Solution of the State Equation of Descriptor Fractional Continuous-Time Linear Systems with Two Different Fractional

Łukasz Sajewski

Bialystok University of Technology, Faculty of Electrical Engineering
Wiejska 45D, 15-351 Białystok
l.sajewski@pb.edu.pl

Abstract. The descriptor fractional continuous-time linear systems with two different fractional orders are considered. The Drazin inverse of matrices is applied to find the solutions of the state equations. Some additional changes to classical Drazin approach for finding solution of the state equation of descriptor systems is proposed. An equality defining the set of admissible initial conditions for given inputs is derived.

Keywords: descriptor, fractional, different order, solution.

1 Introduction

Descriptor (singular) linear systems have been considered in many papers and books [1−3, 5, 6, 8, 10, 15, 16, 20, 28, 29]. The first definition of the fractional derivative was introduced by Liouville and Riemann at the end of the 19^{th} century [22, 23] and another on was proposed in 20^{th} century by Caputo [24]. This idea has been used by engineers for modeling different processes [7, 9]. Mathematical fundamentals of fractional calculus are given in the monographs [19, 21−24]. The positive fractional linear systems have been investigated in [13, 14, 19].

The positive linear systems with different fractional orders have been addressed in [17, 18].

Stability of fractional continuous-time linear systems consisting of n subsystem with different fractional orders [3]. The reachability and minimum energy control problem for systems with two different fractional orders in [25−27].

Drazin inverse matrix method for fractional descriptor continuous-time and discrete-time linear systems have been proposed in [11, 12].

In this paper solution to the state equation of descriptor fractional positive continuous-time linear systems with two different fractional orders will be formulated and solved.

The paper is organized as follows. In section 2 the basic definitions and theorems of the descriptor fractional continuous-time linear systems are recalled. Section 3 gives the problem formulation for systems with two different fractional orders. Solution to the state equation is given in section 4. Concluding remarks are given in section 5.

R. Szewczyk et al. (eds.), *Progress in Automation, Robotics and Measuring Techniques*,
Advances in Intelligent Systems and Computing 350, DOI: 10.1007/978-3-319-15796-2_24

The following notation will be used: $\Re$ – the set of real numbers, $\Re^{n\times m}$ – the set of $n\times m$ real matrices, I_n – the $n\times n$ identity matrix.

2 Descriptor Fractional System

Consider the fractional descriptor continuous-time linear system

$$E\,{}_0D_t^{\alpha}x(t)=Ax(t)+Bu(t)\,,\ n-1<\alpha<n\,,\ n\in W\,, \tag{2.1}$$

where α is fractional order, $x(t)\in\Re^n$ is the state vector $u(t)\in\Re^m$ is the input vector, $E,A\in\Re^{n\times n}$, $B\in\Re^{n\times m}$ and

$${}_0D_t^{\alpha}x(t)=\frac{d^{\alpha}x(t)}{dt^{\alpha}}=\frac{1}{\Gamma(n-\alpha)}\int_0^{\infty}\frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha+1-n}}d\tau\,,\ f^{(n)}(\tau)=\frac{d^nf(\tau)}{d\tau^n} \tag{2.2}$$

where $n-1<\alpha<n$, $n\in W=\{1,2,...\}$ is the Caputo definition of $\alpha\in\Re$ order derivative of $x(t)$ and

$$\Gamma(\alpha)=\int_0^{\infty}e^{-t}t^{\alpha-1}dt \tag{2.3}$$

is the Euler gamma function.

It is well known [19] that the Laplace transform ($\mathcal{L}$) of (2.1) is given by the formula

$$\mathcal{L}\left[\frac{d^{\alpha}f(t)}{dt^{\alpha}}\right]=\int_0^{\infty}\frac{d^{\alpha}f(t)}{dt^{\alpha}}e^{-st}dt=s^{\alpha}F(s)-\sum_{k=1}^{n}s^{\alpha-k}f^{(k-1)}(0+)\,,\ n-1<\alpha<n\,,\ n\in W\,, \tag{2.4}$$

where $F(s)=\mathcal{L}[f(t)]$ and $n-1<\alpha<n$, $n\in W$.

It is assumed that det E = 0 but the pencil (E, A) of (2.1) is regular, i.e.

$$\det[Es-A]\neq 0 \text{ for some } s\in \mathrm{C} \text{ (the field of complex numbers).} \tag{2.5}$$

Assuming that for some chosen $c\in\mathrm{C}$, $\det[Ec-A]\neq 0$ and premultiplying (2.1) by $[Ec-A]^{-1}$ we obtain

$$\overline{E}\,{}_0D_t^{\alpha}x(t)=\overline{A}x(t)+\overline{B}u(t)\,, \tag{2.6a}$$

where

$$\overline{E}=[Ec-A]^{-1}E\,,\ \overline{A}=[Ec-A]^{-1}A\,,\ \overline{B}=[Ec-A]^{-1}B\,. \tag{2.6b}$$

Note that the equations (2.1) and (2.6a) have the same solution $x(t)$.

Definition 2.1. The smallest nonnegative integer q is called the index of the matrix $\overline{E} \in \Re^{n \times n}$ if [4, 16]

$$\text{rank}\, \overline{E}^q = \text{rank}\, \overline{E}^{q+1}. \tag{2.7}$$

Definition 2.1. [4, 16] A matrix $\overline{E}^D$ is called the Drazin inverse of $\overline{E} \in \Re^{n \times n}$ if it satisfies the conditions

$$\overline{E}\overline{E}^D = \overline{E}^D \overline{E}, \tag{2.8a}$$

$$\overline{E}^D \overline{E}\overline{E}^D = \overline{E}^D, \tag{2.8b}$$

$$\overline{E}^D \overline{E}^{q+1} = \overline{E}^q, \tag{2.8c}$$

where q is the index of $\overline{E}$ defined by (2.6).

The Drazin inverse $\overline{E}^D$ of a square matrix $\overline{E}$ always exists and is unique [4, 16]. If $\det \overline{E} \neq 0$ then $\overline{E}^D = \overline{E}^{-1}$.

Lemma 2.1. [4, 16] The matrices $\overline{E}$ and $\overline{A}$ defined by (2.5b) satisfy the following equalities

1. $\overline{A}\overline{E} = \overline{E}\overline{A}$ and $\overline{A}^D \overline{E} = \overline{E}\overline{A}^D$, $\overline{E}^D \overline{A} = \overline{A}\overline{E}^D$, $\overline{A}^D \overline{E}^D = \overline{E}^D \overline{A}^D$, (2.9a)
2. $\ker \overline{A} \cap \ker \overline{E} = \{0\}$, (2.9b)
3. $\overline{E} = T\begin{bmatrix} J & 0 \\ 0 & N \end{bmatrix} T^{-1}$, $\overline{E}^D = T\begin{bmatrix} J^{-1} & 0 \\ 0 & 0 \end{bmatrix} T^{-1}$, $\overline{A} = T\begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} T^{-1}$ (2.9c)

   $\det T \neq 0$, $J \in \Re^{n_1 \times n_1}$, is nonsingular, $N \in \Re^{n_2 \times n_2}$ is nilpotent, $n_1 + n_2 = n$,
4. $(I_n - \overline{E}\overline{E}^D)\overline{A}\overline{A}^D = I_n - \overline{E}\overline{E}^D$ and $(I_n - \overline{E}\overline{E}^D)(\overline{E}\overline{A}^D)^q = 0$. (2.9d)

The solution to the equation (2.1) is given by

$$x(t) = \Phi_0(t)\overline{E}\overline{E}^D v + \overline{E}^D \int_0^t \Phi(t-\tau)\overline{B}u(\tau)d\tau + (\overline{E}\overline{E}^D - I_n)\sum_{k=0}^{q-1} (\overline{E}\overline{A}^D)^k \overline{A}^D \overline{B}u^{(k\alpha)}(t), \tag{2.10a}$$

where

$$\Phi_0(t) = \sum_{k=0}^{\infty} \frac{(\overline{E}^D \overline{A})^k t^{k\alpha}}{\Gamma(k\alpha+1)},\ \Phi(t) = \sum_{k=0}^{\infty} \frac{(\overline{E}^D \overline{A})^k t^{(k+1)\alpha-1}}{\Gamma[(k+1)\alpha]}, \tag{2.10b}$$

$$u^{(k\alpha)}(t) = {}_0D_t^{k\alpha} u(t) \tag{2.10c}$$

and the vector $v \in \Re^n$ is arbitrary [12].

3 Systems with Two Different Fractional Orders

Consider a fractional linear system with two different fractional orders $\alpha \neq \beta$ described by the equation [18, 26]

$$\begin{bmatrix} \frac{d^{\alpha} x_1(t)}{dt^{\alpha}} \\ \frac{d^{\beta} x_2(t)}{dt^{\beta}} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u(t), \tag{3.1}$$

and $p-1 < \alpha < p; q-1 < \beta < q;\ p,q \in W$ where $x_1(t) \in \Re^{n_1}$, $x_2(t) \in \Re^{n_2}$, $u(t) \in \Re^{m}$ and $y(t) \in \Re^{p}$ are the state, input and output vectors respectively, $A_{ij} \in \Re^{n_i \times n_j}$, $B_i \in \Re^{n_i \times m}$; $i, j = 1,2$.

Initial conditions for (3.1) have the form

$$x_1(0) = x_{10},\ x_2(0) = x_{20} \text{ and } x_0 = \begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix}. \tag{3.2}$$

It is well-known [17, 18] that the solution of the equation (3.1) for $0 < \alpha < 1;\ 0 < \beta < 1$ with initial conditions (3.2) has the form

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \Phi_0(t) \begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix} + \int_0^t \Phi_1(t-\tau) \begin{bmatrix} B_1 \\ 0 \end{bmatrix} u(\tau) d\tau + \int_0^t \Phi_2(t-\tau) \begin{bmatrix} 0 \\ B_2 \end{bmatrix} u(\tau) d\tau \tag{3.3a}$$

where

$$T_{k,l} = \begin{cases} I_n & \text{for } k = l = 0 \\ \begin{bmatrix} A_{11} & A_{12} \\ 0 & 0 \end{bmatrix} & \text{for } k = 1, l = 0 \\ \begin{bmatrix} 0 & 0 \\ A_{21} & A_{22} \end{bmatrix} & \text{for } k = 0, l = 1 \\ T_{10} T_{k-1,l} + T_{01} T_{k,l-1} & \text{for } k + l > 0 \end{cases} \tag{3.3b}$$

$$\Phi_0(t) = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} T_{k,l} \frac{t^{k\alpha + l\beta}}{\Gamma(k\alpha + l\beta + 1)},$$

$$\Phi_1(t) = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} T_{k,l} \frac{t^{(k+1)\alpha + l\beta - 1}}{\Gamma[(k+1)\alpha + l\beta]},\ \Phi_2(t) = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} T_{k,l} \frac{t^{k\alpha + (l+1)\beta - 1}}{\Gamma[k\alpha + (l+1)\beta]}. \tag{3.3c}$$

Now let consider the fractional descriptor continuous-time linear system with different fractional orders

$$E\begin{bmatrix} \frac{d^{\alpha}x_1(t)}{dt^{\alpha}} \\ \frac{d^{\beta}x_2(t)}{dt^{\beta}} \end{bmatrix} = A\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + Bu(t), \tag{3.4}$$

and $p-1<\alpha<p; q-1<\beta<q;\ \ p,q\in W$, where $E=\begin{bmatrix} E_1 & 0 \\ 0 & E_2 \end{bmatrix}\in\Re^{(n_1+n_2)\times(n_1+n_2)}$,

$A=\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}\in\Re^{(n_1+n_2)\times(n_1+n_2)}$, $B=\begin{bmatrix} B_1 \\ B_2 \end{bmatrix}\in\Re^{(n_1+n_2)\times m}$.

It is assumed that $\det E = 0$ but the pencil (E, A) of (3.4) is regular, i.e.

$$\det\left[\begin{bmatrix} E_1 & 0 \\ 0 & E_2 \end{bmatrix}\begin{bmatrix} s^{\alpha} & 0 \\ 0 & s^{\beta} \end{bmatrix} - \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}\right] \neq 0 \text{ for some } s^{\alpha}, s^{\beta}\in \mathrm{C} \tag{3.5}$$

where C is the field of complex numbers.

Similar as for (2.1) assuming that for some chosen $c_1, c_2 \in \mathrm{C}$,
$\det[E\operatorname{diag}(c_1,c_2)-A]\neq 0$ and premultiplying (3.4) by $[E\operatorname{diag}(c_1,c_2)-A]^{-1}$ we obtain

$$\overline{E}\begin{bmatrix} \frac{d^{\alpha}x_1(t)}{dt^{\alpha}} \\ \frac{d^{\beta}x_2(t)}{dt^{\beta}} \end{bmatrix} = \overline{A}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \overline{B}u(t), \tag{3.6a}$$

where

$$\begin{aligned}
\overline{E} &= [E\operatorname{diag}(c_1,c_2)-A]^{-1}E = \begin{bmatrix} \overline{E}_{11} & \overline{E}_{12} \\ \overline{E}_{21} & \overline{E}_{22} \end{bmatrix}, \\
\overline{A} &= [E\operatorname{diag}(c_1,c_2)-A]^{-1}A = \begin{bmatrix} \overline{A}_{11} & \overline{A}_{12} \\ \overline{A}_{21} & \overline{A}_{22} \end{bmatrix} = \overline{T}_{10}+\overline{T}_{01}, \\
\overline{B} &= [E\operatorname{diag}(c_1,c_2)-A]^{-1}B = \begin{bmatrix} \overline{B}_1 \\ \overline{B}_2 \end{bmatrix} = \overline{B}_{10}+\overline{B}_{01}, \\
\overline{T}_{10} &= \begin{bmatrix} \overline{A}_{11} & \overline{A}_{12} \\ 0 & 0 \end{bmatrix},\ \overline{T}_{01} = \begin{bmatrix} 0 & 0 \\ \overline{A}_{21} & \overline{A}_{22} \end{bmatrix},\ \overline{B}_{10} = \begin{bmatrix} \overline{B}_1 \\ 0 \end{bmatrix},\ \overline{B}_{01} = \begin{bmatrix} 0 \\ \overline{B}_2 \end{bmatrix}.
\end{aligned} \tag{3.6b}$$

Note that the equations (3.4) and (3.6a) have the same solution $x(t)$.
In case of system with two different fractional order the Definition 2.1 takes the form:

Definition 3.1. The smallest nonnegative integer q_i, $i = 1,2$ is called the index of the matrix $\overline{E}_{ii} \in \Re^{n_i \times n_i}$ if

$$\text{rank}\,\overline{E}_{ii}^{q_i} = \text{rank}\,\overline{E}_{ii}^{q_i+1} \tag{3.7}$$

and $q = q_1 + q_2$ is the index of $\overline{E}$.

Remark 3.1. According to system (3.6), condition $\alpha \neq \beta$ and formulation of solution (3.3) of the state equation (3.1) (case of the system (3.6) with $\det E \neq 0$), impose additional conditions to Lemma 2.1.

4 Solution to the State Equation by the Use of Drazin Inverse

In this section the solution to the state equation (3.6) will be presented by the use of the Drazin inverses of the matrices $\overline{E}$ and $\overline{A}$.

Assumption 4.1. For system with two different fractional orders the first condition of the Lemma 2.1 takes the form $\overline{T}_{k,l}\overline{E} = \overline{E}\,\overline{T}_{k,l}$ and $\overline{T}_{k,l}^D\overline{E} = \overline{E}\,\overline{T}_{k,l}^D$, $\overline{E}^D\overline{T}_{k,l} = \overline{T}_{k,l}\overline{E}^D$, $\overline{T}_{k,l}^D\overline{E}^D = \overline{E}^D\overline{T}_{k,l}^D$.

Theorem 4.1. If the Assumption 4.1 is true then the solution to the equation (3.6) is given by

$$\begin{aligned} x(t) &= \Phi_0(t)\overline{E}\,\overline{E}^D v + \overline{E}^D \int_0^t [\Phi_1(t-\tau)B_{10} + \Phi_2(t-\tau)B_{01}]u(\tau)d\tau \\ &+ (\overline{E}\,\overline{E}^D - I_{n_1+n_2}) \sum_{k=0}^{q_1-1}\sum_{l=0}^{q_2-1} \overline{E}^{k+l}\overline{T}_{k,l}^D\overline{A}^D\overline{B}u^{(k\alpha+l\beta)}(t) \end{aligned} \tag{4.1a}$$

where

$$\overline{T}_{k,l} = \begin{cases} I_n & \text{for } k = l = 0 \\ \begin{bmatrix} \overline{A}_{11} & \overline{A}_{12} \\ 0 & 0 \end{bmatrix} & \text{for } k = 1, l = 0 \\ \begin{bmatrix} 0 & 0 \\ \overline{A}_{21} & \overline{A}_{22} \end{bmatrix} & \text{for } k = 0, l = 1 \\ \overline{T}_{10}\overline{T}_{k-1,l} + \overline{T}_{01}\overline{T}_{k,l-1} & \text{for } k + l > 0 \end{cases} \tag{4.1b}$$

$$\Phi_0(t) = \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} (\overline{E}^D)^{k+l}\overline{T}_{k,l} \frac{t^{k\alpha+l\beta}}{\Gamma(k\alpha+l\beta+1)},$$

$$\Phi_1(t) = \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} (\overline{E}^D)^{k+l}\overline{T}_{k,l} \frac{t^{(k+1)\alpha+l\beta-1}}{\Gamma[(k+1)\alpha+l\beta]},$$

$$\Phi_2(t)=\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}(\overline{E}^D)^{k+l}\overline{T}_{k,l}\frac{t^{k\alpha+(l+1)\beta-1}}{\Gamma[k\alpha+(l+1)\beta]} \tag{4.1c}$$

$$u^{(k\alpha+l\beta)}(t)={}_0D_t^{k\alpha+l\beta}u(t) \tag{4.1d}$$

and the vector $v=\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}\in\Re^{n_1+n_2}$ is arbitrary.

Proof. Proof will be accomplished by showing that (4.1a) satisfies the equation (3.6a).

Since the system (4.1) is linear then we can split proof on two cases:

1) For $u(t)=0$ we have $\overline{E}\,{}_0D_t^{\alpha}x(t)=\overline{A}[\Phi_0(t)\overline{E}\overline{E}^Dv]$ since

$$\begin{aligned}
&\overline{E}\,{}_0D_t^{\alpha}x(t)=\overline{E}\,{}_0D_t^{\alpha}[\Phi_0(t)\overline{E}\overline{E}^Dv]=\\
&=\overline{E}\,{}_0D_t^{\alpha}\left[\overline{E}\overline{E}^Dv+\sum_{\substack{k=0\\k+l\ge 1}}^{\infty}\sum_{l=0}^{\infty}(\overline{E}^D)^{k+l}\overline{T}_{k,l}\frac{t^{k\alpha+l\beta}}{\Gamma(k\alpha+l\beta+1)}\overline{E}\overline{E}^Dv\right]=\\
&=\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\overline{E}(\overline{E}^D)^{k+l+1}\overline{T}_{k+1,l+1}\frac{t^{k\alpha+l\beta}}{\Gamma(k\alpha+l\beta+1)}\overline{E}\overline{E}^Dv,
\end{aligned}$$

$$\begin{aligned}
&\overline{A}[\Phi_0(t)\overline{E}\overline{E}^Dv]=\overline{A}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}(\overline{E}^D)^{k+l}\overline{T}_{k,l}\frac{t^{k\alpha+l\beta}}{\Gamma(k\alpha+l\beta+1)}\overline{E}\overline{E}^Dv=\\
&=\overline{T}_{10}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}(\overline{E}^D)^{k+l}\overline{T}_{k,l}\frac{t^{k\alpha+l\beta}}{\Gamma(k\alpha+l\beta+1)}\overline{E}\overline{E}^Dv+\\
&\quad+\overline{T}_{01}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}(\overline{E}^D)^{k+l}\overline{T}_{k,l}\frac{t^{k\alpha+l\beta}}{\Gamma(k\alpha+l\beta+1)}\overline{E}\overline{E}^Dv=\\
&=\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}(\overline{E}^D)^{k+l}\overline{T}_{k+1,l}\frac{t^{k\alpha+l\beta}}{\Gamma(k\alpha+l\beta+1)}\overline{E}\overline{E}^Dv+\\
&\quad+\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}(\overline{E}^D)^{k+l}\overline{T}_{k,l+1}\frac{t^{k\alpha+l\beta}}{\Gamma(k\alpha+l\beta+1)}\overline{E}\overline{E}^Dv=\\
&=\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\overline{E}(\overline{E}^D)^{k+l+1}\overline{T}_{k+1,l+1}\frac{t^{k\alpha+l\beta}}{\Gamma(k\alpha+l\beta+1)}\overline{E}\overline{E}^Dv
\end{aligned} \tag{4.2}$$

with

$$
\begin{aligned}
&{}_0D_t^{\alpha}\overline{E}\overline{E}^D v = 0,\quad \overline{E}(\overline{E}^D)^{k+l+1}\overline{T}_{k+1,l+1} = \overline{T}_{k+1,l+1}(\overline{E}^D)^{k+l},\quad \overline{A} = \overline{T}_{10} + \overline{T}_{01},\\
&\Phi_1(t) = \sum_{k=0}^{\infty}\sum_{l=0}^{\infty}(\overline{E}^D)^{k+l}\overline{T}_{k,l}\frac{t^{(k+1)\alpha+l\beta-1}}{\Gamma[(k+1)\alpha+l\beta]} =\\
&\quad = \frac{t^{\alpha-1}}{\Gamma(\alpha)} + \sum_{k=0}^{\infty}\sum_{l=0}^{\infty}(\overline{E}^D)^{k+l+1}\overline{T}_{k+1,l}\frac{t^{(k+2)\alpha+l\beta-1}}{\Gamma[(k+2)\alpha+l\beta]},\\
&\Phi_2(t) = \sum_{k=0}^{\infty}\sum_{l=0}^{\infty}(\overline{E}^D)^{k+l}\overline{T}_{k,l}\frac{t^{k\alpha+(l+1)\beta-1}}{\Gamma[k\alpha+(l+1)\beta]} =\\
&\quad = \frac{t^{\beta-1}}{\Gamma(\beta)} + \sum_{k=0}^{\infty}\sum_{l=0}^{\infty}(\overline{E}^D)^{k+l+1}\overline{T}_{k,l+!}\frac{t^{k\alpha+(l+2)\beta-1}}{\Gamma[k\alpha+(l+2)\beta]}
\end{aligned}
\tag{4.3}
$$

and (2.9d) holds.

2) For $v = 0$ we have

$$
\begin{aligned}
&\overline{E}\,{}_0D_t^{\alpha}x(t) - \overline{A}x = \overline{E}^D\overline{B}u(t) + (\overline{E}\overline{E}^D - I_{n_1+n_2})\sum_{k=0}^{q_1-1}\sum_{l=0}^{q_2-1}\overline{E}^{k+l+1}\overline{T}_{k+1,l+1}^D\overline{B}u^{(k\alpha+l\beta+1)}(t)\\
&\quad - \overline{A}(\overline{E}\overline{E}^D - I_{n_1+n_2})\sum_{k=0}^{q_1-1}\sum_{l=0}^{q_2-1}\overline{E}^{k+l}\overline{T}_{k,l}^D\overline{A}^D\overline{B}u^{(k\alpha+l\beta)}(t) = \overline{B}u(t).
\end{aligned}
\tag{4.4}
$$

In general, substituting (4.1a) in the left side of the equation (3.6a), using (4.1b), (4.1c), Definition 2.2 and Assumption 4.1 we obtain

$$
\begin{aligned}
&\overline{E}\begin{bmatrix}\dfrac{d^{\alpha}x_1(t)}{dt^{\alpha}}\\[2mm] \dfrac{d^{\beta}x_2(t)}{dt^{\beta}}\end{bmatrix} = \overline{E}\begin{bmatrix}\dfrac{d^{\alpha}}{dt^{\alpha}}\\[2mm] \dfrac{d^{\beta}}{dt^{\beta}}\end{bmatrix}\begin{bmatrix}\Phi_0(t)\overline{E}\overline{E}^D v + \overline{E}^D\int_0^t[\Phi_1(t-\tau)B_{10} + \Phi_2(t-\tau)B_{01}]u(\tau)d\tau\\ +(\overline{E}\overline{E}^D - I_{n_1+n_2})\sum_{k=0}^{q_1-1}\sum_{l=0}^{q_2-1}\overline{E}^{k+l}\overline{T}_{k,l}^D\overline{A}^D\overline{B}u^{(k\alpha+l\beta)}(t)\end{bmatrix}\\
&= \overline{E}\begin{bmatrix}\dfrac{d^{\alpha}}{dt^{\alpha}}\\[2mm] \dfrac{d^{\beta}}{dt^{\beta}}\end{bmatrix}\begin{bmatrix}\overline{E}\overline{E}^D v + \sum_{\substack{k=1\\k+l\geq 1}}^{\infty}\sum_{l=1}^{\infty}(\overline{E}^D)^{k+l}\overline{T}_{k,l}\dfrac{t^{k\alpha+l\beta}}{\Gamma(k\alpha+l\beta+1)}\quad v + \overline{E}^D\int_0^t\left[\dfrac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)}\overline{B}_{10} + \dfrac{(t-\tau)^{\beta-1}}{\Gamma(\beta)}\overline{B}_{01}\right]u(\tau)d\tau\\ +\overline{E}^D\int_0^t\begin{bmatrix}\sum_{\substack{k=0\\k+l\geq 1}}^{\infty}\sum_{l=0}^{\infty}(\overline{E}^D)^{k+l+1}\overline{T}_{k+1,l}\dfrac{t^{(k+2)\alpha+l\beta-1}}{\Pi[(k+2)\alpha+l\beta]}\quad \overline{B}_{10}\\ + \sum_{\substack{k=0\\k+l\geq 1}}^{\infty}\sum_{l=0}^{\infty}(\overline{E}^D)^{k+l+1}\overline{T}_{k,l+1}\dfrac{t^{(k+2)\beta+k\alpha-1}}{\Pi[(l+2)\beta+k\alpha]}\quad \overline{B}_{01}\end{bmatrix}u(\tau)d\tau\\ +(\overline{E}\overline{E}^D - I_{n_1+n_2})\sum_{k=0}^{q_1-1}\sum_{l=0}^{q_2-1}\overline{E}^{k+l}\overline{T}_{k,l}^D\overline{A}^D\overline{B}u^{(k\alpha+l\beta)}(t)\end{bmatrix}\\
&= \sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\overline{E}(\overline{E}^D)^{k+l+1}\overline{T}_{k+1,l+1}\frac{t^{k\alpha+l\beta}}{\Gamma(k\alpha+l\beta+1)}\overline{E}\overline{E}^D v + \overline{E}^D\overline{B}u(t)\\
&\quad + \overline{E}^D\int_0^t[\overline{T}_{10}\Phi_1(t-\tau)B_{10} + \overline{T}_{01}\Phi_2(t-\tau)B_{01}]u(\tau)d\tau + (\overline{E}\overline{E}^D - I_{n_1+n_2})\sum_{k=0}^{q_1-1}\sum_{l=0}^{q_2-1}\overline{E}^{k+l+1}\overline{T}_{k+1,l+1}^D\overline{B}u^{(k\alpha+l\beta+1)}(t).
\end{aligned}
\tag{4.2}
$$

Therefore, the solution (4.1a) satisfies the equation (3.6a). □

From (4.1a) for $t = 0$ we have

$$x(0) = \overline{E}\overline{E}^D v + (\overline{E}\overline{E}^D - I_{n_1+n_2}) \sum_{k=0}^{q_1-1} \sum_{l=0}^{q_2-1} \overline{E}^{k+l} \overline{T}_{k,l}^D \overline{A}^D \overline{B} u^{(k\alpha+l\beta)}(0). \quad (4.6)$$

Therefore, for given admissible $u(t)$ the consistent initial conditions should satisfy the equality (4.6). In particular case for $u(t) = 0$ we have $x_0 = \overline{E}\overline{E}^D v$ and $x_0 \in \mathrm{Im}(\overline{E}\overline{E}^D)$ where Im denotes the image of $\overline{E}\overline{E}^D$.

5 Concluding Remarks

The descriptor fractional continuous-time linear systems with two different fractional orders has been considered. The Drazin inverse of matrices has been applied to find the solutions of the state equations of the considered system. Some additional changes to classical Drazin approach for finding the solution of the state equation of descriptor systems is proposed. An equality defining the set of admissible initial conditions for given inputs has been derived.

Acknowledgments. This work was supported by National Science Centre in Poland.

References

1. Bru, R., Coll, C., Romero-Vivo, S., Sanchez, E.: Some problems about structural properties of positive descriptor systems. LNCIS, vol. 294, pp. 233–240. Springer, Berlin (2003)
2. Bru, R., Coll, C., Sanchez, E.: About positively discrete-time singular systems, System and Control: theory and applications. Electr. Comput. Eng. Ser. World Sci. Eng. Soc. Press, Athens, 44-48 (2000)
3. Busłowicz, M.: Stability analysis of continuous-time linear systems consisting of n subsystem with different fractional orders. Bull. Pol. Ac. Tech. 60(2), 279–284 (2012)
4. Campbell, S.L., Meyer, C.D., Rose, N.J.: Applications of the Drazin inverse to linear systems of differential equations with singular constant coefficients. SIAMJ Appl. Math. 31(3), 411–425 (1976)
5. Dai, L.: Singular control systems. LNCIS. Springer, Berlin (1989)
6. Dodig, M., Stosic, M.: Singular systems state feedbacks problems. Linear Algebra and its Applications 431(8), 1267–1292 (2009)
7. Dzieliński, A., Sierociuk, D., Sarwas, G.: Ultracapacitor parameters identification based on fractional order model. Proc ECC, Budapest (2009)
8. Fahmy, M.M., O'Reill, J.: Matrix pencil of closed-loop descriptor systems: infinite-eigenvalues assignment. Int. J. Control 49(4), 1421–1431 (1989)

9. Ferreira, N.M.F., Machado, J.A.T.: Fractional-order hybrid control of robotic manipulators. In: Proc. 11th Int. Conf. Advanced Robotics, Coimbra, Portugal. ICAR, pp. 393–398 (2003)
10. Guang-Ren, D.: Analysis and Design of Descriptor Linear Systems. Springer, New York (2010)
11. Kaczorek, T.: Application of Drazin inverse to analysis of descriptor fractional discrete-time linear systems with regular pencils. Int. J. Appl. Math. Comput. Sci. 23(1), 29–33 (2013)
12. Kaczorek, T.: Drazin inverse matrix method for fractional descriptor continuous-time linear systems. Bull 62(3), 409–412 (2014), doi:10.2478/bpasts-2014-0042.
13. Kaczorek, T.: Fractional positive continuous-time systems and their reachability. Int. J. Appl. Math. Comput. Sci. 18(2), 223–228 (2008)
14. Kaczorek, T.: Fractional positive linear systems. Kybernetes: The International Journal of Systems & Cybernetics 38(7/8), 1059–1078 (2009)
15. Kaczorek, T.: Infinite eigenvalue assignment by output-feedbacks for singular systems. Int. J. Appl. Math. Comput. Sci. 14(1), 19–23 (2004)
16. Kaczorek, T.: Research Studies Press J, vol. 1. Wiley, New York (1992)
17. Kaczorek, T.: Positive linear systems consisting of n subsystems with different fractional orders. IEEE Trans. Circuits and Systems 58(6), 1203–1210 (2011)
18. Kaczorek, T.: Positive linear systems with different fractional orders. Bull. Pol. Ac. Tech. 58(3), 453–458 (2010)
19. Kaczorek, T.: Selected Problems in Fractional Systems Theory. Springer, Berlin (2011)
20. Kucera, V., Zagalak, P.: Fundamental theorem of state feedback for singular systems. Automatica 24(5), 653–658 (1988)
21. Miller, K.S., Ross, B.: An Introduction to the Fractional Calculus and Fractional Differenctial Equations. Willey, New York (1993)
22. Nishimoto. K., Fractional Calculus. Decartess Press, Koriama (1984)
23. Oldham, K.B., Spanier, J.: The Fractional Calculus. Academmic Press, New York (1974)
24. Podlubny, I.: Fractional Differential Equations. Academic Press, San Diego (1999)
25. Sajewski, Ł.: Reachability of fractional positive continuous-time linear systems with two different fractional orders. In: Szewczyk, R., Zieliński, C., Kaliczyńska, M. (eds.) Recent Advances in Automation, Robotics and Measuring Techniques. AISC, vol. 267, pp. 239–250. Springer, Heidelberg (2014)
26. Sajewski, Ł.: Reachability, observability and minimum energy control of fractional positive continuous-time linear systems with two different fractional orders. Multidim. Syst. Sign. Process 25, doi:10.1007/s11045-014-0287-2
27. Sajewski, Ł.: Minimum energy control of fractional positive continuous-time linear systems with two different fractional orders and bounded inputs. In: Latawiec, K.J., Łukaniszyn, M., Stanisławski, R. (eds.) Advances in Modeling and Control of Non-integer Order Systems. LNEE, vol. 320, pp. 171–181. Springer, Heidelberg (2015)
28. Van Dooren, P.: The computation of Kronecker's canonical form of a singular pencil. Linear Algebra and its Applications 27, 103–140 (1979)
29. Virnik, E.: Stability analysis of positive descriptor systems. Linear Algebra and its Applications 429, 2640–2659 (2008)