space, then $\upsilon(C_pC_p(X))$ is a Lindelöf Σ-space.

248. Prove that, if X is realcompact and $\upsilon(C_p(X))$ is a Lindelöf Σ-space, then $\upsilon(C_pC_p(X))$ is a Lindelöf Σ-space.

249. Let ω_1 be a caliber of a space X. Prove that $C_p(X)$ is a Lindelöf Σ-space if and only if X has a countable network.

250. Prove that there exists a space X such that ω_1 is a precaliber of X, the space $C_{p,n}(X)$ is a Lindelöf Σ-space for all $n \in \omega$, while X does not have a countable network.

251. Let X be a Lindelöf Σ-space with ω_1 a caliber of X. Prove that any Lindelöf Σ-subspace of $C_p(X)$ has a countable network.

252. Prove that a Lindelöf Σ-space Y has a small diagonal if and only if it embeds into $C_p(X)$ for some X with ω_1 a caliber of X.

253. Prove that, if $C_p(X)$ is a Lindelöf Σ-space and has a small diagonal then X has a countable network.

254. Suppose that a space X has a dense subspace which is a continuous image of a product of separable spaces. Prove that any Lindelöf Σ-subspace of $C_p(X)$ has a countable network.

255. Prove that any first countable space is a Preiss–Simon space.

256. Prove that any Preiss–Simon space is Fréchet–Urysohn.

257. Give an example of a compact Fréchet–Urysohn space which does not have the Preiss–Simon property.

258. Let X be a space which has the Preiss–Simon property. Prove that each pseudocompact subspace of X is closed in X.

259. Suppose that X is a Preiss–Simon compact space. Prove that, for any proper dense $Y \subset X$, the space X is not the Čech-Stone extension of Y.

260. Prove that the following properties are equivalent for any countably compact space X:

(i) X is a Preiss–Simon space;
(ii) each pseudocompact subspace of X is closed in X;
(iii) for each closed $F \subset X$ and any non-isolated $x \in F$, the space $F\backslash\{x\}$ is not pseudocompact.

261. Let X be a Lindelöf Σ-space. Suppose that $Y \subset C_p(X)$ and the set of non-isolated points of Y is Lindelöf Σ. Prove that $C_p(Y, \mathbb{I})$ is Lindelöf Σ.

262. Let X be an Eberlein–Grothendieck space. Suppose that the set of non-isolated points of X is σ-compact. Prove that $C_p(X, \mathbb{I})$ is $K_{\sigma\delta}$.

263. Let X be a second countable space. Prove that, for any $M \subset X$, the space $C_p(X_M, \mathbb{I})$ is Lindelöf Σ.

264. Let X be a σ-compact Eberlein–Grothendieck space. Prove that $C_p(X)$ is a $K_{\sigma\delta}$-space.

265. Give an example of a Lindelöf space X such that $C_p(X, \mathbb{I})$ is Lindelöf Σ and $X \times X$ is not Lindelöf.

266. Suppose that X is a space such that $\upsilon(C_p(X))$ is Lindelöf Σ and we have the equality $s(C_p(X)) = \omega$. Prove that $nw(X) = \omega$.

267. Suppose that $C_p(X)$ is hereditarily stable and υX is a Lindelöf Σ-space. Prove that $nw(X) = \omega$.

268. Show that if $C_p(X)$ is hereditarily stable then $nw(Y) = \omega$ for any Lindelöf Σ-subspace $Y \subset X$.

269. Suppose that $\upsilon(C_p(X))$ is a Lindelöf Σ-space and ω_1 is a caliber of $C_p(X)$. Prove that $nw(Y) = \omega$ for any Lindelöf Σ-subspace $Y \subset X$.

270. Give an example of a space X which has a weakly σ-point-finite family $\mathcal{U} \subset \tau^*(X)$ such that $\mathcal{U}$ is not σ-point-finite.

271. Let X be an arbitrary space with $s(X) \leq \kappa$. Prove that any weakly σ-point-finite family of non-empty open subsets of X has cardinality $\leq \kappa$.

272. Give an example of a non-cosmic Lindelöf Σ-space X such that any closed uncountable subspace of X has more than one (and hence infinitely many) non-isolated points.

273. Suppose that $C_p(X)$ is a Lindelöf Σ-space. Prove that, if all closed uncountable subspaces of $C_p(X)$ have more than one non-isolated points, then $C_p(X)$ has a countable network.

274. Let X be a Lindelöf Σ-space with a unique non-isolated point. Prove that any subspace of $C_p(X)$ has a weakly σ-point-finite T_0-separating family of cozero sets.

275. Let X be a space of countable spread. Prove that $C_p(X)$ is a Lindelöf Σ-space if and only if X has a countable network.

276. Show that, under CH, there exists a space X of countable spread for which there is a Lindelöf Σ-space $Y \subset C_p(X)$ with $nw(Y) > \omega$.

277. Let X be a space with a unique non-isolated point: $X = \{a\} \cup Y$, where all points of Y are isolated and $a \notin Y$. Prove that, for every infinite cardinal κ, the following conditions are equivalent:

(i) $p(C_p(X)) \leq \kappa$;

(ii) if $\{A_\alpha : \alpha < \kappa^+\}$ is a disjoint family of finite subsets of Y then there is an infinite $S \subset \kappa^+$ such that $a \notin \overline{\bigcup\{A_\alpha : \alpha \in S\}}$;

(iii) if $\{A_\alpha : \alpha < \kappa^+\}$ is a family of finite subsets of Y then there is an infinite $S \subset \kappa^+$ such that $a \notin \overline{\bigcup\{A_\alpha : \alpha \in S\}}$.

278. Let X be a space with a unique non-isolated point. Prove that, if X has no non-trivial convergent sequences, then the point-finite cellularity of $C_p(X)$ is countable.

279. Call a family γ of finite subsets of a space X *concentrated* if there is no infinite $\mu \subset \gamma$ such that $\bigcup \mu$ is discrete and C^*-embedded in X. Prove that, if every concentrated family of finite subsets of X has cardinality $\leq \kappa$, then $p(C_p(X)) \leq \kappa$.

280. Prove that there exists a Lindelöf Σ-space X with a unique non-isolated point such that $C_p(X)$ is a Lindelöf Σ-space, $p(C_p(X)) = \omega$, all compact subsets of X are countable and $nw(X) = \mathfrak{c}$.

281. Prove that there exists a space X such that $C_p(X)$ is Lindelöf Σ-space, $nw(X) = \mathfrak{c}$ and $p(X) = \omega$.

282. Prove that any continuous image and any closed subspace of a Gul'ko compact space is a Gul'ko compact space.

283. Prove that any countable product of Gul'ko compact spaces is a Gul'ko compact space.

284. Let X be a Gul'ko compact space. Prove that for every second countable M, the space $C_p(X, M)$ is Lindelöf Σ.

285. Prove that if $C_p(X)$ is a Lindelöf Σ-space then X can be condensed into a Σ-product of real lines. Deduce from this fact that every Gul'ko compact space is Corson compact.

286. Prove that if X is Corson compact then the space $C_p(X)$ condenses linearly into a Σ_*-product of real lines. As a consequence, for any Gul'ko compact X the space $C_p(X)$ condenses linearly into a Σ_*-product of real lines.

287. Let X be a Corson compact space. Prove that, if $p(C_p(X)) = \omega$ then X is metrizable. Therefore if X is a Gul'ko compact space and $p(C_p(X)) = \omega$ then X is metrizable.

288. Suppose that X and $C_p(X)$ are Lindelöf Σ-spaces and $p(C_p(X)) = \omega$. Prove that $|X| \leq \mathfrak{c}$.

289. Prove that a compact space X is Gul'ko compact if and only if X has a weakly σ-point-finite T_0-separating family of cozero sets.

290. Prove that a compact X is Gul'ko compact if and only if there exists a set A such that X embeds into $\Sigma_s(A)$ for some family $s = \{A_n : n \in \omega\}$ of subsets of A with $\bigcup s = A$.

291. Suppose that X is a space, $n \in \mathbb{N}$ and a non-empty family $\mathcal{U} \subset \tau^*(X)$ has order $\leq n$, i.e., every $x \in X$ belongs to at most n elements of $\mathcal{U}$. Prove that there exist disjoint families $\mathcal{V}_1, \dots, \mathcal{V}_n$ of non-empty open subsets of X such that $\mathcal{V} = \bigcup\{\mathcal{V}_i : i \leq n\}$ is a π-base for $\mathcal{U}$.

292. Suppose that a space X has the Baire property and $\mathcal{U}$ is a weakly σ-point-finite family of non-empty open subsets of X. Prove that there exists a σ-disjoint family $\mathcal{V} \subset \tau^*(X)$ which is a π-base for $\mathcal{U}$.

293. Prove that every Gul'ko compact space has a dense metrizable subspace.

294. Let X be a Gul'ko compact space. Prove that $w(X) = d(X) = c(X)$. In particular, each Gul'ko compact space with the Souslin property is metrizable.

295. Let X be a pseudocompact space with the Souslin property. Prove that any Lindelöf Σ-subspace of $C_p(X)$ has a countable network.

296. Let X be a Lindelöf Σ-space. Suppose that Y is a pseudocompact subspace of $C_p(X)$. Prove that Y is compact and metrizable if and only if $c(Y) = \omega$.

297. Prove that every Gul'ko compact space is hereditarily d-separable.

298. Let X be a compact space. Prove that $C_p(X)$ is a K-analytic space if and only if X has a T_0-separating family $\mathcal{U}$ of open F_σ-subsets of X and subfamilies $\{\mathcal{U}_s : s \in \omega^{<\omega}\}$ of the family $\mathcal{U}$ with the following properties:

(a) $\mathcal{U}_\emptyset = \mathcal{U}$ and $\mathcal{U}_s = \bigcup\{\mathcal{U}_{s^\frown k} : k \in \omega\}$ for any $s \in \omega^{<\omega}$;
(b) for every $x \in X$ and every $f \in \omega^\omega$, there exists $m \in \omega$ such that the family $\mathcal{U}_{f|n}$ is point-finite at x for all $n \geq m$.

299. Let X be a compact space. Prove that $C_p(X)$ is a K-analytic space if and only if X can be embedded into some $\Sigma(A)$ in such a way that, for some family $\{A_s : s \in \omega^{<\omega}\}$ of subsets of A, the following conditions are fulfilled:

(a) $A_\emptyset = A$ and $A_s = \bigcup\{A_{s^\frown k} : k \in \omega\}$ for any $s \in \omega^{<\omega}$;
(b) for any point $x \in X$ and any $f \in \omega^\omega$, there exists $m \in \omega$ such that the set $A_{f|n} \cap \operatorname{supp}(x)$ is finite for all $n \geq m$.

300. (Talagrand's example) Show that there exists a Gul'ko compact space X such that $C_p(X)$ is not K-analytic. In other words, not every Gul'ko compact space is Talagrand compact.

1.4 Eberlein Compact Spaces

All spaces are assumed to be Tychonoff. Given an arbitrary set A, we will need the spaces $\Sigma(A) = \{x \in \mathbb{R}^A : |\{a \in A : x(a) \neq 0\}| \leq \omega\}$, $\Sigma_*(A) = \{x \in \mathbb{R}^A :$ for any $\varepsilon > 0$, the set $\{a \in A : |x(a)| \geq \varepsilon\}$ is finite$\}$ and $\sigma(A) = \{x \in \mathbb{R}^A :$ the set $\{a \in A : x(a) \neq 0\}$ is finite$\}$. Suppose that we have a product $X = \prod\{X_t : t \in T\}$ and $x \in X$. Let $\Sigma(X, x) = \{y \in X : |\{t \in T : y(t) \neq x(t)\}| \leq \omega\}$. The space $\Sigma(X, x)$ is called *the Σ-product of $\{X_t : t \in T\}$ centered at the point x* and the point x is *the center* of the relevant Σ-product. If we have a product $X = \prod\{X_t : t \in T\}$ and $x \in X$, then $\sigma(X, x) = \{y \in X : |\{t \in T : y(t) \neq x(t)\}| < \omega\}$. The set $\sigma(X, x)$ is called *the σ-product of $\{X_t : t \in T\}$ centered at the point x* and the point x is *the center* of the relevant σ-product. If no center of a Σ-product or a σ-product is specified, then the formulated statements are valid (or must be proved) for any center of the mentioned Σ-product or σ-product. If X is a set, then $\Delta = \Delta_X = \{(x, x) : x \in X\}$ is the diagonal of X.

The space $\mathbb{D}$ is the two-point set $\{0, 1\}$ with the discrete topology. A space X is called *functionally perfect* if there is a compact $K \subset C_p(X)$ which separates the points of X. A functionally perfect compact space is called *Eberlein compact.* A compact space K is called *a uniform Eberlein compact* if, for some set A, the space K embeds into $\Sigma_*(A)$ in such a way that there exists a function $N : \mathbb{R}^+ \to \mathbb{N}$ such that $|\{a \in A : |x(a)| \geq \varepsilon\}| \leq N(\varepsilon)$ for all $x \in K$ and $\varepsilon > 0$. Here $\mathbb{R}^+$ is the set of all positive real numbers. Call X *strong Eberlein compact* if it is homeomorphic to a compact subspace of $\sigma_0(A) = \{x \in \mathbb{D}^A : |x^{-1}(1)| < \omega\}$ for some A. A space is called *k-separable* if it has a dense σ-compact subspace. The expression $X \simeq Y$ says that X and Y are homeomorphic. Given a locally compact non-compact X, let $\alpha(X) = X \cup \{a_0\}$, where $a_0 \notin X$. Let $\mu = \tau(X) \cup \{\{a_0\} \cup U : X \backslash U$ is compact$\}$. The space $(\alpha(X), \mu)$ is called the *one-point compactification of the space X*.

If $\mathcal{P}$ is a class of spaces, then $\mathcal{P}_\sigma$ consists of spaces representable as a countable union of elements of $\mathcal{P}$. The class $\mathcal{P}_\delta$ contains the spaces which are countable intersections of elements of $\mathcal{P}$ in some larger space. More formally, $X \in \mathcal{P}_\sigma$ if $X = \bigcup\{X_n : n \in \omega\}$ where $X_n \in \mathcal{P}$ for any $n \in \omega$. Analogously, $X \in \mathcal{P}_\delta$ if there exists a space Y and $Y_n \subset Y$ such that $Y_n \in \mathcal{P}$ for all $n \in \omega$ and $\bigcap\{Y_n : n \in \omega\} \simeq X$. Then $\mathcal{P}_{\sigma\delta} = (\mathcal{P}_\sigma)_\delta$. Say that X is *a $K_{\sigma\delta}$-space* if $X \in \mathcal{K}_{\sigma\delta}$ where $\mathcal{K}$ is the class of compact spaces.

Let T be an infinite set. An arbitrary family $\mathcal{A} \subset \exp T$ is called *adequate* if $\bigcup \mathcal{A} = T$, $\exp A \subset \mathcal{A}$ for any $A \in \mathcal{A}$, and $A \in \mathcal{A}$ whenever all finite subsets of A belong to $\mathcal{A}$. Given $A \subset T$, let $\chi_A(t) = 1$ if $t \in A$ and $\chi_A(t) = 0$ if $t \notin A$. The map $\chi_A : T \to \{0, 1\}$ is called *the characteristic function of A in the set T*. If we have a set T and an adequate family $\mathcal{A}$ on T, let $K_{\mathcal{A}} = \{\chi_A \in \mathbb{D}^T : A \in \mathcal{A}\}$. Another object associated with $\mathcal{A}$, is the space $T^*_{\mathcal{A}}$ whose underlying set is $T \cup \{\xi\}$, where $\xi \notin T$, all points of T are isolated in $T^*_{\mathcal{A}}$ and the basic neighbourhoods of ξ are the complements of finite unions of elements of $\mathcal{A}$. A subspace $X \subset \mathbb{D}^T$ is called *adequate* if $X = K_{\mathcal{A}}$ for some adequate family $\mathcal{A}$ on T.

A family γ of subsets of X is T_0-*separating* if, for any distinct $x, y \in X$, there is $A \in \gamma$ such that $|A \cap \{x, y\}| = 1$. Now, γ is T_1-*separating* if, for any distinct $x, y \in X$, there are $A, B \in \gamma$ such that $A \cap \{x, y\} = \{x\}$ and $B \cap \{x, y\} = \{y\}$. A set $U \subset X$ is *cozero* if there is $f \in C_p(X)$ such that $U = f^{-1}(\mathbb{R}\backslash\{0\})$. Say that X is *a Preiss–Simon space* if, for any closed $F \subset X$ and for any non-isolated $x \in F$, there exists a sequence $\{U_n : n \in \omega\}$ of open non-empty subsets of F such that $U_n \to x$, i.e., any neighbourhood of x contains all but finitely many of U_n's. A space X is called *homogeneous* if, for every $x, y \in X$, there exists a homeomorphism $f : X \to X$ such that $f(x) = y$. A space X is σ-metacompact if any open cover of X has a σ-point-finite open refinement, i.e., a refinement which is a countable union of point-finite families.

All linear spaces in this book are considered over the space $\mathbb{R}$ of the reals. Let L be a linear topological space. A set $M \subset L$ is called *a linear subspace of* L if $\alpha x + \beta y \in M$ whenever $x, y \in M$ and $\alpha, \beta \in \mathbb{R}$. A linear space L, equipped with a topology τ, is called *linear topological space* if (L, τ) is a T_1-space and the linear operations $(x, y) \to x + y$ and $(t, x) \to tx$ are continuous with respect to τ. A subset A of a linear space L is called *convex* if $x, y \in A$ implies $tx + (1 - t)y \in A$ for any number $t \in [0, 1]$. A *convex hull* $\mathrm{conv}(A)$ of a set $A \subset L$ is the set $\{t_1 x_1 + \ldots + t_n x_n : n \in \mathbb{N},\ x_1, \ldots, x_n \in A,\ t_1, \ldots, t_n \in [0, 1],\ t_1 + \ldots + t_n = 1\}$. A linear topological space L is called *locally convex* if it has a base which consists of convex sets. If L is a linear space and $A \subset L$ then *the linear span of* A *in* L is the intersection of all linear subspaces of L which contain A. A real-valued function $x \to ||x||$, defined on a linear space L, is *a norm* if it has the following properties:

(N1) $||x|| \geq 0$ for any $x \in L$; besides, $||x|| = 0$ if and only if $x = 0$;
(N2) $||\alpha x|| = |\alpha| \cdot ||x||$ for any $x \in L$ and $\alpha \in \mathbb{R}$;
(N3) $||x + y|| \leq ||x|| + ||y||$ for any $x, y \in L$.

If $||\cdot||$ is a norm on a linear space L, the pair $(L, ||\cdot||)$ is called *a normed space*. If the norm is clear, we write L instead of $(L, ||\cdot||)$. If L is a normed space, then the function $d_L(x, y) = ||x - y||$ is a metric on L. Every normed space L is considered to carry the topology generated by the metric d_L. If (L, d_L) is a complete metric space, L is called *a Banach space*. Given a linear space L, a function $f : L \to \mathbb{R}$ is called *a linear functional* if $f(\alpha x + \beta y) = \alpha f(x) + \beta f(y)$ for any $x, y \in L$ and $\alpha, \beta \in \mathbb{R}$; the functional f is called *trivial* if $f(x) = 0$ for any $x \in L$. The set of all continuous linear functionals on L is denoted by L^*; clearly, $L^* \subset C(L)$. Given a set X and $\mathcal{F} \subset \mathbb{R}^X$, let $\tau_{\mathcal{F}}$ be the topology generated by the family $\{f^{-1}(U) : f \in \mathcal{F},\ U \in \tau(\mathbb{R})\}$ as a subbase. We call $\tau_{\mathcal{F}}$ *the topology generated by* $\mathcal{F}$. If L is a linear topological space, denote by L_w the space (L, τ_{L^*}). We call τ_{L^*} *the weak topology of* L. Given a topological property $\mathcal{P}$, a set $K \subset L$ is called *weakly* $\mathcal{P}$ if the topology, induced in K from L_w, has the property $\mathcal{P}$. For example, a set $K \subset L$ is *weakly compact* if the topology, induced in K from L_w, is compact.

301. Prove that the following conditions are equivalent for any space X:

(i) X is functionally perfect;
(ii) X condenses onto a subspace of $C_p(Y)$ for some compact Y;
(iii) X condenses onto a subspace of $C_p(Y)$ for some σ-compact Y;
(iv) there exists a σ-compact $H \subset C_p(X)$ which separates the points of X;
(v) the space $C_p(X)$ is k-separable.

302. Show that neither $\Sigma(A)$ nor $\mathbb{R}^A$ is functionally perfect whenever the set A is uncountable.
303. Prove that the spaces $\sigma(A)$ and $\Sigma_*(A)$ are functionally perfect for any set A.
304. Prove that $C_p(\omega_1)$ is functionally perfect.
305. Show that, if a space condenses onto a functionally perfect space, then it is functionally perfect.
306. Prove that any subspace of a functionally perfect space is functionally perfect.
307. Prove that a countable product of functionally perfect spaces is a functionally perfect space. In particular, a countable product of Eberlein compact spaces is Eberlein compact.
308. Prove that any σ-product of functionally perfect spaces is a functionally perfect space.
309. Prove that any product of k-separable spaces is k-separable.
310. Prove that a space X is hereditarily k-separable (i.e., every $Y \subset X$ is k-separable) if and only if X is hereditarily separable.
311. Suppose that $f : X \to Y$ is an irreducible perfect map. Show that the space X is k-separable if and only if so is Y.
312. Prove that, for any k-separable X, the space $C_p(X)$ is functionally perfect. In particular, the space $C_p(X)$ is functionally perfect for any compact X.
313. Give an example of a non-k-separable space X for which $C_p(X)$ is functionally perfect.
314. Prove that, for an arbitrary space X, the space $C_p(X)$ is a continuous image of $C_p(C_p(C_p(X)))$.
315. Prove that $C_p(X)$ is k-separable if and only if $C_pC_p(X)$ is functionally perfect. As a consequence, X is functionally perfect if and only if $C_p(C_p(X))$ is functionally perfect.
316. Prove that any metrizable space is functionally perfect. In particular, any second countable space is functionally perfect and hence any metrizable compact space is Eberlein compact.
317. Let X be a metrizable space. Prove that $C_p(X)$ is functionally perfect if and only if X is second countable.
318. Prove that any paracompact space with a G_δ-diagonal can be condensed onto a metrizable space. Deduce from this fact that any paracompact space with a G_δ-diagonal is functionally perfect.
319. Observe that any Eberlein–Grothendieck space is functionally perfect. Give an example of a functionally perfect space which is not Eberlein–Grothendieck.
320. Prove that every metrizable space embeds into an Eberlein compact space.

321. Prove that $C_p(X)$ is a $K_{\sigma\delta}$-space for any Eberlein compact X. In particular, each Eberlein compact space is Gul'ko compact and hence Corson compact.

322. Prove that a compact space X is Eberlein if and only if it embeds into $\Sigma_*(A)$ for some A.

323. Prove that a compact space is metrizable if and only if it has a T_1-separating σ-point-finite family of cozero sets.

324. Prove that a compact space X is Eberlein compact if and only if X has a T_0-separating σ-point-finite family of cozero sets.

325. Give an example of a scattered compact space which fails to be Corson compact and has a T_0-separating σ-point-finite family of open sets.

326. Suppose that a compact X has a T_0-separating point-finite family of open sets. Prove that X is Eberlein compact.

327. Prove that a non-empty compact X is Eberlein if and only if there is a compact $F \subset C_p(X)$ which separates the points of X and is homeomorphic to $A(\kappa)$ for some cardinal κ.

328. Say that a (not necessarily continuous) function $x : \mathbb{I} \to \mathbb{I}$ is increasing (decreasing) if $x(s) \leq x(t)$ (or $x(s) \geq x(t)$ respectively) whenever $s, t \in \mathbb{I}$ and $s \leq t$. A function $x : \mathbb{I} \to \mathbb{I}$ is called monotone if it is either increasing or decreasing. Prove that the *Helly space* $X = \{x \in \mathbb{I}^{\mathbb{I}} : x$ is a monotone function$\}$ is closed in $\mathbb{I}^{\mathbb{I}}$ and hence compact. Is it an Eberlein compact space?

329. Prove that a compact space X is Eberlein compact if and only if $C_p(X)$ is a continuous image of $(A(\kappa))^\omega \times \omega^\omega$ for some infinite cardinal κ.

330. Prove that a compact space X is Eberlein compact if and only if there is a compact space K and a separable space M such that $C_p(X)$ is a continuous image of $K \times M$.

331. Prove that any infinite Eberlein compact space X is a continuous image of a closed subspace of $(A(\kappa))^\omega \times M$, where $\kappa = w(X)$ and M is a second countable space.

332. Prove that each Eberlein compact space is a Preiss–Simon space.

333. Prove that, if a pseudocompact space X condenses onto a subspace of $C_p(K)$ for some compact K, then this condensation is a homeomorphism and X is Eberlein compact. In particular, any functionally perfect pseudocompact space is Eberlein compact.

334. Prove that a zero-dimensional compact space X is Eberlein compact if and only if X has a T_0-separating σ-point-finite family of clopen sets.

335. Let X be a zero-dimensional compact space. Prove that X is Eberlein if and only if $C_p(X, \mathbb{D})$ is σ-compact.

336. Prove that any Eberlein compact space is a continuous image of a zero-dimensional Eberlein compact space.

337. Prove that any continuous image of an Eberlein compact space is Eberlein compact.

338. Prove that there exists a compact X such that X is a union of countably many Eberlein compact spaces while $C_p(X)$ is not Lindelöf. In particular, a compact countable union of Eberlein compact spaces need not be Corson compact.

339. Prove that it is consistent with ZFC that there exists a Corson compact space which does not map irreducibly onto an Eberlein compact space.

340. Suppose that $\mathcal{P}$ is a class of topological spaces such that, for any $X \in \mathcal{P}$, all continuous images and all closed subspaces of X belong to $\mathcal{P}$. Prove that it is impossible that a compact space X be Eberlein compact if and only if $C_p(X) \in \mathcal{P}$, i.e., either there exists an Eberlein compact space X such that $C_p(X) \notin \mathcal{P}$ or there is a compact Y which is not Eberlein and $C_p(Y) \in \mathcal{P}$.

341. Let X be a Gul'ko compact space. Prove that there exists a countable family $\mathcal{F}$ of closed subsets of X such that $\bigcup \mathcal{F} = X$ and $K_x = \bigcap\{A : x \in A \in \mathcal{F}\}$ is Eberlein compact for any $x \in X$.

342. Let X be an Eberlein compact space with $|X| \leq \mathfrak{c}$. Prove that there exists a countable family $\mathcal{F}$ of closed subsets of X such that $\bigcup \mathcal{F} = X$ and the subspace $K_x = \bigcap\{A : x \in A \in \mathcal{F}\}$ is metrizable for any $x \in X$.

343. Observe that $c(X) = w(X)$ for any Eberlein compact space X. Prove that, for any infinite compact X, we have $c(X) = \sup\{w(K) : K \subset C_p(X)$ and K is compact$\}$.

344. Given a pseudocompact space X and functions $f, g \in C(X)$, let $d(f, g) = \sup\{|f(x) - g(x)| : x \in X\}$. Prove that d is a complete metric on the set $C(X)$ and the topology of $C_u(X)$ is generated by d.

345. Prove that, for any pseudocompact X, the space $C_u(X)$ is separable if and only if X is compact and metrizable.

346. Suppose that X is a compact space and let $||f|| = \sup\{|f(x)| : x \in X\}$ for any $f \in C(X)$. Assume additionally that $h \in C(X)$, $r > 0$ and $H = \{h_n : n \in \omega\} \subset C(X)$ is a sequence such that $||h_n|| \leq r$ for all $n \in \omega$ and $h_n(x) \to h(x)$ for any $x \in X$ (i.e., the sequence H converges to h in the space $C_p(X)$). Prove that h belongs to the closure of the convex hull $\operatorname{conv}(H)$ of the set H in the space $C_u(X)$.

347. Suppose that X is a Čech-complete space and we are given a continuous map $\varphi : X \to C_p(K)$ for some compact space K. Prove that there exists a dense G_δ-set $P \subset X$ such that $\varphi : X \to C_u(K)$ is continuous at every point of P.

348. Prove that any Eberlein–Grothendieck Čech-complete space has a dense G_δ-subspace which is metrizable.

349. Prove that if X is an Eberlein–Grothendieck Čech-complete space then $c(X) = w(X)$.

350. Let X be a compact space. Assume that $X = X_1 \cup \ldots \cup X_n$, where every X_i is a metrizable (not necessarily closed) subspace of X. Prove that $\overline{X_1} \cap \ldots \cap \overline{X_n}$ is metrizable. In particular, if all X_i's are dense in X then X is metrizable.

351. Suppose that X is a compact space which is a union of two metrizable subspaces. Prove that X is Eberlein compact which is not necessarily metrizable.

352. Observe that there exists a compact space K which is not Eberlein while being a union of three metrizable subspaces. Suppose that X is a compact space such that $X \times X$ is a union of its three metrizable subspaces. Prove that X is Eberlein compact.

353. Prove that, if X is compact and X^ω is a union of countably many of its Eberlein compact subspaces then X is Eberlein compact.

354. Give an example of an Eberlein compact space which cannot be represented as a countable union of its metrizable subspaces.

355. Let X be a Corson compact space such that X is a countable union of Eberlein compact spaces. Prove that $C_p(X)$ is K-analytic and hence X is Gul'ko compact.

356. Let X be a σ-product of an arbitrary family of Eberlein compact spaces. Prove that $C_p(X)$ is a $K_{\sigma\delta}$-space.

357. Prove that the one-point compactification of an infinite discrete union of non-empty Eberlein compact spaces is an Eberlein compact space.

358. Prove that the Alexandroff double of an Eberlein compact space is an Eberlein compact space.

359. Recall that a space X is homogeneous if, for any $x, y \in X$, there is a homeomorphism $h : X \to X$ such that $h(x) = y$. Construct an example of a homogeneous non-metrizable Eberlein compact space.

360. Give an example of a hereditarily normal but not perfectly normal Eberlein compact space.

361. Let X be an Eberlein compact space such that $X \times X$ is hereditarily normal. Prove that X is metrizable.

362. Prove that there exists an Eberlein compact space X such that $X^2 \backslash \Delta$ is not metacompact.

363. Prove that any Eberlein compact space is hereditarily σ-metacompact.

364. Prove that, for any compact X, the subspace $X^2 \backslash \Delta \subset X^2$ is σ-metacompact if and only if X is Eberlein compact.

365. Prove that a compact space X is Eberlein compact if and only if $X \times X$ is hereditarily σ-metacompact.

366. Prove that a compact space X has a closure-preserving cover by compact metrizable subspaces if and only if it embeds into a σ-product of compact metrizable spaces. In particular, if X has a closure-preserving cover by compact metrizable subspaces then it is an Eberlein compact space.

367. Construct an Eberlein compact space which does not have a closure-preserving cover by compact metrizable subspaces.

368. Observe that every strong Eberlein compact is Eberlein compact. Prove that a metrizable compact space is strong Eberlein compact if and only if it is countable.

369. Prove that a compact X is strong Eberlein compact if and only if it has a point-finite T_0-separating cover by clopen sets.

370. Prove that every σ-discrete compact space is scattered. Give an example of a scattered compact non-σ-discrete space.

371. Prove that every strong Eberlein compact space is σ-discrete and hence scattered.

372. Prove that a hereditarily metacompact scattered compact space is strong Eberlein compact.

373. Prove that the following conditions are equivalent for any compact X:

(i) X is σ-discrete and Corson compact;
(ii) X is scattered and Corson compact;
(iii) X is strong Eberlein compact.

374. Prove that any continuous image of a strong Eberlein compact space is a strong Eberlein compact space.

375. Prove that any Eberlein compact space is a continuous image of a closed subset of a countable product of strong Eberlein compact spaces.

376. Let X be a strong Eberlein compact space. Prove that the Alexandroff double of X is also strong Eberlein compact.

377. Suppose that X_t is strong Eberlein compact for each $t \in T$. Prove that the Alexandroff one-point compactification of the space $\bigoplus\{X_t : t \in T\}$ is also strong Eberlein compact.

378. Observe that any uniform Eberlein compact is Eberlein compact. Prove that any metrizable compact space is uniform Eberlein compact.

379. Observe that any closed subspace of a uniform Eberlein compact space is uniform Eberlein compact. Prove that any countable product of uniform Eberlein compact spaces is uniform Eberlein compact.

380. Prove that if X is a uniform Eberlein compact space then it is a continuous image of a closed subspace of $(A(\kappa))^\omega$ for some infinite cardinal κ.

381. Prove that any continuous image of a uniform Eberlein compact space is uniform Eberlein compact. Deduce from this fact that a space X is uniform Eberlein compact if and only if it is a continuous image of a closed subspace of $(A(\kappa))^\omega$ for some infinite cardinal κ.

382. Given an infinite set T suppose that a space $X_t \neq \emptyset$ is uniform Eberlein compact for each $t \in T$. Prove that the Alexandroff compactification of the space $\bigoplus\{X_t : t \in T\}$ is also uniform Eberlein compact.

383. Let T be an infinite set. Suppose that $\mathcal{A}$ is an adequate family on T. Prove that the space $K_\mathcal{A}$ is Eberlein compact if and only if $T^*_\mathcal{A}$ is σ-compact.

384. Let T be an infinite set. Suppose that $\mathcal{A}$ is an adequate family on T. Prove that the space $K_\mathcal{A}$ is Eberlein compact if and only if there exists a disjoint family $\{T_i : i \in \omega\}$ such that $T = \bigcup\{T_i : i \in \omega\}$ and $x^{-1}(1) \cap T_i$ is finite for every $x \in K_\mathcal{A}$ and $i \in \omega$.

385. Let T be an infinite set and $\mathcal{A}$ an adequate family on T. Prove that the adequate compact $K_\mathcal{A}$ is uniform Eberlein compact if and only if there exists a disjoint family $\{T_i : i \in \omega\}$ and a function $N : \omega \to \omega$ such that $T = \bigcup\{T_i :\in \omega\}$ and $|x^{-1}(1) \cap T_i| \leq N(i)$ for any $x \in K_\mathcal{A}$ and $i \in \omega$.

386. For the set $T = \omega_1 \times \omega_1$ let us introduce an order $<$ on T declaring that $(\alpha_1, \beta_1) < (\alpha_2, \beta_2)$ if and only if $\alpha_1 < \alpha_2$ and $\beta_1 > \beta_2$. Denote by $\mathcal{A}$ the family of all subsets of T which are linearly ordered by $<$ (the empty set and the one-point sets are considered to be linearly ordered). Prove that $\mathcal{A}$ is an adequate family and $X = K_\mathcal{A}$ is a strong Eberlein compact space which is not uniform Eberlein compact.

387. (Talagrand's example) For any distinct $s, t \in \omega^\omega$, consider the number $\delta(s,t) = \min\{k \in \omega : s(k) \neq t(k)\}$. For each $n \in \omega$, let $\mathcal{A}_n = \{A \subset \omega^\omega :$ for any distinct $s, t \in A$ we have $\delta(s,t) = n\}$. Prove that $\mathcal{A} = \bigcup\{\mathcal{A}_n : n \in \omega\}$ is an adequate family and $X = K_\mathcal{A}$ is a Talagrand compact space (i.e., $C_p(X)$ is K-analytic and hence X is Gul'ko compact) while X is not Eberlein compact.

388. Given a compact space X let $||f|| = \sup\{|f(x)| : x \in X\}$ for any $f \in C(X)$. Prove that $(C(X), ||\cdot||)$ is a Banach space.

389. (Hahn–Banach Theorem) Assume that M is a linear subspace of a normed space $(L, ||\cdot||)$ and suppose that $f : M \to \mathbb{R}$ is a linear functional such that $|f(x)| \leq ||x||$ for any $x \in M$. Prove that there exists a linear functional $F : L \to \mathbb{R}$ such that $F|M = f$ and $|F(x)| \leq ||x||$ for all $x \in L$.

390. Given a normed space $(L, ||\cdot||)$ let $S = \{x \in L : ||x|| \leq 1\}$ be the unit ball of L. Prove that a linear functional $f : L \to \mathbb{R}$ is continuous if and only if there exists $k \in \mathbb{N}$ such that $|f(x)| \leq k$ for any $x \in S$.

391. Given a normed space $(L, ||\cdot||)$, let $S = \{x \in L : ||x|| \leq 1\}$ and consider the set $S^* = \{f \in L^* : f(S) \subset [-1, 1]\}$. Prove that S^* separates the points of L.

392. Let L be a linear space without any topology. Suppose that $\mathcal{F}$ is a family of linear functionals on L which separates the points of L. Prove that the topology on L generated by $\mathcal{F}$, is Tychonoff and makes L a locally convex linear topological space.

393. Let L be a linear space. Denote by L' the set of all linear functionals on L. Considering L' a subspace of $\mathbb{R}^L$, prove that L' is closed in $\mathbb{R}^L$.

394. Given a normed space $(L, ||\cdot||)$, consider the sets $S = \{x \in L : ||x|| \leq 1\}$ and $S^* = \{f \in L^* : f(S) \subset [-1, 1]\}$. Prove that, for any point $x \in L$, the set $S^*(x) = \{f(x) : f \in S^*\}$ is bounded in $\mathbb{R}$.

395. Given a normed space $(L, ||\cdot||)$, let $S = \{x \in L : ||x|| \leq 1\}$ and consider the set $S^* = \{f \in L^* : f(S) \subset [-1, 1]\}$. Denote by L_w the set L with the topology generated by L^*. Observe that $S^* \subset C(L_w)$ and give S^* the topology τ induced from $C_p(L_w)$. Prove that (S^*, τ) is a compact space.

396. Prove that L_w is functionally perfect for any normed space $(L, ||\cdot||)$. As a consequence, any compact subspace of L_w is Eberlein compact.

397. Let L be a linear topological space. Given a sequence $\{x_n : n \in \omega\} \subset L$, prove that $x_n \to x$ in the weak topology on L if and only if $f(x_n) \to f(x)$ for any $f \in L^*$.

398. For an arbitrary compact space K let $||f|| = \sup\{|f(x)| : x \in K\}$ for any $f \in C(K)$. Denote by $C_w(K)$ the space $C(K)$ endowed with the weak topology of the normed space $(C(K), ||\cdot||)$. Prove that $\tau(C_w(K)) \supset \tau(C_p(K))$ and show that, in the case of $K = \mathbb{I}$, this inclusion is strict, i.e., $\tau(C_w(\mathbb{I})) \neq \tau(C_p(\mathbb{I}))$.

399. Suppose that K is a compact space and let $||f|| = \sup\{|f(x)| : x \in K\}$ for any $f \in C(K)$. Denote by $C_w(K)$ the space $C(K)$ endowed with the weak topology of the normed space $(C(K), ||\cdot||)$. Prove that, for any $X \subset C(K, \mathbb{I})$, if X is compact as a subspace of $C_p(K)$ then the topologies, induced on X from the spaces $C_p(K)$ and $C_w(K)$, coincide.

400. (The original definition of an Eberlein compact space) Prove that X is an Eberlein compact space if and only if it is homeomorphic to a weakly compact subset of a Banach space.

1.5 Special Embeddings and Extension Operators

A space X is called *splittable* if, for every $f \in \mathbb{R}^X$, there exists a countable set $A \subset C_p(X)$ such that $f \in \overline{A}$ (the closure is taken in $\mathbb{R}^X$). Splittable spaces are also called *cleavable*, but we won't use the last term in this book. A space X is *weakly splittable* if, for every function $f \in \mathbb{R}^X$, there exists a σ-compact $A \subset C_p(X)$ such that $f \in \overline{A}$ (the closure is taken in $\mathbb{R}^X$). A compact weakly splittable space is called *weak Eberlein compact.*

A space X is *strongly splittable* if, for every $f \in \mathbb{R}^X$, there exists a sequence $S = \{f_n : n \in \omega\} \subset C_p(X)$ such that $f_n \to f$. A space X is *functionally perfect* if there exists a compact $K \subset C_p(X)$ which separates the points of X. Given a space X let $L_p(X)$ be the set of all continuous linear functionals on $C_p(X)$ with the topology induced from $C_p(C_p(X))$.

A space X has *a small diagonal* if, for any uncountable set $A \subset (X \times X)\backslash \Delta$, there exists an uncountable $B \subset A$ such that $\overline{B} \cap \Delta = \emptyset$. Here, as usual, the set $\Delta = \{(x,x) : x \in X\} \subset X \times X$ is the diagonal of the space X. A subspace $Y \subset X$ is *strongly discrete* if there exists a discrete family $\{U_y : y \in Y\} \subset \tau(X)$ such that $y \in U_y$ for each $y \in Y$. A space is *strongly σ-discrete* if it is a union of countably many of its closed discrete subspaces. The space $\mathbb{D}$ is the two-point set $\{0,1\}$ endowed with the discrete topology.

An uncountable regular cardinal κ is *a caliber* of a space X if, for any family $\mathcal{U} \subset \tau^*(X)$ of cardinality κ, there exists $\mathcal{U}' \subset \mathcal{U}$ such that $|\mathcal{U}'| = \kappa$ and $\bigcap \mathcal{U}' \neq \emptyset$. An uncountable regular cardinal κ is called *a precaliber* of a space X if, for any family $\mathcal{U} \subset \tau^*(X)$ of cardinality κ, there exists $\mathcal{U}' \subset \mathcal{U}$ such that $|\mathcal{U}'| = \kappa$ and $\mathcal{U}'$ has finite intersection property, i.e., $\bigcap \mathcal{V} \neq \emptyset$ for any finite $\mathcal{V} \subset \mathcal{U}'$. Now, $p(X) = \sup\{|\mathcal{U}| : \mathcal{U}$ is a point-finite family of non-empty open subsets of $X\}$. The cardinal $p(X)$ is called *the point-finite cellularity* of X. Given a space X and $x \in X$, call a family $\mathcal{B} \subset \tau^*(X)$ *a π-base* of X at x if, for any $U \in \tau(x,X)$, there is $V \in \mathcal{B}$ such that $V \subset U$. Note that the elements of a π-base at x need not contain the point x. Now, $\pi\chi(x,X) = \min\{|\mathcal{B}| : \mathcal{B}$ is a π-base of X at $x\}$ and $\pi\chi(X) = \sup\{\pi\chi(x,X) : x \in X\}$. Let $iw(X) = \min\{|\kappa| :$ there is a condensation of X onto a space of weight $\leq \kappa\}$. The cardinal $iw(X)$ is called *the i-weight* of X.

A subspace Y of a space X is *C-embedded in* X if, for any function $f \in C(Y)$, there is $g \in C(X)$ such that $g|Y = f$. If any bounded continuous function on Y can be extended to a continuous function on X, we say that Y *is C^*-embedded in* X.

If X is a space, $Y \subset X$ and $E \subset C_p(Y)$, say that a map $\varphi : E \to C_p(X)$ is *an extender* if $\varphi(f)|Y = f$ for any $f \in E$. A set $Y \subset X$ is *t-embedded in* X, if there exists a continuous extender $\varphi : C_p(Y) \to C_p(X)$. Note that any t-embedded subspace of X is C-embedded in X. It is said that Y is *l-embedded in* X if there exists a linear continuous extender $\varphi : C_p(Y) \to C_p(X)$. Note that any l-embedded subspace of X is t-embedded in X.

A space X is called *extendial or l-extendial* if every closed subspace of X is l-embedded in X. If every closed subspace of X is t-embedded in X, then X is

t-extendial. A space X is *extral or l-extral* if, for any space Y which contains X as a closed subspace, X is l-embedded in Y. Analogously, X is *t-extral* if, for any space Y which contains X as a closed subspace, X is t-embedded in Y. Given a space X and a subspace $F \subset X$, we say that F *is a retract of* X if there exists a continuous map $f : X \to F$ such that $f(x) = x$ for any $x \in F$. The map f is called *retraction of* X *onto* F. A compact space is called *dyadic* if it is a continuous image of the Cantor cube $\mathbb{D}^\kappa$ for some κ. A compact space X is *l-dyadic* if it is l-embedded in some dyadic space. If X can be t-embedded in some dyadic space, it is called *t-dyadic.*

Recall that a space X is *radial* if, for every $A \subset X$ and every $x \in \overline{A}$, there exists a transfinite sequence $s = \{x_\alpha : \alpha < \kappa\} \subset A$ such that $s \to x$. A space is *k-separable* if it has a dense σ-compact subspace. The symbol $\mathfrak{c}$ denotes the cardinality of the continuum. A space X is *Baire* if any countable intersection of open dense subsets of X is dense in X. The space X is Čech-complete if X is a G_δ-set in βX. A sequence $\{\mathcal{U}_n : n \in \omega\}$ of covers of X is called *complete* if, for any filter $\mathcal{F}$ on the set X such that $\mathcal{F} \cap \mathcal{U}_n \neq \emptyset$ for all $n \in \omega$, we have $\bigcap\{\overline{F} : F \in \mathcal{F}\} \neq \emptyset$.

A family $\mathcal{U}$ of subsets of a space X is *weakly σ-point-finite* if there exists a sequence $\{\mathcal{U}_n : n \in \omega\}$ of subfamilies of $\mathcal{U}$ such that, for every $x \in X$, we have $\mathcal{U} = \bigcup\{\mathcal{U}_n : n \in M_x\}$ where $M_x = \{n \in \omega : \mathcal{U}_n$ is point-finite at $x\}$.

401. Prove that every subspace of a splittable space is splittable.
402. Prove that every second countable space is splittable.
403. Prove that $\psi(X) \leq \omega$ for every splittable space X.
404. Prove that, if X condenses onto a splittable space, then X is splittable. In particular, any space of countable i-weight is splittable.
405. Give an example of a splittable space which does not condense onto a second countable space.
406. Give an example of a metrizable space which is not splittable.
407. Give an example of a splittable space whose square is not splittable.
408. Prove that a space X with a unique non-isolated point is splittable if and only if $\psi(X) \leq \omega$.
409. Let X be a non-discrete space. Prove that, for any $f \in \mathbb{R}^X$, there exists a countable $A \subset \mathbb{R}^X \backslash C_p(X)$ such that $f \in \overline{A}$ (the closure is taken in $\mathbb{R}^X$).
410. Let X be a splittable space. Prove that every regular uncountable cardinal is a caliber of $C_p(X)$.
411. Prove that every splittable space has a small diagonal.
412. Prove that $C_p(X)$ is splittable if and only if it condenses onto a second countable space.
413. Show that an open continuous image of a splittable space can fail to be splittable.
414. Let X be a space of cardinality $\leq \mathfrak{c}$. Prove that X is splittable if and only if the i-weight of X is countable.
415. Prove that a space X is splittable if and only if, for every $f \in \mathbb{D}^X$, there is a countable $A \subset C_p(X)$ such that $f \in \overline{A}$ (the closure is taken in $\mathbb{R}^X$).
416. Prove that a space X is splittable if and only if, for any $A \subset X$, there exists a continuous map $\varphi : X \to \mathbb{R}^\omega$ such that $A = \varphi^{-1}\varphi(A)$.
417. Prove that any pseudocompact splittable space must be compact and metrizable.
418. Prove that a Lindelöf space X is splittable if and only if $iw(X) \leq \omega$.
419. Prove that a Lindelöf Σ-space is splittable if and only if it has a countable network.
420. Prove that a Lindelöf p-space is splittable if and only if it is second countable.
421. Prove that any Čech-complete splittable paracompact space is metrizable.
422. Let X be a complete metrizable dense-in-itself space. Prove that X is splittable if and only if $|X| \leq \mathfrak{c}$.
423. Suppose that $X = \bigcup\{X_n : n \in \omega\}$, where $X_n \subset X_{n+1}$ for each $n \in \omega$, the subspace X_n is splittable and C^*-embedded in X for every n. Prove that X is splittable.
424. Prove that any normal strongly σ-discrete space is strongly splittable.
425. Give an example of a strongly σ-discrete space which is not splittable.
426. Show that, for any cardinal κ, there exists a normal strongly σ-discrete (and hence splittable) space X with $c(X) = \omega$ and $|X| \geq \kappa$.
427. Show that there exists a splittable space which cannot be condensed onto a first countable space.

428. Assuming the Generalized Continuum Hypothesis prove that, if X is a splittable space and $A \subset X$, $|A| \leq \mathfrak{c}$ then $|\overline{A}| \leq \mathfrak{c}$.
429. Prove that any Čech-complete splittable space has a dense metrizable subspace.
430. Prove that every subspace of a weakly splittable space must be weakly splittable.
431. Prove that, if X condenses onto a weakly splittable space then X is weakly splittable.
432. Give an example of a weakly splittable non-splittable space.
433. Prove that a separable weakly splittable space is splittable.
434. Let X be a space with ω_1 caliber of X. Prove that, under CH, X is splittable if and only if it is weakly splittable.
435. Show that every functionally perfect space is weakly splittable. In particular, every Eberlein compact space is weakly splittable.
436. Prove that every metrizable space is weakly splittable.
437. Let X be a weakly splittable space of cardinality $\leq \mathfrak{c}$. Prove that $C_p(X)$ is k-separable.
438. Prove that if X is a weak Eberlein compact space and $|X| \leq \mathfrak{c}$ then X is Eberlein compact.
439. Let X be a weak Eberlein compact space. Prove that $w(X) = c(X)$. In particular, a weak Eberlein compact space is metrizable whenever it has the Souslin property.
440. Prove that any weak Eberlein compact space X is ω-monolithic, Fréchet–Urysohn and $C_p(X)$ is Lindelöf.
441. Give an example of a Gul'ko compact space which fails to be weakly splittable.
442. Prove that any subspace of a strongly splittable space must be strongly splittable.
443. Prove that, if X condenses onto a strongly splittable space then X is strongly splittable.
444. Prove that, under MA+$\neg$CH, there is a strongly splittable space which is not σ-discrete.
445. Prove that every subset of a strongly splittable space is a G_δ-set.
446. Show that there exists a space X in which every subset is a G_δ-set while X is not splittable.
447. Let X be a normal space in which every subset is G_δ. Prove that X is strongly splittable.
448. Suppose that $Y \subset X$ and $\varphi : C_p(Y) \to C_p(X)$ is a continuous (linear) extender. For $I = \{f \in C_p(X) : f(Y) \subset \{0\}\}$, define a map $\xi : C_p(Y) \times I \to C_p(X)$ by $\xi(f, g) = \varphi(f) + g$ for any $(f, g) \in C_p(Y) \times I$. Prove that ξ is a (linear) embedding and hence $C_p(Y)$ embeds in $C_p(X)$ as a closed (linear) subspace.
449. Given a space X define a map $e : X \to C_p(C_p(X))$ by $e(x)(f) = f(x)$ for any $x \in X$ and $f \in C_p(X)$. If X is a subspace of a space Y prove that X is t-embedded in Y if and only if there exists a continuous map φ :

$Y \to C_p(C_p(X))$ such that $\varphi|X = e$. Deduce from this fact that $e(X)$ is t-embedded in $C_p(C_p(X))$ and hence X is homeomorphic to a t-embedded subspace of $C_p(C_p(X))$.

450. Prove that, for any space X every t-embedded subspace of X is closed in X.

451. Prove that $\beta\omega\backslash\omega$ is not t-embedded in $\beta\omega$.

452. Suppose that Y is t-embedded in a space X. Prove that $p(Y) \leq p(X)$ and $d(Y) \leq d(X)$.

453. Suppose that Y is t-embedded in X and a regular cardinal κ is a caliber of X. Prove that κ is a caliber of Y.

454. Prove that any closed subspace of a t-extendial space is t-extendial.

455. Let X be a t-extendial space. Prove that $s(X) = p(X)$.

456. Let X be a t-extendial Baire space. Prove that $s(X) = c(X)$. In particular, if X is a pseudocompact t-extendial space, then $c(X) = s(X)$.

457. Prove that, for any compact t-extendial space X, we have $t(X) \leq c(X)$.

458. Prove that, under MA+$\neg$CH, if $X \times X$ is a t-extendial compact space and $c(X) \leq \omega$ then X is metrizable.

459. Suppose that X is a t-extendial Čech-complete space such that ω_1 is a caliber of X. Prove that X is hereditary separable.

460. Assuming MA+$\neg$CH prove that any t-extendial Čech-complete space with the Souslin property is hereditarily separable.

461. Prove that a t-extendial compact space cannot be mapped onto $\mathbb{I}^{\omega_1}$.

462. Prove that the set $\{x \in X : \pi\chi(x, X) \leq \omega\}$ is dense in any t-extendial compact space X.

463. Give an example of a countable space which is not t-extendial.

464. Prove that any strongly discrete subspace $A \subset X$ is l-embedded in X.

465. Prove that, if Y is a retract of X, then Y is l-embedded in X.

466. Given a space X define a map $e : X \to C_p(C_p(X))$ by $e(x)(f) = f(x)$ for any $x \in X$ and $f \in C_p(X)$. Observe that $e(X) \subset L_p(X)$; prove that $e(X)$ is l-embedded in $L_p(X)$ and hence any space X is homeomorphic to an l-embedded subspace of $L_p(X)$.

467. Given a space X define a map $e : X \to C_p(C_p(X))$ by $e(x)(f) = f(x)$ for any $x \in X$ and $f \in C_p(X)$. Observe that $e(X) \subset L_p(X)$ so we can consider that $e : X \to L_p(X)$. If X is a subspace of a space Y prove that X is l-embedded in Y if and only if there exists a continuous map $\varphi : Y \to L_p(X)$ such that $\varphi|X = e$.

468. Prove that any closed subspace of an extendial space is extendial.

469. Prove that every metrizable space is extendial.

470. Prove that, for any zero-dimensional linearly ordered compact space X any closed $F \subset X$ is a retract of X; in particular, the space X is extendial.

471. Give an example of a perfectly normal, hereditarily separable non-metrizable extendial compact space.

472. Give an example of an extendial compact space X such that $X \times X$ is not t-extendial.

473. Show that there exist extendial compact spaces of uncountable tightness.

474. Give an example of a non-linearly orderable extendial compact space.

475. Prove that every t-extral (and hence every extral) space X is compact and every uncountable regular cardinal is a caliber of X.
476. Prove that X is t-extral if and only if it can be t-embedded in $\mathbb{I}^\kappa$ for some cardinal κ.
477. Prove that any retract of a t-extral space is a t-extral space.
478. Let X be a t-extral space such that $w(X) \leq \mathfrak{c}$. Prove that X is separable.
479. Suppose that an ω-monolithic space X is t-extral and has countable tightness. Prove that X is metrizable.
480. Prove that a t-extral space X is metrizable whenever $C_p(X)$ is Lindelöf.
481. Give an example of a t-extral space which is not extral.
482. Prove that every metrizable compact space is extral.
483. Prove that X is extral if and only if it can be l-embedded in $\mathbb{I}^\kappa$ for some cardinal κ.
484. Prove that any retract of an extral space is an extral space.
485. Given an extral space X and an infinite cardinal κ prove that $w(X) > \kappa$ implies that $\mathbb{D}^{\kappa^+}$ embeds in X.
486. Let X be an extral space. Prove that $w(X) = t(X) = \chi(X)$.
487. Assuming that $\mathfrak{c} < 2^{\omega_1}$ prove that any extral space X with $|X| \leq \mathfrak{c}$ is metrizable.
488. Prove that every extral t-extendial space is metrizable.
489. Suppose that every closed subspace of X is extral. Prove that X is metrizable.
490. Prove that any extral linearly orderable space is metrizable.
491. Give an example of an extral space which is not dyadic.
492. Give an example of an extral space X such that some continuous image of X is not extral.
493. Prove that any zero-dimensional extral space is metrizable.
494. Prove that any continuous image of a t-dyadic space is a t-dyadic space.
495. Prove that any continuous image of an l-dyadic space is an l-dyadic space.
496. Given an l-dyadic space X prove that, for any infinite cardinal κ such that $\kappa < w(X)$, the space $\mathbb{D}^{\kappa^+}$ embeds in X.
497. Prove that, if βX is an l-dyadic space then X is pseudocompact.
498. Prove that any hereditarily normal l-dyadic space is metrizable.
499. Prove that any radial l-dyadic space is metrizable.
500. Give an example of an l-dyadic space which is not extral.

Bibliographic Notes to Chapter 1

The material of Chapter 1 consists of problems of the following types:

(1) textbook statements which give a gradual development of some topic;
(2) folkloric statements that might not be published but are known by specialists;
(3) famous theorems cited in textbooks and well-known surveys;
(4) comparatively recent results which have practically no presence in textbooks.

We will almost never cite the original papers for statements of the first three types. We are going to cite them for a very small sample of results of the fourth type. The selection of theorems to cite is made according to the preferences of the author and *does not mean that all statements of the fourth type are mentioned.* I bring my apologies to readers who might think that I did not cite something more important than what is cited. The point is that a selection like that has to be biased because it is impossible to mention all contributors. As a consequence, *there are quite a few statements of the main text, published as results in papers, which are never mentioned in our bibliographic notes.* A number of problems of the main text cite published or unpublished results of the author. However, those are treated exactly like the results of others: some are mentioned and some aren't. On the other hand, the Bibliography contains (to the best knowledge of the author) the papers and books of *all contributors to the material of this book.*

Section 1.1 introduces the technique for representing the whole $C_p(X)$ by means of subspaces which separate the points of a compact space X. A considerable part of the material of this section is covered in Arhangel'skii's book [1992a]. The theorem on equivalence of countability of fan tightness of $C_p(X)$ and Hurewicz property of all finite powers of X (Problem 057) was proved in Arhangel'skii (1986). The result on ω-monolithity of a compact X when $C_p(X)$ is normal (Problem 080) was proved under MA+$\neg$CH by Reznichenko (see Arhangel'skii (1992a)). The fact that PFA implies countable tightness of all compact subspaces of $C_p(X)$, for a Lindelöf X (Problem 089), was established in Arhangel'skii (1992a). A very non-trivial example of a "big" separable σ-compact space X with $(C_p(X))^\omega$ Lindelöf (Problem 094) was constructed by Okunev and Tamano (1996).

Section 1.2 brings in the most important results on Corson compact spaces. Again, the book of Arhangel'skii (1992a) covers a significant part of this section. The invariance of the class of Corson compact spaces under continuous mappings (Problem 151) was proved in Michael and Rudin (1977) and Gul'ko (1977). The Lindelöf property in all iterated C_p's of any Corson compact space (Problem 167) was proved in Sokolov (1986). The example of a Corson compact space which is not Gul'ko compact (Problem 175) was constructed in Leiderman and Sokolov (1984). A very deep result stating that all iterated function spaces of a Corson compact space can be condensed into a Σ-product of real lines (Problem 200) was established in Gul'ko (1981).

Section 1.3 shows that the class of Lindelöf Σ-spaces is of great importance in C_p-theory. A good coverage of the topic is done in the book of Arhangel'skii (1992a) and his survey [1992b]. Okunev's theorem on Lindelöf Σ-property in iterated function spaces (Problems 218 and 219) was a crucial breakthrough. It was published in Okunev (1993a). A complete classification of the distributions of the Lindelöf Σ-property in iterated function spaces (Problem 243) was given in Tkachuk (2000). A very subtle example of a Gul'ko compact space which is not Preiss–Simon (Problem 222) was constructed by Reznichenko]. It is a famous theorem of Gul'ko (1979) that every Gul'ko compact space is Corson compact (Problem 285). Finally, it was proved in Leiderman (1985) that every Gul'ko compact space has a dense metrizable subspace (Problem 293).

Section 1.4 presents the present-day state of knowledge about Eberlein compact spaces. They were originally defined as weakly compact subsets of Banach spaces. Amir and Lindenstrauss (1968) proved that a compact X is an Eberlein compact iff it embeds into $\Sigma_*(A)$ (Problem 322). Rosenthal gave in [1974] a topological criterion in terms of T_0-separating families (Problem 324). The theorem on invariance of the class of Eberlein compact spaces under continuous maps (Problem 337) was proved in Benyamini et al. (1977) and Gul'ko [1977]. The characterization of Eberlein compacta in terms of the topology of X^2 (Problem 364) was established in Gruenhage (1984a). Talagrand constructed in [1979a] an example of a Gul'ko compact space which is not Eberlein compact (Problem 387). A theorem of Namioka (1974) on joint continuity and separate continuity has a great number of applications in functional analysis, topology and topological algebra; an important consequence of this theorem is formulated in Problem 347. The equivalence of the original definition of Eberlein compact spaces to the topological one (Problem 400) was proved by Grothendieck in [1952].

In Section 1.5 we study approximations of arbitrary functions by continuous ones and the classes of spaces with nice properties for extensions of continuous functions. Arhangel'skii and Shakhmatov proved in [1988] that any pseudocompact splittable space is compact and metrizable (Problem 417). Arhangel'skii and Choban introduced and studied in [1988] the classes of extral spaces and extendial spaces which could be considered far-reaching generalizations of absolute retracts and metrizable spaces, respectively.