# Using Different Norms in Packing Circular Objects


Igor Litvinchev(✉), Luis Infante, and Lucero Ozuna

Faculty of Mechanical and Electrical Engineering, UANL
Pedro de Alba S/N, 66450 San Nicolas de los Garza, NL, Mexico
igorlitvinchev@gmail.com



**Abstract.** A problem of packing unequal circles in a fixed size rectangular container is considered. The circle is considered in a general sense, as a set of points that are all the same distance (not necessary Euclidean) from a given point. An integer formulation is proposed using a grid approximating the container and considering the nodes of the grid as potential positions for assigning centers of the circles. The packing problem is then stated as a large scale linear 0-1 optimization problem. Valid inequalities are proposed to strengthening the original formulation. Nesting circles inside one another is considered tacking into account the thickness of the circles. Numerical results on packing circles, ellipses, rhombuses and octagons are presented to demonstrate the efficiency of the proposed approach.


## 1 Introduction

Packing problems generally consist of packing a set of items of known dimensions into one or more large objects or containers to minimize a certain objective (e.g. the unused part of the container or waste). Packing problems constitute a family of natural combinatorial optimization problems applied in computer science, industrial engineering, logistics, manufacturing and production processes (see e.g. [2, 5, 7, 10, 11] and the references therein).

Along with industrial applications one may find packing problems, e.g. in health-care issues. In [22] automated radiosurgical treatment planning for treating brain and sinus tumours was considered. Radiosurgery uses the gamma knife to deliver a set of extremely high dose ionizing radiation, called “shots” to the target tumour area. For large target regions multiple shots of different intensity are used to cover different parts of the tumour. However, this procedure may result in large doses due to overlap of the different shots. Optimizing the number, positions and individual sizes of the shots can reduce the dose to normal tissue and achieve the required coverage.

Packing problems for regular shapes (circles and rectangles) of objects and/or containers are well studied [12]. In circle packing problem the aim is to place a certain number of circles, each one with a fixed known radius inside a container. The circles must be totally placed in the container without overlapping. The shape of the container may vary from a circle, a square, a rectangular, etc. Many variants of packing circular objects have been formulated as nonconvex (continuous) optimization problems with decision variables being coordinates of the centres [12]. Non-overlapping typically is assured by nonconvex constraints representing that the Euclidean distance separating



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the centres of the circles is greater than a sum of their radii. The nonconvex problems can be tackled by available nonlinear programming (NLP) solvers, however most NLP solvers fail to identify global optima and global optimization techniques have to be used [5]. The nonconvex formulations of circular packing problem give rise to a large variety of algorithms which mix local searches with heuristic procedures in order to widely explore the search space. We will refer the reader to review papers presenting the scope of techniques and applications for regular packing problem [1, 4, 6, 18, 19, 20].

Irregular packing problems involve non standard shapes of objects and/or containers. Irregular shapes are those that require non-trivial handling of the geometry. One of the most common representations for irregular shape is a polyhedral domain which may by nonconvex o multi-connected. Heuristic and metaheuristic algorithms are the basis for the solution approaches [7, 21].

In this paper we study approximate packing of circular-like objects using a regular grid to approximate the container. The circular-like object is considered in a general sense, as a set of points that are all the same distance (not necessary Euclidean) from a given point. Thus different shapes, such as ellipses, rhombuses, rectangles, octagons can be treated the same way by simply changing the norm used to define the distance. In a sense, we demostrate that packing some irregular objects is as simple as packing circles. The nodes of the grid are considered as potential positions for assigning centers of the circles. The packing problem is then stated as a large scale linear 0-1 optimization problem. Valid inequalities are proposed to strengthening the original formulation. Nesting circles inside one another is considered tacking into account the thickness of the circles. Numerical results on packing circles, ellipses, rhombuses and octagons are presented to demonstrate the efficiency of the proposed approach.

To the best of our knowledge, the idea to use a grid was first implemented in [3] in the context of cutting problems. This approach was recently applied in [9, 14-17, 21] for packing problems. This work is a continuation of [14]. The rest of the paper is structured as follows. In Section 2 the main integer programming model for packing problem is presented. Section 3 is related to the experimental results on packing circles, ellipses, rhombuses and octagons to show that our methodology is efficient. A final section concludes this work.

## 2 The Principal Model

Suppose we have non-identical circles $C_k$ of known radius $R_k$, $k \in K = \{1, 2, ...K\}$ which have to be packed in a container $G$. It is assumed that no two objects overlap with each other and each packed object lies entirely in the container. Here we consider the circle as a set of points that are all the same distance $R_k$ (not necessary Euclidean) from a given point. In what follows we will use the same notation $C_k$ for the figure bounded by the circle, $C_k = \{z \in \mathbb{R}^2 : \|z - z_{0k}\| \le R_k\}$, assuming that it is easy to understand from the context whether we mean the curve or the figure. Denote by $S_k$ the area of $C_k$.

Let at most $M_k$ circles $C_k$ are available for packing and at least $m_k$ of them have to be packed. Denote by $i \in I = \{1, 2..., n\}$ the node points of a regular grid covering the rectangular container. Denote by $d_{ij}$ the distance (in the sence of norm used to define the circle) between points $i$ and $j$ of the grid. Define binary variables $x_i^k = 1$ if centre of a circle $C_k$ is assigned to the point $i$; $x_i^k = 0$ otherwise. In what follows we will say that the object is assigned to the node $i$ if the corresponding reference point is assigned to that node and will denote this as $C_k^i$.

In order to the circle $C_k$ assigned to the point $i$ be non-overlapping with other circles being packed, it is necessary that $x_j^l = 0$ for $j \in I$, $l \in K$, such that $d_{ij} < R_k + R_l$. For fixed $i,k$ let $N_{ik} = \{ j,l : i \neq j, d_{ij} < R_k + R_l \}$. Let $n_{ik}$ be the cardinality of $N_{ik}$: $n_{ik} = |N_{ik}|$. Then the problem of maximizing the area covered by the circles can be stated as follows:

$$\max \sum_{i \in I} \sum_{k \in K} S_k^2 x_i^k \tag{1}$$

subject to

$$m_k \leq \sum_{i \in I} x_i^k \leq M_k, \quad k \in K, \tag{2}$$

$$\sum_{k \in K} x_i^k \leq 1, \quad i \in I, \tag{3}$$

$$x_i^k = 0 \text{ for } C_k^i \setminus (G \cap C_k^i) \neq \varnothing \text{ for } i \in I,\, k \in K, \tag{4}$$

$$x_i^k + x_j^l \leq 1, \text{ for } i \in I; k \in K; (j,l) \in N_{ik}, \tag{5}$$

$$x_i^k \in \{0,1\}, \quad i \in I, k \in K. \tag{6}$$

Constraints (2) ensure that the number of circles packed is between $m_k$ and $M_k$; constraints (3) that at most one centre is assigned to any grid point; constraints (4) that $C_k$ can not be assigned to the node $i$ if $C_k^i$ is not totally placed inside $G$; pair-wise constraints (5) guarantee that there is no overlapping between the circles; constraints (6) represent the binary nature of variables.

Similar to plant location problems [22] we can state non-overlapping conditions in a more compact form. Summing up pair-wise constraints (5) over $(j,l) \in N_{ik}$ we get

$$n_{ik} x_i^k + \sum_{j,l \in N_{ik}} x_j^l \leq n_{ik} \text{ for } i \in I,\, k \in K \tag{7}$$

Note that constraints similar to (7) were used in [9] for packing equal circles ( $K = 1$ ).

**Proposition 1.** [15,17]**.** Constraints (5), (6) are equivalent to constraints (6), (7).

Thus the problem (1)-(6) is equivalent to the problem (1)-(4), (6), (7). To compare two equivalent formulations, let

$$P_1 = \{x \geq 0 : x_i^k + x_j^l \leq 1, \text{ for } i \in I, k \in K, (j,l) \in N_{ik}\},$$
$$P_2 = \{x \geq 0 : n_{ik} x_i^k + \sum_{j,l \in N_{ik}} x_j^l \leq n_{ik} \text{ for } i \in I, k \in K\}.$$

**Proposition 2.** [15, 17]. $P_1 \subset P_2$.

As follows from Proposition 2, the pair-wise formulation (1)-(6) is stronger [23] than the compact one. Numerical experiments presented in [17] demonstrate that the pair-wise formulation is also computationally more attractive since it provides a tighter LP-bound. Bearing in mind these reasons we restrict ourselves by considering below only pair-wise formulations.

By the definition, $N_{ik} = \{j,l : i \neq j, d_{ij} < R_k + R_l\}$ and hence if $(j,l) \in N_{ik}$, then $(i,k) \in N_{jl}$. Thus a half of the constraints in (5) are redundant since we have

$$x_i^k + x_j^l \leq 1, \text{ for } i \in I, k \in K, (j,l) \in N_{ik},$$
$$x_j^l + x_i^k \leq 1, \text{ for } j \in I, l \in K, (i,k) \in N_{jl}.$$

The redundant constraints can be eliminated without changing the quality of LP-bound giving a reduced pair-wise non overlapping formulation.

In many applied problems nesting circles inside one another is permitted. For example, in [8, 10, 11] nesting is considered in the context of packing pipes of different diameters into a shipping container. Nesting is also essential for storing different cylinders one over another in the form of cylindrical towers.

To consider nesting circles inside one another, we only need to modify the non-overlapping constraints. In order to the circle $C_k$ assigned to the point $i$ be non-overlapping with other circles being packed (including circles placed inside this circle), it is necessary that $x_j^l = 0$ for $j \in I, l \in K$, such the $R_k - R_l < d_{ij} < R_k + R_l$ for $R_k > R_l$. Let

$$\Omega_{ik} = \{j,l : i \neq j, R_k - R_l < d_{ij} < R_k + R_l, R_k > R_l\}$$

Then the non-overlapping constraints for packing circles with nesting can be stated in the form

$$x_i^k + x_j^l \leq 1, \text{ for } i \in I; k \in K; (j,l) \in \Omega_{ik} \tag{8}$$

Constraints (3) have to be relaxed in case of nesting.

If nesting is permitted, e.g., in the case of packing plastic pipes [8,10,11], it may be necessary to take into account the thickness of the pipe, i.e. the difference between external and internal size of the object. To consider nesting-subject-to-thickness we need only to redefine the set $\Omega_{ik}$. Let $g_k$ be the thickness of the circle $C_k$. For $R_k - g_k > R_l$ For $\Omega_{ik}$ defined as

$$\Omega_{ik} = \{j,l : i \neq j, R_k - g_k - R_l < d_{ij} < R_k + R_l, R_k - g_k > R_l\}, \tag{9}$$

we get non-overlapping constraints (8) for the case of "nesting-subject-to-thickness". The rest of the optimization model stated above remains unchanged.

Note that all constructions proposed above, including Propositions 1,2, remain valid for any norm used to define the circular-like object. In fact, changing the norm affects only the distance $d_{ij}$ used in the definitions of the sets $N_{ik}, \Omega_{ik}$ in the non-overlapping constraints (5), (8). That is, by simple pre-processing we can use the basic model (1)-(6) for packing different geometrical objects of the same shape. It is important to note that the non-overlapping condition has the form $d_{ij} \geq R_k + R_l$ no matter which norm is used. For example, a circular object in the maximum norm $\|z\|_\infty := \max_i\{|z_i|\}$ is represented by a square, taxicab norm $\|y\|_1 := \sum_r |y_r|$ yields a rhombus. In a similar way we may manage rectangles, ellipses, etc. Using a superposition of norms, we can consider more complex circular objects. For $\|y\| := \max_r\{|y_r|, \gamma\sum_r |y_r|\}$ and a suitable $0.5 < \gamma < 1$ we get an octagon, an intersection of a square and a rhombus.

We may expect that the linear programming relaxation of the problem (1)-(6) provides a poor upper bound for the optimal objective. For example, for $K = 1$ and suitable $M_k, m_k$ the point $x_i^k = 0.5$ for all $i \in I$ may be feasible to the relaxed problem with the corresponding objective growing linearly with respect to the number of grid points.

To tightening the LP-relaxation for (1)-(6) without nesting we consider valid inequalities aimed to ensure that no grid point is covered by two circles. Define matrix $\left[\alpha_{ij}^k\right]$ as follows. Let $\alpha_{ij}^k = 1$ for $d_{ij} < R_k$, $\alpha_{ij}^k = 0$ otherwise. By this definition, $\alpha_{ij}^k = 1$ if the circle $C_k$ centred at $i$ covers point $j$. The following constraints ensure that no points of the grid can be covered by two circles:

$$\sum_{k\in K}\sum_{j\in I}\alpha_{ij}^k x_j^k \leq 1, \quad i \in I. \tag{10}$$

Note that (10) is not equivalent to non-overlapping constraints (5). Constraints (10) ensure that there is no overlapping in grid points, while (5) guarantee that there is no overlapping at all. The valid inequality (10) holds for any norm used to define the circular object.

## 3 Numerical Results

In this section we present a numerical study on packing equal circles, ellipses, rhombuses and octagons by varying the definition of the norm in (1)-(6). It is assumed that the supply of the objects is unlimited. The standard Euclidean and taxicab norms were used to define circles and rhombuses, $\|z\| := (2z_1^2 + z_2^2)^{1/2}$ and $\|z\| := \max\{|z_1|, |z_2|, (1/\sqrt{2})(|z_1| + |z_2|)\}$ were used for ellipses and octagons. A rectangular uniform grid of size $\Delta$ along both sides of the container was implemented. It is not hard to verify that for these particular shapes constraints (4) constraints (4) can be relaxed by reducing correspondingly the size of the container. The test bed set of 9 instances was used for packing maximal number of equal circular objects into a rectangular container of width 3 and height 6. The values of radii and grid size are the same as in [9, Table 3]. All optimization problems were solved by the system CPLEX 12.6. The runs were executed on a desktop computer with CPU AMD FX 8350 8-core processor 4 Ghz and 32Gb RAM.

The results of the numerical experiment are given in Table 1. Here the first four columns present instance number, radius $R$ of the circular object, size of the grid $\Delta$ and the number of binary variables (dim). The following columns give the number of objects packed and corresponding CPU time (in seconds) for circles (C), ellipses (E), rhombuses (R) and octagons (O). For all problem instances $mipgap = 0$ was set as a stopping criterion for running CPLEX. The asterisk indicates that the computation was interrupted after the computation time exceeded 12-hours CPU time. For problem instances where optimality was not achieved within time limit the number of objects corresponds to the best integer solution and the number in parenthesis indicates the value of $mipgap$ in % obtained to the moment of interruption. Packings for instance 7 are presented in Fig. 1.

**Table 1.** Packing circles, ellipses, rhombuses and octagons

| No. | $R$ | $\Delta$ | dim | O | CPU | R | CPU | C | CPU | E | CPU |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 0.5 | 0.125 | 697 | 18 | 1 | 28 | 1 | 18 | 1 | 34 | 11 |
| 2 | 0.625 | 0.15625 | 1403 | 9 | 52 | 15 | 11 | 10 | 41 | 21 | 25 |
| 3 | 0.5625 | 0.0703125 | 2449 | 12 | 202 | 20 | 312 | 13 | 186 | 27 | 288 |
| 4 | 0.375 | 0.046875 | 1425 | 26 | 49 | 39 | 399 | 32 | 4 | 59 | 2 |
| 5 | 0.3125 | 0.078125 | 2139 | 41 | 6850 | 76 | 6 | 45 | 114 | 99 | 3 |
| 6 | 0.4375 | 0.0546875 | 3666 | 20 | 1430 | 35 | 2829 | 21 | 17654 | 44 (4) | * |
| 7 | 0.25 | 0.0625 | 3649 | 72 | 22 | 127 | 17 | 74 (5) | * | 137 (7) | * |
| 8 | 0.275 | 0.06875 | 2880 | 50 | 20495 | 75 (4.6) | * | 61 | 177 | 108 (6) | * |
| 9 | 0.1875 | 0.046875 | 6897 | 106 (12.5) | * | 167 (9) | * | 140 (5) | * | 261 (4) | * |

As can be seen from Table 1 for all types of the objects the large instances were not solved to optimality within the time limit. The computation was interrupted obtaining a feasible solution within 5-10% of proven relative suboptimality.

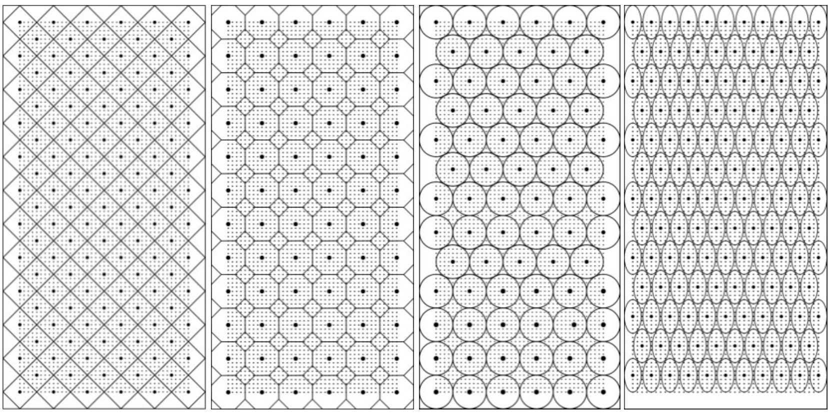

**Fig. 1.** Packing equal objects for instance 7

However, the most of the computation time was used to tightening the dual bound to prove optimality/suboptimality of the feasible solution obtained on early stages of CPLEX. For example, for the case of octagons and instance 9 a feasible solution with 105 objects packed was obtained within 300 sec., for instance 5 - 41 objects in 30 sec, for instance 6 – 19 objects in 30 sec. A similar behaviour was noticed for other objects and instances where the computation time exceeds 1000 sec. For all these instances at most 600 sec. was necessary to get a feasible solution with relative difference 1-2% from the best feasible or optimal solution. Thus we may consider CPLEX as a sufficiently fast tool to find good feasible solutions and in this sense it is comparable with heuristics proposed, e.g., for circle packing problems in [9].

Table 2 presents an effect of introducing valid inequalities (10) in the problem formulation (1)-(6). The columns here indicate the type of the object (C, E, O, R), the value of the LP-relaxation before and after introducing valid inequalities, LP and LPC, correspondingly. The value of LP-relaxation without valid inequalities (second column in Table 2) is just the same for all types of circular objects and equals to the half of the instance dimension (all variables are 0.5). We see that introducing valid inequalities (10) improves significantly the quality of LP bound for all shapes of the objects. The detailed study of this subject for the case of circles one can find in [17] for the same test bed instances.

**Table 2.** LP-relaxations

| No. | LP | O | LPC | R | LPC | C | LPC | E | LPC |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 348.5 | **18** | 19 | **28** | 33.43 | **18** | 19 | **34** | 36 |
| 2 | 701.5 | **9** | 10 | **15** | 16.87 | **10** | 10 | **21** | 25 |
| 3 | 1224.5 | **12** | 14.0743 | **20** | 22.25 | **13** | 14.07 | **27** | 29.91 |
| 4 | 712.5 | **26** | 30.9485 | **39** | 41.37 | **32** | 36.33 | **59** | 68.86 |
| 5 | 1069.5 | **41** | 53.4043 | **76** | 94.76 | **45** | 53.4 | **99** | 110 |
| 6 | 1833.5 | **20** | 22.5537 | **35** | 39.72 | **21** | 23.86 | **43** | 49.787 |
| 7 | 1824.5 | **72** | 90.9767 | **127** | 157.96 | **74** | 90.98 | **137** | 182 |
| 8 | 1440 | **50** | 59.014 | **75** | 79.53 | **61** | 72 | **108** | 134.56 |
| 9 | 3448.5 | **106** | 134.342 | **167** | 182.28 | **140** | 162 | **261** | 273.61 |

The results of a small computational experiment for packing two octagons in a square 30x30 container maximizing the total area of the objects are presented in Table 3.

**Table 3.** Packing 2 different octagons

| No. | $R_1, R_2$ | dim | N- | CPU | N+ | CPU | N+T | CPU |
|---|---|---|---|---|---|---|---|---|
| 1 | 0.6, 6.3 | 441 | 627.48 | 1 | 842.21 | 1 | 804.37 | 1 |
| 2 | 0.6, 6.3 | 961 | 699.06 | 6 | 971.05 | 3 | 910.209 | 5 |
| 3 | 1, 5.3 | 441 | 699.35 | 1 | 952.82 | 1 | 922.99 | 1 |
| 4 | 1, 5.3 | 961 | 750.09 | 57 | 1158.27 | 129 | 1019.1 | 49 |

Here the first three columns give instance number, radii, and a number of grid points (integer variables). The last columns give the total area without nesting (N-), with nesting (N+) and with nesting and thickness (N+T), as well as corresponding CPU time in sec. The thickness $g_k$ in (1.9) was defined as $0.1R_k$. The packings obtained for instance 4 are presented in Fig. 2.

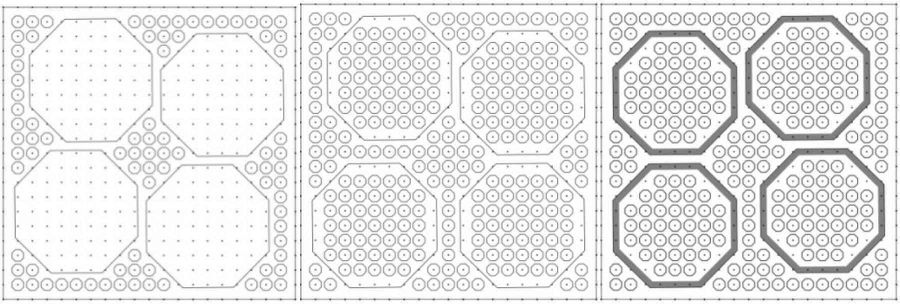

**Fig. 2.** Packing two octagons for instance 4

## 4 Conclusions

Integer formulations were proposed for approximate packing circular-like objects with nesting and taking into account the thickness of objects (the difference between external and internal size of the object). It was demonstrated that by simply changing the definition of the distance (preprocessing) it is possible to use the same basic models for packing different circular-like objects such as circles, ellipses, rhombuses, octagons, etc.

Valid inequalities were considered to strengthening the original formulation. The results of our numerical experiment indicate that the valid inequalities improve significantly the LP-bound. Note that these inequalities can be used for packing different shapes. An interesting area for future research is the generalization of valid inequalities for the case of nesting. To cope with large dimension of arising problems it is interesting to study the use of Lagrangian relaxation and corresponding heuristics [13, 23].

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