

# Positive Operator-Valued Measures in Quantum Decision Theory

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**Abstract.** We show that the correct mathematical foundation of quantum decision theory, dealing with uncertain events, requires the use of positive operator-valued measure that is a generalization of the projection-valued measure. The latter is appropriate for operationally testable events, while the former is necessary for characterizing operationally uncertain events. In decision making, one has to distinguish composite non-entangled events from composite entangled events. The mathematical definition of entangled prospects is based on the theory of Hilbert-Schmidt spaces and is analogous to the definition of entangled statistical operators in quantum information theory. We demonstrate that the necessary condition for the appearance of an interference term in the quantum probability is the occurrence of entangled prospects and the existence of an entangled strategic state of a decision maker. The origin of uncertainties in standard lotteries is explained.

**Keywords:** Decision theory · Quantum information processing · Decisions under uncertainty · Quantum probability · Positive operator-valued measure · Entangled prospects

## 1 Introduction

Techniques of quantum theory are nowadays widely employed not only for physics problems but also in such fields as quantum information processing and quantum computing [1–5]. Another example is the theory of quantum games [6]. A scheme of artificial quantum intelligence was suggested [7, 8]. Applications of quantum techniques to cognitive sciences are also quickly growing.

Actually, the idea that human decision making could be characterized by quantum techniques was advanced long ago by Bohr [9, 10]. Since then, a number

of publications have discussed the possibility of using quantum techniques in cognitive sciences, as is summarized in the recent books [11–13].

Von Neumann [14] mentioned that the theory of quantum measurements can be interpreted as decision theory. There is, however, an important difference between the standard situation in quantum measurements and the often occurring case in realistic decision making. Usual measurements in physics problems are operationally testable, resulting in well defined numerical values of the measured observables. In decision making, however, it is common to deal with composite events requiring to take decisions in uncertain situations. While the final decision should also be operationally testable, intermediate steps often involve uncertainty that is not operationally testable. The correct description of such uncertain composite prospects requires the use of more elaborate mathematics than the projection-valued measure commonly employed for standard physics problems.

In our previous papers [15–19], we have developed Quantum Decision Theory (QDT) whose mathematical basis rests on the theory of quantum measurements and quantum information theory. The strategic state of a decision maker was represented by a wave function. As is well known, a wave function characterizes an isolated quantum system. However, in real life, no decision maker can be absolutely isolated from the society where he/she lives. That is, the characterization of a strategic decision-maker state by a wave function is oversimplified. Simple methods of quantum mechanics are not sufficient for a realistic description of a decision maker that is a member of a society. Being an open system, a decision maker has to be described by a statistical operator, similarly to any non-isolated system in quantum theory, which requires the use of the ramified techniques of quantum statistical theory [20–22].

One could think that, for describing simple psychological laboratory experiments, there is no need of invoking statistical operators and it would be sufficient to just use pure states characterized by wave functions, since most lab-based tests on cognition deal with subjects that are typically isolated and the experiments have limited durations. However, in quantum theory, as is well known, the necessity of using density matrices is dictated by the existence of interactions not merely at the present moment of time, but also at any previous times, which is always relevant for any alive being. In addition, a system is termed open if interactions have been present not only with similar systems, but with any surrounding. Thus, humans are always subject to interactions with many other people as well as with various information sources, such as TV, radio, telephone, internet, newspapers, and so on. All these numerous interactions always influence decision makers making them, without doubt, open systems. In order to reduce a decision maker to a pure quantum-mechanical system for a pure laboratory experiment, it would be necessary to make a surgical operation deleting all memory and information from the brains of these poor decision makers. The usual lab tests, fortunately, do not require this, so that decision makers have to be always treated as open systems described by statistical operators.

It is worth stressing that a theory based on statistical operators includes, as a particular case, the pure-state description. So that all results obtained in the general consideration can be straightforwardly reduced to the latter case by taking the statistical operator in the pure form  $|\psi\rangle\langle\psi|$ .

The goal of the present report is threefold. First, we present the extension of QDT to the most general case when the decision-maker strategic state is characterized by a statistical operator. Second, we demonstrate that the correct mathematical description of uncertain composite events has to be described by a positive operator-valued measure generalizing the projection-valued measure used in quantum mechanics. Third, we show that decision making under uncertainty is a rather common phenomenon that happens in the delusively simple problem of choosing between lotteries and illustrate this by an explicit example.

The important feature of our approach is that we employ the rigorous mathematical techniques developed in the theory of quantum measurements and quantum information theory. In order that the reader with background in psychology would not confuse mathematical notions used in our paper, we give the necessary definitions, trying at the same time to keep the exposition concise. Details can be found in the books [1, 2, 4, 5, 23–26] and review articles [3, 27].

## 2 Operationally Testable Events

The operationally testable events in QDT can be characterized by analogy with operationally testable measurements in quantum theory. Quantum events, obeying the Birkhoff-von Neumann quantum logic [28], form a non-commutative non-distributive ring  $\mathcal{R}$ . The nonempty collection of all subsets of the event ring  $\mathcal{R}$ , including  $\mathcal{R}$ , which is closed with respect to countable unions and complements, is the event sigma algebra  $\Sigma$ . The algebra of quantum events is the pair  $\{\Sigma, \mathcal{R}\}$  of the sigma algebra  $\Sigma$  over the event ring  $\mathcal{R}$ . An elementary event  $A_n$  is represented by a basic state  $|n\rangle$  generating the event operator defined as a projector

$$\hat{P}_n \equiv |n\rangle\langle n|. \quad (1)$$

The space of mind of a decision maker is a Hilbert space  $\mathcal{H}$  that is a closed linear envelope of the basis  $\{|n\rangle\}$  spanning all admissible basic states. The strategic state of a social decision maker is a statistical operator  $\hat{\rho}$  that is a trace-class positive operator normalized to one. The algebra of observables in QDT is the family  $\{\hat{P}_n\}$  of the event operators whose expected values define the event probabilities

$$p(A_n) \equiv \text{Tr}\hat{\rho}\hat{P}_n \equiv \langle\hat{P}_n\rangle, \quad (2)$$

with the trace over  $\mathcal{H}$ . From this definition, it follows:

$$\sum_n p(A_n) = 1, \quad 0 \leq p(A_n) \leq 1,$$

hence the family  $\{p(A_n)\}$  forms a probability measure. By the Gleason theorem [29], this measure is unique for a Hilbert space of dimensionality larger than

two. In the theory of quantum measurements, the projectors  $\hat{P}_n$  play the role of observables, so that, for an event  $A_n$ , one has the correspondence

$$A_n \rightarrow \hat{P}_n \equiv |n\rangle\langle n|. \quad (3)$$

For a union of mutually orthogonal events, there is the correspondence

$$\bigcup_n A_n \rightarrow \sum_n \hat{P}_n, \quad (4)$$

which results in the additivity of the probabilities:

$$p\left(\bigcup_n A_n\right) = \sum_n p(A_n). \quad (5)$$

The procedure described above is called the standard measurement.

Let us emphasize that all formulas of this and following sections are valid for an arbitrary statistical operator, which includes the pure form  $\hat{\rho} = |\psi\rangle\langle\psi|$  as a trivial particular case.

### 3 Operationally Uncertain Events

It may happen that one cannot tell whether a particular event has occurred, but it is only known that some of the events  $A_n$  could be realized. This is what is called an uncertain or inconclusive event.

Assume that the observed event  $A$  is a set  $\{A_n\}$  of possible events. Although the events  $A_m$  and  $A_n$  are orthogonal for  $m \neq n$ , in the case of the uncertain event  $A$ , they form not a standard union but an *uncertain union* that we shall denote as

$$A \equiv \{A_n\} \equiv \biguplus_n A_n \quad (6)$$

in order to distinguish it from the standard union  $\bigcup_n A_n$ . The uncertain event  $A$  is characterized by the wave function

$$|A\rangle = \sum_n a_n |n\rangle, \quad (7)$$

where  $|a_n|^2$  play the role of weights for the events  $A_n$ . Now, instead of the correspondence (4) for the standard union, we have the correspondence

$$\biguplus_n A_n \rightarrow \hat{P}_A \equiv |A\rangle\langle A|. \quad (8)$$

Note that  $\hat{P}_A$  is not a projector.

The probability of the uncertain event  $A$  reads as

$$p(A) = p\left(\biguplus_n A_n\right) = \sum_n |a_n|^2 p(A_n) + q(A), \quad (9)$$

where the second term

$$q(A) \equiv \sum_{m \neq n} a_m^* a_n \langle m | \hat{\rho} | n \rangle \quad (10)$$

is caused by the interference of the uncertain subevents  $A_n$  that are called *modes*. The probability  $p(A)$  of the uncertain event  $A$ , represented by the uncertain union (6), does not equal the sum of the event probabilities  $p(A_n)$ . In that sense, the uncertain union (6) is not additive with respect to partial events  $A_n$ , contrary to the probability of the standard union (5).

## 4 Composite Non-entangled Prospects

Composite events are termed prospects. These can be sorted in two classes, entangled and non-entangled [30]. This classification is based on the theory of Hilbert-Schmidt spaces [31, 32], as is explained below.

It is well known that quantum-mechanical wave functions, pertaining to a Hilbert space, can be either entangled, or non-separable, and non-entangled, or separable. Similarly, by constructing the appropriate Hilbert-Schmidt space, it is possible to introduce the notions of entangled, or non-separable operators, and of non-entangled, or separable operators. We need this classification for a bipartite system, that is, consisting of two parts, although the definition can be straightforwardly generalized for a multi-partite case.

Let us consider a bipartite quantum system, with one subsystem corresponding to a Hilbert space  $\mathcal{H}_A$  and the other, to a Hilbert space  $\mathcal{H}_B$ . The subsystem, defined in  $\mathcal{H}_A$ , is characterized by a set of operators, acting on  $\mathcal{H}_A$ . For what follows, we keep in mind the operators of observables represented by the projectors of operationally testable events. The set of operators on  $\mathcal{H}_A$  forms an algebra of observables  $\mathcal{A}$ . Respectively, the subsystem in  $\mathcal{H}_B$  is characterized by an algebra  $\mathcal{B}$  acting on  $\mathcal{H}_B$ . For any two operators  $\hat{A}_1$  and  $\hat{A}_2$  in the algebra  $\mathcal{A}$ , it is possible to introduce a scalar product

$$(\hat{A}_1, \hat{A}_2) \equiv \text{Tr}_A \hat{A}_1^+ \hat{A}_2,$$

where the trace is over the space  $\mathcal{H}_A$ , inducing the Hilbert-Schmidt norm

$$\|\hat{A}\| \equiv \sqrt{(\hat{A}, \hat{A})}.$$

Then the operator algebra  $\mathcal{A}$ , complemented by the above scalar product, becomes a Hilbert-Schmidt space. The same can be done for the algebra  $\mathcal{B}$  becoming a Hilbert-Schmidt space with the scalar product defined in the same way.

The system, composed of two parts, is a composite system defined in the tensor-product Hilbert space  $\mathcal{H}_A \otimes \mathcal{H}_B$ . The operator algebra  $\mathcal{A} \otimes \mathcal{B}$  acts on this tensor-product space. For any two operators  $\hat{C}_1$  and  $\hat{C}_2$  of the latter algebra, one defines the scalar product

$$(\hat{C}_1, \hat{C}_2) \equiv \text{Tr}_{AB} \hat{C}_1^+ \hat{C}_2,$$

with the trace over the space  $\mathcal{H}_A \otimes \mathcal{H}_B$ . Thus, the algebra  $\mathcal{A} \otimes \mathcal{B}$ , complemented by this scalar product, becomes a composite Hilbert-Schmidt space. In this way, the operators of a Hilbert-Schmidt space can be treated similarly to the vectors of a Hilbert space.

One tells that the operator  $\hat{C}$ , acting on the composite Hilbert-Schmidt space, is separable, or not-entangled, if and only if it can be represented as

$$\hat{C} = \sum_{\gamma} \hat{C}_{\gamma A} \otimes \hat{C}_{\gamma B},$$

where  $\hat{C}_{\gamma A}$  and  $\hat{C}_{\gamma B}$  are the operators from the related algebras of observables, acting on  $\mathcal{H}_A$  and  $\mathcal{H}_B$ , respectively. Such separable operators have been widely used in scattering theory [33]. On the contrary, if the operator cannot be reduced to the separable form, it is termed non-separable, or entangled.

The classification of the operators onto separable and entangled is intensively employed in quantum information theory [1–5], where one considers statistical operators. In quantum decision theory [15–19], this classification is applied to prospect operators. A prospect operator that cannot be represented in the separable form is called entangled or non-separable, while when it can be reduced to that form, it is termed non-entangled, or separable. Similarly, the prospects, represented by the corresponding prospect operators, can be distinguished onto entangled and non-entangled. Exactly this classification will be used below.

Suppose we consider two elementary events  $A_n$ , represented by a vector  $|n\rangle$  from a Hilbert space  $\mathcal{H}_A$ , and  $B_\alpha$ , represented by a vector  $|\alpha\rangle$  from a Hilbert space  $\mathcal{H}_B$ . The composite event, formed by these two elementary events, is treated as a tensor product  $A_n \otimes B_\alpha$ . In this notation, the event  $A_n$  is assumed to happen after the event  $B_\alpha$ . The composite event  $A_n \otimes B_\alpha$  is represented by the vector

$$|n\alpha\rangle = |n\rangle \otimes |\alpha\rangle \quad (11)$$

from the tensor-product Hilbert space

$$\mathcal{H}_{AB} \equiv \mathcal{H}_A \otimes \mathcal{H}_B. \quad (12)$$

The composite event of observing  $A_n$  and  $B_\alpha$  induces the correspondence

$$A_n \otimes B_\alpha \rightarrow \hat{P}_n \otimes \hat{P}_\alpha, \quad (13)$$

where

$$\hat{P}_\alpha \equiv |\alpha\rangle\langle\alpha|$$

is a projector in  $\mathcal{H}_B$ .

The strategic state is now a statistical operator  $\hat{\rho}$  on the tensor-product space (12). The joint probability of the composite event (13) becomes

$$p(A_n \otimes B_\alpha) = \text{Tr} \hat{\rho} \hat{P}_n \otimes \hat{P}_\alpha \equiv \langle \hat{P}_n \otimes \hat{P}_\alpha \rangle, \quad (14)$$

with the trace over the space (12). The composite event (13) is the simplest composite event, which enjoys the factor form, being composed of two elementary events, and being called *non-entangled*.

More complicated structures arise when at least one of the events is a union. It is important to emphasize the difference between the standard union and the uncertain union introduced in (6).

When the composite event is a product of an elementary event  $A_n$  and a standard union of mutually orthogonal events  $B_\alpha$ , we can employ the known property of composite events:

$$A_n \otimes \bigcup_{\alpha} B_{\alpha} = \bigcup_{\alpha} A_n \otimes B_{\alpha}.$$

In the right-hand side here, we have the union of mutually orthogonal composite events, since  $B_\alpha$  are assumed to be mutually orthogonal. Therefore

$$p \left( A_n \otimes \bigcup_{\alpha} B_{\alpha} \right) = \sum_{\alpha} p(A_n \otimes B_{\alpha}). \quad (15)$$

That is, the probability of a composite event, with one of the factors being the standard union of mutually orthogonal events, is additive. Such events are also termed non-entangled.

It is important to stress that the used terminology is in agreement with the notions of separable and non-entangled operators, as is formulated at the beginning of this section. Really, in the present case, the bipartite system is described by the tensor product of the algebras of observables  $\mathcal{A} \equiv \{\hat{P}_n\}$  and  $\mathcal{B} \equiv \{\hat{P}_\alpha\}$ . The event  $A_n \otimes \bigcup_{\alpha} B_{\alpha}$  induces the event operator that has the form

$$\hat{P} \left( A_n \otimes \bigcup_{\alpha} B_{\alpha} \right) = \sum_{\alpha} \hat{P}_n \otimes \hat{P}_{\alpha},$$

corresponding to the definition of a separable, or non-entangled operator.

## 5 Composite Entangled Prospects

The situation is essentially different when dealing with an uncertain union. In that case, composite events are represented by prospect operators that cannot be represented in the separable form.

Let us have such an uncertain union

$$B \equiv \{B_{\alpha}\} \equiv \biguplus_{\alpha} B_{\alpha} \quad (16)$$

corresponding to a vector

$$|B\rangle = \sum_{\alpha} b_{\alpha} |\alpha\rangle. \quad (17)$$

The composite event, or prospect

$$\pi_n = A_n \otimes B = A_n \otimes \bigoplus_{\alpha} B_{\alpha}, \quad (18)$$

corresponds to the prospect state

$$|\pi_n\rangle = |n\rangle \otimes |B\rangle = \sum_{\alpha} b_{\alpha} |n\alpha\rangle. \quad (19)$$

The composite event (18) induces the correspondence

$$\pi_n \rightarrow \hat{P}(\pi_n) \equiv |\pi_n\rangle\langle\pi_n| \quad (20)$$

defining the prospect operator  $\hat{P}(\pi_n)$ .

At this point, it is necessary to make an important comment clarifying the notion of entanglement. The latter is correctly defined when it is explicitly stated what object is considered and with respect to which parts it is entangled or not. Recall that, in quantum mechanics, one considers the entanglement of a wave function of a composite system with respect to the wave functions of its parts, which are not arbitrary functions. Wave functions can be defined as eigenfunctions of Hamiltonians. In the case of statistical operators, one considers their entanglement or separability with respect to statistical operators of subsystems, but not with respect to arbitrary operators [1–5]. In the general case of operators from a Hilbert-Schmidt space, one considers their entanglement with respect to the operators from the corresponding Hilbert-Schmidt subspaces, but not with respect to arbitrary operators not pertaining to the prescribed subspaces.

The prospect state (19) could be qualified as not entangled with respect to arbitrary functions from the Hilbert spaces  $\mathcal{H}_A$  and  $\mathcal{H}_B$ . However, the function  $|B\rangle$  does not correspond to an operationally testable event, while exactly the latter are of our interest. Hence the formal separability of (19) in the Hilbert space is of no importance. What is important is the separability or entanglement of operationally testable events.

In our case, the operationally testable events correspond to the related projectors forming the algebras of observables  $\mathcal{A} \equiv \{\hat{P}_n\}$  and  $\mathcal{B} \equiv \{\hat{P}_{\alpha}\}$ . Complementing them by the appropriate scalar products, we get the corresponding Hilbert-Schmidt spaces. A prospect  $\pi_n$  is characterized by the prospect operator (20). According to the general theory, a prospect operator is separable if and only if it can be reduced to the linear combination of the tensor products of operators from the Hilbert-Schmidt subspaces. However, the prospect operator (20) reads as

$$\hat{P}(\pi_n) = \sum_{\alpha} |b_{\alpha}|^2 \hat{P}_n \otimes \hat{P}_{\alpha} + \sum_{\alpha \neq \beta} b_{\alpha} b_{\beta}^* \hat{P}_n \otimes |\alpha\rangle\langle\beta|.$$

The first term in the right-hand side does correspond to the definition of separability, while the second term does not, since  $|\alpha\rangle\langle\beta|$ , with  $\alpha \neq \beta$ , does not pertain to the algebra of observables  $\mathcal{B} \equiv \{\hat{P}_{\alpha}\}$ . Hence, the prospect operator (20) is not

separable, that is, it is entangled. Respectively, prospect (18), corresponding to this prospect operator can also be termed entangled, since it cannot be represented as a union of mutually orthogonal events. The entangling properties of operators can be quantified by the measure of entanglement production [34–36]. The amount of entanglement produced in the process of decision making can be calculated as shown in Ref. [37].

The prospect states (19) are not necessarily orthogonal and do not need to be normalized to one. Because of this, the prospect operators, generally, are not projectors. However, the resolution of unity is required:

$$\sum_n \hat{P}(\pi_n) = \hat{1}_{AB}, \tag{21}$$

where  $\hat{1}_{AB}$  is the unity operator in the space (12). The family  $\{\hat{P}(\pi_n)\}$  composes a *positive operator-valued measure*.

The prospect probability

$$p(\pi_n) \equiv \text{Tr} \hat{\rho} \hat{P}(\pi_n) \equiv \langle \hat{P}(\pi_n) \rangle, \tag{22}$$

with the trace over space (12), becomes the sum of two terms:

$$p(\pi_n) = f(\pi_n) + q(\pi_n). \tag{23}$$

The first term

$$f(\pi_n) \equiv \sum_{\alpha} |b_{\alpha}|^2 p(A_n \otimes B_{\alpha}) \tag{24}$$

contains the diagonal elements with respect to  $\alpha$ , while the second term

$$q(\pi_n) \equiv \sum_{\alpha \neq \beta} b_{\alpha}^* b_{\beta} \langle n\alpha | \hat{\rho} | n\beta \rangle \tag{25}$$

is formed by the nondiagonal elements. By construction and due to the resolution of unity (21), the prospect probability (22) satisfies the properties

$$\sum_n p(\pi_n) = 1, \quad 0 \leq p(\pi_n) \leq 1, \tag{26}$$

which makes the family  $\{p(\pi_n)\}$  a probability measure.

Expression (25) is caused by the quantum nature of the considered events producing interference of the modes composing the uncertain union (16). Because of this, the term (25) can be called the *quantum factor*, *interference factor*, or *coherence factor*. The quantum term (25) may be nonzero only when prospect (18) is *entangled* in the sense of the nonseparability of the prospect operator in the Hilbert-Schmidt space.

Classical probability has to be a marginal case of quantum probability. To this end, we have to remember the quantum-classical correspondence principle advanced by Bohr [38]. This principle tells us that classical theory is to be the limiting case of quantum theory, when quantum effects vanish. In the present

case, this implies that when the quantum interference factor tends to zero, the quantum probability has to tend to a classical probability. Such a process is also called *decoherence*. According to the principle of the quantum-classical correspondence, we have

$$p(\pi_n) \rightarrow f(\pi_n), \quad q(\pi_n) \rightarrow 0, \quad (27)$$

which means that the decoherence process leads to the classical probability  $f(\pi_n)$ . Being a probability, this classical factor needs to be normalized, so as to satisfy the conditions

$$\sum_n f(\pi_n) = 1, \quad 0 \leq f(\pi_n) \leq 1. \quad (28)$$

As a consequence of the above equations, the interference factor enjoys the properties

$$\sum_n q(\pi_n) = 0, \quad -1 \leq q(\pi_n) \leq 1. \quad (29)$$

The first of these equations is called the *alternation condition*.

One should not confuse the effect of decoherence, based on the quantum-classical correspondence principle, when quantum measurements are reduced to classical, with the Kochen-Specker theorem [39] stating the impossibility of simultaneous embedding of all commuting sub-algebras of the algebra of quantum observables in one commutative algebra, assumed to represent the classical structure of the hidden-variables theory, if the Hilbert space dimension is at least three. This theorem places certain constraints on the permissible types of hidden-variable theories, which try to explain the apparent randomness of quantum mechanics as a deterministic model featuring hidden states. The theorem excludes hidden-variable theories that require elements of physical reality to be non-contextual, i.e., independent of the measurement arrangement. The exclusion of such hidden variables is exactly due to the existence of quantum entanglement.

Our consideration has nothing to do with hidden variables. We do not intend to replace quantum theory by a classical theory with hidden variables. Vice versa, the whole of our approach is completely based on the standard techniques of quantum theory and all results are in full agreement with the known properties of quantum theory. Being always in the frame of quantum theory, we consider a very well known effect called *decoherence* that manifests the transition from quantum to classical behavior. This effect is intimately connected with the quantum-classical correspondence principle, formulated by Bohr and widely used in quantum theory. According to this principle, the results of quantum theory reduce to those of classical theory, when quantum effects, such as entanglement and interference, are washed out. The effect of decoherence is well understood and described in the frame of quantum theory [40–42].

As is mentioned above, the quantum term arises only when the considered prospect is entangled in the sense of the nonseparability of the prospect operator in the Hilbert-Schmidt space. The other necessary condition for the existence of

the quantum term is the entanglement in the strategic state  $\hat{\rho}$ . To illustrate that a disentangled strategic state does not produce interference, let us take the system state in the disentangled product form

$$\hat{\rho} = \hat{\rho}_A \otimes \hat{\rho}_B.$$

Then, the quantum interference term becomes

$$q(\pi_n) = \sum_{\alpha \neq \beta} b_\alpha^* b_\beta \langle n | \hat{\rho}_A | n \rangle \langle \alpha | \hat{\rho}_B | \beta \rangle.$$

Taking into account the normalization condition

$$\text{Tr}_A \hat{\rho}_A = \sum_n \langle n | \hat{\rho}_A | n \rangle = 1,$$

we get

$$\sum_n q(\pi_n) = \sum_{\alpha \neq \beta} b_\alpha^* b_\beta \langle \alpha | \hat{\rho}_B | \beta \rangle = 0.$$

As a result, we find

$$q(\pi_n) = \langle n | \hat{\rho}_A | n \rangle \sum_n q(\pi_n) = 0.$$

So, the disentangled strategic state does not allow for the appearance of a non-trivial quantum interference term.

The quantum term (25) is a random quantity satisfying a very important property called the *quarter law* [16–19]. For a prospect lattice,  $\mathcal{L} \equiv \{\pi_n : n = 1, 2, \dots, N_L\}$  the absolute value of the aggregate quantum factor can be estimated as

$$|\bar{q}| \equiv \frac{1}{N_L} \sum_{j=1}^{N_L} |q(\pi_j)| = \frac{1}{4}. \quad (30)$$

The value 1/4 for the aggregate attraction factor can be shown [43] to arise for a large class of distributions characterizing the attraction factors of different decision makers.

Expression (24), corresponding to classical probability, is defined as an objective term, whose value is prescribed by the prospect utility, justifying to call this term the *utility factor*. The quantum term (25) describes the attractiveness of the prospect to a decision maker, so that it is named the *attraction factor* [15–19]. A prospect  $\pi_1$  is more useful than  $\pi_2$ , if and only if  $f(\pi_1) > f(\pi_2)$ . A prospect  $\pi_1$  is more attractive than  $\pi_2$ , when and only when  $q(\pi_1) > q(\pi_2)$ . And a prospect  $\pi_1$  is preferable to  $\pi_2$ , if and only if  $p(\pi_1) > p(\pi_2)$ . Hence, a prospect can be more useful but less attractive, as a result being less preferable.

## 6 Uncertainty in Standard Lotteries

It can be shown [30], that the necessary condition for the quantum term to be nonzero requires that the considered prospect be entangled and the strategic state  $\hat{\rho}$  also be entangled. This implies that the decision is made under uncertainty [44].

A typical situation in decision making is when one chooses between several lotteries. One may ask what kind of uncertainty is ascribed to such a choice between the lotteries.

Suppose we consider a family  $\{L_n\}$  of lotteries enumerated with the index  $n = 1, 2, \dots, N_L$ . Each lottery is the set

$$L_n = \{x_i, p_n(x_i) : i = 1, 2, \dots\} \quad (31)$$

of payoffs  $x_i$  and payoff probabilities  $p_n(x_i)$ . A decision maker has to choose one of these lotteries.

The choice between the lotteries is a random procedure involving uncertainty. First of all, when choosing a lottery, one does not know exactly what would be a payoff whose occurrence is characterized by the related probability. Moreover, in each choice, there always exists uncertainty caused by two reasons. One reason is the decision-maker doubt about the objectivity of the setup suggesting the choice. The other origin of uncertainty is caused by subjective hesitations of the decision maker with respect to his/her correct understanding of the problem and his/her knowledge of what would be the best criterion for making a particular choice. Let us denote by  $B_1$  the decision-maker confidence in the empirical setup as well as in his/her ability of making a correct decision. Then  $B_2$  corresponds to the disbelief of the decision maker in the suggested setup and/or in his/her understanding of the appropriate criteria for the choice. The combination of belief and disbelief is the set

$$B = \{B_1, B_2\} = \bigoplus_{\alpha} B_{\alpha} \quad (\alpha = 1, 2).$$

In this way, even choosing between simple lotteries  $L_n$ , one actually confronts the composite prospects

$$\pi_n = L_n \otimes B, \quad (32)$$

where the event of selecting a lottery  $L_n$  is denoted by the same letter, which should not lead to confusion. The choice is made under uncertainty incorporated into the set  $B = \{B_1, B_2\}$  of belief and disbelief.

The prospect probability is given by (23). The utility factor, characterizing the objective part of the probability, in the case of the choice between the lotteries can be defined [16, 19] as

$$f(\pi_n) = \frac{U(\pi_n)}{\sum_n U(\pi_n)}, \quad (33)$$

ordering the prospects according to their expected utilities

$$U(\pi_n) = \sum_i u(x_i)p_n(x_i), \quad (34)$$

where  $u(x_i)$  is a utility function. The attraction factors can be evaluated as is explained in the previous section.

For example, dealing with the prospect lattice  $\mathcal{L} = \{\pi_1, \pi_2\}$ , in which the prospect  $\pi_i$  is more attractive than  $\pi_j$ , the prospect probabilities are estimated by the expressions

$$\begin{aligned} p(\pi_i) &= f(\pi_i) + 0.25, \\ p(\pi_j) &= f(\pi_j) - 0.25. \end{aligned} \tag{35}$$

To illustrate how the procedure described above works, let us consider the lotteries discussed by Kahneman and Tversky [45]. Consider, for instance, the lotteries

$$L_1 = \{6, 0.45 \mid 0, 0.55\}, \quad L_2 = \{3, 0.9 \mid 0, 0.1\}.$$

Calculating their expected utilities, we assume, for simplicity, linear utility functions  $u(x) = cx$ . The corresponding utility factors of both the lotteries are equal,

$$f(\pi_1) = 0.5, \quad f(\pi_2) = 0.5.$$

The second prospect is more attractive, being more certain. Then, employing rule (35), we have

$$p(\pi_1) = 0.25, \quad p(\pi_2) = 0.75.$$

The experimental results of Kahneman and Tversky [45] are

$$p_{exp}(\pi_1) = 0.14, \quad p_{exp}(\pi_2) = 0.86,$$

where  $p_{exp}(\pi_i)$  is the ratio of the number of the decision makers, choosing the lottery  $L_i$ , to the total number of participants. Within the statistical errors of  $\pm 0.1$  of these experiments, our theoretical prediction agrees with the empirical results.

Another example by Kahneman and Tversky [45] is the choice between the lotteries

$$L_1 = \{6, 0.001 \mid 0, 0.999\}, \quad L_2 = \{3, 0.002 \mid 0, 0.998\},$$

which enjoy the same utility factors

$$f(\pi_1) = 0.5, \quad f(\pi_2) = 0.5,$$

as in the previous example. The uncertainties of the two lotteries are close to each other. However, the gain in the first prospect is essentially larger, which makes it more attractive, hence, the second prospect less attractive. As a result, the prospect preference reverses, as compared to the previous case, with the prospect probabilities

$$p(\pi_1) = 0.75, \quad p(\pi_2) = 0.25.$$

The experimental data of Kahneman and Tversky [45] are

$$p_{exp}(\pi_1) = 0.73, \quad p_{exp}(\pi_2) = 0.27.$$

Thus our theoretical prediction practically coincides with the empirical data.

We also have analyzed a number of other experimental examples, obtaining good agreement of our theoretical predictions with empirical results. However, we shall not overload the present report by the description of all these experiments, which will be the topic of a separate paper.

## 7 Conclusion

Decision making very often meets the necessity of deciding under uncertainty. Applying quantum techniques to decision making, one has to use the appropriate mathematical tools. We have shown that the correct mathematical foundation of quantum decision theory, dealing with uncertain events, requires the use of positive operator-valued measure that is a generalization of the projection-valued measure. The latter is used for operationally testable events, but cannot be applied to uncertain events typical of decision making under uncertainty. Such operationally uncertain events require the use of the operator-valued measure. In decision making, one has to distinguish composite non-entangled events from composite entangled events. The accurate mathematical formulation of entangled events is based on the notion of entangled or nonseparable prospect operators in a Hilbert-Schmidt space. This should not be confused with the entanglement of functions in a Hilbert space. The operationally testable events are called modes. Therefore the entanglement of such events can be termed *mode entanglement*.

According to the principle of quantum-classical correspondence, classical probabilities can be treated as a limiting case of quantum probabilities, when the effect of decoherence is present. We consider the occurrence of classical probabilities exactly in this sense, which is completely in the frame of quantum theory. We stress that the Kochen-Specker theorem, proving the impossibility of non-contextual hidden variables has no relation to our approach.

Quantum probability can be essentially different from the form of classical probability only for entangled events, defined through the mode entanglement in a Hilbert-Schmidt space. The necessary condition for the appearance of a quantum interference term in the quantum probability is the occurrence of entangled prospects and the existence of an entangled strategic state of a decision maker. The origin of uncertainties in standard lotteries is explained. Our approach makes it possible to provide theoretical predictions that are in good numerical agreement with the results of empirical observations.

Our approach is principally different from those of other authors [11] in several basic points. First, we develop a general theory based on rigorous mathematics of quantum measurement theory and quantum information theory, which can be applied to any decision-making processes. Different from [11] and others, we do not construct special schemes for studying some particular problems. Second, we give a mathematically correct definition of quantum joint probabilities as the probabilities of composite events realized in different measurement channels and represented in tensor-product Hilbert spaces. In contrast, other authors usually consider a single Hilbert space and deal with the Lüders probabilities

of consecutive events, which are, actually, transition probabilities, but cannot be treated as conditional probabilities [30]. Third, we emphasize the necessity of entangled events for the appearance of quantum effects, such as the arising interference. Fourth, we define the entanglement of operationally testable events as mode entanglement described by the nonseparable prospect operators in a Hilbert-Schmidt space. Fifth, our theory allows for quantitative predictions of decision making, without any fitting parameters, including quantitative explanations of classical decision-making paradoxes. This makes our approach unique, since there is no other approach that could be compared with empirical results without invoking fitting parameters.

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