# Chapter 15
# Regularization of Descriptor Systems

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**Abstract** Implicit dynamic-algebraic equations, known in control theory as descriptor systems, arise naturally in many applications. Such systems may not be regular (often referred to as singular). In that case the equations may not have unique solutions for consistent initial conditions and arbitrary inputs and the system may not be controllable or observable. Many control systems can be "regularized" by proportional and/or derivative feedback. We present an overview of mathematical theory and numerical techniques for regularizing descriptor systems using feedback controls. The aim is to provide stable numerical techniques for analyzing and constructing regular control and state estimation systems and for ensuring that these systems are robust. State and output feedback designs for regularizing linear time-invariant systems are described, including methods for disturbance decoupling and mixed output problems. Extensions of these techniques to time-varying linear and nonlinear systems are discussed in the final section.

## 15.1 Introduction

Singular systems of differential equations, known in control theory as *descriptor systems* or *generalized state-space systems*, have fascinated Volker Mehrmann throughout his career. His early research, starting with his habilitation [33, 35], concerned autonomous linear-quadratic control problems constrained by descriptor systems. Descriptor systems arise naturally in many applications, including aircraft guidance, chemical processing, mechanical body motion, power generation, network fluid flow and many others, and can be considered as continuous or discrete implicit dynamic-algebraic systems [32, 41]. Such systems may not be regular (often

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referred to as singular). In that case unique solutions to initial value problems consistent with the system may not exist and the system may not be controllable or observable. An important aspect of control system design is therefore to ensure regularity of the system.

In this chapter we review the work of Volker and his colleagues on mathematical theory and numerical techniques for regularizing descriptor systems using feedback controls. Two key elements contributed initially to the research: the establishment of conditions for the regularizability of descriptor systems by feedback [25, 30] and the development of stable numerical techniques for the reduction of descriptor systems to condensed matrix forms [33, 34, 36]. Following a stimulating meeting at the International Conference on Linear Algebra and Applications in Valencia in 1987, these two research threads were brought together in a report on feedback design for descriptor systems [5] and later published in [6] and [7].

Since that time, Volker has contributed to a whole sequence of exciting results on the regularization of descriptor systems [3, 8–12, 15, 20–22, 24, 31, 37]. The development of sound numerical methods for system design, as well as techniques for guaranteeing the *robustness* of the systems to model uncertainties and disturbances, has formed the main emphasis throughout this research. We describe some of this work in the next sections.

We start with preliminary definitions and properties of descriptor systems and then discuss regularization by state feedback for linear time-invariant systems. Disturbance decoupling by state feedback is also discussed. The problem of regularization by output feedback is then considered. Further developments involving mixed output feedback regularization are given next, and finally work on time-varying and nonlinear systems is briefly described.

## 15.2 System Design for Descriptor Systems

We consider linear dynamical control systems of the form

$$\begin{aligned} E\dot{x}(t) &= Ax(t) + Bu(t), \quad x(t_0) = x_0, \\ y(t) &= Cx(t), \end{aligned} \tag{15.1}$$

or, in the discrete-time case,

$$\begin{aligned} Ex(k+1) &= Ax(k) + Bu(k), \quad x(0) = x_0, \\ y(k) &= Cx(k), \end{aligned} \tag{15.2}$$

where $E, A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$. Here $x(\cdot)$ is the state, $y(\cdot)$ is the output, and $u(\cdot)$ is the input or control of the system. It is assumed that $m, p \leq n$ and that the matrices $B, C$ are of full rank. The matrix $E$ may be *singular*. Such systems are known as *descriptor* or *generalized state-space* systems. In the case $E = I$, the identity matrix, we refer to (15.1) or (15.2) as a *standard* system.

We assume initially that the system is time-invariant; that is, the system matrices $E, A, B, C$ are constant, independent of time. In this context, we are interested in proportional and derivative feedback control of the form $u(t) = Fy(t) - G\dot{y}(t) + v(t)$ or $u(k) = Fy(k) - Gy(k+1) + v(k)$, where $F, G \in \mathbb{R}^{m \times p}$ are selected to give the closed-loop system

$$(E + BGC)\dot{x}(t) = (A + BFC)x(t) + Bv(t) \tag{15.3}$$

or

$$(E + BGC)x(k+1) = (A + BFC)x(k) + Bv(k) \tag{15.4}$$

desired properties. *Proportional output* feedback control is achieved in the special case $G = 0$. *Derivative output* feedback control corresponds to the special case $F = 0$ and derivative and proportional *state* feedback control corresponds to the special case $C = I$. The dual of the control system, an *observer* (or state-estimator), is attained with an appropriate choice for $v$ in the special case $B = I$. The aim of the feedback designs is to alter the behaviour of the system response. Proportional feedback acts to modify the system matrix $A$, whilst derivative feedback alters the system matrix $E$. Different properties of the system can, therefore, be achieved using different feedback combinations.

### *15.2.1 Structure of the System Response*

The response of the descriptor system (15.1) or (15.2) can be described in terms of the eigenstructure of the matrix pencil $\alpha E - \beta A$, which we denote by $(E, A)$. The system is *regular* if the pencil $(E, A)$ is regular, that is,

$$\det(\alpha E - \beta A) \neq 0 \ \text{ for some } \ (\alpha, \beta) \in \mathbb{C}^2. \tag{15.5}$$

The generalized eigenvalues of a regular pencil are defined by the pairs $(\alpha_j, \beta_j) \in \mathbb{C}^2 \backslash \{0, 0\}$ such that

$$\det(\alpha_j E - \beta_j A) = 0, \quad j = 1, 2, \ldots, n. \tag{15.6}$$

If $\beta_j \neq 0$, the eigenvalue pair is said to be *finite* with value given by $\lambda_j = \alpha_j / \beta_j$ and otherwise, if $\beta_j = 0$, then the pair is said to be an *infinite* eigenvalue. The maximum number of finite eigenvalues that a pencil can have is less than or equal to the rank of $E$.

If the system (15.1) or (15.2) is regular, then the existence and uniqueness of classical smooth solutions to the dynamical equations is guaranteed for sufficiently smooth inputs and consistent initial conditions [14, 43]. The solutions are characterized in terms of the Kronecker Canonical Form (KCF) [26]. Nonsingular matrices

$X$ and $Y$ (representing right and left generalized eigenvectors and principal vectors of the system pencil, respectively) then exist such that

$$XEY = \begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix}, \quad XAY = \begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix}, \tag{15.7}$$

where the eigenvalues of the Jordan matrix $J$ coincide with the finite eigenvalues of the pencil and $N$ is a nilpotent Jordan matrix such that $N^i = 0$, $N^{i-1} \neq 0$, $i > 0$, corresponding to the infinite eigenvalues. The *index* of a descriptor system, denoted by $\text{ind}(E, A)$, is defined to be the degree $i$ of nilpotency of the matrix $N$, that is, the index of the system is the dimension of the largest Jordan block associated with an infinite eigenvalue of the KCF (15.7). The index is a fundamental characteristic of a descriptor system, determining the existence and smoothness of solutions.

By convention, a descriptor system is regular and of index 0 if and only if $E$ is nonsingular. In this case the system can be reformulated as a standard system. However, the reduction to standard form can be numerically unreliable if $E$ is ill-conditioned with respect to inversion. Therefore it is desirable to work directly with the generalized state-space form even where $E$ is nonsingular.

A descriptor system is regular and has index at most one if and only if it has exactly $q = \text{rank}(E)$ finite eigenvalues and $n - q$ *non-defective* infinite eigenvalues. Conditions for the system to be regular and of index $\leq 1$ are given by the following important result.

**Theorem 1 ([25, 30])** *Let $E, A \in \mathbb{R}^{n\times n}$ and let $S_\infty(E)$ and $T_\infty(E)$ be full rank matrices whose columns span the null spaces $\mathcal{N}(E)$ and $\mathcal{N}(E^T)$ respectively. Then the following are equivalent:*

*(i)* $\alpha E - \beta A$ *is regular and of index* $\leq 1$;
*(ii)* $rank([E, AS_\infty(E)]) = n$;
*(iii)* $rank\left(\begin{bmatrix} E \\ T_\infty^T(E)A \end{bmatrix}\right) = n$;
*(iv)* $rank(T_\infty^T(E)AS_\infty(E)) = n - rank(E)$.

Systems that are regular and of index at most one can be separated into purely dynamical and algebraic parts (fast and slow modes) [14, 23] and in theory the algebraic part can be eliminated to give a reduced-order standard system. The reduction process, however, may be ill-conditioned for numerical computation and lead to large errors in the reduced order system [28]. If the system is not regular or if $\text{ind}(E, A) > 1$, then impulses can arise in the response of the system if the control is not sufficiently smooth [27, 42]. Since the linear constant coefficient system is usually only a model that approximates a nonlinear model, disturbances in the real application will in general lead to impulsive solutions if the system is of index higher than one.

### *15.2.2 Controllability and Observability*

If the descriptor system (15.1) or (15.2) is *regular*, then the following controllability and observability conditions are sufficient for most classical design aims. To simplify the notation, we hereafter denote a matrix with orthonormal columns spanning the right nullspace of the matrix $M$ by $S_\infty(M)$ and a matrix with orthonormal columns spanning the left nullspace of $M$ by $T_\infty(M)$. The controllability conditions are defined to be:

$$\begin{array}{ll} \mathbf{C0}: & \operatorname{rank}([\alpha E - \beta A, B]) = n \text{ for all } (\alpha, \beta) \in \mathbb{C}^2 \backslash \{(0,0)\}. \\ \mathbf{C1}: & \operatorname{rank}([\lambda E - A, B]) = n \text{ for all } \lambda \in \mathbb{C}. \\ \mathbf{C2}: & \operatorname{rank}([E, AS_\infty(E), B]) = n, \text{ where the columns of } S_\infty(E) \text{ span} \\ & \text{the null space of } E. \end{array} \tag{15.8}$$

The observability conditions are defined as the dual of the controllability conditions:

$$\begin{array}{ll} \mathbf{O0}: & \operatorname{rank}(\begin{bmatrix} \alpha E - \beta A \\ C \end{bmatrix}) = n \text{ for all } (\alpha, \beta) \in \mathbb{C}^2 \backslash \{(0,0)\}. \\ \mathbf{O1}: & \operatorname{rank}(\begin{bmatrix} \lambda E - A \\ C \end{bmatrix}) = n \text{ for all } \lambda \in \mathbb{C}. \\ \mathbf{O2}: & \operatorname{rank}(\begin{bmatrix} E \\ T_\infty^T(E) A \\ C \end{bmatrix}) = n, \text{ where the columns of } T_\infty(E) \text{ span} \\ & \text{the right null space of } E. \end{array} \tag{15.9}$$

For systems that are regular, these conditions characterize the controllability of the system. The condition **C0** ensures that for any given initial and final states of the system, $x_0$, $x_f$, there exists an admissible control that transfers the system from $x_0$ to $x_f$ in finite time [43]. Condition **C1** ensures the same for any given initial and final states $x_0$, $x_f$ belonging to the solution space of the descriptor system [5, 7]. A regular system that satisfies the conditions **C0** and **O0** is said to be *completely controllable* (C–controllable) and *completely observable* (C–observable) and has properties similar to those of standard control systems. A regular system is *strongly controllable* (S–controllable) if **C1** and **C2** hold and *strongly observable* (S–observable) if **O1** and **O2** hold. Regular systems that satisfy condition **C2** are *controllable at infinity* or *impulse controllable* [27, 42]. For these systems, impulsive modes can be excluded. Condition **C2** is closely related to the second condition in Theorem 1, which characterizes regular systems of index at most one. By the definition, a regular descriptor system of index at most one is controllable at infinity.

The controllability and observability conditions **C0**, **C1**, **C2**, and **O0**, **O1**, **O2** are all preserved under non-singular "equivalence" transformations of the pencil and under proportional state and output feedback, but **C2** is not necessarily preserved under derivative feedback. Therefore, if derivative feedback is used to modify the system dynamics, it is necessary to avoid losing controllability at infinity [5, 7].

Whilst regularity is required for controllability and observability, it is *not* needed in order to regularize the system by feedback. Many descriptor systems that are not regular can be regularized by proportional and/or derivative feedback. Conversely, systems that are regular can easily be transformed by feedback into closed-loop systems that are not regular. It is important, therefore, to establish conditions that ensure the regularity of systems under feedback and to develop numerically reliable techniques for constructing regular feedback systems of index at most one.

Theorem 1 defines conditions that must be satisfied by a closed-loop system pencil (15.3) or (15.4) for it to be regular and of index $\leq 1$. These conditions are closely related to the properties **C1**, **C2**, **O1**, **O2**, but regularity is needed for controllability and observability, whereas it is not required for regularization. In [25, 30] it was first shown that these conditions can be used to determine a closed-loop descriptor feedback system that is both regular and of index at most one, using proportional feedback. The system itself does not need to be regular to achieve this result.

In a standard system, derivative feedback does not alter the system behaviour in any way that could not be achieved by proportional feedback alone. However, for descriptor systems, it is possible that derivative feedback can decrease the susceptibility to noise and change the dynamic order of the descriptor system. One of the applications of derivative feedback is to shift infinite frequencies to finite frequencies in order to regularize and control the system. These possibilities together with the implications of Theorem 1, provided a challenge to Volker and his colleagues and motivated their initial work on feedback design for descriptor systems [5–7]. The work is based on numerically stable methods for reducing descriptor systems to condensed forms using unitary transformations. In the next section we summarize this research.

## 15.3 Regularization by Feedback for Time-Invariant Systems

The problem of regularizing a descriptor system of form (15.1) or (15.2) by feedback is defined as:

**Problem 1** Given real system matrices $E, A, B, C$, find real matrices $F$ and $G$ such that the closed-loop pencil

$$(E + BGC, A + BFC) \tag{15.10}$$

is regular and $\text{ind}(E + BGC, A + BFC) \leq 1$.

If $C = I$ this is the *state* feedback regularization problem and otherwise it is the *output* regularization feedback problem.

In the report [5], both the output and the state feedback regularization problems are investigated initially, but the published version [7] treats only the state feedback problem. A complete solution to the state feedback problem was achieved, but the output case proved to be more elusive, and a number of papers tackling this problem followed later. The state feedback problem has its own importance in real applications, so here we consider first the state feedback problem and then the output feedback problem separately.

### 15.3.1 Regularization by State Feedback

In the papers [5–7], two major contributions are made. The first provides conditions for the existence of solutions to the state feedback regularization problem. This is achieved by numerically stable transformations to condensed forms that enable the required feedback matrices to be constructed accurately in practice. The second establishes 'robust' system design techniques for ensuring that the properties of the closed-loop system pencil are insensitive to perturbations in the system matrices $E + BG, A + BF, B$.

The following theorem gives the complete solution to the state feedback regularization problem.

**Theorem 2 ([7])** *Given a system of the form (15.1) or (15.2), if rank*$([E, AS_\infty(E), B])=n$*, that is, if* **C2** *holds, then there exist real feedback matrices* $F, G \in \mathbb{R}^{m\times n}$ *such that the pencil* $(E + BG, A + BF)$ *is regular,* $\mathrm{ind}(E + BG, A + BF) \le 1$*, and rank*$(E + BG) = r$*, where* $0 \le rank([E, B]) - rank(B) \le r \le rank([E, B])$.

To establish the theorem, we compute the QR factorization of $B$ and the URV factorization [28] of $T_\infty^T(B)E$ to obtain orthogonal matrices $P$ and $Q$ such that

$$PEQ = \begin{bmatrix} E_{11} & 0 & 0 \\ E_{21} & E_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad PB = \begin{bmatrix} 0 \\ B_2 \\ 0 \end{bmatrix}, \quad PAQ = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}. \qquad (15.11)$$

Here $E_{11}$ and $B_2$ are nonsingular and $E_{22}$ is full column rank. Both $E_{11}$ and $B_2$ can be further reduced by orthogonal transformations to full-rank positive diagonal matrices. The theorem then follows by selecting feedback matrices to ensure that the closed-loop pencil

$$(E + BG, A + BF) \qquad (15.12)$$

satisfies condition (ii) of Theorem 1. If **C1** holds as well as **C2**, the resulting closed-loop system is then strongly controllable [7]. This system could be reduced further to a standard system, but in this case the feedback matrices would have to be selected with care to ensure that the reduction is numerically stable.

Additional results on state feedback regularization using only proportional or derivative feedback are also given in [5–7]. The existence of regularizing proportional state feedback designs is easily shown in the case where **C2** holds using the condensed form (15.11). For the derivative feedback case, the results are the same as in Theorem 2, with the exception that the potential rank of the matrix $(E + BG)$ is now restricted from below. The maximum rank that can be obtained remains equal to rank$([E, B])$.

In general the feedback designs that regularize the system (15.1) or (15.2) are not uniquely determined by Theorem 2 and additional degrees of freedom in the design can be exploited to obtain robustness and stability of the system as well as regularity. For robustness we want the system to remain regular and of index at most one under perturbations to the closed-loop system matrices. From Theorem 1 the closed-loop pencil (15.12) is regular and of index $\leq 1$ if and only if

$$\text{rank}(\begin{bmatrix} E + BG \\ T_\infty^T(E + BG)(A + BF) \end{bmatrix}) = n. \tag{15.13}$$

It is well-known that for a matrix with full rank, the distance to the nearest matrix of lower rank is equal to its minimum singular value [28]. Hence for robustness of the closed-loop pencil (15.12) we aim to select $F$ and $G$ such that the pencil is unitarily equivalent to a pencil of the form $\alpha S_1 - \beta S_2$ where

$$S_1 = \begin{bmatrix} \Sigma_R & 0 \\ 0 & 0 \end{bmatrix}, \quad S_2 = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & \Sigma_L \end{bmatrix}, \tag{15.14}$$

and the assigned singular values of $\Sigma_R, \Sigma_L$ are such that the condition numbers of $\Sigma_R$ and $\Sigma_L$ are minimal. This choice ensures regularity of the system and maximizes a lower bound on the minimum singular value of (15.13), whilst retaining an upper bound on the magnitude of the gains $F$ and $G$. Details of the algorithm to achieve these results are given in [5, 7, 39]. This choice also ensures that the reduction of the closed-loop descriptor system to a standard form is as well-conditioned as possible. In practice such robust systems also have improved performance characteristics (see [40]).

In addition to regularity, it is desirable to ensure that a system design has stability and even that it has specified finite eigenvalues. The following result, shown in [5, 7], holds for descriptor systems.

**Theorem 3 ([5, 7])** *Given a system of the form (15.1) or (15.2), if the conditions* **C1** *and* **C2** *hold and $r$ is an integer such that $0 \leq rank([E, B]) - rank(B) \leq r \leq rank([E, B])$, then for any arbitrary set $\mathcal{S}$ of $r$ self-conjugate finite poles there exist feedback matrices $F, G \in \mathbb{R}^{m \times n}$ such that the pencil $(E + BG, A + BF)$ is regular,* $\text{ind}(E + BG, A + BF) \leq 1$, *$rank(E + BG) = r$ and all pairs in $\mathcal{S}$ are the finite generalized eigenvalues of the pencil* $(E + BG, A + BF)$.

For robustness of the closed-loop system, we require the maximum number of finite eigenvalues to be assigned and both the finite and infinite eigenvalues to be insensitive to perturbations in the closed-loop system matrices. One strategy for obtaining a robust solution to the eigenvalue assignment problem for a descriptor system is to apply derivative feedback alone to obtain a robust, regular index-one system with $\text{rank}(E + BG) = r = \text{rank}([E, B])$ using singular value assignment, and then to use *robust* proportional state feedback to assign $r$ finite eigenvalues to the system. The problem of eigenvalue assignment by proportional state feedback in descriptor systems is treated in [17, 25, 30]. Techniques for robust eigenstructure assignment ensuring that the assigned eigenvalues of the closed-loop system are insensitive to perturbations in the system matrices are established in [29, 30, 38].

The problem of designing an observer, or state-estimator, is the dual of the state feedback control problem. An observer is an auxiliary dynamical system designed to provide estimates $\hat{x}$ of all the states $x$ of the system (15.1) or (15.2) using measured output data $y$ and $\dot{y}$. The estimator is a closed-loop system that is driven by the differences between the measured outputs and derivatives of the system and their estimated values. The system pencil is given by

$$(E + GC, A + FC), \tag{15.15}$$

where the matrices $F$ and $G$ must be selected to ensure that the response $\hat{x}$ of the observer converges to the system state $x$ for any arbitrary starting condition; that is, the system must be asymptotically stable. By duality with the state feedback problem, it follows that if the condition **O2** holds, then the matrices $F$ and $G$ can be chosen such that the corresponding closed-loop pencil (15.15) is regular and of index at most one. If condition **O1** also holds, then the closed-loop system is S-observable. Furthermore, the remaining freedom in the system can be selected to ensure the stability and robustness of the system and the finite eigenvalues of the system pencil can be assigned explicitly by the techniques described for the state feedback control problem.

### *15.3.2 Disturbance Decoupling by State Feedback*

In practice control systems are subject to disturbances that may include modelling or measurement errors, higher order terms from linearization, or unknown inputs to the system. For such systems it is important to design feedback controllers and observers that suppress the disturbance so that it does not affect the input-output of the system. In research strongly inspired by the earlier work of Volker and his colleagues on state feedback regularization, the problem of disturbance decoupling is treated in [20, 21].

In the case that disturbances are present, the linear time-invariant system takes the form

$$\begin{aligned} E\dot{x}(t) &= Ax(t) + Bu(t) + Hq(t), \quad x(t_0) = x_0, \\ y(t) &= Cx(t), \end{aligned} \tag{15.16}$$

or

$$Ex(k+1) = Ax(k) + Bu(k) + Hq(k), \quad x(0) = x_0,$$
$$y(k) = Cx(k), \tag{15.17}$$

where $E, A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{q\times n}$, $H \in \mathbb{R}^{n\times p}$, and $q(\cdot)$ represents a vector of disturbances.

To suppress the disturbances, a state feedback controller is used to modify the input-output map, or transfer function, of the system. The disturbance decoupling problem for the descriptor system (15.16) or (15.17) is then to find proportional and derivative feedback matrices $F, G$ such that the closed-loop pencil $(E+BG, A+BF)$ is regular and of index at most one and

$$T(s) \equiv C(s(E+BG) - (A+BF))^{-1}H \equiv 0, \tag{15.18}$$

where $T(s)$ defines the transfer function of the closed-loop system from the input disturbance $q(\cdot)$ to the output $y(\cdot)$. This condition ensures that the disturbance does not affect the input-output response of the closed-loop system for any choice of the input control $u(\cdot)$. Necessary and sufficient conditions for the existence of a solution to this problem are established in [21]. In addition, conditions are derived under which the feedback matrices can be chosen such that the closed-loop system is also stable. The derivations are constructive and a numerically stable algorithm is given for implementing the procedure.

In [20] the problem of designing a disturbance-decoupled observer system for estimating (a subset of) the states of the system (15.16) or (15.17) is developed. The aim is to select feedback matrices such that the closed-loop observer is regular and of index at most one and such that the disturbances have no influence on the error in the estimated states of the system. Necessary and sufficient conditions are derived for the existence of disturbance-decoupled observers of this form and also for the observer to be stable, ensuring that the estimated states converge over time to the corresponding states of the original system. The main results are established constructively and are again based on a condensed form that can be computed in a numerically stable way using unitary matrix transformations.

### *15.3.3 Regularization by Output Feedback*

The output feedback regularization problem is to find derivative and state output feedback matrices $F, G$ such that the closed-loop system pencil (15.10) is regular and has index at most one.

Meeting at the Institute for Mathematics and Its Applications in Minnesota in 1992 and following up the earlier research on regularization, Volker and his colleagues tackled the difficult output feedback problem in earnest. The results of the research are published in an extensive report [8] and in later papers [9, 10]. In these papers, a condensed form of the descriptor system pencil is derived that displays the conditions under which the system can be transformed into a regular system of index at most one by output feedback using numerically stable orthogonal transformations. For proportional output feedback the solution to the design problem follows immediately from this condensed form. Necessary and sufficient conditions for a feedback matrix $F \in \mathbb{R}^{m\times p}$ to exist such that the pencil $(E, A + BFC)$ is regular and has index at most one are given by **C2** and **O2**. The closed-loop system is then S-controllable and S-observable if **C1** and **O1** also hold [8, 10].

For combined derivative and proportional output feedback, it is also established in [8, 10], using the condensed form, that if **C2** and **O2** hold, then there exist matrices $F, G \in \mathbb{R}^{m\times p}$ such that the closed-loop pencil $(E + BGC, A + BFC)$ is regular, has index at most one, and $\text{rank}(E + BGC)$ lies in a given range. Techniques such as those used for the state feedback problem to ensure optimal conditioning, or robustness of the closed-loop system to perturbations, are also described in [8, 39].

With proportional output feedback alone, if the system has index $\leq 1$, then the number of finite eigenvalues of the closed-loop pencil $(E, A + BFC)$ is fixed at $r = \text{rank}(E)$. With derivative and proportional feedback, the system pencil becomes $(E + BGC, A + BFC)$ and the system properties that depend on the left and right null spaces of $E$, such as **C2** and **O2**, may be altered and the rank of $E + BGC$ may be increased or decreased from that of $E$. If the closed-loop system is regular with index $= 1$, then the system may be separated into $r = \text{rank}(E + BGC)$ differential or difference equations and $n - r$ purely algebraic equations. In applications, it may be useful to have more or fewer differential or difference equations. A complete characterization of the achievable ranks $r$ for systems that are regular and of index at most one is, therefore, desirable.

Variations of the condensed form of [8, 10] that can be obtained by stable orthogonal transformations have subsequently been derived in [11, 18, 19, 22] and different approaches to the output feedback problem have been developed. A comprehensive summary of the extended results, based on these condensed forms, is given in [3]. The main result can be expressed as follows.

**Theorem 4 ([3, 11, 18, 19, 22])** *Let* $T_a = T_\infty(ES_\infty(C))$, $\quad S_a = S_\infty(T_\infty^T(B)E)$, *and*

$$T_b = T_\infty([E, AS_\infty(\begin{bmatrix} E \\ C \end{bmatrix}), B]), \quad S_b = S_\infty(\begin{bmatrix} E \\ T_\infty^T([E, B])A) \\ C \end{bmatrix}).$$

*Then the following statements are equivalent:*

*(i) There exist feedback matrices* $F, G \in \mathbb{R}^{m\times p}$ *such that the closed-loop pencil* $(E + BGC, A + BFC)$ *is regular and of index at most one.*

*(ii)* $T_a^T A S_b$ *has full column rank,* $T_b^T A S_a$ *has full row rank and*

$$rank(T_\infty^T([E, B])AS_\infty(\begin{bmatrix} E \\ C \end{bmatrix})) \geq n - rank(\begin{bmatrix} E & B \\ C & 0 \end{bmatrix}).$$

*Moreover, if the closed-loop pencil* $(E + BGC, A + BFC)$ *is regular and of index at most one with* $r = rank(E + BGC)$ *then*

$$rank\ ([E, B]) + rank(\begin{bmatrix} E \\ C \end{bmatrix}) - rank(\begin{bmatrix} E & B \\ C & 0 \end{bmatrix}) \leq r \leq$$

$$\leq rank([E, B]) - rank(T_a^T AS_b) \equiv rank(\begin{bmatrix} E \\ C \end{bmatrix}) - rank(T_b^T AS_a).$$

The matrices in the theorem and their ranks are easily obtained from the following condensed form [3, 18, 22], where $U, V, \in \mathbb{R}^{n\times n}$, $P \in \mathbb{R}^{m\times m}$ and $W \in \mathbb{R}^{p\times p}$ are orthogonal matrices:

$$UEV = \begin{array}{c} \\ t_1 \\ t_2 \\ t_3 \\ t_4 \\ t_5 \end{array} \begin{array}{c} \begin{array}{ccccc} t_1 & t_2 & t_3 & s_4 & s_5 \end{array} \\ \begin{bmatrix} E_{11} & 0 & 0 & 0 & 0 \\ E_{21} & E_{22} & 0 & 0 & 0 \\ E_{31} & E_{32} & E_{33} & E_{34} & 0 \\ E_{41} & E_{42} & 0 & E_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array},$$

$$UBP = \begin{array}{c} \\ t_1 \\ t_2 \\ t_3 \\ t_4 \\ t_5 \end{array} \begin{array}{c} \begin{array}{cc} t_3 & t_4 \end{array} \\ \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ B_{31} & B_{32} \\ 0 & B_{42} \\ 0 & 0 \end{bmatrix} \end{array}, \tag{15.19}$$

$$WCV = \begin{array}{c} \\ s_4 \\ t_1 \end{array} \begin{array}{c} \begin{array}{ccccc} t_1 & t_2 & t_3 & s_4 & s_5 \end{array} \\ \begin{bmatrix} C_{11} & C_{12} & 0 & C_{14} & 0 \\ C_{21} & 0 & 0 & 0 & 0 \end{bmatrix} \end{array},$$

where the blocks $E_{11}$, $C_{21}$, $E_{22}$, $E_{33}$, $B_{31}$, $B_{42}$, and $C_{14}$ are nonsingular.

Theorem 4 follows directly from the condensed form (15.19). The theorem gives a complete characterization of the possible ranks of $E + BGC$ for systems that are regular and of index at most one. Additional results on output feedback regularization using only proportional or derivative feedback are also presented in the references. Corresponding results for observer designs can be determined directly by duality.

In practice, it is desirable not only that the closed-loop descriptor system is regular and has index at most one, but also that it is robust in the sense that it is insensitive to perturbations in the system matrices. As in the state feedback case, the aim is to choose $F$ and $G$ such that the closed-loop pencil is unitarily equivalent to a pencil of the form (15.14) where the matrices $\Sigma_R$ and $\Sigma_L$ are well-conditioned for inversion. This choice ensures that the reduction of the closed-loop system to a standard system is computationally reliable. Partial solutions to this problem are provided in [8, 9], based on the results of [24], and an algorithm is given for minimizing upper bounds on the conditioning of $\Sigma_R$ and $\Sigma_L$ using unitary transforms to condensed forms. This procedure generally improves the conditioning of the closed-loop system.

### 15.3.4 Regularization by Mixed Output Feedback

Systems where different states and derivatives can be output arise commonly in mechanical multi-body motion. In such systems, velocities and accelerations can often be measured more easily than states (e.g. by tachometers or accelerometers). Time-invariant systems of this type can be written in the form:

$$\begin{aligned} E\dot{x}(t) &= Ax(t) + Bu(t), \quad x(t_0) = x_0, \\ y_1(t) &= Cx(t), \\ y_2(t) &= \Gamma\dot{x}, \end{aligned} \tag{15.20}$$

or, in the discrete time case

$$\begin{aligned} Ex(k+1) &= Ax(k) + Bu(k), \quad x(0) = x_0, \\ y_1(k) &= Cx(k), \\ y_2(k+1) &= \Gamma x(k+1), \end{aligned} \tag{15.21}$$

where $E, A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times m}, C \in \mathbb{R}^{p\times n}, \Gamma \in \mathbb{R}^{q\times n}$. In this case we are interested in proportional and derivative control of the form $u(t) = Fy_1(t) - G\dot{y}_2(t)$ or $u(k) = Fy_1(k) - Gy_2(k+1)$, where $F$ and $G$ are chosen to give the closed-loop system pencil

$$(E + BG\Gamma, A + BFC) \tag{15.22}$$

desired properties. In particular the aim is to ensure that the closed-loop system is regular and of index at most one. The mixed output feedback regularization problem for this system is stated explicitly as follows.

**Problem 2** For a system of the form (15.20) or (15.21), give necessary and sufficient conditions to ensure the existence of feedback matrices $F \in \mathbb{R}^{m\times p}$ and $G \in$

$\mathbb{R}^{m\times q}$ such that the closed-loop system pencil $(E + BG\Gamma, A + BFC)$ is regular and $\mathrm{ind}(E + BG\Gamma, A + BFC) \leq 1$.

The mixed feedback regularization problem and its variants, which are significantly more difficult than the state and output feedback regularization problems, have been studied systematically by Volker and his colleagues in [22, 37]. These have not been investigated elsewhere, although systems where different states and derivatives are output arise commonly in practice.

Examples frequently take the second order form

$$M\ddot{z} + K\dot{z} + Pz = B_1\dot{u} + B_2u \tag{15.23}$$

and can be written in the generalized state space form

$$\begin{bmatrix} M & 0 \\ K & I \end{bmatrix}\begin{bmatrix} \dot{z} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} 0 & I \\ -P & 0 \end{bmatrix}\begin{bmatrix} z \\ v \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}u. \tag{15.24}$$

If the velocities $\dot{z}$ of the states of the system can be measured, then the states $v = M\dot{z} - B_1u$ are also available and the outputs

$$y_1 = Cx = \begin{bmatrix} 0 & I \end{bmatrix}\begin{bmatrix} z \\ v \end{bmatrix}, \; y_2 = \Gamma\dot{x} = \begin{bmatrix} I & 0 \end{bmatrix}\begin{bmatrix} \dot{z} \\ \dot{v} \end{bmatrix} \tag{15.25}$$

can be used separately to modify the system by either proportional or derivative feedback, respectively. The corresponding closed-loop state-space system matrices then take the form

$$E + BG\Gamma = \begin{bmatrix} M + B_1G & 0 \\ K + B_2G & I \end{bmatrix}, \quad A + BFC = \begin{bmatrix} 0 & I + B_1F \\ -P & B_2F \end{bmatrix}. \tag{15.26}$$

Different effects can, therefore, be achieved by feeding back either the derivatives $\dot{z}$ or the states $v$. In particular, in the case where $M$ is singular, but $\mathrm{rank}[M, B_1] = n$, the feedback $G$ can be chosen such that $M + B_1G$ is invertible and well-conditioned [7], giving a *robust* closed-loop system that is regular and of index zero. The feedback matrix $F$ can be chosen separately to assign the eigenvalues of the system [30], for example, or to achieve other objectives.

The complete solution to the mixed output feedback regularization problem is given in [22]. The theorem and its proof are very technical. Solvability is established using condensed forms derived in the paper. The solution to the output feedback problem given in Theorem 4 is a special case of the complete result for the mixed output case given in [22]. The required feedback matrices are constructed directly from the condensed forms using numerically stable transformations.

Usually the design of the feedback matrices still contains freedom, however, which can be resolved in many different ways. One choice is to select the feedbacks such that the closed-loop system is robust, or insensitive to perturbations, and, in particular, such that it remains regular and of index at most one under perturbations (due, for example, to disturbances or parameter variations). This choice can also be shown to maximize a lower bound on the stability radius of the closed-loop system [13]. Another natural choice would be to use minimum norm feedbacks, which would be a least squares approach based on the theory in [24]. This approach is also investigated in [22, 37]. The conclusion is that although minimum norm feedbacks are important in other control problems, such as eigenvalue assignment or stabilization because they remove ambiguity in the solution in a least squares sense, for the problem of regularization they do not lead to a useful solution, unless the rank of E is decreased. Heuristic procedures for obtaining a system by output feedback that is robustly regular and of index at most one are discussed in [8, 9, 39].

## 15.4 Regularization of Time-Varying and Nonlinear Descriptor Systems

Feedback regularization for time-varying and nonlinear descriptor systems provided the next target for Volker's research. Extending the previous work to the time-varying case was enabled primarily by the seminal paper on the analytic singular value decomposition (ASVD) published by Volker and colleagues in 1991 [4]. The ASVD allows condensed forms to be derived for the time-varying problem, just as the SVD does for the time-invariant case, and it provides numerically stable techniques for determining feedback designs.

The continuous form of the time-varying descriptor system is given by the implicit system

$$\begin{aligned} E(t)\dot{x}(t) &= A(t)x(t) + B(t)u(t), \quad x(t_0) = x_0, \\ y(t) &= C(t)x(t), \end{aligned} \tag{15.27}$$

where $E(t), A(t) \in \mathbb{R}^{n\times n}, B(t) \in \mathbb{R}^{n\times m}, C(t) \in \mathbb{R}^{p\times n}$ are all *continuous* functions of time and $x(t)$ is the state, $y(t)$ is the output, and $u(t)$ is the input or control of the system. (Corresponding discrete-time systems with time-varying coefficients can also be defined, but these are not considered here.)

In this general form, complex dynamical systems including constraints can be modelled. Such systems arise, in particular, as linearizations of a general nonlinear control system of the form

$$\begin{aligned} \mathscr{F}(t, x, \dot{x}, u) &= 0, \qquad x(t_0) = x_0, \\ y &= \mathscr{G}(t, x), \end{aligned} \tag{15.28}$$

where the linearized system is such that $E(t), A(t), B(t)$ are given by the Jacobians of $\mathscr{F}$ with respect to $\dot{x}, x, u$, respectively, and $C(t)$ is given by the Jacobian of $\mathscr{G}$ with respect to $x$ (see [31]).

For the time-varying system (15.27) and the nonlinear system (15.28), the system properties can be modified by time-varying state and output feedback as in the time-invariant case, but the characterization of the system, in particular the solvability and regularity of the system, is considerably more complicated to define than in the time-invariant case and it is correspondingly more difficult to analyse the feedback problem. The ultimate goal remains, however, to obtain stable numerical approaches to the problem using time-varying orthogonal transformations to condensed forms.

If time-varying orthogonal transformations $U(t), V(t), W(t), Y(t)$ are applied to the system (15.27), and all variables are assumed to be time-dependent, then the system becomes

$$\begin{aligned} U^T EV\dot{z} &= (U^T AV - U^T EVS)z + U^T BWw, \\ \tilde{y} &= YCVz, \end{aligned} \tag{15.29}$$

where $x(t) = V(t)z(t)$, $u(t) = W(t)w(t)$, $\tilde{y}(t) = Y(t)y(t)$ and $S(t) = V(t)^T \dot{V}(t)$ is a skew-symmetric matrix. We see that applying time-varying transformations alters the system matrix $A$, and this must be taken into account where reducing the system to equivalent condensed forms.

In [1, 2] it is shown that the ASVD can be used to produce a condensed form for system (15.27), similar to the form derived in [10]. A time-varying system is defined here to be regular and of index at most one if the conditions of Theorem 1 hold for all $t$ and the system can be decoupled into purely dynamic and algebraic parts. In order to establish regularizability of system (15.27), the strong assumption is made that rank$(E(t))$ is constant and that ranks in the condensed form are also constant. Time-varying output feedback matrices are then constructed to produce a closed-loop pointwise regular pencil of the form (15.10) with index at most one. The rank assumptions ensure the solvability of the closed-loop system. The system matrices $E, A, B, C$, are assumed to be analytic functions of $t$, but these conditions can be relaxed provided the ASVD decompositions remain sufficiently smooth.

In the papers [12, 31], a much deeper analysis of the regularization problem is developed. Detailed solvability conditions for the time-varying system (15.27) are established and different condensed forms are derived, again using the ASVD. Constant rank assumptions do not need to be applied, although the existence of smooth ASVDs are required. The analysis covers a plethora of different possible behaviours of the system. One of the tasks of the analysis is to determine redundancies and inconsistencies in the system in order that these may be excluded from the design process. The reduction to the condensed forms displays all the invariants that determine the existence and uniqueness of the solution. The descriptor system is then defined to be regularizable if there exist proportional or derivative feedback matrices such that the closed-loop system is uniquely solvable for every consistent initial state vector and any given (sufficiently smooth) control. Conditions for the system to be regularizable then follow directly from the condensed forms.

In [31] a behaviour approach is taken to the linear time-varying problem where state, input and output variables are all combined into one system vector and the combined system is studied. This approach allows inhomogeneous control problems also to be analysed. Instead of forming a derivative array from which the system invariants and the solutions of the original system can be determined, as in [14, 16], the behaviour approach allows the invariants to be found without differentiating the inputs and thus avoids restrictions on the set of admissible controls. Reduction of the behaviour system to condensed form enables an underlying descriptor system to be extracted and the conditions under which this system can be regularized by proportional and derivative feedback are determined. The construction of the feedback matrices is also described. The reduction and construction methods rely on numerically stable equivalence transformations.

More recent work of Volker and his colleagues [15] extends the behaviour approach to a general implicit nonlinear model of the form

$$\mathscr{F}(t, x, \dot{x}, u, y) = 0, \quad x(t_0) = x_0. \tag{15.30}$$

The property of ‘strangeness-index’ is defined and used in the analysis. This property corresponds to ‘index’, as defined for a linear time-invariant descriptor system, and ‘strangeness-free’ corresponds to the condition that a time-invariant system is of index at most one. Conditions are established under which a behaviour system can be reduced to a differential-algebraic system, and after reinterpretation of the variables, to a typical implicit nonlinear system consisting of differential and algebraic parts. Locally linear state feedback can then be applied to ensure that the system is regular and strangeness-free. Standard simulation, control, and optimization techniques can be applied to the reformulated feedback system. Further details of Volker’s work on nonlinear differential–algebraic systems can be found in other chapters in this text.

## 15.5 Conclusions

We have given here a broad-brush survey of the work of Volker Mehrmann on the problems of regularizing descriptor systems. The extent of this work alone is formidable and forms only part of his research during his career. We have concentrated specifically on results from Volker’s own approaches to the regularity problem. The primary aim of his work has been to provide stable numerical techniques for analyzing and constructing control and state estimation systems and for ensuring that these systems are robust. The reduction of systems to condensed forms using orthogonal equivalence transformations forms the major theme in this work. Whilst some of the conclusions described here can also be obtained via other canonical or condensed forms published in the literature, these cannot be derived by sound numerical methods and the required feedbacks cannot be generated from these by backward stable algorithms. Volker’s work has therefore had a real practical impact on control system design in engineering as well as producing some beautiful theory. It has been a pleasure for us to be involved in this work.

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