Chapter 6
Canonical Forms of Structured Matrices and Pencils

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Abstract This chapter provides a survey on the development of canonical forms for matrices and matrix pencils with symmetry structures and on their impact in the investigation of application problems. The survey mainly focuses on the results from three topics that have been developed during the past 15 years: structured canonical forms for Hamiltonian and related matrices, structured canonical forms for doubly structured matrices and pencils, and singular value-like decompositions for matrices associated with two sesquilinear forms.

6.1 Introduction

Eigenvalue problems frequently arise in several applications from natural sciences and industry and therefore the corresponding theory is a fundamental topic in Linear Algebra, Matrix Theory, and Numerical Analysis. The practical applications typically lead to matrices, matrix pencils, or matrix polynomials with additional symmetry structures that reflect symmetries in the underlying physics. As a consequence also the *eigenstructures* (i.e., eigenvalues, eigenvectors, root vectors, Jordan blocks, singular blocks and other invariants as, e.g., algebraic, geometric, and partial multiplicities) of such matrices, matrix pencils, and matrix polynomials inherit certain symmetries or patterns. As these reflect the nature and characteristics of the original application problems, they play critical roles both in theory and practice.

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Typically, the solution of structured eigenvalue problems is a challenge, because there is demand for the design of new algorithms that are structure-preserving in each step, so that the corresponding symmetry in the spectrum is maintained in finite precision arithmetic and the obtained results are physically meaningful [48]. Simple variations of the QR algorithm or methods based on standard Krylov subspaces may not be sufficient to achieve this goal so that new ideas and concepts need to be developed. This requires a deeper understanding of the corresponding eigenstructures and therefore the derivation of structured canonical forms is essential. It is the aim of this chapter to review such forms for some particular classes of structured matrices or matrix pencils.

The most important and well-known matrices with symmetry structures are probably real or complex Hermitian, skew-Hermitian, and unitary matrices. Still, there are many other kinds of important structured matrices like complex symmetric, skew-symmetric, and orthogonal matrices as well as nonnegative matrices all of which are discussed in the classical books [13, 14]. In this chapter, we focus on structured matrices that are self-adjoint, skew-adjoint, or unitary with respect to an inner product associated with a possibly indefinite Hermitian or skew-Hermitian matrix and give a brief review on their theory, also including the corresponding matrix pencils that generalize those structures. We do not consider the corresponding structured matrix polynomials in this chapter, but refer the reader to Chap. 12of this book instead.

Let $\mathbb{F}$ be either the real field $\mathbb{R}$ or the complex field $\mathbb{C}$. Suppose $M \in \mathbb{F}^{m\times m}$ is an invertible Hermitian or skew-Hermitain matrix, and define the bilinear or sesquilinear form

$$[x, y]_M = x^* M y =: [x, y], \qquad x, y \in \mathbb{F}^m, \tag{6.1}$$

where $*$ is the conjugate transpose, which reduces to just T, the transpose, if $\mathbb{F} = \mathbb{R}$. Then three sets of structured matrices can be defined:

1. The set of M-Hermitian matrices or M-selfadjoint matrices:
   $\mathbb{H}_M = \{A \mid A^*M = MA\} = \{A \mid [Ax, y]_M = [x, Ay]_M \text{ for all } x, y \in \mathbb{F}^m\}$.
2. The set of M-skew-Hermitian matrices or M-skew-adjoint matrices:
   $\mathbb{S}_M = \{K \mid K^*M = -MK\} = \{K \mid [Kx, y]_M = -[x, Ky]_M \text{ for all } x, y \in \mathbb{F}^m\}$.
3. The set of M-unitary matrices:
   $\mathbb{U}_M = \{U \mid U^*MU = M\} = \{U \mid [Ux, Uy]_M = [x, y]_M \text{ for all } x, y \in \mathbb{F}^m\}$.

The concept of M-Hermitian and M-skew-Hermitian matrices can be generalized to matrix pencils via

$$\lambda M - B; \quad \text{with} \quad M = \pm M^*, \quad B = \pm B^*. \tag{6.2}$$

In fact, if M is invertible, the generalized eigenvalue problem with underlying matrix pencil as in (6.2) is equivalent to the eigenvalue problem for the matrix $A = M^{-1}B$, which is M-Hermitian or M-skew-Hermitian, depending on whether

M and B are Hermitian or skew-Hermitian. M-unitary matrices may be related to structured pencils indirectly by using a Cayley-transformation [24, 28, 38].

Another structured matrix pencil of the form

$$\lambda A^* - A,$$

which is called *palindromic* [22, 29, 43, 44], can also be transformed to a Hermitian/skew-Hermitian pencil with a Cayley-transformation and can therefore be considered a generalization of M-unitary matrices as well.

The study of matrices and matrix pencils with the symmetry structures outlined above started about one and a half centuries ago (we refer to the review article [25] and the references therein for more details) and continues to be of strong interest as there are many important applications in several areas of science and engineering, see, e.g., [16, 24, 38, 41, 45, 48, 49, 55].

A particular example is given by *Hamiltonian matrices* that arise, e.g., in systems and control theory [27, 38, 45, 55] and in the theory of dynamical and Hamiltonian systems [19–21]. These matrices are J-skew-Hermitian, where the skew-symmetric matrix J is given by

$$\mathrm{J} := \mathrm{J}_n := \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix} \in \mathbb{R}^{2n \times 2n}. \tag{6.3}$$

(We drop the subscript n whenever it is clear from the context.) Due to their many applications, in particular those in system and control theory, the investigation of Hamiltonian matrices has been an important part of Volker Mehrmann's research interest and he and his coauthors have contributed many results to their theory, like discovering the reason for the difficulty in computing Hamiltonian Hessenberg forms [1], finding necessary and sufficient conditions for the existence of the Hamiltonian Schur form [28], and developing several algorithms for the Hamiltonian eigenvalue problem [4, 8, 38, 39]. For the understanding of the underlying theory, it was crucial to be aware of the presence of additional invariants besides the eigenvalues, eigenvectors, and root vectors of Hamiltonian matrices, the so called *signs* in the *sign characteristic* of purely imaginary eigenvalues. The classical Jordan canonical form cannot display these additional invariants, because it is obtained under general similarity transformations that ignore the special structure of Hamiltonian matrices. Therefore, it was important to develop a canonical form that is obtained under structure-preserving transformations, so that additional information like the sign characteristic is preserved and can be read off.

The phenomenon of presence of a sign characteristic not only occurs for the special case of Hamiltonian matrices, but for all three types of matrices structured with respect to the inner product (6.1) induced by M. To be more precise, it occurs for real eigenvalues of M-Hermitian, purely imaginary eigenvalues of M-skew-Hermitian, and unimodular eigenvalues of M-unitary matrices, as well as for the classes of related matrix pencils as in (6.2). In all cases, the sign characteristic has proven to play a fundamental role in theory and applications, like in the analysis of structured dynamic systems [19–21], in perturbation analysis of structured matrices [31–33, 40], and in the investigation of solutions of Riccati equations [12, 24, 28], to name a few examples.

After introducing the well-known canonical forms for Hermitian pencils and M-Hermitian matrices in the next section, we will give a survey on three related topics in the following sections:

(a) Structured canonical forms for Hamiltonian and related matrices.
(b) Canonical forms for doubly structured matrices.
(c) Singular value-like decompositions for matrices associated with two sesquilinear forms.

Throughout the chapter, we will use the following notation. $A_1 \oplus \cdots \oplus A_m$ is the block diagonal matrix $\mathrm{diag}(A_1, \ldots, A_m)$. The $n \times n$ identity matrix is denoted by I_n and $0_{m\times n}$ (0_n) stand for the $m \times n$ $(n \times n)$ zero matrix. If the size is clear from the context, we may use I and 0 instead for convenience. We denote by e_j the jth unit vector, i.e., the jth column of I.

The $n \times n$ reverse identity will be denoted by R_n while $J_n(\alpha)$ stands for the upper triangular $n \times n$ Jordan block with eigenvalue α, that is

$$R_n = \begin{bmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{bmatrix} \in \mathbb{F}^{n\times n}, \quad J_n(\alpha) = \begin{bmatrix} \alpha & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \alpha \end{bmatrix} \in \mathbb{F}^{n\times n}$$

Finally, the $m \times (m+1)$ singular block in the Kronecker canonical form of matrix pencils is denoted by

$$L_m(\lambda) = \begin{bmatrix} \lambda & 1 & & \\ & \ddots & \ddots & \\ & & \lambda & 1 \end{bmatrix}.$$

6.2 Canonical Forms for Hermitian Pencils and M-Hermitian Matrices

For all the structured pencils of one of the forms in (6.2), the theory of structured Kronecker canonical forms is well-established, see, e.g., [9, 25, 26, 46, 47], following the work from the second half of the nineteenth century [23, 51, 52]. These forms are obtained under congruence transformations $(\lambda M - B) \mapsto X^*(\lambda M - B)X$ with X invertible, because those preserve both the Hermitian and the skew-Hermitian structure of matrices and thus the structure of pencils $\lambda M - B$ of the forms in (6.2). For instance, a *Hermitian pencil* $\lambda M - B$, i.e., a pencil such that both M and

B are Hermitian, has the following structured Kronecker canonical form under congruence.

Theorem 1 *Let $\lambda M - B$ be a complex $n \times n$ Hermitian pencil. Then there exists an invertible matrix X such that*

$$X^*(\lambda M - B)X = \mathscr{J}_C(\lambda) \oplus \mathscr{J}_R(\lambda) \oplus \mathscr{J}_\infty(\lambda) \oplus \mathscr{L}(\lambda), \tag{6.4}$$

where

$$\begin{aligned}
\mathscr{J}_C(\lambda) &= \left(\lambda \begin{bmatrix} 0 & R_{m_1} \\ R_{m_1} & 0 \end{bmatrix} - \begin{bmatrix} 0 & R_{m_1}J_{m_1}(\lambda_1) \\ R_{m_1}J_{m_1}(\bar{\lambda}_1) & 0 \end{bmatrix}\right) \oplus \cdots \\
&\qquad \oplus \left(\lambda \begin{bmatrix} 0 & R_{m_p} \\ R_{m_p} & 0 \end{bmatrix} - \begin{bmatrix} 0 & R_{m_p}J_{m_p}(\lambda_p) \\ R_{m_p}J_{m_p}(\bar{\lambda}_p) & 0 \end{bmatrix}\right), \\
\mathscr{J}_R(\lambda) &= s_1 R_{n_1}\big(\lambda I_{n_1} - J_{n_1}(\alpha_1)\big) \oplus \cdots \oplus s_q R_{n_q}\big(\lambda I_{n_q} - J_{n_q}(\alpha_q)\big), \\
\mathscr{J}_\infty(\lambda) &= s_{q+1} R_{k_1}\big(\lambda J_{k_1}(0) - I_{k_1}\big) \oplus \cdots \oplus s_{q+r} R_{k_r}\big(\lambda J_{k_r}(0) - I_{k_r}\big), \\
\mathscr{L}(\lambda) &= \begin{bmatrix} 0 & L_{\ell_1}(\lambda) \\ L_{\ell_1}(\lambda)^T & 0 \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} 0 & L_{\ell_t}(\lambda) \\ L_{\ell_t}(\lambda)^T & 0 \end{bmatrix} \oplus 0_{\nu \times \nu},
\end{aligned}$$

with $\operatorname{Im} \lambda_j > 0,\ j = 1, \ldots, p;\ \alpha_j \in \mathbb{R},\ j = 1, \ldots, q;\ s_j = \pm 1,\ j = 1, \ldots, q+r$ *and* $p, q, r, t \in \mathbb{N}$.

If two pencils $C(\lambda)$ and $D(\lambda)$ are equivalent, i.e., $X_1 C(\lambda) X_2 = D(\lambda)$ for some invertible matrices X_1, X_2 independent of λ, we use the notation $C(\lambda) \sim D(\lambda)$. It is easy to show that the blocks in (6.4) satisfy

$$\begin{aligned}
\mathscr{J}_C(\lambda) &\sim \big(\lambda I_{m_1} - J_{m_1}(\lambda_1)\big) \oplus \big(\lambda I_{m_1} - J_{m_1}(\bar{\lambda}_1)\big) \oplus \\
&\qquad \cdots \oplus \big(\lambda I_{m_p} - J_{m_p}(\lambda_p)\big) \oplus \big(\lambda I_{m_p} - J_{m_p}(\bar{\lambda}_p)\big) \\
\mathscr{J}_R(\lambda) &\sim \big(\lambda I_{n_1} - J_{n_1}(\alpha_1)\big) \oplus \big(\lambda I_{n_q} - J_{n_q}(\alpha_q)\big) \\
\mathscr{J}_\infty(\lambda) &\sim \big(\lambda J_{k_1}(0) - I_{k_1}\big) \oplus \cdots \oplus \big(\lambda J_{k_r}(0) - I_{k_r}\big) \\
\mathscr{L}(\lambda) &\sim L_{\ell_1}(\lambda) \oplus L_{\ell_1}(\lambda)^T \oplus \cdots \oplus L_{\ell_t}(\lambda) \oplus L_{\ell_t}(\lambda)^T \oplus 0_{\nu \times \nu}.
\end{aligned}$$

Therefore, the classical Kronecker canonical form of the pencil $\lambda M - B$ can easily be read off from the structured version (6.4). In particular, the pairing of blocks elegantly displays the corresponding symmetry in the spectrum: the block $\mathscr{J}_C(\lambda)$ contains the nonreal eigenvalues that occur in complex conjugate pairs $\lambda_j, \bar{\lambda}_j$, both having exactly the same Jordan structures. If the pencil is singular, then the singular blocks – contained in $\mathscr{L}(\lambda)$ – are also paired: each *right singular block* $L_{\ell_j}(\lambda)$ has a corresponding *left singular block* $L_{\ell_j}(\lambda)^T$.

However, the structured canonical form (6.4) has an important advantage over the classical Kronecker canonical form of a Hermitian pencil. It displays additional invariants that are present under congruence transformations, the *signs* $s_1, \dots, s_{q+r}$ attached to each Jordan block of a real eigenvalue and each Jordan block of the eigenvalue infinity. The collection of these signs is referred to as the *sign characteristic* of the Hermitian pencil [25], see also [15].

As a corollary of Theorem 1, one obtains a canonical form for M-Hermitian matrices, also known as M-selfadjoint matrices, see [15, 25].

Corollary 1 *Let $M \in \mathbb{C}^{n\times n}$ be Hermitian and invertible and let $A \in \mathbb{C}^{n\times n}$ be M-Hermitian. Then there exists an invertible matrix $X \in \mathbb{C}^{n\times n}$ such that*

$$X^{-1}AX = \mathscr{J}_R \oplus \mathscr{J}_C, \quad X^*MX = \mathscr{M}_R \oplus \mathscr{M}_C,$$

where

$$\mathscr{J}_R = J_{n_1}(\alpha_1) \oplus \cdots \oplus J_{n_q}(\alpha_q), \quad \mathscr{M}_R = s_1 R_{n_1} \oplus \cdots \oplus s_q R_{n_q}$$

$$\mathscr{J}_C = \begin{bmatrix} J_{m_1}(\lambda_1) & 0 \\ 0 & J_{m_1}(\bar{\lambda}_1) \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} J_{m_p}(\lambda_p) & 0 \\ 0 & J_{m_p}(\bar{\lambda}_p) \end{bmatrix}, \; \mathscr{M}_C = R_{2m_1} \oplus \cdots \oplus R_{2m_p},$$

where $\alpha_j \in \mathbb{R}$, $s_j = \pm 1$, $j = 1, \dots, q$, $\operatorname{Im}\lambda_j > 0$, $j = 1, \dots, p$, and $p, q \in \mathbb{N}$.

Indeed, the form is easily obtained by recalling that a matrix A is M-Hermitian if and only if the pencil $\lambda M - MA$ is a Hermitian pencil and applying Theorem 1 to this pencil. By convention, we will call s_j in Corollary 1 the *sign* of the Jordan block $J_{n_j}(\alpha_j)$.

For the other three types of matrix pencils in (6.2), structured canonical forms can be derived directly from (6.4). If $\lambda M - B$ is Hermitian/skew-Hermitian, skew-Hermitian/Hermitian, or skew-Hermitian/skew-Hermitian, then Theorem 1 can be applied to the Hermitian pencils $\lambda M - (-iB)$, $\lambda(-iM) - B$ or $\lambda(-iM) - (-iB)$, respectively, to obtain (6.4). As a consequence, these pencils also have a sign characteristic. In the case of pencils of "mixed" structure, i.e., one matrix being Hermitian and the other skew-Hermitian, now the purely imaginary eigenvalues (including the eigenvalue infinity) have signs.

For the case of real pencils of the form (6.2), also real structured Kronecker canonical forms under real congruence transformations are known. We refer the reader to [26, 47] for details.

6.3 Structured Canonical Forms for Hamiltonian Matrices

When the matrix defining the inner product (6.1) is the skew-symmetric matrix J from (6.3), then a J-Hermitian matrix is called *skew-Hamiltonian*, a J-skew-Hermitian is called *Hamiltonian*, and a J-unitary matrix is called *symplectic*.

In many applications, in particular in systems and control, *invariant Lagrangian subspaces* are of interest. A *Lagrangian subspace* is an n-dimensional subspace $\mathscr{L} \subseteq \mathbb{F}^{2n}$ that is J-neutral, i.e., $[x, y]_{\mathrm{J}} = 0$ for all $x, y \in \mathscr{L}$. Suppose the columns of the matrix $W_1 \in \mathbb{F}^{2n\times n}$ span an invariant Lagrangian subspace $\mathscr{L}$ of a Hamiltonian matrix $H \in \mathbb{F}^{n\times n}$. Then there exists a $2n \times n$ matrix W_2 such that $W = [W_1, W_2]$ is symplectic. Indeed, one may choose $W_2 = \mathrm{J}^T W_1 (W_1^* W_1)^{-1}$. Then

$$W^*\mathrm{J}W = \begin{bmatrix} W_1^*\mathrm{J}W_1 & W_1^*\mathrm{J}W_2 \\ W_2^*\mathrm{J}W_1 & W_2^*\mathrm{J}W_2 \end{bmatrix} = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix} = \mathrm{J},$$

because $W_1^*\mathrm{J}W_1 = 0 = (W_1^* W_1)^{-1} W_1^*\mathrm{J}\mathrm{J}\mathrm{J}^T W_1 (W_1^* W_1)^{-1} = W_2^*\mathrm{J}W_2$ as $\mathscr{L}$ is J-neutral. Since $\mathscr{L}$ is also an invariant subspace of H, we obtain

$$W^{-1}HW = \begin{bmatrix} T & D \\ 0 & -T^* \end{bmatrix}, \quad D = D^*. \tag{6.5}$$

From the decomposition (6.5), we can easily see that a necessary condition for the existence of an invariant Lagrangian subspace is that the algebraic multiplicities of all purely imaginary eigenvalues must be even, because any purely imaginary eigenvalue $i\alpha$ of T is also an eigenvalue of $-T^*$. This condition, however, is not sufficient as the following example shows.

Example 1 Consider the Hamiltonian matrices

$$H_1 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} = \mathrm{J}_2, \quad H_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix} = \begin{bmatrix} \mathrm{J}_1 & 0 \\ 0 & -\mathrm{J}_1^T \end{bmatrix}$$

Then the matrix H_1 does not have a decomposition (6.5) since for any symplectic matrix W, by definition, $W^* H_1 W = W^*\mathrm{J}_2 W = \mathrm{J}_2$. The matrix H_2 on the other hand already is of the form (6.5). Surprisingly, the matrices H_1 and H_2 are similar. It is easy to check that they both have the semi-simple eigenvalues i and $-i$, which both have the algebraic multiplicity two.

To explain this surprising behavior, a closer look at a structured canonical form of Hamiltonian matrices is necessary. One way is to consider instead of a Hamiltonian matrix H the iJ-Hermitian matrix iH and to apply Corollary 1. This yields the existence of an invertible matrix X such that

$$X^{-1}HX = \mathscr{H}_I \oplus \mathscr{H}_C, \quad X^*JX = \mathscr{M}_I \oplus \mathscr{M}_C$$

where

$$\mathscr{H}_I = iJ_{n_1}(\alpha_1) \oplus \cdots \oplus iJ_{n_q}(\alpha_q), \quad \mathscr{M}_I = s_1 iR_{n_1} \oplus \cdots \oplus s_q iR_{n_q}$$

$$\mathscr{H}_C = i\begin{bmatrix} J_{m_1}(\lambda_1) & 0 \\ 0 & J_{m_1}(\bar{\lambda}_1) \end{bmatrix} \oplus \cdots \oplus i\begin{bmatrix} J_{m_p}(\lambda_p) & 0 \\ 0 & J_{m_p}(\bar{\lambda}_p) \end{bmatrix},$$

$$\mathscr{M}_C = iR_{2m_1} \oplus \cdots \oplus iR_{2m_p}.$$

Here, s_j is the sign of the Jordan block $J_{n_j}(i\alpha_j)$ of H, for $j = 1, \ldots, q$, i.e., the purely imaginary eigenvalues come with a sign characteristic. Although this canonical form reveals these additional invariants, one cannot tell immediately whether a decomposition as (6.5) exists. One possible way to proceed is to apply further transformations to transform $X^*\mathrm{J}X$ back to J, say by constructing an invertible matrix Y such that

$$\mathscr{H} := (XY)^{-1}H(XY), \quad (XY)^*J(XY) = J$$

Then, the matrix XY is symplectic, because $(XY)^*J(XY) = J$. Clearly, there are many such transformations and then the task is to choose among all these transformations a particular one so that $\mathscr{H}$ is as close to a block upper triangular form (6.5) as possible. In [28] such an *optimal* canonical form is presented in the sense that the (2,1) block of $\mathscr{H}$ has the lowest possible rank. The result is given in the following theorem.

Theorem 2 ([28]) *Let $H \in \mathbb{C}^{2n\times 2n}$ be a Hamiltonian matrix. Then there exists a symplectic matrix $W \in \mathbb{C}^{2n\times 2n}$ such that*

$$W^{-1}HW = \left[\begin{array}{cccc|cccc} T_c & & & & 0 & & & \\ & T_{ie} & & & & D_{ie} & & \\ & & T_{io} & & & & D_{io} & \\ & & & T_{ior} & & & & D_{ior} \\ \hline & & & & -T_c^* & & & \\ & & & & & -T_{ie}^* & & \\ & & & & & & -T_{io}^* & \\ & & & M_{ior} & & & & -T_{ior}^* \end{array}\right],$$

where the blocks have the following properties:

(i)

$$T_c = J_{m_1}(\lambda_1) \oplus J_{m_2}(\lambda_2) \oplus \cdots \oplus J_{m_p}(\lambda_p),$$

where $\lambda_1, \ldots, \lambda_p \in \mathbb{C}$ *with* $\operatorname{Re}\lambda_1, \ldots, \operatorname{Re}\lambda_p > 0.$

(ii)

$$T_{ie} = J_{n_1}(i\alpha_1) \oplus \cdots \oplus J_{n_q}(i\alpha_q), \quad D_{ie} = s_1 e_{n_1} e_{n_1}^* \oplus \cdots \oplus s_q e_{n_q} e_{n_q}^*,$$

where $\alpha_1, \ldots, \alpha_q \in \mathbb{R}$ *and* $s_1, \ldots, s_q = \pm 1$. *Each sub-matrix*

$$\begin{bmatrix} J_{n_j}(i\alpha_j) & s_j e_{n_j} e_{n_j}^* \\ 0 & -(J_{n_j}(i\alpha_j))^* \end{bmatrix}$$

corresponds to an even-sized Jordan block $J_{2n_j}(i\alpha_j)$ *of* H *with sign* s_j.

(iii)

$$T_{io} = T_{io}^{(1)} \oplus \cdots \oplus T_{io}^{(r)}, \quad D_{io} = D_{io}^{(1)} \oplus \cdots \oplus D_{io}^{(r)},$$

and

$$T_{io}^{(j)} = \begin{bmatrix} J_{\ell_j}(i\beta_j) & 0 & -\frac{\sqrt{2}}{2} e_{\ell_j} \\ 0 & J_{k_j}(i\beta_j) & -\frac{\sqrt{2}}{2} e_{k_j} \\ 0 & 0 & i\beta_j \end{bmatrix}, \quad D_{io}^{(j)} = \frac{\sqrt{2}i}{2} \sigma_j \begin{bmatrix} 0 & 0 & e_{\ell_j} \\ 0 & 0 & -e_{k_j} \\ -e_{\ell_j}^* & e_{k_j}^* & 0 \end{bmatrix},$$

where $\beta_1, \ldots, \beta_r \in \mathbb{R}$ *and* $\sigma_1, \ldots, \sigma_r = \pm 1$. *For each* $j = 1, \ldots, r$, *the sub-matrix*

$$\begin{bmatrix} T_{io}^{(j)} & D_{io}^{(j)} \\ 0 & -(T_{io}^{(j)})^* \end{bmatrix}$$

corresponds to two odd-sized Jordan blocks of H *associated with the same purely imaginary eigenvalue* $i\beta_j$. *The first is* $J_{2\ell_j+1}(i\beta_j)$ *with sign* σ_j *and the second is* $J_{2k_j+1}(i\beta_j)$ *with sign* $-\sigma_j$.

(iv)

$$T_{ior} = T_{ior}^{(1)} \oplus \cdots \oplus T_{ior}^{(t)}, \quad M_{ior} = M_{ior}^{(1)} \oplus \cdots \oplus M_{ior}^{(t)}, \quad D_{ior} = D_{ior}^{(1)} \oplus \cdots \oplus D_{ior}^{(t)},$$

where

$$T_{ior}^{(j)} = \begin{bmatrix} J_{\eta_j}(i\gamma_j) & 0 & -\frac{\sqrt{2}}{2} e_{\eta_j} \\ 0 & J_{\nu_j}(i\delta_j) & -\frac{\sqrt{2}}{2} e_{\nu_j} \\ 0 & 0 & \frac{i}{2}(\gamma_j + \delta_j) \end{bmatrix}, \quad M_{ior}^{(j)} = \sigma_{r+j} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2}(\gamma_j - \delta_j) \end{bmatrix},$$

$$D_{ior}^{(j)} = \frac{\sqrt{2}i}{2} \sigma_{r+j} \begin{bmatrix} 0 & 0 & e_{\eta_j} \\ 0 & 0 & -e_{\nu_j} \\ -e_{\eta_j}^* & e_{\nu_j}^* & -\frac{\sqrt{2}i}{2}(\gamma_j - \delta_j) \end{bmatrix}$$

and $\gamma_j, \delta_j \in \mathbb{R}$, $\gamma_j \neq \delta_j$, $\sigma_{r+j} = \pm 1$ *for* $j = 1, \ldots, t$. *The submatrix*

$$\begin{bmatrix} T_{ior}^{(j)} & D_{ior}^{(j)} \\ M_{ior}^{(j)} & -(T_{ior}^{(j)})^* \end{bmatrix}$$

corresponds to two odd-sized Jordan blocks of H *associated with two distinct purely imaginary eigenvalues* $i\gamma_j$ *and* $i\delta_j$. *The first one is* $J_{2\eta_j+1}(i\gamma_j)$ *with sign* σ_{r+j} *and the second one is* $J_{2\nu_j+1}(i\delta_j)$ *with sign* $-\sigma_{r+j}$.

Thus, the spectrum of H can be read off from the Hamiltonian submatrices

$$H_c := \begin{bmatrix} T_c & 0 \\ 0 & -T_c^* \end{bmatrix}, \quad H_{ie} := \begin{bmatrix} T_{ie} & D_{ie} \\ 0 & -T_{ie}^* \end{bmatrix},$$

$$H_{io} := \begin{bmatrix} T_{io} & D_{io} \\ 0 & -T_{io}^* \end{bmatrix}, \quad H_{ior} := \begin{bmatrix} T_{ior} & D_{ior} \\ M_{ior} & -T_{ior}^* \end{bmatrix}.$$

The submatrix H_c contains all Jordan blocks associated with eigenvalues that are not purely imaginary. To be more precise, T_c contains all the Jordan blocks of eigenvalues of H with positive real parts, and $-T_c^*$ contains all the Jordan blocks of eigenvalues of H with negative real parts. The submatrix H_{ie} contains all even-sized Jordan blocks associated with purely imaginary eigenvalues of H, whereas H_{io} and H_{ior} contain all Jordan blocks associated with purely imaginary eigenvalues of H that have odd sizes. Here, H_{io} consists of pairs of Jordan blocks of (possibly different) odd sizes that are associated with the same purely imaginary eigenvalue, but have opposite signs. On the other hand, H_{ior} consists of the remaining Jordan blocks that do not allow such a pairing. In particular, if H_{ior} contains more than one Jordan block associated to a particular purely imaginary eigenvalue, then all such blocks must have the same sign in the sign characteristic.

While the canonical form in Theorem 2 looks quite complicated at first sight, its advantage is that the conditions for the existence of a decomposition of the form (6.5) can now be trivially derived by requesting the submatrix H_{ior} being void. Thus, with the interpretation of H_{ior} in terms of the sign characteristic, we immediately obtain the following result that is in accordance with a corresponding result in [42] in terms of M-selfadjoint matrices.

Theorem 3 ([28]) *A Hamiltonian matrix* H *has a decomposition (6.5) if and only if for each purely imaginary eigenvalue of* H, *it has an even number of odd-sized Jordan blocks half of which have sign* $+1$ *and half of which have sign* -1.

The theorem also gives necessary and sufficient conditions for the existence of the *Hamiltonian Schur form*. A Hamiltonian matrix H is said to have a Hamiltonian Schur form, if it allows a decomposition of the form (6.5) with T being upper triangular and W being both symplectic and unitary, i.e., satisfying $W^*JW = J$ and $W^*W = I$. Under the same conditions as in Theorem 3, we obtain the existence of a symplectic matrix W such that $W^{-1}HW$ is in the Hamiltonian canonical form of

Theorem 2 without the blocks from H_{ior}. Since the blocks T_c, T_{ie}, and T_{io} are upper triangular, we find that $W^{-1}HW$ has the form (6.5) with T being upper triangular. A Hamiltonian Schur form can then be derived by performing a symplectic QR-like decomposition to the symplectic matrix W, see [7, 28].

Corollary 2 ([28]) *Let H be a Hamiltonian matrix. Then there exists a unitary and symplectic matrix W such that $W^{-1}HW$ has the form (6.5) with T being upper triangular if and only if for each purely imaginary eigenvalue, H has an even number of odd-sized Jordan blocks half of which have sign $+1$ and half of which have sign -1.*

The following example, borrowed from [30], shows that the two Jordan blocks that are paired in one of the particular submatrices of H_{io} in Theorem 2 may indeed have different sizes.

Example 2 Consider the two matrices

$$H = \begin{bmatrix} i & 1 & 1 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & -1 & i \end{bmatrix} \quad \text{and} \quad X = \frac{1}{\sqrt{2}} \begin{bmatrix} 2i & 0 & i & 2i \\ 0 & 1 & -i & -i \\ 0 & 0 & 1 & 0 \\ 0 & -i & 1 & 1 \end{bmatrix}.$$

Then H is a Hamiltonian matrix in Hamiltonian Schur form and X is the transformation matrix that brings the pair $(iH, i\mathrm{J})$ into the canonical form of Corollary 1:

$$X^{-1}(iH)X = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \quad X^*(iJ)X = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix},$$

Thus, H has the eigenvalue i and two corresponding Jordan blocks with sizes 3 and 1. The Jordan block of size 3 has the sign $+1$ and the Jordan block of size 1 has the sign -1 thus satisfying the condition of Theorem 3.

Example 3 Revisiting the matrices H_1 and H_2 from Example 1, one can easily check that the eigenvalues i and $-i$ of H_2 have one Jordan block with sign $+1$ and one Jordan block with sign -1 each. In fact, H_2 is a matrix of the form H_{io} as in Theorem 2. On the other hand, for the matrix H_1 the signs corresponding to i are both $+1$ and the signs corresponding to $-i$ are both -1. In fact, H_1 is in the canonical form of Theorem 2 corresponding exactly to a matrix in the form H_{ior}.

However, for the matrix $X = [e_1, e_3, e_2, e_4]$, which is not symplectic, we obtain that $X^{-1}H_1X = H_2$ is in the form (6.5). Although the transformation with X maps the Hamiltonian matrix H_1 to the Hamiltonian matrix H_2, it is not a structure-preserving transformation in the sense that for small Hamiltonian perturbations $H_1 + \Delta H$ the transformed matrix $H_2 + X^{-1}\Delta HX$ is in general not Hamiltonian. This fact in a sense allows the similarity transformation with X to take a bypass by ignoring the sign constraints shown in Theorem 3. It was shown in [28] that the existence of the decomposition (6.5) with a non-symplectic similarity transformation only requires the algebraic multiplicities of all purely imaginary eigenvalues of H to be even.

In the case that the Hamiltonian matrix under consideration is real, there is also a canonical form under real symplectic similarity, see [28, Theorem 22]. In this case, the eigenvalues of a Hamiltonian matrix are not only symmetric with respect to the imaginary axis, but also with respect to the real axis. Thus, in particular the Jordan blocks associated with purely imaginary eigenvalues $i\alpha$, $\alpha > 0$ occur in complex conjugate pairs and it turns out that their signs in the sign characteristic are related. It can be shown that if $J_{m_1}(i\alpha), \ldots, J_{m_p}(i\alpha)$ are the Jordan blocks of a Hamiltonian matrix H associated with the eigenvalue $i\alpha$ and having the signs $s_1, \ldots, s_p$, then the signs of the corresponding Jordan blocks $J_{m_1}(-i\alpha), \ldots, J_{m_p}(-i\alpha)$ are $-s_1, \ldots, -s_p$, respectively. Another key difference between the real and the complex case is the behavior of the eigenvalue 0 when H is singular. While in the complex case this eigenvalue can be treated as any other purely imaginary eigenvalue, it has a special Jordan structure in the real case: each odd-sized Jordan block associated with zero must have an even number of copies and in the corresponding sign characteristic, half of the signs must be $+1$ and half of the signs must be -1. In contrast, there is no such pairing for Jordan blocks associated with zero that have even sizes. This extraordinary behavior of the eigenvalue zero leads to a real version of Theorem 3 that yields slightly different conditions in comparison with the complex case.

Theorem 4 ([28]) *A real Hamiltonian matrix H has a decomposition (6.5) with a real symplectic transformation matrix W if and only if for each nonzero purely imaginary eigenvalue, H has an even number of odd-sized Jordan blocks half of which have sign $+1$ and half of which have sign -1.*

For most of the problems arising from systems and control, one actually is interested in special invariant Lagrangian subspaces of Hamiltonian matrices. For instance, for the existence of solutions of algebraic Riccati equations [24, 38, 55] one is interested in the invariant Lagrangian subspaces of a Hamiltonian matrix corresponding to the eigenvalues in the closed or open left half complex plane. A more general question is the following: if H is a $2n \times 2n$ Hamiltonian matrix and a list Λ of n of its eigenvalues (counted with multiplicities) is prescribed, does there exists an invariant Lagrangian subspace associated with the eigenvalues in Λ, and if so, is this subspace unique? This question can be answered with the help of Theorem 3 or its corresponding real version. As we already know, the existence of an invariant Lagrangian subspace for a Hamiltonian matrix H is equivalent to the existence of a decomposition of the form (6.5). From (6.5), the spectrum of H is the union of the spectra of both T and $-T^*$. So one may assume that H has pairwise distinct non purely imaginary eigenvalues

$$\lambda_1, -\bar{\lambda}_1, \ldots, \lambda_p, -\bar{\lambda}_p, \quad \text{with algebraic multiplicities} \quad \nu_1, \nu_1, \ldots, \nu_p, \nu_p$$

and pairwise distinct purely imaginary eigenvalues

$$i\alpha_1, \dots, i\alpha_q \quad \text{with algebraic multiplicities} \quad 2\mu_1, \dots, 2\mu_q.$$

In order to have an invariant Lagrangian subspace, or, equivalently, a decomposition (6.5), it is necessary that the spectrum of T contains λ_j and $-\bar{\lambda}_j$ with algebraic multiplicities k_j and $\nu_j - k_j$, respectively, for each $j = 1, \dots, p$, and μ_j copies of $i\alpha_j$ for each $j = 1, \dots, q$. Let $\Omega(H)$ denote the set of all possible spectra for T in a decomposition of the form (6.5) of H. Then this set contains $\prod_{j=1}^{p}(\nu_j + 1)$ different selections, because k_j can be any number from 0 to ν_j for each j. Among them there are 2^p selections that contain either λ_j or $-\bar{\lambda}_j$, but not both, for all j. This subset of $\Omega(H)$ is denoted by $\tilde{\Omega}(H)$.

The answer to the question of existence of invariant Lagrangian subspaces with a prescribed spectrum is then given in the following theorem.

Theorem 5 ([12]) *A Hamiltonian matrix H has an invariant Lagrangian subspace corresponding to every $\omega \in \Omega(H)$ if and only if the conditions Theorem 3 (or Theorem 4 in real case) hold. Concerning uniqueness, we have the following conditions.*

(i) For every $\omega \in \Omega(H)$, H has a unique corresponding invariant Lagrangian subspace if and only if for every non purely imaginary eigenvalue λ_j (and $-\bar{\lambda}_j$) H has only a single Jordan block, and for every purely imaginary eigenvalue $i\alpha_j$, H only has even-sized Jordan blocks all of them having the same sign.

(ii) For every $\omega \in \tilde{\Omega}(H)$, H has a unique corresponding invariant Lagrangian subspace if and only if for every purely imaginary eigenvalue $i\alpha_j$, H has only even-sized Jordan blocks all of them having the same sign.

When the Lagrangian invariant subspaces corresponding to the eigenvalues in $\Omega(H)$ or $\tilde{\Omega}(H)$ are not unique, then it is possible to parameterize their bases. Moreover, the results in Theorem 5 can be used to study Hermitian solutions to algebraic Riccati equations, see [12].

We will now turn to skew-Hamiltonian matrices. Analogously to the case of Hamiltonian matrices, it can be shown that if the columns of W_1 span an invariant Lagrangian subspace of a skew-Hamiltonian matrix K, then there exists a symplectic matrix $W = [W_1, W_2]$ such that

$$W^{-1}KW = \begin{bmatrix} T & D \\ 0 & T^* \end{bmatrix}, \quad D = -D^*. \tag{6.6}$$

Structured canonical forms for complex skew-Hamiltonian matrices can be constructed in the same way as for complex Hamiltonian matrices, using the fact that K is skew-Hamiltonian if and only if iK is Hamiltonian. Thus, the conditions for the existence of invariant Lagrangian subspaces are the same as in Theorem 3 replacing "purely imaginary eigenvalues" with "real eigenvalues".

Interestingly, for any real skew-Hamiltonian matrix the real version of the decomposition (6.6) always exists, see [50]. Also, it is proved in [10] that for a real skew-Hamiltonian matrix K, there always exists a real symplectic matrix W such that

$$W^{-1}KW = N \oplus N^T,$$

where N is in real Jordan canonical form. The result shows clearly that every Jordan block of K has an even number of copies.

Finally, if $S \in \mathbb{F}^{n \times n}$ is a symplectic matrix and if the columns of W_1 span an invariant Lagrangian invariant subspace of S, then similar to the Hamiltonian case one can show that there exists a symplectic matrix $W = [W_1, W_2]$ such that

$$W^{-1}SW = \begin{bmatrix} T & D \\ 0 & T^{-*} \end{bmatrix}, \quad DT^* = (DT^*)^*. \tag{6.7}$$

The case of symplectic matrices can be reduced to the case of Hamiltonian matrices with the help of the *Cayley transformation*, see Chap. 2 in this book for details on the Cayley transformation. Therefore, structured Jordan canonical forms for symplectic matrices can be derived using the structured canonical forms for Hamiltonian matrices in Theorem 2 and its real version. Then, conditions for the existence of a decomposition of the form (6.7) can be obtained which are essentially the same as in Theorems 3 and 4, with purely imaginary eigenvalues replaced by unimodular eigenvalues in the symplectic case, see [28].

6.4 Doubly Structured Matrices and Pencils

In this section we discuss canonical forms of doubly structured matrices and pencils. This research was mainly motivated by applications from quantum chemistry [2, 3, 16, 41, 49]. In linear response theory, one has to solve a generalized eigenvalue problem with a pencil of the form

$$\lambda \begin{bmatrix} C & Z \\ -Z & -C \end{bmatrix} - \begin{bmatrix} E & F \\ F & E \end{bmatrix}, \tag{6.8}$$

where C, E, F are $n \times n$ Hermitian matrices and Z is skew-Hermitian. The simplest response function model is the time-dependent Hartree-Fock model (also called random phase approximation) in which the pencil (6.8) takes the simpler structure $C = I$ and $Z = 0$ so that the corresponding eigenvalue problem can be reduced to a standard eigenvalue problem with a matrix of the form

$$A = \begin{bmatrix} E & F \\ -F & -E \end{bmatrix}$$

with E and F being Hermitian. It is straightforward to check that A is Hamiltonian (or J-skew-Hermitian) and M-Hermitian, where

$$M = \begin{bmatrix} I_n & 0 \\ 0 & -I_n \end{bmatrix}.$$

In the general setting, we consider matrices that are structured with respect to two invertible Hermitian or skew-Hermitian matrices K and M. Because any skew-Hermitian matrix K can be transformed to the Hermitian matrix iK and any K-skew-Hermitian matrix A can be transformed to the K-Hermitian matrix iA, we may assume that both K and M are invertible and Hermitian and consider two cases only:

(a) A is K-Hermitian and M-Hermitian, i.e., $KA = A^*K,\ MA = A^*M$,
(b) A is K-Hermitian and M-skew-Hermitian, i.e., $KA = A^*K,\ MA = -A^*M$.

The task is now to find an invertible matrix X to perform a transformation

$$\mathscr{A} = X^{-1}AX, \quad \mathscr{K} = X^*KX, \quad \mathscr{M} = X^*MX$$

so that the canonical form of Corollary 1 (or the corresponding version for M-skew-Hermitian matrices) for both pairs (A, K) and (A, M) can simultaneously be recovered. As shown in [34], this is not always possible, because the situation is too general. So it is reasonable to restrict oneself to the situation where the pencil $\lambda K - M$ is *nondefective*, meaning that all the eigenvalues of the Hermitian pencil $\lambda K - M$ are semisimple. (This assumption is satisfied in the case $K = i\,\mathrm{J}_n$ and $M = \mathrm{diag}(I_n, -I_n)$ which is relevant for the applications in quantum chemistry.) Then by (6.4), there exists an invertible matrix Q such that

$$Q^*(\lambda K - M)Q = (\lambda K_1 - M_1) \oplus \cdots \oplus (\lambda K_p - M_p), \tag{6.9}$$

where, for each $j = 1, \ldots, p$, either

$$\lambda K_j - M_j = \lambda \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} 0 & \lambda_j \\ \bar{\lambda}_j & 0 \end{bmatrix}$$

containing a pair of nonreal eigenvalues $\lambda_j, \bar{\lambda}_j$, or

$$\lambda K_j - M_j = s_j\big(\lambda\,[\,1\,] - [\,\alpha_j\,]\big), \quad s_j = \pm 1,$$

containing a single real eigenvalue α_j with a sign s_j. (We highlight that the same eigenvalues $\lambda_j, \bar{\lambda}_j$ or α_j, respectively, may appear multiple times among the blocks $\lambda K_1 - M_1, \ldots, \lambda K_p - M_p$.)

Under this assumption, the following structured canonical form can be obtained for a matrix that is doubly structured in the sense of case (a) above.

Theorem 6 ([34]) *Suppose K, M are Hermitian and invertible, such that the pencil $\lambda K - M$ is nondefective. Suppose A is both K-Hermitian and M-Hermitian. Then there exists an invertible matrix X such that*

$$\begin{aligned} \mathscr{A} &:= X^{-1}AX = A_1 \oplus A_2 \oplus \cdots \oplus A_p \\ \mathscr{K} &:= X^*KX \;\, = K_1 \oplus K_2 \oplus \cdots \oplus K_p \\ \mathscr{M} &:= X^*MX \;\, = M_1 \oplus M_2 \oplus \cdots \oplus M_p, \end{aligned}$$

where for each $j = 1, 2, \ldots, p$ the blocks A_j, K_j, M_j are in one of the following forms.

(i) Blocks associated with a pair of conjugate complex eigenvalues of A:

$$A_j = \begin{bmatrix} J_{m_j}(\lambda_j) & 0 \\ 0 & J_{m_j}(\bar{\lambda}_j) \end{bmatrix}, \quad K_j = \begin{bmatrix} 0 & R_{m_j} \\ R_{m_j} & 0 \end{bmatrix}, \quad M_j = \begin{bmatrix} 0 & \gamma_j R_{m_j} \\ \bar{\gamma}_j R_{m_j} & 0 \end{bmatrix},$$

where $\lambda_j \in \mathbb{C} \setminus \mathbb{R}$, $\gamma_j = c_j + i d_j \neq 0$ with $c_j, d_j \in \mathbb{R}$ and $d_j \geq 0$.

(ii) Blocks associated with real eigenvalues of A and real eigenvalues of $\lambda K - M$:

$$A_j = J_{n_j}(\alpha_j), \quad K_j = s_j R_{n_j}, \quad M_j = s_j \eta_j R_{n_j},$$

where $s_j = \pm 1$, $0 \neq \eta_j \in \mathbb{R}$, and $\alpha_j \in \mathbb{R}$. The sign of the block A_j with respect to K is s_j and the sign with respect to M is $\operatorname{sign}(s_j \eta_j)$.

(iii) Blocks associated with real eigenvalues of A and a pair of conjugate complex eigenvalues of $\lambda K - M$:

$$A_j = \begin{bmatrix} J_{n_j}(\alpha_j) & 0 \\ 0 & J_{n_j}(\alpha_j) \end{bmatrix}, \quad K_j = \begin{bmatrix} 0 & R_{n_j} \\ R_{n_j} & 0 \end{bmatrix}, \quad M_j = \begin{bmatrix} 0 & \gamma_j R_{n_j} \\ \bar{\gamma}_j R_{n_j} & 0 \end{bmatrix},$$

where $\alpha_j \in \mathbb{R}$ and $\gamma_j = c_j + i d_j$ with $c_j, d_j \in \mathbb{R}$ and $d_j > 0$. Thus A_j contains a pair of two $n_j \times n_j$ Jordan blocks of A associated with the same real eigenvalue α_j. The pairs of corresponding signs are $(+1, -1)$ *with respect to both K and M.*

It is easily seen that the structured canonical forms for A with respect to K and M, respectively, can immediately be read off from the canonical form in Theorem 6. In addition, the structured canonical form of $\lambda K - M$ as in (6.9) can easily be derived from $\lambda \mathscr{K} - \mathscr{M}$. Therefore, Theorem 6 combines three different structured canonical forms into one.

On the other hand, Theorem 6 shows that the presence of two structures in A leads to additional restrictions in the Jordan structure of A which can be seen from the blocks of type (iii) of Theorem 6. This block is *indecomposable* in a sense that there does not exist any transformation of the form $(A_j, K_j, M_j) \mapsto (Y^{-1}A_jY, Y^*K_jY, Y^*M_jY)$ that simultaneously block-diagonalizes all three matrices. As a consequence, the Jordan structure of a matrix A that is both K-Hermitian and M-Hermitian is rather restricted if the pencil $\lambda K - M$ (is defective and) has only nonreal eigenvalues. In that case, each Jordan block associated with a real eigenvalue of A must occur an even number of times in the Jordan canonical form of A. In particular, all real eigenvalues of A must have even algebraic multiplicity.

In case (b), i.e., when A is K-Hermitian and M-skew-Hermitian, then the eigenstructure of A has even richer symmetry than in case (a), because now the spectrum has to be symmetric to both the real and the imaginary axes. Also, the Jordan blocks associated with real eigenvalues of A will have signs with respect to K while the ones associated with purely imaginary eigenvalues will have signs with respect to M. Thus, the eigenvalue zero will play a special role, because it will have signs both with respect to K and to M. A structured canonical form for this case will be given in the next theorem, for which we need extra notation. By Σ_n, we denote the $n \times n$ anti-diagonal matrix alternating sign matrix, i.e.,

$$\Sigma_n = \begin{bmatrix} & & & (-1)^0 \\ & & (-1)^1 & \\ & \cdot^{\cdot^{\cdot}} & & \\ (-1)^{n-1} & & & \end{bmatrix}.$$

Theorem 7 ([34]) *Suppose K, M are Hermitian and invertible, and $\lambda K - M$ is nondefective. Suppose A is both K-Hermitian and M-skew-Hermitian. Then there exists an invertible matrix X such that*

$$\begin{aligned} \mathscr{A} &:= X^{-1}AX = A_1 \oplus A_2 \oplus \cdots \oplus A_p \\ \mathscr{K} &:= X^*KX \ = K_1 \oplus K_2 \oplus \cdots \oplus K_p \\ \mathscr{M} &:= X^*MX = M_1 \oplus M_2 \oplus \cdots \oplus M_p, \end{aligned}$$

where for $j = 1, 2, \ldots, p$ the blocks A_j, K_j, M_j are in one of the following forms.

(i) Blocks associated with nonreal, non purely imaginary eigenvalues of A:

$$A_j = \begin{bmatrix} J_{m_j}(\lambda_j) & 0 & 0 & 0 \\ 0 & -J_{m_j}(\lambda_j) & 0 & 0 \\ 0 & 0 & J_{m_j}(\bar{\lambda}_j) & 0 \\ 0 & 0 & 0 & -J_{m_j}(\bar{\lambda}_j) \end{bmatrix},$$

$$K_j = \begin{bmatrix} 0 & 0 & R_{m_j} & 0 \\ 0 & 0 & 0 & R_{m_j} \\ R_{m_j} & 0 & 0 & 0 \\ 0 & R_{m_j} & 0 & 0 \end{bmatrix}, \quad M_j = \begin{bmatrix} 0 & 0 & 0 & \gamma_j R_{m_j} \\ 0 & 0 & \gamma_j R_{m_j} & 0 \\ 0 & \bar{\gamma}_j R_{m_j} & 0 & 0 \\ \bar{\gamma}_j R_{m_j} & 0 & 0 & 0 \end{bmatrix},$$

where $\lambda_j = a_j + ib_j$ with $a_j, b_j \in \mathbb{R}$ and $a_j b_j > 0$, and the parameter γ_j satisfies one of the following three mutually exclusive conditions: (a) $\gamma_j = \beta_j$ with $\beta_j > 0$, (b) $\gamma_j = i\beta_j$ with $\beta_j > 0$, or (c) $\gamma_j = c_j + id_j$ with $c_j, d_j \in \mathbb{R}$ and $c_j d_j > 0$.

(ii) Blocks associated with a pair of real eigenvalues $\pm\alpha_j$ *of* A *and nonreal non purely imaginary eigenvalues of* $\lambda K - M$*:*

$$A_j = \begin{bmatrix} J_{n_j}(\alpha_j) & 0 & 0 & 0 \\ 0 & -J_{n_j}(\alpha_j) & 0 & 0 \\ 0 & 0 & J_{n_j}(\alpha_j) & 0 \\ 0 & 0 & 0 & -J_{n_j}(\alpha_j) \end{bmatrix},$$

$$K_j = \begin{bmatrix} 0 & 0 & R_{n_j} & 0 \\ 0 & 0 & 0 & R_{n_j} \\ R_{n_j} & 0 & 0 & 0 \\ 0 & R_{n_j} & 0 & 0 \end{bmatrix}, \quad M_j = \begin{bmatrix} 0 & 0 & 0 & \gamma_j R_{n_j} \\ 0 & 0 & \gamma_j R_{n_j} & 0 \\ 0 & \bar{\gamma}_j R_{n_j} & 0 & 0 \\ \bar{\gamma}_j R_{n_j} & 0 & 0 & 0 \end{bmatrix},$$

where $0 < \alpha_j \in \mathbb{R}$ *and* $\gamma_j = c_j + i d_j$ *with* $c_j, d_j \in \mathbb{R}$ *and* $c_j d_j > 0$*. The two Jordan blocks associated with* α_j *have the signs* 1 *and* -1 *with respect to* K *and the two Jordan blocks associated with* $-\alpha_j$ *also have the signs* 1 *and* -1 *with respect to* K*.*

(iii) Blocks associated with a pair of real eigenvalues $\pm\alpha_j$ *of* A *and real or purely imaginary eigenvalues of* $\lambda K - M$*:*

$$A_j = \begin{bmatrix} J_{n_j}(\alpha_j) & 0 \\ 0 & -J_{n_j}(\alpha_j) \end{bmatrix},$$

$$K_j = s_j \begin{bmatrix} R_{n_j} & 0 \\ 0 & \left(\frac{\gamma_j}{|\gamma_j|}\right)^2 R_{n_j} \end{bmatrix}, \quad M_j = \begin{bmatrix} 0 & \gamma_j R_{n_j} \\ \bar{\gamma}_j R_{n_j} & 0 \end{bmatrix},$$

where $0 < \alpha_j \in \mathbb{R}$, $s_j = \pm 1$, $\gamma_j = \beta_j$ *or* $\gamma_j = i\beta_j$ *with* $0 < \beta_j \in \mathbb{R}$*. The Jordan block of* A *associated with* α_j *has the sign* s_j *with respect to* K *and the one associated with* $-\alpha_j$ *has the sign* $(-1)^{n_j+1} s_j (\gamma_j/|\gamma_j|)^2$ *with respect to* K*.*

(iv) Blocks associated with a pair of purely imaginary eigenvalues $\pm i\alpha_j$ *of* A *and nonreal non purely imaginary eigenvalues of* $\lambda K - M$*:*

$$A_j = \begin{bmatrix} iJ_{n_j}(\alpha_j) & 0 & 0 & 0 \\ 0 & -iJ_{n_j}(\alpha_j) & 0 & 0 \\ 0 & 0 & iJ_{n_j}(\alpha_j) & 0 \\ 0 & 0 & 0 & -iJ_{n_j}(\alpha_j) \end{bmatrix},$$

$$K_j = \begin{bmatrix} 0 & 0 & 0 & R_{n_j} \\ 0 & 0 & R_{n_j} & 0 \\ 0 & R_{n_j} & 0 & 0 \\ R_{n_j} & 0 & 0 & 0 \end{bmatrix}, \quad M_j = \begin{bmatrix} 0 & 0 & \gamma_j R_{n_j} & 0 \\ 0 & 0 & 0 & \gamma_j R_{n_j} \\ \bar{\gamma}_j R_{n_j} & 0 & 0 & 0 \\ 0 & \bar{\gamma}_j R_{n_j} & 0 & 0 \end{bmatrix},$$

where $0 < \alpha_j \in \mathbb{R}$ *and* $\gamma_j = c_j + i d_j$ *with* $c_j, d_j \in \mathbb{R}$ *and* $c_j d_j > 0$. *The two Jordan blocks associated with* $i\alpha_j$ *have the signs* 1 *and* -1 *with respect to* M *and the two Jordan blocks associated with* $-i\alpha_j$ *also have the signs* 1 *and* -1 *with respect to* M.

(v) *Blocks associated with a pair of purely imaginary eigenvalues* $\pm i\alpha_j$ *of* A *and real or purely imaginary eigenvalues of* $\lambda K - M$:

$$A_j = \begin{bmatrix} iJ_{n_j}(\alpha_j) & 0 \\ 0 & -iJ_{n_j}(\alpha_j) \end{bmatrix},$$

$$K_j = \begin{bmatrix} 0 & R_{n_j} \\ R_{n_j} & 0 \end{bmatrix}, \quad M_j = s_j|\gamma_j| \begin{bmatrix} R_{n_j} & 0 \\ 0 & \left(\frac{|\gamma_j|}{\gamma_j}\right)^2 R_{n_j} \end{bmatrix},$$

where $0 < \alpha_j \ in\, \mathbb{R}$, $s_j = \pm 1$, $\gamma_j = \beta_j$ *or* $\gamma_j = i\beta_j$ *with* $0 < \beta_j \in \mathbb{R}$. *The Jordan block of* A *associated with* $i\alpha_j$ *has sign* s_j *with respect to* M *and the one associated with* $-\alpha_j$ *has sign* $(-1)^{n_j+1} s_j (\gamma_j/|\gamma_j|)^2$ *with respect to* M.

(vi) *A pair of blocks associated with the eigenvalue zero of* A *and nonreal, non purely imaginary eigenvalues of* $\lambda K - M$:

$$A_j = \begin{bmatrix} J_{n_j}(0) & 0 \\ 0 & J_{n_j}(0) \end{bmatrix}, \quad K_j = \begin{bmatrix} 0 & R_{n_j} \\ R_{n_j} & 0 \end{bmatrix},$$

$$M_j = s_j \begin{bmatrix} 0 & \gamma_j \Sigma_{n_j} \\ (-1)^{n_j+1}\bar{\gamma}_j \Sigma_{n_j} & 0 \end{bmatrix},$$

where $s_j = \pm 1$, $\gamma_j = c_j + i d_j$ *with* $c_j, d_j \in \mathbb{R}$ *and* $c_j d_j > 0$. *The two Jordan blocks of* A *associated with the eigenvalue* 0 *have the signs* 1 *and* -1 *with respect to both* K *and* M.

(vii) *A pair of blocks associated with the eigenvalue zero of* A *and real or purely imaginary eigenvalues of* $\lambda K - M$:

$$A_j = \begin{bmatrix} J_{n_j}(0) & 0 \\ 0 & J_{n_j}(0) \end{bmatrix}, \quad K_j = \begin{bmatrix} 0 & R_{n_j} \\ R_{n_j} & 0 \end{bmatrix}, \quad M_j = \begin{bmatrix} 0 & \gamma_j \Sigma_{n_j} \\ -\gamma_j \Sigma_{n_j} & 0 \end{bmatrix},$$

where $\gamma_j = \beta_j$ *if* n_j *is even and* $\gamma_j = i\beta_j$ *if* n_j *is odd for some* $0 < \beta_j \in \mathbb{R}$. *The two Jordan blocks of* A *associated with the eigenvalue zero of* A *have the signs* 1 *and* -1 *with respect to both* K *and* M.

(viii) *A single block associated with the eigenvalue zero of* A *and real or purely imaginary eigenvalues of* $\lambda K - M$:

$$A_j = J_{n_j}(0), \quad K_j = s_j R_{n_j}, \quad M_j = \sigma_j \gamma_j \Sigma_{n_j},$$

where $s_j, \sigma_j = \pm 1$; *and* $\gamma_j = \beta_j$ *if* n_j *is odd and* $\gamma_j = i\beta_j$ *if* n_j *is even for some* $0 < \beta_j \in \mathbb{R}$. *The Jordan block of* A *associated with the eigenvalue zero has the sign* s_j *with respect to* K *and the sign* $\frac{\gamma_j}{|\gamma_j|}\sigma_j\, i^{n_j-1}$ *with respect to* M.

Theorem 7 shows the intertwined connection of the three different structures: the double structure of A with respect to K and M and the structure of the Hermitian pencil $\lambda K - M$. The property of being K-Hermitian forces the spectrum of A to be symmetric with respect to the real axis and the property of being M-skew-Hermitian forces the spectrum to be symmetric with respect to the imaginary axis. The particular structure of blocks, however, depends in addition on the eigenvalues of the Hermitian pencil $\lambda K - M$. Interestingly, there is not only a distinction between real and nonreal eigenvalues of $\lambda K - M$, but also the purely imaginary eigenvalues of $\lambda K - M$ play a special role. This effect can in particular be seen in the blocks associated with the eigenvalue zero, the only point in the complex plane that is both real and purely imaginary. Depending on the type of the corresponding eigenvalues of $\lambda K - M$, we have the following cases:

(a) Real eigenvalues of $\lambda K - M$: in this case, *even*-sized Jordan blocks of A associated with zero must occur in pairs (vii), but *odd*-sized Jordan blocks need not (viii);
(b) Purely imaginary eigenvalues of $\lambda K - M$: in this case, *odd*-sized Jordan blocks of A associated with zero must occur in pairs (vii), but *even*-sized Jordan blocks need not (viii);
(c) Nonreal, non purely imaginary eigenvalues of $\lambda K - M$: in this case, all Jordan blocks of A associated with the eigenvalue zero must occur in pairs (vi).

Structured canonical forms for A as a K-Hermitian matrix, for A as an M-Hermitian matrix and for the Hermitian pencil $\lambda K - M$ can be easily derived from the canonical form in Theorem 7, so again the result combines three different canonical forms into one.

As the particular application from quantum chemistry shows, there is also interest in doubly structured generalized eigenvalue problems. In general, we can consider a matrix pencil $\lambda A - B$ with both A, B being doubly structured with respect to two invertible Hermitian or skew-Hermitian matrices K and M. It turns out that a structured Weierstraß canonical form for a regular doubly structured pencil can easily be derived by using the results of the matrix case as in Theorems 6 and 7.

Theorem 8 ([34]) *Suppose K, M are both invertible and each is either Hermitian or skew-Hermitian, i.e.,*

$$K^* = \sigma_K K, \quad M^* = \sigma_M M, \qquad \sigma_K, \sigma_M = \pm 1.$$

Let $\lambda A - B$ be a regular pencil (that is $\det(\lambda A - B) \not\equiv 0$*) with A, B satisfying*

$$A^* K = \varepsilon_A KA, \quad A^* M = \delta_A MA, \qquad \varepsilon_A, \delta_A = \pm 1$$
$$B^* K = \varepsilon_B KB, \quad B^* M = \delta_B MB, \qquad \varepsilon_B, \delta_B = \pm 1.$$

Then there exist invertible matrices X, Y such that

$$Y^{-1}(\lambda A - B)X = \lambda \begin{bmatrix} I & 0 \\ 0 & E \end{bmatrix} - \begin{bmatrix} H & 0 \\ 0 & I \end{bmatrix}$$

$$X^*KY = \begin{bmatrix} K_1 & 0 \\ 0 & K_2 \end{bmatrix}, \qquad X^*MY = \begin{bmatrix} M_1 & 0 \\ 0 & M_2 \end{bmatrix},$$

where E is nilpotent and all three matrices in (H, K_1, M_1) and (E, K_2, M_2), respectively, have the same sizes. Furthermore, we have that

$$K_1^* = \sigma_K \varepsilon_A K_1, \quad M_1^* = \sigma_M \delta_A M_1; \quad H^* K_1 = (\varepsilon_A \varepsilon_B) K_1 H, \quad H^* M_1 = (\delta_A \delta_B) M_1 H,$$
$$K_2^* = \sigma_K \varepsilon_B K_2, \quad M_2^* = \sigma_M \delta_B M_2; \quad E^* K_2 = (\varepsilon_A \varepsilon_B) K_2 E, \quad E^* M_2 = (\delta_A \delta_B) M_2 E.$$

Clearly, H is a doubly structured matrix associated with the Hermitian or skew-Hermitian matrices K_1, M_1, and E is a doubly structured matrix associated with the Hermitian or skew Hermitian matrices K_2, M_2. Thus, the pencil $\lambda A - B$ is decoupled and becomes $(\lambda I - H) \oplus (\lambda E - I)$. Hence a structured Weierstraß canonical form of $\lambda A - B$ can be derived by applying the results in Theorems 6 and 7 to H and E separately.

Note that in Theorem 8 one does not require $\lambda K - M$ to be nondefective. However, in order to apply Theorems 6 or 7 to obtain structured Jordan canonical forms for the matrices H and E, the condition that both $\lambda K_1 - M_1$ and $\lambda K_2 - M_2$ are nondefective is necessary.

Finally, we point out that for the special type of doubly structured matrices and matrix pencils from linear response theory [16, 41, 49], necessary and sufficient conditions for the existence of structured Schur-like forms (obtained under unitary transformations) were provided in [36].

6.5 Structured Singular Value Decompositions

The singular value decomposition (SVD) is an important tool in Matrix Theory and Numerical Linear Algebra. For a given matrix $A \in \mathbb{C}^{m \times n}$ it computes unitary matrices X, Y such that Y^*AX is diagonal with nonnegative diagonal entries. The condition that X and Y are unitary can be interpreted in such a way that the standard Euclidean inner product is preserved by the transformation with X and Y. Thus, to be more precise, we have a transformation on the matrix triple (A, I_n, I_m) that yields the canonical form

$$Y^*AX = \begin{bmatrix} \Delta & 0 \\ 0 & 0 \end{bmatrix}, \quad X^* I_n X = I_n, \quad Y^* I_m Y = I_m,$$

where Δ is diagonal with positive diagonal entries. But the singular value decomposition even yields more information as the nonzero singular values are the square roots of the positive eigenvalues of the matrices AA^* and A^*A. Thus, in addition to a canonical form for A under unitary equivalence, the SVD simultaneously provides two spectral decompositions

$$Y^*(AA^*)Y = \begin{bmatrix} \Delta^2 & 0 \\ 0 & 0 \end{bmatrix}, \quad X^*(A^*A)X = \begin{bmatrix} \Delta^2 & 0 \\ 0 & 0 \end{bmatrix},$$

for the Hermitian (positive semi-definite) matrices AA^* and A^*A.

This concept can be generalized to the case of possibly indefinite inner products. Suppose that the two spaces $\mathbb{C}^n$ and $\mathbb{C}^m$ are equipped with inner products given by the Hermitian invertible matrices $K \in \mathbb{C}^{n\times n}$ and $M \in \mathbb{C}^{m\times m}$, respectively. Then the task is to find invertible matrices X and Y such that

$$\mathscr{A} = Y^*AX, \quad \mathscr{K} = X^*KX, \quad \mathscr{M} = Y^*MY, \tag{6.10}$$

are in a canonical form, so that also the canonical forms of the K-Hermitian matrix T and the M-Hermitian matrix Z can easily be derived, where

$$T = K^{-1}A^*M^{-1}A, \qquad Z = M^{-1}AK^{-1}A^*. \tag{6.11}$$

Equivalently, we obtain structured canonical forms for the two Hermitian pencils $\lambda K - A^*M^{-1}A$ and $\lambda M - AK^{-1}A^*$

The transformation (6.10) has several mathematical applications. For instance, the existence of a generalization of a polar decompositions for a matrix A in a space equipped with an indefinite inner product as in (6.1) given by the invertible Hermitian matrix $\widetilde{M} \in \mathbb{C}^{n\times n}$ is related to the matrices $AA^{[*]}$ and $A^{[*]}A$, where $A^{[*]} := \widetilde{M}^{-1}A^*\widetilde{M}$. By definition, a matrix $A \in \mathbb{C}^{n\times n}$ is said to have an $\widetilde{M}$*-polar decomposition*, if there exists an $\widetilde{M}$-Hermitian matrix H and an $\widetilde{M}$-unitary matrix U, such that $A = UH$, see [5, 6]. In contrast to the classical polar decomposition in the case of the Euclidean inner product, an $\widetilde{M}$-polar decomposition need not exist for a given matrix $A \in \mathbb{C}^{n\times n}$. In [37], it was proved that a matrix $A \in \mathbb{C}^{n\times n}$ allows an $\widetilde{M}$-polar decomposition if and only if the two $\widetilde{M}$-Hermitian matrices $AA^{[*]}$ and $A^{[*]}A$ have the same canonical forms (as in Corollary 1) – a fact that was already conjectured in [18]. If a canonical form under a transformation as in (6.10) is given with the matrices T and Z as in (6.11), then we have that $AA^{[*]} = MZM^{-1}$ and $A^{[*]}A = T$ with $K = \widetilde{M}$ and $M = \widetilde{M}^{-1}$. Thus, structured canonical forms can easily be derived from the canonical form under the transformation (6.10).

On the other hand, the simultaneous transformation (6.10) provides more flexibility in solving the eigenvalue problem of a structured matrix as $B = A^*M^{-1}A$ from a numerical point of view. That is, instead of performing similarity transformations on B, one may use two-sided transformations on A. For example, when $K = \mathrm{J}$ and $M = I$, a structured condensed form for a matrix A was proposed in [53] and a numerical method was given in [54].

In the case when A is invertible (hence square), the following theorem provides the desired canonical form for the transformation in (6.10). Here, we use the notation $J_m^2(\alpha)$ for the square of a Jordan block $J_m(\alpha)$ of size m associated with the eigenvalue α.

Theorem 9 ([35]) *Let $A \in \mathbb{C}^{n\times n}$ be nonsingular and let $K, M \in \mathbb{C}^{n\times n}$ be Hermitian and invertible. Then there exist invertible matrices $X, Y \in \mathbb{C}^{n\times n}$ such that*

$$Y^*AX = A_c \oplus A_r, \qquad X^*KX = K_c \oplus K_r, \qquad Y^*MY = M_c \oplus M_r. \tag{6.12}$$

Consequently, for the K-Hermitian matrix $T = K^{-1}A^*M^{-1}A$ and the M-Hermitian matrix $Z = M^{-1}AK^{-1}A^*$, one has

$$X^{-1}TX = T_c \oplus T_r, \qquad Y^{-1}ZY = Z_c \oplus Z_r. \tag{6.13}$$

The diagonal blocks in (6.12) and (6.13) have the following forms.

(i)

$$\begin{aligned}
A_c &= \begin{bmatrix} J_{m_1}(\mu_1) & 0 \\ 0 & J_{m_1}(\bar{\mu}_1) \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} J_{m_p}(\mu_p) & 0 \\ 0 & J_{m_p}(\bar{\mu}_p) \end{bmatrix}, \\
K_c &= \begin{bmatrix} 0 & R_{m_1} \\ R_{m_1} & 0 \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} 0 & R_{m_p} \\ R_{m_p} & 0 \end{bmatrix}, \\
M_c &= \begin{bmatrix} 0 & R_{m_1} \\ R_{m_1} & 0 \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} 0 & R_{m_p} \\ R_{m_p} & 0 \end{bmatrix}, \\
T_c &= \begin{bmatrix} J_{m_1}^2(\mu_1) & 0 \\ 0 & J_{m_1}^2(\bar{\mu}_1) \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} J_{m_p}^2(\mu_p) & 0 \\ 0 & J_{m_p}^2(\bar{\mu}_p) \end{bmatrix}, \\
Z_c &= \begin{bmatrix} J_{m_1}^2(\mu_1) & 0 \\ 0 & J_{m_1}^2(\bar{\mu}_1) \end{bmatrix}^* \oplus \cdots \oplus \begin{bmatrix} J_{m_p}^2(\mu_p) & 0 \\ 0 & J_{m_p}^2(\bar{\mu}_p) \end{bmatrix}^*,
\end{aligned}$$

where $\mu_j = a_j + ib_j$ with $0 < a_j, b_j \in \mathbb{R}$ for $j = 1, \ldots, p$. For each j, both the diagonal block $\operatorname{diag}\big(J_{m_j}^2(\mu_j), J_{m_j}^2(\bar{\mu}_j)\big)$ *of T_c as well as the diagonal block* $\operatorname{diag}\big(J_{m_j}^2(\mu_j), J_{m_j}^2(\bar{\mu}_j)\big)^*$ *of Z_c are similar to a matrix consisting of two $m_j \times m_j$ Jordan blocks, one of them associated with the nonreal and non purely imaginary eigenvalue μ_j^2 and the other one with $\bar{\mu}_j^2$.*

(ii)

$$\begin{aligned}
A_r &= J_{n_1}(\beta_1) \oplus \cdots \oplus J_{n_q}(\beta_q), \\
K_r &= s_1 R_{n_1} \oplus \cdots \oplus s_q R_{n_q}, \\
M_r &= \sigma_1 R_{n_1} \oplus \cdots \oplus \sigma_q R_{n_q}, \\
T_r &= s_1\sigma_1 J_{n_1}^2(\beta_1) \oplus \cdots \oplus s_q\sigma_q J_{n_q}^2(\beta_q), \\
Z_r &= s_1\sigma_1 \big(J_{n_1}^2(\beta_1)\big)^* \oplus \cdots \oplus s_q\sigma_q \big(J_{n_q}^2(\beta_q)\big)^*,
\end{aligned}$$

where $\beta_j > 0$, *and* $s_j, \sigma_j = \pm 1$ *for* $j = 1, \ldots, q$. *For each* j, *the block* $s_j \sigma_j J_{n_j}^2(\beta_j)$ *of* T_r *is similar to an* $n_j \times n_j$ *Jordan block associated with a real eigenvalue* $s_j \sigma_j \beta_j^2$ *of* T *with the sign with respect to* K *being*

$$\begin{cases} s_j & \text{if } n_j \text{ is odd, or if } n_j \text{ is even and } s_j\sigma_j = 1, \\ \sigma_j & \text{if } n_j \text{ is even and } s_j\sigma_j = -1, \end{cases}$$

and the block $s_j \sigma_j \left(J_{n_j}^2(\beta_j)\right)^*$ *of* Z_r *is similar to an* $n_j \times n_j$ *Jordan block associated with a real eigenvalue* $s_j \sigma_j \beta_j^2$ *of* Z *with the sign with respect to* M *being*

$$\begin{cases} \sigma_j & \text{if } n_j \text{ is odd, or if } n_j \text{ is even and } s_j\sigma_j = 1, \\ s_j & \text{if } n_j \text{ is even and } s_j\sigma_j = -1. \end{cases}$$

For a general rectangular matrix $A \in \mathbb{C}^{m \times n}$, the situation is more complicated because of (a) the rectangular form of A and (b) the presence of the eigenvalue 0 in T or Z. Indeed, note that these two matrices T and Z can be represented as products of the same two factors, but with different order, i.e., $T = BC$ and $Z = CB$, where $B = K^{-1}A^*$ and $C = M^{-1}A$. By a well-known result [11], the Jordan structures of the nonzero eigenvalues of the two matrix products BC and CB are identical, while this is not the case for the eigenvalue zero. Despite this additional complexity in the problem of finding a canonical form under the transformation (6.10), a complete answer is still possible as shown in the next theorem.

Theorem 10 ([35]) *Let* $A \in \mathbb{C}^{m \times n}$, *and let* $K \in \mathbb{C}^{n \times n}$ *and* $M \in \mathbb{C}^{m \times m}$ *be Hermitian and invertible. Then there exist invertible matrices* $Y \in \mathbb{C}^{m \times m}$ *and* $X \in \mathbb{C}^{n \times n}$ *such that*

$$\begin{aligned} Y^*AX &= A_c \oplus A_r \oplus A_1 \oplus A_2 \oplus A_3 \oplus A_4, \\ X^*KX &= K_c \oplus K_r \oplus K_1 \oplus K_2 \oplus K_3 \oplus K_4, \\ Y^*MY &= M_c \oplus M_r \oplus M_1 \oplus M_2 \oplus M_3 \oplus M_4. \end{aligned} \tag{6.14}$$

Moreover, for the K*-Hermitian matrix* $T = K^{-1}A^*M^{-1}A \in \mathbb{C}^{n \times n}$ *and for the* M*-Hermitian matrix* $Z = M^{-1}AK^{-1}A^* \in \mathbb{C}^{m \times m}$ *we have that*

$$\begin{aligned} X^{-1}TX &= T_c \oplus T_r \oplus T_1 \oplus T_2 \oplus T_3 \oplus T_4, \\ Y^{-1}ZY &= Z_c \oplus Z_r \oplus Z_1 \oplus Z_2 \oplus Z_3 \oplus Z_4. \end{aligned}$$

The blocks $A_c, A_r, K_c, K_r, M_c, M_r$ *have the same forms as in* (6.12). *Therefore, the blocks* T_c, T_r *and* Z_c, Z_r *have the same forms as in* (6.13). *The remaining blocks are associated with the eigenvalue* 0 *of* T *and* Z *and have the following forms.*

(i)

$$A_1 = 0_{\ell\times k},\ \ K_1 = \mathrm{diag}(I_{k_1}, -I_{k_2}),\ \ M_1 = \mathrm{diag}(I_{\ell_1}, -I_{\ell_2}),\ T_1 = 0_k,\ Z_1 = 0_\ell,$$

where $k_1 + k_2 = k$ *and* $\ell_1 + \ell_2 = \ell$. *So there are* k *copies of* 1×1 *Jordan blocks associated with the eigenvalue* 0 *of* T *such that* k_1 *of them have the sign* $+1$ *and* k_2 *of them the sign* -1 *with respect to* K, *and there are* ℓ *copies of* 1×1 *Jordan blocks associated with the eigenvalue* 0 *of* Z *such that* ℓ_1 *of them have the sign* $+1$ *and* ℓ_2 *of them the sign* -1.

(ii)

$$\begin{aligned}
A_2 &= J_{2r_1}(0) \oplus J_{2r_2}(0) \oplus\cdots\oplus J_{2r_u}(0)\ ,\\
K_2 &= R_{2r_1} \oplus R_{2r_2} \oplus\cdots\oplus R_{2r_u}\ ,\\
M_2 &= R_{2r_1} \oplus R_{2r_2} \oplus\cdots\oplus R_{2r_u}\ ,\\
T_2 &= J^2_{2r_1}(0) \oplus J^2_{2r_2}(0) \oplus\cdots\oplus J^2_{2r_u}(0)\ ,\\
Z_2 &= \big(J^2_{2r_1}(0)\big)^T \oplus \big(J^2_{2r_2}(0)\big)^T \oplus\cdots\oplus \big(J^2_{2r_u}(0)\big)^T.
\end{aligned}$$

For each $j = 1, \dots, u$, *the block* $J^2_{2r_j}(0)$ *of* T_2 *is similar to a matrix consisting of two copies of the Jordan block* $J_{r_j}(0)$ *of* T *with one of them having the sign* $+1$ *and the other having the sign* -1 *with respect to* K, *and the block* $\big(J^2_{2r_j}(0)\big)^T$ *is similar to a matrix consisting of two copies of the Jordan block* $J_{r_j}(0)$ *of* Z *with one of them having the sign* $+1$ *and the other having the sign* -1 *with respect to* M.

(iii)

$$\begin{aligned}
A_3 &= \begin{bmatrix} I_{s_1} \\ 0 \end{bmatrix}_{(s_1+1)\times s_1} \oplus \begin{bmatrix} I_{s_2} \\ 0 \end{bmatrix}_{(s_2+1)\times s_2} \oplus\cdots\oplus \begin{bmatrix} I_{s_v} \\ 0 \end{bmatrix}_{(s_v+1)\times s_v},\\
K_3 &= \phi_1 R_{s_1} \oplus \phi_2 R_{s_2} \oplus\cdots\oplus \phi_v R_{s_v}\ ,\\
M_3 &= \psi_1 R_{s_1+1} \oplus \psi_2 R_{s_2+1} \oplus\cdots\oplus \psi_v R_{s_v+1}\ ,\\
T_3 &= \phi_1\psi_1 J_{s_1}(0) \oplus \phi_2\psi_2 J_{s_2}(0) \oplus\cdots\oplus \phi_v\psi_v J_{s_v}(0)\ ,\\
Z_3 &= \phi_1\psi_1 J^T_{s_1+1}(0) \oplus \phi_2\psi_2 J^T_{s_2+1}(0) \oplus\cdots\oplus \phi_v\psi_v J^T_{s_v+1}(0),
\end{aligned}$$

where for $j = 1, \dots, v$, $\phi_j = 1$ *and* $\psi_j = \pm 1$ *if* s_j *is even, and* $\phi_j = \pm 1$ *and* $\psi_j = 1$ *if* s_j *is odd. Hence, for each* j, *the block* $\phi_j\psi_j J_{s_j}(0)$ *of* T_3 *is a modified* $s_j \times s_j$ *Jordan block associated with the eigenvalue* 0 *of* T *with sign* ϕ_j *if* s_j *is odd and* ψ_j *if* s_j *is even; the block* $\phi_j\psi_j J^T_{s_j+1}(0)$ *of* Z_3 *is a modified* $(s_j + 1) \times (s_j + 1)$ *Jordan block associated with the eigenvalue* 0 *of* Z *with sign* ϕ_j *if* s_j *is odd and* ψ_j *if* s_j *is even.*

(iv)

$$
\begin{aligned}
A_4 &= \begin{bmatrix}0 & I_{t_1}\end{bmatrix}_{t_1\times(t_1+1)} \oplus \begin{bmatrix}0 & I_{t_2}\end{bmatrix}_{t_2\times(t_2+1)} \oplus \cdots \oplus \begin{bmatrix}0 & I_{t_w}\end{bmatrix}_{t_w\times(t_w+1)},\\
K_4 &= \theta_1 R_{t_1+1} \oplus \theta_2 R_{t_2+1} \oplus \cdots \oplus \theta_w R_{t_w+1},\\
M_4 &= \rho_1 R_{t_1} \oplus \rho_2 R_{t_2} \oplus \cdots \oplus \rho_w R_{t_w},\\
T_4 &= \theta_1\rho_1 J_{t_1+1}(0) \oplus \theta_2\rho_2 J_{t_2+1}(0) \oplus \cdots \oplus \theta_w\rho_w J_{t_w+1}(0),\\
Z_4 &= \theta_1\rho_1 J_{t_1}^T(0) \oplus \theta_2\rho_2 J_{t_2}^T(0) \oplus \cdots \oplus \theta_w\rho_w J_{t_w}^T(0),
\end{aligned}
$$

where for $j = 1,\ldots,w$, $\theta_j = 1$ and $\rho_j = \pm 1$ if t_j is odd, and $\theta_j = \pm 1$ and $\rho_j = 1$ if t_j is even. Hence, for each j, the block $\theta_j\rho_j J_{t_j+1}(0)$ of T_4 is a modified $(t_j + 1) \times (t_j + 1)$ Jordan block associated with the eigenvalue 0 of T with sign ρ_j if t_j is odd and θ_j if t_j is even; the block $\theta_j\rho_j J_{t_j}^T(0)$ of Z_4 is a modified $t_j \times t_j$ Jordan block associated with the eigenvalue 0 of Z with sign ρ_j if t_j is odd and θ_j if t_j is even.

Theorem 10 shows that the sizes of the Jordan blocks and signs associated with the eigenvalue zero may be different for the matrices T and Z. Still, they are related and the canonical form for A exactly explains in which way.

As mentioned earlier, the investigation of the canonical forms of the matrices $AA^{[*]}$ and $A^{[*]}A$ is crucial if one wants to check if $A \in \mathbb{C}^{n\times n}$ has an M-polar decomposition with respect to the invertible Hermitian matrix M. Therefore, the possible difference in the canonical forms of $AA^{[*]}$ and $A^{[*]}A$ has been analyzed in [17]. With the canonical form from Theorem 10 there is now a complete classification of all possible canonical forms for the matrices $AA^{[*]}$ and $A^{[*]}A$ for a general matrix A.

A real version of (6.14) for real A, K, M can be derived essentially in the same way. In the case that all A, K, and M are real and at least one of K and M is skew-symmetric, the real canonical forms of the simultaneous transformation (6.10) can be derived too, but with some additional techniques. The details can be found in [35].

6.6 Conclusion

Applications in different areas provide a variety of eigenvalue problems with different symmetry structures that lead to symmetries in the spectra of the corresponding matrices or matrix pencils. It is crucial to use structure-preserving algorithms so that the symmetry in the spectra is not lost due to roundoff errors in the numerical computation and that the computed results are physically meaningful. For the understanding of the behavior of these algorithms and the effect in the corresponding perturbation theory, structured canonical forms are an essential tool.

In this survey, we have presented three particular structured canonical forms with respect to matrices that carry one or two structures with respect to possible indefinite inner products. Moreover, we have highlighted the important role that the sign characteristic plays in the understanding of the behavior of Hamiltonian matrices under structure-preserving transformations.

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