Chapter 19
A Particular Case of Evans-Hudson Diffusion

Cristina Serbănescu

Abstract We know that the Markov processes are the solutions of certain stochastic equations. In this article we will construct a noncommutative Markov process by noncommutative stochastic calculus. We will also show that these are particular cases of Evans-Hudson diffusions. At the end we will present two examples starting from the classical theory of probabilities (the Brownian motion and the Poisson process) which lead to particular cases of the noncommutative Markov processes.

Keywords Noncommutative Markov process • C*-algebra • Brownian motion • Poisson process • Stochastic equation

19.1 Introduction

Studies in Quantum mechanics have posed the problem of completely positive applications on C*-algebra of continuous linear operators on a Hilbert space. We consider completely positive applications because they describe the evolution of a quantum system, a high-physics energy system and we assume that this evolution is not affected by the existence of other systems that do not interact with the given one. Details concerning the way that the high-physics energy has come to pose this problem may be found in [2] and [3]. Starting from a semigroup of positive operators or from its infinitesimal operator, we can construct a homogenous Markov process. The construction of these processes is done through different methods of which we emphasize on solving stochastic integral equations [16]. Hence the theory of quantum probabilities has developed as a noncommutative theory of probabilities in [1] with motivations in high-physics energy [10]. The corresponding stochastic processes were constructed only in the case of infinitesimal operators and are expressed as finite sums. These are called Evans-Hudson diffusions [5]. This article builds these processes on the antisymmetric Fock space (called fermionic) in which the infinitesimal operator is an infinite sum. The case of the symmetric Fock space

C. Serbănescu (✉)
Department of Mathematics, University "Politehnica" Bucharest, Splaiul Independentei nr. 313, Sector 6, Bucharest, Romania
e-mail: serbanescuc@hotmail.com

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(called bosonic) was treated in [12]. This article shows how the obtained processes as solutions of certain stochastic integral equations are noncommutative Markov processes and appear as a particular case of certain Evans-Hudson diffusions [14] with an infinite number of components, a notion yet to be defined.

This paper shows how noncommutative Markov processes are obtained as solutions of certain stochastic equations, being particular cases of Evans-Hudson Diffusions with an infinity of components.

The case of Markov processes on symmetric Fock spaces for infinitesimal operator as an infinite sum was studied by Hudson and Parthasarathy [10]. This paper aims to build noncommutative Markov processes on antisymmetric Fock spaces where we do not have exponential commutative vectors and where the commutative property does not occur between operators describing disjoint time intervals. Unlike the symmetric Fock space, the defined operators are continuous and the integral is a particular case of Bochner integral [8].

The Brownian Motion and the Poisson Process were given as examples.

19.2 Fermion Stochastic Integrals of Simple Processes

First we construct a stochastic integral on Fermion Fock space [9, 13] by analogy with the same kind of integral on Boson Fock space [4], first of simple processes. We define the Fermion stochastic integral for square-integrable integrands. We present the infinitesimal operators like infinite sums, but we assume they are continuous. Because of the canonical anticommutation relation we have left, right and mixed stochastic integrals.

The noncommutative stochastic calculus was developed on Fermion Fock space.

Definition 2.1. *Let H be a Hilbert space. We define the antisymmetric Fock space $\Gamma_a(H)$ over H as the linear hull of all $x_1 \wedge x_2 \wedge \cdots \wedge x_n,\ n \geq 0,\ x_i \in H$ (where for $n = 0$, we have the unit element, namely 1) with the following inner product*:

$$\langle x_1 \wedge x_2 \wedge \cdots \wedge x_n, y_1 \wedge y_2 \wedge \cdots \wedge y_k \rangle = \delta_{n,k} \quad \det\left(\langle x_i, y_i \rangle\right)_{i,j=1,\ldots,n}$$

for $n = k = 0$ the determinant is considered to be 1.

About this space we mention the following:

(i) $\Gamma_a(H) = \bigoplus_{n \geq 0} H_\wedge^n$, *where $H_\wedge^n$ is the closed linear hull of all $x_1 \wedge x_2 \wedge \cdots \wedge x_n,\ x_i \in H$*

(ii) $x_{y(1)} \wedge \cdots \wedge x_{y(n)} = \varepsilon(\gamma)\, x_{y(1)} \wedge \cdots \wedge x_{y(n)}$ *where $\varepsilon(\gamma)$ is 1 or -1 if γ is even or odd.*

If two x_i with different indexes i are equal this product is null.

(iii) $\Gamma_a(H) = \left\{\sum_{n\geq 0} x_n : x_n \in H_\wedge^n, \ \{n : x_n \neq 0\} \ finite\right\}$ *is an associative algebra, with unit element 1 and* $x_1 \wedge \cdots \wedge x_n$ *is the product of* $x_1, \ldots, x_n, \ x_i \in H = H_\wedge^1$, *in established order.*

(iv) *If* $H \subset K$, *then* $\Gamma_a(H) \subset \Gamma_a(K)$.

Definition 2.2. *Let H be a Hilbert space*

(a) *By a filtration in H we mean a family* $(H_t)_{t\in[0,\infty)}$ *of closed subspaces of H such that*

$$H_s \subset H_t, \quad \forall s < t.$$

(b) *We say that it is a right continuous filtration if* $H_t = \underset{u>t}{\cap} H_u$

(c) *We say that it is a left continuous filtration if* H_t *is the closure of* $\underset{u>s}{\cup} H_s$

(d) *We say that* $(H_t)_{t>0}$ *is continuous if the filtration is right and left continuous.*

The idea of defining these processes may be found in [6].

Definition 2.3. *An adapted process is a family of operators* $F = (F(t); t \geq 0)$ *on h such that for each* $t \geq 0$:

(a) $D\left(F(t)\right) = h_0 \underline{\otimes} \varepsilon_t \underline{\otimes} h^t$.

(b) *There is an operator* $F^+(t) : h_0 \underline{\otimes} \varepsilon_t \underline{\otimes} h^t \to h_0$ *such that*

$$\langle F(t)\zeta, \eta\rangle = \left\langle \zeta, F(t)^+\eta\right\rangle \text{for } \forall \zeta\eta \in h_0 \underline{\otimes} \varepsilon_t \underline{\otimes} h^t$$

(c) *There are operators* $F_1(t)$ *and* $F_1^+(t)$ *on* $h_0 \otimes \varepsilon_t$ *such that*:

$$F(t) = F_1(t) \otimes 1$$

$$F^+(t) = F_1^+(t) \otimes 1$$

(d) *For each* t_0 *and* $x \in \varepsilon$ *we have*:

$$\sup_{\|u\|\leq 1} \left\|\left(F\left(t_0 + h\right) - F(t)\right)(u \otimes x)\right\| \overset{h\to 0}{\to} 0$$

hence $\forall x \in \varepsilon$ *and* $t \in [0, \infty)$ $\lim_{s\to t}\|F(t) - F(s)\|_x = 0$
where $\|T\|_x = \sup_{\|u\|\leq 1} \|T\left(u \otimes x\right)\|$

Definition 2.4. *A simple process is an adapted process of the form*:

$F(t) = \sum_{n=0}^{\infty} F_n \chi_{[t_n, t_{n+1})}(t); t \geq 0$ *for some sequence* $0 = t_0 < t_1 < \cdots < t_n \to \infty$

We denote by A_0 *and* A, *respectively, the sets of simple and adapted processes.*

Definition 2.5. *Let* $F, G, H \in A_0$ *and write*

$$F = \sum_{n=0}^{\infty} F_n \chi_{[t_n, t_{n+1})} \ , G = \sum_{n=0}^{\infty} G_n \chi_{[t_n, t_{n+1})} \ , H = \sum_{n=0}^{\infty} H_n \chi_{[t_n, t_{n+1})}$$

$$0 = t_0 < t_1 < \cdots < t_n \to \infty$$

The family of operators $M = (M(t), t \geq 0)$ *with* $D\,(M(t)) = h_0 \underline{\otimes}\, \varepsilon_t \underline{\otimes}\, h^t$ *defined by* $M(0) = 0$

$M(t) = M\,(t_n) + \left(A_L^+(t) - A_L^+\,(t_n)\right) F_n + G_n\,(A_L(t) - A_L\,(t_n)) + (t - t_n)\,H_n$ *for* $t_n < t < t_{n+1}$ *is called stochastic integral of* (F, G, H) *and are denoted by*:

$$M(t) = \int_0^t dA_L^+ F + G dA_L + H ds$$

19.3 Stochastic Integrals of Continuous Processes

Now we want to estimate the norm of $M(t)\,(u \otimes x)$, in order to define the stochastic integrals [11].

We consider three possibilities, where the first is:

$M(t) = \sum_{n=0}^{b} \left(A_L^+\,(s_{n+1}) - A_L^+\,(s_n)\right) F_n$ for $t_b < t < t_{b+1}$, $s_i = t_i$ for $i = 0, \cdots, b$ and $s_{b+1} = t$.

We denote $F(t) = \sum_{n=0} \chi_{[t_n, t_{n+1}]}(t) F_n$ and we write briefly

$$dM = \left(dA_L^+\right) F \text{ or } M(t) = \int_0^t \left(dA_L^+\right) F.$$

We write as follows:

$$\|M(t)\,(u \otimes x)\|^2 \leq \left\| \sum_p L_p * L_p \right\|^2 \int_0^t \|F\,(u \otimes x)\|^2 da + \int_0^t \|F\,(\theta u \otimes x)\|^2 da$$
$$+ \sum_k \sup_{a \leq t} \|M(a)\,(u \otimes x_{ck})\|^2 \|x_k\|^2$$

Now we deduce that if $F_c = 1, 2, \ldots$ are "simple integrands" like before and if every $u \in h_0$, $x \in \varepsilon$ and $t > 0$, $\int_0^t \left\| F_c\,(u \otimes x) - F_{c'}\,(u \otimes x) \right\|^2 da \to 0$ for

$c, c' \to \infty$, then for every $t > 0$, $u \in h_0$ and $x \in \varepsilon$,
$\sup_{a \leq t} \left\| M_c\,(u \otimes x) - M_{c'}\,(u \otimes x) \right\|^2 \to 0$, where $dM_c = \left(dA_L^+\right) F_c$.

We consider $x = x_1 \wedge \cdots \wedge x_r$ and by induction on r, the term
$\sum_k \sup_{a \leq t} \| M(a)\,(u \otimes x_{ck}) \|^2 \| x_k \|^2$ vanishes for $r = 0$.

This is the way we define $\int_0^t \left(dA_L^+\right) F$ for those F for which it is a sequence F_c of simple integrands with $\int_0^t \left\| F_c\,(u \otimes x) - F_{c'}\,(u \otimes x) \right\|^2 da \to 0$ for $c' \to \infty$ for every $t > 0$, $u \in h_0$ and $x \in \varepsilon$.

We remark that if $F(t) = F_1(t) \otimes 1$ with respect to $h = h_t \otimes h^t$ and if $F(t)\,(u \otimes x)$ is continuous in t for every $u \in h_0$, $x \in \varepsilon$, then there exists a sequence F_c namely $F_c(t) = \sum_{k \geq 0} \chi_{\left[\frac{k}{2^n}, \frac{k+1}{2^n}\right]} F\left(\frac{k}{2^n}\right)$.

We also mention that:

$$\| M(t) \|_x^2 \leq \left\| \sum_p L_p * L_p \right\|^2 \int_0^t \| F(a) \|_x^2 da + \int_0^t \| F(a) \|_x^2 da + \sum_k \sup_{a \leq t} \| M(a) \|_{x_{ck}}^2 \| x_k \|^2.$$

Writing the formulas $\| M(t)\,(u \otimes x) \|^2$ for $M(t) - M(s)$, we shall obtain:

$$\| M(t) - M(s) \|_x^2 \leq \left\| \sum_p L_p * L_p \right\|^2 \int_s^t \| F(a) \|_x^2 da + r \int_s^t \| F(a) \|_x^2 da + \sum_k \sup_{a \leq t} \| M(a) - M(s) \|_{x_{ck}}^2 \| x_k \|^2$$

and by induction we show that F is continuous hence $\lim_{s \to t} \| F(s) - F(t) \|_x = 0$ for every x, then $M(t)$ follows continuous similarly.

Definition 3.1. *The integrals can be defined separately and we have*:

$$M(t) = \int_0^t \left(dA_L^+ F + G dA_L + H ds\right) = \int_0^t dA_L^+ F + \int_0^t G dA_L + \int_0^t H ds.$$

19.4 Stochastic Equations

Theorem 4.1. *Let be the operators* $X(0)$, B *and* D *in* $L(h_0)$.

We show that the stochastic differential equation:

$$X(t) = X(0) + \int_0^t dA_L^* \,(B_F X D_F) + \int_0^t (B_G X D_G)\, dA_L + \int_0^t (B_H X D_H)\, ds$$

has a unique solution which is a continuous process.

Proof. We remark that the integrands are "allowable", hence the stochastic integrals are well defined.

Unicity: if X and Y are two solutions, with $X(0) = Y(0)$, then $Z = X - Y$ will be a solution of the equation with $Z(0) = 0$.

Since $\|BFD\|_x \leq \|B\| \, \|F\|_x \, \|D\|$ for $B, D \in L\,(h_0)$, we have for $t \leq T$:

$$\|Z(t)\|_x^2 \leq c \int_0^t \|Z(a)\|_x^2 da + \sum_k \sup_{a \leq t} \int_0^a \|B_F Z(a) D_F\|^2 X C_k \|x_k\|^2 da$$
$$+ \left(\int_0^t \sum_k \|(B_G Z(a) D_G)\|^2 X C_k da \right) e^{\sum_k \|x_k\|^2}$$

We give the proof by induction on r, if $x = x_1 \wedge x_2 \wedge \cdots \wedge x_r$.

Knowing that

$Z(a)\,(u \otimes (x_1 \wedge x_2 \wedge \cdots \wedge x_{r-1})) = 0$ for all u and x_i, we deduce:

$\|Z(t)\|_x^2 \leq c \int_0^t \|Z(a)\|_x^2 da + 0$ (for $k = 0$ this is obvious) and using Gronwall's lemma, we obtain $\|Z(t)\|_x^2 \leq 0e^{ct}$, hence $Z(a)\,(u \otimes (x_1 \wedge x_2 \wedge \cdots \wedge x_{r-1})) = 0$ for all u and x_i.

Existence: We establish the existence iteratively.

We fix T > 0 and we consider $X(0)(t) = X(0)$ for every $t \leq T$ and then inductively:

$$X_{n+1}(t) = X(0) + \int_0^t dA_L^* \,(B_F X_n D_F) + \int_0^t (B_G X_n D_G)\, dA_L + \int_0^t (B_H X_n D_H)\, ds$$

We have:

$$\|X_{n+1}(t) - Xn(t)\|_y$$

$$= \left\| \int_0^t dA_L^* \left(B_F \left(X_n - X_{n-1}\right) D_F\right) + \int_0^t \left(B_G \left(X_n - X_{n-1}\right) D_G\right) dA_L \right.$$

$$\left. + \int_0^t \left(B_H \left(X_n - X_{n-1}\right) D_H\right) ds \right\|_y$$

$$\leq d^{'} q_{n-1}{}^{(p-1)} + d^{'} q_{n-2}{}^{(p-1)} cT + \cdots + d^{'} q_0{}^{(p-1)} \left(\left(cT\right)^{n-1} / \left(n-1\right)!\right)$$

$$+ c^{'} \left(c^n T^n / n!\right)$$

If we denote with q_n the last expression which doesn't depend on t, we have, if k and $n - k$ converge to ∞ with n:

$$q_n = d' \sum_{j=1}^{k-1} q_{n-j}{}^{(p-1)} (cT)^{j-1} / \left(j - 1\right)!$$

$$+ d' \left(q_{n-k}{}^{(p-1)} (cT)^{k-1} / \left(k-1\right)! + q_{n-k-1}{}^{(p-1)} (cT)^{k} / k!\right)$$

$$+ \sum_{j=k+2}^{n} q_{n-j}{}^{(p-1)} (cT)^{j-1} / \left(j - 1\right)! + c' \left(c^n T^n / n!\right)$$

Now we use $\left(\sum_k a_k\right) \Big/ \left(\sum_k b_k\right) \leq \max\left(a_k / b_k\right)$, and we have:

$$\max_{k+2 \leq j \leq n} \left(\left(cT\right) \Big/ \left(j - 1\right), cT/n\right) =$$

$$\max\left(\max_{j \geq n-k} \left(q_{j+1}^{(p-1)} \Big/ q_j^{(p-1)}\right), \left(q_{n-k}^{(p-1)} / q_{n-1-k}^{(p-1)}\right) + cT/k, \max_{k+1 \leq j} \left(cT/j\right)\right)$$

which converges to 0.

Hence $\sum \left(\left(X_n - X_{n-1}\right)(t)\right)$ is the solution of the equation.

Theorem 4.2. *We consider the stochastic integral equation*: $U(t) = 1 + \int_0^t \left(U\theta\left(dA_L^+\right) + U\theta\left(dA_L\right) + UXds\right)$,

where $X = -\left(\sum_p L_p + L_p\right) \Big/ 2.$

Then there exists a unique unitary process satisfying this equation.

Proof. We have $U^+(t) = 1 + \int_0^t \left(\left(dA_L^+\right) \theta U^+ + (dA_L)\, \theta U^+ + XU^+ ds \right)$ (since $X = X^*$).

19.5 Noncommutative Markov Processes as Stochastic Equation Solutions

Definition 5.1. *A noncommutative Markov process is a system which includes*:

(i) *A Hilbert space* h_0.
(ii) *A* C^**-algebra* $A \subset L(h_0)$ *with* 1.
(iii) *A family of completely positive mappings*: $T_1 : A \to A,\ \ t \geq 0$ *with* $T_t 1 = 1$, $T_0 = 1$ *and* $T_{t+s} = T_t T_s$ *(briefly a semigroup of completely positive mappings on A with* $T_t 1 = 1$).
(iv) *Another Hilbert space h, in which* h_0 *is a closed subspace.*
(v) *A family* $(j_t)_{t\geq 0}$ *of* $*$*-homomorphisms* $j_t : A \to L(h)$, *such that*:

1. $j_0(x) = x \oplus 0$ *relatively to* $h = h_0 \oplus h_0^{\perp}$.
2. $j_s(1) j_{s+t}(x) j_s(1) = j_s(T_t x)$.

Remark.

(a) $j_s(1)$ is a projector.
(b) $j_s(1) \leq j_{s+t}(1)$ for $t \geq 0$, hence denoting $h_t = \operatorname{Im} j_t(1)$, notation which is not incompatible with h_0, we obtain a filtration (h_t).
(c) There results that T_t is completely positive:
$j_s(T_t x) = j_s(1) j_{s+t}(x) j_s(1)$ for $s = 0$ we have $j_0(T_t x) = j_0(1) j_t(x) j_0(1)$ and $\langle j_0(T_t x)\, u, v\rangle = \langle j_0(1) j_t(x) j_0(1) u, v\rangle$ and j_0 is a projector.
We also have:

$$\langle (T_t x)\, u, v\rangle = \langle j_t(x)\,(u \otimes 1), (v \otimes 1)\rangle .$$

Let be $S_i \in A$, $V_i \in A$, then we have:

$$\left\langle \sum\nolimits_{i,j} S_i * T_t\left(V_i^* V_j\right) S_j u, v \right\rangle = \sum\nolimits_{i,j} \left\langle T_t\left(V_i^* V_j\right) S_j u, S_i v\right\rangle$$
$$= \sum\nolimits_{i,j} \left\langle j_t\left(V_i^* V_j\right)\left(S_j u \otimes 1\right), (S_i v \otimes 1)\right\rangle$$
$$= \sum\nolimits_{i,j} \left\langle j_t * (V_i)\, j_t(V)_j \left(S_j u \otimes 1\right), S_i v \otimes 1\right\rangle$$
$$= \sum\nolimits_{i,j} \left\langle j_t\left(V_j\right)\left(S_j u \otimes 1\right), j_t(V_i)\, S_i v \otimes 1\right\rangle = \left\| \sum\nolimits_j \langle j_t(V_i)\, S_i v \otimes 1\rangle \right\|^2 \geq 0$$

(d) If $S : A \to A$ is continuous and linear, then $T_t = e^{tS}$ defines a semigroup, but generally T_t are not completely positive. If $S1 = 0$ then $T_t 1 = 1$.
(e) If $U \in L(h)$ is unitary, then $T \to UTU^*$ is a *-homomorphism:

$$L(h) \to L(h).$$

We shall use the following formulas:

I. If $M(t) = M(0) + \int_0^t dA_L^+ F + (dA)_L G + (ds)H$ then

$$\begin{aligned}
&\langle M(t)(u \otimes x), M(t)(v \otimes y)\rangle = \langle M(0)(u \otimes x), M(0)(v \otimes y)\rangle \\
&+ \int_0^t \Big(\sum\nolimits_p \big\langle L_p F(a)(u \otimes x), L_p F(a)(v \otimes y)\big\rangle \\
&+ \sum\nolimits_{p,j} (-1)^{j+r+w} \big(x_{jp}(a) \big\langle M(a)\big(u \otimes x_{cj}\big),\ \ L_p I F(a) I\ (v \otimes y)\big\rangle \\
&+ \overline{y_{jp}}(a) \big\langle L_p I F(a) I\ (u \otimes x), M(a)\big(v \otimes y_{cj}\big)\big\rangle \\
&+ (-1)^{j-1} \big(\overline{y_{jp}}(a) \langle M(a)(u \otimes x),\ \ L_p I G(a) I\ \big(v \otimes y_{cj}\big)\big\rangle \\
&+ x_{jp}(a) \big\langle L_p I G(a) I\ \big(u \otimes x_{cj}\big), M(a)(v \otimes y)\big\rangle\big) \\
&+ \langle H(a)(u \otimes x), M(a)(v \otimes y)\rangle\big)\, da
\end{aligned}$$

II. For $S \in L(h_0)$, we have $(S \otimes 1) = \int_0^t dA_L^+ F + (dA_L) G + H ds = \int_0^t \left(dA_{SL}^+\right) F + (dA_{LS^*}) G + SH ds$ as we know from the definition of the stochastic integral.

III. From $U^*(t) = 1 + \int_0^t \left(\left(dA_L^+\right)\theta U^* + (dA_L)\theta U^* + XU^* ds\right)$ we deduce that

$$(S \otimes 1) U^*(t) = (S \otimes 1) + \int_0^t \left(\left(dA_L^+\right)\theta U^* + (dA_{LS^*})\theta U^* + XU^* ds\right)$$

We consider

$$U(t) = 1 + \int_0^t \left(U\theta\left(dA_L^+\right) + U\theta\left(dA_L\right) + UXds\right)$$

we write it as follows:

$$U\left(t+s\right) = U(s) + \int_0^{s+t} \left(U\theta\left(dA_L^+\right) + U\theta\left(dA_L\right) + UXds\right)$$

The integral can be considered as $\int_0^t$ of the same integrant with h_s instead of h_0 and h^s instead of h^0. Using "III.", the equation can be written: $U(s)^{-1}U\left(t+s\right) = 1 + \int_0^{s+t} \left(U(s)^{-1}U\theta\left(dA_L^+\right) + U(s)^{-1}U\theta\left(dA_L\right) + U(s)^{-1}UXds\right)$ and then $U(s)^{-1}$ $U\left(.+s\right)$ appears as $U(.)$.

Lemma 5.2. *We consider the equation* $U(t) = 1 + \int_0^t \left(U\theta\left(dA_L^+\right) + U\theta\left(dA_L\right) + UXds\right)$ *where* $X^* = X = -\left(\sum L_p + L_p\right)/2$.

If we define:

$$A = \{S; \in L\left(h_0\right), S\theta = \theta S\}, \quad T_t(S) = e^{ty}(S)$$

where $Y(S) = \left(\sum_p L_p{}^* S L_p\right) + XS + SX$.

Then we have $\langle T_t(S)u, v\rangle = \langle U(t)\left(S \otimes 1\right) U(t)\left(u \otimes 1\right), \left(v \otimes 1\right)\rangle$.

Proposition 5.3. *We consider the equation*

$$U(t) = 1 + \int_0^t \left(U\theta\left(dA_L^+\right) + U\theta\left(dA_L\right) + UXds\right)$$

where $X = -\left(\sum L_p + L_p\right)/2$.

Then, if we define

$$A = \{S; \in L\left(h_0\right), S\theta = \theta S\}, \quad T_t(S) = e^{ty}(S)$$

where
$Y(S) = \left(\sum_p L_p{}^* S L_p\right) + XS + SX$ *and* $j_t(S) = (U_t\,(S \otimes 1)\,U_t{}^*)\,P_t$, *where* P_t *is the projector on* h_t, *the system* h_0, $L(h_0)$, T_t *and* j_t *is a noncommutative Markov process.*

Proof. Writing the equation

$$U^+(t) = 1 + \int_0^t \left(\left(dA_L^+\right)\theta U^+ + (dA_L)\,\theta U^+ + XU^+ds\right) \text{ and since } U^*(t)$$

appears as $\otimes 1$ relatively to $h = h_t \otimes h^t$ and $P_{s+t} = 1\otimes$ our relation becomes

$$\left\langle (S \otimes 1)\,U(s+t)^*\,(u \otimes x)\,, U(s+t)^*\,(v \otimes y)\right\rangle =$$
$$\left\langle (T_t(S) \otimes 1)\,U(s)^*\,(u \otimes x)\,, U(s)^*\,(v \otimes y)\right\rangle$$

We have $U(s)^*\,(u \otimes x)\,, U(s)^*\,(v \otimes y) \in h_s$ and it suffices to show that $\left\langle (S \otimes 1)\,U(s+t)^*\,(u \otimes x)\,, U(s+t)^*\,(v \otimes y)\right\rangle = \langle (T_t(S) \otimes 1)\,u, v\rangle$.

Hence we obtain the formula from Lemma 5.2, that is

$$\langle T_t(S)u, v\rangle = \left\langle U(t)\,(S \otimes 1)\,U(t)^*\,(u \otimes 1)\,, (v \otimes 1)\right\rangle.$$

Then $T \to UTU^*$ is a *-homomorphism: $L(h) \to L(h)$.

19.5.1 The Brownian Motion as Noncommutative Markov Process

We consider H a Hilbert space, a Brownian x_t on a probability space (E, K, P), $A = A^* \in L(H)$ and $U\,(t, \omega) = e^{i x_t(\omega) A}$. Hence $U(t, \omega)$ is a unitary operator of $L(H)$. Let be $T_t : L(H) \to L(H)$ defined as $T_t(x) = \int U(t) X U(t)^* dP$.

We have $U(t) = \sum_{n \geq 0} i^n (x_t)^n A^n / n!$, hence

$$U(t)^* = \sum_{n \geq 0} (-1)^n (x_t)^n A^n / n!, U(t)X(t)U(t)^*$$
$$= \sum_{n,k} (-1)^{n-k} (x_t)^{n+k} A^n X A^k / n!k!$$
$$= \sum_u \sum_{n+k=u} (x_t)^{n+k} i^u i^{-2k} A^n X A^k / n!k!$$
$$= \sum_u (i x_t)^u \sum_{n+k=u} (-1)^k A^n X A^k / n!k! = \sum_u (i x_t)^u D^u(X)/u!$$

where $D(X) = AX - XA$.

Indeed $D = P - Q$.

$P(X) = AX$, $Q(X) = XA$, hence

$D^u(X) = (P-Q)^u(X) = \sum_{n+k=u} C_u^n (-1)^k P^n Q^k X$, since P and Q commute.

Hence $T_t(X) = \sum_u i^u E\left((x_t)^u\right) D^u(X)/u!$.

From $E\left(e^{i\lambda x t}\right) = e^{-t\lambda^2/2} E\left(e^{i\lambda x_t}\right) = e^{-t\lambda^2/2}$ we deduce that

$$\sum_n i^n E\left((x_t)^n\right) \lambda^n/n! = \sum_n \left(-\lambda^2 t/2\right)^n/n!$$

and replacing $\lambda = D$ we obtain $T_t(X) = e^{-tD^2/2}(X)$, that is $T_t = e^{-tD^2/2}$.

We have

$-\left(D^2/2\right)(X) = -\left(A^2X - XA^2\right)/2 + AXA$, hence it is of the considerate form with only one term $L = L^* = A$.

19.5.2 The Poisson Process as Noncommutative Markov Process

We consider H a Hilbert space, a sequence U_n of unitary operators, a convergent sum with positive terms $\sum_n \lambda_n = \lambda$, $p_n = \lambda_n/\lambda$, a probability space (E, K, P) and a particular composite Poisson process on it, that is $x_t = y_{z_t}$, where (z_t) is a Poisson process of parameter λ and $(y_n)_{n\geq 1}$ is a sequence of independent variables, independent of (z_t), all having the repartition $\Lambda = \Sigma\, p_n \varepsilon_n$. We consider $Y_0 = 1$.

For every t we consider $U_t(\omega) = U_{y_{z_t}} \dots U_0$ and we define for $X \in L(H)$,

$$T_t(X) = \int U(t) X U(t)^* dP.$$

We have

$$T_t(X) = \sum_k \int \chi_{(z_t=k)} U(t) X U(t)^* dP$$
$$= \sum_k \left((\lambda t)^k e^{-\lambda t}/k!\right) \int \chi_{(z_t=k)} U_{y_k} \dots U_{y_0} X U^*{}_{y_0} dP =$$
$$\sum_k \left((\lambda t)^k e^{-\lambda t}/k!\right) \sum_{n_1,\dots,n_k} p_{n_1} \dots p_{n_k} U_{n_k} \dots U_{n_1} X U^*{}_{n_1} \dots U^*{}_{n_k}$$

We denote $L_i(X) = U_i X U_i{}^*$ and we have:

$$T_t(X) = \sum_k \left((t)^k e^{-\lambda t}/k!\right) \sum_{n_1,\dots,n_k} \lambda_{n_1} \dots \lambda_{n_k} L_{n_k} \dots L_{n_1}(X)$$
$$= \sum_k \left((t)^k e^{-\lambda t}/k!\right) \left(\sum_n \lambda_n L_n\right)^k (X) = e^{t\left(-\lambda + \sum_n \lambda_n L_n\right)}(X)$$

hence $T_t = e^{t\left(-\lambda + \sum_n \lambda_n L_n\right)}$.

We remark now that

$$\left(-\lambda + \sum_n \lambda_n L_n\right)(X) = \sum_n \left(\lambda_n{}^{1/2} U^*{}_n\right)^* X \left(\lambda_n{}^{1/2} U^*{}_n\right) + ZX + XZ,$$

where $Z = -\lambda/2 = -\sum_n \left(-\lambda_n^{1/2} U^*{}_n\right)^* \left(\lambda_n^{1/2} U^*{}_n\right)/2$, hence T_t is a particular case of the considerate semigroups.

19.6 Conclusions

The need to build the non-commutative Markov processes was given by the evolution of probabilities in quantum mechanics. This paper aims to build these processes on antisymmetric Fock space where we do not have exponential commutative vectors and where the commutative property does not occur between operators describing disjoint time intervals. For this reason the processes are obtained as solutions of stochastic integral equations. This mathematical model creates the possibility to construct physical processes as stochastic integral equations solutions, being at the same time a new method of proving that certain processes are noncommutative. The model may be used in diffusion processes.

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