Edge-Colorings of Weighted Graphs
(Extended Abstract)

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Abstract. Let G be a graph with a positive integer weight $\omega(v)$ for each vertex v. One wishes to assign each edge e of G a positive integer $f(e)$ as a color so that $\omega(v) \le |f(e) - f(e')|$ for any vertex v and any two edges e and e' incident to v. Such an assignment f is called an ω-edge-coloring of G, and the maximum integer assigned to edges is called the span of f. The ω-chromatic index of G is the minimum span over all ω-edge-colorings of G. In the paper, we present various upper and lower bounds on the ω-chromatic index, and obtain three efficient algorithms to find an ω-edge-coloring of a given graph. One of them finds an ω-edge-coloring with span smaller than twice the ω-chromatic index.

1 Introduction

An ordinary *edge-coloring* of a graph G assigns different colors to any two adjacent edges. The paper extends the concept to an edge-coloring of a weighted graph.

Let $G = (V, E)$ be a graph with a positive integer weight $\omega(v) \in \mathbb{N}$ for each vertex $v \in V$, where $\mathbb{N}$ is the set of all positive integers. Indeed G may be a multigraph. Figure 1 illustrates such a graph G, in which each vertex v is drawn as a circle and the weight $\omega(v)$ is written in it. One wishes to assign each edge $e \in E$ a positive integer $f(e)$ as a color so that $\omega(v) \le |f(e) - f(e')|$ for any vertex $v \in V$ and any two edges e and e' incident to v. Such a function $f : E \to \mathbb{N}$ is called an *edge-coloring of a graph G with a weight function* ω or simply an *ω-edge-coloring* of G. An ω-edge-coloring f of a graph G is illustrated in Fig. 1, where $f(e)$ is attached to each edge e.

The *span* $\mathrm{span}(f)$ of an ω-edge-coloring f of a graph G is the maximum integer assigned to edges by f, that is, $\mathrm{span}(f) = \max_{e \in E} f(e)$. An ω-edge-coloring f of G is called *optimal* if $\mathrm{span}(f)$ is minimum among all ω-edge-colorings of G. The ω-edge-coloring in Fig. 1 is optimal, and its span is 7. The span of an optimal ω-edge-coloring of a graph G is called the *ω-chromatic index* $\chi'_\omega(G)$ of G. The *ω-edge-coloring problem* is to find an optimal ω-edge-coloring of a given graph.

An ω-edge-coloring often appears in a task scheduling problem [12]. Each vertex v of a graph G represents a processor, while each edge $e = (u, v)$ of G represents a task, which can be executed within a unit time with the cooperation of the two processors represented by vertices u and v. Each processor v needs an

M.S. Rahman and E. Tomita (Eds.): WALCOM 2015, LNCS 8973, pp. 32–43, 2015.

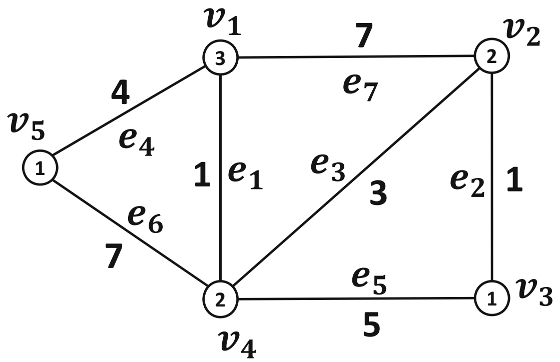

Fig. 1. An optimal ω-edge-coloring f of a graph G

idle time $\omega(v)$ between any two tasks executed by v. Then an optimal ω-edge-coloring of G corresponds to a scheduling with the minimum makespan.

If $\omega(v) = 1$ for every vertex v of a graph G, then an ω-edge-coloring of G is merely an ordinary edge-coloring of G and the ω-chromatic index $\chi'_\omega(G)$ of G is equal to the ordinary chromatic index $\chi'(G)$ of G. Since an ordinary edge-coloring problem is NP-complete [4], the ω-edge-coloring problem is strongly NP-complete and does not look to be solved in polynomial time or in pseudo polynomial time. So it is desired to obtain an efficient approximation algorithm for the ω-edge-coloring problem.

In this paper we present various upper and lower bounds on the ω-chromatic index, and obtain three efficient approximation algorithms for the ω-edge-coloring problem. The first algorithm **Delta** finds an ω-edge-coloring f of a given graph G such that $\mathrm{span}(f) \leq \Delta'_{1\omega}(G) + 1$, where $\Delta'_{1\omega}(G)$ is the maximum "uni-directional ω-edge-degree" of G. The second algorithm **Degenerate** finds an ω-edge-coloring f such that $\mathrm{span}(f) \leq k+1$ for any "k-edge-degenerated graph." **Delta** and **Degenerate** have approximation ratios smaller than two and four, respectively. We also show that an optimal ω-edge-coloring can be easily obtained for a graph G with the maximum degree $\Delta(G)$ at most two. The third algorithm **Factor** first decomposes a given graph G into several subgraphs $G_1, G_2, ..., G_r$, each having the maximum degree at most two, then finds optimal ω-edge-colorings of $G_1, G_2, ..., G_r$, and finally combines them to an ω-edge-coloring of G. The approximation ratio of **Factor** is near to $3/2$ for many graphs.

2 Preliminaries

In this section, we define several terms, present simple lower and upper bounds on the ω-chromatic index, and show that an optimal ω-edge-coloring of a graph G can be easily obtained if $\Delta(G) \leq 2$.

We denote by $G = (V, E)$ a graph with vertex set V and edge set E. G is a so-called *multigraph*, which has no selfloops but may have multiple edges.

We denote by (u, v) an edge joining vertices u and v. Let $n = |V|$ and $m = |E|$ throughout the paper. One may assume that G has no isolated vertex and hence $m \geq n/2$. Let $\omega : V \rightarrow \mathbb{N}$ be a *weight function* of G.

We denote by $E(v)$ the set of all edges incident to a vertex v in a graph $G = (V, E)$. The *degree* of a vertex v is $|E(v)|$ and is denoted by $d(v, G)$ or simply $d(v)$. The maximum degree of vertices in G is called the *maximum degree* of G, and denoted by $\Delta(G)$ or simply Δ. Every ω-edge-coloring f of G satisfies

$$1 + (d(v) - 1)\omega(v) \leq \max_{e \in E(v)} f(e)$$

for every vertex v. We thus define the ω*-degree* $d_\omega(v)$ of a vertex v as follows:

$$d_\omega(v) = 1 + (d(v) - 1)\omega(v). \tag{1}$$

The maximum ω-degree of vertices in G is called the *maximum* ω*-degree* $\Delta_\omega(G)$ of G. It should be noted that $\Delta_\omega(G) = \Delta(G)$ if $\omega(v) = 1$ for every vertex v. Clearly $\Delta_\omega(G) \leq \text{span}(f)$ for every ω-edge-coloring f of G. Therefore, the following lower bound holds for the ω-chromatic index $\chi'_\omega(G)$:

$$\Delta_\omega(G) \leq \chi'_\omega(G). \tag{2}$$

The graph G in Fig. 1 satisfies $\Delta_\omega(G) = d_\omega(v_1) = 7$, the ω-edge-coloring f in Fig. 1 has span 7, and hence f is optimal. In Section 3 we will present an upper bound: $\chi'_\omega(G) \leq 2\Delta_\omega(G) - 1$ for every graph G.

Since the weight of a vertex of degree 1 is meaningless, we define the *largest weight* $\omega_l(G)$ of a graph G as follows:

$$\omega_l(G) = \max\{\omega(v) \mid v \in V, d(v) \geq 2\}$$

where $\omega_l(G)$ is defined to be zero if $\Delta(G) = 1$. Since $1 + \omega_l(G) \leq \Delta_\omega(G)$, Eq. (2) implies the following lower bound:

$$1 + \omega_l(G) \leq \chi'_\omega(G) \tag{3}$$

We often denote $\omega_l(G)$ simply by ω_l.

Suppose that a graph G is ordinarily edge-colored by colors 1, 2, ..., c, where $c \geq \chi'(G)$. Replace colors 1, 2, ..., c by 1, $1 + \omega_l$, ..., $1 + (c - 1)\omega_l$, respectively. Then the resulting coloring is an ω-edge-coloring of G. Thus we have an upper bound:

$$\chi'_\omega(G) \leq 1 + (\chi'(G) - 1)\omega_l \tag{4}$$

for every graph G.

V. G. Vizing showed that $\chi'(G) \leq \Delta(G) + 1$ for every *simple* graph G, which has no multiple edges [14,15]; and it is known that such an edge-coloring of G can be found in time $O(mn)$, $O(m\Delta \log n)$ or $O(m\sqrt{n \log n})$ [3]. Therefore, by Eq. (4) we have

$$\chi'_\omega(G) \leq 1 + \Delta\omega_l$$

for every simple graph G, and such an ω-edge-coloring can be found in time $O(mn)$, $O(m\Delta \log n)$ or $O(m\sqrt{n \log n})$.

D. König showed that $\chi'(G) = \Delta(G)$ for every bipartite graph G [14,15], and it is known that such an edge-coloring can be found in time $O(m \log \Delta)$ [1]. Therefore, by Eq. (4) we have

$$\chi'_\omega(G) \leq 1 + (\Delta - 1)\omega_l$$

for every bipartite graph G, and such an ω-edge-coloring can be found in time $O(m \log \Delta)$. Since $\Delta_\omega(G) \leq \chi'_\omega(G)$ by Eq. (2), such an ω-edge-coloring of a bipartite graph G is optimal if

$$\Delta_\omega(G) = 1 + (\Delta - 1)\omega_l. \tag{5}$$

A graph G satisfies Eq. (5) if and only if G has a vertex v such that $d(v) = \Delta$ and $\omega(v) = \omega_l$, and does for example if either G is a regular graph or $\omega(v) = 1$ for every vertex v.

We then present another lower bound $1 + \omega_s(G)$ on $\chi'_\omega(G)$. An *odd cycle* C in G has an odd number of vertices. We define $\omega_s(C)$ as follows:

$$\omega_s(C) = \min\{\omega(u) + \omega(v) \mid \text{ vertices } u \text{ and } v \text{ consecutively appear in } C\}.$$

We define $\omega_s(G)$ as follows:

$$\omega_s(G) = \max\{\omega_s(C) \mid C \text{ is an odd cycle in } G\}$$

where $\omega_s(G)$ is defined to be zero if G has no odd cycle. We often denote $\omega_s(G)$ simply by ω_s. One can easily prove the following lemma for a lower bound on $\chi'_\omega(G)$.

Lemma 1. *For every graph G*

$$1 + \max\{\omega_l, \omega_s\} \leq \chi'_\omega(G).$$

We then show that if $\Delta(G) \leq 2$ then $\chi'_\omega(G)$ is equal to the rather trivial lower bound in Lemma 1 and an optimal ω-edge-coloring of G can be easily obtained. One may assume that G is connected. Then G is a path or cycle. If G is a path or an even cycle, then a coloring of G in which edges are colored alternately by 1 and $1+\omega_l$ is an optimal ω-edge coloring and hence $\chi'_\omega(G) = 1+\omega_l$. One may thus assume that G is an odd cycle. Let the vertices $v_1, v_2, ..., v_n$ appear in G in this order, where n (≥ 3) is an odd number. One may further assume that $\omega_s(G) = \omega(v_2) + \omega(v_3)$. Color the consecutive three edges $e_1 = (v_1, v_2), e_2 = (v_2, v_3)$ and $e_3 = (v_3, v_4)$ by 1, $1 + \omega(v_2)$ and $1 + \max\{\omega_l, \omega_s\}$, respectively, and color the remaining $n-3$ edges alternately by 1 and ω_l. Then the resulting coloring f of G is obviously an ω-edge-coloring of G, and $\text{span}(f) = 1 + \max\{\omega_l, \omega_s\}$. Since $\chi'_\omega(G) \geq 1 + \max\{\omega_l, \omega_s\}$ by Lemma 2.1, f is optimal and $\chi'_\omega(G) = 1 + \max\{\omega_l, \omega_s\}$.

We thus have the following theorem.

Theorem 1. *If G is a graph with $\Delta(G) \leq 2$, then $\chi'_\omega(G) = 1 + \max\{\omega_l, \omega_s\}$ and an optimal ω-edge-coloring of G can be found in linear time.*

For two integers α and β, we denote by $[\alpha,\ \beta]$ the set of all integers z with $\alpha \leq z \leq \beta$. Let f be an ω-edge-coloring of a graph G. Let $e = (u, v)$ be an edge in G, let e' be an edge adjacent to e, and let x be a vertex to which both e and e' are incident. Then x is u or v. Neither the consecutive $\omega(x)$ integers greater than or equal to $f(e')$ nor those smaller than or equal to $f(e')$ can be assigned to e. Therefore

$$f(e) \notin B(e, e', x)$$

where

$$B(e, e', x) = [f(e') - \omega(x) + 1, f(e') + \omega(x) - 1].$$

Clearly $|B(e, e', x)| = 2\omega(x) - 1$. G has $d(u) - 1$ edges adjacent to e at end u and $d(v) - 1$ edges adjacent to e at end v. Therefore, there are at most $(d(u) - 1)$ $(2\omega(u) - 1) + (d(v) - 1)(2\omega(v) - 1)$ integers that cannot be assigned to e. This number is called a *bi-directional ω-edge-degree* $d_{2\omega}(e, G)$ of e, and hence

$$d_{2\omega}(e, G) = (d(u) - 1)(2\omega(u) - 1) + (d(v) - 1)(2\omega(v) - 1). \tag{6}$$

The maximum bi-directional ω-edge-degree of edges in G is called the *maximum bi-directional edge-degree* $\Delta'_{2\omega}(G)$ of G. Then one can easily prove by induction on the number m of edges that the following upper bound on $\chi'_\omega(G)$ holds for every graph G:

$$\chi'_\omega(G) \leq \Delta'_{2\omega}(G) + 1.$$

For the graph G in Fig. 1, $\Delta'_{2\omega}(G) = d_{2\omega}(e_1) = 19$.

Let f be an ω-edge-coloring of a graph G. Let $e = (u, v)$, let e' be adjacent to e, and let x be a vertex to which both e and e' are incident. Suppose that $f(e') < f(e)$. Then

$$f(e) \notin B_1(e, e', x)$$

where

$$B_1(e, e', x) = [f(e'), f(e') + \omega(x) - 1]$$

and

$$|B_1(e, e', x)| = \omega(x).$$

Therefore we have

$$f(e) \notin \Big(\bigcup_{e'} B_1(e, e', u)\Big) \bigcup \Big(\bigcup_{e'} B_1(e, e', v)\Big) \tag{7}$$

where e' runs over every edge such that e' is adjacent to e and $f(e') < f(e)$. In this sense we define the *uni-directional ω-edge-degree $d_{1\omega}(e)$ of an edge* $e = (u, v)$ as follows:

$$d_{1\omega}(e) = (d(u) - 1)\omega(u) + (d(v) - 1)\omega(v). \tag{8}$$

The maximum uni-directional ω-degree of edges in G is called the *maximum uni-directional ω-edge-degree* $\Delta'_{1\omega}(G)$ of G:

$$\Delta'_{1\omega}(G) = \max_{e \in E} d_{1\omega}(e).$$

Clearly $\Delta'_{1\omega}(G) \leq \Delta'_{2\omega}(G)$ for every graph G. For the graph G in Fig. 1, $\Delta'_{1\omega}(G) = d_{1\omega}(e_1) = 12$. We will show in Section 3 that the following upper bound holds for every graph G:

$$\chi'_\omega(G) \leq \Delta'_{1\omega}(G) + 1.$$

3 Algorithm Delta

In this section we present an algorithm **Delta** to find an ω-edge-coloring f of a given graph G such that $\mathrm{span}(f) \leq \Delta'_{1\omega}(G) + 1$, and show that the approximation ratio of **Delta** is smaller than two.

For an ω-edge-coloring f of a graph $G = (V, E)$, one may assume that

$$f(e_1) \leq f(e_2) \leq \ldots \leq f(e_m) \tag{9}$$

for some numbering $e_1, e_2, \ldots, e_m$ of the edges in E. Let $2 \leq i \leq m$, and let $e_i = (u, v)$. We define $E_i(u)$ as follows:

$$E_i(u) = \{e_j \in E(u) \mid 1 \leq j < i\}.$$

We similarly define $E_i(v)$. Then Eq. (7) implies that

$$f(e_i) \geq \max\{\max_{e_j \in E_i(u)} (f(e_j) + \omega(u)), \max_{e_j \in E_i(v)} (f(e_j) + \omega(v))\}. \tag{10}$$

Algorithm **Delta** finds a numbering $e_1, e_2, \ldots, e_m$ satisfying Eq. (9) and determines $f(e_1), f(e_2), \ldots, f(e_m)$ in this order so that $f(e_1) = 1$ and Eq. (10) holds in equality, that is,

$$f(e_i) = \max\{\max_{e_j \in E_i(u)} (f(e_j) + \omega(u)), \max_{e_j \in E_i(v)} (f(e_j) + \omega(v))\}.$$

Delta is similar to the Dijkstra's shortest path algorithm [2], and its details are as follows, where P is the set of edges e for which $f(e)$ have been decided.

Algorithm. Delta(G, f)

for every edge $e \in E$, let $f(e) := 1$; (initialization)
$P{:=}\emptyset$;
for $i1$ **until** m **do**
{
let $e_i = (u, v)$ be an edge $e \in E \backslash P$ with minimum $f(e)$;
$P := P \cup \{e_i\}$; ($f(e_i)$ is decided)
for every edge $e \in E(u) \backslash P$, let $f(e) := \max\{f(e), f(e_i) + \omega(u)\}$; (update $f(e)$)
for every edge $e \in E(v) \backslash P$, let $f(e) := \max\{f(e), f(e_i) + \omega(v)\}$; (update $f(e)$)
}
end for

Clearly **Delta** correctly finds an ω-edge-coloring f of G. For the graph G in Fig. 1, **Delta** finds the coloring f in Fig. 1 such that $\mathrm{span}(f) = 7 = \Delta_\omega(G)$, and hence f happens to be optimal. **Delta** decides $f(e_1)$, $f(e_2)$, ..., $f(e_7)$ in this order for the edge-numbering $e_1, e_2, ..., e_7$ depicted in Fig. 1.

We then prove that the coloring f obtained by **Delta** satisfies

$$\mathrm{span}(f) \le \Delta'_{1\omega}(G) + 1.$$

Obviously $f(e_1) = 1$ and $\mathrm{span}(f) = f(e_m)$. Let $e_m = (u, v)$, and let j be any integer in $[1, \mathrm{span}(f) - 1]$. Since j is not assigned to e_m by f, either $f(e_i) \le j \le f(e_i) + \omega(u) - 1$ for some edge $e_i \in E_m(u)$ or $f(e_i) \le j \le f(e_i) + \omega(v) - 1$ for some edge $e_i \in E_m(v)$. Therefore,

$$[1, \mathrm{span}(f) - 1] \subseteq \left(\bigcup_{e_i \in E_m(u)} B_1(e_m, e_i, u) \right) \bigcup \left(\bigcup_{e_i \in E_m(v)} B_1(e_m, e_i, v) \right)$$

and hence

$$\mathrm{span}(f) - 1 \le (d(u) - 1)\omega(u) + (d(v) - 1)\omega(v) = d_{1\omega}(e_m).$$

We have thus proved

$$\mathrm{span}(f) \le d_{1\omega}(e_m) + 1 \le \Delta'_{1\omega}(G) + 1. \tag{11}$$

From Eqs. (1), (2), (8) and (11) we have

$$\begin{aligned} \mathrm{span}(f) &\le d_{1\omega}(e_m) + 1 \\ &\le d_\omega(u) + d_\omega(v) - 1 \\ &\le 2\Delta_\omega(G) - 1 \\ &\le 2\chi'_\omega(G) - 1. \end{aligned}$$

Thus **Delta** has an approximation ratio smaller than two.

Using a binary heap [2], one can implement **Delta** so that it takes time $\mathrm{O}(m\Delta \log m)$, similarly as the Dijkstra's shortest path algorithm.

We thus have the following theorem.

Theorem 2. *For every graph G*

$$\chi'_\omega(G) \leq \Delta'_{1\omega}(G) + 1 \leq 2\Delta_\omega(G) - 1.$$

Algorithm ***Delta*** *finds in time* $\mathrm{O}(m\Delta \log m)$ *an* ω*-edge-coloring of* G *such that* $\mathrm{span}(f) \leq \Delta'_{1\omega}(G) + 1$*, and its approximation ratio is smaller than two.*

4 Edge-Degenerated Graphs

It is known that a "k-degenerated graph" has a vertex-coloring with $k+1$ colors [5]. In this section, we define a "k-edge-degenerated graph," and present an algorithm **Degenerate** to find an ω-edge-coloring f of a k-edge-degenerated graph such that $\mathrm{span}(f) \leq k+1$.

A graph G is called *k-edge-degenerated* for a non-negative integer k if G has an edge-numbering $e_1, e_2, ..., e_m$ such that $d_{2\omega}(e_i, G_i) \leq k$ for every index i, $1 \leq i \leq m$, where G_i is a subgraph of G induced by edges $e_1, e_2, ..., e_i$.

Since G_1 consists of a single edge e_1, we have $d_{2\omega}(e_1, G_1) = 0 \leq k$ and hence $\mathrm{span}(f) = 1 \leq k+1$ for an ω-edge-coloring f of G_1 such that $f(e_1) = 1$. This coloring f of G_1 can be extended to an ω-edge-coloring f of G_2 such that $\mathrm{span}(f) \leq k+1$. Repeating such an extention, **Degenerate** obtains an ω-edge-coloring f of $G = G_m$ such that $\mathrm{span}(f) \leq k+1$.

We shall prove that an ω-edge-coloring f of G_i, $i \geq 1$, with $\mathrm{span}(f) \leq k+1$ can be extended to an ω-edge-coloring f of G_{i+1} with $\mathrm{span}(f) \leq k+1$. Let $e_{i+1} = (u, v)$, then an integer $j \in [1, k+1]$ can be chosen as $f(e_{i+1})$ for the extention if and only if

$$j \notin \Big(\bigcup_{e_l} B(e_{i+1}, e_l, u)\Big) \bigcup \Big(\bigcup_{e_l} B(e_{i+1}, e_l, v)\Big) \tag{12}$$

where the unions are taken over all edges e_l of G_{i+1} that are adjacent to e_{i+1}, and hence $1 \leq l \leq i$. The cardinality of the set in the right hand side of Eq. (12) is bounded above by

$$d_{2\omega}(e_{i+1}, G_{i+1}) = (d(u, G_{i+1}) - 1)(2\omega(u) - 1) + (d(v, G_{i+1}) - 1)(2\omega(v) - 1),$$

and $d_{2\omega}(e_{i+1}, G_{i+1}) \leq k$ since G is k-edge-degenerated. Therefore, there always exists an integer $j \in [1, k+1]$ which can be chosen as $f(e_{i+1})$, and hence f can be extended to an ω-edge-coloring of G_{i+1} with $\mathrm{span}(f) \leq k+1$.

Algorithm **Degenerate** successively finds ω-edge-colorings of G_1, G_2, ..., $G_m (= G)$ in this order. Indeed it employs a simple greedy technique; when extending an ω-edge-coloring of G_i to that of G_{i+1}, $1 \leq i \leq m-1$, **Degenerate** always chooses, as $f(e_{i+1})$, the *smallest* positive integer j satisfying Eq. (12). For every edge e_l adjacent to e_{i+1} in G_{i+1}, let

$$B(e_{i+1}, e_l, x) = [\alpha(e_l, x), \beta(e_l, x)]$$

where x is u or v, $\alpha(e_l, x) = f(e_l) - \omega(x) + 1$ and $\beta(e_l, x) = f(e_l) + \omega(x) - 1$. Sorting the set $\{\alpha(e_l, x) \mid x \text{ is } u \text{ or } v,\ e_l \text{ is adjacent to } e_{i+1} \text{ in } G_{i+1}\}$ of $d(u, G_{i+1}) +$

$d(v, G_{i+1})-2$ integers, one can find the smallest integer j above in time $\mathrm{O}((d(u)+d(v))\log(d(u)+d(v)))$. Thus **Degenerate** takes time $\mathrm{O}(m\Delta \log \Delta)$.

The *ω-edge-degeneracy* $k_\omega(G)$ of a graph G is defined to be the minimum integer k such that G is k-edge-degenerated. Then, similarly as the case of the "vertex-degeneracy" [5], one can compute $k_\omega(G)$ as follows. Let $G_m = G$, and let e_m be an edge e in G_m with minimum $d_{2\omega}(e, G_m)$. Let G_{m-1} be the graph obtained from G_m by deleting e_m, and let e_{m-1} be an edge e in G_{m-1} with minimum $d_{2\omega}(e, G_{m-1})$. Repeating the operation, one can obtain an edge-numbering $e_1, e_2, ..., e_m$ of G, and $k_\omega(G) = \max_{1\le i\le m} d_{2\omega}(e_i, G_i)$.

Using a binary heap, one can compute $k_\omega(G)$ in time $\mathrm{O}(m\Delta \log m)$. Using a Fibonacci heap [2], one can improve the time complexity to $\mathrm{O}(m\Delta + m\log m)$.

Clearly $k_\omega(G) \le \Delta'_{2\omega}(G)$. Let $\Delta'_{2\omega}(G) = d_{2\omega}(e)$ for an edge $e = (u, v)$, then by Eqs. (1), (2) and (6) we have

$$\begin{aligned}\Delta'_{2\omega}(G) + 1 &= (d(u)-1)(2\omega(u)-1) + (d(v)-1)(2\omega(v)-1) + 1 \\ &= 2(d_\omega(u) + d_\omega(v)) - d(u) - d(v) - 1 \\ &< 4\Delta_\omega(G) \\ &\le 4\chi_\omega(G).\end{aligned}$$

We thus have the following theorem.

Theorem 3. *Algorithm **Degenerate** finds in time* $\mathrm{O}(m\Delta \log \Delta)$ *an ω-edge-coloring f of a k-edge-degenerated graph G such that* $\mathrm{span}(f) \le k+1$. *When $k = k_\omega(G)$, the approximation ratio of **Degenerate** is smaller than four.*

5 Algorithm Factor

C. E. Shannon showed that every graph G can be edge-colored with at most $3\Delta(G)/2$ colors [13], and it is known that such a coloring can be found in time $\mathrm{O}(m(n+\Delta))$ [9]. Therefore, by Eq. (4) we have

$$\chi'_\omega(G) \le 1 + (3\Delta/2 - 1)\omega_l$$

for every graph G, and an ω-edge-coloring f of G with $\mathrm{span}(f) \le 1+(3\Delta/2-1)\omega_l$ can be found in time $\mathrm{O}(m(n+\Delta))$. In this section we present an algorithm **Factor** of time complexity $\mathrm{O}(m\log \Delta)$.

One may assume that a graph $G = (V, E)$ is connected. Our third algorithm **Factor** finds an ω-edge-coloring f of G as follows.

(Step 1)

Partition E into $r(=\lceil \Delta/2 \rceil)$ subsets E_i, $1 \le i \le r$, so that the subgraph G_i of G induced by E_i satisfies $\Delta(G_i) \le 2$, and hence G_i consists of vertex-disjoint paths and cycles. (Such a partition is called a *factorization* of G to subgraphs G_i with $\Delta(G_i) \le 2$.)

(Step 2)

Using the algorithm in Section 2, obtain an optimal ω-edge-coloring f_i of G_i for each index i, $1 \le i \le r$.

(Step 3)
Obtain an ω-edge-coloring f of G by combining f_i, $1 \le i \le r$.

We now describe the details of these three steps.

[Step 1]
G contains an even number of vertices of odd degree. Join them pairwise by dummy edges, and let G' be the resulting Eulerian graph. (G' may have multiple edges even if G has no multiple edges.) Then the maximum degree $\Delta(G')$ of G' is an even number. More precisely, $\Delta(G') = 2r$ for an integer

$$r = \lceil \Delta(G)/2 \rceil. \tag{13}$$

Let C be an Eulerian circuit of G', which passes through every edge of G' exactly once. We then construct a bipartite graph $B = (V_B, E_B)$ according to the direction of edges in C. The left vertices of B are the vertices of G, and the right vertices are their copies. All edges of B are copies of the edges of G. B has an edge joining a left vertex u and a right vertex v if and only if the Eulerian circuit C passes through an edge (u, v) of G from u to v. (A similar construction of B has appeared in [6].) For every vertex $v \in V$, at most r edges emanate from v in C and at most r edges enter to v. Therefore, $\Delta(B) \le r$ and hence B has an ordinary edge-coloring with r colors. Let $E_{B_1}, E_{B_2}, ..., E_{B_r}$ be the color classes of the edge-coloring of B. Let $E_1, E_2, ..., E_r$ be the subsets of E which correspond to $E_{B_1}, E_{B_2}, ..., E_{B_r}$, respectively. Then the subgraph G_i, $1 \le i \le f$, of G induced by E_i satisfies $\Delta(G_i) \le 2$ since E_{B_i} is a matching in B.

[Step 2]
By Theorem 1 one can find an optimal ω-edge-coloring $f_i : E_i \to \mathbb{N}$ of G_i in linear time, and f_i satisfies

$$\mathrm{span}(f_i) = 1 + \max\{\omega_l(G_i), \omega_s(G_i)\} \tag{14}$$

for every index i, $1 \le i \le r$.

[Step 3]
When combining f_i, $1 \le i \le r$, to f, we shift up $f_i(e)$ uniformly for every edge $e \in E_i$. More precisely, let

$$\begin{aligned} f_i(e) := f_i(e) + \mathrm{span}(f_1) + (\omega_l(G) - 1) + \mathrm{span}(f_2) + (\omega_l(G) - 1) \\ + ... + \mathrm{span}(f_{i-1}) + (\omega_l(G) - 1) \end{aligned}$$

for each index i, $2 \le i \le r$. Then, simply superimposing $f_1, f_2, ..., f_r$, one can obtain an ω-edge-coloring f of G; $f(e) = f_i(e)$ if $e \in E_i$.

We then evaluate $\mathrm{span}(f)$ for the coloring f obtained by **Factor**. Clearly

$$\mathrm{span}(f) = \sum_{i=1}^{r} \mathrm{span}(f_i) + (r - 1)(\omega_l(G) - 1). \tag{15}$$

Since $\omega_s(G) \le 2\omega_l(G)$ and $\omega_l(G_i) \le \omega_l(G)$ and $\omega_s(G_i) \le \omega_s(G)$ for every index i, $1 \le i \le r$, by Eqs. (13), (14) and (15) we have

$$\begin{aligned} \mathrm{span}(f) &\le r(1 + \max\{\omega_l(G), \omega_s(G)\}) + (r-1)(\omega_l(G) - 1) \\ &= 1 + r(\omega_l(G) + \max\{\omega_l(G), \omega_s(G)\}) - \omega_l(G) \\ &\le 1 + (3r-1)\omega_l(G) \\ &= 1 + (3\lceil \Delta(G)/2 \rceil - 1)\omega_l(G). \end{aligned} \tag{16}$$

Assume now that G satisfies Eq. (5). Then, by Eqs. (5) and (16) we have

$$\mathrm{span}(f) \le \begin{cases} 3\Delta_\omega/2 + (\omega_l(G) - 1)/2 & \text{if } \Delta \text{ is even;} \\ 3\Delta_\omega/2 + 2\omega_l(G) - 1/2 & \text{otherwise.} \end{cases} \tag{17}$$

Since $\Delta_\omega \le \chi'_\omega$ by Eq. (2), the approximation ratio of **Factor** is near to 3/2. Especially when Δ is even, one may assume that $\Delta \ge 4$, and hence by Eqs. (5) and (17) we have

$$\mathrm{span}(f) \le (5\Delta_\omega - 2)/3 < 5\chi'_\omega/3$$

and hence the approximation ratio is smaller than 5/3.

The most time-consuming part of **Factor** is Step 1, in which one must find an ordinary edge-coloring of a bipartite graph $B = (V_B, E_B)$ with $\Delta(B)$ colors. The coloring can be found in time $\mathrm{O}(|E_B| \log \Delta(B))$ [1]. Since $|E_B| = m$ and $\Delta(B) \le r = \lceil \Delta(G)/2 \rceil$, **Factor** takes time $\mathrm{O}(m \log \Delta)$.

We thus have the following theorem.

Theorem 4. *For every graph G, algorithm **Factor** finds in time $\mathrm{O}(m \log \Delta)$ an ω-edge-coloring f of G such that $\mathrm{span}(f) \le 1 + (3\lceil \Delta/2 \rceil - 1)\omega_l$. If $\Delta_\omega(G) = 1 + (\Delta - 1)\omega_l$, then*

$$\mathrm{span}(f) \le \begin{cases} 3\Delta_\omega/2 + (\omega_l - 1)/2 & \textit{if } \Delta \textit{ is even;} \\ 3\Delta_\omega/2 + 2\omega_l - 1/2 & \textit{otherwise.} \end{cases}$$

If $\Delta_\omega(G) = 1 + (\Delta - 1)\omega_l$ and Δ is even, then the approximation ratio is smaller than 5/3.

References

1. Cole, R., Ost, K., Schirra, S.: Edge-coloring bipartite multigraphs in $O(E \log D)$ time. Combinatorica 21(1), 5–12 (2001)
2. Corman, T.H., Leiserson, C.E., Rivest, R.L., Stein, C.: Introduction to Algorithms. MIT Press and McGraw Hill, Cambridge (2001)
3. Gabow, H.N., Nishizeki, T., Kariv, O., Leven, D., Terada, O.: Algorithms for edge-coloring graphs, Tech. Rept. TRECIS 41-85, Tohoku Univ. (1985)
4. Holyer, I.J.: The NP-completeness of edge coloring. SIAM J. on Computing 10, 718–721 (1981)
5. Jensen, T.R., Toft, B.: Graph Coloring Problems. John Wiley & Sons, New York (1995)

6. Karloff, H., Shmoys, D.B.: Efficient parallel algorithms for edge-coloring problems. J. of Algorithms 8(1), 39–52 (1987)
7. McDiamid, C.: On the span in channel assignment problems: bounds, computing and counting. Discrete Math 266, 387–397 (2003)
8. McDiamid, C., Reed, B.: Channel assignment on graphs of bounded treewidth. Discrete Math 273, 183–192 (2003)
9. Nakano, S., Nishizeki, T.: Edge-coloring problems for graphs. Interdisciplinary Information Sciences 1(1), 19–32 (1994)
10. Nishikawa, K., Nishizeki, T., Zhou, X.: Bandwidth consecutive multicolorings of graphs. Theoretical Computer Science 532, 64–72 (2014)
11. Obata, Y., Nishizeki, T.: Approximation Algorithms for Bandwidth Consecutive Multicolorings. In: Chen, J., Hopcroft, J.E., Wang, J. (eds.) FAW 2014. LNCS, vol. 8497, pp. 194–204. Springer, Heidelberg (2014)
12. Pinedo, M.L.: Scheduling: Theory. Springer Science, New York (2008)
13. Shannon, C.E.: A theorem on coloring the lines of a network. J. Math. Physics 28, 148–151 (1949)
14. Stiebitz, M., Scheide, D., Toft, B., Favrholdt, L.M.: Graph Edge Coloring. Wiley, Hoboken (2012)
15. West, D.B.: Introduction to Graph Theory. Prentice-Hall, Englewood Cliffs (1996)