Fractional Standard and Positive Descriptor Time-Varying Discrete-Time Linear Systems

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Abstract. The Weierstrass-Kronecker theorem on the decomposition of the regular pencil is extended to the fractional descriptor time-varying discrete-time linear systems. A method for computing the solutions of the fractional systems is proposed. Necessary and sufficient conditions for the positivity of the systems are established.

Keywords: fractional, descriptor, time-varying, positive, discrete-time, solution.

1 Introduction

A dynamical system is called positive if its trajectory starting from any nonnegative initial condition state remains forever in the positive orthant for all nonnegative inputs. An overview of state of the art in positive system theory is given in the monographs [7, 8] and in the papers [9−12]. Models having positive behavior can be found in engineering, economics, social sciences, biology and medicine, etc.

The Laypunov, Bohl and Perron exponents and stability of time-varying discrete-time linear systems have been investigated in [1−6]. The positive standard and descriptor systems and their stability have been analyzed in [8−11].The positive linear systems with different fractional orders have been addressed in [9, 14] and the singular discrete-time linear systems in [10].The switched discrete-time systems have been considered in [16−18] and the extremal norms for positive linear inclusions in [15].

The positivity and stability of time-varying discrete-time linear systems have been investigated in [13].

In this paper the Weierstrass-Kronecker decomposition theorem will be applied to fractional descriptor time-varying discrete-time linear systems with regular pencils to find their solutions and necessary and sufficient conditions for the positivity of the systems will be established.

The paper is organized as follows. In section 2 the Weierstrass-Kronecker decomposition theorem is applied to find solutions to standard fractional descriptor time-varying discrete-time linear systems. Necessary and sufficient conditions for the

R. Szewczyk et al. (eds.), *Progress in Automation, Robotics and Measuring Techniques*,
Advances in Intelligent Systems and Computing 350, DOI: 10.1007/978-3-319-15796-2_11

positivity of the descriptor systems are established in section 3. Concluding remarks are given in section 4.

The following notation will be used: $\Re$ – the set of real numbers, $\Re^{n\times m}$ – the set of $n\times m$ real matrices, $\Re_+^{n\times m}$ – the set of $n\times m$ matrices with nonnegative entries and $\Re_+^n = \Re_+^{n\times 1}$, I_n – the $n\times n$ identity matrix.

2 Standard Fractional Descriptor Systems

Consider the fractional descriptor time-varying discrete-time linear system

$$E(i)\Delta^\alpha x_{i+1} = A(i)x_i + B(i)u_i\,,\ i\in Z_+ = \{0,1,...\} \tag{2.1a}$$

$$y_i = C(i)x_i \tag{2.1b}$$

where $x_i \in \Re^n$, $u_i \in \Re^m$, $y_i \in \Re^p$ are the state, input and output vectors and $A(i)\in \Re^{n\times n}$, $B(i)\in \Re^{n\times m}$, $C(i)\in \Re^{p\times n}$ are matrices with entries depending on $i\in Z_+$ and the fractional difference of the order α is defined by

$$\Delta^\alpha x_i = \sum_{j=0}^{i}(-1)^j \binom{\alpha}{j} x_{i-j} \tag{2.1c}$$

$$\binom{\alpha}{j} = \begin{cases} 1 & \text{for} \quad j=0 \\ \dfrac{\alpha(\alpha-1)...(\alpha-j+1)}{j!} & \text{for} \quad j=1,2,... \end{cases}. \tag{2.1d}$$

It is assumed that $\det E(i) = 0$, $i\in Z_+$ and

$$\det[E(i)\lambda - A(i)] \neq 0 \tag{2.2}$$

for some $\lambda \in \mathrm{C}$ (the field of complex numbers) and $i\in Z_+$.

Substituting (2.1c) into (2.1a) we obtain

$$E(i)x_{i+1} = [E(i)\alpha - A(i)]x_i - \sum_{j=2}^{i+1} c_j E(i)x_{i-j+1} + B(i)u_i \tag{2.3a}$$

where

$$c_j = (-1)^{j+1}\binom{\alpha}{j}. \tag{2.3b}$$

It is well-known [11, 15] that if (2.2) holds then there exists a pair of nonsingular matrices $P(i),Q(i)\in \Re^{n\times n}$ such that

$$P(i)[E(i)\lambda - A(i)]Q(i) = \begin{bmatrix} I_{n_1} & 0 \\ 0 & N \end{bmatrix}\lambda - \begin{bmatrix} A_1(i) & 0 \\ 0 & I_{n_2} \end{bmatrix},\ i \in Z_+ \tag{2.4}$$

where $u_1 = \deg\det[E(i)\lambda - A(i)]$, $A_1(i) \in \Re^{n_1 \times n_1}$, $N \in \Re^{n_2 \times n_2}$ is the nilpotent matrix with the index μ (i.e. $N^{\mu} = 0$ and $N^{\mu-1} \neq 0$).

The matrices $P(i), Q(i), A_1(i)$ can be found by for example the use of elementary row and column operations [15].

Premultiplying (2.1a) by the matrix $P(i)$, introducing the new state vector

$$\bar{x}_i = Q^{-1}(i)x_i = \begin{bmatrix} \bar{x}_{1,i} \\ \bar{x}_{2,i} \end{bmatrix},\ \bar{x}_{1i} = \begin{bmatrix} \bar{x}_{11,i} \\ \bar{x}_{12,i} \\ \vdots \\ \bar{x}_{1n_1,i} \end{bmatrix},\ \bar{x}_{2i} = \begin{bmatrix} \bar{x}_{21,i} \\ \bar{x}_{22,i} \\ \vdots \\ \bar{x}_{2n_2,i} \end{bmatrix} \tag{2.5}$$

and using (2.4) we obtain

$$\bar{x}_{1,i+1} = A_{1\alpha}(i)\bar{x}_{1,i} - \sum_{j=2}^{i+1} c_j \bar{x}_{1,i-j+1} + B_1(i)u_i, \tag{2.6a}$$

$$N\bar{x}_{2,i+1} = (N_\alpha + I_{n_2})\bar{x}_{2,i} - \sum_{j=2}^{i+1} c_j N\bar{x}_{2,i-j+1} + B_2(i)u_i \tag{2.6b}$$

where

$$A_{1\alpha}(i) = A_1(i) + \alpha I_{n_1} \in \Re^{n_1 \times n_1},\ P(i)B(i) = \begin{bmatrix} B_1(i) \\ B_2(i) \end{bmatrix},\ B_1(i) \in \Re^{n_1 \times m},\ B_2(i) \in \Re^{n_2 \times m}. \tag{2.6c}$$

The solution $\bar{x}_{1,i}$ of equation (2.6a) for known initial condition $\bar{x}_{10} \in \Re^{n_1}$ and input $u_i \in \Re^m$, $i \in Z_+$ can be computed iteratively using the formula

$$\bar{x}_{1,i} = A_{1\alpha}(i-1)\bar{x}_{1,i-1} - \sum_{j=2}^{i} c_j \bar{x}_{1,i-j} + B(i-1)u_j,\ i \in Z_+ \tag{2.7}$$

where c_j is defined by (2.6b).

To simplify the notation it is assumed that the matrix N in (2.6b) has the form

$$N = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix} \in \Re^{n_2 \times n_2}. \tag{2.8}$$

From (2.6b) and (2.8) we have

$$\begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix} \begin{bmatrix} \bar{x}_{21,i+1} \\ \bar{x}_{22,i+1} \\ \vdots \\ \bar{x}_{2n_2,i+1} \end{bmatrix} = \begin{bmatrix} 1 & \alpha & 0 & \dots & 0 \\ 0 & 1 & \alpha & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \alpha \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix} \begin{bmatrix} \bar{x}_{21,i} \\ \bar{x}_{22,i} \\ \vdots \\ \bar{x}_{2n_2,i} \end{bmatrix}$$
$$+ \sum_{j=2}^{i+1} \begin{bmatrix} 0 & c_j & 0 & \dots & 0 \\ 0 & 0 & c_j & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & c_j \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix} \begin{bmatrix} \bar{x}_{21,i-j+1} \\ \bar{x}_{22,i-j+1} \\ \vdots \\ \bar{x}_{2n_2,i-j+1} \end{bmatrix} + \begin{bmatrix} B_{21}(i) \\ \vdots \\ B_{2n_2}(i) \end{bmatrix} u_i, \quad i \in Z_+ \tag{2.9}$$

and

$$\begin{aligned} 0 &= \bar{x}_{2n_2,i} + B_{2n_2}(i)u_i, \\ \bar{x}_{2n_2,i+1} &= \bar{x}_{2n_2-1,i} + \alpha\bar{x}_{2n_2,i} + \sum_{j=2}^{i+1} c_j \bar{x}_{2n_2,i-j+1} + B_{2n_2-1}(i)u_i, \\ &\vdots \\ \bar{x}_{22,i+1} &= \bar{x}_{21,i} + \alpha\bar{x}_{22,i} + \sum_{j=2}^{i+1} c_j \bar{x}_{22,i-j+1} + B_{21}(i)u_i \end{aligned}, \quad i \in Z_+. \tag{2.10}$$

Solving the equations (2.10) with respect to the components of the vector $\bar{x}_{2,i}$ we obtain

$$\begin{aligned} \bar{x}_{2n_2,i} &= -B_{2n_2}(i)u_i, \\ \bar{x}_{2n_2-1,i} &= -B_{2n_2}(i+1)u_{i+1} + \alpha B_{2n_2}(i)u_i + \sum_{j=2}^{i+1} c_j B_{2n_2}(i-j+1)u_{i-j+1} - B_{2n_2-1}(i)u_i, \\ &\vdots \\ \bar{x}_{21,i} &= -B_{2n_2}(i+n_2-1)u_{i+n_2-1} + \alpha B_2(i+n_2-2)u_i + \sum_{j=2}^{i+1} c_j B_{2n_2}(i+n_2-j-1)u_{i-j+1} + \dots - B_{21}(i)u_i. \end{aligned} \tag{2.11}$$

The admissible initial conditions for the system (2.6b) are given by (2.11) for $i = 0$.

The solution of the equation (2.6b) for known $u_i \in \Re^m$ and admissible initial conditions $\bar{x}_{20} \in \Re^{n_2}$ is given by (2.11).

The considerations can be easily extended to the case when the matrix N in (2.6b) has the form

$$N = \text{blockdiag}[N_1, ..., N_q], \quad q > 1 \tag{2.12}$$

and N_k for $k = 1, 2, \ldots, q$ has the form (2.7).

Example 2.1. Consider the fractional descriptor time-varying system described by the equation (2.1a) with the matrices

$$E(i) = \begin{bmatrix} 0 & 0 & 0 & \frac{e^{2i}}{\cos(i)+2} \\ -\frac{(i+2)(\sin(i)+1)}{i+1} & e^i & 0 & -\frac{e^{2i}(e^{-i}+1)}{\cos(i)+2} \\ \frac{i+2}{i+1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$B(i) = \begin{bmatrix} \frac{1}{\cos(i)+2} & 0 \\ e^{-i} - \frac{e^{-i}+1}{\cos(i)+2} & \frac{2i(i+2)(\cos(i)+1)(\sin(i)+1)}{i+1} - \sin(i)(\sin(i)+1) \\ 0 & \sin(i) - \frac{2i(i+2)(\cos(i)+1)}{i+1} \\ 0 & \frac{2i(i+2)}{i+1} \end{bmatrix}, \tag{2.13}$$

$$A(i) = \begin{bmatrix} 0 & 0 & a_{13}(i) & 0 \\ a_{21}(i) & a_{22}(i) & a_{23}(i) & a_{24}(i) \\ a_{31}(i) & 0 & 0 & a_{34}(i) \\ 0 & 0 & 0 & a_{44}(i) \end{bmatrix},$$

where

$$a_{13}(i) = \frac{1}{\cos(i)+2},$$

$$a_{21}(i) = \frac{(i+2)(i+2\cos(i)+2\sin(i)+i\sin(i)+\cos(i)\sin(i)+3)}{(i+1)(\sin(i)+2)},$$

$$a_{22}(i) = 1 - 2e^i, \quad a_{23}(i) = -\frac{e^{-i}+1}{\cos(i)+2}, \quad a_{24}(i) = \frac{e^{2i}(i+2)(\cos(i)+1)(\sin(i)+1)}{i+1},$$

$$a_{31}(i) = -\frac{i+2}{\sin(i)+2},\quad a_{34}(i) = -\frac{e^{2i}(i+2)(\cos(i)+1)}{i+1},\quad a_{44}(i) = \frac{e^{2i}(i+2)}{i+1}.$$

The condition (2.2) is satisfied since

$$\det[E(i)\lambda - A(i)] = -\frac{(i+2)^2(2e^i + \lambda e^i - 1)(2\lambda + i + \lambda\sin(i)+1)e^{2i}}{(i+1)^2(\cos(i)+2)(\sin(i)+2)} \neq 0\,. \tag{2.14}$$

In this case

$$P(i) = \begin{bmatrix} 1+e^{-i} & 1 & 1+\sin(i) & 0 \\ 0 & 0 & 1 & 1+\cos(i) \\ 2+\cos(i) & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{i+1}{i+2} \end{bmatrix},\quad Q(i) = \begin{bmatrix} 0 & \frac{i+1}{i+2} & 0 & 0 \\ e^{-i} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & e^{-2i} \end{bmatrix} \tag{2.15}$$

and

$$\begin{bmatrix} I_{n_1} & 0 \\ 0 & N \end{bmatrix} = P(i)E(i)Q(i) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\begin{bmatrix} A_1(i) & 0 \\ 0 & I_{n_2} \end{bmatrix} = P(i)A(i)Q(i) = \begin{bmatrix} e^{-i}-2 & 1+\cos(i) & 0 & 0 \\ 0 & -\frac{i+1}{2+\sin(i)} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{2.16}$$

$$\begin{bmatrix} B_1(i) \\ B_2(i) \end{bmatrix} = P(i)B(i) = \begin{bmatrix} e^{-i} & 0 \\ 0 & \sin(i) \\ 1 & 0 \\ 0 & 2i \end{bmatrix},$$

$(n_1 = n_2 = 2)$.

The equations (2.6) have the form

$$\begin{bmatrix} \bar{x}_{11,i+1} \\ \bar{x}_{12,i+1} \end{bmatrix} = \begin{bmatrix} e^{-i}-2 & 1+\cos(i) \\ 0 & -\frac{i+1}{2+\sin(i)} \end{bmatrix} \begin{bmatrix} \bar{x}_{11,i} \\ \bar{x}_{12,i} \end{bmatrix} + \sum_{j=2}^{i+1} c_j \begin{bmatrix} \bar{x}_{11,i-j+1} \\ \bar{x}_{12,i-j+1} \end{bmatrix} + \begin{bmatrix} e^{-i} & 0 \\ 0 & \sin(i) \end{bmatrix} \begin{bmatrix} u_{1,i} \\ u_{2,i} \end{bmatrix} \tag{2.17a}$$

and

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \bar{x}_{21,i+1} \\ \bar{x}_{22,i+1} \end{bmatrix} = \begin{bmatrix} 1 & \alpha \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \bar{x}_{21,i} \\ \bar{x}_{22,i} \end{bmatrix} + \sum_{j=2}^{i+1} c_j \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \bar{x}_{21,i-j+1} \\ \bar{x}_{22,i-j+1} \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 2i \end{bmatrix} \begin{bmatrix} u_{1,i} \\ u_{2,i} \end{bmatrix}. \tag{2.17b}$$

The solution of (2.17a) is given by (2.7) with the matrices $A_1(i)$ and $B_1(i)$ defined by (2.16).

From (2.17b) we have

$$\begin{aligned} \bar{x}_{22,i} &= -2iu_{2,i}, \\ \bar{x}_{21,i} &= -2(i+1)u_{2,i+1} + \alpha 2iu_{2,i} + \sum_{j=2}^{i+1} c_j 2(i-j+1)u_{2,i-j+1} - u_{1,i}, \quad i \in Z_+. \end{aligned} \tag{2.18}$$

The solution of the equation (2.1a) with (2.13) is given by

$$x(i) = \begin{bmatrix} x_1(i) \\ x_2(i) \\ x_3(i) \\ x_4(i) \end{bmatrix} = Q(i) \begin{bmatrix} \bar{x}_{11,i} \\ \bar{x}_{12,i} \\ \bar{x}_{21,i} \\ \bar{x}_{22,i} \end{bmatrix}, \quad i \in Z_+ \tag{2.19}$$

where $Q(i)$ is defined by (2.14) and the components of the state vector $\bar{x}(i)$ by (2.7) with $A_1(i)$ and $B_1(i)$ defined by (2.16).

3 Positive Systems

Definition 3.1. The fractional descriptor time-varying discrete-time linear system (2.1) is called the (internally) positive if and only if $x_i \in \Re_+^n$ and $y_i \in \Re_+^p$, $i \in Z_+$ for any admissible initial conditions $x_0 \in \Re_+^n$ and all inputs $u_i \in \Re_+^m$, $i \in Z_+$.

The matrix $Q(i) \in \Re^{n \times n}$, $i \in Z_+$ is called monomial if in each row and column only one entry is positive and the remaining entries are zero for $i \in Z_+$.

It is well-known [8] that $Q^{-1}(i) \in \Re_+^{n \times n}$, $i \in Z_+$ if and only if the matrix is monomial.

It is assumed that for the positive system (2.1) the decomposition (2.4) is positive for the monomial matrix $Q(i)$. In this case

$$x_i = Q(i)\bar{x}_i \in \Re_+^n \text{ if and only if } \bar{x}_i \in \Re_+^n, \; i \in Z_+. \tag{3.1}$$

It is also well-known that premultiplication of the equation (2.1a) by the matrix $P(i)$ does not change its solution x_i, $i \in Z_+$.

From (2.11) it follows that $\bar{x}_{2,i} \in \Re_+^{n_2}$, $i \in Z_+$ for $u_i \in \Re_+^m$, $i \in Z_+$ if and only if

$$-B_2(i) \in \Re_+^{n_2 \times m} \text{ for } i \in Z_+ . \tag{3.2}$$

Therefore, the following theorem has been proved.

Theorem 3.1. Let the decomposition (2.4) of the system be possible for a monomial matrix $Q(i) \in \Re_+^{n \times n}$, $i \in Z_+$. The substitution (2.6b) is positive if and only if the condition (3.2) is satisfied.

Theorem 3.2. Let the decomposition (2.4) of the system be possible for a monomial matrix $Q(i) \in \Re_+^{n \times n}$, $i \in Z_+$. The substitution (2.6a) for $0 < \alpha < 1$ is positive if and only if

$$A_{1\alpha}(i) \in \Re_+^{n_1 \times n_1}, \quad B_1(i) \in \Re_+^{n_1 \times m}, \ i \in Z_+ . \tag{3.3}$$

Proof. Sufficiency. If $0 < \alpha < 1$ then from (2.3b) and (2.1d) we have

$$c_2 = (-1)^2 \frac{\alpha(\alpha - 1)}{2} < 0 \tag{3.4a}$$

and

$$c_{j+!} = (-1)^{j+1} \binom{\alpha}{j+1} = \frac{(j-\alpha)}{j+1} c_j < 0, \quad j = 2, 3, \ldots . \tag{3.4b}$$

From (2.7) and (3.4) it follows that $\bar{x}_{1,i} \in \Re_+^{n_1}$, $i \in Z_+$ for $x_0 \in \Re_+^n$ and $u_i \in \Re_+^m$, $i \in Z_+$ if the condition (3.3) is satisfied.

The necessity can be shown in a similar way as for standard descriptor systems in [11]. ■

Theorem 3.3. Let the decomposition (2.4) of the system be possible for a monomial matrix $Q(i) \in \Re_+^{n \times n}$, $i \in Z_+$. The system (2.1) for $0 < \alpha < 1$ is positive if and only if:

1) the conditions (3.3) are satisfied,
2) (3.2) holds,
3) $C(i) \in \Re_+^{p \times n}$ for $i \in Z_+$.

Proof. By Theorem 3.2 and 3.1 the solutions (2.6a) and (2.6b) are positive if and only if the conditions (3.2) and (3.3) are met. From (2.1b) and (2.5a) we have

$$y_i = C(i)Q(i)Q^{-1}(i)x_i = \overline{C}(i)\bar{x}_i , \ i \in Z_+ . \tag{3.5a}$$

where

$$\overline{C}(i) = C(i)Q(i) . \tag{3.5b}$$

For monomial matrix $Q(i) \in \Re_+^{n \times n}$ from (3.3) we have

$$\overline{C}(i) \in \Re_+^{p \times n}, \ i \in Z_+ \text{ if and only if } C(i) \in \Re_+^{p \times n}, \ i \in Z_+ \tag{3.6}$$

and

$$y_i \in \Re_+^p,\ i \in Z_+ \text{ if and only if } C(i) \in \Re_+^{p\times n},\ i \in Z_+ . \tag{3.7}$$

■

Example 3.1. Consider the fractional descriptor time-varying system described by the equation (2.1) with the matrices

$$E(i) = \begin{bmatrix} 0 & 0 & 0 & \frac{1}{2\sin(i)+4} \\ -\cos(i)-1 & \frac{1}{\cos(i)+2} & 0 & -\frac{e^{-i}+2}{2\sin(i)+4} \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$B(i) = \begin{bmatrix} -\frac{1}{\sin(i)+2} & 0 \\ e^{-i} + \frac{e^{-i}+2}{\sin(i)+2} & -(\cos(i)+1)(e^{-i}+\sin(i)+2) \\ 0 & e^{-i}+\sin(i)+2 \\ 0 & -1 \end{bmatrix}, \tag{3.8}$$

$$C(i) = \begin{bmatrix} 0 & \frac{1}{\cos(i)+2} & 0 & 0.5 \\ \frac{i+2}{i+1} & 0 & \frac{e^{-i}}{e^{-i}+1} & 0 \end{bmatrix},$$

$$A(i) = \begin{bmatrix} 0 & 0 & a_{13}(i) & 0 \\ a_{21}(i) & a_{22}(i) & a_{23}(i) & a_{24}(i) \\ a_{31}(i) & 0 & 0 & a_{34}(i) \\ 0 & 0 & 0 & a_{44}(i) \end{bmatrix},$$

where

$$a_{13}(i) = \frac{1}{(\sin(i)+2)(e^{-i}+1)},\quad a_{21}(i) = -e^{-i} - \cos(i) - \sin(i) - e^{-i}\cos(i),$$

$$a_{22}(i) = \frac{i+1}{(i+2)(\cos(i)+2)},\quad a_{23}(i) = -\frac{e^{-i}+2}{(\sin(i)+2)(e^{-i}+1)},$$

$$a_{24}(i) = \frac{(i+2)(\cos(i)+1)(e^{-i}+1)}{2(i+1)},$$

$$a_{31}(i)=e^{-i}+1,\ \ a_{34}(i)=-\frac{(i+2)(e^{-i}+1)}{2(i+1)},\ \ a_{44}(i)=\frac{i+2}{2(i+1)}.$$

The condition (2.2) is satisfied since

$$\det[E(i)\lambda-A(i)]=\frac{(e^{-i}-\lambda+1)(i-2\lambda-\lambda i+1)}{2(i+1)(\cos(i)+2)(\sin(i)+2)(e^{-i}+1)}\neq 0 \tag{3.9}$$

In this case

$$P=\begin{bmatrix} 2+e^{-i} & 1 & 1+\cos(i) & 0 \\ 0 & 0 & 1 & 1+e^{-i} \\ 2+\sin(i) & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{i+1}{i+2} \end{bmatrix},\ \ Q=\begin{bmatrix} 0 & 1 & 0 & 0 \\ 2+\cos(i) & 0 & 0 & 0 \\ 0 & 0 & 1+e^{-i} & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} \tag{3.10}$$

and

$$\begin{aligned} \begin{bmatrix} I_{n_1} & 0 \\ 0 & N \end{bmatrix} &= P(i)E(i)Q(i)=\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \\ \begin{bmatrix} A_1(i) & 0 \\ 0 & I_{n_2} \end{bmatrix} &= P(i)A(i)Q(i)=\begin{bmatrix} \frac{i+1}{i+2} & 1-\sin(i) & 0 & 0 \\ 0 & 1+e^{-i} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \\ \begin{bmatrix} B_1(i) \\ B_2(i) \end{bmatrix} &= P(i)B(i)=\begin{bmatrix} e^{-i} & 0 \\ 0 & 1+\sin(i) \\ -1 & 0 \\ 0 & -\frac{i+1}{i+2} \end{bmatrix}, \\ \overline{C}(i) &= C(i)Q(i)=\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & \frac{i+2}{i+1} & e^{-i} & 0 \end{bmatrix} \end{aligned} \tag{3.11}$$

The descriptor system is positive since the tree conditions of Theorem 3 are satisfied. The matrix $Q(i)$ defined by (3.10) is monomial, the conditions (3.2) and (3.3) are met

$$-B_2(i)=\begin{bmatrix}1 & 0\\ 0 & \frac{i+1}{i+2}\end{bmatrix}\in R_+^{2\times 2},$$

$$A_1(i)=\begin{bmatrix}\frac{i+1}{i+2} & 1-\sin(i)\\ 0 & 1+e^{-i}\end{bmatrix}\in R_+^{2\times 2} \text{ and } B_1(i)=\begin{bmatrix}e^{-i} & 0\\ 0 & 1+\sin(i)\end{bmatrix}\in R_+^{2\times 2},\ \ i\in Z_+ \tag{3.12}$$

and

$$C(i)=\begin{bmatrix}0 & \frac{1}{\cos(i)+2} & 0 & 0.5\\ \frac{i+2}{i+1} & 0 & \frac{e^{-i}}{e^{-i}+1} & 0\end{bmatrix}\in R_+^{2\times 4} \text{ for } Z_+ . \tag{3.13}$$

The solution to the equation (2.1) with the matrices $E(i)$, $A(i)$, $B(i)$ given by (3.8) can be found in a similar way as in Example 2.1.

4 Concluding Remarks

The Weierstrass-Kronecker theorem on the decomposition of the regular pencil has been extended to the fractional descriptor time-varying discrete-time linear systems. A method for computing the solutions of the fractional systems has been proposed. Necessary and sufficient conditions for the positivity of the systems have been established. The effectiveness of the test are demonstrated on examples. The considerations can be extended to the fractional descriptor time-varying continuous-time linear systems.

Acknowledgments. This work was supported by National Science Centre in Poland.

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