Is Logic Universal or Hierarchical?

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Abstract Some conceptions of logic claim that they are universal. By contrast, I assume that the applications of any logic are central to its conception, so that it has to comprise a hierarchy of its metalogics, metametalogics, ..., indefinitely extended but never capped off with some universal logic. I also advocate for the distinction between parts and moments of a multitude as key to this conception, and I query the assumption that set theory provides the most general means of handling collections of objects.

Keywords Logics · Hierarchies of metalogics · Logical pluralism · Self-reference · Multisets · Universal logic · Paraconsistent logics

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> [...] the attempt to formulate the foundations of logic is rendered arduous by a corresponding 'logocentric' predicament. *In order to give an account of logic, we must presuppose and employ logic.*
> —Henry Sheffer [22, pp. 227–228]

1 On Logical Knowledge

Logic is a strange subject. It seems to be present, to a greater or lesser extent, in all discourses on any topic, especially concerning the development of a line of reasoning and comparisons between rival lines; yet it is elusive, hard to disentangle from the discourse itself, which usually is not logic but anything else (for example, proofs of theorems in geometry, or assessing the sweetness of wines, or studying the history of violin-making in Belgium ...). I take logic to be concerned with the properties of correct and incorrect reasoning from premises to conclusions in a discourse, whether in a formalised or a natural language. My approach may be called 'naturalistic', since I am much influenced by previous attempts to characterise logic; every suggestion that I offer below has been advocated somewhere. However, the combination that I offer might be new, especially for this reason.

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I do not ask ‘what is logic?’, for it is subject to the usual criticisms concerning essentialism; instead I pose the much more modest question ‘where is logic?’, especially in discourses and in applications (for which I use the word ‘settings’). The difference between the two questions is *very* considerable; in particular, the fact that the role of logic in its settings is as important as its ‘pure’ characteristics makes this approach quite different from others. Another basic difference is that I reject the common claim that ‘logic dictates that ...’ to justify some line of reasoning in a setting; for, with the important exception presented in Sect. 3, in my characterisation logic contains *none* of its settings, and so cannot dictate anything about any of them. Naturally this dependent status does not discourage or invalidate the development of logic ‘purely’ on its own, without appeal to a setting. However, I am struck by the paucity or superficiality of the discussion of logic as such in many logic textbooks!

Some assumptions and limitations need to be confessed before the characterisation of logic is described. Firstly, the word ‘logic’ refers to the subject, not to other uses such as in ‘the logic of the situation’ or ‘logistics’. Secondly, well-formed propositions are translinguistic objects; languages have ‘sentences’, which are uttered as ‘statements’. Thirdly, I shall consider only propositions in declarative mood, ignoring logics concerned with questions or commands (which are ignored too often). Fourthly, I grant epistemological priority to propositions over terms. Fifthly, I do not treat probability logic, often called ‘inductive logic’ in the older literature, for it is only one (or some) important logic among many. Sixthly, logical pluralism obtains, so that I use ‘logics’, ‘a logic’ and ‘any logic’, and ‘logical knowledge’ for the total ensemble of logics.

2 Specification

2.1 List

Although I do not try to ‘capture’ logic, I need to furnish a working list of topics that a logic should contain. Each topic is necessary for logichood, none is sufficient.

1) Well-formed propositions as translinguistic objects and their calculi (languages have ‘sentences’, uttered as ‘statements’).
2) Connectives, including their inter-definability.
3) Propositional functions, covering also relations and attributes, and their calculi.
4) Existential and universal quantification of individuals, propositions and propositional functions, to any order; each quantifier to reign over some specified universe of discourse.
5) Assertion or denial of the truth-values assigned to its propositions.
6) (In)valid reasoning, covering (at least) deduction, implication, entailment, inference, derivation, proof, natural deduction and logical consequence.

2.2 Comments

Many issues, especially philosophical ones, play important roles in discussions of logic. Cases include whether a logic is construed epistemologically or ontologically or conventionally or predicatively . . . ; whether propositions are to be understood positivistically or Platonically or . . . ; whether quantification is construed objectually or substitutionally: the deployment or avoidance of non-monotonic reasoning; the statuses that may be imposed upon symbols (schematic letters, quantifiable variables, constants, semiotic families of signs, and so on); on the extra-logical factors that may be associated with the specification of truth-values (for example, empirical testing). I hope that my modest question allows me to remain neutral on them.

I also hope that the various kinds of definitions needed in logics (which again are often ignored) can be formulated; not only nominal ones but also contextual definitions, definitions under hypothesis, creative definitions within an axiom system (that can help prove an otherwise unprovable theorem that contains neither the defined term nor any term dependent upon them), definition by mathematical induction and more generally by recursion, and allegedly essentialist ones.

3 Logics as Momental Theories

Logics have long been considered as concerned with forms; important examples include the classification of (in)valid syllogistic inferences in Aristotle and many later authors, and the distinction between form and matter that was emphasised by various logicians in the 19th century. 'Schema(ta)' and 'logical skeleton' have also been used; also 'structure' is sometimes used, although there are various theories outside logics that exhibit structures; for example, abstract algebras in mathematics. This status is often exhibited by writing a proposition in forms such as 'all . . . are . . . '.

This approach is attractive, but lacks a clear means of distinguishing forms from non-forms in the first place; so I call upon a powerful philosophical distinction made by phenomenologists between independent and dependent parts (or 'moments') of a totality. It was applied widely to logics and mathematics especially by the philosopher Edmund Husserl from the 1890s, and is explored in detail in [24]. An example is the tail of a dog and its weight: the tail can be considered on its own, but its weight *necessarily* pertains to it as the totality. Both parts and moments may have their own parts and moments (the fur on the tail, the weight of that fur, the owner's surprise over that weight, and so on). While parts of parts of . . . of a totality can all be expressed in set theory using membership and inclusion; moments cannot, for they do not belong to the totality in the same way. However, one can speak of a set of moments for a given totality.

Here a logic is taken to be 'momental' relative to its setting and always subordinate to it in the way just described; talk of 'forms' and 'structures' is momental. To emulate a characterisation of philosophy made by Popper [17, p. 73], any logic is 'deeply rooted in non-[logical] problems', namely, its settings. This status helps to explain *why* a logic is so elusive; although it is omnipresent, it is always momentally dependent upon its settings for expression.

The distinction between parts and moments occurs also within a logic. For instance, the propositions 'P' and 'Q' are parts of their conjunction 'P & Q', but '&' is a moment of it. So are all other connectives, and also quantifiers, with their laws of combination indicated by their scopes.

4 The Central Importance of Hierarchies

Unlike a part, no 'host' logic L exists in isolation. Instead it *unavoidably* requires a metalogic $\mathrm{M}(L)$ which, among other duties, is to express inference and assertion, and to (fail to) show consistency and completeness. When it takes itself as a setting, it is the site not only for metapropositions such as the consistency of L but also, for example, for paradoxes like 'all propositions are untrue'. I use the plural 'metalogics' which allows for the fact that L can generate several: the formalist kind that David Hilbert developed is a major one, but there are others, often not formalised, such as comparisons with other logic, ways of teaching L, and its history. One reason why hierarchies cannot be avoided is that they are needed already in the list; assertion and inference belong to $\mathrm{M}(L)$. Each $\mathrm{M}(L)$ has metametalogics $\mathrm{MM}(L), \ldots$, and so on up, thereby generating a denumerable 'first-order' hierarchy

$$\mathrm{H}(L) := \langle \mathrm{M}(L), \mathrm{MM}(L), \ldots \rangle.$$

$\mathrm{H}(L)$ is a well-ordered sequence of collections of logics; while each member is a logic momental of all its predecessors. It is not itself a logic, nor is any subset. It is discussed in its own momental metatheory $\mathrm{MH}(L)$, which generates a 'second-order' hierarchy

$$\mathrm{HH}(L) := \langle \mathrm{MH}(L), \mathrm{MMH}(L), \ldots \rangle, \quad \text{and so on,}$$

in which, again, each member logic is a moment of all its predecessors. 'Meta... logic' names any logic.

As usual, the logic that forms each meta... logic in a hierarchy can, but does not have to, differ from each other. Among examples, the metalogic of bivalent logic is usually also bivalent and that of fuzzy logic is fuzzy; but the 'mathematics of the second order' of L.E.J. Brouwer's intuitionistic mathematics was bivalent [4, p. 61].

The assumed existence of the hierarchy will please supporters of Tarskian theories, Tr of truth, who will use it to lay out their solutions to paradoxes. However, it does not endorse or reject any proposed solutions to paradoxes; all may be tried, not only Tr but also types, say, or limitation of size. Each logic in a hierarchy may contain paradoxes of its own, awaiting attention of some kind.

5 No Universal Logic(s)

Some logicians advocate a 'universal' logic U, which is maximal, incapable of extension, embracing all known logic; but there cannot be such a logic, *since U cannot contain* $\mathrm{H}(U)$. The reply might be that $\mathrm{H}(U)$ is universal; but this is no defense, since H(U) has to be talked about, and this can only be done in its own momental metalogic M(U), which generates its own hierarchy HH(U) of H(U), and so on forever, but with *no* universal theory at the 'end'. Each logic has limits of various kinds (of settings, for example), but logical knowledge has no limits: the only appropriate talk of universes in logics is about expanding ones!

My doubts about universal logic resemble the difference between Georg Cantor and Bertrand Russell on transfinite arithmetic. In his theory of transfinite ordinals and cardinals Cantor found it basically a mistake to imagine that there could be a largest of either kind of number; neither has an immediate predecessor, just as no $(\aleph_0 - 1)$ or $(\omega - 1)$ can be found just before his transfinite sequences. The reason is that if one offered him any set S, then he would invoke his power-set proof-method and exhibit its power set, which has a larger cardinality or ordinality than S. It was Russell who proposed the (sic) set of all sets as a universal set, applied Cantor's power-set argument to it, and thereby found his paradox [8]. Cantor's reply to him could have been: 'serve you right'. In the same way, *whatever* logic a universal logician proposes as universal, I invoke its own hierarchy, which it cannot contain.

There is also no possibility of finding some universal category that embraces all the roles that were assigned to the list, as some Kantian and Hegelian logicians hoped to find in judgements. On the contrary, the roles include both logical and metalogical notions, so that the hierarchy is already in use.

6 Logics and/in Mathematics

A major concern for many logics, especially since the 1870s or so, is their relationships to branches of mathematics, in particular, those with a strong philosophical component and/or some generality. An interest among universal logicians is the characterisation of logics relative to mathematical theories: '*Universal logic* [...] is not new logic, but a general theory of logics, considered as mathematical structures' [3, p. vii]. However, take a logic L and consider the mathematical theories T relative to which it is considered: they have *their own* metatheories, which are bound to involve some logical notions. Mathematicians seldom explicitly list the logical needs of their theories, but in this context it is essential, and the charge of vicious circles must be examined; any item in the hierarchy H(T) of T cannot be in L. In particular, it is likely that H(T) includes some basic gear from bivalent (meta)logic, such as consistency and the law of excluded middle, and/or some tools from set theory. For example, Lutz Straßburger [25] seeks to characterise logic (and also proof) by attractive means E, which include ordering and several elements from set theory; but they appear to assume bivalence, and so cannot capture any logic (or theory of proof) that contains elements of E.

A reply might appeal to a distinction between 'working' everyday arithmetic that, among other duties, numbers the pages of a book on the foundations of arithmetic in which, without begging any questions, the author lays out on page 58 some foundational definition of cardinals from 0, including 58. One might make an analogous distinction between everyday set theory and topology and their foundational formulations, and remove them from the logic underlying T. However, none of these every-day theories in the elements of E helps the universalists: for on the contrary, the influence must go in the opposite direction, *from L to* them.

7 On Paraconsistent Logics: Gaps or Gluts?

Those logics with aspirations for universality need special attention. One influential case is paraconsistency, an example of the 'truth-glut' kind of logic, where in the context of bivalent logics some propositions, especially those that generate paradoxes, are both true and untrue. The reaction to paradoxes is accommodation: rejected is the 'explosion' rule

$$P \text{ and not } P; \quad \text{therefore } Q$$

for any propositions P and Q. An impressive range of theories has been found [2]. 'Dialethism' is the name given to the version of these theories that holds further that contradictions and paradoxes are both true and untrue, and permits P to be substituted by (P is true) and vice versa in well-formed contexts. Self-reference is handled similarly: the property 'is untrue of itself' is true of itself and also untrue of itself. While generality as such is not a prime property or claim for either logic, the scale of their potential application is extensive enough to excite quiet widespread doubts similar to those over universal logics: if a paraconsistent logic is to be really logical, then it has to be accompanied by its hierarchy, just like any other logic, including metaproposals such as paradoxes being both true and untrue. It is advocated especially by Graham Priest ([18] and elsewhere).

'Suppose, then', says the dialethist, 'that we allow sentences to take as semantic values one of the corresponding four *subsets* of the set {true, false}'; then the liar paradox quickly shows that if any proposition a is both true and untrue, then so are both not a and (a and not a) [18, pp. 4–5]. However, the assumption of truth-glut, especially in dialethic contexts, is dubious. The 'truth-gap' alternative, where some propositions are neither true nor untrue, is much more convincing: from 'it is true that this proposition is untrue' we deduce that 'it is untrue that this proposition is untrue' and also vice versa, suggesting that *neither* of these propositions take any truth-value, although each of them is interpretable in its metalogic. In a fine comparison of truth-glut and truth-gap theories Hartrey Field is strongly critical of the gluttists, especially the dialethists [6, especially Chapts. 7–8 and Part 5].

8 Sortality and Syntax

The preference for gaps over gluts benefits two important but often-overlooked aspects of the predicate calculus.

Firstly, gaphood accords with the theory of 'sortal terms', which Russell captured beautifully in a letter to Louis Couturat in 1906: 'To say, for example, that the principle of the excluded middle is not red, would be to utter a nonsense, and not a truth' [23, p. 604]. That is, the subject of this ill-suited proposition does not belong to the range of significance (or 'sort') of its predicate. The same can be said of the negation of the proposition, so that neither proposition takes a truth-value. Elaborating sortality is a challenging task, but it constitutes an important point of interaction between logics and semantics [15].

Secondly, gaps bear upon another much underrated aspect, namely, specifying ranges of significance of predicates called 'syntactic relativisation'. Here changing the range, or the universe of discourse of a logical argument, can convert an ill-sorted proposition to an untrue one.[1] For example, the pair of propositions 'all raccoons (dis)like Sundays' is ill-suited if its range of significance contains only raccoons; but if the range were extended to, say, all entities, then the propositions become 'for all x, if x is a raccoon, then x (does not hate) hates Sundays', which can be regarded as untrue.

9 On Collections, Especially Multisets

Throughout the history of logics collections have been handled. From antiquity up to the algebraic logicians such as George Boole and C.S. Peirce the usual theory was based upon some version of the part–whole distinction; the name 'mereology' is attached to some versions. I name this type of collection 'class'. From the late 19th century onwards it was challenged by Cantor's set theory, in which things belonged to sets and their membership was distinguished from the inclusion of subsets. A major difference was that in set theory an object is distinguished from its unit set, whereas in part–whole theory this distinction does not hold. Since around 1900 mathematical logicians have assumed that set theory is the only way to handle collections and they seem to have devised various axiomatic versions. However, other theories should be noted; not only mereologies but also, for instance, constructive set theories and fuzzy set theory.

Thus we have a meta-collection of theories of collections, which constitutes a body of knowledge of its own, *not* reducible to *any* logic. Indeed, as with mathematical theories, each will have to rely upon a logic and its hierarchy.

Thus I reject the claims of the universality of set theory, whether in general or embodied in laicisms, or whether related to mathematics ('set theory is the foundation of all mathematics [. . .]' according to Felix Hausdorff [12, p. 1], a noted despiser of logic!). However, my position does not rely on any of the considerations treated in this paper, but on pointing out a severe limitation in set theory itself: namely, that an object can be a member of a set *only once* (and analogously in mereologies, that a part is a part only once). This restriction severely limits the applicability of both kinds of theory, for it is very easy to find contexts in which multiple partship or membership is required, and no order is imposed upon the repeated occurrences: for example, repeated zeroes of polynomials and of latent roots of matrices, the repeated appearance as prime factors of many integers such as 16 and 28, and tallying the value of the coins in my purse. We cannot express such properties in terms of either theory; for example, the factors of 16 are 2, 2, 2 and 2, which forms neither a class

[1] See [21]. [13] contrasts the need for sortal terms in linguistics with their superfluity in mathematical logic; I regard them as relevant to any context involving predicates.

nor a set. It is just as easy to find cases in ordinary life or circumstances: for instance, words are multisets of letters and propositions of words.

Whatever logic is being developed with some wide-ranging generality or omnipresence intended, multisets *must* be used, that is, collections to which a member or part can pertain *more than once*. It is a total mystery to me that such an obvious point is not 'universally' known and accepted long ago; but multisets are still almost unknown among mathematicians and logicians [11].[2] Computer scientists are rather better served, due to the advocacy by Donald Knuth ([14] and elsewhere).

10 On Relevance Logics

One of Fields's criticisms of dialethism was Priest's rogue conditional [6, pp. 371–375]. This is unfortunate, because implication provides a link across to relevance logics, where the antecedent and consequent must be related in some way, such as having a proposition or a propositional function in common; for if in addition the antecedent is contradictory, then explosion cannot occur. The range of these logics provides a framework capable of formulating not only bivalent but also intuitionistic and modal logics[3] (always accompanied by hierarchies, of course); perhaps sufficiently large to accommodate a universal logic. They are paraconsistent logics, for the existence of a contradiction will not cause explosion. This follows from the fact that a conditional with a contradictory antecedent that does not share any propositional or predicate letters with the consequent cannot be true (or derivable).

Another aspect of relevance logic is its relationship to *logic diagrams*, which Dejnocka argues is much closer than has been realised [5]. They have a substantial history, and in recent years have gained a remarkably wide range of roles (see, for example, [1]). Some diagrams are partly representational (such as Euler and Venn diagrams), some are schematic (such as arrow diagrams). So what is the status of diagrams in logics?

Self-reference arises again, as *self-referencing self-reference*, which is so elusive that it seems to escape all hierarchies (of hierarchies of ...). Does it need a third dimension? If so, comprising what? Does it arise within some other logics that may have potential for universality?

11 On the History of Logical Hierarchies and Logical Pluralism

A.N. Whitehead and Russell were universal logicists in two respects: firstly, they regarded bivalent logic monistically as the only one; secondly, they made no effort to isolate its metalogical components (as we could construe them): [22] recognised the consequences at the heart of this article. Understandably puzzled, Ludwig Wittgenstein had already at-

[2] A good survey of applications of multisets is given in [27]. A short historical review of their presence and especially absence is provided in [11].

[3] A good formulation of relevance logics is available in [16] and [28].

tempted in his *Tractatus* to specify logic independently of the logicist thesis and imposed his own monism. Pondering the limitations of thought and logic, he denied that one could think beyond the limits; in particular, 'It is clear: the logical laws may not themselves fall again under logical laws'. He also distinguished between what can only be shown and what can be said (including this distinction, apparently!) [29, clause 6.123, see also especially clauses 4.1212, 5.6, 5.61, 5.621, 6.21, 7].

In his introduction to the essay, Russell noted this feature and proposed as an alternative position 'that every language has, as Mr. Wittgenstein says, a structure concerning which, *in the language*, nothing can be said, but that there may be another language dealing with the structure of the first language, and having itself a new structure, and that to this hierarchy of languages there may be no limit' [20, p. xxii]. He rejected the enterprise of thinking about limits and limitations; in contrast, Wittgenstein's commitment to it was to be an important motivation for forming dialethic logics in the 1980s, especially with Priest [18, especially Chapt. 12].

Wittgenstein was not the first philosopher to focus upon limitations of various kinds; in an excellent survey Priest discusses several predecessors, including Aristotle, Immanuel Kant and G.F. Hegel. In those days, the usual view of logics was monistic, in favour of some version of bivalent logic, and metalogics were not distinguished from their host logic. However, now that hierarchies are available, the doubt over a supposedly universal logic carries over to paraconsistent logics: each of them has its hierarchy, and there is no limiting case.

By contrast to Russell's bewilderment and Wittgenstein's disdain, great improvements were made by many others from the late 1920s and early 1930s, when it was recognized not only that metalogic was distinct from its host logic, but also that this distinction was of central importance. Among the leaders in this direction were David Hilbert with various papers in the 1920s, Kurt Gödel's paper [7] on incompletability (after reading it Rudolf Carnap proposed 'metalogic' as a technical term in 1931), and Alfred Tarski (and maybe some other Polish logicians as well) on 'metalanguage'.

It would be a pleasure to judge Russell as one of the father figures of this change, but this is not possible. His rebuff of Wittgenstein was one of his finest contributions to philosophy, but he *never* recognised its importance; in particular, it did not suggest to him any analogous hierarchy of logics and metalogics. Neither did his hierarchy of propositions in – that is, within – the theory of types that he used to solve the paradoxes. Thus he never understood Gödel's first (meta)theorem; not only did he always misstate it as applicable to all mathematical theories (a frequent error!) but especially he thought that his hierarchy of languages solved some 'puzzle' that it posed rather than being essential to its statement.[4] He always remained a logical monist with no hierarchy – one of the most striking ironies in the history of philosophy and logics.[5]

From the 1930s onwards metalogic, and metatheorising in general, became a central feature of logics and the foundations of mathematics. However, hardly any study seems to

[4] For discussion and evidence of this situation see [9, pp. 565, 592–593]. Russell seems to have ignored Gödel's second theorem.

[5] A similar position may be predicated of Gottlob Frege, but it differs from the absolutism imposed on him by Jean Van Heijenoort.

have been made of metametalogics, metametametalogics, . . . , metametametamathematics; search engines find very few occurrences of these words. I think this is the territory to explore; it is capable of indefinite extension and iteration, but with no 'universal' logic embracing it 'all' . . .

The reluctance of logicians to explore hierarchies contrasts strongly with the keenness for logical pluralism [10]. I share this enthusiasm, to the extent that I find the lateness of its arrival and slow progress in the early 1900s hard to understand.[6] The millennia-long history of over-looking temporal logics is especially extraordinary. Surely it is obvious, not only now but some millennia ago, that we very often make statements about sequences of actions and decisions effected *in some order in time*, with 'and' meaning 'and then'; for example, 'he opened the door and walked down the corridor', which is not logically equivalent to 'he walked down the corridor and opened the door'. However, this is not bivalent logic because conjunction is not commutative.[7] Yet temporal logics were not even among the non-bivalent logics to develop first.

12 The Glut of Foundational Notions

The main features of this conception are as follows. A logic has six offices to fulfil, involving propositions, connectives, propositional functions, quantification, deduction, assertion and truth-values; it comes with an attached hierarchy of meta. . . logics, which is momental internally and also relative to the setting in which it is used; there is no universal logic; the proposition is the prime concept; no logic includes any theory of collections (including multisets); logical pluralism abounds but hierarchies rarely feature above the meta level.

Numerous notions are now involved in the study of foundational theories in logics and mathematics. Taking in a few topics that have not been explicitly mentioned, at least the following are involved (where the grouping of topics is not necessarily defensible):

- logics: metalogics, hierarchies, form and matter, structure, parts and moments;
- logical monism and pluralism, universality, paraconsistency, generality;
- propositions, connectives, assertion, truth-values, theories of truth;
- implication, (rules of) inference, deduction, entailment, relevance, semantic tableaux;

[6] I am struck by the conformities between my advocacy of logical pluralism and hierarchies and the criticisms of the biologist Robert Rosen of the dominant status assigned to physics among the sciences, which for him forbade adequate engagement with biology and the life sciences in general. He associated physicalism with, for example, the foundationalists' preference for syntax over semantics and the advocacy of Hilbert's metamathematics. (He could have added the philosophy behind Russell's logicism, but not Whitehead's.) In his alternative approach Rosen used other devices from logic and foundations, especially relations and category theory, in order to specify minimal capacities that a material system should possess in order to be one of the simplest functional organisms normally regarded as being 'alive' [26].

[7] This difference is obtained independently of any construal of time itself (whether relative or absolute, for example, or as conceived in relativity theories). One must distinguish temporal logics from the expression of the temporal order of events in terms of ordered sets *within* bivalent logic. W.V. Quine does not do this in his treatment of tense in [19, pp. 30–31], the only attention paid to time in his book on the 'Philosophy of Logic'. He staunchly supported monism for bivalent logic while rejecting Russell's all-purpose 'implication'.

- non-monotonic reasoning, logical consequence;
- propositional functions, relations, attributes;
- quantification, relativisation of syntax, universe of discourse, (un)sortal predicates;
- self- and cross-reference, paradoxes, contradictions, fallacies, vicious circles;
- metamathematics, axiomatisation, proof theory, formalism;
- various senses of completeness, independence, consistency, constructivity;
- modality, impredicativity, temporality, vagueness;
- logicism, types, definite descriptions;
- various theories of collections, infinities, order-types, diagonalisation;
- vagueness, fuzziness;
- intension and extension;
- various senses of existence, judgments, analyticity, identity;
- model theory: categoricity, compactness, soundness;
- symbolism, well-formedness, semiotics, abstract algebras, operator algebras;
- syntax, logic diagrams, various kinds of definition;
- various kinds of foundations of mathematics.

The situation shows the merits of pluralism; but when there is *such* prolificacy, how do topics relate to each other?[8] Take a few of the principal topics: set theories, metamathematics, model theory, axiomatisation, mathematical logics, arithmetic, abstract and operator algebras and category theory. I have tried to find a comprehensible representation or summary of their relationships, but in vain; they seem to be extraordinarily complicated, with no powerful strategies for simplification.

Further, as was mentioned in Sect. 4 in connection with mathematics in logics and logics in mathematics, a serious danger lies in vicious circles, especially if a logic contains and also uses some logic. The only clear conclusion is that no universal logic will accommodate all, including itself. There does not appear to be much effort to tackle either question.

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[8] While working on this point I received the last catalogue of books on ‘Logic and Foundations’ published by the American Mathematical Society of America. About 50 books were listed, from classics of the early 20th century to a section of recent works. The panorama of topics covered was beyond assessment.

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